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Dissertation

On Some Categories of L-valued Sets and Many-valued Topologies: Theoretical Foundations

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Anotation

This work is performed in the field of the so called "Mathematics of fuzz sets". Concerning pure mathematics, the research in this field could be conditionally dividend into two mainstreams: the research if mathematical structures on the L-powerset of subsets of ordinary sets and the research of sets endowed with many-valued equalities. he subject of our work is a certain synthesis of the both approaches. Namely, we develop foundations of topological theories in the context of L-(fuzzy) subsets of many valued sets. To realize this programme we had to develop a construction allowing to extend many-valued relations from a set to its L-powerset. In our opinion this construction is interesting by itself and could be important also for other merits.

MSC: Primary: 54A40, 04A72; **Secondary:** 06F05, 18B30, 18D20.

Key words and phrases: L-valued relations, L-valued equalities, categories of L-valued sets, L-subsets, categories of L-topologies and L-fuzzy topologies on L-valued sets, fuzzy categories; lattices, cl-monoids, MV-algebras.

Anotācija

Darbs ir izstrādāts tā saucamajā "nestriktās matemātikas" jomā. Nestriktajā matemātikā nosacīti var izdalīt divas atšķirīgas pieejas: pirmās pieejas izejas punkts ir parastas kopas nestriktas apakškopas, bet otrās - parasta kopa, kurā definēta daudzvērtīga vienādība. Darba galvenais mērķis ir attīstīt teoriju, kas būtu abu iepriekš minēto pieeju sintēze, konkrētāk, aprakstīt nestriktas topoloģijas teotijas pamatus daudzvērtīgas kopas L-(nestriktās) apakškopās. Lai iecerēto īstenotu, ir izstrādāta konstrukcija, ar kuras palīdzību daudzvērtīgas attiecības parastā kopā tiek pārceltas uz kopas L-eksponenti. Šī konstrukcija ir interesanta pati par sevi un varētu tikt izmantota arī citiem mērķiem.

MSC: 54A40, 04A72, 06F05, 18B30, 18D20.

Atslēgas vārdi: L-vērtīgas attiecības, L-vērtīgas vienādības, L-vērtīgu kopu kategorijas, L-apakškopas, L-vērtīgu kopu L-topoloģiju un L-nestriktu topoloģiju kategorijas, nestriktas kategorijas, režģi, cl-monoīdi, MV-algebras.

Contents

Introduction

The aim of this work is to make a certain contribution in the so called "Matehmatics of fuzzy sets". To be more concrete, we are interested in the research of topological-type structures in the context of fuzzy sets and fuzzy enviroment. In order to go into some details of the work, we first have to outline the subject of what we assume under the term "Mathematics of fuzzy sets."

0.1 Fuzzy sets

The concept of a fuzzy set (in the modern terminology, an $[0,1]$-set) was introduced by L. Zadeh in 1965 [?]; two years later J.A. Goguen [?] generalized it by introducing the concept of an L-fuzzy set where L is an arbitrary infinitely distributive lattice L (or even a *cl*-monoid).

The idea of the concept of a fuzzy set is that very often *sets* which appear both in theoretical mathematics and especially in its applications, are not "real" sets, but set-like congromelates wihich have a vague, uncertain or inprecise border. Also the logical statements characterizing such sets are not formulated according to the laws of classic logic. For example, how should we understan a statement like "*John is rather young"*? The whole problem was not new and can be traced already in Aristotel's works. However the interest, (and not only in purely theoretical, but also in applied problems) essentially increased in the 20-th centure. In particular, in the first half of the 20-th century there were published important works by J. Łukasiewicz, [?], [?] M. Blank [?] et al, where such problems and ideas were discussed. However, the scientist, who put the cornstone in the the systematic mathematical study of such congromelate-like quantities, to develop the corresponding theory and to investigate its prossible applications, was an American scientist of Azerbajdzhan descent Lotfi A. Zadeh [?]. Also the term *a fuzzy set* (more precisely, a fuzzy subset of a set) as a mapping $A : X \to [0,1]$ was introduced by L.A. Zadeh in [?]. Two years later J.A. Goguen [?] extended this concept to the concept of an L-fuzzy set where the closed interval $[0,1]$ is replaced by an arbitrary infinitely distributive lattice L. In this work we accept the modern terminology, saying,
an L-set instead of an L-fuzzy set, and in particular, an $[0,1]$-set, instead of an $[0,1]$-fuzzy set or just a fuzzy set.
However, the adjective "fuzzy" will remain for the whole subject. For the precise definitions of an L-set and related concepts see the preliminaries.

0.2 Mathematical structures on the basis of L-sets

The concept of an L-set aroused interest among mathematicians, who intended to extend the classic mathematical concepts and to develop corresponding theories to the context of mathematics of fuzzy stes, and among specialists in other areas (engeniers, economists, et. al) who hoped to use the new concepts and theories in their work. Concerning theoretical mathematics, already some years after the concept of a fuzzy set was introduced in 1965, there were published papers on fuzzy topologies [?], [?], fuzzy groups and other fuzzy algebraic structures [?], [?], etc., later also on fuzzy measures [?], fuzzy integrals, etc.

To give just an impression on the idea of this approach, we recall that according to C.L. Chang [?], cf also [?] *a fuzzy topology* on a set X is a family τ of its fuzzy subsets ($\tau \subseteq [0,1]^X$) such that

$$0_X, 1_X \in \tau; \text{ if } U, V \in \tau \text{ then } U \wedge V \in \tau; \text{ if } U_i \in \tau \ \forall i \in \mathcal{I}, \text{ then } \bigvee_{i \in \mathcal{I}} U_i \in \tau.$$

A. Rosenfeld [?], cf also [?] defines a *a fuzzy group* as a mapping $H : G \to [0,1]$ where $G = (G, \cdot, e)$ is a group and

$$H(x \cdot y) \geq H(x) \cdot H(y) \quad \forall x, y \in G.$$

Already at this moment we would like to emphasize that when extending classical concepts to the "fuzzy enviroment" different authors had different viewpoint and different methods how this shoud be done. Concerning the "Fuzzy topology" cf e.g. essentially different approaches in [?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?]. Also in this work two different approaches to structures in Fuzzy topology will be reflected (see Sections 4 and 5).

0.3 L-valued equalities and L-valued sets

Up to now we spoke about fuzzy mathematical structures on the basis of a *classical* set X. However some authors considered it to be more interesting and important to study sets endowed with fuzzy equalities, that is pairs (X, E) where X is a set and $E : X \times X \to L$ is the so called L-valued equality, that is a mapping satisfying certain conditions (see subsection 3.1 for the precise definitions). The informal meaning of the value $E(x, x')$ is *the extent, to which elements x and x' ($x, x' \in X$) are equal.* Although the idea that it could be important to measure the extent, to which two elements of a set X are *equal* can be trased in the papers of several authors, as the main source here we relay on U. Höhle's fundamental work [?]. The corresponding pair (X, E) will be refered to as an L-valued set.

The main concrete goal of this work is to develop an approach which would be a synthesis of the two above mentioned approaches: namely, to develop foundations of the theories of fuzzy topology in the context of *L*-valued sets.

0.4 The structure and the main results of the work

The work consists of short abstracts in English and Latvian, the contents of the work, an Introduction (you are reading it at present), six sections, the list of references, and the list of publications of the author.

0.4.1 Preliminaries

The work starts with section 1: Preliminaries. Here we recall the basic concepts which make the context for our research. Namely we recall here:

- the well-known concept of a lattice and make precisation of the terminology used in the work concerning lattices. This terminology is taken from classical sources: the well-known books by G.Birhoff, see e.g. [?], and the Compedium on Continuous Lattices written by several authors [?];
- the concept of a GL-monoid, introduced by U. Höhle, [?], which we view as the basic background for our work, and which is a special kind of a more general concept of a cl-monoid, introduced much earlier in Birhoff's works see [?], etc, and used already in Goguen's works [?];
- a special class of GL-monoids, namely MV-algebras (see e.g. [?], [?], [?]) and a more general concept of a Girard monoid, having most important properties of a MV-algebra and be helpfull in the class of cl-monoids, thus be helpfull in a context wider than our main context - the class of GL-monoids.
- finally we make precisation of the concept of an L-set (cf [?], [?]) and operatiorns with L-sets, in particular, in the case when L is a GL-monoid.

Properties of the introduced concepts used in our main text (sections 2-6) are also collected here. In particular, the proofs are given for the resultes which we need, in cases when we did not find the appropriate references.

Although the theory of categories is not in the center of this work, we use the basic concepts and results of the Category theory. This is convenient, since it allows us to formulate the main concepts introduced here and the obtained results as well as the mainstream of our study more clearly. We do not collect the (not many) notions and results from the category theory used here and refer the to the classic sources, see [?], [?].

0.4.2 Categories of L-valued L-sets

In Section 2 we consider categories of L-valued L-sets. In subsection 2.1 we start with the category $\mathrm{SET}^{loc}(\mathrm{L})$ of local L-valued L-sets introduced by U.Höhle [?], however soon reduce our interest to its full subcategory $\mathrm{SET}(L)$ consisting of what U. Höhle calls global many valued sets, and which in this work are called just *L-valued sets*. Essentially they are pairs (X, E) where X is a set and E is an L-valued equality on it.

Further, in subsection 2.2 we introduce the concept of an *L-valued L-set* and define extensionality of mappings between such sets. In the result we get a category L-$\mathrm{SET}(L)$ generalizing the category SET(L) and closely related to the above mentioned category $\mathrm{SET}^{loc}(L)$.

In subsection 2.3 we consider the family $\mathfrak{E}$(X,A) of all L-valued equalities on the L-subset X, define, in a natural way, an order $\preceq$ on it and show, that ($\mathfrak{E}$(X,A)$\preceq$) is a complete infinitely distributive lattice. This allows us to show in the subsequent subsection 2.4 the existence of final and initial L-valued equalities on L-sets. This result will be important further in our work. However, in our opinion, it is interesting also by itself.

0.4.3 L-valued order relations and their extensions to L-powersets

When studying mathematical structures on L-valued sets (X, E) it is often necessary to extend the L-valued equalities $E : X \times X \to L$ from a set to its L-powerset and to get L-valued equalities on the L-powerset L^X. In particular, such extension is crucial when considering topological-type structures on L-valued sets. This problem is the subject studied in Section 3.

Although the problem of extension L-valued equalities was auxiliary for the principal goals of our work, the results obtained in this direction in Section 3 seem to be quite interesting by themselves. Besides, when studying this problem we came to conclusion, that it is natural to start with the situation when a set X is equiped *not with an L-valued equality $E : X \times X \to L$*, but with a more general, L-valued *preoder-type structure* $R : X \times X \to L$, to extend it to a certain *preoder-type structure* on the L-powerset L^X

$$\mathcal{R} : L^X \times L^X \to L,$$

and then, by symmetrizing it, to come to an L-valued equality

$$\mathcal{E} : L^X \times L^X \to L.$$

Section 3 starts with subsection 3.1 where we consider different (known) L-valued ordered type relations R on a set X, define corresponding categories, and give several examples of such relations. Subsection 3.2 is the central one in this part of the work: here we construct extension of a relation $R : X \times X \to L$ to a relation $\mathcal{R} : L^X \times L^X \to L$, study basic properties of such extension and give some examples. In Subsection 3.3. we show that the obtained construction assigning to a L-valued preodered set, that is to a pair (X, R), an L-valued preodered powerset $(L^X, \mathcal{R})$, is a contravariant functor of the corresponding categories. In subsection 3.4 we study lattice properties of L-valued relation-type structures on the L-powerset L^X of a set X. In particular, it is shown that such structures make a complete lattice. A series of examples can be found in this subsection, too. In the final subsection 3.5 the construction of symmetrizing of the relation $\mathcal{R} : L^X \times L^X \to L$ is elaborated. As a result we can, starting with an L-valued set (X, E) get an L-valued powerset $(L^X, \mathcal{E})$ - the fact which will be crucial in Section 5.

Basic results of this subsection are published in the author's works[**?**] and [**?**].

0.4.4 Categories of L-topologies on L-valued sets and their subcategories

In Section 4 we introduce and start the study of two categories of L-topologies on L-valued sets and their L-subsets. Recall, that an L-topology (L-fuzzy topology in the original terminology [**?**],[**?**]) on a set X is a family $\tau \subseteq L^X$ such that

$$0_X, 1_X \in \tau;\ U, V \in \tau \Rightarrow U \wedge V \in \tau;\ U_i \in \tau\ \forall i \in \mathcal{I} \Rightarrow \bigvee_{i \in \mathcal{I}} U_i \in \tau.$$

In subsection 5.1 we define and study an L-valued analogue of an L-topology. Namely, we introduce the concept of an L-topology on an L-valued set (X, E) and consider the corresponding category TOP(L) of L-topological spaces and continuous mappings between them.[1] The general structure of an L-valued L-topological space is studied. It is important to emphasize that in the part of this subsection where closed L-subsets of an L-valued L-topological space

[1]Note that the idea of this approach is not new and can be traced, in particular, in [**?**]

are involved we need an additional assumption that the lattice L is an MV-algebra, or more generally, a Girard monoid (in case we extend our research to the case of *cl*-monoids). As the main results in subsection 4.1 we consider the description and the properties of the closure operator of an L-topology on an L-valued set.

In subsection 4.1 we started with the category SET(L) as the ground category (cf [**?**]) to define a topological category TOP(L) over SET(L). In the subsequent subsection 4.2 we start with the category L-SET(L) to define the corresponding category L-TOP(L) of L-valued L-topological L-sets. The main attention here is payed to the lattice-type properties of the family of all L-topologies on L-subsets of L-valued sets. It is proved that the family $\mathfrak{T}$ of L-valued L-topologies on an L-valued L-set (X, A, E_A) is a complete lattice. Its top element is an L-valued L-topology $\tau_1 = L^X_{E_A}$ consisting of all extensional L-valued L-subsets of A, and its bottom element is the indiscrete L-valued L-topology $\tau_0 = \{A, 0_X\}$. The existence of final and initial L-topologies on L-subsets of L-valued sets is established. Basing on these results we prove that the category L-TOP(L) is *topological* over the category L-SET(L) of L-subsets of L-valued sets with respect to the forgetfull functor

$$\mathfrak{F} : \ L - TOP(L) \to L - SET(L),$$

and the category TOP(L) is its complete subcategory. Several examples of L-topologies on L-subsets of L-valued sets are given.

Basic results of this subsection are published in the author's works [**?**]

0.4.5 Categories of L-fuzzy topologies on L-valued subsets L-valued sets and their subcategories

In 1985 T.Kubiak [**?**] and A.Šostak [**?**] independently introduced a more general and, as it is considered now by many specialists) a more consistent concept (to compare with Chang-Goguen's L-topology) of a topological structure in the context of L-sets. According to the modern terminology (accepted also in this work) it is called *an L-fuzzy topology.* Namely, while Chang-Goguen's L-topology on a set is *a usual subset τ of the family L^X of L-subsets of a set X*, an L-fuzzy topology is *an L-subset $\mathcal{T}$ of L^X*, that is a mapping $\mathcal{T} : L^X \to L$ subjected to the following properties:

$$\mathcal{T}(0_X) = \mathcal{T}(1_X) = 1; \quad \mathcal{T}(U \wedge V) \geq \mathcal{T}(U) \wedge \mathcal{T}(V); \quad \mathcal{T}\left(\bigvee_{i \in \mathcal{I}} U_i\right) \geq \bigwedge_{i \in \mathcal{I}} \mathcal{T}(U_i).$$

Later there were published many papers where this approach was developed (see, e.g. [?] for the survey). In Section 5 of our work we initiate the study of L-fuzzy topologies on L-powersets of L*-valued sets.* [2]

The section consists of two subsections. In subsection 5.1 we introduce the concept of an extensional L-fuzzy topology on an L-valued set, introduce the corresponding category FTOP(L), study the lattice of all L-fuzzy topologies on a fixed L-valued set (X, E), and apply the obtained results in order to get the principal result of this subsection: namely, to prove that the category FTOP(L) is topological over the category SET(L) with respect to the forgetful functor $\mathcal{F} : FTOP(\mathrm{L}) \to SET(\mathrm{L})$. The last part of subsection 5.1 is devoted to two important full subcategories of FTOP(L): namely the category EFTOP(L) of enriched L-fuzzy topological spaces and the category SFTOP(L) of stratified L-fuzzy topological spaces on L-valued sets. [3]

In subsection 5.2 we extend the concept of an L-topology from the case of an L-valued set to its extensional L-subset, and introduce the corresponding category L-FTOP(L). The results here are analogous to the ones in the previous subsection and therefore given here without proofs. As the main result here we mention Theorem 5.17, stating that the category L-FTOP(L) is topological over the category L-SET(L) with respect to the forgetful functor

$$\mathfrak{F} : \; L - FTOP(L) \to L - SET(L).$$

Also relations between the category $L - FTOP(L)$ and its subcategories FTOP(L), L-TOP(L) et. al. are briefly discussed.

0.4.6 L-fuzzy categories

The last, 6th Section is devoted to fuzzification of the categories studied in the work. Although fuzzy structures were in the center of interest in all these categories, the categories themselves were ordinary, crisp categories. The concept of an (L)-fuzzy category was introduced in [?] and later was studied in a series of papers see, e.g. [?], [?] etc. These papers contain also many examples of L-fuzzy categories which appear in the natural way, by *"fuzzifying"* classic categories. Actually an L-fuzzy category is a triple $(\mathcal{C}, \mu, \omega)$ where $\mathcal{C}$ is an ordinary category with the class of objects $\mathcal{O}(\mathcal{C})$ and the class of morphisms $\mathcal{M}(\mathcal{C})$, and $\omega : \mathcal{O}(\mathcal{C}) \to L$, $\mu : \mathcal{M}(\mathcal{C}) \to L$ are respectively L-subclasses of the class objects $\mathcal{O}(\mathcal{C})$ and the class of morphism,

[2] Note that a structure of such type was mentioned already in [?]

[3] Note that in case when X is a crisp set, these concepts can be traced back to the work [?], and in case of L*-topological* spaces when $L = [0, 1]$ viewed as a Heyting algebra, already in Pu and Liu papers published in 1980 [?], [?].

$\mu : \mathcal{M}(\mathcal{C})$ subjected to certain properties. The intuitive meaning of the values $\omega(X)$ and $\mu(f)$ are the measures to which X and f are respectively the object and the morphism of the corresponding category.

The sixth section contains two subsections. In subsection 6.1 we develop fuzzifications of the categories SET(L) and L-SET(L); the resulting fuzzy categories are $\mathcal{F}$-SET(L) and $\mathcal{F}$-L-SET(L). In subsection 6.2 the fuzzifications of the categories TOP(L) and L-TOP(L) are presented; the resulting fuzzy categories are $\mathcal{F}$-TOP(L) and $\mathcal{F}$-L-TOP(L).

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1 Preliminaries

1.1 Lattices

By a lattice, following [**?**], we mean a quadruple $(L, \leq, \wedge, \vee)$ where $(L, \leq)$ is a partially ordered set and for every two elements $\alpha, \beta \in L$ their infimum $\alpha \wedge \beta$ and supremum $\alpha \vee \beta$ are defined. In particular, $\alpha \leq \beta$ iff $\alpha \wedge \beta = \alpha$ and $\alpha \vee \beta = \beta$. A lattice L is called complete, if for every its subset $A \subseteq L$ there exists supremum

$$\sup A = \bigvee \{a_i \mid a_i \in A\} \in L$$

and infimum

$$\inf A = \bigwedge \{a_i \mid a_i \in A\} \in L.$$

In particular $\sup L =: 1$ and $\inf L =: 0$ are respectively the greatest and the smallest elements of the lattice L. As usually, for an empty set we prescribe $\sup \emptyset := 0$ and $\inf \emptyset := 1$.

A complete lattice is called infinitely distributive [**?**] if

$$(\bigvee A) \wedge \beta = \bigvee \{a \wedge \beta \mid a \in A\}$$

and

$$(\bigwedge A) \vee \beta = \bigwedge \{a \vee \beta \mid a \in A\}$$

for every $\beta \in L$ and every $A \subseteq L$.

A mapping ${}^c : L \to L$ is called an order reversing involution if

$$a \leq b \Longleftrightarrow a^c \geq b^c \quad \forall a,\ b \in L \text{ and } (a^c)^c.$$

Here we give some examples of lattices which will be used in the main text:

1. The two-point lattice $\{0, 1\} =: \mathbf{2}$ with natural order $\leq$, and operations $\wedge$ and $\vee$ defined in an obvious way. Involution is given by $0^c = 1$ and $1^c = 0$.

2. The unit interval $[0, 1]$ with natural order $\leq$ and operations $\wedge$ and $\vee$ defined in an obvious way. Involution is given by $a^c = 1 - a$ for all $a \in [0, 1]$.

3. Given a set Z let $\mathbf{2}^Z$ stand for the set of all subsets of Z. Defining relation $\leq$ by setting

$$A \leq B \Longleftrightarrow A \subseteq B \quad \forall A, B \in \mathbf{2}^Z,$$

operations as

$$A \vee B = A \cup B, \quad A \wedge B = A \cap B$$

and involution as the complement $A^c = Z \setminus A$, we get an infinitely distributive lattice.

4. Generalizing the previous example, consider a set Z and an infinitely distributive lattice $(L, \leq, \wedge, \vee)$. If we extend the order relation $\leq$ and operations $\wedge$ and $\vee$ from L to the the L-powerset L^Z pointwise, $(L^Z, \leq, \wedge, \vee)$ becomes an infinitely distributive lattice.

Remark 1.1 Our main source for the concepts and results from lattice theory are classic monographs [**?**], and [**?**].

1.2 GL-monoids

A $GL-$monoid is an infinitely distributive lattice $(L, \leq, \wedge, \vee)$ enriched with a monotone, commutative and associative binary operation $*$ such that

1. $a * 1 = a$ and $a * 0 = 0$ for all $a \in L$;
2. $a * \left(\bigvee_{j \in J} b_j \right) = \bigvee_{j \in J} (a * b_j) \quad \forall a \in L, \quad \forall \{b_j : j \in J\} \subset L$;
3. If $a \leq b$, then there exists $c \in L$ such that $a = b * c$.

Such operation will be refered to as the conjunction in the GL-monoid.

It is known that every $GL-$monoid is residuated, i.e. there exists a further binary operation " $\mapsto$ " (implication) on L satisfying the following condition:

$$a * b \leq c \Longleftrightarrow a \leq (b \mapsto c) \qquad \forall a, b, c \in L.$$

Explicitely implication is given by $a \mapsto b = \bigvee \{\lambda \in L \mid a * \lambda \leq b\}$. In the sequel we shall use the following properties of a GL-monoid (see e.g. [**?**]):

1. $a \mapsto b = 1 \Longleftrightarrow a \leq b \quad \forall a, b \in L$;
2. $a \mapsto \left(\bigwedge_{i \in \mathcal{I}} b_i \right) = \bigwedge_{i \in \mathcal{I}} (a \mapsto b_i) \quad \forall a \in L, \ \{b_i \mid i \in \mathcal{I}\} \subseteq L$;
3. $\left(\bigvee_{i \in \mathcal{I}} a_i \right) \mapsto b = \bigwedge_{i \in \mathcal{I}} (a_i \mapsto b) \quad \forall b \in L, \ \{a_i \mid i \in \mathcal{I}\} \subseteq L$;
4. $(a \mapsto c) * (c \mapsto b) \leq a \mapsto b \quad \forall a, b, c \in L$;

5. $a \leq b \Longrightarrow c \mapsto a \leq c \mapsto b \ \forall a, b, c \in L$;

6. $(a * b \mapsto c) = a \mapsto (b \mapsto c) \ \forall a, b, c \in L$.

Consider some examples of GL-monoids.

1. Any complete infinitely distributive lattice $(L, \leq, \wedge, \vee)$ (Heyting algebra) can be viewed as a GL-monoid, if we set $* = \wedge$, that is, consider it as a 5-tuple $(L, \leq, \wedge, \vee, \wedge.)$ In particular, lattices considered in **??**, **??**, **??** and **??** are GL-monoids.

2. Let $L = [0,1]$ be a unit interval (**??**) and let $a * b = a \cdot b$, that is conjunction is defined as the product operation in $[0,1]$

Some more examples will be given in the next subsections.

Remark 1.2 (Historical comments) The concept of a GL-monoid was introduced by U. Höhle in [**?**], as a special type of the concept of a complete-lattice monoid or a *cl*-monoid introduced by Birkhof (see e.g. [**?**]) and later used by many authors (see e.g. [**?**]). As different from the general concept of a *cl*-monoid, in the definition of a GL-monoid there is the requirement that the top element of it acts as a unit: $a * 1 = a$ for each $a \in L$; and the divisibility property:
If $a \leq b, \ a, b \in L$, then there exists $c \in L$ such that $a = b * c$. Many of the results obtained in this work could be extended to the case when L is a *cl*-monoid. However in the work we restrict to the case of a GL-monoid in order to make exposition more homogeneous.

Note that monotone commutative associative operations

$$* : [0,1] \times [0,1] \to [0,1]$$

such that $a * 1 = 1$ and $a * 0 = 0$ were studied by many authors under the name of *a triangular norm* or *a t-norm*, see e.g. [**?**], [**?**], [**?**], [**?**]. In particular, especially important is the class of t-norms satisfying condition

$$\left(\bigvee_{i \in \mathcal{I}} a_i\right) * b = \bigvee_{i \in \mathcal{I}} (a_i * b) \forall \ \{a_i \mid \in \mathcal{I}\} \subseteq L, \ \forall b \in L.$$

Such t-norms are known under the name *left continuous t-norms,* and they describe a large class of GL-monoids.

1.2.1 MV-algebras. Girard monoids

When studying the structure of L-valued L-topological spaces and L-valued L-fuzzy topological spaces sometimes we will need to restrict our subject to the special case of GL-monoids – the so called MV-algebras. Here we consider this concept.

Definition 1.3 *A GL-monoid $(L, \wedge, \vee, *)$ is called an MV-algebra if for every $a, b \in L$ it holds*

$$(a \mapsto b) \mapsto b = a \vee b.$$

An important example of an MV-algebra is the unit interval $[0, 1]$ considered as a lattice (**??**) nd enriched with the Łukasiewicz t-norm:

$$([0, 1], \wedge, \vee, *_L),$$

where

$$a *_L b = \max\{a + b - 1, 0\}.$$

The corresponding residuation is given by

$$a \mapsto b = \min\{1 - a + b, 1\}.$$

From the definition of an MV-algebra it follows that

$$(a \mapsto 0) \mapsto 0 = a.$$

Exactly this property will be crucial for us when studying the closed structure of L-valued topological spaces and L-valued L-fuzzy topological spaces, since it allows to introduce an order reversing involution $^c : L \to L$ (and in the long run logical negation) on L in a natural way, namely, by setting

$$a^c := a \mapsto 0 \quad \forall a \in L.$$

In [**?**] it is proved that GL-monoids, satisfying the last property are MV-algebras.

When studying the closed structure of an L-topology and L-fuzzy topology on an L-valued set and its L-subset we will need the propeties of MV-algebras, which are discribed in the following Lemma:

Lemma 1.4 *Let $(L, \leq, \wedge, \vee, *)$ be an MV-algebra and $a, b \in L$. Then*

$$b \mapsto a = (a \mapsto 0) \mapsto (b \mapsto 0).$$

Proof: By a propery of a GL-monoid we have

$$(b \mapsto a) * (a \mapsto 0) \leq b \mapsto 0,$$

and, applying the Galois connection, it follows

$$b \mapsto a \leq (a \mapsto 0) \mapsto (b \mapsto 0).$$

To get the opposite inequality, replace in the above inequality a by $b \mapsto 0$ and b by $a \mapsto 0$. We get

$$(a \mapsto 0) \mapsto (b \mapsto 0) \leq ((b \mapsto 0) \mapsto 0) \mapsto ((a \mapsto 0) \mapsto 0).$$

Now, appying the double negation law, we get

$$b \mapsto a \geq (a \mapsto 0) \mapsto (b \mapsto 0).$$

□

Also the following Lemma proved in [**?**] will be needed in the sequel:

Lemma 1.5 [**?**], pp. 38-39 *Let* $(L, \leq, \wedge, \vee, *)$ *be an* MV*-algebra. Then*

1. $\left(\bigwedge_{i \in \mathcal{I}} a_i\right) \mapsto 0 = \bigvee_{i \in \mathcal{I}} (a_i \mapsto 0) \quad \forall \{a_i \mid i \in \mathcal{I}\} \subseteq L,$

2. $a * \left(\bigwedge_{i \in \mathcal{I}} b_i\right) = \bigwedge_{i \in \mathcal{I}} (a * b_i) \quad \forall \{b_i \mid i \in \mathcal{I}\} \subseteq L, \quad \forall a \in L.$

Remark 1.6 (Historical remarks and comments on terminology) The concept of an MV-algebra was introduced by C.C Chang [**?**], and later studied and used by many authors, see e.g. [**?**], [**?**].

If we replace the concept of GL-monoid by omitting the divisibility condition, in particular in the class of cl-monoids the property

$$a^c := a \mapsto 0 \quad \forall a \in L$$

does not imply the property

$$(a \mapsto b) \mapsto b = a \vee b$$

in the definition of an MV-algebra. cl-monoids satisfying the condition

$$a^c := a \mapsto 0 \quad \forall a \in L$$

are known as Girard cl-monoids see e.g. [**?**]. Actually our results, where L was assumed to be an MV-algebra in Section 5.1 could be extended to the case when $(L, \leq, \wedge, \vee, *)$ is a Girard cl-monoid.

Here we give an example of a Girard cl-monoid which fails to be a GL-monoid, since the operation $*$ is not divisible. It is defined on the basis of the unit interval $[0, 1]$. This example is taken from Jenei's work [**?**], p. 285:

Example 1.7 Let operation $* : [0,1] \times [0,1] \to [0,1]$ be defined by

$$a * b = \begin{cases} 0 \text{ if } b \leq 1 - a \\ \min(a, b) \text{ otherwise} . \end{cases}$$

Then $([0,1], \wedge, \vee, *)$ is a Girard monoid.

1.3 L-sets

1.3.1 L-sets: Basic definition

Let L be a complete infinitely distributive lattice, *in particular*, a GL-monoid.

Definition 1.8 *Given a set X, by an L-set in X (or more precisely, by an L-subset of a set X) we mean a mapping $A : X \to L$, cf [**?**], [**?**]. The value $A(x)$ is interpreted as the degree to which a point $x \in X$ "belongs" to the L-set A.*

Remark 1.9 A usual (crisp) subset $A \subseteq X$ is identified with its characteristic function

$$1_A : X \to \{0; 1\} \subseteq L$$

and thus can be considered as a particular case of an L-set for every lattice L. In particular, crisp subsets of X can be viewed as $\{0,1\}$-subsets of X by identifying a set $A \subseteq X$ with its characteristic function 1_A. The family of all L-subsets of X is denoted L^X (the so called L-powerset of X); in particular $\{0,1\}^X =: \mathbf{2}^X$ will stand for the family of all crisp subsets of X interpreted by their characteristicfunctions, see also **??**, **??**.

1.3.2 Operations on L-sets

Definition 1.10 *Given a family of L-subsets of a set X,*

$$\mathcal{M} = \{M_j : j \in J\} \subseteq L^X,$$

its union (join), and its intersection (meet) are defined pointwise, i.e.

$$\bigvee \mathcal{M}(x) = \bigvee_{j \in J} M_j(x) \text{ and } \bigwedge \mathcal{M}(x) = \bigwedge_{j \in J} M_j(x),$$

respectively.

Remark 1.11 Along with the definitions of operations on L-sets presented above, sometimes in "Fuzzy Mathematics" and especialy in its aplications, other definitions are used. For example, in case $L = [0,1]$ the union and intersection of $A, B \in [0,1]^X$ are sometimes defined as $A + B - A \cdot B$ and $A \cdot B$ respectively. However, in this work, as in an overwhelming majority of the papers on the subject of fuzzy topology, the operations on L-sets are always realised as defined above.

Definition 1.12 *In case L is equipped with an involution ${}^c : L \times L \to L$, the* pseudo*complement $M^c \in L^X$ of an L-fuzzy set $M \in L^X$ can be defined by the equality $M^c(x) = (M(x))^c$, $x \in X$.*[4]

Definition 1.13 *Given a family of sets $\{X_j : j \in J\}$ and their L-subsets $M_j \in L^{X_j}$, the product $M = \prod_{j \in J} M_j$ is the L-subset of the set $X = \prod_{j \in J} X_j$ defined by the equality*

$$M(x) = \bigwedge_{j \in J} M_j(x_j)$$

where x_j stands for the j-th coordinate of the point $x \in X$.

1.3.3 Preimages and images of L-sets

Definition 1.14 *Let X, Y be sets and $f : X \to Y$ be a mapping. Then the preimage $f^{-1}(N) \in L^X$ of a L-set $N \in L^Y$ is defined by equality*

$$f^{-1}(N)(x) = (N \circ f)(x)(= N(f(x))).$$

The image $f(M) \in L^Y$ of an$L$-set $M \in L^X$ is defined by

$$f(M)(y) = sup\{M(x) : x \in f^{-1}(y)\},$$

(recall that the supremum of the empty set is 0).

In the case $L = \{0,1\}$ these definitions are equivalent to the usual i.e. crisp ones (of course to realize this equivalence we have to interpret a subset $A \subseteq X$ as its characteristic function $1_A : A \to \mathbf{2}$.
The properties of images and preimages of L-sets are quite similar to properties of images and preimages of ordinary sets. For example:

$$f^{-1}\left(\bigvee_{j \in J} N_j\right) = \bigvee_{j \in J} f^{-1}(N_j) \text{ and } f^{-1}\left(\bigwedge_{j \in J} N_j\right) = \bigwedge_{j \in J} f^{-1}(N_j)$$

[4]Note, that in case when L is not a Boolean, the involution mail fail to be a *real* complement, since the equalities $M \vee M^c = 1$ and $M \wedge M^c = 0$ generally do not hold.

for any family $\{N_j \mid i \in J\} \subseteq L^Y$ and

$$f\left(\bigvee_{j\in J} M_j\right) = \bigvee_{j\in J} f(M_j)$$

for any family $\{M_j \mid i \in J\} \subseteq L^X$.

1.4 Categories

When studying different mathematical structures in this work, we often use the terminology from the category theory. Although it is not the main goal of this work to investigate the categorical aspects of concepts studied here, we consider it to be very convenient and useful to use the basic terms and some results from the category theory. In particular, this is essential for the last, 6th Section. For the terminology concerning category theory our sources of references are [**?**] and [**?**].

2 Categories of L-valued L-sets.

2.1 Category SET(L) and its subcategories

2.1.1 Local L-valued sets

Definition 2.1 Following U. Höhle (cf. e.g. [?]) *by an L-valued set we call a pair (X, E) where X is a set and E is an L-valued equality, that is a mapping $E : X \times X \to L$ such that*

(1eq) $E(x, y) = E(x, x) \wedge E(y, y) \quad \forall x, y \in X;$

(2eq) $E(x, y) = E(y, x) \quad \forall x, y \in X;$

(3eq) $E(x, y) * (E(y, y) \mapsto E(y, z)) \leq E(x, z) \quad \forall x, y, z \in X.$

Further, recall that an L-set, or more precisely, an L-subset of a set X is just a mapping $A : X \to L$. In case (X, E) is an L-valued set, its L-subset A is called *extensional* if

$$A(x) * E(x, x') \leq A(x') \quad \forall x, x' \in X.$$

Equivalently, extensionality of an L-set can be defined by the inequality

$$\bigvee_{x \in X} A(x) * E(x, x') \leq \ A(x') \ \ \forall x' \in X.$$

A mapping $f : (X, E_X) \to (Y, E_Y)$ where $(X, E_X), (Y, E_Y)$ are L-valued sets, is called *extensional* if

$$E_X(x, x') \leq E_Y(f(x), f(x')) \ \forall x, x' \in X.$$

In the sequel we shall need the following result:

Lemma 2.2 *Let $f : (X, E_X) \to (Y, E_Y)$ be an extensional mapping and let $B \in L^Y$ be an extensional L-subset of (Y, E_Y). Then its preimage $f^{-1}(B) \in L^X$ is an extensional subset of (X, E_X)*

Proof: follows from the next sequence of inequalities:

$$f^{-1}(B)(x) * E_X(x, x') = B(f(x)) * E_X(x, x') \leq$$

$$B(f(x)) * E_Y(f(x), f(x')) \leq B(f(x')) = f^{-1}(B(x')).$$

U. Höhle interprets the valued $\mathbb{E}(x) = E(x, x)$ as the degree of extent of the element x in an L-valued set (X, E), see e.g. [?], [?].

Further U.Höhle starts to call L-valued equalities and L-valued sets as they are defined above, *local L-valued sets* as different from a special type of L-valued sets, called global L-valued sets:

Definition 2.3 *An L-valued set (X, E) is called global, if the first condition in definition* **??** *is fulfilled in the following, stronger form:*

1'eq $E(x,x) = 1 \quad \forall x \in X$.
In case of a global L-valued set the third condition in definition **??** can be also simplified by replacing it with a simpler condition:

3'eq $E(x,y) * E(y,z) \leq E(x,z)$.

In this work we leave the terms *an L-valued equality* and*an L-valued set*, for global L-valued sets, calling *local L-valued equalities* and *local L-valued sets*, as they are defined in definition **??**

The category of local L-valued sets and extensional mappings between them will be denoted $\mathrm{SET}^{loc}(L)$ and its full subcategory of (global) L-valued sets and extensional mappings between them will be denoted $\mathrm{SET}(L)$.

2.2 L-SET(L) and its subcategories

At the beginning of the previous subsection we recalled U. Höhle's definition of a *local* L-valued equality and a local L-valued set. In this section we introduce and start studying the concept of an L-subset of an L-valued set, which is actually an alternative viewpoint on the concept of a local L-valued set in the sense of U. Höhle.

Definition 2.4 [L-valued L-set] [?], *. Let X be a set and A be its L-subset. An L-valued equality on A is a mapping $E : X \times X \to L$, such that*

1. $E(x,x) = A(x) \quad \forall x \in X$;
2. $E(x,y) \leq A(x) \wedge A(y)$ *for all* $x, y \in X$;
3. $E(x,y) = E(y,x)$ *for all* $x, y \in X$;
4. $E(x,y) * \big(A(y) \to E(y,z)\big) \leq E(x,z)$ *for all* $x, y, z \in X$.

The triple (X, A, E) is called an L-valued L-set.

Remark 2.5 Notice, that if $E : X \times X \to L$ is a local L-valued equality then defining an L-set $A : X \to L$ by $A(x) = E(x,x)$ for all $x \in X$ we obtain an L-valued L-set (X, A, E). Hence the value $A(x)$ can be interpreted as the extent of an element x in the L-valued set (X, E).

Besides the L-set A thus obtained is extensional. Indeed, by applying the first axiom in Definition 2.1. we have

$$A(x)*E(x,x') = E(x,x)*E(x,x') = E(x,x)*(E(x,x)\wedge E(x',x')) \leq E(x',x').$$

Definition 2.6 [?] By a mapping f from an L-valued L-set (X, A, E_A) into an L-valued L-set (Y, B, E_B) (in notation $f : (X, A, E_A) \to (Y, B, E_B)$) we call a mapping $f : X \to Y$ such that $A(x) \leq B(f(x))$. A mapping

$$f : (X, A, E_A) \to (Y, B, E_B)$$

is called extensional if $E_A(x, x') \leq E_B(f(x), f(x'))$ for all $x, x' \in X$.

If $f : (X, A, E_A) \to (Y, B, E_B)$ and $g : (Y, B, E_B) \to (Z, C, E_C)$ are mappings of the corresponding L-valued L-sets, then obviously their set-theoretic composition is a mapping

$$g \circ f : (X, A, E_A) \to (Z, C, E_C)$$

of the corresponding L-valued L-sets. Besides, if f and g are extensional, then the composition

$$g \circ f : (X, A, E_A) \to (Z, C, E_C)$$

is extensional. Indeed, if $x, x' \in X$, then

$$E_B(f(x), f(x')) \geq E_A(x, x')$$

by the extensionality of f and, further,

$$E_C(g(f(x)), g(f(x'))) \geq E_B(f(x), f(x')) \geq E_A(x, x')$$

by the extensionality of g. Besides, the identity mapping $id_X : X \to X$ is obviously extensional and can be considered also as the identity mapping

$$id_{(X,A,E_A)} : (X, A, E_A) \to (X, A, E_A).$$

From these observations we get the following

Theorem 2.7 *L-valued L-sets and mappings between them form a category. This category will be denoted L-SET(L).*

To get more non-trivial examples of categories of L-SET(L)-type, it is often useful to consider its full subcategories subcategories of L_0-SET(L_1, Æ, Ł$_2$)-type. To define a category of L_0-SET(L_1, Æ, Ł$_2$)-type we first specify the notations. Given a GL-monoid L, let L_0, L_1 and L_2 be its sublattices, in particular sub-GL-monoids. We request that the top elements of all lattices coincide with the top element of the lattice L:

$$\top_{L_0} = \top_L = 1, \top_{L_1} = \top_L = 1, \top_{L_2} = \top_L = 1,$$

and the bottom element of the lattice L_2 coincides with the bottom element of the lattice L:

$$\bot_{L_0} = \bot_L = 0.$$

Let $\mathbf{1} = \{1\}$ be a one-point lattice;
$\mathbf{2} = \{0, 1\}$ be a two-point lattice and
$[0, 1]$ be the unit interval.
In particular, as L_0 and L_1 we can take $\mathbf{1} = \{1\}$, $\mathbf{2} = \{0, 1\}$, or $\mathbf{I}$; while as L_2 we can take $\mathbf{2} = \{0, 1\}$ or $\mathbf{I}$, but not $\mathbf{1} = \{1\}$. Further let X be a set and Æ be a family of L_2 valued equalities on L_1^X:

$$E : L_1^X \to L_2.$$

In particular, we write $\mathfrak{E}$ to denote the class of all L_2-valued equalities on L_1^X,
$\mathfrak{E}_s$ to denote the class of separated L_2-valued equalities on L_1^X,
$\mathfrak{E}_c$ to denote the class of crisp L_2-valued equalities on L_1^X,
and $\mathfrak{E}_{cs}$ to denote the class of crisp separated L_2-valued equalities on L_1^X.

Then objects of the category L_0-SET(L_1, Æ, L_2) are qudruples

$$(X, A, L_1^X, E, L_2)$$

where $A \in L_0^X$, $E : L_1^X \to L_2$ is taken from Æ and (X, A, E') is an L_2-valued L_0-set where E' is the restriction of E to X. (Note that a set X naturally can be viewed as a subset $\mathbf{1}^X$ of L_1^X.

The morphisms in the category L_0-SET(L_1, Æ, Ł$_2$) are defined in the same way as in the category L-SET(L).

Examples

1. Our category L-SET(L) in these notations is L-SET($\mathbf{1}, \mathfrak{E}, L$).
2. $\mathbf{1}$ $-SET(\mathbf{1},\ \mathfrak{E},\ L)$ is the category SET(L);
3. $\mathbf{1}$ $-SET(\mathbf{1},\ \mathfrak{E}_{cs},\ L) = \mathbf{1}$ -SET($L, \mathfrak{E}_s,\ \mathbf{2}$)
   is the Goguen's category of L-sets, see [?];
4. $\mathbf{1}$ $-SET(\mathbf{1},\ \mathfrak{E}_{cs},\ [0, 1]) = \mathbf{1}$ -SET($[0, 1], \mathfrak{E}_s,\ \mathbf{2}$)
   is the original Zadeh's category of fuzzy sets, see [?];
5. $\mathbf{1}$ $-SET(\mathbf{1},\ \mathfrak{E}_{cs},\ \mathbf{2}) = \mathbf{1}$-SET $\mathbf{2}, \mathfrak{E}_s,\ \mathbf{2}$)
   is the category of sets.
6. $\mathbf{2}$ $-SET(\mathbf{1},\ \mathfrak{E}_{cs},\ \mathbf{2}) = \mathbf{1}$ -SET $\mathbf{2}, \mathfrak{E}_s,\ \mathbf{2}$)
   is obviously the so called category of pairs of sets (see, e.g. [?]).

2.3 Lattices of L-valued equalities

Let a $GL - monoid$ $(L, \leq, \wedge, \vee, *)$ and a set X be given. On the family $\mathfrak{E}(X,A)$ of all L-valued equalities on an L-set (X,A) we introduce a partial order by setting $E_1 \preceq E_2$ iff

$$E_1 \preceq E_2 \iff E_1(x_1,x_2) \leq E_2(x_1,x_2) \quad \forall x_1,x_2 \in X.$$

Our next aim is to show that $(\mathfrak{E}(X,A), \preceq)$ is a complete lattice.

Theorem 2.8 *Given a family* $Æ = \{E_i \mid i \in \mathcal{I}\}$ *of* L*-valued equalities on an* L*-set* (X,A)*, let* $E_Æ : X \times X \to L$ *be defined by*

$$E_Æ(x,y) = \bigwedge\{E_i(x,y) \mid i \in \mathcal{I}\}.$$

Then $E_Æ$ *is an* L*-valued equality on* (X,A)*, and* $E_Æ = \inf\{Æ\}$.

Proof: The validity of the first three axioms of an L-valued equality on (X,A) for $E_Æ$ is clear. The validity of the fourth axiom follows from the next chain of (in)equalities:

$$E_Æ(x,y) * \big(A(y) \mapsto E_Æ(y,z)\big) =$$

$$= \bigwedge_{i\in\mathcal{I}} (E_{i\in\mathcal{I}}(x,y)) * \left(A(y) \mapsto \bigwedge_{i\in\mathcal{I}}(E_{i\in\mathcal{I}}(y,z))\right) \leq$$

$$\leq \bigwedge_{i\in\mathcal{I}}(E_{i\in\mathcal{I}}(x,y)) * \bigwedge_{i\in\mathcal{I}}((A(y) \mapsto E_{i\in\mathcal{I}}(y,z)) \leq$$

$$\leq (E_{i\in\mathcal{I}}(x,y)) * (A(y) \mapsto (E_{i\in\mathcal{I}}(y,z))) \quad \forall i \in \mathcal{I}.$$

Since this is true for every $i \in \mathcal{I}$ we conclude that

$$E_Æ(x,y) * \big(A(y) \mapsto E_Æ(y,z)\big) \leq$$

$$\leq \bigwedge_{i\in\mathcal{I}}(E_{i\in\mathcal{I}}(x,z)) = E_Æ(x,z).$$

From the previous theorem we have

Corollary 2.9 *The family* $\mathfrak{E}(X,A)$ *of all* L*-valued equalities on an* L*-set* (X,A) *is a complete lattice. The infimum of a family* $Æ = \{E_i \mid i \in \mathcal{I}\} \subseteq \mathfrak{E}(X,A)$ *is*

$$E_Æ(x,y) = \bigwedge\{E_i(x,y) \mid i \in \mathcal{I}\},$$

and its supremum is

$$E^Æ(x,y) = \bigwedge\{E_j(x,y) \mid E_j \geq E_i \ \forall i \in \mathcal{I}\}.$$

Theorem 2.10 *The bottom element of* $\mathfrak{E}(X, A)$ *is defined by*

$$E_\perp(x,y) = 0 \text{ if } x \neq y, x, y \in X \text{ and } E_\perp(x,x) = A(x)\ \forall x \in X$$

and its top element is defined by

$$E^\top(x,y) = A(x) \wedge A(y) \text{ for all } x, y \in X.$$

Proof:

- To show that $E_\perp(x,y)$ thus defined is the bottom element in (X, A) it suffices to show that $E_\perp(x,y)$ is an L-valued equality. The first three axioms are obvious. To show the last axiom, notice, that the only non-trivial case to be verified is

  $$E(x,x) * (A(x) \mapsto E(x,z)) \leq E(x,z).$$

  However, this is clear, since from the properties of GL-monoid, it follows that

  $$a * (a \mapsto 0) = a * \bigvee\{\lambda \mid \lambda * a \leq 0\} = 0.$$

- To show that

  $$E^\top(x,y) = A(x) \wedge A(y) \text{ for all } x, y \in X$$

  is the top element in (X, A) it suffices to notice that

  $$(A(x) \wedge A(y)) * (A(y) \mapsto (A(y) \wedge A(z))) \leq A(x) \wedge A(z).$$

  □

Remark 2.11 The above statements can be extended to the case of the categories of L_0-SET$(L_1, Æ, L_2)$-type. However in this case we need to make certain natural restrictions on the sublattices L_0, L_1 and L_2 of the GL-monoid L and on the class $\mathcal{E}$ of many-valued equalities.

2.4 Final and initial L-valued equalities on L-sets

Now we can construct initial and final L-valued equalities on L-sets for a given family of extensional mappings. Consider a co-structured source

$$\mathcal{F} = \{f_i : (X_i, A_i, E_i) \to (Y, B)\}.$$

We start with the case when the family $\mathcal{F}$ consists of a single co-structured map

$$f : (X, A, E_A) \to (Y, B).$$

Let $\mathcal{E}_f$ be the family of all such L-valued equalities E_j on (Y, B) for which $f : (X, A, E) \to (Y, B, E_j)$ is a morphism in L-SET(L). Then applying Theorem **??** we know that

$$E_f := f(E_A) := \bigwedge \mathcal{E}_f := \bigwedge \{E_j | E_j \in \mathcal{E}_f\}$$

is an L-valued equality on (Y, B). Besides it is easy to notice that $f : (X, A, E_A) \to (Y, B, E_f)$ is a morphism in L-SET(L) and that E_f is the weakest of all L-valued equalities with this property.

Coming now to the general case of the family

$$\mathcal{F} = \{f_i : (X_i, A_i, E_i) \to (Y, B)\},$$

let $f_i(E_i)$ be defined as above and let

$$E_B := \bigvee_{i \in \mathcal{I}} f_i(E_i)$$

be the supremum of this family in the lattice $\mathfrak{E}(Y, B)$ of all L-valued equalities on (Y, B). Then, E_B is exactly the final L-valued equality on (Y, B) for the family $\mathcal{F}$. Indeed, the validity of the inequalities

$$E_i(x_i, x_i') \le E_{f_i}(f_i(x_i), f_i(x_i')) \le E_B(f_i(x_i), f_i(x_i'))$$

for all $x_i, x_i' \in X_i$ $\forall i \in \mathcal{I}$ is clear from the construction of E_B and the minimality of the L-valued equality E_B among all L-valued equalities E_{f_i} on (Y, B) for which all

$$f_i : (X_i, A_i, E_i) \to (Y, B, E_{f_i})$$

are morphisms is obvious.
□

We consider now the dual problem, namely the existence of the initial L-valued equality for a family of mappings. Explicitely, let

$$\mathcal{F} := \{f_i : (X, A) \to (Y_i, B_i, E_i) \mid i \in \mathcal{I}\}$$

be a structured sink of mappings. Our goal is to find the initial L-valued equality on (X, A) for this family, that is the largest L-valued equality E_A

on (X, A) for which all $f_i : (X, A, E_A) \to (Y_i, B_i, E_i)$ are morphisms in the category L-SET(L).

Again, we start with the case when there is only one structured map in the family $\mathcal{F}$:

$$f : (X, A) \to (Y, B, E_B).$$

Let $\mathfrak{E}^f$ be the family of all L-valued equalities E_i^f on X for which

$$f : (X, A, E_i^f) \to (Y, B, E_B)$$

is a morphism in L-SET(L). Let

$$E^f = \bigvee_{j \in J} \{E_j^f \mid E_j^f \in \mathfrak{E}^f\}.$$

(The supremum, naturally is taken in the lattice $\mathfrak{E}(X, A)$.)
Then it is easy to notice that $f : (X, A, E^f) \to (Y, B, E_B)$ is a morphism in the category L-SET(L) and besides E^f is the greatest one with this property.

Notice that in case when $f : (X, A) \to (Y, B, E_B)$ preserves the belongness degree, that is when

$$A(x) = B(f(x)) \quad \forall x \in X,$$

then $E^f = f^{-1}(E_B)$ can be defined explicitely, by the equality:

$$E^f(x_1, x_2) = E_B(f(x_1), f(x_2)) \quad \forall x_1, x_2 \in X.$$

It is easy to see that E^f thus defined is an L-valued equality on (X, A) and the mapping

$$f : (X, A, E^f) \to (Y, B, E_B)$$

is extensional. Indeed, for any $x, x' \in X$ we have:

$$E_B(f(x), f(x')) * B(f(x) = E^f(x, x') * A(x) \leq A(x') = B(f(x'))$$

and besides E^f is the greatest L-valued equality with this property, that is for which the mapping $f : (X, A, E_f) \to (Y, B, E_B)$ is a morphism in L-SET(L).

Coming now to the general case of a co-structured sink, that is a family of mappings

$$\mathcal{F} := \{f_i : (X, A) \to (Y_i, B_i, E_i) \mid i \in \mathcal{I}\},$$

we define $E^{f_i} = f_i^{-1}(E_i)$ as above. From Theorem **??** it follows that

$$E_A := \bigwedge_{i \in \mathcal{I}} E^{f_i}$$

is the largest one of the L-valued equalities on (X, A) for which all

$$f_i : (X, A, E_A) \to (Y_i, B_i, E_i)$$

are morphisms in L-SET(L). Thus E_A is the initial L-valued equality for this family.
□

The existence of initial L-valued equalities guarantees that the operation of product is well defined in L-SET(L). Here we give the explicite description of the product.

Let $\{(X_i, A_i, E_i) \mid i \in \mathcal{I}\}$ be a family of L-valued L-sets, let

$$(X, A) = \prod_{i \in \mathcal{I}} (X_i, A_i)$$

be the product of L-sets and let

$$p_i : (X, A) \to (X_i, A_i, E_i)$$

be the corresponding projections. Further, let

$$E = \bigwedge_{i \in \mathcal{I}} (p_i)^{-1}(E_i),$$

where $p_i^{-1}(E_i) = E^{p_i}$ is the initial L-valued equality for the projection $p_i : (X, A) \to (X, A_i, E_i$. Then (X, A, E) is the product of the family

$$\{(X_i, A_i, E_i) \mid i \in \mathcal{I}\}$$

of mappings in the category L-SET(L).

In a similar way, the existence of final equalities guarantees that the operation of co-product is well defined in this category.

3 Extension of L-valued relations and L-valued equalities to L-powersets.

As it was mentioned in the Intoduction, one of our main goals is to develop foundations of a theory of L-fuzzy topologies on L-valued sets and their L-subsets. To achieve this merit we have first to evolve a procedure, allowing to extend L-valued equalities from L-valued sets to their L-powersets. This will be the subject of this section.

3.1 L-valued preodered sets, category PROSET(L) and some related categories

Definition 3.1 *An L-valued relation (or a fuzzy relation) on a set X is a map* $R : X \times X \to L$.

An L-valued relation R is called

1. *reflexive if* $R(x,x) = 1$ *for all* $x \in X$;
2. *transitive, if* $R(x,y) * R(y,z) \le R(x,z)$ *for all* $x,y,z \in X$;
3. *symmetric, if* $R(x,y) = R(y,x)$ *for all* $x,y \in X$;
4. *separated, if* $R(x,y) = R(y,x) = 1$ *implies that* $x = y$ *for all* $x,y \in X$.

Different authors have used different terminology to describe fuzzy relations with special properties. We shall use the following names:

A transitive L-valued relation is called an L-valued quasipreoder. A reflexive transitive L-valued relation is called an L-valued preoder. A separated L-valued preoder is called an L-valued partial order. A symmetric L-valued preorder is called an L-valued equality. The corresponding pair (X,R) *will be refereed to as an L-valued quasipreodered set, L-valued preodered set, an L-valued partially ordered set, and an L-valued set resp.*

If R is an L-valued preoder on a set, then given $x,y \in X$ the value $R(x,y)$ is interpreted as the degree to which x is greater than or equal to y. In case R is an L-valued equality on X, the intuitive meaning of the value $R(x,y)$ is the degree to which x and y are equal.

Remark 3.2 L-valued relations, usually in case when $L = [0,1]$ and when $*$ is a left-semicontinuous t-norm (see e.g. [?]) were considered by many authors and they used different terminology. In particular, a fuzzy relation $R : X \times X \to [0,1]$ satisfying (1), (2) and (3) is called *a fuzzy equality* in [?],

[?] *a fuzzy equivalence* in [?], [?], or an *indistinguishability operator* [?]. In [?], [?], a fuzzy relation $R : X \times X \to L$ is called *a fuzzy equality* if it satisfies all conditions (1) – (4).

Example 3.3

1. *Let $X = L$. Then by setting $R(x,y) = x \mapsto y$ we define a canonical L-valued partial oder on X and by setting $E(x,y) = R(x,y) \wedge R(y,x)$ we define a canonical L-valued separated equality on X (cf. e.g. [?]).*

2. *Let (X, ρ) be a pseudo-quasimetric space such that $\rho(x,y) \leq 1$ for all $x, y \in X$. Then by setting $R(x,y) = 1 - \rho(x,y)$ we define an L-valued preoder on X where L is the unit interval [0,1] endowed with the Łukasiewicz conjunction $*$. Moreover, if ρ is a pseudometric, then R is an L-valued equality, and in case ρ is a metric, the L-valued equality R is separated (cf e.g. [?]).*

3. *Let $\mathcal{A} \subseteq L^X$ be a family of L-subsets of X. Then, by setting*

$$R(\mathcal{A})(x,y) = \bigwedge_{A \in \mathcal{A}} (A(x) \mapsto A(y))$$

we obtain an L-valued preoder on X.

Definition 3.4 *Given L-valued (quasi)preodered sets (X, R_X) and (Y, R_Y) a mapping $f : X \to Y$ is called extensional if*

$$R_X(x_1, x_2) \leq R_Y(f(x_1), f(x_2)) \text{ for all } x_1, x_2 \in X.$$

L-valued quasi-preodered sets and extensional mappings between them form a category which will be denoted **QPROSET**(L). Its full subcategories consisting of L-valued preodered sets and L-valued sets will be denoted resp. by **PROSET**(L) and **SET**(L). To denote the subcategories of these categories determined by separated L-valued relation we use notations **SQPROSET**(L), **SPROSET**(L) and **SSET**(L) resp. However for the category of separated L-valued partial ordered sets **SPROSET**(L) which are separated by definition and which play a special role in our work an alternative notation **PAOSET**(L) will be also used. In the sequel our main interest here will be in categories **PROSET**(L) and **PAOSET**(L). Categories **SET**(L) and **SSET**(L) will be discussed in subsection 3.5.

Proposition 3.5 *Let X be a set and $\mathfrak{R}(X, L)$ be the family of L-valued preoders on X. Then $\mathfrak{R}(X, L)$ is a complete lattice. Its bottom* $\inf \mathfrak{R}$ *is the discrete (or crisp) (L-valued) preoder*

$$R_{dis}(x, y) = \begin{cases} 1 \text{ if } x = y; \\ 0 \text{ if } x \neq y. \end{cases}$$

The top $\sup \mathfrak{R}$ *of the lattice $\mathfrak{R}(X, L)$ is the indiscrete (L-valued) preoder*

$$R_{ind}(x, y) = 1 \text{ for all } x, y \in X.$$

3.2 L-valued preoder on the L-powerset of an L-valued preodered set

Let (X, R) be an L-valued preodered set. Our first aim is to lift the L-valued preoder R from X to the L-valued quasipreoder $\mathcal{R}$ on the L-powerset L^X of X. We do it as follows.

Given $A, C \in L^X$ we set

$$\mathcal{R}(A, C) = \bigwedge_{x,z \in X} \left((R(x, z) * A(x)) \mapsto C(z)\right).$$

Thus we obtain an L-valued relation

$$\mathcal{R} : L^X \times L^X \to L.$$

From the properties of a GL-monoid it follows that equivalently $\mathcal{R}(A, C)$ can be defined by

$$\mathcal{R}(A, C) = \bigwedge_{x,z \in X} \left(R(x, z) \mapsto \big(A(x) \mapsto C(z)\big)\right).$$

Remark 3.6 The "defuzzified" meaning of the formulae

$$(R(x, z) * A(x)) \mapsto C(z) \text{ and } R(x, z) \mapsto (A(x) \mapsto C(z))$$

can be explained as follows:
If x is greater than or equal to z and x belongs to A then z should belong to C. In particular, in this case, taking $x = z$ we get $A(x) \mapsto C(x)$ for every $x \in X$. By verifying this condition for all $x, z \in X$ we conclude whether A is less than or equal to C – this is the "defuzzified" meaning of the value $\mathcal{R}(A, C)$.

In case $A, C \subseteq X$, that is A, C are crisp subsets of X

$$\mathcal{R}(A, C) = \begin{cases} 1 \text{ if } x \in A \text{ and } R(x, z) > 0 \text{ implies } z \in C \\ 0 \text{ otherwise.} \end{cases}$$

In particular, in case R is a crisp preoder $\leq$ on X, then

$$\mathcal{R}(A, C) = 1 \text{ iff } x \in A \text{ and } z \leq x \text{ implies that } z \in C$$

and $\mathcal{R}(A, C) = 0$ otherwise.

Proposition 3.7 *If $R : X \times X \to L$ is an L-valued reflexive relation on X, then*

$$\mathcal{R}(A, C) \geq \mathcal{R}(A, B) * \mathcal{R}(B, C) \text{ for all } A, B, C \in L^X,$$

and hence $\mathcal{R} : L^X \times L^X \to L$ is an L-valued quasipreorder on L^X.

Proof
To prove the statement we define an auxiliary relation

$$\mathcal{Q} : L^X \times L^X \to L$$

as follows: given $A, C \in L^X$ let

$$\mathcal{Q}(A, C) = \bigwedge_{x,y,z \in X} ((R(x, y) * R(y, z)) \mapsto (A(x) \mapsto C(z)))\,.$$

Obviously $\mathcal{Q}(A, C) \leq \mathcal{R}(A, C)$: just take $y = z$ and apply reflexivity of R according to which $R(z, z) = 1$. On the other hand

$$\mathcal{Q}(A, C) \geq \mathcal{R}(A, B) * \mathcal{R}(B, C) \text{ for any } B \in L^X.$$

Indeed, fix any $x, y, z \in X$. Then

$$(R(x, y) * R(y, z)) \mapsto (A(x) \mapsto C(z)) \geq$$

$$\geq (R(x, y) * R(y, z)) \mapsto ((A(x) \mapsto B(y)) * (B(y) \mapsto C(z))) \geq$$

$$\geq (R(x, y) \mapsto (A(x) \mapsto B(y))) * (R(y, z) \mapsto (B(y) \mapsto C(z)))\,.$$

Now, taking infimum on the both sides of the obtained inequalities by $x, y, z \in X$ and taking into account that $\mathcal{Q}(A, C) \leq \mathcal{R}(A, C)$, we get the required inequality

$$\mathcal{R}(A, C) \geq \mathcal{R}(A, B) * \mathcal{R}(B, C) \quad \forall\ A, B, C \in L^X.$$

□

Corollary 3.8 *If $R : X \times X \to L$ is an L-valued preoder on X, then $\mathcal{R} : L^X \times L^X \to L$ is an L-valued quasipreorder on L^X.*

Remark 3.9 In case R is an L-valued preoder, then $\mathcal{R} = \mathcal{Q}$. Indeed, the equality $\mathcal{Q} \leq \mathcal{R}$ is proved above. Conversly, by transitivity of R we have $R(x,y) * R(y,z) \leq R(x,z)$, and hence

$$(R(x,y) * R(y,z)) \mapsto (A(x) \mapsto C(z)) \geq R(x,z) \mapsto (A(x) \mapsto C(z)).$$

By taking infimum on $x, y, z \in X$ we get the inequality $\mathcal{Q} \geq \mathcal{R}$. Hence $\mathcal{R} = \mathcal{Q}$.

Remark 3.10 In analogy with $\mathcal{Q} : L^X \times L^X \to L$, we can define a relation $\mathcal{R}_n : L^X \times L^X \to L$ by setting

$$\mathcal{R}_n(A,C) = \bigwedge_{y_0,\dots y_n} \left((R(y_0,y_1) * \ldots * R(y_{n-1},y_n)) \mapsto (A(x) \mapsto C(z)) \right),$$

where $y_0 = x, \ldots, y_n = z$. In these notations $\mathcal{R} = \mathcal{R}_1$ and $\mathcal{Q} = \mathcal{R}_2$.
Analogously, as above, one can show that for every $n \geq 2$ and for every $k, 1 < k < n$ the inequality

$$\mathcal{R}_k(A,C) \geq \mathcal{R}_n(A,C) \geq \mathcal{R}_k(A,B) * \mathcal{R}_{n-k}(B,C)$$

holds for all $A, B, C \in L^X$ and hence, in particular $\mathcal{R}_n = \mathcal{R}$ for all n in case R is an L-valued preoder.

Remark 3.11 Let us call an L-set A R-extensional, if

$$R(x,z) * A(x) \leq A(z) \text{ for all } x, z \in X.$$

(A similar property, in case R is an L-valued equality was considered by U. Höhle see e.g. [**?**] and other authors.)
The intuitive "defuzzified" meaning of this condition is the requirement that z should belong to A whenever x belongs to A and z is less than or equal to x.

Let R ve an L-valued quasipreoder on X and let L^X_R be the set of all R-extensional L-sets. In case $A, B, C \in L^X_R$ we have additionally that

$$\mathcal{R}(A,C) = \mathcal{Q}(A,C) \quad \forall\ A, C \in L^X.$$

Indeed, in the obtained inequality

$$\mathcal{R}(A,C) \geq \mathcal{Q}(A,C) \geq \mathcal{R}(A,B) * \mathcal{R}(B,C)$$

just take $B = A$.

In the proposition 3.2., we have proved that the relation $\mathcal{R}$ on L^X is an L-valued quasipreoder. Unfortunately, the reflexivity cannot be ensured by this relation if all L-sets were considered (even if R itself was reflexive). Nevertheless, the reflexivity can be proved if we restrict the domain of $\mathcal{R}$ to the set L_R^X of all R-extensional L-sets.

Theorem 3.12 *If*

$$R : X \times X \to L$$

is an L-valued preoder on X, then

$$\mathcal{R} : L_R^X \times L_R^X \to L$$

is a separated L-valued preoder on L_R^X.
Moreover $\mathcal{R} = \mathcal{Q}$ when restricted to L_R^X.

Proof From proposition 3.2 it follows that $\mathcal{R} : L_R^X \times L_R^X \to L$ is transitive. Further, by definitions and known properies, we conclude that under these assumptions for every $A \in L_R^X$

$$\mathcal{R}(A, A) = \bigwedge_{x,z\in X} ((R(x, z) * A(x)) \mapsto A(z)) \geq \bigwedge_{x\in X} (A(x) \mapsto A(x)) = 1,$$

and hence $\mathcal{R}$ is reflexive.
Finally, to prove that $\mathcal{R} : L_R^X \times L_R^X \to L$ is separated let $A, C \in L^X$ and assume that $\mathcal{R}(A, C) = 1$. Then

$$\mathcal{R}(A, C) = \bigwedge_{x,z\in X} ((R(x, z) * A(x)) \mapsto C(z)) = 1.$$

This means that

$$\forall x, z \in X \quad (R(x, z) * A(x)) \mapsto C(z) = 1,$$

and in particular

$$\forall x \in X \quad (R(x, x) * A(x)) \mapsto C(x) = 1,$$

however this means that $A(x) \leq C(x)$ for all $x \in X$, that is $A \leq C$. In a similar way from the assumption $\mathcal{R}(C, A) = 1$ we conclude that $C \leq A$. Thus if $\mathcal{R}(A, C) = \mathcal{R}(C, A) = 1$, then $A = C$.
Now from the inequality

$$\mathcal{R}(A, C) \geq \mathcal{Q}(A, C) \geq R(A, B) * \mathcal{R}(B, C)$$

we get

$$\mathcal{R}(A,C) = \mathcal{Q}(A,C):$$

just take $B = A$.
□

From Propositions **??** and **??** we get

Theorem 3.13 *If*

$$R : X \times X \to L$$

is an L-valued preoder on X then

$$\mathcal{R} : L^X \times L^X \to L$$

is an L-valued quasipreoder on the powerset L^X and an L-valued partial oder on the extensional powerset L^X_R.

Example 3.14 In all these examples

$$\mathcal{R} : L^X \times L^X \to L$$

is an L-valued quasipreoder on L^X induced by an L-valued preoder

$$R : X \times X \to L$$

unless specified. By α_X we denote the constant function $\alpha_X : X \to L$ with value $\alpha \in L$.

1. Let $A \in L^X_R$. Then $\mathcal{R}(A, 0_X) = \left(\bigvee_{x \in X} A(x) \right) \to 0.$

2. $\mathcal{R}(A, 1_X) = 1$ for any $A \in L^X$.

3. $\mathcal{R}(1_X, A) = 1 \to \bigwedge_{x \in X} A(x)$.

4. Given $a \in X$ let 1_a stand for the characteristic function of the set $\{a\}$. Then
$$\mathcal{R}(A, 1_a) = \left(\bigvee_{x \neq a} A(x) \right) \to 0.$$
In particular, if $a \neq b, a, b \in X$, then $\mathcal{R}(1_a, 1_b) = 0$.

5. For every $a \in X$ we define an L-set

$$s_a : X \to L \quad \text{by } s_a(x) = R(a, x).$$

This is the so called singleton generated by a. Since

$$s_a(x) * R(x, z) = R(a, x) * R(x, z) \leq R(a, z) = s_a(z),$$

singletons are extensional. Moreover, it is easy to notice that s_a is the smallest one of all extensional L-sets, which are greater than or equal to the L-set 1_a.

Let $a, b \in X$. Then

$$\mathcal{R}(s_a, s_b) = \bigwedge_{x,z \in X} ((R(a, x) * R(x, z)) \mapsto R(b, z)) =$$

$$\bigwedge_{z \in X} (R(a, z) \mapsto R(b, z)) \leq$$

$$\leq R(a, a) \mapsto R(b, a) = R(b, a).$$

On the other hand, since

$$R(a, b) * R(b, z) \leq R(a, z)$$

from the Galois connection we conclude that for all $a, b \in X$ and every $z \in X$ it holds

$$R(b, z) \mapsto R(a, z) \geq R(a, b),$$

and, since this holds for any $z \in X$, by taking infimum on x we obtain:

$$\mathcal{R}(s_a, s_b) \geq R(b, a),$$

and hence

$$\mathcal{R}(s_a, s_b) = R(b, a).$$

This equality can be interpreted as follows. Let $\mathcal{R}^c$ stand for the order on L^X obtained by reversing of $\mathcal{R}$. That is

$$\mathcal{R}^c(A, C) = \mathcal{R}(C, A).$$

Now the obtained equality means that by assigning to each $a \in X$ its singleton $s_a \in L_E^X$ we may identify (X, R) with the L-valued partially odered subset $(S, \mathcal{R}_S^c)$ of the L-valued partially ordered set $(L_R^X, \mathcal{R})$ where $S = \{s_a : a \in X\}$ and $\mathcal{R}_S^c$ is the restriction of $\mathcal{R}^c$ to S.

3.3 Powerset functor $\Phi : \mathbf{PROSET}(L) \to \mathbf{PAOSET}(L)^{op}$

In this section we show that the construction assigning to an L-valued preodered set (X, R) its extensional powerset $(L^X_R, \mathcal{R})$ can be considered as a contravariant functor Φ from the category $\mathbf{PROSET}(L)$ into the category $\mathbf{PAOSET}(L)$ that is as a functor

$$\Phi : \mathbf{PROSET}(L) \to \mathbf{PAOSET}(L)^{op}.$$

We shall discuss some properties of this functor. We start with the following

Proposition 3.15 *Let (X, R_X), (Y, R_Y) be L-valued preodered sets and*

$$f : X \to Y$$

be an extensional mapping. Then for every $C, D \in L^Y$ it holds

$$\mathcal{R}_X(f^{-1}(C), f^{-1}(D)) \geq \mathcal{R}_Y(C, D).$$

Recall that the preimage of an L-set $C : Y \to L$ under a function $f : X \to Y$ is defined by the equality $f^{-1}(C)(x) = (f \circ C)(x)$.

Proof follows from the next series of inequalities:

$$\mathcal{R}_X(f^{\leftarrow}(C), f^{\leftarrow}(D)) =$$

$$= \bigwedge_{x,x' \in X} \big(R_X(x, x') \mapsto \big(f^{-1}(C)(x) \mapsto f^{-1}(D)(x')\big)\big) =$$

$$= \bigwedge_{x,x' \in X} (R_X(x, x') \mapsto (C(f(x)) \mapsto D(f(x')))) \geq$$

$$\geq \bigwedge_{x,x' \in X} (R_Y(f(x), f(x')) \mapsto (C(f(x)) \mapsto D(f(x'))) \geq$$

$$\geq \bigwedge_{y,y' \in Y} (R_Y(y, y') \mapsto (C(y) \mapsto D(y'))) = \mathcal{R}_Y(C, D).$$

From Proposition **??** and Theorem **??** we get

Theorem 3.16 *By assigning to each L-valued preodered set*

$$(X, R) \in Ob(\mathbf{PROSET}(L))$$

its extensional powerset $(L_E^X, \mathcal{R})$ *and to each extensional mapping*

$$f : (X, R_X) \to (Y, R_Y)$$

the mapping

$$f^{\leftarrow} : (L_R^Y, \mathcal{R}_X) \to (L_R^X, \mathcal{R}_Y)$$

we define a functor

$$\Phi : PROSET(L) \to PAOSET(L)^{op}.$$

(Here $f^{\leftarrow}(C) = f^{-1}(C)$ for $C \in L^Y$, cf. e.g. [**?**].)

Theorem 3.17 *Functor*

$$\Phi : \mathbf{PROSET}(L) \to \mathbf{PAOSET}(L)^{op}$$

is one-to-one on objects. The restriction Φ' *of the functor* Φ *to* $\mathbf{PAOSET}(L)$*, that is the functor*

$$\Phi' : \mathbf{PAOSET}(L) \to \mathbf{PAOSET}(L)^{op}$$

is an embedding.

Proof Let R_1 and R_2 be L-valued relations on a set X and $R_1 \neq R_2$. Then there exist $a, b \in X$ such that $R_1(a,b) \neq R_2(a,b)$. However, as it was shown above, $\mathcal{R}_1(s_a, s_b) = R_1(b,a)$ and $\mathcal{R}_2(s_a, s_b) = R_2(b,a)$ (where s_a, s_b are singletons corresponding to the points a, b).
Hence $\mathcal{R}_1 \neq \mathcal{R}_2$.
□

Remark 3.18 In a similar way as functor Φ one can consider a functor

$$\tilde{\Phi} : \mathbf{PROSET}(L) \to \mathbf{QPROSET}(L)^{op}$$

assigning to each (X, R) the L-valued quasipreoder set $(L^X, \mathcal{R})$. The image $\tilde{\Phi}(\mathbf{PROSET}(L))$ is a subcategory of the category $\mathbf{QPROSET}(L)^{op}$. We shall not go into details of this construction here.

Remark 3.19 *Functors* Φ *and* $\tilde{\Phi}$ *are order reversing.*

Indeed, assume that R_1 and R_2 are two L-valued preoders on X and $R_1 \leq R_2$. Then for any $A, C \in L^X$

$$\mathcal{R}_1(A, C) = \bigwedge_{x,z \in X} ((R_1(x,z) * A(x)) \mapsto C(z)) \geq$$

$$\geq \bigwedge_{x,z \in X} ((R_2(x,z) * A(x)) \mapsto C(z)) = \mathcal{R}_2(A, C)$$

and hence $\mathcal{R}_1 \geq \mathcal{R}_2$.

Applying previous remark it is easy to prove the following two theorems:

Theorem 3.20 *Let Z be a set, (X_i, R_i) be a family of sets endowed with some order type relation, and*

$$f_i : Z \to X_i, i \in \mathcal{I}$$

be a family of mappings. Further, let R_0 be an order-type relation on Z, initial for this family of mappings. Then the corresponding L-valued relation on the powerset L^Z (or L^Z_R) $\mathcal{R}_0$ is the final order type relation for the family of mappings

$$f_i^{\leftarrow} : (L^X, \mathcal{R}_i) \to L^Z.$$

Theorem 3.21 *Let Z be a set, (X_i, R_i) be a family of sets endowed with some order type relation, and*

$$f_i : X_i \to Z, i \in \mathcal{I}$$

be a family of mappings. Further, let R^0 be an order-type relation on Z, final for this family of mappings. Then the corresponding L-valued relation on the powerset L^Z (or L^Z_R) $\mathcal{R}^0$ is the initial order type relation on L^Z (or L^Z_R) for the family of mappings

$$f_i^{\leftarrow} : L^Z \to (L^X, \mathcal{R}_i).$$

From these theorems we have the following corollaries:

Corollary 3.22 *Let*

$$\mathcal{X} = \{(X_i, R_i) \mid i \in \mathcal{I}\}$$

be a family of sets endowed with some order type relations R_i and let

$$\prod_{i \in \mathcal{I}} (X_i, R_i) = (X, R)$$

be the product of these sets. Further, let $(L^{X_i}, \mathcal{R}_i)$ be the L-exponent of (X_i, R_i). Then $(\coprod_{i \in \mathcal{I}} L^{X_i}, \mathcal{R})$ is coproduct of the family

$$\{(L^{X_i}, \mathcal{R}_i) \mid i \in \mathcal{I}\}.$$

Corollary 3.23 *Let*

$$\mathcal{X} = \{(X_i, R_i) \mid i \in \mathcal{I}\}$$

be a family of sets endowed with some order type relations R_i and let

$$\coprod_{i \in \mathcal{I}} (X_i, R_i) = (X, R)$$

be the coproduct of these sets. Further, let $(L^{X_i}, \mathcal{R}_i)$ be the L-exponent of (X_i, R_i). Then $(\prod_{i \in \mathcal{I}} L^{X_i}, \mathcal{R})$ is the product of the family

$$\{(L^{X_i}, \mathcal{R}_i) \mid i \in \mathcal{I}\}.$$

3.4 Lattices $QPR(L^X)$ and $PR(L^X)$

Given a set X we denote by $PR(L^X)$ the family of all L-valued preoders $\mathcal{R}$ on L^X obtained from L-valued preoders R on X. In other words $\mathcal{S} \in PR(L^X)$ if and only if $(L^X_E, \mathcal{S}) \in Ob(\Phi(\mathbf{PROSET}(L)))$. In a similar way $\mathcal{S} \in QPR(L^X)$ if and only if $(L^X, \mathcal{S}) \in Ob(\Phi(\mathbf{QPROSET}(L)))$. From the previous results it follows, that $QPR(L^X)$ and $PR(L^X_R)$ are bounded lattices where the greatest element $\mathcal{R}_\top$ is induced by the discrete (L-valued) preoder R_{dis} on X and the smallest element $\mathcal{R}_\bot$ is induced by indiscrete L-valued preoder R_{ind} on X. Explicitely, for the largest element $\mathcal{R}_\top$: given $A, C \in L^X_R$

$$\mathcal{R}_\top(A,C) = \bigwedge_{x\in X} \left(A(x) \mapsto C(x)\right).$$

Indeed,

$$\mathcal{R}_\top(A,C) = \bigwedge_{x,z\in X} (R_{dis}(x,z) \mapsto (A(x) \mapsto C(z)))$$

and

$$R_{dis}(x,z) \mapsto (A(x) \mapsto C(z)) = 1 \text{ if } x \neq z$$

while

$$R_{dis}(x,x) \mapsto (A(x) \mapsto C(z)) = A(x) \mapsto C(z).$$

For the smallest element $\mathcal{R}_\bot$: given $A, C \in L^X_R$

$$\mathcal{R}_\bot(A,C) = \bigvee_{x\in X} A(x) \mapsto \bigwedge_{z\in X} C(z).$$

Indeed

$$\mathcal{R}(A,C) = \bigwedge_{x,z\in X} (R_{ind}(x,z) \mapsto (A(x) \mapsto C(z))) =$$

$$= \bigwedge_{x,z\in X} (1 \mapsto (A(x) \mapsto C(z))) = \bigwedge_{x,z\in X} (A(x) \mapsto C(z)) =$$

$$= \bigvee_{x\in X} A(x) \mapsto \bigwedge_{z\in X} C(z).$$

Note that in case A is R_{ind}-extensional, then

$$\mathcal{R}_\bot(A,A) = \bigwedge_{x\in X} (A(x) \mapsto A(x)) = 1,$$

and hence $\mathcal{R}_\bot$ is an L-valued preoder, but generally $\mathcal{R}_\bot$ is only a quasi-preoder.

Example 3.24

1. Let $L = [0,1]$ and $* = \wedge$ in $(L, \leq, \wedge, \vee, *)$, that is

$$(L, \leq, \wedge, \vee)$$

is viewed as a Heyting algebra. Recall that the corresponding residium is defined by

$$\alpha \mapsto \beta = \begin{cases} 1 \text{ if } \alpha \leq \beta \text{ and} \\ 0 \text{ otherwise} \end{cases}$$

for $\alpha, \beta \in L$.

   (a) Let $R = R_{ind}$ be the indiscrete L-valued preoder on X and $A, C \in L^X$. Then

$$\mathcal{R}(A, C) = \bigvee_{x \in X} A(x) \mapsto \bigwedge_{z \in X} C(z).$$

   Hence

$$\mathcal{R}(A, C) = \begin{cases} 1 \text{ if } \sup_{x \in X} A(x) \leq \inf_{x] \in X} C(x) \text{ and} \\ \inf_{x \in X} C(x) \text{ otherwise .} \end{cases}$$

   In particular, for $A, C \subseteq X$

$$\mathcal{R}(A, C) = \begin{cases} 1 \text{ if } A = \emptyset \text{ or } C = X \text{ and} \\ 0 \text{ otherwise .} \end{cases}$$

   (Note that X and $\emptyset$ are the only extensional sets in this case.)

   (b) Let $R = R_{dis}$ be the discrete L-valued preoder on X and $A, C \in L^X$. Then

$$\mathcal{R}(A, C) = \bigwedge_{x \in X} (A(x) \mapsto C(x)).$$

   Hence

$$\mathcal{R}(A, C) = \begin{cases} 1 \text{ if } A(x) \leq C(x) \ \ \forall x \in X \text{ and} \\ \inf_x \{C(x) \mid x \in X, A(x) \geq C(x)\} \ \ \text{otherwise.} \end{cases}$$

   In particular, for $A, C \subseteq X$

$$\mathcal{R}(A, C) = \begin{cases} 1 \text{ if } A \subseteq C \text{ and} \\ 0 \text{ if } A \not\subseteq C. \end{cases}$$

2. Let $L = [0,1]$ and $*$ be the Łukasiewicz conjunction that is

$$\alpha * \beta = \max\{\alpha + \beta - 1, 0\} \text{ for } \alpha, \beta \in [0,1]$$

and hence $(L, \leq, \wedge, \vee, *)$ is an MV-algebra. Recall that the corresponding residium is defined by

$$\alpha \mapsto \beta = \min\{1 - \alpha + \beta,\ 1\}.$$

(a) Let $R = R_{ind}$ be the indiscrete L-valued preoder on X and $A, C \in L^X$. Then

$$\mathcal{R}(A, C) = \bigwedge_{x,z \in X} \min\{1 - A(x) + C(z),\ 1\}.$$

Hence

$$\mathcal{R}(A, C) = \begin{cases} 1 \text{ if } \sup\limits_{x \in X} A(x) \leq \inf\limits_{x \in X} C(x) \text{ and} \\ 1 - \sup\limits_{x \in X} A(x) + \inf\limits_{x \in X} C(x) \quad \text{otherwise.} \end{cases}$$

(b) Let $R = R_{dis}$ be the discrete L-valued preoder on X and $A, C \in L^X$. Then

$$\mathcal{R}(A, C) = \bigwedge_{x,z \in X} ((R(x,z) * A(x)) \mapsto C(z)) =$$

$$= \bigwedge_{x \in X} (A(x) \mapsto\ C(x))$$

Hence

$$\mathcal{R}(A, C) = \begin{cases} 1 \text{ if } A(x) \leq C(x)\ \forall x \in X \text{ and} \\ \inf\limits_{x \in X} \{1 - A(x) + C(x)\} \text{ otherwise.} \end{cases}$$

3. Let $L = [0,1]$ and $*$ be the product on $[0,1]$ that is $\alpha * \beta = \alpha \cdot \beta$ for $\alpha, \beta \in [0,1]$.
Recall that the corresponding residium in this case is defined by

$$\alpha \mapsto \beta = \begin{cases} 1 \text{ if } \alpha \leq \beta \text{ and} \\ \frac{\beta}{\alpha} \text{ otherwise .} \end{cases}$$

(a) Let $R = R_{ind}$ be the indiscrete L-valued preoder on X and $A, C \in L^X$. Then

$$\mathcal{R}(A,C) = \bigvee_{x \in X} A(x) \mapsto \bigwedge_{x \in X} C(x).$$

Hence

$$\mathcal{R}(A,C) = \begin{cases} 1 \text{ if } \sup\limits_{x \in X} A(x) \leq \inf\limits_{x \in X} C(x) \text{ and} \\ \frac{\bigwedge_{x \in X} C(x)}{\bigvee_{x \in X} A(x)} \text{ otherwise.} \end{cases}$$

(b) Let $R = R_{dis}$ be the discrete L-valued preoder on X and $A, C \in L^X$. Then

$$\mathcal{R}(A,C) = \bigwedge_{x \in X} (A(x) \mapsto C(x))$$

Hence

$$\mathcal{R}(A,C) = \begin{cases} 1 \text{ if } A(x) \leq C(x) \ \forall x \in X \text{ and} \\ \frac{\bigwedge_{x \in X : A(x) \geq C(x)} C(x)}{\bigwedge_{x \in X : A(x) \geq C(x)} A(x)} \text{ otherwise.} \end{cases}$$

3.5 L-valued equality on the L-powerset of an L-valued set

Let X be a set and $E : X \times X \to L$ be an L-valued equality on X, that is a symmetric preoder. Refering to Section 3 by setting

$$\mathcal{R}(A,C) = \bigwedge_{x,z \in X} (E(x,z) \mapsto (A(x) \mapsto C(z)))$$

we obtain a separated L-valued preoder on L_E^X (where L_E^X is the family of all extensional L-subsets of X) and an L-valued quasipreoder on L^X. In the next theorem we symmetrize this relation in order to get an L-valued equality on L_E^X.

Theorem 3.25 *For $A, C \in L^X$ let*

$$\mathcal{E}(A,C) = \mathcal{R}(A,C) \wedge \mathcal{R}(C,A).$$

Then $\mathcal{E} : L_E^X \times L_E^X \mapsto L$ is an L-valued equality on L_E^X.

Proof The reflexivity of $\mathcal{E}$ follows from the reflexivity of $\mathcal{R}$.
The symmetry of $\mathcal{E}$ is obvious from the definition.

The transitivity follows from the next series of (in)equalities (see Proposition 1.1):

$$\mathcal{E}(A,B) * \mathcal{E}(B,C) =$$
$$= (\mathcal{R}(A,B) \wedge \mathcal{R}(B,A)) * (\mathcal{R}(B,C) \wedge \mathcal{R}(C,B)) \le$$
$$\le (\mathcal{R}(A,B) * \mathcal{R}(B,C)) \wedge (\mathcal{R}(C,B) * \mathcal{R}(B,A)) \le$$
$$\le \mathcal{R}(A,C) \wedge \mathcal{R}(C,A) = \mathcal{E}(A,C).$$

Hence the pair $(L_E^X, \mathcal{E})$ is a separated L-valued set.
□

Thus, assigning to an L-valued set (X,E) the pair $(L_E^X, \mathcal{E})$ we obtain a functor

$$\Psi : \mathbf{SET}(L) \to \mathbf{SSET}(L)^{op},$$

where $\mathbf{SSET}(L)$ is the category of all separated L-valued L-sets.

One can get results about L-valued equalities on the L-powerset and the funcor Ψ analogous to the results about L-valued preoders on the L-powersets and the functor Φ discussed in sections 3, 4 and 5.

Remark 3.26 There are alternative ways how one can extend an L-valued equality $E : X \times X \to L$ to the L-powerset L_E^X. In particular, let

$$\mathcal{E}' : L_E^X \times L_E^X \to L$$

be defined by setting $\mathcal{E}'(A,C) = \mathcal{R}(A,C) * \mathcal{R}(C,A)$. One can easily notice that $\mathcal{E}'$ is an L-valued equality on L_E^X and that $\mathcal{E}' \le \mathcal{E}$. However, the equality generally does not hold.

4 Categories of L-valued L-topological spaces

4.1 Category TOP(L) and its subcategories

4.1.1 L-valued L-topological spaces and category TOP(L): Basic definitions

Here we introduce the concept of an L-valued L-topological space and study the structure of such spaces. Also continuity of mappings between L-valued L-topological spaces will be defined. As the result we come to the category TOP(L) of L-valued L-topological spaces. We shall not go into studying details of this category since, as it will be clear, TOP(L) is a full subcategory of a more general category L-TOP(L) which will be more thorouhly studied in the next subsection.

Definition 4.1 *Let $L = (L, \wedge, \vee, *)$ be a GL-monoid and X be a set. Further, let $E : L \times L \to L$ be an L-valued equality on X. A family $\tau \subseteq L^X$ is called an L-topology on an L-valued set (X, E) if*

1. $0_X \in \tau; 1_X \in \tau$;
2. *if* $U, V \in \tau$, *then* $U \wedge V \in \tau$;
3. *if* $U_i \in \tau \quad \forall i \in \mathcal{I}$, *then* $\bigvee_{i \in i} U_i \in \tau$;
4. *if* $U \in \tau$, *then* $U(x) * E(x, x') \leq U(x') \; \forall x, x' \in X$.
   The last condition means that all L-sets in τ are extensional and hence $\tau \subseteq L^X_E$.

The triple (X, E, τ) where τ is an L-topology on an L-valued set will be refered to as an L-valued L-topological space. Respectively, the elements $U \in \tau$ will be refered to as open L-sets in this L-valued L-topological space.

Definition 4.2 *Let (X, E_X, τ_X) and (Y, E_Y, τ_Y) be L-valued Ltopological spaces and $f : X \to Y$ be a mapping. This mapping will be called continuous as a mapping between L-valued L-topological spaces (X, E_X, τ_X) and (Y, E_Y, τ_Y):*

$$f : (X, E_X, \tau_X) \to (Y, E_Y, \tau_Y) \quad \textit{if}$$

1. $E_X(x, x') \leq E_Y(f(x), f(x'))$ *for all* $x, x' \in X$,
   that is f is an extensional mapping between the corresponding L-valued sets (X, E_X) and (Y, E_Y), and
2. $f^{-1}(V) \in \tau_X \quad$ *whenever* $\quad V \in \tau_Y$.

Obviously, L-valued L-topological spaces and continuous mappings between them form a category TOP(L). As it was said above, we postpone description of properties of this category to the next section. Here we focus our int erest on the structure of an L-valued L-topological space.

4.1.2 Interior operator in L-valued topological spaces

Let (X, E, τ) be an L-valued L-topological space. Notice first, that by lower-semicontinuity of conjunction for extensional L-sets U_i we have

$$\left(\bigvee_{i\in\mathcal{I}} (U_i)\,(x)\right) * E(x,x') = \left(\bigvee_{i\in\mathcal{I}} U_i(x)\right) * E(x,x') \leq$$

$$\leq \bigvee_{i\in\mathcal{I}} (U_i(x) * E(x,x')) \leq \bigvee_{i\in\mathcal{I}} (U_i(x')) \quad \forall x, x' \in X.$$

Thus the supremum of extensional L-subsets of an L-valued set is extensional itself. Therefore, in an analogy with classical topology [5] we can define an interior $int(A)$ of an L-subset A of an L-valued L-topological space (X, E, τ) as the largest ($\geq$) one of all open L-subsets of the L-valued L-topological space (X, E, τ) contained ($\leq$) in A. Equivalently, it can be defined by the formula

$$int(A) = \bigvee \{U \in \tau \mid U \leq A\}.$$

One can easily verify that the resulting operator $int : L^X \to L^X$ satisfies all properties, analogous to the properties of the interior operator in classical topology, and, as the letter, can be used as an alternative way to introduce an L-topology on an$L$-valued set.

4.1.3 Closed structure of an L-valued L-topological space

In order to define the concept of a closed L-set in an L-valued L-topological space in a coherent way (that is in such a way that the analogy with the classic case would hold for our situation) we need to assume additionally that L is a MV-algebra (or, if we extend to a more general case of *cl*-monoids, to assume that L is a Girard monoid). Thus in this subsection L is an MV-algebra. As we saw above, see Lemma **??**, if L is a GL-monoid, then by setting

$$a \Rightarrow a^c \text{ where } a^c := a \mapsto 0,$$

we obtain an order reversing involution, that is:

$$a \leq b \Longrightarrow b^c \leq a^c \text{ and } (a^c)^c = a.$$

[5] see e.g.[**?**], [**?**] cf also interior operator for L-topological spaces [**?**], [**?**]

We extend this involution pointwise to the L-powerset, by setting for a given $A \in L^X$

$$A^c(x) := (A(x))^c \ \forall x \in X.$$

Since $(A^c)^c = A$, the L-set A^c is interpreted as the complement of the L-fuzzy set A. We do not want to lose extensionality, and therefore we need the following lemma:

Lemma 4.3 *If $A \in L^X$ is an extensional L-subset of an L-valued set (X, E), then its complement A^c is also extensional.*

Proof: Let $A \in L^X$ be extensional and let A^c be its complement. We have to show that

$$A^c(x) * E(x, x') \leq A^c(x') \quad \forall x, x' \in X.$$

The required inequality can be rewritten in an equivalent way as follows:

$$E(x, x') \leq (A(x) \mapsto 0) \mapsto (A(x') \mapsto 0).$$

Now, referrring to Lemma **??**, this is equivalent to the inequality

$$E(x, x') \leq A(x) \mapsto A(x'),$$

which, by the Galois connection means just the condition of extenssionality for a given L-set A:

$$A(x) * E(x, x') \leq A(x').$$

□

Definition 4.4 *An (extensional) L-subset A in an L-valued L-topological space is called closed, if its complement A^c is open.*

The relations between open and closed L-sets in an L-valued L-topological space are quite analogous to the relations between open and closed sets in an ordinary topological space (of course, in case when L is a Girard monoid).[6] In particular, the following result can be easily obtained, refering to the properties of a Girard monoid:

Theorem 4.5 *Let (X, E, τ) be an L-valued L-topological space and let σ stand for the family of all its closed L-subsets. Then:*

1. $0_X, 1_X \in \sigma$,

[6] see e.g. [**?**], [**?**], cf also properties of closed sets in an L-topological space, see e.g. [**?**], [**?**].

2. $A, B \in \sigma \Longrightarrow A \vee B \in \sigma$,

3. $A_i \in \sigma \quad \forall i \in \mathcal{I} \Longrightarrow \bigwedge\limits_{i \in \mathcal{I}} A_i \in \sigma$.

Proof:

- The equality $0 \mapsto 1 = 1$ holds in any GL-monoid. The equality $1 \mapsto 0 = 0$ holds in any Girard monoid, in particular, in an MV-algebra, since operation $a \mapsto 0$ in a Girard monoid is order reversing and, obviously, bijective. Thus $0_X, 1_X \in \sigma$.

- Let $A, B \in \sigma$. Then $A \mapsto 0 \in \tau,\ B \mapsto 0 \in \tau$ and hence

  $$(A \mapsto 0) \wedge (B \mapsto 0) \in \tau.$$

  Refering to Lemma **??** we have

  $$(A \mapsto 0) \wedge (B \mapsto 0) = (A \vee B) \mapsto 0 \in \tau,$$

  and hence $A \vee B \in \sigma$.

- Let $A_i \in \sigma\ \forall i \in \mathcal{I}$, then $A_i \mapsto 0 \in \sigma\ \forall i \in \mathcal{I}$, and hence $\bigvee\limits_{i \in \mathcal{I}} (A_i \mapsto 0) \in \tau$. Referring to Lemma **??** again, we have

  $$\left(\bigwedge_{i \in \mathcal{I}} A_i \mapsto 0\right) = \bigvee_{i \in \mathcal{I}} (A_i \mapsto 0) \in \tau,$$

  and hence $\bigwedge\limits_{i \in \mathcal{I}} A_i \in \sigma$.

□

4.1.4 Closure operators in L-valued L-topological spaces

As in the previous subsection we assume that L is an MV-algebra. Let (X, E, τ) be an L-valued L-topological space, and σ be the family of its closed L-sets. Since σ is invariant under taking arbitrary intersections, for any $M \in L^X$, in particular, for $M \in L^X_E$, by setting

$$cl(M) := \bigwedge \{A \mid A \in \sigma, A \geq M\},$$

we obtain the closure operator in the space (X, E, τ):

$$cl : L^X \to \sigma (\subseteq L^X_E).$$

Reasoning in the same way as in classical topology, we obtain the following result:

Theorem 4.6 *Let (X, E, τ) be an L-valued L-topological space and $cl : L^X \to L^X_E$ its closure operator. Then*

1. $M \leq N \Longrightarrow cl(M) \leq cl(N)$;
2. $cl(0_X) = 0_X$,
3. $cl(M \vee N) = cl(M) \vee cl(N)$;
4. $cl(cl(M)) = cl(M)$.

Obviously, also closure operators can be used as primary concepts to introduce an L-topology on an L-valued set.

4.2 Category L-TOP(L) and its subcategories

4.2.1 Basic definitions

Definition 4.7 *By an L-topology on an L-valued L-set (X, A, E_A) we call a family of L-subsets $\tau = \{U_i \in \tau; \forall i \in I\}$, such that $U_i \leq A \quad \forall U_i \in \tau$ and*

1. $0 \in \tau; A \in \tau$;
2. *if* $U, V \in \tau$, *then* $U \wedge V \in \tau$;
3. *if* $U_i \in \tau \ \forall i \in \mathcal{I}$, *then* $\bigvee_{i \in i} U_i \in \tau$
4. *if* $U \in \tau$, *then* $U(x) * E_A(x, x') \leq U(x') \ \forall x, x' \in X$. Thus all L-sets in τ are extensional.

The quadruple (X, A, E_A, τ) is called an L-valued L-topological space.

Remark 4.8 The fourth axiom means that every L-subset of an L-set A from the family τ satisfies the extensionality type condition with respect to the L-valued equality E_A on (X, A).

Definition 4.9 *By a continuous mapping from an L-valued L-topological space (X, A, E_A, τ_A) into an L-valued L-topological space (Y, B, E_B, τ_B) we mean a mapping $f : X \to Y$ such that*

1. $A(x) \leq B(f(x)) \ \forall x \in X$;
2. $E_A(x, x') \leq E_B(f(x), f(x')) \ \forall x, x' \in X$;
3. $\forall \ V \in \tau_B \Rightarrow f^{-1}(V) \in \tau_A$.

Since composition of continuous mappings is obviously continuous and the identity mapping $f : (X, A, E_A, \tau_A) \to (X, A, E_A, \tau_A)$ is continuous, we obtain the following

Theorem 4.10 *L-valued L-topological spaces as objects and continuous mappings between them as morphisms form a category. This category will be denoted L-TOP(L).*

Definition 4.11 *A family $\mathcal{B} \subseteq L_E^X$ is called a base for an L-valued L-topology τ on an L-set if τ is obtained by taking all suprema of L-sets in $\mathcal{B}$. A family $\mathcal{C} \subseteq L_E^X$ is called a subbase for an L-topology τ if the family of all finite intersections of L-sets from $\mathcal{C}$ is a base for τ*

As in classic topology it is easy to see that $\mathcal{C}$ is a subbase for an L-valued L-topology τ if and only if τ is the weakest ($\leq$) L-valued L-topology containing ($\supseteq$) $\mathcal{C}$.

One can also easily prove the following analogue of the well known result from the classic topology:

Theorem 4.12 *Let $(X, A, E_A \tau_A)$ and (Y, B, E_B, τ_B) be L-valued L-topological spaces, $\mathcal{C}_B$ a subbase of τ_B and let*

$$f : (X, A, E_A) \to (Y, B, E_B)$$

be an extensional mapping. Then the following conditions are equivalent:

1. *The mapping $f : (X, A, E_A, \tau_A) \to (Y, B, E_B, \tau_B)$ is continuous.*
2. *$f^{-1}(V) \in \tau_A$ for every $V \in \mathcal{C}_B$*

From the definitions immediately follows

Theorem 4.13 *Category TOP(L) considered in the previous subsection is isomorphic to a full subcategory of the present category L-TOP(L)[7], whose objects are of the form $(X, 1_X, E, \tau)$.*

4.2.2 The lattice of L-valued L-topologies. Final and initial structures in the categoty L-TOP(L)

One can easily prove the following

[7]just for this reason we disided to leave the same term, namely, L-valued L-topological space, for the objects of the category L-TOP(L)

Definition 4.14 *Let $\mathfrak{T}$ be a family of L-topologies on an L-valued L-set (X, A, E). We introduce a partial order $\preceq$ on $\mathfrak{T}$ by setting*

$$\tau_A \preceq \tau_B \Longleftrightarrow \tau_A \subseteq \tau_B.$$

Since the infimum of L-valued L-topologies, is obviously, an L-valued L-topology, we get the following theorem:

Theorem 4.15 *The family $\mathfrak{T}$ of L-topologies on an L-valued L-set (X, A, E_A) is a complete lattice. Its top element is an L-valued L-topology $\tau_1 = L^X_{E_A}$ consisting of all extensional L-valued L-subsets of A, and its bottom element is the indiscrete L-valued L-topology $\tau_0 = \{A, 0_X\}$.*

Consider a family of L-valued L-topological spaces $\{(X_i, A_i, E_i, \tau_i) | i \in I\}$, an extensional L-set (Y, B, E_B) and a family of mappings

$$\Phi := \{f_i : (X_i, A_i, E_i, \tau_i) \to (Y, B, E_B)\}.$$

Our aim is to find the final L-valued L-topology for this family. We start with the case when the family Φ consists of a single mapping

$$f : (X, A, E_A, \tau_A) \to (Y, B, E_B).$$

Let L^B_E be the family of all extentional L-subsets of (Y, B, E_B), and let

$$T_B = \{V \in L^B_E \mid f^{-1}(V) \in \tau_A\}.$$

Further, let an L-valued L-topology $\tau_B := f(\tau_A)$ be defined by T_B as a subbase. It is easy to notice that τ_B is the smallest L-valued L-topology on (Y, B, E_B), such that the mapping f is a morphism in L-TOP(L) and hence it is final for this mapping.

Consider now the general case of a family of functions

$$\Phi := \{f_i : (X_i, A_i, E_i, \tau_i) \to (Y, B, E_B) \mid i \in \mathcal{I}\}.$$

Let for each $i \in \mathcal{I}$ $f(\tau_i)$ be defined as above, then the subbase of the final L-valued L-topology τ_B on (Y, B, E_B) will be defined as the join of all L-valued L-topologies $T_B = \bigwedge_{i \in I} f(\tau_i)$ in the lattice of L-valued L-topoloies. Again, this is the smallest L-valued L-topology on (Y, B, E_B), such that all mappings f_i are morphism in L-TOP(L).

We consider now the dual problem, namely the initial L-valued L-topology for a family of mappings. Explisily, let

$$\Phi := \{f_i : (X, A, E_A) \to (Y_i, B_i, E_i, \tau_i) \mid i \in \mathcal{I}\}$$

be a family of mappings from an L-valued L-set (X, A, E_A) into an L-valued L-topological space (Y_i, B_i, E_i, τ_i). Our goal is to find the initial L-topology on an L-valued L-set (X, A, E_A) for this family. Again we start with the case when there is only one maping in the family Φ:

$$f : (X, A, E_A) \to (Y, B, E_B, \tau_B).$$

The initial L-valued topology $\tau_A := f^{-1}(\tau_B)$ on (X, A, E_A) is defined as

$$f^{-1}(\tau_B) = \{U = f^{-1}(V), \text{ where } V \in \tau_B\}.$$

One can esily verify that it is indeed an L-valued L-topology, and besides the weakest one, for which

$$f : (X, A, E_A, \tau_A) \to (Y, B, E_B, \tau_B)$$

is continuous.

Consider now a family of mappings

$$\Phi := \{f_i : (X, A, E_A) \to (Y_i, B_i, E_i, \tau_i) \mid i \in \mathcal{I}\}.$$

Let $f^{-1}(\tau_{B_i})$ be defined as above for each $i \in \mathcal{I}$, then τ_A will be obtained from the subbase

$$\mathcal{C}_A = \bigcup_{i \in I} f_i^{-1}(\tau_{B_i}).$$

Again, from the definition of τ_A it is clear that it is the weakest L-valued L-topology on (X, A, E_A), for which all mappings of the family

$$\Phi := \{f_i : (X, A, E_A) \to (Y_i, B_i, E_i, \tau_i) \mid i \in \mathcal{I}\}$$

are continuous.

Thus we have established that both final and initial L-valued L-topologies for a family of mappings exist in L-TOP(L), and, moreover gave an explicite way how they can be constructed. From here it follows that both products and co-products and moreover, an explicite way of their construction is presented.

Now we can prove the following result:

Theorem 4.16 *The category L-TOP(L) is topological over the category L-SET(L) with respect to the forgetful functor* $\mathfrak{F} : L-TOP(L) \to L-SET(L)$.

Proof: Since $\mathfrak{T}$ is a complete lattice, to prove the theorem we have to show that for every L-valued L-topological space (Y, B, E_B, τ_B) an extensional map $f : (X, A, E_A) \to (Y, B, E_B, \tau_B)$ has a unique initial lift

$$f : (X, A, E_A, \tau_A) \to (Y, B, \mathcal{E}_B, \tau_B).$$

Let

$$\tau_A := f^{-1}(\tau_B) \subseteq L^X_{E_A}$$

be defined as above. Then, refering to the previous subsection we conclude that

$$f : (X, A, E_A, \tau_A) \to (Y, B, \mathcal{E}_B, \tau_B)$$

is continuous. In order to show that f is the initial morphism, we consider an extensional continuous map

$$h : (Z, C, E_C, \tau_C) \to (Y, B, E_B \tau_B)$$

and let

$$g : (Z, C, E_C, \tau_C) \to (X, A, E_A, \tau_A)$$

be such that $h = g \circ f$. From the definition of τ_A and the continuity of h we conclude that $g : (Z, C, E_C, \tau_C) \to (X, A, E_A, \tau_A)$ is continuous, and hence $f : (X, A, E_A, \tau_A) \to (Y, B, \mathcal{E}_B, \tau_B)$ is the initial lift. The uniqueness of the lift is obvious.

4.2.3 Subcategories of L-TOP(L)

Now we shall consider some subcategories of the category L-TOP(L). In order to get a more lucid description, we *formally* generalize the category L-TOP(L) in a way, analogous as it is done in the case of the category L-SET(L). Namely, To get more non-trivial examples of categories of L-SET(L)-type, it is often useful to consider its full subcategories subcategories of L_0-TOP(L_1, Æ, Ł$_2$)-type. To define a category of L_0-TOP(L_1, Æ, Ł$_2$)-type we first specify the notations. Given a GL-monoid L, let L_0, L_1 and L_2 be its sublattices, in particular sub-GL-monoids. We request that the top elements of all lattices coincide with the top element of the lattice L:

$$\top_{L_0} = \top_L = 1, \top_{L_1} = \top_L = 1, \top_{L_2} = \top_L = 1,$$

and the bottom element of the lattice L_2 coincides with the bottom element of the lattice L:

$$\bot_{L_0} = \bot_L = 0.$$

Let $\mathbf{1} = \{1\}$ be a one-point lattice;
$\mathbf{2} = \{0, 1\}$ be a two-point lattice and
$[0, 1]$ be the unit interval.
In particular, as L_0 and L_1 we can take $\mathbf{1} = \{1\}$, $\mathbf{2} = \{0, 1\}$, or $\mathbf{I}$; while as L_2 we can take $\mathbf{2} = \{0, 1\}$ or $\mathbf{I}$, but not $\mathbf{1} = \{1\}$. Further let X be a set and Æ be a family of L_2 valued equalities on L_1^X:

$$E : L_1^X \to L_2.$$

In particular, we write $\mathfrak{E}$ to denote the class of all L_2-valued equalities on L_1^X,
$\mathfrak{E}_s$ to denote the class of separated L_2-valued equalities on L_1^X,
$\mathfrak{E}_c$ to denote the class of crisp L_2-valued equalities on L_1^X,
and $\mathfrak{E}_{cs}$ to denote the class of crisp separated L_2-valued equalities on L_1^X.

Then objects of the category L_0-TOP$(L_1, Æ, L_2)$ are 6-tuples

$$(X, A, L_1^X, E, L_2, \tau)$$

where $A \in L_0^X$, $E : L_1^X \to L_2$ is taken from Æ and (X, A, E') is an L_2-valued L_0-set where E' is the restriction of E to X. (Note that a set X naturally can be viewed as a subset $\mathbf{1}^X$ of L_1^X, and $\tau \subseteq L_1^X$.

The morphisms in the category L_0-TOP$(L_1, Æ, Ł_2)$ are defined in the same way as in the category L-TOP(L).

Examples

1. Our category L-TOP(L) in these notations is L-TOP$(\mathbf{1}, \mathfrak{E}, L)$.
2. $\mathbf{1}$ $-TOP(\mathbf{1},\ \mathfrak{E},\ L)$ is the category TOP(L);
3. Category $\mathbf{1}$ -TOP$(\mathbf{1},\ \mathfrak{E},\ L)$ where L where L is a Heyting algebra was recently considered by U. Höhle and T. Kubiak is the category TOP(L);
4. $\mathbf{1}$ $-TOP(\mathbf{1},\ \mathfrak{E}_{cs},\ L) = \mathbf{1}$-TOP$(L, \mathfrak{E}_s,\ \mathbf{2})$
   is the Goguen's category of L-topological spaces, see [**?**];
5. $\mathbf{1}$ $-TOP(\mathbf{1},\ \mathfrak{E}_{cs},\ [0,1]) = \mathbf{1}$-TOP$([0,1], \mathfrak{E}_s,\ \mathbf{2})$
   is the original Chang's category of fuzzy topological spaces, see [**?**];
6. $\mathbf{1}$ $-TOP(\mathbf{1},\ \mathfrak{E}_{cs},\ \mathbf{2}) = \mathbf{1}$-TOP$\mathbf{2}, \mathfrak{E}_s,\ \mathbf{2})$
   is the category of topological spaces.

5 Categories of L-valued L-fuzzy topological spaces.

5.1 Category FTOP(L) of L-valued L-fuzzy topological spaces.

5.1.1 Basic definition

Let (X, E) be an L-valued set and let $(L^X_E, \mathcal{E})$ be its powerset of extensional L-subsets.

Definition 5.1 *By an L-fuzzy topology on an L-valued set (X, E) or an L-valued L-fuzzy topology for short, we call an extensional mapping*

$$\mathcal{T} : L^X_E \to L,$$

satisfying the following properties:

1. $\mathcal{T}(0_X) = \mathcal{T}(1_X) = 1$,
2. $\mathcal{T}(U \wedge V) \geq \mathcal{T}(U) \wedge \mathcal{T}(V) \quad \forall U, V \in L^X_E$,
3. $\mathcal{T}\left(\bigvee_{i\in\mathcal{I}} U_i\right) \geq \bigwedge_{i\in\mathcal{I}} \mathcal{T}(U_i)$ *for any family* $\{U_i \mid i \in \mathcal{I}\} \subseteq L^X_E\}$.

A triple $(X, E, \mathcal{T})$ is called an L-valued L-fuzzy topological space.

Given an L-valued L-fuzzy topological space $(X, E, \mathcal{T})$ and $U \in L^X_E$, the value $\mathcal{T}(U)$ is interpreted as the degree of openess of an extensional L-set U in an L-valued L-fuzzy topological space $(X, E, \mathcal{T})$.

Definition 5.2 *Let $(X, E_X, \mathcal{T}_X)$ and $(Y, E_Y, \mathcal{T}_Y)$ be L-valued L-fuzzy topological spaces and*

$$f : (X, E_X, \mathcal{T}_X) \to (Y, E_Y, \mathcal{T}_Y)$$

an extensional mapping. It is called continuous if

$$\forall V \in L^Y_E \text{ it holds } \mathcal{T}_X(f^{-1}(V)) \geq \mathcal{T}_Y(V).$$

Since composition of continuous mappings is obviously continuous and the identity mapping is continuous, we conclude that L-valued L-fuzzy topological spaces as objects and continuous mappings between them as morphisms form a category which will be denoted FTOP(L) and called the category of L-valued L-fuzzy topological spaces.

Remark 5.3 Since we are mainly working in the context of extensional sets, it is important to notice that the preimage of an extensional L-set under an extensional mapping is extensional. Indeed, we have to show that

$$f^{-1}(V)(x) * E_X(x,x') \le f^{-1}(V)(x') \quad \forall\, V \in L_E^Y \quad \forall x, x' \in X.$$

However, this follows from the next series of inequalities which are guaranteed by the extensionality of f:

$$f^{-1}(V)(x) * E(x,x') = V(f(x)) * E_X(x,x')) \le E_Y(f(x), f(x')) * V(f(x)) \le$$
$$\le V(f(x')) = f^{-1}(V)(x').$$

5.1.2 Lattice $\mathfrak{T}_L E(X)$ of extensional L-valued L-fuzzy topologies

Let L be a GL-monoid and let an L-valued set (X, E) be fixed. On the set $\mathfrak{T}_{LE(X)}$ of all extensional L-fuzzy topologies, we introduce an order by setting

$$\mathcal{T}_1 \preceq \mathcal{T}_2 \Longleftrightarrow \mathcal{T}_1(U) \le \mathcal{T}_2(U) \quad \forall U \in L_E^X.$$

Proposition 5.4 *$\mathfrak{T}_{LE(X)}$ is a complete lattice.*

Proof: Let $\mathcal{T}_{dis} : L_E^X \to L$ be defined by

$$\mathcal{T}_{dis}(U) = 1 \quad \forall U \in L_E^X.$$

Obviously

$$\mathcal{T}_{dis}(U) * \mathcal{E}(U,U') \le 1 = \mathcal{T}_{dis}(U') \;\forall U, U' \in L_E^X,$$

that is $\mathcal{T}_{dis} : L_E^X \to L$ is extensional, and hence, obviously, the upper bound in $\mathfrak{T}_{LE(X)}$.

Further, let $\{\mathcal{T}_i \mid i \in \mathcal{I}\}$ be a family of L-fuzzy topologies on an L-valued set (X, E). Noticing that

$$\Big(\bigwedge \mathcal{T}_i(U)\Big) * E(U,V) \le \Big(\bigwedge \mathcal{T}_i(U) * E(U,V)\Big) \le \bigwedge \mathcal{T}_i(V) \quad \forall\, U, V \in L_E^X,$$

we conclude that $\bigwedge_{i\in\mathcal{I}} \mathcal{T}_i$ thus defined is an extensional L-fuzzy topology on the L-valued set (X, E) which is the infimum of the family $\{\mathcal{T}_i \mid i \in \mathcal{I}\}$.
□

Corollary 5.5 *Let $\mathcal{R} : L_E^X \to L$ be a mapping and let*

$$\mathfrak{T}_{\mathcal{R}} = \{\mathcal{T}_i : L_E^X \to L \mid \mathcal{T}_i \ge \mathcal{R}\}$$

be a family of L-fuzzy topologies on an L-valued set (X, E). Then

$$\mathcal{T}_{\mathcal{R}} := \bigwedge_{i\in\mathcal{I}} \mathcal{T}_i$$

is the infimum of the family $\mathfrak{T}$.

$\mathcal{T}_{\mathcal{R}}$ will be referred to as the L-valued L-fuzzy topology generated by $\mathcal{R}$ and resp. $\mathcal{R}$ as the subbase of the L-valued L-fuzzy topology $\mathcal{T}_{\mathcal{R}}$.

Proposition 5.6 *Let* $(X_1, E_1, \mathcal{T}_1)$, $(X_2, E_2, \mathcal{T}_2)$ *be* L*-valued* L*-fuzzy topological spaces and let*

$$f : (X_1, E_1) \to (X_2, E_2)$$

be an extensional mapping. Then the following statements are equivalent:

1. $f : (X, E_1, \mathcal{T}_1) \to (X, E_2, \mathcal{T}_2)$ *is continuous,*

2. *for each* $V \in L_E^{X_2}$ *it holds*

$$\mathcal{R}(V) \leq \mathcal{T}_1(f^{-1}(V)) \quad \forall\ V \in L_E^{X_2}.$$

Proof: Implication (1) $\Longrightarrow$ (2) is obvious. To prove the converse implication let $\bar{\mathcal{T}}_1 : L_E^{X_2} \to L$ be defined by

$$\bar{\mathcal{T}}_1(V) = \mathcal{T}_1(f^{-1}(V)) \quad V \in L_E^{X_2}.$$

Since $\mathcal{T}_1 : L_{E_1}^{X_1} \to L$ is extensional one can easily notice that $\bar{\mathcal{T}}_1 : L_{E_2}^{X_2} \to L$ is also extensional. Besides, since $\mathcal{T}_1 : L_{E_1}^{X_1} \to L$ is an L-valued L-fuzzy topology, it is clear that $\bar{\mathcal{T}}_1 : L_{E_2}^{X_2} \to L$ is an L-valued L-topology. Moreover, from its definition it is clear that it is the smallest ($\leq$) one for which the mapping $f : (X_1, E_1, \mathcal{T}_1) \to (X_2, E_2, \mathcal{T}_2)$ is continuous. Now from (1) we conclude that

$$\mathcal{R}(V) \leq \bar{\mathcal{T}}_1(V) \qquad \forall V \in L_E^{X_2}.$$

Since $\mathcal{R}$ is a subbase of $\mathcal{T}_2$, the assertion (2) follows.
□

5.1.3 Properties of the category FTOP(L)

Theorem 5.7 *Let*

$$\mathfrak{F} : FTOP(L) \to SET(L)$$

be the forgetful functor. Then FTOP(L) is a topological category over the category SET(L) with respect to $\mathfrak{F}$.

Proof:

Since $\mathfrak{T}_{LE(X)}$ is a complete lattice, to prove the theorem we have to show that for every L-valued L-fuzzy topological space $(Y, E_Y, \mathcal{T}_Y)$: an extensional map $f : (X, E_X) \to (Y, E_Y, \mathcal{T}_Y)$ has a unique initial lift

$$f : (X, E_X, \mathcal{T}_X) \to (Y, E_Y, \mathcal{T}_Y).$$

Define a map $\mathcal{R} : L_E^X \to L$ by setting

$$\mathcal{R}(U) = \bigvee \{\mathcal{T}_Y(V) \mid V \in L_E^Y, U = f^{-1}(V)\} \quad \forall U \in L_E^X,$$

and let $\mathcal{T}_X$ be an L-valued L-fuzzy topology generated by the subbase $\mathcal{R}$. Then, referring to Proposition **??** we conclude that

$$f : (X, E_X, \mathcal{T}_X) \to (Y, E_Y, \mathcal{T}_Y).$$

is continuous. In order to show that f is the initial lift, we consider an extensional continuous map

$$h : (Z, E_Z, \mathcal{T}_Z) \to (Y, E_Y, \mathcal{T}_Y)$$

and let

$$g : (Z, E_Z, \mathcal{T}_Z) \to (X, E_X, \mathcal{T}_X)$$

be such that $h = g \circ f$. From the definition of $\mathcal{R}$ and the continuity of h we conclude that

$$\mathcal{R}(U) \leq \mathcal{T}_Z(g^{-1}(U)) \quad \forall\ U \in L_E^X.$$

Since $\mathcal{R}$ is a subbase of the L-valued L-fuzzy topology $\mathcal{T}_X$, again applying proposition **??** we conclude that $g : (Z, E_Z, \mathcal{T}_Z) \to (X, E_X, \mathcal{T}_X)$ is continuous. However this means that $f : (X, E_X, \mathcal{T}_X) \to (Y, E_Y, \mathcal{T}_Y)$ is the initial lift. The uniqueness of the lift is obvious.

The next proposition presents a level decomposition of L-valued L-fuzzy topologies into L-valued L-topologies.:

Proposition 5.8 *Let $(X, E, \mathcal{T})$ be an L-valued L-fuzzy topological space and let $\alpha \in L$. Then*

$$\mathcal{T}_\alpha := \{U \in L_E^X \mid \mathcal{T}(U) \geq \alpha\}$$

is an L-valued L-topology on an L-valued X.
A function

$$f : (X, E_X, \mathcal{T}_X) \to (Y, E_Y, \mathcal{T}_Y)$$

is continuous (that is a morphism in FTOP(L)) if and only if the mappings

$$f : (X, E_X, \mathcal{T}_\alpha^X) \to (Y, E_Y, \mathcal{T}_\alpha^Y)$$

are continuous (that is are morphisms in TOP(L) for every $\alpha \in L$.

Proof follows easily from the axioms of an L-fuzzy topology on an L-valued set and the definitions of continuity in the categories TOP(L) and FTOP(L). □

5.1.4 Enriched and stratified L-valued L-fuzzy topologies.

The concept of a stratified L-fuzzy topology first appeared in [?] (in [?] they are called weakly enriched), while the class of enriched L-fuzzy topology was introduced in [?]. Here we consider these concepts developed in the context of L-fuzzy topologies on L-valued sets.

Let α_X be the constant mapping $\alpha_X : X \to L$ with the value α. Obviously

$$\alpha_X(x) * E(x, x') \leq \alpha_X(x') = \alpha,$$

and hence constant mappings are extensional.

Definition 5.9 *An L-fuzzy topology $\mathcal{T}$ on an L-valued set (X, E) is called enriched if*

$$\mathcal{T}(U) \leq \mathcal{T}(\alpha * U) \qquad \forall \alpha \in L, \quad \forall U \in L_E^X.$$

Definition 5.10 *An L-fuzzy topology $\mathcal{T}$ on an L-valued set (X, E) is called stratified if*

$$\mathcal{T}(\alpha_X) = 1 \qquad \forall \alpha \in L.$$

Since

$$\mathcal{T}(\alpha_X) = \mathcal{T}(\alpha * 1_X) \geq \mathcal{T}(1_X) = 1,$$

and hence $\mathcal{T}(\alpha_X) = 1$ for every α, we conclude that every enriched L-fuzzy topology on an L-valued set is stratified.
Obviously, the discrete L-valued L-fuzzy topology is enriched.

One can easily prove the following

Proposition 5.11 *Intersection of every family of enriched (stratified) L-fuzzy topologies on an L-valued set are enriched (resp. stratified).*

Let $E\mathcal{T}_{L_E(X)}$ and $S\mathcal{T}_{L_E(X)}$ denote subsets of the lattice $\mathcal{T}_{L_E(X)}$ consisting by all extensional and all stratified L-valued L-fuzzy topologies resp. Since intersection of any family of enriched (resp. stratified) L-valued L-fuzzy topologies is enriched (resp. stratified), it follows, that $E\mathcal{T}_{L_E(X)}$ and $S\mathcal{T}_{L_E(X)}$ are complete sublattices of the lattice $\mathcal{T}_{L_E(X)}$.

Let EFTOP(L) and SFTOP(L) denote the full subcategories of the category FTOP(L) the objects of which are enriched and stratified L-fuzzy L-valued topological spaces respectively. From the above proposition **??**we obtain the following

Theorem 5.12 *EFTOP(L) and SFTOP(L) are coreflective subcategories of the category FTOP(L).*

5.2 Category L-FTOP(L) of L-valued L-fuzzy topological L-spaces

Definition 5.13 *Let (X, A, E_A) be an L-valued L-set. By an L-fuzzy topology on an L-valued L-set (X, E, A) we call an extensional mapping $\mathcal{T} : L^X_{E_A} \to L$ such that*

1. $\mathcal{T}(0_X) = \mathcal{T}(A) = 1$,
2. $\mathcal{T}(U \wedge V \wedge A) \geq \mathcal{T}(U \wedge A) \wedge \mathcal{T}(V \wedge A) \quad \forall\, U, V \in L^X_{E_A}$,
3. $\mathcal{T}\left(\bigvee_{i\in\mathcal{I}} (U_i \wedge A)\right) \geq \bigwedge_{i\in\mathcal{I}} \mathcal{T}(U_i \wedge A) \quad \forall\, U_i \in L^X_{E_A}$.

The quadruple $(X, A, E_A, \mathcal{T})$ is referred to as an L-valued L-fuzzy topological L-space.

Definition 5.14 *Given two L-valued L-fuzzy topological L-spaces $(X, A, E_A, \mathcal{T}_A)$ and (Y, B, E_B, T_B), an extensional mapping $f : (X, A, E_A) \to (Y, B, E_B)$ is called continuous if*

1. $f(A) \leq B$ *(that is $f : (X, E_X, A) \to (Y, E_Y, B)$ is a morphism in the category $L - SET(L)$), and*
2. $\mathcal{T}_A(f^{-1}(V \wedge A)) \geq \mathcal{T}_B(V) \quad \forall\, V \in L^Y_{E_B}$,

Since composition of continuous mappings is obviously continuous and the identity mapping $f : (X, A, E_A) \to (X, A, E_A)$ is continuous, we conclude that L-fuzzy L-valued topological L-spaces as objects and continuous mappings between them as morphisms form a category which will be denoted L-FTOP(L) and called the category of L-valued L-fuzzy L-valued topological L-spaces.

Remark 5.15 The category FTOP(L) described in the previous section can be identified with the full subcategory of the category L-FTOP(L), whose objects are of the form $(X, 1_X, E, \mathcal{T})$.

Let L be a GL-monoid and let (X, A, E_A) be an L-valued L-set.

On the set $\mathfrak{T}_{LE(X,A)}$ of all L-fuzzy topologies on an L-valued L-set (X, A, E_A) we introduce an order by setting

$$\mathcal{T}_1 \preceq \mathcal{T}_2 \Longleftrightarrow \mathcal{T}_1(U) \leq \mathcal{T}_2(U) \quad \forall U \in L^X_{E_A}.$$

Proposition 5.16 $\mathfrak{T}_{LE(X,A)}$ *is a complete lattice.*

Proof can be done verbatim as the proof of Proposition **??**

Corollary 5.17 *Let* $\mathcal{R} : L^X_{E_A} \to L$ *be a mapping such that* $\mathcal{R}(A) = 1$ *and let*

$$\mathfrak{T} = \{\mathcal{T}_i : L^X_{E_A} \to L \ \mid \mathcal{T}_i \geq \mathcal{R}\},$$

be a family of L-fuzzy topologies on an L-valued L-set (X, A, E_A). *Then*

$$\mathcal{T}_{\mathcal{R}} := \bigwedge_{i \in \mathcal{I}} (\mathcal{T}_i)$$

is the infimum of the family $\mathfrak{T}$.

Theorem 5.18 *Let*

$$\mathfrak{F} : L - FTOP(L) \to L - SET(L)$$

be the forgetful functor. Then L-FTOP(L) is a topological category over the category L-SET(L) with respect to $\mathfrak{F}$.

Proof can be done similar, as the proof of Theorem **??**.

6 L-fuzzy categories

6.1 An L-fuzzy category: basic concepts

The concept of an (L)-fuzzy category was introduced by A.Šostak in [?] and later was studied in a series of papers see, e.g. [?], [?], etc. These papers contain also many examples of L-fuzzy categories which appear in a natural way, by *"fuzzifying"* classic categories.

The aim of this section is to consider several examples of L-fuzzy categories which we obtain from the categories studied in this work. However, first we introduce the concept of an L-fuzzy category as it was given in [?]. An L-fuzzy category $\mathcal{C}$ consists of:

(1) A class $\mathcal{O}(\mathcal{C})$ of potential objects.

(2) An L-fuzzy subclass ω of $\mathcal{O}b(\mathcal{C})$: that is $\omega : \mathcal{O}b(\mathcal{C}) \to L$.

(3) A class $\mathcal{M}(\mathcal{C}) = \bigcup\{\mathcal{M}_{\mathcal{C}}(X,Y) | X, Y \in \mathcal{O}b(\mathcal{C})\}$ of pairwise disjoint sets $\mathcal{M}_{\mathcal{C}}(X,Y)$ for each pair of potential objects $X, Y \in \mathcal{O}b(\mathcal{C})$; the members of $\mathcal{M}_{\mathcal{C}}(X,Y)$ are called potential morphisms from X to Y and the members of $\mathcal{M}(\mathcal{C})$ are called potential morphisms of the category $\mathcal{C}$.

(4) An L-fuzzy subclass μ of $\mathcal{M}(\mathcal{C})$: that is $\mu : \mathcal{M}(\mathcal{C}) \to L$, such that if $f \in \mathcal{M}_{\mathcal{C}}(X,Y)$, then $\mu(f) \leq \omega(X) \wedge \omega(Y)$.

(5) A composition $\circ$ of morphisms is defined, i.e. for each triple $X, Y, Z \in \mathcal{O}b(\mathcal{C})$ of potential objects there exists a map

$$\circ : \mathcal{M}_{\mathcal{C}}(X,Y) \times \mathcal{M}_{\mathcal{C}}(Y,Z) \to \mathcal{M}_{\mathcal{C}}(X,Z) \quad ((f,g) \to g \circ f),$$

such that the following axioms are satisfied:

(i) preservation of morphisms: $\mu(g \circ f) \geq \mu(g) * \mu(f)$;

(ii) associativity:
if $f \in \mathcal{M}_{\mathcal{C}}(X,Y), g \in \mathcal{M}_{\mathcal{C}}(Y,Z)$ and $h \in \mathcal{M}_{\mathcal{C}}(Z,U)$, then
$h \circ (g \circ f) = (h \circ g) \circ f$;

(iii) existence of identities:
for each $X \in \mathcal{O}(\mathcal{C})$ there exists an identity $e_X \in \mathcal{M}_{\mathcal{C}}(X,Y)$ such that $\mu(e_X) = \omega(X)$ and for all $X, Y, Z \in \mathcal{O}(\mathcal{C})$, all $f \in \mathcal{M}_{\mathcal{C}}(X,Y)$ and $g \in \mathcal{M}_{\mathcal{C}}(Z,X)$ it holds $f \circ e_X = f$ and $e_X \circ g = g$.

As in the classic situation, it is easy to see that the identity morphism is uniquely determined by X.

6.2 Fuzzification of the category SET(L): Fuzzy category $\mathcal{F}$-SET(L).

In all L-fuzzy categories considered in the sequel we take

$$\omega(\mathcal{O}(\mathcal{C})) = 1,$$

that is the class of objects is crisp, and only the class of morphisms will be fuzzy.

Let the objects of the category $\mathcal{F}$-SET(L) be the same as in the category SET(L), that is L-valued sets. As (potential) morphisms in $\mathcal{F}$-SET(L) $f : (X, E_X) \to (Y, E_Y)$ we take *all mappings* between the corresponding sets, that is $f : X \to Y$. We define the measure of extensionality for a mapping

$$f : (X, E_X) \to (Y, E_Y)$$

by

$$\mu(f) = \bigwedge_{x \in X} (E_X(x, x') \mapsto E_Y(f(x), f(x'))) \,.$$

Obviously, μ is a mapping from the class of all functions between sets, (that is from the class of all morphisms in the category SET) into L.

Theorem 6.1 *The triple*

$$(\mathcal{O}(SET(L)), \mathcal{M}(SET), \mu)$$

is an L-fuzzy category. It will be denoted by $\mathcal{F}$-SET(L)and called the L-fuzzification of the category SET(L).

Proof: Let $id_X : (X, E) \to (X, E)$ be the identity mapping. Then, obviously, $\mu(id_X) = 1$. Hence, to prove the theorem, we have to show that if $f : (X, E_X) \to (Y, E_Y)$ and $g : (Y, E_Y) \to (Z, E_Z)$ are potential morphisms, then $\mu(g \circ f) \geq \mu(g) * \mu(f)$. Indeed,

$$\mu(g \circ f) = \bigwedge_{x, x' \in X} (E_X(x, x') \mapsto E_Z((g \circ f)(x), (g \circ f)(x'))) \geq$$

$$\geq \bigwedge_{x, x' \in X} (E_X(x, x') \mapsto E_Y(f(x), f(x'))) *$$

$$* \bigwedge_{x, x' \in X} (E_Y(f(x), f(x') \mapsto E_Z(g(f(x)), g(f(x')))) \geq$$

$$\geq \mu(f) * \bigwedge_{y, y' \in Y} (E_Y(y, y') \mapsto E_Z(g(y), g(y'))) = \mu(f) * \mu(g).$$

□

In a similar way we can fuzzify the category L-SET(L) of L-valued L-sets. In this case we take $\mathcal{O}(L\text{-SET}(L))$ as the class of objects,

$$\mathcal{M}(L - SET) := \{f : (X, A) \to (Y, B)\}$$

as the class of potential morphisms and given

$$(X, A, E_X), (Y, B, E_B) \in L - FTOP(L)$$

and given

$$f : (X, A, E_X) \to (Y, B, E_B) \in L - FTOP(L)$$

define

$$\mu(f) = \bigwedge_{x \in X} (E_X(x, x') \mapsto E_Y(f(x), f(x'))) \, .$$

In the result we obtain an L-fuzzy category:

$$\mathcal{F} - L - SET(L) = (\mathcal{O}(L - SET(L)), \mathcal{M}(L - SET), \mu).$$

The same method allows to construct fuzzifications of the categories TOP(L) and L-TOP(L). In particular, in case of fuzzification of the category L-TOP(L) we take objects of L-TOP(L), that is L-valued L-topological spaces (X, A, τ_A) as the class of objects for $\mathcal{F}$-TOP(L) and the class of morphisms in the category L-TOP(L), that is continuous mappings $f : (X, A, \tau_A) \to (Y, B, \tau_B)$, as the class of potential mappings. We set

$$\mu(f) = \bigwedge_{x \in X} (E_X(x, x') \mapsto E_Y(f(x), f(x'))) \, .$$

The resulting L-fuzzy categories are

$$\mathcal{F} - TOP(L) = (\mathcal{O}(TOP(L)), \mathcal{M}(TOP), \mu) \quad \text{and}$$

$$\mathcal{F} - L - TOP(L) = (\mathcal{O}(L - TOP(L)), \mathcal{M}(L - TOP), \mu),$$

respectively.

6.3 Fuzzification of the category FTOP(L): Fuzzy category $\mathcal{F}$-FTOP(L).

As different from the categories considered in the previous subsection, in case of the category FTOP(L) one can consider two properties of the morphisms for which the measures are naturally defined. Namely, this is the degree of extensionality and the degree of continuity.

Let $(X, E_X, \mathcal{T}_X)$ and $(Y, E_Y, \mathcal{T}_Y)$ be two L-valued L-fuzzy topological spaces and $f : X \to Y$ be a mapping of the corresponding underlying sets. We introduce the measure of the extensionality $\mu_1(f)$ of the mapping

$$f : (X, E_X) \to (Y, E_Y)$$

in the same way, as in the previous subsection. The measure of continuity μ_2 of the mapping

$$f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$$

will be defined, following [?] by the formula

$$\mu_2(f) = \bigwedge_{V \in L_E^Y} \left(\mathcal{T}_Y(V) \mapsto \mathcal{T}_X(f^{-1}(V)) \right).$$

Further, for a mapping

$$f : (X, E_X, \mathcal{T}_X) \to (Y, E_Y, \mathcal{T}_Y)$$

we introduce the measure of continuity $\mu(f)$ by setting

$$\mu(f) = \mu_1(f) \wedge \mu_2(f).$$

Theorem 6.2

$$\mathcal{F} - FTOP(L) = (\mathcal{O}(FTOP(L), \mathcal{M}(SET), \mu)$$

is an L-fuzzy category.

Proof: Since for the identity morphism $id_X : (X, E_X, \mathcal{T}_X) \to (X, E_X, \mathcal{T}_X)$ obviously $\mu_1(id_X) = 1$ and $\mu_2(id_X) = 1$, we have $\mu(id_X) = 1$. Therefore to prove the theorem we have to establish that given two functions

$$f : (X, E_X, \mathcal{T}_X) \to (Y, E_Y, \mathcal{T}_Y) \text{ and } g : (Y, E_Y, \mathcal{T}_Y) \to (Z, E_Z, \mathcal{T}_Z),$$

viewd as the mappings of the corresponding underlying sets it holds $\mu(g \circ f) \geq \mu(g) * \mu(f)$.

The inequality $\mu_1(g \circ f) \geq \mu_1(g) * \mu_1(f)$ was established in the previous subsection. The inequality $\mu_2(g \circ f) \geq \mu_2(g) * \mu_2(f)$ was established in [?], see also [?]. Now, referring to the properties of a GL-monoid, we get

$$\mu(g \circ f) = \mu_1(g \circ f) \wedge \mu_2(g \circ f) \geq (\mu_1(g) * \mu_1(f)) \wedge \mu_2(g) * \mu_2(f) \geq$$

$$(\mu_1(g) \wedge \mu_2(g)) * (\mu_1(f) * \mu_2(f)) = \mu(g) * \mu(f).$$

□

In a similar way we can fuzzify the category $L - TOP(L)$ and get the L-fuzzy category

$$\mathcal{F} - L - FTOP(L) = (\mathcal{O}(F - L - TOP(L), \mathcal{M}(L - SET), \mu).$$

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