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Asymptotic methods for linear Markovian iterative convergence analysis

Doctor thesis

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Annotation

The doctor thesis is devoted to the asymptotic methods for Markovian iterative procedures convergence analysis, which are presented in a form of linear difference equation $x_t = A(y_t)x_{t-1},\ t \in \mathbb{N}$ in $\mathbb{R}^n$, where $\{A(y),\ y \in \mathbb{Y}\}$ is continuous $n \times n$ matrix function on the metric compact $\mathbb{Y}$, and $\{y_t,\ t \in \mathbb{N}\}$ is a homogeneous ergodic Feller Markov chain with phase space $\mathbb{Y}$. The proposed method and algorithm are based on construction of discrete semigroup for the covariance matrices and spectral analysis of the semigroup generator. This permits to apply well developed spectral theory of positive operators in Banach space and to elaborate a special version of the second Lyapunov method for mean square asymptotic stability analysis. Especially a semigroup conception may be successfully used for asymptotical analysis of difference equations with near to constant coefficients. In that case one can employ the powerful tools of Kato perturbation theory for spectral projector decomposition and succeed in a calculation of mean square Lyapunov index for the above discrete dynamical system. The proposed method and algorithm of asymptotical analysis of small random perturbations are illustrated not only by theoretical examples, but also by analysis of fourth moments GARCH model, which is high-usage tool of contemporary financial econometrics.

Anotācija

Promocijas darbs ir veltīts lineāru Markova iterāciju, kuras var izteikt lineāru diferenču vienādojumu formā $x_t = A(y_t)x_{t-1},\ t \in \mathbb{N}$ telpā $\mathbb{R}^n$, konverģences analīzes asimptotiskām metodēm, kur $\{A(y),\ y \in \mathbb{Y}\}$ ir nepārtraukta $n \times n$ matricu funkcija metriskā kompaktā telpā $\mathbb{Y}$, $\{y_t,\ t \in \mathbb{N}\}$ ir homogēna ergodiska Fellera Markova ķēde ar fāzu telpu $\mathbb{Y}$. Piedāvātā metode un algoritms ir balstīti uz diskrētas pusgrupas konstruēšanu kovariācijas matricai un pusgrupas ģeneratora spektrālo analīzi. Tas ļauj pielietot labi attīstīto spektrālo teoriju pozitīviem operatoriem Banaha telpā un izstrādāt otrās Ļapunova metodes speciālu versiju asimptotiskās stabilitātes vidējā kvadrātiskā nozīmē analīzei. Galvenokārt pusgrupas jēdziens var veiksmīgi tikt pielietots diferenču vienādojumu ar gandrīz konstantiem koeficientiem asimptotiskai analīzei. Tādā gadījumā spektrālā projektora dekompozīcijai var izmantot Kato perturbācijas teorijas efektīvos līdzekļus un veiksmīgi aprēķināt vidējo kvadrātisko Ļapunova indeksu iepriekš minētajai diskrētai dinamiskai sistēmai. Piedāvātā metode un algoritms mazu gadījuma perturbāciju asimptotiskai analīzei ir ilustrēti ne tikai ar teorētiskiem piemēriem, bet arī ar ceturto momentu analīzi GARCH modelim, kas ir plaši lietots instruments mūsdienu finansu ekonometrijā.

1 Introduction

This doctor thesis is devoted to the asymptotic methods for linear Markovian iterative convergence analysis. The asymptotic behavior of linear difference equations with almost constant Markov coefficients subjected by a small parameter is investigated.

The asymptotical analysis of stochastic dynamical systems is mainly based on such qualitative methods as: the second Lyapunov method, limit theorems of probability theory, perturbation theory and others. For mean square equilibrium stability analysis the second Lyapunov method and perturbation theory of linear continuous operators in Banach space can be applied.

The first results on asymptotic theory of random processes appeared in the publications of Gikhman and Skorokhod ([20], [22], [23], [24], [25]) at the end of 1950s. Discrete dynamical systems with random parameters in mathematical nowadays literature appear relatively recently. In 1972 Vazan [58] described iterative methods for algebraic equations in case a noise exists. Also in 1972 the authors Nevelson and Hasminskij [51] successfully used the idea of the second Lyapunov method for asymptotic analysis of iterative stochastic procedures. In more details the use of limit theorems in the asymptotic analysis of solutions of difference equations with random parameters is introduced by Anisimov [1].

Stability of Markov processes are investigated by such authors as I. I. Gikhman, R. Z. Hasminskij, M. B. Nevelson, H. J. Kushner, A. V. Skorokhod, Ye. F. Carkovs, M. L. Sverdan and others (for example, [20], [22], [23], [39], [51], [56]). Theory of asymptotic stability analysis developed substantially thanks to works of L. Arnold, V. I. Oseledets, M. B. Nevelson, R. Z. Hasminskij, H. J. Kushner. The interplay between characterization and approximation or convergence problems for Markov processes is the central theme of Ethier and Kurtz [20]. H. J. Kushner has contributed to many areas of stochastic systems theory and applications (for example, [39], [40]). He has developed the main current numerical methods for stochastic control problems in continuous time [38]. Many researches on random dynamical systems are performed by L. Arnold [3]-[5].

Kesten in his work [35] studied the limit distribution of the solution Y_n of the difference equation $Y_n = M_n Y_{n-1} + Q_n$, $n \ge 1$, where M_n and Q_n are random matrices. The conditions for the exponential convergence of $M_1 M_2 \dots M_n$ to 0 in the special case is given by Konstantinov and Nevelson [36].

L. Aceto, R. Pandolfi and D. Trigiante studied the linear difference equations depending on a complex parameter [1]. By using the fact that the associated polynomials are solutions of a difference equation, they carried out a complete analysis for the class of linear multistep methods.

Stochastic difference equations are one of the basic tools for analysis of time series. In most cases it is assumed that time series have conditional Gaussian distribution with constant variance. Therefore its mathematical model can be represented in a form of linear inhomogeneous iteration procedure in $\mathbb{R}^n$:

$$X_t = FX_{t-1} + \eta_t, \tag{1.1}$$

where $\{\eta_t, t \in \mathbb{Z}\}$ - a sequence of identically distributed random variables in $\mathbb{R}^n$ with mean value zero and covariance matrix Σ (so called "residuals").

It is well known [28], [42], [45] that under condition $\sigma(F) \cap \{|z| = 1\} = \varnothing$ (here and further $\sigma(\cdot)$ denotes a spectrum of a matrix or an operator) there exists unique satisfying to (1.1) stationary time series $\{\hat{x}_t, t \in \mathbb{Z}\}$. Let X_t be an arbitrary satisfying to (1.1) iteration and $Y_t := X_t - \hat{x}_t$. This random sequence satisfies deterministic recurrent procedure $Y_t = FY_{t-1}$ and therefore the difference $Y_t := X_t - \hat{x}_t$ converges to zero with $t \to \infty$ if and only if $\sigma(F) \subset \{|z| < 1\}$. In this case one says that (1.1) defines converging iterative procedures or the above stationary solution of (1.1) is asymptotically stable.

However many problems of contemporary econometrics have to model residuals in (1.1) as a product $\eta_t = \Sigma_t \xi_t$, where $\{\xi_t, t \in \mathbb{Z}\}$ is a sequence of identically distributed random variables in $\mathbb{R}^n$ with mean value zero and unit covariance matrix, but matrices $\{\Sigma_t^2, t \in \mathbb{Z}\}$ (conditional covariance) are defined as a solution of difference equations with coefficients linear dependent on ξ_t (models VecGARCH [28]). For example, in scalar models like GARCH(p,q) (*Generalized Auto Regressive Conditional Heteroskedasticity*) conditional variance of residuals satisfies an equality

$$\sigma_t^2 = \varphi_0 + \sum_{k=1}^{q} \varphi_k \sigma_{t-k}^2 + \sum_{j=1}^{q} \theta_j \sigma_{t-j}^2 \xi_{t-j} \ , \qquad (1.2)$$

where $\varphi_0 > 0$, $\varphi_k \geq 0$, $\theta_j \geq 0$, $j = 1, \ldots, q$. It is proved [7] that the stationary time series $\left\{ \hat{\sigma}_t^2, \, t \in \mathbb{Z} \right\}$ defined by (1.2) exists, and $\mathrm{E}\left\{ \hat{\sigma}_t^2 \right\} < \infty$, if and only if for any other satisfies the equality (1.2) time series $\left\{ \sigma_t^2, \, t \in \mathbb{Z} \right\}$, mathematical expectation of deviation $x_t := \hat{\sigma}_t^2 - \sigma_t^2$ tends to zero with $t \to \infty$. One can find the above mentioned second moment in turn-key form $\mathrm{E}\left\{ \hat{\sigma}_t^2 \right\} = \varphi_0 \left(1 - \sum_{k=1}^{q} \varphi_k - \sum_{j=1}^{q} \theta_j \right)^{-1}$ and to insure that satisfying (1.2) stable stationary solution with second moment exists if and only if $\sum_{k=1}^{p} \varphi_k + \sum_{j=1}^{q} \theta_j < 1$. Not so difficult to write scalar iterative equation for x_t or vector equation for $X_t := \left\{ x_t, x_{t-1}, \ldots, x_{t-q} \right\}$ with random matrix $F_t := F\left(\xi_t, \xi_{t-1}, \ldots, \xi_{t-q} \right)$ and to formulate the above problem as a convergence problem for defined by equation $X_t = F_t X_{t-1}$ iterative procedures.

It should be mentioned that all parameters of above defined equations (1.1) and (1.2) can be determined by given sampling, using the least square method [49], [59]. Of course, the existence of asymptotic stable stationary solution having the second moment for variance equation is a main assumption, that is, convergence of matrices $M_4(t) := E\left\{ X_t X_t^T \right\}$ to some constant matrix M_4, if $t \to \infty$. This question leads to the analysis of moment behavior for corresponding homogeneous equation. The assumption about independence of sequence $\left\{ \xi_t, t \in \mathbb{Z} \right\}$ elements allows rather easy to obtain its necessary and sufficient conditions in a form convenient for use [13]. However in contemporary finance econometrics mostly are used regression models with uncertainty given in a form of random sequences in discrete state spaces. For example, in Cox-Rubinstein model [14] in the analysis of options' prices fixing possibility a market dynamics are determined by behavior of stocks, which can be represented in a form of iteration procedure $S_t = \zeta_t S_{t-1}$, where interest rates $\left\{ \zeta_t, t \in \mathbb{Z} \right\}$ are independent and have only two values in each time moment: either up or down. Even in such a simple case a sequence, which defines dynamics of such derivative securities like options, futures and so on at a securities market, is not only the sequence with random coefficients as independent multipliers, but with coefficients having Markov property. If it would be interpreted using regression models with type GARCH residuals, it will be as modeling of a sequence of residuals' conditional variances in a form of linear difference equations with Markov coefficients in $\mathbb{R}^n$:

$$x_t = A(\xi_t)\,x_{t-1} \tag{1.3}$$

where $\{\xi_t, t \in \mathbb{Z}\}$ is a homogeneous Markov chain with phase space $\mathbb{Y}$ and transition probabilities $P(y,dz)$. The problem we have to deal with is not only convergence to zero of any iteration defined by (1.3), but also convergence to zero of the unconditional second moments of the above stochastic recurrent procedure as *t* tends to infinity.

The research object is a linear difference equation with Markov coefficients in space $\mathbb{R}^n$:

$$x_t = A(y_t)\,x_{t-1},\ t \in \mathbb{N} \tag{1.4}$$

where $\{A(y),\ y \in \mathbb{Y}\}$ is continuous $n \times n$ matrix function on the metric compact $\mathbb{Y}$, $\sup\limits_y \|A(y)\| = const < \infty$; $\{y_t,\ t \in \mathbb{N}\}$ is a homogeneous exponentially ergodic Feller Markov chain with phase space $\mathbb{Y}$, invariant measure $\mu(dy)$ and transition probability $p(y,dz)$. Under initial conditions $x_k = x$, $y_k = y$ the vector function $x_t(k,x,y) = X(t,k,y)\,x$, where $X(t,k,y) := \prod\limits_{m=k+1}^{t} A(y_m)$, satisfies the given difference equation, and it is called as a solution of this equation and matrix function $X(t,k,y)$ as a Cauchy matrix.

The investigated problem is the second moments behavior of the above defined linear difference equation (1.4) with Markov coefficients as *t* tends to infinity, that is, convergence to zero of the unconditional second moments of the above stochastic recurrent procedure as *t* tends to infinity $E\left\{\left|x_t(k,x,y)\right|^2\right\} \xrightarrow[t\to\infty]{} 0$.

The aim of the doctor thesis is elaborate a methodology which can be used for the dynamics analysis of the second moments matrix of the above defined difference equation (1.4) solution, that is, behavior of a matrix as matrix function of argument *t*:

$E_{x,y}\left\{x_t x_t^T\right\} =: Q_t$, in case if the matrix function $\left\{A(y),\ y \in \mathbb{Y}\right\}$ is near to constant and can be given in a form of uniformly converging sequence $A(y) := A_0 + \varepsilon \sum_{k=0}^{\infty} \varepsilon^k A_{k+1}(y)$, where $\varepsilon \in (0,1)$ is a small positive parameter.

The research theme is **actual**, because difference equations with random coefficients are widely used in such contemporary applications of dynamical system theory as regressive financial time series analysis. These models one can find for example, in Bera and Higgins [6]; Bollerslev, Engle and Nelson [8]; Li with co-authors [41]-[45], [47], [49]; Wong and Li [59]-[64]; Engle [19]; Gourieroux [26]; Heynen and Kat [29]; Pantula [53]. The analysis of the time varying stock returns and investigate return volatility is crucially important for many issues in macroeconomics and finance, such as for irreversible investments, option pricing, the term structure of interest rates, and general dynamic asset pricing relationships. The proposed in our dissertation methods and algorithms make possible asymptotical analysis of residuals of GARCH models. The proposed method allows to analyze the second moments behavior of the iteration procedures defined by (1.4) as *t* tends to infinity, that is, to analyze asymptotical stability. Moreover this method is convenient for use.

Scientific innovation. The analysis of stochastic dynamical systems is an important research topic in the contemporary econometrics. A number of researchers worked on these problems, for example, [3], [4], [5], [9], [11], [13], [15], [20], [22], [23], [24], [25], [28], [30], [31], [32], [35], [36], [38], [39], [40], [51]. The methods and algorithms proposed of the above mentioned papers are mainly based on random coefficients independence. Our proposal methods and algorithms allow to take account of data correlation assuming the perturbed sequence as discrete Markov process.

The second section contains auxiliary results regarding iteration procedures with Markov coefficients. Markov chain defined by stochastic difference equations is introduced. Stability analysis method – the second Lyapunov method – for solutions of difference equations with random coefficients is described. Necessary and sufficient conditions for mean square stability of linear systems with independent coefficients in finite space are given.

The third section is devoted to asymptotical methods. As auxiliary results in this section the necessary issues from the perturbation theory for linear operators in a finite-dimensional space are considered. The main question is how the eigenvalues and eigenvectors change with the operator, in particular when the operator depends on a parameter analytically.

For equation (1.4) with almost constant coefficients a convenient for application asymptotic algorithm of mean square stability analysis by the second Lyapunov method is elaborated involving Laurent series decomposition by small parameter powers of specially constructed quadratic Lyapunov functions. The given algorithm is expounded using two examples: exponentially mean square stable difference equation and exponentially mean square unstable difference equation.

A method for simplified analysis of linear difference equations in *n*-dimensional real space with near to constant coefficients dependent on homogeneous ergodic Markov chain is given. The difference equation in $\mathbb{R}^n$ with constant coefficients is constructed to approximate the covariation semigroup of correlation matrix family. The proposal method is based on decomposition of specially constructed spectral projector for generating operator of the above mentioned semigroup.

In the fourth section the first and the second moments of linear difference equations with coefficients dependent on homogeneous ergodic Markov chain are analyzed. A convenient for application method of the first moment analysis is elaborated. This method is adapted to the analysis of the dynamics of the second moment matrix of difference equation (1.4) solution. In case if random perturbations are independent the proposal method enables to write necessary and sufficient stability conditions involving system coefficients.

2 Iteration procedures with Markov coefficients

2.1 Markov chain defined by stochastic difference equations

Let assume some probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ is given. Let $\mathbb{X}$ be a metric space and $\Sigma_{\mathbb{X}}$ - Borel σ-algebra on its subsets. A function $P: \mathbb{N} \times \mathbb{N} \times \mathbb{X} \times \Sigma_{\mathbb{X}} \to [0,1]$ is called [18] transition function to $(\mathbb{X}, \Sigma_{\mathbb{X}})$ if for all natural $n \ge s > 0$:

1. $P(s,n,x,\mathbb{X}) = 1$ for all $x \in \mathbb{X}$;
2. $P(s,s,x,A) = \begin{cases} 1, & if \;\; x \in A \\ 0, & if \;\; x \notin A \end{cases}$ for all $x \in \mathbb{X}$ and $A \in \Sigma_{\mathbb{X}}$;
3. $P(s,n,x,A)$, as a function of argument *A*, is a probability measure on $\Sigma_{\mathbb{X}}$;
4. $P(s,n,x,A)$, as a function of argument *x*, is $\Sigma_{\mathbb{X}}$-measurable;
5. for all $x \in \mathbb{X}$, $A \in \Sigma_{\mathbb{X}}$ and $s < m < n$ Chapman-Kolmogorov equation
$$P(s,n,x,A) = \int_{\mathbb{X}} P(s,m,x,dy) P(m,n,y,A)$$
is fulfilled.

If a series of elements $\{x_n(\omega), n \in \mathbb{N}\}$ is given on $(\Omega, \mathfrak{F}, P)$ with values in measurable space $(\mathbb{X}, \Sigma_{\mathbb{X}})$ and if such a transition function $P(s,n,x,A)$ exists that for all $A \in \Sigma_{\mathbb{X}}$ and natural $s < k < n$ the following equality is fulfilled

$$P\left(x_n(\omega) \in A \,\middle|\, x_s(\omega), x_{s+1}(\omega), \ldots, x_k(\omega)\right) = P\left(k, n, x_k(\omega), A\right), \tag{2.1}$$

this series has a Markov property. The equation (2.1) together with the Chapman-Kolmogorov equation allows to define distribution for any finite sample of random variables $\{x_{n_0}(\omega), x_{n_0+1}(\omega), \ldots, x_{n_0+k}(\omega)\}$ in case the distribution $\mu_{n_0}(dx)$ of element $x_{n_0}(\omega)$ is known. For this reason for any Borel $A_0, A_1, \ldots, A_k$ a recurrent expression

$$\begin{aligned} &P\left(x_{n_0+k}(\omega) \in A_k, x_{n_0+k-1}(\omega) \in A_{k-1}, \ldots, x_{n_0}(\omega) \in A_0\right) = \\ &= P\left(x_{n_0+k}(\omega) \in A_k \,\middle|\, x_{n_0+k-1}(\omega) \in A_{k-1}\right) P\left(x_{n_0+k-1}(\omega) \in A_{k-1}, \ldots, x_{n_0}(\omega) \in A_0\right), \end{aligned} \tag{2.2}$$

the Markov property (2.1) and an equality

$$\begin{aligned} &P\left(x_{n_0+1}(\omega)\in A_1, x_{n_0}(\omega)\in A_0\right)= \\ &=P\left(x_{n_0+1}(\omega)\in A_1 \middle| x_{n_0}(\omega)\in A_0\right)P\left(x_{n_0}(\omega)\in A_0\right)= \\ &=\int_{A_0} P\left(n_0, n_0+1, x, A_1\right)\mu_{n_0}(dx) \end{aligned} \tag{2.3}$$

can be used. These formulas will be needed for the description of series $\{x_n, n\in\mathbb{N}\}$ using the initial distribution of random variable $x_1(\omega)$ and transition probability $P(s,n,x,A)$.

Let look in more details to a stochastic recurrent procedure. The difference equation in a form

$$x_{n+1} = f_n\left(x_n, \xi_{n+1}(\omega)\right) \tag{2.4}$$

will be investigated, where $\{\xi_n, n\in\mathbb{N}\}$ is a series of independent identically distributed random variables with values in $(\mathbb{Y}, \Sigma_{\mathbb{Y}})$, and $\{f_n(x,y), n\in\mathbb{N}\}$ is a series of $\Sigma_{\mathbb{X}}\times\Sigma_{\mathbb{Y}}$-measurable functions $f_n:\mathbb{X}\times\mathbb{Y}\to\mathbb{X}$. Let's denote $\mathfrak{F}_k^n$ a minimal σ-algebra concerning which random variables $\xi_k, \xi_{k+1}, \dots, \xi_n$ are measurable, for $n\ge k>0$. If $x_n = x$ let define an operator $X_k^n x := X_{n-1}^n X_k^{n-1} x$, $n\ge k>0$, where $X_k^{k+1}x := f_k(x, \xi_{k+1})$.

Theorem 2.1 [11] Equality $P(k,n,x,A) = P\left(X_k^n x\in A\right)$ defines a transition function of Markov chain.

Let $\{g(n,x),\ n\in\mathbb{N},\ x\in\mathbb{X}\}$ be a series of scalar continuous functions on $\mathbb{X}$. Using (2.4) an operator can be defined

$$\begin{aligned} &(Lg)(n,x) = \mathbb{E}\left\{g\left(n+1, X_n^{n+1}x\right)\right\} - g(n,x) = \\ &=\int_{\mathbb{X}} P(n,n+1,x,dy)\, g(n+1,y) - g(n,x) \end{aligned} \tag{2.5}$$

if the right part exists for all $x\in\mathbb{X}$ and $n\ge 0$. This operator L is called as discrete Lyapunov operator for (2.4) and its definition area let denote $\mathfrak{D}(L)$.

2.2 *Second Lyapunov method*

2.2.1 Definition of stability

The equation (2.4) will be investigated in real *n*-dimension space $\mathbb{R}^n$. It is assumed that a condition $f_n(0, y) \equiv 0$ is fulfilled for all $n \in \mathbb{N}$. A series $\{\xi_n\}_{n\geq 1}$ contains independent identically distributed random variables. The equation (2.4) has a trivial solution $x_n = 0,\ n \in \mathbb{N}$. An item for investigation is behavior of a solution of (2.4) in some neighbourhood of zero in $\mathbb{R}^n$.

A trivial solution of (2.4) is called:

- *p*-stable, if for any $\varepsilon > 0$ such a $\delta > 0$ exists that for all $x \in U_\delta = \{x \in \mathbb{R}^n : |x| < \delta\}$ and $n \geq s > 0$:

$$E\left\{\left|X_s^n x\right|^p\right\} < \varepsilon; \tag{2.6}$$

- stable by probability, if for any $\varepsilon > 0$, $\gamma > 0$ such a $\delta > 0$ exists that for all $x \in U_\delta$ and $n \geq s > 0$:

$$P\left\{\left|X_s^n x\right| \geq \varepsilon\right\} < \gamma; \tag{2.7}$$

- stable almost sure, if for any $\varepsilon > 0$, $\gamma > 0$ such a $\delta > 0$ exists that for all $x \in U_\delta$ and $n \geq s > 0$:

$$P\left\{\sup_{k\geq n}\left|X_s^k x\right| \geq \varepsilon\right\} < \gamma; \tag{2.8}$$

- stable in whole in meaning of previous stability definitions, if inequalities (2.6), (2.7) and (2.8) are fulfilled for all $x \in \mathbb{R}^n$ starting with some $n = n(x, \varepsilon, \gamma)$;
- asymptotically stable in meaning of previous stability definitions, if it is stable and such a $\delta_1 > 0$ exists that the left side of inequalities (2.6), (2.7) and (2.8) tend to zero if $n \to \infty$ for all $x \in U_{\delta_1}$;
- exponentially *p*-stable (*p*>*0*) , if such $\delta > 0$, $M > 0$, $\gamma > 0$ exist that for all $t \geq s > 0$ and $x \in U_\delta$:

$$E\left\{\left|X_s^t x\right|^p\right\} \leq M\,|x|^p\, e^{-\gamma(t-s)}; \tag{2.9}$$

- asymptotically stochastic stable, if it is stable almost sure and for any $\varepsilon > 0$ such a $\delta > 0$ can be found that for all $x \in U_\delta$ and $s > 0$:

$$P\left\{\lim_{t\to\infty} X_s^t x = 0\right\} \geq 1-\varepsilon . \quad (2.10)$$

Stability in some meaning if *p=2* is called mean square stability. From *p*-stability for $p>0$ αp-stability follows for any $\alpha \in (0,1]$ as

$$E\left\{\left|X_s^n x\right|^p\right\} \geq \left(E\left\{\left|X_s^n x\right|^{\alpha p}\right\}\right)^{1/\alpha} .$$

From exponentially *p*-stability for some $p>0$ asymptotically stability follows almost sure.

Investigation of the stability of (2.4) trivial solution analyzing $(Lg)(n,x)$ behavior in some zero neighbourhood using a series of functions $g(n,x)$ is called the second Lyapunov method. The function $g(n,x)$ is called Lyapunov function if $g(n,x) \geq 0$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}^m$, and such number *N* exists that

$$\sup_{\substack{n\geq N \\ x\in U_r}} g(n,x) = \hat{g}(r) \to 0 \quad \text{if } r \to 0 , \quad (2.11)$$

$$\sup_{\substack{n\geq N \\ x\notin U_r}} g(n,x) = g(r) \to \infty \quad \text{if } r \to \infty . \quad (2.12)$$

$\hat{g}$ and *g* are monotone functions.

In the behavior analysis of Markov chain defined using an iteration procedure (2.4), it is important to know the conditions when the trajectory goes out of open area *G* with probability 1 in finite time. Let $\{x_n, n \in \mathbb{N}\}$ be a Markov chain with reproducing operator *L* and arbitrary initial distribution, $g(n,x)$ is a positive function. If $g(n,x) \in \mathfrak{D}(L)$, then for $n > S \geq n_0$

$$Eg\left(n+1, X_{n_0}^{n+1} x\right) - Eg\left(n, X_{n_0}^{n} x\right) = E\left(Lg\left(n, X_{n_0}^{n} x\right)\right),$$

$$Eg\left(n+1, X_{n_0}^{n+1} x\right) = Eg\left(s, X_{n_0}^{s} x\right) + \sum_{k=s}^{n} E\left(Lg\left(k, X_{n_0}^{k} x\right)\right).$$

Let denote $\tau_G(\omega)$ the moment of the first sequence's $X_{n_0}^n x_0$ outlet of area *G*, where x_0 is a random variable, σ-algebra $B(x_0)$ measurable and not dependent on $\mathfrak{F}_{n_0}^{\infty}$. Let define a process $\tau_G(t) = \min(t, \tau_G) = \begin{cases} t, & t < \tau_G \\ \tau_G, & t \geq \tau_G \end{cases}$. This process belongs to σ-algebra $N_t = B(x_0) \times \mathfrak{F}_{n_0}^{t}$. For all *t* also $\tau_G(t+1) = \begin{cases} t+1, & t < \tau_G \\ \tau_G, & t \geq \tau_G \end{cases}$ is N_t-measurable.

Theorem 2.2 [51] If a mathematical expectation exists, then for all $n \ge n_0$

$$Eg\left(\tau_G(n+1), X_{n_0}^{\tau_G(n+1)}x_0\right) = Eg(n_0, x_0) + E\left[\sum_{k=n_0}^{\tau_G(n)} Lg\left(k, X_{n_0}^{k}x_0\right)\right].$$

If the conditions of Theorem 2.2 are true and $Lg(n,x) \le 0$ in G for all n, then a process given by $y(t) = g\left(\tau_G(t), X_{n_0}^{\tau_G(t)}x_0\right)$ is non-negative supermartingale.

Lemma 2.1 [51] Let function $g(t,x) \ge 0$ exists, $t \ge 0,\ x \in G$. For some $Lg(t,x) \le -\alpha(t)$ in this area, where $\alpha(t)$ is a sequence satisfying condition $\alpha(t) > 0,\ \sum_{t=0}^{\infty}\alpha(t) = \infty$. Then process x_t goes out of G in a finite time with probability 1, that is, $P(\tau_G = \infty) = 0$.

Let B be a closed subset in $\mathbb{R}^m$; $U_\varepsilon(B) = \{x : \rho(x,B) < \varepsilon\}$ its ε-neighbourhood, where $\rho(x,B) = \inf_{y\in B}\rho(x,y)$ and $\bar{U}_\varepsilon(B) = \mathbb{R}^m \setminus U_\varepsilon(B)$; $S(0,R) = \{x : \rho(0,x) < R\}$. Let denote $U_{\varepsilon,R}(B) := \bar{U}_\varepsilon(B) \cap S(0,R)$. Let say that function $\varphi(t,x) \in \mathfrak{D}(L)$ belongs to class $\Phi(B)$, if it is non-negative and for some $N(\varepsilon,R) = N$ $\inf_{t\in N, x\in U_{\varepsilon,R}(B)}\varphi(t,x) > 0$ for all $0 < \varepsilon < R$.

If such R exists that $\{x_n, n \in \mathbb{N}\} \subset S(0,R)$, $\varphi(n,x) \in \Phi(B)$ and $\lim_{n\to\infty}\varphi(n,x_n) = 0$ is valid, then $\lim_{n\to\infty}\rho(x_n,B) = 0$.

Lemma 2.2 [51] Let such a function $g(t,x) \ge 0$ and a closed subset $B \subset \mathbb{R}^m$ exist that $\inf_{t\ge 0} g(t,x) = \underset{|x|\to\infty}{\underset{\sim}{g}}(x) \to \infty$ (the Lyapunov condition) and $Lg(t,x) \le -\alpha(t)\varphi(t,x)$, $t \ge 0$, $x \in \mathbb{R}^m$, $\varphi(t,x) \in \Phi(B)$ are valid, hereto a sequence $\{\alpha(t),\ t \in \{0,1,2,\ldots\}\}$ satisfies conditions $\alpha(t) > 0$, $\sum_{t=0}^{\infty}\alpha(t) = \infty$. Then

$$P\left(\sup_{t\ge 0}\left|X_0^t x\right| < \infty\right) = 1, \tag{2.13}$$

$$P\left(\sum_{t=0}^{\infty}\alpha(t)\varphi\left(t, X_0^t x\right) < \infty\right) = 1, \tag{2.14}$$

$$P\left(\lim \rho\left(X_0^t x, B\right) = 0\right) = 1 \text{ (lower limit).} \tag{2.15}$$

Theorem 2.3 [51] Let such a function $g(t,x) \geq 0$ and a subset $B \subset \mathbb{R}^m$ exist for which

$$\tilde{g}(x) = \sup_{t \geq 0} g(t,x) \to 0, \quad if\, \rho(x,B) \to 0,$$
$$\underset{\sim}{g}(x) = \inf_{t \geq 0} g(t,x) \to \infty, \quad if\, \rho(x,B) \to \infty,$$
$$Lg(t,x) \leq -\alpha(t) g(t,x),\ t \geq 0,\ x \in \mathbb{R}^m,$$

where $\alpha(t) > 0$, $\beta(t) = \sum_{k=0}^{t} \alpha(k) \underset{t\to\infty}{\to} \infty$. Then $P\left(\lim_{t\to\infty} \rho\left(X_0^t x, B\right) = 0\right) = 1$.

2.2.2 Sufficient conditions for stability

Theorem 2.4 [11] If Lyapunov function $g(n,x)$ exists and such a number $N > 0$ exists, that for all $n \geq N$ and $x \in \mathbb{R}^m$

$$(Lg)(n,x) \leq 0, \qquad (2.16)$$

then a trivial solution of (2.4) is stable almost sure.

Corollary 2.1 [11] If Lyapunov function exists satisfying the condition

$$(Lg)(n,x) \leq -cg(n,x) \qquad (2.17)$$

for all $n \in \mathbb{N}$, $x \in \mathbb{R}^m$ and some $c \in (0,1)$, then a trivial solution of (2.4) is asymptotically stable almost sure.

2.2.3 Stability of linear systems almost sure

Let suppose that a difference equation in $\mathbb{R}^m$ has a form

$$x_{n+1} = A(\xi_{n+1}) x_n, \qquad (2.18)$$

where $\{\xi_n\}$ is a series of independent identically distributed random variables with values in metric space $\mathbb{Y}$, $A(y)$ is continuous by $y \in \mathbb{Y}$ matrix function, hereto $\sup_{y\in\mathbb{Y}} \|A(y)\| = const < \infty$.

Theorem 2.5 [56] If a trivial solution of (2.18) is stable almost sure, then it is *p*-stable for all sufficiently small positive *p*.

If trivial solution is asymptotically stable almost sure, then it is asymptotically *p*-stable for sufficiently small positive *p*.

Theorem 2.6 [56] If (2.18) trivial solution is asymptotically *p*-stable, then it is exponentially *p*-stable.

Exponential p-stability for sufficiently small $p > 0$ follows from asymptotical stability almost sure of (2.18) trivial solution.

Theorem 2.7 [56] If (2.18) trivial solution is asymptotically stable almost sure, then Lyapunov function $g(x)$ exists such, that

$$|x|^p \le g(x) \le \hat{\gamma}|x|^p, \tag{2.19}$$

$$(Lg)(x) \le -cg(x), \tag{2.20}$$

$$g(x_1 + x_2) \le g(x_1) + g(x_2) \tag{2.21}$$

for all $x, x_1, x_2 \in \mathbb{R}^m$ and sufficiently small $p > 0$, $\hat{\gamma} > 0$ and $c \in (0,1)$.

2.3 Mean square stability of linear systems with independent coefficients

2.3.1 Necessary and sufficient conditions for stability in finite space

Let consider linear difference equation in $\mathbb{R}^m$ what has form

$$x_{n+1} = Ax_n + \xi_{n+1}Bx_n, \tag{2.22}$$

where $\{\xi_n, n \in \mathbb{N}\}$ is a series of scalar identically distributed random variables, $\mathbb{E}\{\xi_1\} = 0$, $\mathbb{E}\{\xi_1^2\} = 1$. Let define a following matrix for $n \ge k > 0$:

$$X_k^n = \begin{cases} (A + \xi_n B)\ldots(A + \xi_{k+1}B), & \text{if } n > k, \\ \mathrm{I}, & \text{if } n = k. \end{cases} \tag{2.23}$$

A solution of (2.22) has a form $\{X_{n_0}^n x, n \ge n_0\}$ for initial conditions $x_{n_0} = x$. It is $\mathfrak{F}_{n_0+1}^n$- measurable and defines a transition probability

$$P(n, x, C) = \mathbb{P}\left(X_k^{k+n} x \in C\right)$$

for all $n > 0$, $x \in \mathbb{R}^m$ and $C \in B^m$, where B^m is σ-algebra of Borel sets in $\mathbb{R}^m$. In a set of $m \times m$ real matrices $\mathbb{M}_m(\mathbb{R})$ a subset of symmetric matrices let denote $\mathbb{V}$ and a set $\mathbb{K} \subset \mathbb{V}$ - positive defined matrices, that is,

$$\mathbb{K} = \left\{q \in \mathbb{V} : (qx, x) \ge 0, \forall x \in \mathbb{R}^m\right\}.$$

Let define in space $\mathbb{V}$ a norm by equality

$$\|q\| = \sup_{|x|=1} |(qx, x)|. \tag{2.24}$$

The following items from linear algebra will be used. Let $q \in \mathbb{V}$. Then

1) the spectrum of q consists of real values, hereto for all $x \in \mathbb{R}^m$

$$|x|^2 \lambda_{\min} \le (qx, x) \le |x|^2 \lambda_{\max},$$

where

$$\lambda_{\max} = \max\{\lambda, \lambda \in \sigma(q)\},$$

$$\lambda_{\min} = \min\{\lambda, \lambda \in \sigma(q)\};$$

2) $\|q\| = \max\limits_{|x|=1} |qx|$;

3) such a nonsingular transform $U \in \mathbb{M}_m(\mathbb{R})$ exists, that $U^T = U^{-1}$ (orthogonal transform) and

$$U^T q U = diag\{\lambda_1, \dots, \lambda_m\},$$

where $\lambda_j \in \sigma(q)$, $j = 1, 2, \dots, m$;

4) $q \in \mathbb{K}$ then and only then, if all its corresponding eigenvalues non-negative;

5) $\mathbb{K}$ is closed, that is, from $\{q_n, n \in \mathbb{N}\} \subset \mathbb{K}$, $q = \lim\limits_{n\to\infty} q_n$ follows that $q \in \mathbb{K}$;

6) from $q_1, q_2 \in \mathbb{K}$ and $\alpha_1 \ge 0$, $\alpha_2 \ge 0$ follows $\alpha_1 q_1 + \alpha_2 q_2 \in \mathbb{K}$;

7) a set $\overset{\circ}{\mathbb{K}} = \{q \in \mathbb{K} : (qx, x) > 0, \forall x \ne 0\}$ consists of inner points of $\mathbb{K}$;

8) for any $q \in \mathbb{V}$ such $q_1, q_2 \in \mathbb{K}$ exist, that $q = q_1 - q_2$;

9) $q \in \overset{\circ}{\mathbb{K}}$ then and only then, if such a positive number β exists, that $(qx, x) \ge \beta |x|^2$ for all $x \in \mathbb{R}^m$.

A set satisfying properties 5) and 6) is called a cone [37].

A correlation matrix of m-dimensional random variable ξ is defined by equality

$$q = E\{\xi \xi^T\},$$

and it is an element of $\mathbb{K}$.

A system of difference equations (2.22) defines an linear operator family $\{T(n), n \ge 0\}$ by equality

$$(T(n) qx, x) = E\{(qX_k^{k+n} x, X_k^{k+n} x)\} \tag{2.25}$$

for any $k \ge 0$, $q \in \mathbb{V}$, $x \in \mathbb{R}^m$, hereto $T(0) = I$.

Lemma 2.3 [11] For any $n \in \mathbb{N}$ an operator $T(n)$ leaves as invariant $\mathbb{K}$**.**

Lemma 2.4 [11] For any $n \geq 0$ and $k \geq 0$

$$T(n+k) = T(n)T(k). \tag{2.26}$$

Let denote $T(1) = T$, then from (2.26) follows equality $T(n) = T^n$ for all $n \in \mathbb{N}$. It is easy to calculate, that

$$Tq = A^T qA + B^T qB \tag{2.27}$$

for all $q \in \mathbb{V}$.

Let define a scalar product in $\mathbb{V}$ by equality

$$[q, p] := Sp(qp) = Sp(pq). \tag{2.28}$$

Let assume that initial condition x_0 - a random variable having a correlation matrix $q_0 \in \mathbb{K}$ and not dependent on $\mathfrak{F}_0^\infty$. Then a correlation matrix q_1 of random variable $x_1 = X_0^1 x_0$ can be found using equality

$$\begin{aligned} q_1 &= E\left\{x_1 x_1^T\right\} = E\left\{X_0^1 x_0 x_0^T \left(X_0^1\right)^T\right\} = \\ &= E\left\{X_0^1 q_0 \left(X_0^1\right)^T\right\} = Aq_0 A^T + Bq_0 B^T. \end{aligned} \tag{2.29}$$

If a linear operator $\mathbb{A}$ is defined as

$$\mathbb{A}q := AqA^T + BqB^T, \tag{2.30}$$

then from (2.29) follows $q_1 = \mathbb{A}q_0$.

Lemma 2.5 [11] $\mathbb{A} = T^*$.

From linear operator properties in Banach space follows that

$$\sigma(\mathbb{A}) = \sigma(T). \tag{2.31}$$

Theorem 2.8 [11] A trivial solution of (2.22) is exponentially mean square stable if and only if spectrum of operator *T* is located inside circle $\{z \in \mathbb{C} : |z| < 1\}$.

Let define operator *G* by equality

$$Gq = \sum_{k=0}^{\infty} T^k q \tag{2.32}$$

and let write $q \in \mathfrak{D}(G)$, if this series converges for given $q \in \mathbb{V}$.

Corollary 2.2 [11] A trivial solution of (2.22) is exponentially mean square stable if and only if for any $r \in \overset{\circ}{\mathbb{K}}$ such a $q \in \overset{\circ}{\mathbb{K}}$ exists that $(T - J)q = -r$.

Corollary 2.3 [11] A trivial solution of (2.22) is exponentially mean square stable if and only if $\mathfrak{D}(G) \supset \mathbb{K}$.

Theorem 2.9 [11] For exponentially stability of (2.22) trivial solution it is necessary and sufficiently that for any $a \in \mathbb{R}^m$ the following inequality is valid

$$\sum_{n=0}^{\infty} E\left\{\left(X_0^n x, a\right)^2\right\} \leq cE\left\{|x|^2\right\}|a|^2$$

for some $c > 0$ and all random vectors x having the second moment and not dependent on $\mathfrak{F}_0^\infty$.

3 Asymptotical methods

3.1 λ-group and projection for the λ-group

In this section the necessary issues from the perturbation theory [33] for linear operators in a finite-dimensional space are considered. The main question is how the eigenvalues and eigenvectors change with the operator, in particular when the operator depends on a parameter analytically.

Let turn to the perturbation theory for the eigenvalues problem in a finite-dimensional vector space $\mathbb{X}$, where $0 < \dim \mathbb{X} = N < \infty$. A typical problem of this theory is to investigate how the eigenvalues and eigenvectors of a linear operator T change when T is subjected to a small perturbation. In dealing with such a problem, it is often convenient to consider a family of operators of the form

$$T(\varepsilon) = T + \varepsilon T' \tag{3.1}$$

where ε is a scalar parameter supposed to be small. $T(0) = T$ is called unperturbed operator and $\varepsilon T'$ the perturbation.

A question arises whether the eigenvalues and the eigenvectors of $T(\varepsilon)$ can be expressed as power series in ε, that is, whether they are holomorphic functions of ε in the neighborhood of $\varepsilon = 0$. When $T(\varepsilon)$ is defined and differentiable everywhere in its domain, $T(\varepsilon)$ is said to be holomorphic. If this is a case, the change of the eigenvalues and eigenvectors is of the same order of magnitude as the perturbation $\varepsilon T'$ itself for small $|\varepsilon|$. However, this is not always the case. (3.1) can be generalized to

$$T(\varepsilon) = T + \varepsilon T^{(1)} + \varepsilon^2 T^{(2)} + \ldots. \tag{3.2}$$

Let suppose that an operator-valued function $T(\varepsilon)$ is given, which is holomorphic in a given domain D_0 of the complex ε-plane. The eigenvalues of $T(\varepsilon)$ satisfy the characteristic equation

$$\det\left(T(\varepsilon) - \xi\right) = 0\,. \tag{3.3}$$

This is an algebraic equation in ξ of degree $N = \dim \mathbb{X}$ with coefficients, which are holomorphic in ε. The number of eigenvalues of $T(\varepsilon)$ is a constant s independent of

ε, with the exception of some special values of ε. There are only a finite number of such exceptional points ε in each compact subset of D_0. This number *s* is equal to *N* if these analytic functions are all distinct; in this case $T(\varepsilon)$ is simple and therefore diagonable for all non-exceptional ε. If, on the other hand, there happen to be identical ones among these analytic functions, then $s < N$; in this case $T(\varepsilon)$ is said to be permanently degenerate.

Let consider the eigenvalues of $T(\varepsilon)$ in more detail. Since these are in general multiple-valued analytic functions of ε, some care is needed in their notation. If ε is restricted to a simply-connected subdomain *D* of the fundamental domain D_0 containing no exceptional point (for brevity such a subdomain will be called a simple subdomain), the eigenvalues of $T(\varepsilon)$ can be written

$$\lambda_1(\varepsilon), \lambda_2(\varepsilon), \ldots, \lambda_s(\varepsilon), \tag{3.4}$$

all *s* functions $\lambda_h(\varepsilon),\ h = 1, \ldots, s$ being holomorphic in *D* and $\lambda_h(\varepsilon) \neq \lambda_k(\varepsilon),\ h \neq k$.

Next the behavior of the eigenvalues in the neighborhood of one of the exceptional points, which can be taken as $\varepsilon = 0$, is considered. Let *D* be a small disk near $\varepsilon = 0$ but excluding $\varepsilon = 0$. The eigenvalues of $T(\varepsilon)$ for $\varepsilon \in D$ can be expressed by *s* holomorphic functions of the form (3.4). If *D* is moved continuously around $\varepsilon = 0$, these *s* functions can be continued analytically. When *D* has been brought to its initial position after one revolution around $\varepsilon = 0$, the *s* functions (3.4) will have undergone a permutation among themselves. These functions may therefore be grouped in the manner

$$\{\lambda_1(\varepsilon), \ldots, \lambda_p(\varepsilon)\}, \{\lambda_{p+1}(\varepsilon), \ldots, \lambda_{p+q}(\varepsilon)\}, \ldots, \tag{3.5}$$

in such a way that each group undergoes a cyclic permutation by a revolution of *D* of the kind described. For brevity each group will be called a cycle at the exceptional point $\varepsilon = 0$, and the number of elements of a cycle will be called its period.

The elements of a cycle of period *p* constitute a branch of an analytic function (defined near $\varepsilon = 0$) with a branch point (if $p \geq 2$) at $\varepsilon = 0$, and Puiseux series can be obtained such as

$$\lambda_h(\varepsilon) = \lambda + \alpha_1 \omega^h \varepsilon^{1/p} + \alpha_2 \omega^{2h} \varepsilon^{2/p} + \ldots,\ h = 0, 1, \ldots, p-1, \tag{3.6}$$

where $\omega = \exp(2\pi i/p)$. It should be noticed that here no negative powers of $\varepsilon^{1/p}$ appear, for the coefficient of the highest power ξ^N in (3.3) is $(-1)^N$ so that the $\lambda_h(\varepsilon)$ are continuous at $\varepsilon = 0$[3]. $\lambda = \lambda_h(0)$ is called the center of the cycle under consideration. (3.6) shows that $|\lambda_h(\varepsilon) - \lambda|$ is in general of the order $|\varepsilon|^{1/p}$ for small $|\varepsilon|$ for $h = 1, \ldots, p$. If $p \geq 2$, therefore, the rate of change at an exceptional point of the eigenvalues of a cycle of period *p* is infinitely large compared with change of $T(\varepsilon)$ itself.

In general there are several cycles with the same center λ. All the eigenvalues (3.6) belonging to cycles with center λ are said to depart from the unperturbed eigenvalue λ by splitting at $\varepsilon = 0$. The set of these eigenvalues will be called the λ-group, since they cluster around λ for small $|\varepsilon|$.

The resolvent

$$R(\xi, \varepsilon) = (T(\varepsilon) - \xi)^{-1} \tag{3.7}$$

of $T(\varepsilon)$ is defined for all ξ not equal to any of the eigenvalues of $T(\varepsilon)$ and is a meromorphic function of ξ for each fixed $\varepsilon \in D_0$.

Theorem 3.1 [33] $R(\xi, \varepsilon)$ is holomorphic in the two variables ξ, ε in each domain in which ξ is not equal to any of the eigenvalues of $T(\varepsilon)$.

Let λ be one of the eigenvalues of $T = T(0)$, with multiplicity *m*. Let Γ be a closed positively-oriented curve, say a circle, in the resolvent set $P(T)$ enclosing λ but no other eigenvalues of *T*. The second Neumann series

$$\begin{aligned} R(\xi, \varepsilon) &= R(\xi)\left[1 + A(\varepsilon) R(\xi)\right]^{-1} = \\ &= R(\xi) \sum_{p=0}^{\infty} \left[-A(\varepsilon) R(\xi)\right]^p = R(\xi) + \sum_{n=1}^{\infty} \varepsilon^n R^{(n)}(\xi), \end{aligned} \tag{3.8}$$

where

$$R^{(n)}(\xi) = \sum_{\substack{\nu_1 + \ldots + \nu_p = n \\ \nu_j \geq 1}} (-1)^p R(\xi) T^{(\nu_1)} R(\xi) T^{(\nu_2)} \ldots T^{(\nu_p)} R(\xi), \tag{3.9}$$

is then convergent for sufficiently small $|\varepsilon|$ uniformly for $\xi \in \Gamma$. The existence of the resolvent $R(\xi,\varepsilon)$ of $T(\varepsilon)$ for $\xi \in \Gamma$ implies that there are no eigenvalues of $T(\varepsilon)$ on Γ. The operator

$$P(\varepsilon) = -\frac{1}{2\pi i}\int_{\Gamma} R(\xi,\varepsilon)\,d\xi^2 \tag{3.10}$$

is a projection and is equal to the sum of the eigenprojections for all the eigenvalues of $T(\varepsilon)$ lying inside Γ. In particular $P(0) = P$ coincide with the eigenprojection for the eigenvalue λ of *T*. Integrating (3.8) term by term, one can get

$$P(\varepsilon) = P + \sum_{n=1}^{\infty} \varepsilon^n P^{(n)} \tag{3.11}$$

with

$$P^{(n)} = -\frac{1}{2\pi i}\int_{\Gamma} R^{(n)}(\xi)\,d\xi\,. \tag{3.12}$$

The range $M(\varepsilon)$ of $P(\varepsilon)$ is isomorphic with the (algebraic) eigenspace $M = M(0) = P\mathbb{X}$ of *T* for the eigenvalue λ. In particular

$$\dim P(\varepsilon) = \dim P = m\,. \tag{3.13}$$

Since (3.13) is true for all sufficiently small $|\varepsilon|$, it follows that the eigenvalues of $T(\varepsilon)$ lying inside Γ from exactly the λ-group. For brevity $P(\varepsilon)$ is called the total projection, and $M(\varepsilon)$ the total eigenspace, for the λ-group.

Now let consider a simple subdomain *D* of the ε-plane and the set (3.4) of the eigenvalues of $T(\varepsilon)$ for $\varepsilon \in D$, and let $P_h(\varepsilon)$ be the eigenprojection for the eigenvalue $\lambda_h(\varepsilon)$, $h = 1,\ldots,s$. Each $P_h(\varepsilon)$ is holomorphic in *D* and each $\lambda_h(\varepsilon)$ has constant multiplicity m_h. Here it is essential that *D* is simple (contains no exceptional point); in fact, $P_1(\varepsilon_0)$ is not even defined if, for example, $\lambda_1(\varepsilon_0) = \lambda_2(\varepsilon_0)$ which may happen if ε_0 is exceptional.

Let $M_h(\varepsilon) = P_h(\varepsilon)\mathbb{X}$ be the (algebraic) eigenspace of $T(\varepsilon)$ for the eigenvalue $\lambda_h(\varepsilon)$. Then

$$\mathbb{X} = M_1(\varepsilon) \oplus \ldots \oplus M_s(\varepsilon)\,, \tag{3.14}$$

$$\dim M_h(\varepsilon) = m_h, \quad \sum_{j=1}^{s} m_h = N, \quad \varepsilon \in D.$$

Let assume that the power series for $T(\varepsilon)$ is given:

$$T(\varepsilon) = T + \varepsilon T^{(1)} + \varepsilon^2 T^{(2)} + \dots. \tag{3.15}$$

Let λ be one of the eigenvalues of the unperturbed operator $T = T(0)$ with (algebraic) multiplicity *m*, and let *P* and *D* be the associated eigenprojection and eigennilpotent. Thus

$$\begin{aligned} &TP = PT = PTP = \lambda P + D, \\ &\dim P = m, \quad D^m = 0, \quad PD = DP = D. \end{aligned} \tag{3.16}$$

The eigenvalue λ will in general split into several eigenvalues of $T(\varepsilon)$ for small $\varepsilon \neq 0$ (the λ-group). The total projection $P(\varepsilon)$ for this λ-group is holomorphic at $\varepsilon = 0$

$$P(\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n P^{(n)}, \quad P^{(0)} = P, \tag{3.17}$$

with $P^{(n)}$ given by (3.12). The subspace $M(\varepsilon) = P(\varepsilon)\mathbb{X}$ is *m*-dimensional and invariant under $T(\varepsilon)$. The λ-group eigenvalues of $T(\varepsilon)$ are identical with all the eigenvalues of $T(\varepsilon)$ in $M(\varepsilon)$. In order to determine the λ-group eigenvalues, an eigenvalue problem in the subspace $M(\varepsilon)$, which is in general smaller than the whole space $\mathbb{X}$, should be solved.

The eigenvalue problem for $T(\varepsilon)$ in $M(\varepsilon)$ is equivalent to the eigenvalue problem for the operator

$$T_r(\varepsilon) = T(\varepsilon)P(\varepsilon) = P(\varepsilon)T(\varepsilon) = P(\varepsilon)T(\varepsilon)P(\varepsilon). \tag{3.18}$$

Thus the λ-group eigenvalues of $T(\varepsilon)$ are exactly those eigenvalues of $T_r(\varepsilon)$ which are different from zero, provided that $|\lambda|$ is large enough to ensure that these eigenvalues do not vanish for the small $|\varepsilon|$ under consideration.

It follows that

$$\hat{\lambda}(\varepsilon) = \frac{1}{m} tr\left(T(\varepsilon)P(\varepsilon)\right) = \lambda + \frac{1}{m} tr\left(\left(T(\varepsilon) - \lambda\right)P(\varepsilon)\right) \tag{3.19}$$

is equal to the weighted mean of the λ-group eigenvalues of $T(\varepsilon)$, where weight is the multiplicity of each eigenvalue. If there is no splitting of λ so that the λ-group consists of a single eigenvalue $\lambda(\varepsilon)$ with multiplicity *m* then

$$\hat{\lambda}(\varepsilon)=\lambda(\varepsilon), \tag{3.20}$$

in particular this is always true if $m=1$. In such a case the eigenprojection associated with $\lambda(\varepsilon)$ is exactly the total projection (3.17) and the eigennilpotent is given by

$$D(\varepsilon)=\left(T(\varepsilon)-\lambda(\varepsilon)\right)P(\varepsilon). \tag{3.21}$$

These series give a complete solution to the eigenvalue problem for the λ-group in the case of no splitting, $\lambda(\varepsilon)$, $P(\varepsilon)$ and $D(\varepsilon)$ being all holomorphic at $\varepsilon=0$.

Let consider the series (3.18) for $T_r(\varepsilon)=T(\varepsilon)P(\varepsilon)$. For computation it is more convenient to consider the operator $\left(T(\varepsilon)-\lambda\right)P(\varepsilon)$ instead of $T_r(\varepsilon)$ itself. From (3.10) follows that

$$\left(T(\varepsilon)-\lambda\right)P(\varepsilon)=-\frac{1}{2\pi i}\int_{\Gamma}(\xi-\lambda)R(\xi,\varepsilon)\,d\xi \tag{3.22}$$

since $\left(T(\varepsilon)-\lambda\right)R(\xi,\varepsilon)=1+(\xi-\lambda)R(\xi,\varepsilon)$ and the integral of 1 along Γ vanishes. Noting that $(T-\lambda)P=D$ by (3.16), it can be obtained that

$$\left(T(\varepsilon)-\lambda\right)P(\varepsilon)=D+\sum_{n=1}^{\infty}\varepsilon^n\tilde{T}^{(n)} \tag{3.23}$$

with

$$\tilde{T}^{(n)}=-\frac{1}{2\pi i}\sum_{\substack{\upsilon_1+\ldots+\upsilon_p=n\\ \upsilon_j\geq 1}}(-1)^p\int_{\Gamma}R(\xi)T^{(\upsilon_1)}\ldots T^{(\upsilon_p)}R(\xi)(\xi-\lambda)\,d\xi \tag{3.24}$$

for $n\geq 1$.

If λ is a semisimple eigenvalue of *T*, $D=0$ and (3.23) gives

$$\tilde{T}^{(1)}(\varepsilon)\equiv\frac{1}{\varepsilon}\left(T(\varepsilon)-\lambda\right)P(\varepsilon)=\sum_{n=0}^{\infty}\varepsilon^n\tilde{T}^{(n+1)}. \tag{3.25}$$

Since $M(\varepsilon)=R\left(P(\varepsilon)\right)$ is invariant under $T(\varepsilon)$, there is an obvious relationship between the parts of $T(\varepsilon)$ and $\tilde{T}^{(1)}(\varepsilon)$ in $M(\varepsilon)$. Thus the solution of the eigenvalue problem for $T(\varepsilon)$ in $M(\varepsilon)$ reduces to the same problem for $\tilde{T}^{(1)}(\varepsilon)$. Now (3.25) shows that $\tilde{T}^{(1)}(\varepsilon)$ is holomorphic at ε =0. This process of reducing the problem for $T(\varepsilon)$ to the one for $\tilde{T}^{(1)}(\varepsilon)$ is called the reduction process. The "unperturbed operator" for this family $\tilde{T}^{(1)}(\varepsilon)$ is

$$\tilde{T}^{(1)}(0) = \tilde{T}^{(1)} = PT^{(1)}P. \tag{3.26}$$

It follows that each eigenvalue of $\tilde{T}^{(1)}$ splits into several eigenvalues of $\tilde{T}^{(1)}(\varepsilon)$ for small $|\varepsilon|$. Let the eigenvalues of $\tilde{T}^{(1)}$ in the invariant subspace $M = M(0) = R(P)$ be denoted by $\lambda_j^{(1)},\ j = 1, 2, \ldots$. The spectral representation of $\tilde{T}^{(1)}$ in *M* takes the form

$$\begin{aligned} &\tilde{T}^{(1)} = PT^{(1)}P = \sum_j \left(\lambda_j^{(1)} P_j^{(1)} + D_j^{(1)} \right), \\ &P = \sum_j P_j^{(1)}, \\ &P_j^{(1)} P_k^{(1)} = \delta_{jk} P_j^{(1)}. \end{aligned} \tag{3.27}$$

Suppose for the moment that all the $\lambda_j^{(1)}$ are different from zero. By perturbation each $\lambda_j^{(1)}$ will split into several eigenvalues (the $\lambda_j^{(1)}$-group) of $\tilde{T}^{(1)}(\varepsilon)$, which are power series in $\varepsilon^{\frac{1}{p_j}}$ with some $p_j \geq 1$. The corresponding eigenvalues of $T(\varepsilon)$ have the form

$$\lambda + \varepsilon\lambda_j^{(1)} + \varepsilon^{1+\frac{1}{p_j}} \alpha_{jk} + \ldots,\ k = 1, 2, \ldots. \tag{3.28}$$

If some $\lambda_j^{(1)}$ is zero, the associated eigenspace of $\tilde{T}^{(1)}$ includes the subspace $R(1-P)$. But this inconvenience may be avoided by adding to $T(\varepsilon)$ a term of the form $\alpha\varepsilon$, which amounts to adding to $\tilde{T}^{(1)}(\varepsilon)$ a term $\alpha P(\varepsilon)$. This has only the effect of shifting the eigenvalues of $T^{(1)}(\varepsilon)$ in $M(\varepsilon)$ by the amount α, leaving the eigenprojections and eigennilpotents unchanged. By choosing α appropriately the modified $\lambda_j^{(1)}$ can be made different from zero. Thus the assumption that $\lambda_j^{(1)} \neq 0$ does not affect the generality, and this should be assumed in the following whenever convenient.

The eigenvalues (3.28) of $T(\varepsilon)$ for fixed λ and $\lambda_j^{(1)}$ will be said to from the $\lambda + \varepsilon\lambda_j^{(1)}$-group. From (3.28) follows the following theorem.

Theorem 3.2 [33] If λ is a semisimple eigenvalue of the unperturbed operator *T*, each of the λ-group eigenvalues of $T(\varepsilon)$ has the form (3.28) so that it belongs to some $\lambda+\varepsilon\lambda_j^{(1)}$-group. These eigenvalues are continuously differentiable near $\varepsilon=0$ (even when $\varepsilon=0$ is a branch point). The total projection $P_j^{(1)}(\varepsilon)$ (the sum of eigenprojections) for the $\lambda+\varepsilon\lambda_j^{(1)}$-group and the weighted mean of this group are holomorphic at $\varepsilon=0$.

The reduction process described above can further be applied to the eigenvalue $\lambda_j^{(1)}$ of $\tilde{T}^{(1)}$ if it is semisimple, with the result that the $\lambda_j^{(1)}$-group eigenvalues of $\tilde{T}^{(1)}(\varepsilon)$ have the form $\lambda_j^{(1)}+\varepsilon\lambda_{jk}^{(2)}+o(\varepsilon)$. The corresponding eigenvalues of $T(\varepsilon)$ have the form

$$\lambda+\varepsilon\lambda_j^{(1)}+\varepsilon^2\lambda_{jk}^{(2)}+o\left(\varepsilon^2\right). \tag{3.29}$$

These eigenvalues with fixed *j*, *k* form the $\lambda+\varepsilon\lambda_j^{(1)}+\varepsilon^2\lambda_{jk}^{(2)}$-group of $T(\varepsilon)$. In this way it can be seen that the reduction process can be continued, and the eigenvalues and eigenprojections of $T(\varepsilon)$ can be expanded into formal power series in ε, as long as the unperturbed eigenvalue is semisimple at each stage of the reduction process.

But it is not necessary to continue the reduction process indefinitely, even when this is possible. Since the splitting must end after a finite number, say *n*, of steps, the total projection and the weighted mean of the eigenvalues at the *n*-th stage will give the full expansion of the eigenprojection and the eigenvalue themselves, respectively.

3.2 Operators in Banach spaces

3.2.1 Banach spaces and the adjoint space

A normed space is a vector space $\mathbb{X}$ in which a function $\|\cdot\|$ is defined and satisfies the conditions of a norm. In a normed space $\mathbb{X}$ the convergence of a sequence of vectors $\{u_n\}$ to a $u\in\mathbb{X}$ can be defined by $\|u_n-u\|\to 0$. As in the finite-dimensional case, this implies the Cauchy condition $\|u_n-u_m\|\to 0$. In the infinite-dimensional case, however, a Cauchy sequence $\{u_n\}$ (a sequence that satisfies the Cauchy condition) need not have a limit $u\in\mathbb{X}$. A normed space in which every

Cauchy sequence has a limit is said to be complete. A complete normed space is called a Banach space.

The adjoint space $\mathbb{X}^*$ of $\mathbb{X}$ is defined as the set of all bounded semilinear forms on $\mathbb{X}$, and $\mathbb{X}^*$ is a normed vector space if the norm of $f \in \mathbb{X}^*$ is defined as the bound $\|f\|$ of *f*. Let introduce the scalar product $(f,u) = f[u]$. $\mathbb{X}^*$ is a Banach space.

Theorem 3.3 [33] Let *M* be a closed linear manifold of $\mathbb{X}$ and let $u_0 \in \mathbb{X}$ not belong to *M*. Then there is $f \in \mathbb{X}^*$ such that $(f,u_0)=1$, $(f,u)=0$ for $u \in M$ and $\|f\| = \dfrac{1}{dist(u_0, M)}$.

The adjoint space $\mathbb{X}^{**}$ of $\mathbb{X}^*$ is again a Banach space. As in the finite-dimensional case, each $u \in \mathbb{X}$ may be regarded as an element of $\mathbb{X}^{**}$.

3.2.2 Linear operators in Banach spaces

Let define a linear operator *T* from $\mathbb{X}$ to $\mathbb{Y}$ as a function, which sends every vector *u* in a certain linear manifold *D* of $\mathbb{X}$ to a vector $v = Tu \in \mathbb{Y}$ and which satisfies the linearity condition $T(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 T u_1 + \alpha_2 T u_2$ for all $u_1, u_2 \in D$. *D* is called the domain of definition, or simply the domain, of *T* and is denoted by $D(T)$. The range $R(T)$ of *T* is defined as the set of vectors of the form Tu with $u \in D(T)$. $\mathbb{X}$ and $\mathbb{Y}$ are respectively called the domain and range spaces. If $D(T) = \mathbb{X}$, *T* is said to be defined on $\mathbb{X}$. If $\mathbb{Y} = \mathbb{X}$, it is said that *T* is an operator in $\mathbb{X}$. The null space $N(T)$ of *T* is the set of all $u \in D(T)$ such that $Tu = 0$.

The inverse T^{-1} of an operator *T* from $\mathbb{X}$ to $\mathbb{Y}$ is defined if and only if the map *T* is one to one, which is the case if and only if $Tu = 0$ implies $u = 0$. T^{-1} is by definition the operator from $\mathbb{Y}$ to $\mathbb{X}$ that sends Tu into *u*. Thus

$$D(T^{-1}) = R(T), \quad R(T^{-1}) = D(T) \tag{3.30}$$

$$T^{-1}(Tu) = u, \; u \in D(T), \quad T(T^{-1}v) = v, \; v \in R(T) \tag{3.31}$$

T is said to be invertible if T^{-1} exists.

An operator *T* from $\mathbb{X}$ to $\mathbb{Y}$ is continuous at $u = u_0 \in D(T)$ if $\|u_n - u_0\| \to 0$, $u_n \in D(T)$, implies $\|Tu_n - Tu_0\| \to 0$. *T* is continuous everywhere in its domain if it is continuous at $u = 0$. T is continuous if and only if T is bounded: $\|Tu\| \le M\|u\|$, $u \in D(T)$. The smallest number M with this property is called the bound of T and is denoted by $\|T\|$.

3.2.3 Bounded operators

Let denote by $B(\mathbb{X},\mathbb{Y})$ the set of all bounded operators on $\mathbb{X}$ to $\mathbb{Y}$. Every operator belonging to $B(\mathbb{X},\mathbb{Y})$ has domain $\mathbb{X}$ and range in $\mathbb{Y}$. The resulting operator of the linear combination $\alpha S + \beta T$ of $S,T \in B(\mathbb{X},\mathbb{Y})$ is again linear and bounded. Thus $B(\mathbb{X},\mathbb{Y})$ is a normed space with the norm $\|T\|$ defined as the bound of T:

$$\|T\| = \sup_{0 \ne u \in \mathbb{X}} \frac{\|Tu\|}{\|u\|} = \sup_{\|u\|=1} \|Tu\| = \sup_{\|u\| \le 1} \|Tu\|, \quad T \in B(\mathbb{X},\mathbb{Y}).$$

Similarly, the product TS is defined for $T \in B(\mathbb{Y},\mathbb{Z})$, $S \in B(\mathbb{X},\mathbb{Y})$ by $(TS)u = T(Su)$ for all $u \in \mathbb{X}$ and belongs to $B(\mathbb{X},\mathbb{Z})$.

$B(\mathbb{X},\mathbb{Y})$ is a Banach space. To prove the completeness of $B(\mathbb{X},\mathbb{Y})$, let $\{T_n\}$ be a Cauchy sequence of elements of $B(\mathbb{X},\mathbb{Y})$. Then $\{T_n u\}$ is a Cauchy sequence in $\mathbb{Y}$ for each fixed $u \in \mathbb{X}$, for $\|T_n u - T_m u\| \le \|T_n - T_m\|\|u\| \to 0$. Since $\mathbb{Y}$ is complete, there is a $v \in \mathbb{Y}$ such that $T_n u \to v$. Let define an operator T by setting $v = Tu$. T is linear and bounded so that $T \in B(\mathbb{X},\mathbb{Y})$ and that $\|T_n - T\| \to 0$.

Different kinds of convergence can be introduced into $B(\mathbb{X},\mathbb{Y})$. Let $T, T_n \in B(\mathbb{X},\mathbb{Y})$, $n = 1,2,\ldots$. The convergence of $\{T_n\}$ to T in the sense of $\|T_n - T\| \to 0$ (convergence in the normed space $B(\mathbb{X},\mathbb{Y})$) is called *uniform convergence* or *convergence in norm*. $\{T_n\}$ is said to converge *strongly* to T if $T_n u \to Tu$ for each $u \in \mathbb{X}$. $\{T_n\}$ converges in norm if and only if $\{T_n u\}$ converges uniformly for $\|u\| \le 1$. $\{T_n\}$ is said to converge *weakly* if $\{T_n u\}$ converges weakly for each $u \in \mathbb{X}$, that is, if $(T_n u, g)$ converges for each $u \in \mathbb{X}$ and $g \in \mathbb{Y}^*$. If $\{T_n u\}$ has a weak limit Tu for each $u \in \mathbb{X}$, $\{T_n\}$ has the weak limit T. $\{T_n\}$ converges in norm if and only if $(T_n u, g)$ converges uniformly for $\|u\| \le 1$ and $\|g\| \le 1$. A weakly convergent sequence has a weak limit if $\mathbb{Y}$ is weakly complete. Convergence in norm implies strong convergence, and strong convergence implies weak convergence. Let use the notations $T_n \xrightarrow[u]{} T$ for convergence in norm, $T_n \xrightarrow[s]{} T$ for strong convergence and $T_n \xrightarrow[w]{} T$ for weak convergence. If $\{T_n\}$ is weakly convergent, it is uniformly bounded, that is, $\{\|T_n\|\}$ is bounded.

Lemma 3.1 [33] Let $\{T_n\}$ be uniformly bounded. Then $\{T_n\}$ converges strongly to T if $\{T_n u\}$ converges strongly to Tu for all u of a fundamental subset of $\mathbb{X}$.

Lemma 3.2 [33] Let $\{T_n\}$ be uniformly bounded. Then $\{T_n\}$ converges weakly to T if $\{(T_n u, g)\}$ converges to (Tu, g) for all u of a fundamental subset of $\mathbb{X}$ and for all g of a fundamental subset of $\mathbb{Y}^*$.

Lemma 3.3 [33] If $T_n \xrightarrow[s]{} T$ then $T_n u \xrightarrow[s]{} Tu$ uniformly for all u of a compact subset $\mathbb{S}$ of $\mathbb{X}$.

Lemma 3.4 [33] If $T_n \xrightarrow[s]{} T$ in $B(\mathbb{Y}, \mathbb{Z})$ and $S_n \xrightarrow[s]{} S$ in $B(\mathbb{X}, \mathbb{Y})$, then $T_n S_n \xrightarrow[s]{} TS$ in $B(\mathbb{X}, \mathbb{Z})$.

Lemma 3.5 [33] If $T_n \xrightarrow[w]{} T$ in $B(\mathbb{Y}, \mathbb{Z})$ and $S_n \xrightarrow[s]{} S$ in $B(\mathbb{X}, \mathbb{Y})$, then $T_n S_n \xrightarrow[w]{} TS$ in $B(\mathbb{X}, \mathbb{Z})$.

$B(\mathbb{X}) = B(\mathbb{X}, \mathbb{X})$ is the set of all bounded operators on $\mathbb{X}$ to itself. In $B(\mathbb{X})$ not only the linear combination of two operators S, T but also their product ST is defined and belongs to $B(\mathbb{X})$. Thus $B(\mathbb{X})$ is a *complete normed algebra* (Banach algebra). It should be noted that the completeness of $B(\mathbb{X})$ is essential here; for example, the existence of the sum of an absolutely convergent series of operators depends on completeness.

$T \in B(\mathbb{X})$ is said to be *nonsingular* if T^{-1} exists and belongs to $B(\mathbb{X})$. $1 - T$ is nonsingular if $\|T\| < 1$. It follows that T^{-1} is a continuous function of T on the set of all nonsingular operators, which is open in $B(\mathbb{X})$.

The *spectral radius* $\operatorname{spr}T = \lim \left\| T^n \right\|^{1/n}$ can also be defined for every $T \in B(\mathbb{X})$. The trace and determinant of $T \in B(\mathbb{X})$ are in general not defined, but they can be defined for certain classes of operators of $B(\mathbb{X})$.

For each $T \in B(\mathbb{X}, \mathbb{Y})$, the adjoint T^* is defined and belongs to $B(\mathbb{Y}^*, \mathbb{X}^*)$. For each $g \in \mathbb{Y}^*$, $u \to (g, Tu)$ is a bounded semilinear form on $\mathbb{X}$ by virtue of $\left|(g,Tu)\right| \le \|g\| \|Tu\| \le \|T\| \|g\| \|u\|$, so that it can be written (f,u) with an $f \in \mathbb{X}^*$; T^* is defined by $T^* g = f$. $\left\|T^* g\right\| = \|f\| = \sup_{\|u\| \le 1} \left|(f,u)\right| \le \|T\| \|g\|$ gives $\left\|T^*\right\| \le \|T\|$. $\left\|T^{**}\right\| \le \left\|T^*\right\| \le \|T\|$, but $T^{**} \supset T$ if $\mathbb{X}$ is identified with a subspace of $\mathbb{X}^{**}$, for $\left(T^{**}u, g\right) = \left(u, T^* g\right) = \overline{\left(T^* g, u\right)} = \overline{(g, Tu)}$ shows that the semilinear form $T^{**}u$ on $\mathbb{Y}^*$ is represented by $Tu \in \mathbb{Y}$ and therefore $T^{**}u = Tu$ by identification. Since $T^{**} \supset T$ implies $\left\|T^{**}\right\| \ge \|T\|$, $\left\|T^*\right\| = \|T\|$.

An idempotent operator $P \in B(\mathbb{X})$ $\left(P^2 = P\right)$ is called a *projection*. Let *M* and *N* be two complementary linear manifolds of $\mathbb{X}$, that is

$$\mathbb{X} = M \oplus N, \tag{3.32}$$

where $M = P\mathbb{X}$ and $N = (1-P)\mathbb{X}$. *M* and *N* are closed linear manifolds of $\mathbb{X}$. A decomposition (3.32) of a Banach space into the direct sum of two closed linear manifolds defines a projection *P* on *M* along *N*. *P* is a linear operator on $\mathbb{X}$ to itself.

For a given closed linear manifold *M* of $\mathbb{X}$, it is not always possible to find a complementary subspace *N* such that (3.32) is true. In other words, *M* need not have a projection on it. On the other hand, *M* may have more than one projections.

3.2.4 Resolvents and spectra

An eigenvalue of *T* is defined as a complex number λ such that there exists a nonzero $u \in D(T) \subset \mathbb{X}$, called an eigenvector, such that $Tu = \lambda u$. In other words, λ is an eigenvalue if the null space $N(T - \lambda)$ is not 0; this null space is the geometric eigenspace for λ and its dimension is the geometric multiplicity of the eigenvalue λ. These definitions are often vacuous, however, since it may happen that *T* has no eigenvalue at all or, even if *T* has, there are not "sufficiently many" eigenvectors.

T is assumed to be a closed operator in $\mathbb{X}$. Then the same is true of $T-\zeta$ for any complex number ζ. It $T-\zeta$ is invertible with

$$R(\zeta)=R(\zeta,T)=(T-\zeta)^{-1}\in B(\mathbb{X}), \tag{3.33}$$

ζ is said to belong to the *resolvent set* of *T*. The operator-valued function $R(\zeta)$ thus defined on the resolvent set $P(T)$ is called the *resolvent* of *T*. Thus $R(\zeta)$ has domain $\mathbb{X}$ and range $D(T)$ for any $\zeta\in P(T)$.

Theorem 3.4 [33] Assume that $P(T)$ is not empty. In order that *T* commute with $A\in B(\mathbb{X})$, it is necessary that

$$R(\zeta)A=AR(\zeta) \tag{3.34}$$

for every $\zeta\in P(T)$, and it is sufficient that this hold for some $\zeta\in P(T)$.

The resolvent $R(\zeta)$ satisfies the *resolvent equation*

$$R(\zeta_1)-R(\zeta_2)=(\zeta_1-\zeta_2)R(\zeta_1)R(\zeta_2)$$

for every $\zeta_1,\zeta_2\in P(T)$. $TR(\zeta)$ is defined everywhere on $\mathbb{X}$. From this it follows that the Neumann series

$$R(\zeta)=\left[1-(\zeta-\zeta_0)R(\zeta_0)\right]^{-1}R(\zeta_0)=\sum_{n=0}^{\infty}(\zeta-\zeta_0)^n R(\zeta_0)^{n+1}$$

for the resolvent is valid.

Theorem 3.5 [33] $P(T)$ is an open set in the complex plane, and $R(\zeta)$ is (piecewise) holomorphic for $\zeta\in P(T)$. ("Piecewise" takes into account that $P(T)$ need not be connected.) Each component of $P(T)$ is the natural domain of $R(\zeta)$ ($R(\zeta)$ cannot be continued analytically beyond the boundary of $P(T)$).

The complementary set $\Sigma(T)$ (in the complex plane) of $P(T)$ is called the *spectrum* of *T*. Thus $\zeta\in\Sigma(T)$ if either $T-\zeta$ is not invertible or it is invertible but has range smaller than $\mathbb{X}$. It is possible for $\Sigma(T)$ to be empty or to cover the whole plane. It happens frequently that the spectrum is an uncountable set.

Consider an operator $T\in B(\mathbb{X})$. Then neither $P(T)$ nor $\Sigma(T)$ is empty. More precisely, $P(T)$ contains the exterior of the circle

$$|\zeta| = \operatorname{spr}T = \lim_{n\to\infty}\left\|T^n\right\|^{1/n} = \inf_{n\ge 1}\left\|T^n\right\|^{1/n} \qquad (3.35)$$

(which reduces to the single point $\zeta = 0$ if and only if $\operatorname{spr}T = 0$, that is, T is quasi-nilpotent), whereas there is at least one point of $\Sigma(T)$ on this circle. In particular $\Sigma(T)$ is a subset of the closed disk $|\zeta| \le \|T\|$. Let note also that

$$\left\|\zeta R(\zeta)+1\right\| \to 0, \quad \zeta \to \infty . \qquad (3.36)$$

The Neumann series on the right of

$$R(\zeta) = -\zeta^{-1}\left(1-\zeta^{-1}T\right)^{-1} = -\sum_{n=0}^{\infty}\zeta^{-n-1}T^n$$

converges for ζ outside of the circle (3.35). Since the convergence domain of this series is $|\zeta| > \operatorname{spr}T$, it follows that there is at least one point of $\Sigma(T)$ on (3.35) provided that $\operatorname{spr}T > 0$. If $\operatorname{spr}T = 0$, $\zeta = 0$ belongs to $\Sigma(T)$ because otherwise $R(\zeta)$ would be an entire function, contradicting (3.36) and Liouville's theorem. Liouville's theorem implies that $R(\zeta)$ is constant; since $R(\zeta) \to 0$ for $\zeta \to \infty$, this constant is 0.

3.3 Asymptotic algorithm of mean square stability analysis by the second Lyapunov method

Let analyze a real n-dimension linear stochastic difference equation, which is an iterative procedure in $\mathbb{R}^n$ defined by equality

$$x_t = A(y_t)x_{t-1},\ t \in \mathbb{N}, \qquad (3.37)$$

where $\{A(y),\ y \in \mathbb{Y}\}$ is continuous $n \times n$ matrix function on the metric compact $\mathbb{Y}$, $\sup_y \|A(y)\| = const < \infty$. Let $\{y_t,\ t \in \mathbb{N}\}$ be a homogeneous exponentially ergodic Feller Markov chain with phase space $\mathbb{Y}$, invariant measure $\mu(dy)$ and transition probability $p(y,dz)$. Under initial conditions $x_k = x$, $y_k = y$ the vector function $x_t(k,x,y) = X(t,k,y)x$, where $X(t,k,y) := \prod_{m=k+1}^{t} A(y_m)$, satisfies the equality (3.37). The above defined vector function $x_t(k,x,y)$ is called as a solution of (3.37) and matrix function $X(t,k,y)$ as a Cauchy matrix of (3.37). The equation (3.37) is called as exponentially mean square stable if there exist such a constants $c>0$ and $\lambda \in (0,1)$ that

$$E\left|x_t(k,x,y)\right|^2 \le c\lambda^{t-k}\left|x\right|^2$$

for any $y \in \mathbb{Y}$, $x \in \mathbb{R}^n$, $k \in \mathbb{N}$ and $t \ge k$. To derive mean square stability conditions for (3.37) the spectral properties of linear continuous operator

$$(\mathbf{A}q)(y) := \int_Y A^T(z)q(z)A(z)\,p(y,dz) \tag{3.38}$$

acting in the Banach space $\mathbb{V}$ of symmetric uniformly bounded continuous $n \times n$ matrix functions $\{q(y),\ y \in \mathbb{Y}\}$ with norm

$$\|q\| := \sup_{y\in\mathbb{Y},\ \|x\|=1} \left|(q(y)x,x)\right|,$$

where $\forall x \in \mathbb{R}^n$, $(q(y)x,x) := \sum_j \sum_k q_{kj}(y)x_k x_j$, are analyzed. The operator (3.38) leaves as invariant the reproducing cone [37]

$$\mathbb{K} := \left\{q \in V : \inf_{y\in\mathbb{Y},\|x\|=1} (q(y)x,x) \ge 0\right\}$$

with a set of inner points

$$\mathring{\mathbb{K}} := \left\{q \in V : \inf_{y\in\mathbb{Y},\|x\|=1} (q(y)x,x) > 0\right\}.$$

Let generalize the situation and choose near to constant matrix coefficients in (3.37), that is, matrix

$$A(y,\varepsilon) := M + \sum_{k=1}^{l} \varepsilon^k A_k(y) \tag{3.39}$$

instead of $A(y)$, where ε is small positive parameter. Applying (3.38) to matrix (3.39) an operator family $\mathbf{A}(\varepsilon)$ can be decomposed by power of ε

$$\mathbf{A}(\varepsilon) = \mathbf{A}_0 + \sum_{k=1}^{2l} \varepsilon^k \mathbf{A}_k \tag{3.40}$$

with some bounded operators $\mathbf{A}_k$, $k = 1,2,\dots,2l$ and $\mathbf{A}_0 q = \int_Y M^T q(u) M p(y,du)$.

If the spectrum $\sigma(M)$ of matrix M is situated within the circle $\{|\lambda| \le \gamma < 1\}$ the equation (3.37) is exponentially mean square stable for all sufficiently small positive ε. On the contrary if $\sigma(M) \cap \{|\lambda| > 1\} \ne 0$ there exists such a solution that becomes unrestrictedly large with $n \to \infty$. The problem arises only if spectral radius of matrix M is equal to one. Let suppose that matrix M has spectrum in a following form: $\sigma(M) = \sigma_0(M) \cup \sigma_\nu(M)$ divided to two parts $\sigma_0(M) \subset \{|\lambda| = 1\}$ and $\sigma_\nu(M) \subset \{|\lambda| \le \nu < 1\}$.

This section proposes an algorithm for finding the conditions, which guarantee decreasing of the iterations (3.37) for any sufficiently small ε. The main idea of algorithm is testing of positive definition property of a solution of the specially constructed matrix equation.

Theorem 3.6 [11] The next assertions are equivalent:

(i) equation (3.37) is exponentially mean square stable;

(ii) there exists such $q \in \mathring{\mathbb{K}}$ that

$$\mathbf{A}q - q = -I \tag{3.41}$$

where I is unit matrix;

(iii) real part of the spectrum $\sigma(\mathbf{A})$ of operator $\mathbf{A}$ is situated in the circle $\{z \in C : |z| < 1\}$.

Theorem 3.7 [9] There exists such a positive number ε_0 that for any $\varepsilon \in (0, \varepsilon_0)$ equation (3.37) with matrix (3.39) is exponentially mean square stable if and only if the equation:

$$\mathbf{A}(\varepsilon) q(\varepsilon) - q(\varepsilon) = -I \tag{3.42}$$

has solution in a form of Laurent series by powers of ε, that is,

$$q(y,\varepsilon) = \sum_{k=-d}^{\infty} \varepsilon^k q_k(y), \; d \ge 1 \tag{3.43}$$

with positive defined main part

$$\hat{q}(y,\varepsilon) := \sum_{k=-d}^{0} \varepsilon^k q_k(y). \tag{3.44}$$

Due to the third assertion of Theorem 3.6 and spectrum $\sigma(\mathbf{A}_0)$ property there exists [33] such a positive number $\bar{\varepsilon}$ that a solution of the equation (3.41) has decomposition

$$q(y,\varepsilon) = \sum_{k=-d}^{\infty} \varepsilon^k q_k(y) \tag{3.45}$$

for all $\varepsilon \in (0,\bar{\varepsilon})$, where matrices $\{q_k(y),\ k \ge -d\}$ are defined by the following equations:

$$\begin{aligned}
\mathbf{J}q_{-d} &= 0\\
\mathbf{J}q_{-d+1} &= -\mathbf{A}_1 q_{-d}\\
\mathbf{J}q_{-d+2} &= -\left(\mathbf{A}_1 q_{-d+1} + \mathbf{A}_2 q_{-d}\right)\\
&\dots\\
\mathbf{J}q_0 &= -\left(\mathbf{A}_1 q_{-1} + \mathbf{A}_2 q_{-2} + \ldots + \mathbf{A}_d q_{-d} + I\right)\\
&\dots
\end{aligned} \tag{3.46}$$

where $\mathbf{J}q = \mathbf{A}_0 q - q$. By definition of the above solution one can write

$$\left(\mathbf{A}(\varepsilon) - \mathbf{J}\right)\left(\hat{q}(y,\varepsilon) + \sum_{k=1}^{\infty} \varepsilon^k q_k(y)\right) = -I$$

and therefore the main part of this solution satisfies the equality

$$\left(\mathbf{A}(\varepsilon) - \mathbf{J}\right)\hat{q}(y,\varepsilon) = -\left(I - \left(\mathbf{A}(\varepsilon) - \mathbf{J}\right) + \sum_{k=1}^{\infty} \varepsilon^k q_k(y)\right). \tag{3.47}$$

Because for sufficiently small positive ε the right part of this equation is negative define matrix and analyzed difference equation (3.37) is exponentially mean square stable, the solution (3.44) should be positive defined.

The proposed algorithm uses the method of equating the coefficients corresponding to the same powers of the parameter ε in the equation (3.42) and the Fredholm alternative [37], which will be described below for the equation

$$\mathbf{J}q(y) = f(y) \tag{3.48}$$

in the space $\mathbb{V}$. Following the Fredholm alternative in order to conclude the existence of solution of (3.48) the orthogonality of the right part f of (3.48) to any solution of the equation

$$\mathbf{J}^* p = 0 \tag{3.49}$$

must be verified, where $\mathbf{J}^*$ is the conjugate of $\mathbf{J}$. A linear functional on the space $\mathbb{V}$ can be represented [37] in the form

$$\langle p, q \rangle := \int_{y \in \mathbb{Y}} Tr\, q(y)\, p(dy), \tag{3.50}$$

where the elements of symmetrical matrix valued function $p(dy)$ belong to the space $\mathbb{V}^*$ of matrix valued regular count additive measures. The space $\mathbb{V}$ can be considered as the tensor product $\mathbb{V} = \hat{M}_n(\mathbb{R}) \otimes \mathbb{C}(\mathbb{Y})$ of the space $\hat{\mathbb{M}}_n(\mathbb{R})$ of symmetric $n \times n$-

matrices and space $\mathbb{C}(\mathbb{Y})$. The space $\mathbb{V}^*$ is tensor product $\mathbb{V}^* = \hat{\mathbb{M}}_n(\mathbb{R}) \otimes \mathbb{C}^*(\mathbb{Y})$ where $\mathbb{C}^*(\mathbb{Y})$ is the space of regular count additive measures. Hence the operators $\mathbb{A}_0$ and $\mathbb{A}_0^*$ can be considered as a tensor product $\mathbb{A}_0 = \mathcal{A}_0 \otimes P$ and $\mathbb{A}_0^* = \mathcal{A}_0^* \otimes P^*$ where P is transition operator of Markov chain and the operators $\mathcal{A}_0$ un $\mathcal{A}_0^*$ are defined on each $q \in \hat{\mathbb{M}}_n(\mathbb{R})$ by equalities

$$\mathcal{A}_0 q = M^T qM, \ \mathcal{A}_0^* q = MqM^T .$$

Let suppose that the operator $\mathcal{A}_0$ has 1 as its spectrum point of multiplicity *l* and let $p_k,\ k = 1,\dots,l$ form basis in corresponding to 1 root subspace of adjoint operator $\mathcal{A}_0^*$. Any solution of (3.49) can be represented as the sum of the products $\rho_k(dy) = p_k\mu(dy)$. Thus, the equation (3.48) has solution if and only if

$$\langle \rho_j, f \rangle = \sum_{\mathbb{Y}} Tr\ f(y)\rho_j(y) = 0 \tag{3.51}$$

for all $j = 1,\dots,l$. Due to the assumptions this equation has *l* linearly independent constant solutions $g^{(j)}$, which form the basis in the corresponding to spectrum point 1 of the operator $\mathcal{A}_0$ root subspace of the space $\hat{\mathbb{M}}_n(\mathbb{R})$. Hence any solution of the first equation of (3.46) has a form

$$q_{-d}(y) := \sum_{j=1}^{l} c_j g^{(j)} \tag{3.52}$$

with arbitrary constants c_j.

In the second step at first it must be determined whether *d* is equal or larger than one. If *d=1* the following equation should be solved

$$\mathbf{J}q(y) = -I - \sum_{i=1}^{l} c_i v_i(y) \tag{3.53}$$

where

$$v_i(y) = \left[A_1^T(y) g_i A_1(y)\right] \tag{3.54}$$

for $i = 1,\dots,l$. If such numbers c_i can be chosen that the right part of (3.53) is orthogonal to each of above $p_s\mu(dy)$, where $s = 1,\dots,l$, then *d=1* and

$$\hat{q}(\varepsilon, y) = \frac{1}{\varepsilon}\sum_{i=1}^{l} c_i v_i(y) + q_0(y). \tag{3.55}$$

If the above mentioned c_i do not exist then we should put $d \ge 2$ and deal with equation

$$\mathbf{J}q(y) = -\sum_{i=1}^{l} c_i \mathbf{A}_1 q_i , \tag{3.56}$$

and look for c_i in such a way that right part of this equation be orthogonal to each $p_s\mu(dy),\ s = 1,\dots,l$. Now the solution $q_{-(d-1)}(y)$ of the second equation of (3.46) should be found. The condition of normal solvability of the second equation of (3.46) permits to find some of constants c_i, but some of them will be as an arbitrary. Now on this step the equation

$$\mathbf{J}q(y) = -I - \mathbf{A}_1 q_{-(d-1)}(y) - \mathbf{A}_2 q_{-d}(y) \tag{3.57}$$

will be analyzed with constants that have been found on the previous step. *d=2* if and only if the remaining numbers c_i can be found in such way that the right part of this equation is orthogonal to any $p_s\mu(dy),\ s = 1,\dots,l$. In this case the main part (3.44) of Laurent series (3.43) has form

$$\hat{q}(y,\varepsilon) = \frac{1}{\varepsilon^2} q_{-2} + \frac{1}{\varepsilon} q_{-1}(y) + q_0(y). \tag{3.58}$$

The equation (3.37) is exponentially mean square stable for all sufficiently small positive ε if and only if (3.58) is positive defined. If the right part of (3.57) is not orthogonal some of $p_s\mu(dy)$, $d \ge 3$ should be put and so on. So, step by step all c_i and the main part (3.44) of Laurent series (3.43) can be found.

3.3.1 Example 1

The following difference equation is given in $\mathbb{R}^2$:

$$x_t = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} + \varepsilon y_t \begin{pmatrix} 0 & 1 \\ 1 & b \end{pmatrix} \right) x_{t-1}, \tag{3.59}$$

where y_t is ergodic Markov process with three states $\{-1, 0, 1\}$ and the following transition probability matrix $P = \begin{pmatrix} 1/4 & 3/4 & 0 \\ 1/8 & 1/2 & 3/8 \\ 1/2 & 1/4 & 1/4 \end{pmatrix}$.

In this example $M=\begin{pmatrix}1 & 0\\ 0 & 1/2\end{pmatrix}$ and $A_1=\begin{pmatrix}0 & 1\\ 1 & b\end{pmatrix}$. Using that $\mu P=\mu$, the following equation system should be solved for finding the invariant measure μ:

$$\begin{cases}\frac{1}{4}\mu_1+\frac{1}{8}\mu_2+\frac{1}{2}\mu_3=\mu_1\\ \frac{3}{4}\mu_1+\frac{1}{2}\mu_2+\frac{1}{4}\mu_3=\mu_2\\ \frac{3}{8}\mu_2+\frac{1}{4}\mu_3=\mu_3\end{cases},$$

taking into account that $\mu_1+\mu_2+\mu_3=1$. In this case the invariant measure is equal to $\mu=\{1/4,\,1/2,\,1/4\}$. Now let find the linear continuous operator (3.38):

$$\begin{aligned}(\mathbf{A}q)(y)&=\int_{\mathbb{Y}}A^T(z,\varepsilon)q(z)A(z,\varepsilon)P(y,dz)=\\ &=\int_{\mathbb{Y}}(M+\varepsilon zA_1)^T q(z)(M+\varepsilon zA_1)P(y,dz)=\\ &=\int_{\mathbb{Y}}\left[M^Tq(z)M+\varepsilon\left(M^Tq(z)A_1+A_1^Tq(z)M\right)z+\varepsilon^2A_1^Tq(z)A_1z^2\right]P(y,dz)\end{aligned}$$

Therefore this operator has a form:

$$(\mathbf{A}q)(y)=(\mathbf{A}_0q)(y)+\varepsilon(\mathbf{A}_1q)(y)+\varepsilon^2(\mathbf{A}_2q)(y), \tag{3.60}$$

where

$$(\mathbf{A}_0q)(y)=\int_{\mathbb{Y}}M^Tq(z)Mp(y,dz), \tag{3.61}$$

$$(\mathbf{A}_1q)(y)=\int_{\mathbb{Y}}z\left(M^Tq(z)A_1+A_1^Tq(z)M\right)p(y,dz), \tag{3.62}$$

$$(\mathbf{A}_2q)(y)=\int_{\mathbb{Y}}z^2A_1^Tq(z)A_1p(y,dz). \tag{3.63}$$

Let choose d=1 in the first step. Thereby the following equations system should be solved:

$$\begin{cases}(\mathbf{J}q_{-1})(y)=0\\ (\mathbf{J}q_0)(y)=-((\mathbf{A}_1q_{-1})(y)+I)\end{cases}. \tag{3.64}$$

In the first equation of (3.64)

$$(\mathbf{J}q_{-1})(y)=(\mathbf{A}_0q_{-1})(y)-q_{-1}(y)=\int_{\mathbb{Y}}M^Tq_{-1}(z)MP(y,dz)-q_{-1}(y)=0$$

it can be assumed that $q_{-1}(y)$ is a constant: $q_{-1}(y)=\begin{pmatrix} q_{11}^{(-1)} & q_{12}^{(-1)} \\ q_{12}^{(-1)} & q_{22}^{(-1)} \end{pmatrix}$. It follows that $M^T q_{-1}(y)M - q_{-1}(y)=0$, because $\int_{\mathbb{Y}} P(y,dz)=1$. Consequently the solution of the first equation of system (3.64) has a form $q_{-1}=\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ with arbitrary constant *q*. Since matrices *M* and A_1 in this example are symmetric matrices, then a solution of the adjoint equation $\mathbf{J}^* p=0$ has a form $p(y)=q_{-1}\mu(y)$. Further the second equation of the system (3.64) should be solved:

$$(\mathbf{J}q_0)(y)=-I-\int_{\mathbb{Y}} z\left(M^T q_{-1}A_1 + A_1^T q_{-1}M\right)P(y,dz) := f^{(1)}(y).$$

Simplifying its right part $f_1(y)$ can be found for each value of *y*:

$$f^{(1)}(-1)=\begin{pmatrix} -1 & q/4 \\ q/4 & -1 \end{pmatrix};\ f^{(1)}(0)=\begin{pmatrix} -1 & -q/4 \\ -q/4 & -1 \end{pmatrix};\ f^{(1)}(1)=\begin{pmatrix} -1 & q/4 \\ q/4 & -1 \end{pmatrix}.$$

Now the existence condition for the second equation of the system (3.64) should be verified, that means, the following condition should be true $\left\langle f^{(1)}(y),\mu\right\rangle=0$. In this case $\left\langle f^{(1)}(y),\mu\right\rangle=-1\neq 0$, it means that the solution does not exist and the next step when *d*=2 should be chosen.

In case when *d=2* the following equations system should be analyzed:

$$\begin{cases} (\mathbf{J}q_{-2})(y)=0 \\ (\mathbf{J}q_{-1})(y)=-(\mathbf{A}_1 q_{-2})(y) \\ (\mathbf{J}q_0)(y)=-\left((\mathbf{A}_1 q_{-1})(y)+(\mathbf{A}_2 q_{-2})(y)+I\right) \end{cases} \tag{3.65}$$

Identically as in the first step, the solution of the first equation of the system (3.65)has a form $q_{-2}=\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$, where *q* is arbitrary constant. Using this solution the second equation of this system can be rewritten:

$$(\mathbf{J}q_{-1})(y)=-\int_{\mathbb{Y}} z\left(M^T q_{-2}A_1 + A_1^T q_{-2}M\right)P(y,dz) := f_1^{(2)}(y) \tag{3.66}$$

Solving it, the following three matrices are obtained:

$$f_1^{(2)}(-1)=\begin{pmatrix} 0 & q/4 \\ q/4 & 0 \end{pmatrix};\ f_1^{(2)}(0)=\begin{pmatrix} 0 & -q/4 \\ -q/4 & 0 \end{pmatrix};\ f_1^{(2)}(1)=\begin{pmatrix} 0 & q/4 \\ q/4 & 0 \end{pmatrix}.$$

Since $\left\langle f_1^{(2)}(y),\mu\right\rangle = 0$, then the solution of the equation (3.66) exists. To find the form of this solution, the following equations system are analyzed:

$$\begin{cases} (\mathbf{A}_0 q_{-1})(-1) - q_{-1}(-1) = f(-1) \\ (\mathbf{A}_0 q_{-1})(0) - q_{-1}(0) = f(0) \\ (\mathbf{A}_0 q_{-1})(1) - q_{-1}(1) = f(1) \end{cases}. \tag{3.67}$$

Let note matrices $q_{-1}(y)$ like $q_{-1}(y) = \begin{pmatrix} q_1^{(-1)}(y) & q_2^{(-1)}(y) \\ q_2^{(-1)}(y) & q_3^{(-1)}(y) \end{pmatrix}$. Then from (3.67) follows that:

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \cdot \left[\frac{1}{4} \cdot \begin{pmatrix} q_1^{(-1)}(-1) & q_2^{(-1)}(-1) \\ q_2^{(-1)}(-1) & q_3^{(-1)}(-1) \end{pmatrix} + \frac{3}{4} \cdot \begin{pmatrix} q_1^{(-1)}(0) & q_2^{(-1)}(0) \\ q_2^{(-1)}(0) & q_3^{(-1)}(0) \end{pmatrix} \right] \times$$

$$\times \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} - \begin{pmatrix} q_1^{(-1)}(-1) & q_2^{(-1)}(-1) \\ q_2^{(-1)}(-1) & q_3^{(-1)}(-1) \end{pmatrix} = \begin{pmatrix} 0 & \frac{q}{4} \\ \frac{q}{4} & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \cdot \left[\frac{1}{8} \cdot \begin{pmatrix} q_1^{(-1)}(-1) & q_2^{(-1)}(-1) \\ q_2^{(-1)}(-1) & q_3^{(-1)}(-1) \end{pmatrix} + \frac{1}{2} \cdot \begin{pmatrix} q_1^{(-1)}(0) & q_2^{(-1)}(0) \\ q_2^{(-1)}(0) & q_3^{(-1)}(0) \end{pmatrix} + \frac{3}{8} \cdot \begin{pmatrix} q_1^{(-1)}(1) & q_2^{(-1)}(1) \\ q_2^{(-1)}(1) & q_3^{(-1)}(1) \end{pmatrix} \right] \times$$

$$\times \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} - \begin{pmatrix} q_1^{(-1)}(0) & q_2^{(-1)}(0) \\ q_2^{(-1)}(0) & q_3^{(-1)}(0) \end{pmatrix} = \begin{pmatrix} 0 & -\frac{q}{4} \\ -\frac{q}{4} & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \cdot \left[\frac{1}{2} \cdot \begin{pmatrix} q_1^{(-1)}(-1) & q_2^{(-1)}(-1) \\ q_2^{(-1)}(-1) & q_3^{(-1)}(-1) \end{pmatrix} + \frac{1}{4} \cdot \begin{pmatrix} q_1^{(-1)}(0) & q_2^{(-1)}(0) \\ q_2^{(-1)}(0) & q_3^{(-1)}(0) \end{pmatrix} + \frac{1}{4} \cdot \begin{pmatrix} q_1^{(-1)}(1) & q_2^{(-1)}(1) \\ q_2^{(-1)}(1) & q_3^{(-1)}(1) \end{pmatrix} \right] \times$$

$$\times \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} - \begin{pmatrix} q_1^{(-1)}(1) & q_2^{(-1)}(1) \\ q_2^{(-1)}(1) & q_3^{(-1)}(1) \end{pmatrix} = \begin{pmatrix} 0 & \frac{q}{4} \\ \frac{q}{4} & 0 \end{pmatrix}.$$

Solving all these equations the form of the solution (3.66) can be obtained:

$$q_{-1}(-1) = \begin{pmatrix} 0 & -\frac{2}{11}q \\ -\frac{2}{11}q & 0 \end{pmatrix};\ q_{-1}(0) = \begin{pmatrix} 0 & \frac{8}{33}q \\ \frac{8}{33}q & 0 \end{pmatrix};\ q_{-1}(1) = \begin{pmatrix} 0 & -\frac{10}{33}q \\ -\frac{10}{33}q & 0 \end{pmatrix}.$$

Further the third equation of the system (3.65) should be solved:

$$\begin{aligned}(\mathbf{J}q_0)(y) = -\int_{\mathbb{Y}} z\left(M^T q_{-1}(z) A_1 + A_1^T q_{-1}(z) M\right) P(y,dz) - \\ -\int_{\mathbb{Y}} z^2 A_1^T q A_1 P(y,dz) - I := f_2^{(2)}(y)\end{aligned} \tag{3.68}$$

Solving the equation (3.68) and taking into account the results from the previous equations, as well as the condition for the existence of solution, the following matrices are obtained:

$$f_2^{(2)}(-1) = \begin{pmatrix} -\frac{1}{11}q-1 & -\frac{1}{22}qb \\ -\frac{1}{22}qb & -\frac{13}{44}q-1 \end{pmatrix}; \; f_2^{(2)}(0) = \begin{pmatrix} \frac{2}{11}q-1 & \frac{1}{11}qb \\ \frac{1}{11}qb & -\frac{9}{22}q-1 \end{pmatrix};$$

$$f_2^{(2)}(1) = \begin{pmatrix} -\frac{1}{33}q-1 & -\frac{1}{66}qb \\ -\frac{1}{66}qb & -\frac{101}{132}q-1 \end{pmatrix}.$$

Let assume that in general the matrix $f_2^{(2)}(y)$ can be written as $f_2^{(2)}(y) = \begin{pmatrix} c_1(y)-1 & c_2(y) \\ c_2(y) & c_3(y)-1 \end{pmatrix}$. Then, to make sure whether a solution exists for the third equation of the system (3.65), the following condition should be verified:

$$\int_{\mathbb{Y}} Sp\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} c_1(y)-1 & c_2(y) \\ c_2(y) & c_3(y)-1 \end{pmatrix}\right] d\mu = \int_{\mathbb{Y}} \left(c_1(y)-1\right) d\mu = 0. \tag{3.69}$$

From (3.69) it follows that the constant q should be $q = \frac{33}{2} = 16,5$. Therefore

$$f_2^{(2)}(-1) = \begin{pmatrix} -\frac{5}{2} & -\frac{3}{4}b \\ -\frac{3}{4}b & -\frac{47}{8} \end{pmatrix}; \; f_2^{(2)}(0) = \begin{pmatrix} 2 & \frac{3}{2}b \\ \frac{3}{2}b & -\frac{31}{4} \end{pmatrix}; \; f_2^{(2)}(1) = \begin{pmatrix} -\frac{3}{2} & -\frac{1}{4}b \\ -\frac{1}{4}b & -\frac{109}{8} \end{pmatrix}.$$

Next the following equations should be solved

$$\begin{cases} (\mathbf{A}_0 q_0)(-1) - q_0(-1) = f(-1) \\ (\mathbf{A}_0 q_0)(0) - q_0(0) = f(0) \\ (\mathbf{A}_0 q_0)(1) - q_0(1) = f(1) \end{cases} \tag{3.70}$$

to find the solution of the equation (3.68). Let note matrices $q_0(y)$ like $q_0(y) = \begin{pmatrix} q_1^{(0)}(y) & q_2^{(0)}(y) \\ q_2^{(0)}(y) & q_3^{(0)}(y) \end{pmatrix}$. Then from (3.70) follows that:

$$\begin{pmatrix}1 & 0\\ 0 & \frac{1}{2}\end{pmatrix}\cdot\left[\frac{1}{4}\cdot\begin{pmatrix}q_1^{(0)}(-1) & q_2^{(0)}(-1)\\ q_2^{(0)}(-1) & q_3^{(0)}(-1)\end{pmatrix}+\frac{3}{4}\cdot\begin{pmatrix}q_1^{(0)}(0) & q_2^{(0)}(0)\\ q_2^{(0)}(0) & q_3^{(0)}(0)\end{pmatrix}\right]\times$$

$$\times\begin{pmatrix}1 & 0\\ 0 & \frac{1}{2}\end{pmatrix}-\begin{pmatrix}q_1^{(0)}(-1) & q_2^{(0)}(-1)\\ q_2^{(0)}(-1) & q_3^{(0)}(-1)\end{pmatrix}=\begin{pmatrix}-\frac{5}{2} & -\frac{3}{4}b\\ -\frac{3}{4}b & -\frac{47}{8}\end{pmatrix},$$

$$\begin{pmatrix}1 & 0\\ 0 & \frac{1}{2}\end{pmatrix}\cdot\left[\frac{1}{8}\cdot\begin{pmatrix}q_1^{(0)}(-1) & q_2^{(0)}(-1)\\ q_2^{(0)}(-1) & q_3^{(0)}(-1)\end{pmatrix}+\frac{1}{2}\cdot\begin{pmatrix}q_1^{(0)}(0) & q_2^{(0)}(0)\\ q_2^{(0)}(0) & q_3^{(0)}(0)\end{pmatrix}+\frac{3}{8}\cdot\begin{pmatrix}q_1^{(0)}(1) & q_2^{(0)}(1)\\ q_2^{(0)}(1) & q_3^{(0)}(1)\end{pmatrix}\right]\times$$

$$\times\begin{pmatrix}1 & 0\\ 0 & \frac{1}{2}\end{pmatrix}-\begin{pmatrix}q_1^{(0)}(0) & q_2^{(0)}(0)\\ q_2^{(0)}(0) & q_3^{(0)}(0)\end{pmatrix}=\begin{pmatrix}2 & \frac{3}{2}b\\ \frac{3}{2}b & -\frac{31}{4}\end{pmatrix},$$

$$\begin{pmatrix}1 & 0\\ 0 & \frac{1}{2}\end{pmatrix}\cdot\left[\frac{1}{2}\cdot\begin{pmatrix}q_1^{(0)}(-1) & q_2^{(0)}(-1)\\ q_2^{(0)}(-1) & q_3^{(0)}(-1)\end{pmatrix}+\frac{1}{4}\cdot\begin{pmatrix}q_1^{(0)}(0) & q_2^{(0)}(0)\\ q_2^{(0)}(0) & q_3^{(0)}(0)\end{pmatrix}+\frac{1}{4}\cdot\begin{pmatrix}q_1^{(0)}(1) & q_2^{(0)}(1)\\ q_2^{(0)}(1) & q_3^{(0)}(1)\end{pmatrix}\right]\times$$

$$\times\begin{pmatrix}1 & 0\\ 0 & \frac{1}{2}\end{pmatrix}-\begin{pmatrix}q_1^{(0)}(1) & q_2^{(0)}(1)\\ q_2^{(0)}(1) & q_3^{(0)}(1)\end{pmatrix}=\begin{pmatrix}-\frac{3}{2} & -\frac{1}{4}b\\ -\frac{1}{4}b & -\frac{109}{8}\end{pmatrix}.$$

Now the matrices $q_0(y)$ can be found:

$$q_0(-1)=\begin{pmatrix}\frac{10}{3} & 0\\ 0 & \frac{1090}{129}\end{pmatrix};\ q_0(0)=\begin{pmatrix}0 & 2b\\ 2b & \frac{1408}{129}\end{pmatrix};\ q_0(1)=\begin{pmatrix}\frac{38}{9} & 0\\ 0 & \frac{2114}{129}\end{pmatrix}.$$

The Loran series main part of the given difference equation (3.59) has a form

$$q(y)=\frac{q_{-2}}{\varepsilon^2}+\frac{q_{-1}(y)}{\varepsilon}+q_0(y),$$

therefore:

$$\hat{q}(-1,\varepsilon)=\begin{pmatrix}\dfrac{16,5}{\varepsilon^2}+3,3 & -\dfrac{3}{\varepsilon}\\ -\dfrac{3}{\varepsilon} & 8,4\end{pmatrix};$$

$$\hat{q}(0,\varepsilon)=\begin{pmatrix}\dfrac{16,5}{\varepsilon^2} & \dfrac{4}{\varepsilon}-2b\\ \dfrac{4}{\varepsilon}-2b & 10,9\end{pmatrix}; \tag{3.71}$$

$$\hat{q}(1,\varepsilon)=\begin{pmatrix}\dfrac{16,5}{\varepsilon^2}+4,2 & \dfrac{5}{\varepsilon}\\ \dfrac{5}{\varepsilon} & 16,4\end{pmatrix}.$$

Since the matrices (3.71) are positive defined for all sufficiently small positive ε, the difference equation (3.59) is exponentially mean square stable for any constant *b*.

3.3.2 Example 2

Let analyze the difference equation rather similar to the first example, that is, the following equation is given in $\mathbb{R}^2$:

$$x_t=\left(\begin{pmatrix}1 & 0\\ 0 & 1/2\end{pmatrix}+\varepsilon y_t\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)x_{t-1}, \tag{3.72}$$

where y_t again is ergodic Markov process with three states $\{-1, 0, 1\}$ and with the same transition probability matrix $P=\begin{pmatrix}1/4 & 3/4 & 0\\ 1/8 & 1/2 & 3/8\\ 1/2 & 1/4 & 1/4\end{pmatrix}$. It means that also the invariant measure is the same $\mu=\{1/4, 1/2, 1/4\}$.

Because all calculations are identical to the previous example, only some intermediate results will be given. The first step, when *d=1*, does not give a solution, therefore *d=2* should be chosen. In this case again the equations system type (3.65) should be solved. The first equation of this system has a solution in a form $q_{-2}=\begin{pmatrix}q & 0\\ 0 & 0\end{pmatrix}$, where *q* is arbitrary constant; the second equation – in a form:

$$q_{-1}(-1)=\begin{pmatrix}-\dfrac{2}{9}q & 0\\ 0 & 0\end{pmatrix};\; q_{-1}(0)=\begin{pmatrix}\dfrac{4}{9}q & 0\\ 0 & 0\end{pmatrix};\; q_{-1}(1)=\begin{pmatrix}-\dfrac{2}{3}q & 0\\ 0 & 0\end{pmatrix}.$$

Verifying whether the solution of the system's second equation exists it can be obtained that the constant q should be $q = -\frac{144}{53}$. Therefore

$$q_{-2} = \begin{pmatrix} -\frac{144}{53} & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$q_{-1}(-1) = \begin{pmatrix} \frac{32}{53} & 0 \\ 0 & 0 \end{pmatrix};\ q_{-1}(0) = \begin{pmatrix} -\frac{64}{53} & 0 \\ 0 & 0 \end{pmatrix};\ q_{-1}(1) = \begin{pmatrix} \frac{96}{53} & 0 \\ 0 & 0 \end{pmatrix}.$$

Substituting these results in the system's (3.65) third equation $q_0(y)$ can be found:

$$q_0(-1) = \begin{pmatrix} -\frac{80}{159} & 0 \\ 0 & \frac{4}{5} \end{pmatrix};\ q_0(0) = \begin{pmatrix} \frac{124}{159} & 0 \\ 0 & -\frac{4}{3} \end{pmatrix};\ q_0(1) = \begin{pmatrix} -\frac{168}{159} & 0 \\ 0 & -\frac{1052}{45} \end{pmatrix}.$$

Consequently the Loran series main part of the given difference equation (3.72) has a form

$$\hat{q}(-1,\varepsilon) = \begin{pmatrix} \frac{-432+96\varepsilon-80\varepsilon^2}{159\varepsilon^2} & 0 \\ 0 & \frac{4}{5} \end{pmatrix};$$
$$\hat{q}(0,\varepsilon) = \begin{pmatrix} \frac{-432-192\varepsilon+124\varepsilon^2}{159\varepsilon^2} & 0 \\ 0 & -\frac{4}{3} \end{pmatrix}; \tag{3.73}$$
$$\hat{q}(1,\varepsilon) = \begin{pmatrix} \frac{-432+288\varepsilon-168\varepsilon^2}{159\varepsilon^2} & 0 \\ 0 & -\frac{1052}{45} \end{pmatrix}.$$

Evidently all matrices (3.73) are not positive defined for all sufficiently small positive ε, therefore the difference equation (3.72) is not exponentially mean square stable.

3.4 Spectrum decomposition

The spectrum $\sigma(\mathbf{A})$ of the operator $\mathbf{A}$ introduced in the section 3.3 defines the second moment of solutions of equations (3.37) dynamic. The exponential decreasing of the second moments is equivalent to spectrum $\sigma(\mathbf{A})$ location inside the unit radius circle $S_1 := \{|z|<1\}$ of the complex plane. This spectrum location analysis can be performed [7] for the equation (3.37) with coefficients dependent on independent identically distributed sequence $\{y_t\}$ or constant dealing only with the operator (3.38) restriction

$$\bar{\mathbf{A}}q := \int_Y A^T(z)qA(z)\mu(dz)$$

in the $\frac{(n+1)n}{2}$-dimensional subspace $\mathbb{V}_0 \subset \mathbb{V}$ of constant symmetric $n\times n$ matrices. This assertion greatly simplifies Lyapunov stability analysis of the equation (3.37).

An algorithm has delivered, which reduces the performances of the equation (3.37) with matrices given in a form (3.39) second moments dynamic to analysis of the operator $\mathbf{A}(\varepsilon)$ in the $\frac{(n+1)n}{2}$-dimensional subspace $\mathbb{V}(\varepsilon)\subset\mathbb{V}$. This subspace as well as the restriction matrix $\Lambda(\varepsilon)$ of the operator $\mathbf{A}$ may be defined by the specially constructed basis $\mathrm{B}(\varepsilon)$, analytically dependent on ε. The maximal by modulus real eigenvalue $\rho(\varepsilon)$ of matrix $\Lambda(\varepsilon)$ for sufficiently small $\varepsilon>0$ coincides with similar eigenvalue of operator $\mathbf{A}(\varepsilon)$. By terminology of [56] this number defines mean square Lyapunov index by formula $\lambda_2(\varepsilon)=\limsup_{t\to\infty}\frac{1}{2t}\ln E\left\{\left|x_t(k,x,y)\right|^2\right\}$ and this number defines behavior of the second moment $E\left\{\left|x_t(k,x,y)\right|^2\right\}$ as $t\to\infty$: if $\lambda_2(\varepsilon)<0$ sequence $E\left\{\left|x_t(k,x,y)\right|^2\right\}$ exponentially decreases, if $\lambda_2(\varepsilon)>0$ - exponentially increases.

Owing to exponential ergodicity assertion the defined on $\mathbb{C}(\mathbb{Y})$ Markov operator $\mathbf{P}$ has spectrum $\sigma(\mathbf{P})=\{1\}\cup\sigma_\rho$ where $\sigma_\rho\subset\left\{\lambda\in C:|\lambda|<\rho<1\right\}$. Therefore there exists such a number $\varepsilon_0>0$ that for any $\varepsilon\in(0,\varepsilon_0)$ spectrum $\sigma(\mathbf{A}(\varepsilon))$ of the perturbed operator $\mathbf{A}(\varepsilon)$ consists of two sets σ_γ and σ_ε where spectrum σ_ε is located in ε-neighborhood of the part of the operator $\mathbf{A}_0$ spectrum defined by equality $\hat{\sigma}(\mathbf{A}_0):=\left\{\lambda\nu:\lambda\in\sigma(M),\nu\in\sigma(M)\right\}$ and $\sigma_\gamma\subset\left\{\lambda\in C:|\lambda|<\gamma<1\right\}$. By definition the operator $\mathbf{A}_0$ leaves as invariant the space $\hat{\mathbb{M}}(\mathbb{R}^n)$ of symmetric $n\times n$ matrices with dimension $m = \frac{(n+1)n}{2}$. Hence the operators $\mathbf{A}(\varepsilon)$ have invariant subspaces $\mathbb{V}(\varepsilon) \subset \mathbb{V}$ with $\dim \mathbb{V}(\varepsilon) = m$ and a basis $B(\varepsilon) = \{b^1(\varepsilon, y), \ldots, b^m(\varepsilon, y)\}$ in a form

$$B(\varepsilon) = P(\varepsilon) B^0 \tag{3.74}$$

can be chosen [33], where $P(\varepsilon)$ is the total projector in $\mathbb{V}(\varepsilon)$ and B^0 is basis in $\hat{\mathbb{M}}(\mathbb{R}^n)$. Because of projector $P(\varepsilon)$ is an analytic function of ε [33] one can look for the basis as decomposition

$$B(\varepsilon) = B^0 + \varepsilon B^1 + \varepsilon^2 B^2 + \ldots, \tag{3.75}$$

where $B^0 = \{b^{01}, \ldots, b^{0m}\}$ and $B^j = \{b^{j1}(y), \ldots, b^{jm}(y)\}$ for any $j \in \mathbb{N}$. This means that all B^j are rows with $b^{jk}(y)$, $k = 1, \ldots, m$ as $n \times n$ matrices.

For the shortening of the computations let define some special operations involving a row

$$G = \{g_1, g_2, \ldots, g_m\}$$

of elements of the space $\mathbb{V}$ and a column

$$H = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_m \end{pmatrix}$$

of elements of the space $\mathbb{V}^*$. The first of these operations is

$$H \bullet G := \begin{pmatrix} [h_1, g_1] & \ldots & [h_1, g_m] \\ \ldots & \ldots & \ldots \\ [h_m, g_1] & \ldots & [h_m, g_m] \end{pmatrix},$$

where $[p, q]$ is the scalar product of $p \in \mathbb{V}^*$ and $q \in \mathbb{V}$, that is $[p, q] = \int_{\mathbb{Y}} q(y)\, p(dy)$. The second operation with row G is defined by

$$\mathbf{A}_j * G := \{\mathbf{A}_j g_1, \mathbf{A}_j g_2, \ldots, \mathbf{A}_j g_m\}, \; j = 1, \ldots, 2l$$

Let

$$C = \begin{pmatrix} c_{11} & c_{12} & \ldots & c_{1m} \\ c_{21} & c_{22} & \ldots & c_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \ldots & c_{mm} \end{pmatrix}$$

be a real matrix with m rows and m columns. We also write for above G, H and C

$$G \circ C := \left\{ \sum_{i=1}^{m} g_i c_{i1}, \sum_{i=1}^{m} g_i c_{i2}, \ldots, \sum_{i=1}^{m} g_i c_{im} \right\}$$

and

$$C \circ H := \left\{ \begin{matrix} \sum_{i=1}^{m} h_i c_{1i} \\ \sum_{i=1}^{m} h_i c_{2i} \\ \ldots \\ \sum_{i=1}^{m} h_i c_{mi} \end{matrix} \right\}.$$

Let $\Lambda(\varepsilon)$ be the matrix of restriction of the operator $\mathbf{A}(\varepsilon)$ on the subspace $\mathbb{V}(\varepsilon)$. This matrix can be obtained [33] from the expression

$$\mathbf{A}(\varepsilon) * B(\varepsilon) = B(\varepsilon) \circ \Lambda(\varepsilon), \tag{3.76}$$

where for the matrix $\Lambda(\varepsilon)$ also can be used the decomposition

$$\Lambda(\varepsilon) = \Lambda_0 + \varepsilon\Lambda_1 + \varepsilon^2\Lambda_2 + \ldots$$

Therefore equality (3.76) can be rewritten into the form

$$\left(\mathbf{A}_0 + \varepsilon\mathbf{A}_1 + \varepsilon^2\mathbf{A}_2 + \ldots\right) * \left(B^0 + \varepsilon B^1 + \varepsilon B^2 + \ldots\right) = \left(B^0 + \varepsilon B^1 + \varepsilon B^2 + \ldots\right) \circ \left(\Lambda_0 + \varepsilon\Lambda_1 + \varepsilon\Lambda_2 + \ldots\right).$$

We can look for $\Lambda_0, \Lambda_1, \Lambda_2, \ldots$ by equating the coefficients corresponding to the same powers of ε. At the same time we have to look for the components $B^0, B^1, B^2, \ldots$ of the basis decomposition given by (3.75).

On the first step we have to deal with the system of m equations:

$$\mathbf{A}_0 * B^0 - B^0 \circ \Lambda_0 = 0 \tag{3.77}$$

for the elements $b^{01}, b^{02}, \ldots, b^{0m}$ of the basis B^0. One can satisfy the equations with any basis $B^0 = P(0)\hat{\mathbb{M}}(\mathbb{R}^n) \subset \hat{\mathbb{M}}(\mathbb{R}^n)$ in the root subspace corresponding to the matrix Λ_0 of the operator $\mathbf{A}_0$ in this basis. The matrix Λ_0 is in a form

$$\Lambda_0 = \begin{pmatrix} \lambda_{11}^0 & \ldots & \lambda_{1m}^0 \\ \cdots & \cdots & \cdots \\ \lambda_{m1}^0 & \ldots & \lambda_{mm}^0 \end{pmatrix}.$$

On the second step there is the following system of equations which involves the m components of the basis B^1:

$$\mathbf{A}_0 * B^1 - B^1 \circ \Lambda_0 = B^0 \circ \Lambda_1 - \mathbf{A}_1 * B^0 . \qquad (3.78)$$

This system has solution if and only if the right part is orthogonal to m linearly independent solutions of the adjoint equation. It is obvious that the adjoint homogeneous equation for (3.78) based on the scalar product defined above and using the given notations has the form

$$\mathbf{A}_0^* * H - \Lambda_0 \circ H = 0$$

where the adjoint operator $\mathbf{A}_0^*$ is defined by $\left(\mathbf{A}_0^* h\right)(dz) = \int_{\mathbb{Y}} \tilde{A}_0^T h(dy)\, p(y, dz)$ for any count additive matrix measure $h \in \mathbb{V}^*$. It can be proven that due to ergodicity assertion we can present matrix measures $h_j(dy)$ in a form $h_j(dy) = \hat{h}_j \mu(dy),\ j = 1, \ldots, m$ where $\tilde{A}_0^T \hat{h}_j = \sum_{i=1}^m \lambda_{ji}^0 \hat{h}_i$ and the column $\hat{H}$ of constant matrixes $\hat{h}_j$ can be chosen in such a way that

$$b^{0k} \hat{h}_j = \begin{cases} 0, \ \text{ja } j \neq k, \\ 1, \ \text{ja } j = k. \end{cases} \qquad (3.79)$$

Next the condition of orthogonality of all elements of H with respect to the right part of (3.78) must be verified, that is

$$h_j \perp \left(\sum_{i=1}^m b^{0i} \lambda_{ik}^1 - \mathbf{A}_1 b^{0k} \right),\ \forall\, j, k = \overline{1, m}$$

or using previous notations

$$H \bullet \left(B^0 \circ \Lambda_1 - \mathbf{A}_1 * B^0 \right) = 0 . \qquad (3.80)$$

This condition permits to find the matrix Λ_1:

$$\lambda_{jk}^1 = \bar{\tilde{A}}_1 b^{0k} \hat{h}^j . \qquad (3.81)$$

Using (3.81) and solving the equation (3.78) the basis B^1 can be found.

On the next step the equation for B^2 what is based on the coefficients corresponding to ε^2 in the equation (3.76) should be analyzed:

$$\mathbf{A}_0 * B^2 - B^2 \circ \Lambda_0 = B^0 \circ \Lambda_2 + B^1 \circ \Lambda_1 - \mathbf{A}_1 * B^1 - \mathbf{A}_2 * B^0 \qquad (3.82)$$

Making the similar considerations as on the second step one can find the matrix Λ_2 using the following condition of orthogonality

$$H \bullet \left(B^0 \circ \Lambda_2 + B^1 \circ \Lambda_1 - \mathbf{A}_1 * B^1 - \mathbf{A}_2 * B^0 \right) = 0 . \qquad (3.83)$$

Therefore the elements for matrix Λ_2 can be expressed as

$$\lambda_{jk}^2 = \left(\bar{\tilde{A}}_1 b^{1k} + \bar{\tilde{A}}_2 b^{1k} - \sum_{i=1}^{m} b^{1i} \lambda_{ik}^1 \right) \hat{h}^j \tag{3.84}$$

and the matrix B^2 can be found.

3.4.1 Example

Let consider the following difference equation in $\mathbb{R}^2$:

$$x_t = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} + \varepsilon y_t \begin{pmatrix} 0 & 1 \\ 1 & b \end{pmatrix} \right) x_{t-1} \tag{3.85}$$

where y_t is Markov chain with states $\{-1, 0, 1\}$ and transition probability matrix

$$P = \begin{pmatrix} 1/4 & 3/4 & 0 \\ 1/8 & 1/2 & 3/8 \\ 1/2 & 1/4 & 1/4 \end{pmatrix}$$

It is easy to find that the invariant measure is equal to $\mu = \{1/4, 1/2, 1/4\}$. The linear operator should be analyzed

$$(\mathbf{A}\vec{q})(y) = (\mathbf{A}_0\vec{q})(y) + \varepsilon(\mathbf{A}_1\vec{q})(y) + \varepsilon^2(\mathbf{A}_2\vec{q})(y)$$

where

$$(\mathbf{A}_0\vec{q})(y) = \int_Y \tilde{A}_0 \vec{q}(z)\, p(y, dz),$$

$$(\mathbf{A}_1\vec{q})(y) = \int_Y \tilde{A}_1(z) \vec{q}(z)\, p(y, dz),$$

$$(\mathbf{A}_2\vec{q})(y) = \int_Y \tilde{A}_2(z) \vec{q}(z)\, p(y, dz)$$

and matrices $\tilde{A}_0$, $\tilde{A}_1$, $\tilde{A}_2$ for finding operators $(\mathbf{A}_0\vec{q})(y)$, $(\mathbf{A}_1\vec{q})(y)$, $(\mathbf{A}_2\vec{q})(y)$ are as followings: $\tilde{A}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}$, $\tilde{A}_1(z) = \begin{pmatrix} 0 & 2z & 0 \\ z & bz & z/2 \\ 0 & z & bz \end{pmatrix}$, $\tilde{A}_2(z) = \begin{pmatrix} 0 & 0 & z^2 \\ 0 & z^2 & bz^2 \\ z^2 & 2bz^2 & b^2z^2 \end{pmatrix}$.

In the first equation (3.77) a basis can be chosen as $B_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and therefore $\Lambda_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}$. Substituting known results in the second equation

(3.78) and following the steps described in the algorithm it is possible to obtain $\Lambda_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and corresponding basis B_1 for each y value:

$$B_1(-1) = \begin{pmatrix} 1 & 0 & 0 \\ -2/11 & 1-2b/3 & 0 \\ 0 & -4/11 & 0 \end{pmatrix},\ B_1(0) = \begin{pmatrix} 1 & 2/3 & 0 \\ 8/33 & 1 & 1/3 \\ 0 & -16/33 & 4b/3 \end{pmatrix},$$

$$B_1(1) = \begin{pmatrix} 1 & -4/3 & 0 \\ -10/33 & 1-10b/9 & -2/3 \\ 0 & -76/33 & -8b/9 \end{pmatrix}.$$

Considering all obtained results it is possible to find Λ_2 using equation (3.82). Leaving out calculations Λ_2 can be presented as

$$\Lambda_2 = \begin{pmatrix} -2/33 & -2b/9 & 1/6 \\ -b/33 & -5/66-b^2/9 & 2b/9 \\ 31/66 & 40b/99 & -1/6-5b^2/18 \end{pmatrix}.$$

Therefore

$$\Lambda = \Lambda_0 + \varepsilon\Lambda_1 + \varepsilon^2\Lambda_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} -2/33 & -2b/9 & 1/6 \\ -b/33 & -5/66-b^2/9 & 2b/9 \\ 31/66 & 40b/99 & -1/6-5b^2/18 \end{pmatrix}.$$

For higher accuracy of Λ the algorithm can be continued in the similar way as described steps till Λ_2.

4 Covariance analysis of linear Markov iterations

4.1 Analysis of the first moments

Henceforward a linear n-dimensional column-vector space $\mathbb{R}^n$ will be viewed as Euclid space with a scalar product

$$u \in \mathbb{R}^n,\ v \in \mathbb{R}^n:\ (u,v) = u^T v\,.$$

Let assume that a Markov sequence $\vec{y} := \{y_t,\ t \in \mathbb{N}\}$ is given in a filtrated probability space $(\Omega, \mathfrak{F}, \mathfrak{F}^t, P)$, where $\{\mathfrak{F}^t\}$ is a minimal filtration harmonizing it. The following assumptions are necessary to get formulas convenient for use:

- matrix function $\{A(y), y \in \mathbb{Y}\}$ is continuous;
- Markov chain phase space $\mathbb{Y}$ is a metric and compact space;
- transition probabilities $p(y,dz)$ have Feller property, that is, if functions $\{u(y), y \in \mathbb{Y}\}$ are continuous, continuity of functions $\{(\mathcal{P}u)(y), y \in \mathbb{Y}\}$ follows, where

$$(\mathcal{P}u)(y) =: \int_{\mathbb{Y}} u(z)\, p(y,dz)\,; \tag{4.1}$$

- only one probability measure exists on space $\mathbb{Y}$, which satisfies the equality

$$(\mathcal{P}^*\mu)(dz) =: \int_{\mathbb{Y}} \mu(dy)\, p(y,dz)\,; \tag{4.2}$$

- such a positive number $\rho < 1$ exists, that the spectrum of operator $\mathcal{P}$ defined on $\mathbb{C}(\mathbb{Y})$ can be expressed in a form

$$\sigma(\mathcal{P}) = \{1\} \cup \sigma_\rho,\ \ \sigma_\rho \in \{\lambda \in C : |\lambda| < \rho\} \tag{4.3}$$

(exponential ergodicity).

The following denotations will be used:

$$s \in \mathbb{N}:\ X_s^s = I; \qquad t > s:\ X_s^t := \prod_{k=s+1}^{t} A(y_k)\,. \tag{4.4}$$

The solution of the linear difference equation with Markov coefficients

$$x_t = A(y_t)\, x_{t-1}\,, \tag{4.5}$$

where $\{y_t, t \in \mathbb{N}\}$ is a homogeneous Markov chain with phase space $\mathbb{Y}$ and transition probabilities $p(y, dz)$, can be written in a form $x_t = X_t^s x_s$ for all $s \in \mathbb{N}$, $t \geq s$. As the sequence $\{y_t, t \in \mathbb{N}\}$ has Markov property, probabilistic characteristics of matrix $\{X_s^t, t \geq s\}$ depend only on probabilistic characteristics of y_s. Let define an operator in a space of continuous *n*-dimensional reproductions $\mathbb{C}(\mathbb{Y} \to \mathbb{R}^n) := \mathbb{C}_n(\mathbb{Y})$:

$$y \in \mathbb{Y}, u \in \mathbb{C}_n(\mathbb{Y}): \; (\mathbf{A}u)(y) = \int_{\mathbb{Y}} A^T(z) u(z) p(y, dz) \tag{4.6}$$

Since transition probabilities have Feller property, $\mathbf{A}u \in \mathbb{C}_n(\mathbb{Y})$ is valid. Hereto for all $u \in \mathbb{C}_n(\mathbb{Y})$

$$\|\mathbf{A}u\| \leq \sup_{z \in \mathbb{Y}} \left| A^T(z) u(z) \right| \leq \sup_{z \in \mathbb{Y}} \left\| A^T(z) \right\| \sup_{z \in \mathbb{Y}} \|u(z)\|,$$

follows from (4.6) and therefore

$$\|\mathbf{A}\| := \sup_{\|u\|=1} \|\mathbf{A}u\| \leq \sup_{z \in \mathbb{Y}} \|A(z)\|, \tag{4.7}$$

that is, the operator $\mathbf{A}$ is linear continuous operator in $\mathbb{C}_n(\mathbb{Y})$.

Lemma 4.1 For any $s \in \mathbb{N}$, $t > 0$, $v \in \mathbb{C}_n(\mathbb{Y})$, $x \in \mathbb{R}^n$

$$E\left\{ \left(X_s^{s+t} x, v(y_{s+t}) \right) / \mathfrak{F}^s \right\} = \left(x, \left(\mathbf{A}^t v \right)(y_s) \right) \tag{4.8}$$

Proof. An induction method is used. For $t = 1$ and any $s \in \mathbb{N}$ the equality (4.8) follows from Markov property of sequence $\vec{y}$:

$$E\left\{ \left(X_s^{s+1} x, v(y_{s+1}) \right) / \mathfrak{F}^s \right\} = E\left\{ \left(x, A^T(y_{s+1}) v(y_{s+1}) \right) / y_s \right\} = \left(x, (\mathbf{A}v)(y_s) \right).$$

To be sure about correctness of the lemma's statement for $t = m + 1$, this equality is used taking $s + m$ instead of *s*, assuming that (4.8) is true for $t = m$:

$$E\left\{ \left(X_s^{s+m+1} x, v(y_{s+m+1}) \right) / \mathfrak{F}^s \right\} =$$

$$E\left\{ E\left\{ \left(X_s^{s+m} x, A^T(y_{s+m+1}) v(y_{s+m+1}) \right) / \mathfrak{F}^{s+m} \right\} / \mathfrak{F}^s \right\} =$$

$$E\left\{ E\left\{ \left(z, A^T(y_{s+m+1}) v(y_{s+m+1}) \right) / \mathfrak{F}^{s+m} \right\} \Big|_{z = X_s^{s+m} x} / \mathfrak{F}^s \right\} =$$

$$E\left\{ \left(X_s^{s+m} x, (\mathbf{A}v)(y_{s+m}) \right) / \mathfrak{F}^s \right\} = \left(x, \left(\mathbf{A}^m (\mathbf{A}v) \right)(y_s) \right) = \left(x, \left(\mathbf{A}^{m+1} v \right)(y_s) \right).$$

Lemma is proved.

Theorem 4.1 Let elements of sequence $\{y_t, t \in \mathbb{N}\}$ are independent and identically distributed. Then

(i) operator $\mathbf{A}$ leaves as invariant a subspace $\mathbb{R}^n \subset \mathbb{C}_n(\mathbb{Y})$ and restriction $\bar{\mathbf{A}}$ of operator $\mathbf{A}$ in this subspace is defined by equality

$$v \in \mathbb{R}^n : \ \bar{\mathbf{A}}v = \bar{A}^T v \tag{4.9}$$

where $\bar{A} = E\{A(y_0)\}$;

(ii) for each $s \in \mathbb{N}$, each $t > s$ and each $\mathfrak{F}^t$-adapted solution $\{x_t, t \geq 0\}$ of equation (4.5) the following equality is into force:

$$E\{x_t\} = \bar{A}^{t-s} E\{x_s\}. \tag{4.10}$$

Proof. Since in the theorem conditions $p(y, dz)$ is independent from *y*, that is, $p(y, dz) \equiv p(dz)$, then assertion (i) directly follows from (4.6):

$$v \in \mathbb{R}^n : \ \mathbf{A}v = \int_{\mathbb{Y}} A^T(y) v p(dz) = E\{A^T(y_0)\} v = \bar{\mathbf{A}}v.$$

Further, since elements of the sequence $\{y_t, t \in \mathbb{N}\}$ are independent, according to filtration definition a random variable y_t does not depend on $\mathfrak{F}^{t-1}$-measurable random vector x_{t-1}. Therefore for each $v \in \mathbb{R}^n$

$$\begin{aligned}(E\{x_t\}, v) &= E\left\{E\left\{\left(x_{t-1}, A^T(y_t)v\right)/\mathfrak{F}^{t-1}\right\}\right\} = E\left\{\left(x_{t-1}, E\left\{A^T(y_t)/\mathfrak{F}^{t-1}\right\}v\right)\right\} \\ &= \left(E\{x_{t-1}\}, E\left\{A^T(y_0)\right\}v\right) = \left(E\{x_{t-1}\}, \bar{A}^T v\right)\end{aligned}$$

is into force. Applying this formula sequentially for all $t \in \{s+1, t-1\}$ the theorem is proved.

One of the most common [11] methods for difference equations, which can be represented in a form

$$t \in \mathbb{N}: \ y_t = G_t y_{t-1}, \tag{4.11}$$

analysis in $\mathbb{R}^n$ is variables substitution $y_t = B_t z_t$, where B_t is a basis matrix of some variable in $\mathbb{R}^n$. If such a sequence of matrices can be found that $z_t = H z_{t-1}$, then from the equation (4.11) can be changed to an equation with constant coefficients and each its solution can be written in a form

$$y_t = B_t z_t = B_t H^{t-s} z_s = B_t H^{t-s} B_s^{-1} y_s.$$

This property in theory of difference equations is called reducibility. The equation (4.5) is mean reducible, if such a continuous matrix function $\{B(y), y \in \mathbb{Y}\}$ and such a matrix Λ exist, that for all $s \in \mathbb{N}$ and $t > s$ the following equality is into force

$$E\{B(y_t)x_t / \mathfrak{F}^s\} = \Lambda^{t-s} B(y_s) x_s. \tag{4.12}$$

Further the possibility of (4.5) mean reducibility will be considered in the case when the matrix function $\{A(y), y \in \mathbb{Y}\}$ is near to constant and can be given in a form of uniformly converging sequence:

$$A(y) := A_0 + \varepsilon \sum_{k=0}^{\infty} \varepsilon^k A_{k+1}(y) \tag{4.13}$$

where $\varepsilon \in (0,1)$ is a small parameter.

At the beginning some additional constructions are needed. Let define a tensor product for elements of spaces $\mathbb{C}(\mathbb{Y})$ and $\mathbb{R}^n$ as a product of scalar function with vector, and let represent a space $\mathbb{C}_n(\mathbb{Y})$ as a tensor product of spaces $\mathbb{C}_n(\mathbb{Y}) = \mathbb{C}(\mathbb{Y}) \otimes \mathbb{R}^n$. A tensor product of linear spaces $\mathbb{H}$ and $\mathbb{G}$ defines as a linear span of tensor set $\{h \otimes g, h \in \mathbb{H}, g \in \mathbb{G}\}$. The operator (4.6), which corresponds to matrix (4.13), can be expressed in a form $\mathbf{A}(\varepsilon) = \mathbf{A}_0 + \varepsilon \sum_{k=0}^{\infty} \varepsilon^k \mathbf{A}_{k+1}(\varepsilon)$, hereto the operator $\mathbf{A}_0$ leaves as invariant subspace $\mathbb{R}^n$, and it can be represented as a tensor product of operators $\mathbf{A}_0 = \mathcal{P} \otimes A_0^T$:

$$h \in \mathbb{C}(\mathbb{Y}), g \in \mathbb{R}^n: \ \mathbf{A}_0(h \otimes g) = \mathcal{P}h \otimes A_0^T g,$$

where $\mathcal{P}$ is a Markov operator defined by equality (4.1). The tensor representation of operator allows to simplify finding the spectrum and resolvent using the spectrum and resolvent of operators which define it [34]. Taking into account the assumption about exponential ergodicity of (4.3) the operator $\mathbf{A}_0$ spectrum can be expressed in a form:

$$\sigma(\mathbf{A}_0) = \{\lambda_1\lambda_2 : \lambda_1 \in \sigma(\mathcal{P}), \lambda_2 \in \sigma(A_0)\} = \sigma(A_0) \cup \sigma_\rho \tag{4.14}$$

where $\sigma_\rho := \{\lambda_1\lambda_2 : \lambda_1 \in \sigma(\mathcal{P}), \lambda_2 \in \sigma_\rho\}$. As main assumption for mean reducibility of the equation (4.5) is disjunction of sets in spectrum decomposition (4.14), that is,

$$\sigma(A_0) \cap \sigma_\rho = \varnothing. \tag{4.15}$$

It makes possible to offer an asymptotical method, which is based on the decomposition of operator $\mathbf{A}(\varepsilon)$ spectral projection [33] by powers of a small parameter ε.

Conjugated space of $\mathbb{C}_n(\mathbb{Y})$ [17] is a space of vector-valued measures $\mathbb{C}_n^*(\mathbb{Y})$, and scalar product of elements $v \in \mathbb{C}_n(\mathbb{Y})$ and $g \in \mathbb{C}_n^*(\mathbb{Y})$ is defined by equality

$$\langle g, v\rangle := \int_{\mathbb{Y}} \big(g(dy), v(y)\big). \tag{4.16}$$

Using the definition of conjugated operators $\langle \mathbf{A}_0^* g, v\rangle = \langle g, \mathbf{A}_0 v\rangle$, that is,

$$\int_{\mathbb{Y}} \big(g(dy), A_0^T v(z)\big)\, p(y,dz) = \left(\int_{\mathbb{Y}} A_0 g(dy)\, p(y,dz), v(z)\right)$$

it's form can be found:

$$\big(\mathbf{A}_0^* g\big)(dz) = \int_{\mathbb{Y}} A_0 g(dy)\, p(y,dz). \tag{4.17}$$

The space $\mathbb{C}_n^*(\mathbb{Y})$ can be expressed also as a tensor product of the space consisting scalar count additive measures $\mathbb{C}^*(\mathbb{Y})$ and $\mathbb{R}^n$. Using the definition of tensor product the following equality follows:

$$a \in \mathbb{C}^*(\mathbb{Y}), b \in \mathbb{R}^n : \big(\mathbf{A}_0^*(a \otimes b)\big)(dy) := \big(\mathcal{P}^* a\big)(dy) \otimes A_0 b. \tag{4.18}$$

Lemma 4.2 If all above mentioned assumptions are into force, then for sufficiently small $\bar{\varepsilon} > 0$ and all $|\varepsilon| < \bar{\varepsilon}$ a difference equation is mean reducible, hereto the matrix function $\{B(y,\varepsilon), y \in \mathbb{Y}\}$ is a basis in operator $\mathbf{A}(\varepsilon)$ root subspace which corresponds to the spectrum $\sigma_0(\varepsilon)$ part which is defined by equality $\lim_{\varepsilon \to 0} \sigma_0(\varepsilon) = \sigma_0$, but matrix $\Lambda(\varepsilon)$ is operator $\mathbf{A}(\varepsilon)$ restriction matrix to this root subspace. For each $|\varepsilon| < \bar{\varepsilon}$ $n\times n$-matrix function of basis $\{B(y,\varepsilon), y \in \mathbb{Y}\}$ and constant $n\times n$-matrix $\Lambda(\varepsilon)$ unambiguously are defined by equality

$$y \in \mathbb{Y}, |\varepsilon| < \bar{\varepsilon} : \big(\mathbf{A}(\varepsilon) B\big)(y,\varepsilon) = B(y,\varepsilon)\Lambda^T(\varepsilon) \tag{4.19}$$

Proof. Since the assumptions about exponential ergodicity of Markov process and possibility to express operator $\mathbf{A}_0$ spectrum in a form (4.15) are into force, the dimension of the root subspace of this operator, which corresponds to spectrum part $\sigma(A_0)$, is equal to *n*. Hereto, taking into account the assumption about uniform convergence by $y \in \mathbb{Y}$ of matrices sequence $A(y,\varepsilon)$, such a positive number $\bar{\varepsilon}$ exists, that an operator family $\mathbf{A}(\varepsilon)$ is analytically dependent on parameter ε at $|\varepsilon| < \bar{\varepsilon}$ [33]. The equality $\lim\limits_{\varepsilon \to 0} \sigma_0(\varepsilon) = \sigma_0$ unambiguously defines an isolated operator $\mathbf{A}(\varepsilon)$ spectrum part for all $|\varepsilon| < \bar{\varepsilon}$, but dimension of the root subspace corresponding to this spectrum part is equal to *n*. Therefore a basis can be chosen in this root subspace using *n* elements from $\mathbb{C}_n(\mathbb{Y})$ in a form of matrices function:

$$B(y,\varepsilon) = \{b_1(y,\varepsilon), b_2(y,\varepsilon), \ldots, b_n(y,\varepsilon)\} \tag{4.20}$$

The operator $\mathbf{A}(\varepsilon)$ restriction matrix in this subspace can be obtained consecutive applying operator $\mathbf{A}(\varepsilon)$ for each $\varepsilon \in (-\bar{\varepsilon}, \bar{\varepsilon})$ to the elements of basis (4.20) and to the decomposition by this basis:

$$\begin{aligned}(\mathbf{A}(\varepsilon)B)(y,\varepsilon) &:= \{(\mathbf{A}(\varepsilon)b_j)(y,\varepsilon),\ j = 1,2,\ldots,n\} = \\ &= \left\{\sum_{k=1}^{n} \lambda_{jk}(\varepsilon) b_j(y,\varepsilon),\ j = 1,2,\ldots,n\right\} := B(y,\varepsilon)\Lambda^T(\varepsilon)\end{aligned} \tag{4.21}$$

Substituting (4.20) in the equality (4.12) one can get:

$$\begin{aligned}E\{B(y_t,\varepsilon)x_t / \mathfrak{F}^s\} &= \{E\{(x_t, b_1(y,\varepsilon))/\mathfrak{F}^s\},\ldots,E\{(x_t, b_n(y,\varepsilon))/\mathfrak{F}^s\}\} = \\ &= \left\{\left\{\left(x_s, \mathbf{A}(\varepsilon)^{t-s} b_1\right)(y_s,\varepsilon)\right\},\ldots,\left\{\left(x_s, \mathbf{A}(\varepsilon)^{t-s} b_n\right)(y_s,\varepsilon)\right\}\right\} = \\ &= \left(\mathbf{A}(\varepsilon)^{t-s} B\right)(y_s,\varepsilon)x_s = B(y_s,\varepsilon)\left(\Lambda^T(\varepsilon)\right)^{t-s} x_s\end{aligned}$$

and the lemma is proved.

For the description of the construction algorithm for basis matrix (4.20) and matrix $\Lambda(\varepsilon)$ the decompositions of these matrices in a form of uniformly converged sequences by powers of a small parameter ε:

$$\begin{aligned}\Lambda(\varepsilon) &:= \Lambda_0 + \varepsilon \sum_{k=0}^{\infty} \Lambda_{k+1} \\ B(y,\varepsilon) &:= B_0 + \varepsilon \sum_{k=0}^{\infty} B_{k+1}(y)\end{aligned} \tag{4.22}$$

and also the decomposition of operator $\mathbf{A}(\varepsilon)$ in a form of uniformly converged sequence by powers of a small parameter ε:

$$\mathbf{A}(\varepsilon) := \mathbf{A}_0 + \varepsilon \sum_{k=0}^{\infty} \varepsilon^k \mathbf{A}_{k+1} \tag{4.23}$$

are used, where

$$\left(\mathbf{A}_j v\right)(y) = \int_{\mathbb{Y}} A_j^T(z) v(z)\, p(y, dz). \tag{4.24}$$

For each sufficiently small ε these decompositions can be substituted in the expression (4.21). Equating coefficients of equal powers of ε and taking into account (4.24), the following equations can be obtained:

$$\mathbf{A}_0 B_0 = B_0 \Lambda_0^T \tag{4.25}$$

$$\mathbf{A}_0 B_1 - B_1 \Lambda_0^T = B_0 \Lambda_1^T - \mathbf{A}_1 B_0 \tag{4.26}$$

$$\mathbf{A}_0 B_2 - B_2 \Lambda_0^T = B_0 \Lambda_2^T + B_1 \Lambda_1^T - \mathbf{A}_0 B_2 - \mathbf{A}_1 B_1 \tag{4.27}$$

and so on. These equations can be used for finding the unknown elements of series (4.22). Taking unit matrix $B_0 := I$ as basis in $\mathbb{R}^n$ and substituting it in the equation (4.25), $\Lambda_0^T = A_0^T$ can be find, that is, $\Lambda_0 = A_0$. Let define an operator

$$y \in \mathbb{Y}, v \in \hat{\mathbb{C}}:$$

$$\begin{aligned}(\mathbb{L}v)(y) &:= (\mathbf{A}_0 v)(y) - v(y) A_0^T := \\ &:= \int_{\mathbb{Y}} A_0^T \left(v(z) - v(y)\right) p(y, dz) + A_0^T v(y) - v(y) A_0^T := \\ &:= (\mathbb{H}v)(y) + (\mathbb{G}v)(y)\end{aligned} \tag{4.28}$$

for the elements of continuous matrix functions space $\hat{\mathbb{C}}$. Looking at $\hat{\mathbb{C}}$ as at $\mathbb{R}^{n^2}$, similarly as in $\mathbb{C}_n(\mathbb{Y})$ case, count additive matrix-valued measure $\hat{\mathbb{C}}^*$ can be found, which will be as conjugated space, and a scalar product of elements $g \in \hat{\mathbb{C}}^*$ and $v \in \hat{\mathbb{C}}$ can be defined by formula

$$\langle g, v \rangle := Tr\left\{\int_{\mathbb{Y}} v^T(y) g(dy)\right\}, \tag{4.29}$$

where $Tr\{\ \}$ is a matrix trace. Using this definition, analogically as (4.17), the conjugated operator $\mathbb{L}^*$ of operator $\mathbb{L}$ can be found:

$$\begin{aligned}(\mathbb{L}^* g)(dz) &:= (\mathbf{A}_0^* g)(dz) - g(dz) A_0 := \\ &:= \int_{\mathbb{Y}} A_0 \left(g(dy) - g(dz)\right) p(y, dz) + A_0 g(dz) - g(dz) A_0 := \\ &:= (\mathbb{H}^* g)(dy) + (\mathbb{G}^* g)(dy)\end{aligned} \tag{4.30}$$

Similarly as in $\mathbb{C}_n(\mathbb{Y})$ case, $\hat{\mathbb{C}}$ can be viewed as a tensor product of $\mathbb{C}(\mathbb{Y})$ and the space of constant matrices $\mathbb{M}_n$, but a space $\hat{\mathbb{C}}^*$ - a tensor product of $\mathbb{C}^*(\mathbb{Y})$ and the space of constant matrices $\mathbb{M}_n$. The operator $\mathbb{L}^*$ in accordance with (4.30) is a tensor sum of operator $\mathbb{H}^*$ acting in $\mathbb{C}^*(\mathbb{Y})$ and operator $\mathbb{G}^*$ acting in $\mathbb{M}_n$. Therefore the kernel of operator $\mathbb{L}^*$ consists of elements in a form

$$M \in Ker\{\mathbb{L}^*\} \Leftrightarrow M = \mu(dy)M, \quad M \in \{M_n : A_0 M - MA_0 = 0\}. \tag{4.31}$$

Now (4.26) can be rewritten in a form

$$\mathbf{A}_0 B_1(y) - B_1(y)\Lambda_0^T = \Lambda_1^T - A_1^T(y)$$

and Fredholm theorem about normal solvability can be applied. The right side orthogonality of this equation respecting to the all elements of operator $\mathbb{L}^*$ kernel is necessary and sufficiently to ensure that a solution exists for the equation in a form

$$\mathbf{A}_0 B_2 - B_2\Lambda_0^T = C$$

for all $C \in \hat{\mathbb{C}}$. Consequently for solvability of (4.26), taking into account that $B_0 = I$, it is necessary to fulfill the equality

$$\int_{\mathbb{Y}} \left[\Lambda_1 - A_1(y)\right] M^T \mu(dy) = 0 \tag{4.32}$$

for all $n \times n$ matrices, which satisfy the equality $A_0 M - MA_0 = 0$. The equality (4.32) is into force for each constant matrix M if

$$\Lambda_1 = \bar{A}_1 := \int_{\mathbb{Y}} A_1(y)\mu(dy). \tag{4.33}$$

Now $B_1(y)$ can be found and the next equation can be analyzed for finding Λ_2 and $B_2(y)$. The equation (4.26) has many solutions. It is convenient to choose such a solution $B_1(y)$, that $\bar{B}_1 = 0$. Known matrices can substitute in the equation (4.27) and this equation can be rewritten in a form

$$\mathbf{A}_0 B_2 - B_2\Lambda_0^T = \Lambda_2^T + B_1(y)\bar{A}_1^T - A_1^T(y)B_1(y).$$

Now the Fredholm alternative can be applied

$$\Lambda_2^T = -\bar{B}_1\bar{A}_1^T + \int_{\mathbb{Y}} A_1^T(y)B_1(y)\mu dy \tag{4.34}$$

and matrix Λ_2 can be found, afterwards also $B_2(y)$. Then the next equations can be written for finding Λ_3, $B_3(y)$ and so on until the needed accuracy of matrix $\Lambda(\varepsilon)$

decomposition is obtained. Since $\mathbb{Y}$ is compact and matrices $\{B_j(y), j = 1, 2, \ldots\}$ are continuous, the elements of obtained basis $B := I + \varepsilon B_1 + \varepsilon^2 B_2 + \ldots$ are linearly independent for sufficiently small ε.

4.2 Covariance analysis

In this paragraph dynamics of the second moment matrix of difference equation (4.5) solution will be analyzed, that is, behavior of matrix as matrix function of argument t

$$Q_t := E\{x_t x_t^T\}. \tag{4.35}$$

Let introduce some denotations. At first it should be noted that a real $n \times n$ matrix space $\mathbb{M}_n$ can be viewed as n^2-dimensional Euclid space $\mathbb{R}^{n^2}$ with scalar product $[q, g] := Tr\{qg^T\}$. A set of symmetric $n \times n$ matrices $\hat{\mathbb{M}}_n$ in a form

$$q := \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{12} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{1n} & q_{2n} & \cdots & q_{nn} \end{pmatrix} \tag{4.36}$$

makes linear closed subspace in $\mathbb{M}_n$. Since for $q \in \hat{\mathbb{M}}_n$ an equality

$$\|q\|^2 := [q, q] := \sum_{k=1}^{n} \sum_{j=1}^{n} (q_{kj})^2 = 0$$

is equivalent to

$$\|\vec{q}\|^2 := (q, q) := \sum_{k=1}^{n} \sum_{j=1}^{n} (q_{kj})^2 = 0,$$

then $\hat{\mathbb{M}}_n$ can be identified with Euclid space $\mathbb{R}^{\frac{n(n+1)}{2}}$ with column vectors in form

$$\vec{q} := \left(q_{11}, q_{12}, \ldots, q_{1n};\ q_{22}, q_{23}, \ldots, q_{2n};\ q_{(n-1)(n-1)}, q_{(n-1)n};\ q_{nn}\right)^T \tag{4.37}$$

and scalar product $(q, g) := q^T g$. Using these denotations, equation (4.5) and the results from the section 4.1 for matrices sequence $(xx)_t := x_t x_t^T$ a linear difference equation in space $\mathbb{M}_n$ can be written as

$$(xx)_t = A(y_t)(xx)_{t-1} A^T(y_t) := \vec{A}(y_t)(xx)_{t-1}. \tag{4.38}$$

The above defined linear operator $\vec{A}(y_t)$ family leaves as invariant symmetric matrices space $\hat{\mathbb{M}}_n$ for each fixed value of argument y_t, therefore, if it is more convenient for use, instead of (4.38) the corresponding linear difference equation in space $\mathbb{R}^{\frac{n(n+1)}{2}}$ can be analyzed.

Let denote $\mathbb{V}$ Banach space, which consists of symmetric $n \times n$ matrix functions $\{q(y), y \in \mathbb{Y}\}$ with norm

$$\|q\| := \sup_{y \in \mathbb{Y}, \|x\|=1} \left|\left(q(y)x, x\right)\right|.$$

Let define a linear continuous operator in space $\mathbb{V}$ using matrix function $\{A(y), y \in \mathbb{Y}\}$ and transition probabilities of Markov chain

$$(\mathbf{A}q)(y) := \int_{\mathbb{Y}} A^T(z) q(z) A(z) p(y, dz) \tag{4.39}$$

All results from the section 4.1 can be adapted to the analysis of this operator. The operator (4.39) has a property, which allows to simplify its analysis. The operator (4.39) leaves as invariant a cone of non-negative defined matrix functions [37]

$$\mathbb{K} := \left\{ q \in \mathbb{V} : \inf_{y \in \mathbb{Y}, \|x\|=1} \left(q(y)x, x\right) \geq 0 \right\}$$

with a set of inner points

$$\mathring{\mathbb{K}} := \left\{ q \in \mathbb{V} : \inf_{y \in \mathbb{Y}, \|x\|=1} \left(q(y)x, x\right) > 0 \right\}.$$

A cone $\mathbb{K}$ partially allows to arrange space $\mathbb{V}$ using inequality

$$q_1 << q_2 \Leftrightarrow q_2 - q_1 \in \mathbb{K}.$$

$q \in \mathring{\mathbb{K}}$ if and only if such a $c(q)$ exists that $q >> c(q) I$, where I is unit matrix. This arrangement makes possible easy to analyze behavior of the second moment of (4.5) solution for $t \to \infty$. It is convenient for use to consider a denotation for (4.5) solution $x_{t+k}(k, x, y)$, what satisfies an initial conditions $x_k = x$, $y_k = y$, and $X(t+k, k, y)$ for matrix (4.4) if $y_k = y$. It is understood that $x_{t+k}(k, x, y) = X(t+k, k, y)x$. If (4.35) solution's the unconditional second moment exponentially decreases for $t \to \infty$, that is,

$$\exists C>0, \exists \lambda \in \{\mathcal{C}: |z|<1\}, \forall x \in \mathbb{R}^n, \forall y \in \mathbb{Y}, \forall k \in \mathbb{Z}, \forall t \geq 0:$$
$$E\left\{\left\|x_{t+k}(k,x,y)\right\|^2\right\} \leq C\lambda^t |x|^2 \tag{4.40}$$

then [51] (4.5) is exponentially mean square stable. In this section it will be shown that this property is rather easy to determine analyzing a positive real spectrum of operator (4.39).

Lets define how the operator $\mathbf{A}$ acts using the equation (4.5) solution.

Lemma 4.3 [9] For any $q \in \mathbb{V}$, $t > k \geq 0$, $y \in \mathbb{Y}$ and $x \in \mathbb{R}^n$

$$\left(\left(\mathbf{A}^t q\right)(y)x, x\right) = E\left\{\left(q\left(y_{t+k}\right)x_{t+k}(k,x,y), x_{t+k}(k,x,y)\right)/\, y_k = y\right\}.$$

Analogically than (4.8), this formula follows from Markov property for a sequence $\{x_t, y_t\}$:

$$E\left\{\left(q\left(y_{t+k}\right)x_{t+k}(k,x,y), x_{t+k}(k,x,y)\right)/\, y_k = y\right\} =$$
$$E\left\{\left(A^T\left(y_{t+k}\right)q\left(y_{t+k}\right)A\left(y_{t+k}\right)x_{t+k-1}(k,x,y), x_{t+k-1}(k,x,y)\right)/\, y_k = y\right\} =$$
$$E\left\{\left(\int_{\mathbb{Y}} A^T(z)q(z)A(z)\,p\left(y_{t+k-1}, dz\right)h, h\right)\Bigg|_{h=x_{t+k-1}(k,x,y),\, y_k=y}\right\} =$$
$$E\left\{\left(\left(\mathbf{A}q\right)\left(y_{t+k-1}\right)x_{t+k-1}(k,x,y), x_{t+k-1}(k,x,y)\right)/\, y_k = y\right\} = \ldots =$$
$$E\left\{\left(\left(\mathbf{A}^{t-1}q\right)\left(y_{k+1}\right)x_{k+1}(k,x,y), x_{k+1}(k,x,y)\right)/\, y_k = y\right\} = \left(\left(\mathbf{A}^t q\right)(y)x, x\right).$$

Using denotation (4.4), the statement of lemma can be rewritten in a form of matrix

$$\left(\mathbf{A}^t q\right)(y) = E\left\{X^T(t+k,k,y)\,q\left(y_{t+k}\right)X(t+k,k,y)/\, y_k = y\right\}. \tag{4.41}$$

Theorem 4.2 The following assertions are equivalent:

(i) the equation (4.5) is exponentially mean square stable;

(ii) such a matrix function $q \in \mathring{\mathbb{K}}$ exists that

$$\mathbf{A}q - q = -I; \tag{4.42}$$

(iii) a maximal positive point of operator $\mathbf{A}$ spectrum $r\{\mathbf{A}\}$ is less than unit.

Proof. $(i) \to (ii)$. Using equality

$$\left\|E\left\{X^T(t,0,y)X(t,0,y)\right\}\right\| = \sup_{|x|=1}\left|\left(E\left\{X^T(t,0,y)X(t,0,y)\right\}x, x\right)\right| =$$
$$\sup_{|x|=1}\left|E\left\{\left(X(t,0,y)x, X(t,0,y)x\right)\right\}\right| = \sup_{|x|=1} E\left\{\left|x_t(0,x,y)\right|^2\right\}$$

and conditions for exponentially mean square stability, an existence of matrix function defined by equality

$$q(y) := \sum_{t=0}^{\infty} E\left\{X^T(t,0,y)\,X(t,0,y)\right\},$$

can be proved. Taking into account an identity $X(k,k,y) \equiv I$, from equality

$$\sum_{t=0}^{\infty} E\left\{X^T(t,0,y)\,X(t,0,y)\right\} = I + \sum_{t=1}^{\infty} E\left\{X^T(t,0,y)\,X(t,0,y)\right\}$$

inequality $q >> I$ follows. Therefore $q \in \overset{\circ}{\mathbb{K}}$. To finish the proof of the theorem's first assertion, a formula (4.41) can be applied for matrix function $q(y) \equiv I$:

$$\begin{aligned}\mathbf{A}q(y) - q(y) &= \mathbf{A}\left(\sum_{t=0}^{\infty} \mathbf{A}^t I\right) - \sum_{t=0}^{\infty} \mathbf{A}^t I \\ &= \sum_{t=0}^{\infty} \mathbf{A}^{t+1} I - \sum_{t=0}^{\infty} \mathbf{A}^t I = -I\end{aligned}$$

$(ii) \to (iii)$. If $q \in \overset{\circ}{\mathbb{K}}$, then such a positive number $c(q)$ can be found that $c(q)I << q << \|q\| I$, because $\mathbb{Y}$ is compact space and this matrix function is continuous. Let assume that this matrix function satisfies equality (4.42). Then inequality $\mathbf{A}q - q << -\dfrac{q}{\|q\|}$ or $\mathbf{A}^t q << r^t q$ should be into force for any $t \in \mathbb{N}$, where $r = 1 - \|q\|^{-1} \in (0,1)$. Therefore

$$\mathbf{A}^t I << \frac{1}{c(q)} \mathbf{A}^t q << \frac{r^t}{c(q)} q << \|q\| \frac{r^t}{c(q)} I$$

for any $t \in \mathbb{N}$, that is,

$$\sum_{t=0}^{m} \mathbf{A}^t I << \frac{\|q\|}{c(q)} \sum_{t=0}^{m} r^t I << \frac{\|q\|}{c(q)(1-r)} I$$

for any $m \in \mathbb{N}$ and

$$\lim_{m\to\infty} \sup_{|x|=1,\, y\in\mathbb{Y}} \sum_{t=0}^{m} \left|\left(\left(\mathbf{A}^t g\right)(y)x, x\right)\right| < \infty \tag{4.43}$$

for any matrix function $g \in \mathbb{V}$. Because the linear continuous operator $\mathbf{A}$ leaves as invariant a solid cone $\mathbb{K}$, such a positive spectrum point $\rho(\mathbf{A})$ exists [37] that

$\rho(\mathbf{A}) = \sup\{|z|, z \in \sigma(\mathbf{A})\}$ and for this spectrum point a real eigen-function (matrix function) $q_\rho \in \mathbb{K}$ corresponds, that is, $\mathbf{A}q_\rho = \rho(\mathbf{A})q_\rho$. Therefore, if $\rho(\mathbf{A}) \geq 1$, then

$$\lim_{m\to\infty} \sup_{|x|=1, y\in\mathbb{Y}} \sum_{t=0}^{m} \left(\left(\mathbf{A}^t q_\rho\right)(y)x, x\right) = \infty .$$

This is contradictory with (4.43).

$(iii) \to (i)$. Because the operator $\mathbf{A}$ leaves as invariant a cone $\mathbb{K}$, such a positive spectrum point $r(\mathbf{A})$ exists [37] which satisfies $r(\mathbf{A}) = \max \operatorname{Re}\{\sigma(\mathbf{A})\}$. Consequently, if $r(\mathbf{A}) < 1$, then $\sigma(\mathbf{A}) \subset \{z \in \mathbb{C} : |z| < 1\}$ and then such constants $c > 0$ and $\lambda \in (0,1)$ exist [37], that $\|\mathbf{A}^t\| \leq c\lambda^t$ for all $t \in \mathbb{N}$. Now an inequality

$$\mathbb{E}\left|x_{t+k}(k,x,y)\right|^2 = \left(\left(\mathbf{A}^t I\right)(y)x, x\right) \leq c\lambda^t |x|^2$$

can be used and the theorem is proved.

If the matrix function $\{A(y), y \in \mathbb{Y}\}$ is near constant and can be represented in a form of uniformly convergent series (4.13), where $\varepsilon \in (0,1)$ is a small parameter, then considerations can be made similarly as in the section 4.1 for mean square reducibility of equation, changing a dimension from *n* to $\frac{n(n+1)}{2}$ of matrix $\Lambda(\varepsilon)$ and matrix-basis $B(\varepsilon)$. Due to this increasing of dimension the calculation becomes more complicated and therefore an algorithm for the behavior analysis of the second moments of equation (4.5) solution described below can be useful. This algorithm is based on the application of the statement (*iii*) of Theorem 4.2, since according to (*iii*) an isolated positive major by module eigenvalue $\hat{\lambda}(\varepsilon)$ of operator (4.39) should exist. If matrix (4.13) analytically depends on parameter ε, then this operator in some neighbourhood $|\varepsilon| < \varepsilon_0$ also analytically depends on ε and its isolated eigenvalue $\hat{\lambda}(\varepsilon)$ also can be represented in a form of series by powers of ε. For finding decomposition $\hat{\lambda}(\varepsilon) := \lambda_0 + \varepsilon\lambda_1 + \ldots$ a basis $\hat{B}(\varepsilon)$ in a root subspace of this eigenvalue will be needed. If an eigenvalue has dimension *m*, then a basis consists of *m* elements from space $\mathbb{V}$ and it can be represented as row $\hat{B}(\varepsilon) := \{\hat{b}_1(\varepsilon), \hat{b}_2(\varepsilon), \ldots, \hat{b}_m(\varepsilon)\}$. A basis can be represented in a form of series [33]

$\hat{B}(\varepsilon) := \hat{B}_0 + \varepsilon \hat{B}_1 + \varepsilon^2 \hat{B}_2 + \ldots$ for sufficiently small ε. Because a total projector in a root subspace also is an analytic function of parameter ε, then a matrix $\hat{\Lambda}(\varepsilon)$ of operator $\mathbf{A}(\varepsilon)$ restriction $\hat{\mathbf{A}}(\varepsilon)$ on this subspace also can be represented in a form of series $\hat{\Lambda}(\varepsilon) := \hat{\Lambda}_0 + \varepsilon \hat{\Lambda}_1 + \varepsilon^2 \hat{\Lambda}_2 + \ldots$. Following to the considerations from the section 4.1, the equations for finding λ_j, $\hat{\Lambda}_j$ and $\hat{B}_j$ for all $j = 0,1,2,\ldots$ can be written:

$$\hat{\mathbf{A}}_0 \hat{B}_0 = \hat{B}_0 \hat{\Lambda}_0^T \tag{4.44}$$

$$\hat{\mathbf{A}}_0 \hat{B}_1 - \hat{B}_1 \hat{\Lambda}_0^T = \hat{B}_0 \hat{\Lambda}_1^T - \hat{\mathbf{A}}_1 \hat{B}_0 \tag{4.45}$$

$$\hat{\mathbf{A}}_0 \hat{B}_2 - \hat{B}_2 \hat{\Lambda}_0^T = \hat{B}_0 \hat{\Lambda}_2^T + \hat{B}_1 \hat{\Lambda}_1^T - \hat{\mathbf{A}}_2 \hat{B}_0 - \hat{\mathbf{A}}_1 \hat{B}_1 \tag{4.46}$$

Some remarks in connection with possibilities to simplify the calculation. At first, the representation of matrix $A(\varepsilon)$ in a form of series by powers of parameter ε can be chosen in such a way, that all matrix A_0 eigenvalues $\{v_1, v_2, \ldots, v_k\}$, which are by module equal to spectral radius $\rho_0 := \rho(A_0)$, are simple, where k is total multiplicity of above described eigenvalues. Since the space consisting of continuous $n \times n$-matrix functions can be represented in a form of tensor product $\mathbb{V} := \mathbb{C}(\mathbb{Y}) \otimes \mathbb{R}^n \otimes \mathbb{R}^n$ and operator $\mathbf{A}_0$ as a tensor product $\mathcal{P} \otimes A_0 \otimes A_0$, the largest by module eigenvalues of this operator create a set $\{v_j v_l, l = 1,2,\ldots,k; j = 1,2,\ldots,k\}$. Then the largest by module real positive eigenvalues of operator $\hat{\mathbf{A}}_0$ has a form [34] $\hat{\lambda}_0 = \rho_0^2$ and has a multiplicity $m = 2k$. Therefore, in the root subspace corresponding to this eigenvalue a basis $\hat{B}_0$ can be chosen so, that $\hat{\Lambda}_0^T = \rho_0^2 I$, where I – unit $m \times m$ matrix. Now using this basis the following operator can be defined

$$\left(\hat{\mathbb{L}} v\right)(y) := \hat{\mathbf{A}}_0 v - \rho_0^2 \hat{v} \tag{4.47}$$

and equation (4.45) can be rewritten in a form

$$\left(\hat{\mathbb{L}} \hat{B}_1\right)(y) = \hat{B}_0 \hat{\Lambda}_1^T - \hat{\mathbf{A}}_1 \hat{B}_0 . \tag{4.48}$$

Now the Fredholm theorem about normal solvability should be applied to the equation (4.48), using the conjugated equation in space of matrix-valued measures, as it was described in the section 4.1. The orthogonality of the (4.48) right side to all elements of operator $\mathbb{L}^{*}$ kernel is necessary and sufficiently condition for the existence of the equation's (4.48) solution. Therefore for (4.48) solvability, taking into account $B_0 = I$, it is enough to ensure equality

$$\int_Y \left[\hat{B}_0 \hat{\Lambda}_1^T - \hat{\mathbf{A}}_1 \hat{B}_0 \right] \mu(dy) = 0\,, \tag{4.49}$$

from which $\hat{\Lambda}_1$ can be found. Substituting it in (4.48) $\hat{B}_1$ can be found. Then proceeding with further equations $\hat{\Lambda}_2$ and $\hat{B}_2$ can be found, and so on until necessary accuracy is obtained.

4.3 Equations with independent coefficients

If a sequence $\{y_t, t \in \mathbb{N}\}$ consists of independent random variables having identical distribution $p(dy)$, then analysis of covariance of (4.5) gets simpler. In this case, similarly as in analysis of the first moments, the operator's $\mathbf{A}$ defined by formula (4.39) restriction $\hat{\mathbf{A}}$ on space of constant real symmetric $n \times n$-matrices $\mathbb{M}_n$:

$$\hat{\mathbf{A}} q := E\left\{ A^T(y_t) q A(y_t) \right\} = \int_Y A^T(y) q A(y) p(dy)$$

and the cone of positive defined matrices $\mathring{\mathbf{K}}_n := \hat{\mathbb{M}}_n \cap \mathring{\mathbb{K}}$ can be used.

Corollary 4.1 Let consider that a sequence $\{y_t, t \in \mathbb{N}\}$ consists of independent random variables and the other conditions of Theorem 4.2 are satisfied. Then following statements are equivalent:

(i) the equation (4.5) is exponentially mean square stable;

(ii) such a matrix $q \in \mathring{\mathbf{K}}_n$ exists , that

$$\mathbf{A} q - q = -I\,; \tag{4.50}$$

(iii) maximal by module spectrum point $r\{\mathbf{A}\}$ of operator $\mathbf{A}$ is less than unit.

The statement (*iii*) allows rather easy to analyze the solutions of *m*-dimensional scalar difference equations behavior

$$x_{n+m} = \sum_{k=0}^{m-1} a_{k+1} x_{n+k} + c \sum_{k=0}^{m-1} h_{k+1} y_{n+k+1} x_{n+k}\,, \tag{4.51}$$

where $\{y_k\}$ is a sequence of identically distributed independent random variables with mean value zero and unit variance. This equation can be rewritten in vector form in space $\mathbb{R}^m$:

$$\vec{X}_{n+1} = A\vec{X}_n + c\sum_{k=0}^{m-1} y_{n+k+1} H_{k+1} \vec{X}_n \,, \tag{4.52}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ a_m & a_{m-1} & a_{m-2} & \dots & a_1 \end{pmatrix}, \; H_k = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & h_{m-k} & 0 & \dots & 0 \end{pmatrix}.$$

According to Corollary 4.1 the second moment of any solution of equation (4.52) is exponentially decreasing if and only if for some $0 \le \rho < 1$ a positive defined matrix solution of equation

$$A^T qA + c^2 \sum_{k=1}^{m} H_k^T qH_k = \rho q$$

exists. Therefore, if eigenvalues of matrix A are located inside circle $\{|z| < 1\}$, then such a positive number $c^2 < r^2$ exists, that the second moment of each solution $E|x_n|^2$ of equation (4.51) tends to zero if $n \to \infty$, but in case if $c^2 > r^2$ then unlimited increasing solution exists. Submitting this number r^2 in the previous matrix equation:

$$A^T qA + r^2 \sum_{k=1}^{m} h_k^T qH_k - q = 0 \,. \tag{4.53}$$

It can be rewritten in a form of equations system for matrix elements $q := \{q_{sj}\}$:

$$q_{11} = q_{22}, \;\; q_{12} = q_{23}, \;\; \dots, q_{1m-1} = q_{2m}, q_{1m} = \sum_{i=1}^{m} q_{2i} a_i,$$

$$q_{22} = q_{33}, \;\; q_{23} = q_{34}, \;\; \dots, q_{2m-1} = q_{3m}, q_{2m} = \sum_{i=1}^{m} q_{3i} a_i,$$

$$\dots$$

$$\begin{aligned} q_{mm} = {} & \left(a_1^2 + r^2 h_1^2\right) q_{11} + \dots + \left(a_m^2 + r^2 h_m^2\right) q_{mm} + \\ & +2a_m a_{m-1} q_{12} + \dots + 2a_m a_1 q_{1m} + 2a_{m-1} a_{m-2} q_{23} + \dots + \\ & +2a_{m-1} a_1 q_{2m} + \dots + 2a_2 a_1 q_{m-1m}. \end{aligned}$$

From these equalities form of matrix-solutions can be found:

$$q=\begin{pmatrix} q_{mm} & q_{m-1m} & q_{m-2m} & \cdots & q_{3m} & q_{2m} & q_{1m} \\ q_{m-1m} & q_{mm} & q_{m-1m} & \cdots & q_{4m} & q_{3m} & q_{2m} \\ q_{m-2m} & q_{m-1m} & q_{mm} & \cdots & q_{5m} & q_{4m} & q_{3m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ q_{3m} & q_{4m} & q_{5m} & \cdots & q_{mm} & q_{m-1m} & q_{m-2m} \\ q_{2m} & q_{3m} & q_{4m} & \cdots & q_{m-1m} & q_{mm} & q_{m-1m} \\ q_{1m} & q_{2m} & q_{3m} & \cdots & q_{m-2m} & q_{m-1m} & q_{mm} \end{pmatrix}$$

and then from (4.53) a system of m linear equations for numbers $q_{jm},\ j=1,2,\dots,m$ can be easy found:

$$q_{im}-\sum_{l=1}^{i}a_l q_{(m-i-1+l)m}-\sum_{l=1}^{m-i}a_{m-i-l+1}q_{(m-l+1)m}=0,\ i=1,2,\dots,m-1 \qquad (4.54)$$

$$q_{mm}\left(1-\sum_{i=1}^{m}\left(a_{m-i+1}^2+r^2h_{m-i+1}^2\right)\right)-2\sum_{l=1}^{m-1}\sum_{s=1}^{l}a_{m-l+1}a_{l-s+1}q_{lm}=0 \qquad (4.55)$$

Because of number r^2 existence, this equation should have nontrivial solution and therefore a determinant of equations system (4.54)-(4.55) should be equal to zero. Taking into account the form of analyzed equations system the following conclusion can be made, that its determinant is a linear function of parameter r^2 and this number can be found as ratio of two parameters. Lets illustrate above described algorithm on example analyzing existence of a stable stationary process GARCH(p,q) [13] having the second moment defined by formula

$$\sigma_t^2=\varphi_0+\sum_{k=1}^{p}\varphi_k\sigma_{t-k}^2+\sum_{j=1}^{q}\theta_j\sigma_{t-j}^2\xi_{t-j}\ . \qquad (4.56)$$

Let consider $\hat{\sigma}_t^2$ - a stationary process satisfying the formula (4.56) and $x_t:=\sigma_t^2-\hat{\sigma}_t^2$. If $s^4:=E\left\{\left(\varepsilon_1^2-1\right)^2\right\}$ exists then a difference equation in a form (4.51) for x_t can be written, where $\xi_t:=\left(\varepsilon_t^2-1\right)s^{-2}$, $c=s^2$, $m=\max\{p,q\}$,

$$a_k=\begin{cases}\phi_k+\theta_k, & if\ \ p\le q=m,\ k=1,2,\dots,p,\\ \phi_k, & if\ \ p<q=m,\ k=p+1,p+2,\dots,m,\\ \theta_k, & if\ \ q<p=m,\ k=q+1,q+2,\dots,m,\end{cases}$$

$h_k=\theta_k$ for $k=1,2,\dots,q$ and $h_k=0$ for $k>q$. Any process satisfying (4.56) tends to stationary if and only if the second moment of equation (4.51) tends to zero if $t\to\infty$. A number r^2 can be found and compared to $c^2:=s^4$. If inequality $s^4<r^2$ is into force, then GARCH(p,q) process defined by (4.56) converges to stationary for $t\to\infty$. This condition is also necessary for existence of unconditional second moment of

conditional variance σ_t^2. The proposed algorithm is rather simple to use. It is particularly simple for small values of *p* and *q*. For example, for model GARCH(2,1) equations system for finding r^2 has a form:

$$\begin{aligned} q_{12}\left(a_2-1\right)+a_1 q_{22} &= 0, \\ 2a_1a_2q_{12}-\left(1-a_1^2-a_2^2-r^2b_1^2\right)q_{22} &= 0. \end{aligned}$$

Substituting $a_1=\phi_1+\theta_1$, $a_2=\phi_2$, $b_1=\theta_1$ and equating to zero, a critical value of r^2 can be found:

$$r^2=-\frac{\begin{vmatrix} a_2-1 & a_1 \\ 2a_1a_2 & a_1^2+a_2^2-1 \end{vmatrix}}{\begin{vmatrix} a_2-1 & a_1 \\ 0 & b_1^2 \end{vmatrix}}=\frac{\left(1+\phi_2\right)\left[\left(1-\phi_2\right)^2-\left(\phi_1+\theta_1\right)^2\right]}{\theta_1^2\left(1-\phi_2\right)}.$$

As consequence, a stationary process GARCH(2,1) with the second moment of conditional variance exits if and only if the fourth moment $s^4 := E\left\{\left(\varepsilon_1^2-1\right)^2\right\}$ of random perturbations $\left\{\varepsilon_k\right\}$ satisfies inequality

$$s^4<\frac{\left(1+\phi_2\right)\left[\left(1-\phi_2\right)^2-\left(\phi_1+\theta_1\right)^2\right]}{\theta_1^2\left(1-\phi_2\right)}.$$

Conclusion

In this doctor thesis it is shown that for asymptotical analysis of Markovian iterative procedures, which are presented in a form of linear difference equation, as a base a construction of discrete semigroup for the covariance matrices and spectral analysis of the semigroup generator can be chosen and spectral theory of positive operators in Banach space can be used. Against this background it was possible to elaborate a special version of the second Lyapunov method for mean square asymptotic stability analysis of difference equations with near to constant coefficients.

As main result of the doctor thesis is developed methodology for analyzing convergence to zero of the unconditional second moments of the linear difference equation as t tends to infinity $E\left\{\left|x_t\left(k, x, y\right)\right|^2\right\} \xrightarrow[t\to\infty]{} 0$. For this reason two theorems (Theorem 4.1 and Theorem 4.2) and two lemmas (Lemma 4.1 and Lemma 4.2) were proved. For the simplifying the description at first the algorithm was elaborated for the first moments dynamics analysis and then adapted to the analysis of the second moments matrix of difference equation solution. It is shown that in case if random perturbations are independent identically distributed the proposal method gets simpler. Kato perturbation theory for spectral projector decomposition was used.

Publications

The main results of this research have been published in the following papers:

1. Carkova, V., **Goldšteine, J.**: *Asymptotical decomposition methods for moment stability analysis of Markov difference equations*. Journal of Applied Mathematics, Vol. 1 No. 1, Slovak University of Technology, Bratislava, pp. 125-131, 2008.
2. Carkova, V., **Goldšteine, J.**: *Moment equations for discrete linear Markov dynamical systems*. In Proceedings of the 6th International Conference Aplimat, Slovak University of Technology, Bratislava, pp. 451-456, 2007.
3. Carkova, V., **Goldšteine, J.**: *On mean square Lyapunov index for Markov iterations.* In Proceedings of the International Conference "Decision making intellectual systems and information technology, Chernivtsi, 2006", Ruta, Chernivtsi, pp.102-105, 2006; and in The International Journal "System Research & Information Technologies" No. 3, IASA, pp. 82-99, 2007.
4. **Goldšteine, J.**: *On almost reducibility of linear Markov difference equations.* In Proceedings of the 5th International Conference Aplimat, Slovak University of Technology, Bratislava, pp. 249-254, 2006.
5. **Goldšteine, J.**, Pavļenko, O.: *Asymptotical analysis of continuous time stochastic regression models.* Acta Societatis Mathematicae Latviensis, No. 7, Latvian Mathematical Society, pp. 42, 2006
6. Carkova, V., **Goldšteine, J.**: *Mean square Lyapunov index for linear Markov iterations*. In Proceedings of the 4th International Conference Aplimat, Slovak University of Technology, Bratislava, pp. 49-56, 2005.
7. Carkova, V., **Goldšteine, J.**: *On mean square stability of linear Markov difference equations*. In Proceedings of the 3rd International conference Aplimat, Slovak University of Technology, Bratislava, pp. 313-319, 2004.
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