

UNIVERSITY OF LATVIA



ANITA PILIKSERE

**Mathematical Models and Their Solutions  
for Different Heat Exchange Processes in  
Environment with Layered Structure**

Ph.D. Thesis

Field: Mathematics

Sub-field: Mathematical modelling

Supervisor: Prof., Dr. habil. math. ANDRIS BUIĶIS

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IEGULDĪJUMS TAVĀ NĀKOTNĒ

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## ANOTATION

Deriving of solutions for steady-state 3D, 2D and 1D problems of heat transfer for cylindrical system with fin are considered in this thesis. Also, non steady-state solutions of hyperbolic heat transfer problems for 1D cylinder and 2D cylindrical system with fin are discussed. Semi-analytical solutions for cylindrical system with fin with conservative averaging method are obtained. Analytical solutions with generalised Green function method for several cylindrical systems with fin are constructed. Some of listed models of steady-state problems are constructed and calculation examples are presented. The obtained results are compared and analysed.

The thesis is written in English, it contains 87 pages (including 1 Appendix), 23 figures, 300 formulae, 6 tables and 53 references.

### **Keywords**

Conservative averaging, Green function, mathematical modelling, cylindrical fin, intensive quenching

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## INTRODUCTION

Extended surfaces, or fins, are a popular topic of heat transfer [1], [2] and [3]. Fin is a surface which extends from the heat exchange object to increase the rate of heat transfer to or from the environment. Generally this thesis is dedicated to different heat transfer problems in cylindrical systems with extended surfaces – fins.

In **chapter 1**, some three-dimensional problem for one element of cylindrical wall and fin is defined. It is assumed that the heat transfer process in the wall and the fin is in a steady-state. Some criterions when it is possible to replace three-dimensional formulation of problem with two- or one-dimensional statement are given. Semi analytical 1D and 2D solutions are constructed for domains with a) ideal contact [4], [5] and b) contact resistance. These solutions are obtained by the original method of conservative averaging [6], [7], [8] and [9]. Calculations and results are compared and discussed.

In **chapter 2**, analytical solutions for the steady state problem of cylindrical wall and fin are discussed [10]. At first, the problem is reduced to dimensionless arguments (see appendix A). Then, additional boundary conditions are added. Two different cases – with homogeneous surrounding temperature and with non-homogeneous boundary conditions – are analysed and solved with a similar method. The solution obtained using a combination of two particular solutions by Green function for rectangular domain [11]. Still, solutions for separate domains contain unknown functions. Both domains are joined through conjugation conditions and problem reduced to the inhomogeneous Fredholm equation of the second kind.

In **chapter 3**, hyperbolic heat transfer problems are discussed [12], [13]. To describe the process of intensive quenching [14], [15] hyperbolic heat transfer by professor A. Buiķis is proposed [16]. It is assumed that the heat transfer process in the wall and the fin is in a non steady-state. Solutions for the time inverse problem for one-dimensional cylinder and two-dimensional cylindrical wall with fin are constructed.

### Importance of Subject

Processes where the main or a significant role is being played by heat transfer are important in many different fields: technology, agriculture, medicine, and other fields. In many cases, it is necessary to intensify the heat transfer in a device. For this purpose, both periodic and non-periodic systems with extended surfaces with fins are introduced. There is a wide range of applications for devices of this type and they can be found in many places starting from home heating systems – radiators, refrigerators, to computer hardware, jet engines and spacecrafts [1]. In recent years, systems where porous and non-porous materials are in contact are also considered in many fields, for example, when dealing with geological structures, composite materials [17] etc.

The mathematical models describing thermal processes in real environments are usually characterised by the fact that they contain elements that make

mathematical modelling difficult, for example, the environment can seldom be considered homogenous. Very often in real life applications, a piecewise homogenous environment can be found, when the physical parameters (heat transfer coefficient and heat capacity) differ many times, it is often combined with different geometrical sizes for different sub-domains.

It is important to find solutions for the non-steady processes, including situations where very rapid temperature changes take place at the beginning of the process. This feature is characteristic to the steel quenching process in water that can be described using the hyperbolic heat transfer equation [16].

Examples and methods for 1D cases have been widely discussed in the literature [3], [2], however problems with complicated domains [3], a combination of different domains or the solutions for multidimensional problems [11] with non-linear boundary conditions or problems in cylindrical coordinates with an axially symmetrical domain can rarely be found in the literature, mostly as a specific problem of mathematical physics with a solution ready for use.

The analytical solutions in the literature are often perceived as precise but not flexible and, in the worst case, as overly complicated and difficult to understand. However, recent trends show that the use of a combination of analytical and semi-analytical solutions can be interesting for a wide class of problems, because they can offer greater insight into the solution, require less computer resources and/or provide a more precise solution [18].

## **The Objective of the Thesis**

The main tasks of this Thesis can be divided into two parts. The first part concerns stationary problems of mathematical physics for the classical heat transfer equations, the second part – non-steady solutions for the hyperbolic heat equations.

Some of the main aims for this work are:

- Using different boundary, symmetry and harmonisation conditions that describe physical processes as accurately as possible, to develop a semi-analytical and analytical solution for stationary heat transfer process in systems with cylindrical fin for the classical heat transfer equation,
- To provide a solution for a group of problems of mathematical physics that is as wide as possible,
- To develop a number of mathematical models and compare the results obtained,
- To find analytical solutions to the unsteady time-inverse hyperbolic heat transfer problem in a cylindrical domain,
- To find an analytical solution to the unsteady hyperbolic heat transfer problem for a cylindrical domain with a fin.

## **Research Methodology**

In this thesis, the method of conservative averaging is employed to acquire the semi-analytical solutions [19]. However, the Green function method is employed to acquire the analytical solution in non-canonical domains. All the problems of mathematical physics first are formulated in 3D and then, using the averaging over one of the dimensions, the 2D model is obtained.

## **Scientific Novelty and Main Results**

This work contains the following results:

- A semi-analytical solution for a system with cylindrical fin (with an ideal contact and contact resistance) is obtained using the method of conservative averaging.
- A mathematical model for corresponding solutions obtained with the method of conservative averaging has been developed for 2D and 1D problems. Numerical calculations have been carried out using different values of physical parameters and geometrical sizes of domains.
- A precise analytical solution has been found using the Green's function method for a system with a cylindrical fin for a wide class of problems with different boundary conditions.
- A mathematical model has been developed for solving the 2D homogenous problem using the Green's function. Numerical calculations have been carried out using different values of physical parameters and geometrical sizes of domains.
- An analytical solution for the 2D unsteady time-inverse problem for the hyperbolic heat transfer equation in a cylindrical domain has been found.
- An analytical solution for the 2D unsteady problem for the hyperbolic heat transfer equation for a system with a cylindrical fin has been found.

## **Application**

Any of the results presented in this work can be used both from practical and theoretical aspects.

The advantage of the semi-analytical solution is the speed of calculations of numerical results. Semi-analytical and analytical results require much less computing time than the numerical results using methods of finite differences or elements. During international scientific conferences, foreign scientists have expressed interest many times in the results obtained in this work and their possible application in large scale computational systems.



However, the results obtained with the Green's function method show how to generalise the methodology for a wider range of problems when the geometry consists of a number of canonical domains. The results acquired using the Green's function method have the advantage of being precise analytical solutions.

The precise analytical solution for the stationary heat transfer problem with a non-homogenous surrounding environment allows solving problems for a cylindrical system with an outer hydrodynamic field [20], [21].

The solutions provided by the hyperbolic heat transfer processes are interesting in the analysis of an intensive steel quenching process [14], [15]. Special attention is given to the question of whether the problems considered in this Thesis can be applied to real life situations, for example, whether it is possible to obtain the initial parameters of a mathematical model. A group of Latvian mathematicians has taken part in such a research project since 2005. It is important to develop models for components of complicated shapes such as, for example, the cylindrical system with a fin, described in this work.

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## PUBLICATIONS AND REPORTS

### *Publications*

1. *Steel quenching process as hyperbolic heat equation for cylinder*, **Anita Piliksere, Margarita Buiķe, Andris Buiķis**. Proceeding of 6th Baltic Heat Transfer Conference – BHTC2011, 2011, ISBN 978-952-15-2640-4 (CD-ROM)
2. *Analytical Solution for Intensive Quenching of Cylindrical Sample*, **Piliksere A., Buiķis A.** Proceeding of 6<sup>th</sup> International Scientific Colloquium of University of Latvia, Leibniz University of Hannover. 2010, pp. 181-186. ISBN 978-9984-45-223-4
3. *Exact Analytical 3D Steady-State Solution for Cylindrical Fin*, **Piliksere A., Buiķis A.** WSEAS Press, 2007, Topics in Advanced Theoretical and Applied Mechanics. Proceeding of the 3<sup>rd</sup> WSEAS International Conference on Applied and Theoretical Mechanics, pp. 1-6. ISBN: 978-960-6766-19-0
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### *Reports on International Scientific Conferences*

1. 6<sup>th</sup> Baltic Heat Transfer Conference (BHTC), Tampere, Finland, August 24 – 28, 2011, pp. 89 – 90. *Steel quenching process as hyperbolic heat equation for cylinder*, **Anita Piliksere, Margarita Buiķe, Andris Buiķis**.
2. 16<sup>th</sup> International Conference on Mathematical Modelling and Analysis, Sigulda, Latvia, May 25-28, 2011, *Issues of hyperbolic heat exchange equation in polar coordinate system*, **Andris Buiķis, Anita Piliksere**.
3. European Geosciences Union General Assembly 2011, NH1.3/HS12.7, Vienna, Austria, April 3 – 8, 2011. *A flood risk assessment for Riga city taking account climate changes*, **Anita Piliksere, Aigars Valainis, Juris Seņņikovs**.
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5. 4<sup>th</sup> WSEAS International Conference on Heat Transfer, Thermal Engineering and Environment. Elounda, Greece, August 21-23, 2006. *Some Approximate Analytical Steady-State Solutions For Cylindrical Fin*, **Anita Brūvere, Andris Buiķis**.
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7. Abstracts of the 11<sup>th</sup> International Conference of Mathematical Modelling and Analysis (MMA), Jurmala, Latvia, May 31 – June 3, 2006, p. 68. *3D Heat Transfer In Cylindrical Wall With Fin*, **Anita Brūvere**.
8. International Conference Problems of numerical and applied mathematics, Lviv, Ukraina, 13-16.09, 2004. *Аналитическое и численное сравнение различных решений стационарной задачи для стенки с ребром*, **А.Буйкис, М. Буйке, А.Брувере**.

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1. LU 69. Konference Matemātiskās modelēšanas, skaitliskās analīzes un diferencu vienādojumu sekcija, 17. februārī, 2011, *Tērauda rūdīšanas cilindrisku modeļu risinājumi*, **Piliksere A.**
2. 6<sup>th</sup> Latvian Mathematical Conference, Abstracts, April 7–8, 2006, Liepāja, Latvia, *Three Dimensional Steady-States Heat Transfer Problem For Cylindrical Wall With Fin*, **Brūvere A.**
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## **LIST OF ABBREVIATIONS**

<b>1D</b>	one dimensional
<b>2D</b>	two dimensional
<b>3D</b>	tree dimensional
<b>BC</b>	boundary condition
<b>IQ</b>	Intensive quenching
<b>M-G</b>	Murray - Gardner

# 1. SOME SEMI-ANALYTICAL STEADY-STATE SOLUTIONS FOR CYLINDRICAL FIN

## 1.1 Problem Introduction

Obtaining efficient cooling for the components of devices is a difficult challenge in modern industry. It is a topical issue in refrigerators, radiators, engines and modern electronics, etc. Often a solution using only primary surfaces does not provide expected efficiency. A well-known solution described in literature [1], [2], [22] is extended surfaces. An extended surface (also known as a combined conduction-convection

system or a fin), is a object where heat transfer by conduction is assumed to be one dimensional, while heat is also transferred by convection (and/or radiation) from the surface in a direction transverse to that of conduction [2], [23]. Extended surfaces may exist in many situations but are commonly used as fins to enhance heat transfer by increasing the surface area available for convection (and/or radiation). Figure 1.1 depicts several examples of extended surfaces. Many more different shapes are used in modern industry.

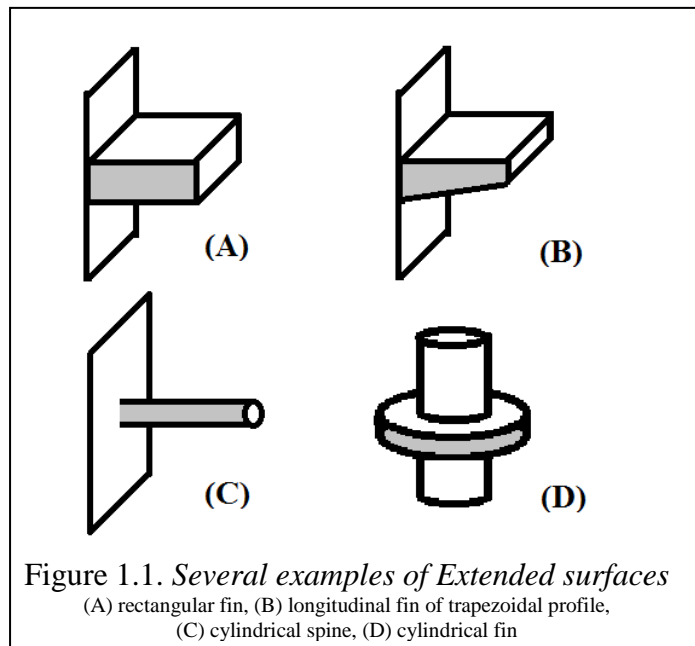


Figure 1.1. *Several examples of Extended surfaces*  
(A) rectangular fin, (B) longitudinal fin of trapezoidal profile,  
(C) cylindrical spine, (D) cylindrical fin

Some analytical solutions and 2D solution with method of Green function for extended surface (Figure 1.1 (A)) with a rectangular wall and a fin in Cartesian coordinates are given in the Master theses of author [9] and PhD thesis of M.Buiķe [8]. In this work, an extended surface (Figure 1.1 (D)) with cylindrical wall with a fin will be discussed. Usually its mathematical modelling is realised by one dimensional steady-state assumptions and, in all cases, the finite element method is offered [1], [24], [25]. Two dimensional analytical approximate solutions for a rectangular fin are constructed in papers [26], [27], [28], [29], but solutions with method of Green function are described in [27]. A number of solutions of the problem with no contact resistance between the primary surface and the fin are offered in papers [4], [5]. In this chapter, a number of new approximate analytical three dimensional solutions by the original method of conservative averaging and some its simplifications (special cases) are obtained.

## 1.2 Problem Statement

Let's discuss the following 3D heat exchange problem in periodical system (Figure 1.2). There is hot fluid inside the cylindrical wall. The heat flows from the inside to the outside. One element of the system is shown in Figure 1.3. Usually for the system so-called Murray – Gardner [1] assumptions are formulated. They are:

1. The heat flow in the fin and the temperature  $\tilde{V}(\tilde{r}, \tilde{\varphi}, \tilde{z})$  at any point on the fin remain constant with time;
2. The fin material is homogeneous; its thermal conductivity  $k$  is the same in all directions and remains constant;
3. The heat transfer coefficient  $h$  between the fin and the surrounding medium is uniform and constant over the entire surface of the fin;
4. The temperature  $T_a$  of the medium surrounding the fin is uniform;
5. The fin height is so small compared with its length that temperature gradients *across* the fin height may be neglected;
6. The temperature at the base of the fin is uniform;
7. There are no heat sources within the fin itself;
8. Heat transfer to or from the fin is proportional to the temperature difference between the fin and the surrounding medium;
9. There is no contact resistance between fins in the configuration or between the fin and the surrounding medium;
10. The heat transferred through the outermost edge of the fin is negligible compared to that through the lateral surfaces (faces) of the fin.

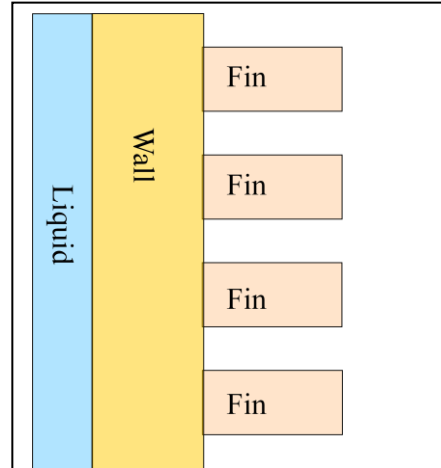


Figure 1.2. Axial section of 3D domain

*On the left side of the domain is hot liquid. Heat transfer is oriented from the inside to the outside.*

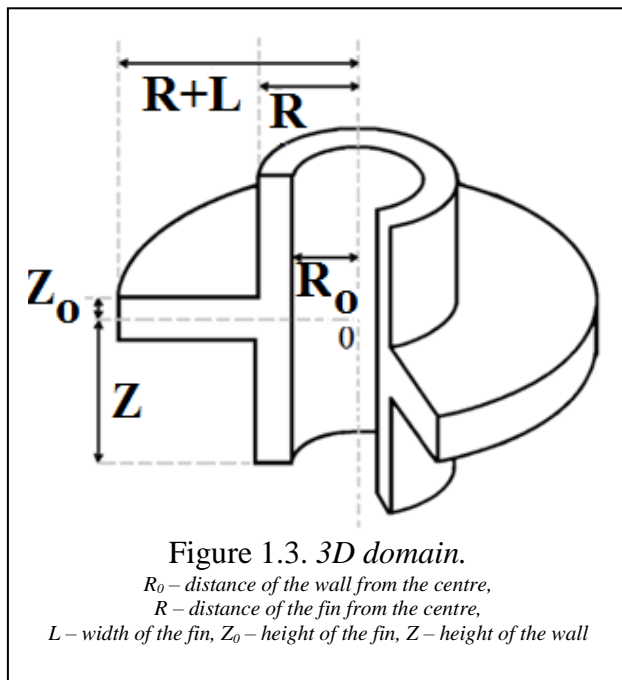


Figure 1.3. 3D domain.

$R_0$  – distance of the wall from the centre,

$R$  – distance of the fin from the centre,

$L$  – width of the fin,  $Z_0$  – height of the fin,  $Z$  – height of the wall

The problem discussed further differs from the assumptions above in point 9 – the contact resistance between the wall and the fin will be taken into account.

### 1.3 Mathematical Formulation of 3D Problem and Reduction to 2D

Let's start with accurate three-dimensional formulation of the steady-state problem for one element of the periodical system for the cylindrical wall and fin (Figure 1.3). Consumptions and boundary conditions of one element in this model describe whole system. The one element of the wall (base) is placed in the domain  $\{\tilde{r} \in [R_0, R], \tilde{z} \in [0, Z], \varphi \in [0, \phi]\}$  and temperature field  $\tilde{V}_0(\tilde{r}, \tilde{z}, \varphi)$  in the wall with Laplace equation [30], [31] is described:

$$\frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left( \tilde{r} \frac{\partial \tilde{V}_0}{\partial \tilde{r}} \right) + \frac{\partial^2 \tilde{V}_0}{\partial \tilde{z}^2} + \frac{1}{\tilde{r}^2} \frac{\partial \tilde{V}_0}{\partial \varphi^2} = 0. \quad (1.1)$$

The cylindrical fin of length  $L$  occupies the domain  $\{\tilde{r} \in [R, R+L], \tilde{z} \in [0, Z_0], \varphi \in [0, \phi]\}$  and the temperature field  $\tilde{V}(\tilde{r}, \tilde{z}, \varphi)$  fulfils the equation:

$$\frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left( \tilde{r} \frac{\partial \tilde{V}}{\partial \tilde{r}} \right) + \frac{\partial^2 \tilde{V}}{\partial \tilde{z}^2} + \frac{1}{\tilde{r}^2} \frac{\partial \tilde{V}}{\partial \varphi^2} = 0. \quad (1.2)$$

Following boundary conditions in accordance with M-G point 5) in  $\varphi$  direction are formulated:

$$\left. \frac{\partial \tilde{V}}{\partial \varphi} \right|_{\varphi=0} = \left. \frac{\partial \tilde{V}}{\partial \varphi} \right|_{\varphi=\phi} = 0; \quad \left. \frac{\partial \tilde{V}_0}{\partial \varphi} \right|_{\varphi=0} = \left. \frac{\partial \tilde{V}_0}{\partial \varphi} \right|_{\varphi=\phi} = 0. \quad (1.3)$$

The problem (1.1) - (1.2) from 3D to 2D is reduced using the following average integral for argument  $\varphi$ :

$$\tilde{U}(\tilde{r}, \tilde{z}) = \frac{1}{\phi} \int_0^\phi \tilde{V}(\tilde{r}, \tilde{z}, \varphi) d\varphi \quad \text{and} \quad \tilde{U}_0(\tilde{r}, \tilde{z}) = \frac{1}{\phi} \int_0^\phi \tilde{V}_0(\tilde{r}, \tilde{z}, \varphi) d\varphi. \quad (1.4)$$

#### 1.3.1 Problem Formulation in Dimensionless Arguments

The following dimensionless arguments and parameters are used, to transform problem (1.1) – (1.3) to a dimensionless problem:

$$r = \frac{\tilde{r}}{Z}, \quad z = \frac{\tilde{z}}{Z}, \quad \delta_0 = \frac{R_0}{Z}, \quad \delta = \frac{R}{Z}, \quad \delta + l = \frac{R+L}{Z}, \quad b = \frac{Z_0}{Z}, \quad \beta_0 = \frac{hZ}{k_0}, \quad \beta = \frac{hZ}{k},$$

$$\beta_0^0 = \frac{h_0 Z}{k_0}. \quad \text{And temperatures: } U(r, z) = \frac{\tilde{U}(r, z) - T_a}{T_b - T_a}, \quad U_0(r, z) = \frac{\tilde{U}_0(r, z) - T_a}{T_b - T_a}.$$

Here  $k(k_0)$  - heat conduction coefficient for the fin (wall),  $h(h_0)$  - heat exchange coefficient for the fin (wall),  $Z_0$  - height of the fin,  $L$  - length of the fin,  $Z$  - height of the wall,  $T_b$  - the surrounding temperature on the left (hot) side of the wall,  $T_a$  - the surrounding temperature on the right (cold) side of the wall and the fin.



Detailed description of formulation problem in dimensionless arguments is discussed in Appendix A, Chapter A.1.

### 1.3.2 Description of Temperature Field in the Wall

One element of the wall (base) (Figure 1.4) placed in the domain is now  $\{r \in [\delta_0, \delta], z \in [0, 1]\}$  and describes the dimensionless temperature field  $U_0(r, z)$  in the wall with the equation:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial U_0}{\partial r} \right) + \frac{\partial^2 U_0}{\partial z^2} = 0 \quad (1.5)$$

Using Fourier's fourth law and Newton's second law, and heat balance, boundary conditions for inner ③ and outward ④ surface of wall are formulated as follow:

$$r \frac{\partial U_0}{\partial r} + \beta_0^0 (1 - U_0) = 0, r = \delta_0, z \in [0, 1], \quad (1.6)$$

$$r \frac{\partial U_0}{\partial r} + \beta_0 U_0 = 0, r = \delta, z \in [b, 1]. \quad (1.7)$$

As we consider that the domain of this problem is periodical, id est., symmetrical against line  $z=0$  and  $z=1$ , then there is no heat exchange on following boundaries of wall  $r \in (\delta_0, \delta), z=0$  and  $r \in (\delta_0, \delta), z=1$  (⊙). Therefore, homogeneous boundary conditions ⊙ are formulated:

$$\frac{\partial U_0}{\partial z} \Big|_{z=0} = \frac{\partial U_0}{\partial z} \Big|_{z=1} = 0, r = [\delta_0, \delta]. \quad (1.8)$$

The conjugation conditions ⑨ on the surface between the wall and the fin are not an ideal thermal contact. There is contact resistance assumed. A solution for an ideal thermal contact with the method discussed further is described in paper [4]. Also heat balance is fulfilled:

$$U_0 \Big|_{\delta-0} = \left( U - \alpha \frac{\partial U}{\partial r} \right) \Big|_{\delta+0}, \beta \frac{\partial U_0}{\partial r} \Big|_{\delta-0} = \beta_0 \frac{\partial U}{\partial r} \Big|_{\delta+0}, \quad (1.9)$$

There,  $\alpha$  is  $\alpha = \frac{k \cdot \Delta l}{k'}$ . The contact resistance is interpreted as arising from a layer of some environment with width  $\Delta l$  and heat conduction coefficient  $k'$ .

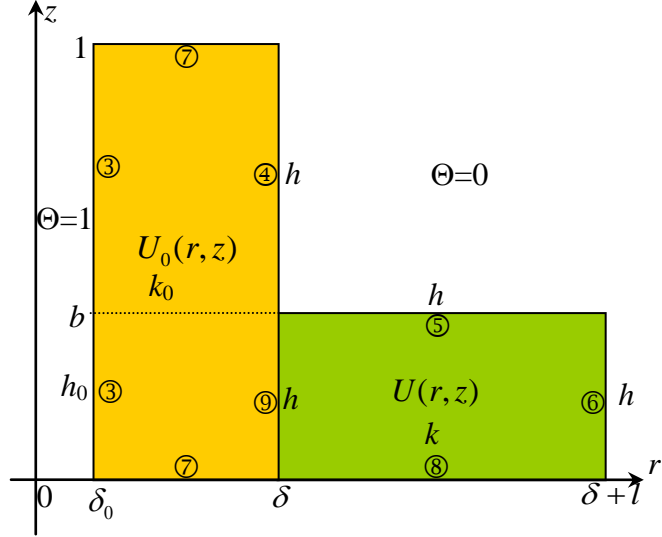


Figure 1.4 Dimensionless domain diagram.

### 1.3.3 Description of Temperature Field in the Fin

The cylindrical fin of length  $l$  occupies the domain  $\{r \in [\delta, \delta + l], z \in [0, b]\}$  (Figure 1.4) and the temperature field  $U(r, z)$  fulfils the equation:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial U}{\partial r} \right) + \frac{\partial^2 U}{\partial z^2} = 0. \quad (1.10)$$

According to Newton's second law, following boundary conditions ⑤ and ⑥ for the fin is valid:

$$\frac{\partial U}{\partial z} + \beta U = 0, \quad r = [\delta, \delta + l], \quad z = b, \quad (1.11)$$

$$r \frac{\partial U}{\partial r} + \beta U = 0, \quad r = \delta + l, \quad z \in [0, b]. \quad (1.12)$$

And homogeneous boundary conditions (⑧):

$$\left. \frac{\partial U}{\partial z} \right|_{z=0} = 0, \quad r = [\delta, \delta + l]. \quad (1.13)$$

## 1.4 Approximate Solution of 2D Problem for a Periodical System

The original method of conservative averaging described in Appendix A, Chapter A.2, is used to obtain a 2D half analytical solution for problem (1.5) – (1.13).

### 1.4.1 Reduction of the 2D Problem for the Fin

Similar to papers [8], [9], [26], [28] the original method of conservative averaging is used. The 2D temperature field  $U(r, z)$  for the fin is reduced in following form:

$$U(r, z) = f_0(r) + (e^{\rho z} - 1) f_1(r) + (1 - e^{-\rho z}) f_2(r), \quad \rho = b^{-1} \quad (1.14)$$

with three unknown functions  $f_i(r), i = 0, 1, 2$ . For this purpose we introduce the integral average value of function  $U(r, z)$  in the  $z$  - direction:

$$u(r) = \rho \int_0^b U(r, z) dz. \quad (1.15)$$

This equality (1.15), together with boundary conditions (1.13 (⑧)), allows expressing function  $f_1(r)$  through  $f_2(r)$  in following way:

$$\left. \frac{\partial U}{\partial z} \right|_{z=0} = \left( \rho e^{\rho z} f_1(r) + \rho e^{-\rho z} f_2(r) \right) \Big|_{z=0} = 0$$

$$\begin{aligned}\rho e^0 f_1(r) + \rho e^0 f_2(r) &= 0 \\ f_1(r) &= -f_2(r).\end{aligned}\tag{1.16}$$

From (1.16) and assumption (1.14) follows:

$$\begin{aligned}U(r, z) &= f_0(r) + (e^{\rho z} - 1)f_1(r) - (1 - e^{-\rho z})f_1(r) \\ U(r, z) &= f_0(r) + (e^{\rho z} + e^{-\rho z} - 2)f_1(r) \\ U(r, z) &= f_0(r) + 2(\cosh(\rho z) - 1)f_1(r)\end{aligned}\tag{1.17}$$

Using integral value (1.15) and equation (1.17) it is possible to express function  $f_1(r)$  through function  $f_0(r)$  in the following way:

$$\begin{aligned}u(r) &= \rho \int_0^b (f_0(r) + 2(\cosh(\rho z) - 1)f_1(r)) dz \\ u(r) &= \rho f_0(r) z \Big|_0^b + 2\rho \left( \frac{\sinh(\rho z)}{\rho} - z \right) f_1(r) \Big|_0^b \\ u(r) &= \rho f_0(r) b + 2 \left( \frac{\sinh(1)}{\rho} - b \right) f_1(r) - 2f_1(r) \sinh(0) \\ u(r) &= f_0(r) + 2(\sinh(1) - 1)f_1(r) \\ f_1(r) &= \frac{u(r) - f_0(r)}{2(\sinh(1) - 1)}.\end{aligned}\tag{1.18}$$

Now two unknown functions  $f_i(r)$ ,  $i = 1, 2$  can be excluded from equation (1.14) putting (1.18) into (1.17) and only two unknown functions  $u(r)$  and  $f_0(r)$  are left:

$$\begin{aligned}U(r, z) &= f_0(r) + \frac{(\cosh(\rho z) - 1)(u(r) - f_0(r))}{\sinh(1) - 1} \\ U(r, z) &= \frac{f_0(r)(\sinh(1) - 1) + (\cosh(\rho z) - 1)(u(r) - f_0(r))}{\sinh(1) - 1} \\ U(r, z) &= \frac{\cosh(\rho z) - 1}{\sinh(1) - 1} u(r) + \frac{\sinh(1) - \cosh(\rho z)}{\sinh(1) - 1} f_0(r).\end{aligned}\tag{1.19}$$

Finally, by using of the boundary condition  $\textcircled{S}$  (1.11)  $f_0(r)$  from the last expression (1.19), function  $f_0(r)$  can be excluded in following form:

$$\begin{aligned}\left( \frac{\partial U}{\partial z} + \beta U \right) \Big|_{z=b} &= \frac{\rho \sinh(\rho b)}{\sinh(1) - 1} u(r) - \frac{\rho \sinh(\rho b)}{\sinh(1) - 1} f_0(r) + \\ &+ \beta \left( \frac{\cosh(1) - 1}{\sinh(1) - 1} u(r) + \frac{\sinh(1) - \cosh(1)}{\sinh(1) - 1} f_0(r) \right) = 0 \\ (\rho \sinh(1) + \beta(\cosh(1) - 1))u(r) &= (\rho \sinh(1) - \beta(\sinh(1) - \cosh(1)))f_0(r) \\ f_0(r) &= \psi u(r), \text{ where } \psi = \frac{\sinh(1) + \beta b(\cosh(1) - 1)}{\sinh(1) + \beta b(\cosh(1) - \sinh(1))}.\end{aligned}\tag{1.20}$$

It can obviously be seen that representation (1.20) pasted in (1.19) gives representation of the 2D solution  $U(r, z)$  for the fin in the form of multiplication of two, one argument functions:

$$U(r, z) = u(r)\Phi(z), \quad (1.21)$$

where  $\Phi(z)$  from equations (1.19) and (1.20) is

$$\begin{aligned} \Phi(z) &= \frac{(1-\psi)\cosh(\rho z) + \psi \sinh(1) - 1}{\sinh(1) - 1} = \\ &= \frac{(1-\psi)\cosh(\rho z) + \psi \sinh(1) - 1 \pm \sinh(1)}{\sinh(1) - 1} = \\ &= 1 + (1-\psi) \frac{\cosh(\rho z) - \sinh(1)}{\sinh(1) - 1}. \end{aligned} \quad (1.22)$$

Putting equation (1.20) into equation (1.22) and unifying denominators, the following equation is obtained:

$$\begin{aligned} \Phi(z) &= 1 - \beta b \frac{\cosh(\rho z) - \sinh(1)}{\sinh(1) + \beta b(\cosh(1) - \sinh(1))} = \\ &= \frac{\sinh(1) + \beta b(\cosh(1) - \cosh(\rho z))}{\sinh(1) + \beta b(\cosh(1) - \sinh(1))}. \end{aligned} \quad (1.23)$$

Obviously that function  $\Phi(z) > 0$  for all  $z \in [0, b]$ .

The second stage for the method of conservative averaging is to transform the partial differential equation (1.15) for the function  $U(r, z)$  to the differential equation for one argument, function  $u(r)$ . To realise this goal, the main differential equation (1.10) in the  $z$ -direction is integrated. And using equation (1.15), the following connection is obtained:

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{du}{dr} \right) + \frac{1}{b} \frac{\partial U}{\partial z} \Big|_{z=b} - \frac{1}{b} \frac{\partial U}{\partial z} \Big|_{z=0} = 0. \quad (1.24)$$

By using the boundary condition (1.13 (⊗)) at  $z=0$  for the function  $U(r, z)$  and expressing the first derivative  $\frac{\partial U}{\partial z}$  through the function  $U(r, z)$  from the boundary condition (1.11 (⊙)) at  $z=b$  the following differential equation can be obtained:

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{du}{dr} \right) - \frac{\beta U}{b} \Big|_{z=b} = 0. \quad (1.25)$$

Expressing in differential equation (1.21) the function  $U(r, z)$  through the function  $u(r)$  with the help of the equality (1.25), then the new differential equation, which describes the 1D dimensional temperature field  $u(r)$  in the fin, is received:

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{du}{dr} \right) - \mu^2 u(r) = 0, \quad (1.26)$$

here

$$\mu^2 = \frac{\beta}{b} \Phi(b). \quad (1.27)$$

By help of substitution  $r_1 = \mu r$  equation (1.26) can be rewritten in following well-known form:

$$r_1^2 u''(r_1) + r_1 u'(r_1) - (r_1^2 + n^2)u(r_1) = 0, \text{ where } n = 0. \quad (1.28)$$

Differential equation (1.28) has a well-known solution [32] through Bessel's modified functions  $I_0, K_0$ :

$$u(r) = C_1 K_0(\mu r) + C_2 I_0(\mu r). \quad (1.29)$$

Now we can exclude constant  $C_2$  from equation (1.29) by using boundary condition (1.12 (©)) and (1.21) in the following way:

$$\begin{aligned} & \left( r \frac{\partial U}{\partial r} + \beta U \right) \Big|_{r=\delta+l} = \\ & = \left( r C_2 \frac{\partial I_0(\mu r)}{\partial r} + r C_1 \frac{\partial K_0(\mu r)}{\partial r} + \beta C_2 I_0(\mu r) + \beta C_1 K_0(\mu r) \right) \Big|_{\delta+l} \cdot \Phi(z) = 0 \\ & C_2 = C_1 \frac{(\delta+l)\mu K_1(\mu(\delta+l)) - \beta K_0(\mu(\delta+l))}{(\delta+l)\mu I_1(\mu(\delta+l)) + \beta I_0(\mu(\delta+l))} = \mu_1 C_1. \end{aligned} \quad (1.30)$$

Solution (1.29) using (1.30) can be written in the following way:

$$u(r) = C_1 (\mu_1 I_0(\mu r) + K_0(\mu r)) \quad (1.31)$$

And from equations (1.21) and (1.31), the solution for the fin which includes only one unknown constant  $C_1$  is obtained:

$$U(r, z) = C_1 (\mu_1 I_0(\mu r) + K_0(\mu r)) \Phi(z). \quad (1.32)$$

#### 1.4.2 Reduction of the 2D Problem for the Wall

The same method of conservative averaging is used to describe the temperature field in the wall. The approximation of the 2D temperature field  $U_0(r, z)$  for the wall is in following form:

$$U_0(r, z) = g_0(z) + \left( \frac{1}{r} - \frac{1}{\delta} \right) g_1(z) + \frac{\delta - r}{\delta - \delta_0} g_2(z), \quad (1.33)$$

with three unknown functions  $g_i(z), i = 0, 1, 2$ . For this purpose, the integral average value of function  $U_0(r, z)$  in the  $r$  - direction is introduced:

$$u_0(z) = \frac{2}{\delta^2 - \delta_0^2} \int_{\delta_0}^{\delta} r U_0(r, z) dr. \quad (1.34)$$

This equality, (1.34), together with equality (1.33), gives following equation:

$$u_0(z) = \frac{2}{\delta^2 - \delta_0^2} \int_{\delta_0}^{\delta} r \left( g_0(z) + \left( \frac{1}{r} - \frac{1}{\delta} \right) g_1(z) + \frac{\delta - r}{\delta - \delta_0} g_2(z) \right) dr$$

$$\begin{aligned}
u_0(z) &= \frac{2}{\delta^2 - \delta_0^2} \left( \frac{r^2}{2} g_0(z) + \left( r - \frac{r^2}{2\delta} \right) g_1(z) + \frac{1}{\delta - \delta_0} \left( \frac{r^2 \delta}{2} - \frac{r^3}{3} \right) g_2(z) \right) \Big|_{\delta_0}^{\delta} \\
u_0(z) &= \frac{2}{\delta^2 - \delta_0^2} \left( \frac{\delta^2 - \delta_0^2}{2} g_0(z) + \frac{(\delta - \delta_0)^2}{2\delta} g_1(z) \right) + \frac{\delta^3 - 2\delta_0^2 \delta + 2\delta_0^3 - \delta_0^2 \delta}{3(\delta - \delta_0)(\delta^2 - \delta_0^2)} g_2(z) \\
u_0(z) &= g_0(z) + \varphi_1 g_1(z) + \varphi_2 g_2(z), \tag{1.35}
\end{aligned}$$

where  $\varphi_1 = \frac{\delta - \delta_0}{(\delta + \delta_0)\delta}$ ,  $\varphi_2 = \frac{\delta + 2\delta_0}{3(\delta + \delta_0)}$ .

Let's find the derivation of equation (1.33):

$$\frac{\partial U_0}{\partial r} = -\frac{g_1(z)}{r^2} - \frac{g_2(z)}{\delta - \delta_0}. \tag{1.36}$$

Now, putting into boundary condition (1.6 (3)) equations (1.36) and (1.33), the following equation is obtained:

$$\begin{aligned}
r \frac{\partial U_0}{\partial r} + \beta_0^0 (1 - U_0) \Big|_{\delta_0} &= \left( -\frac{r g_1(z)}{r^2} - \frac{r g_2(z)}{\delta - \delta_0} \right) \Big|_{\delta_0} + \\
&+ \beta_0^0 \left( 1 - \left( g_0(z) + \left( \frac{1}{r} - \frac{1}{\delta} \right) g_1(z) + \frac{\delta - r}{\delta - \delta_0} g_2(z) \right) \right) \Big|_{\delta_0} = 0 \\
\beta_0^0 (1 - g_0(z)) &= \frac{\delta + \beta_0^0 (\delta - \delta_0)}{\delta \delta_0} g_1(z) + \frac{\delta_0 + \beta_0^0 (\delta - \delta_0)}{\delta - \delta_0} g_2(z). \tag{1.37}
\end{aligned}$$

Function  $g_2(z)$  from equation (1.35) is expressed:

$$g_2(z) = \frac{u_0(z) - g_0(z) - \varphi_1 g_1(z)}{\varphi_2} \tag{1.38}$$

and pasted to expression (1.37) in the following way:

$$\begin{aligned}
\beta_0^0 (1 - g_0(z)) &= \frac{\delta + \beta_0^0 (\delta - \delta_0)}{\delta \delta_0} g_1(z) + \\
&+ \frac{\delta_0 + \beta_0^0 (\delta - \delta_0)}{\delta - \delta_0} \cdot \frac{u_0(z) - g_0(z) - \varphi_1 g_1(z)}{\varphi_2} \\
K_1 g_1(z) &= A_1 g_0(z) - B_1 u_0(z) - D_1, \tag{1.39}
\end{aligned}$$

where

$$\begin{aligned}
K_1 &= \frac{\delta + (\delta - \delta_0) \beta_0^0}{\delta \delta_0} - \frac{\varphi_1 (\delta_0 + (\delta - \delta_0) \beta_0^0)}{\varphi_2 (\delta - \delta_0)}, \\
A_1 &= -\beta_0^0 + \frac{\delta_0 + (\delta - \delta_0) \beta_0^0}{\varphi_2 (\delta - \delta_0)},
\end{aligned}$$

$$B_1 = \frac{\delta_0 + (\delta - \delta_0)\beta_0^0}{\varphi_2(\delta - \delta_0)}, \quad D_1 = -\beta_0^0.$$

Now, function  $g_1(z)$  from equation (1.35) is expressed as:

$$g_1(z) = \frac{u_0(z) - g_0(z) - \varphi_2 g_2(z)}{\varphi_1} \quad (1.40)$$

and pasted into expression (1.37) gives:

$$\begin{aligned} \beta_0^0(1 - g_0(z)) &= \frac{\delta + \beta_0^0(\delta - \delta_0)}{\delta\delta_0} \cdot \frac{u_0(z) - g_0(z) - \varphi_2 g_2(z)}{\varphi_1} + \\ &+ \frac{\delta_0 + \beta_0^0(\delta - \delta_0)}{\delta - \delta_0} g_2(z), \\ K_2 g_2(z) &= -A_2 g_0(z) + B_2 u_0(z) + D_2, \end{aligned} \quad (1.41)$$

where

$$\begin{aligned} K_2 &= \frac{\delta_0 + (\delta - \delta_0)\beta_0^0}{\delta - \delta_0} - \frac{\varphi_2(\delta + (\delta - \delta_0)\beta_0^0)}{\varphi_1\delta\delta_0}, \\ A_2 &= \beta_0^0 - \frac{\delta + (\delta - \delta_0)\beta_0^0}{\delta\delta_0\varphi_1}, \\ B_2 &= -\frac{\delta + (\delta - \delta_0)\beta_0^0}{\delta\delta_0\varphi_1}, \quad D_2 = \beta_0^0. \end{aligned}$$

Equations (1.39) and (1.41) give:

$$\begin{cases} g_1(z) = a_1 g_0(z) - b_1 u_0(z) - d_1 \\ g_2(z) = -a_2 g_0(z) + b_2 u_0(z) + d_2 \end{cases} \quad (1.42)$$

where  $a_i = \frac{A_i}{K_i}$ ,  $b_i = \frac{B_i}{K_i}$ ,  $d_i = \frac{D_i}{K_i}$ .

Putting equation (1.42) into equation (1.33), the following expression is obtained:

$$\begin{aligned} U_0(r, z) &= g_0(z) + \left(\frac{1}{r} - \frac{1}{\delta}\right)(a_1 g_0(z) - b_1 u_0(z) - d_1) + \\ &+ \frac{\delta - r}{\delta - \delta_0}(-a_2 g_0(z) + b_2 u_0(z) + d_2) \\ U_0(r, z) &= \left(1 + \frac{\delta - r}{\delta r} a_1 - \frac{\delta - r}{\delta - \delta_0} a_2\right) g_0(z) + \\ &+ (\delta - r) \left(\frac{b_2}{\delta - \delta_0} - \frac{b_1}{\delta r}\right) u_0(z) + (\delta - r) \left(\frac{d_2}{\delta - \delta_0} - \frac{d_1}{\delta r}\right). \end{aligned} \quad (1.43)$$

Equation (1.43) still has two unknown functions  $g_0(z)$  and  $u_0(z)$ . Therefore different boundary and conjugations conditions on the wall to exclude these functions will be used.

### 1.4.2.1 Solution for upper Wall

Let's find the derivation of (1.43):

$$\left. \frac{\partial U_0}{\partial r} \right|_{\delta} = - \left( \frac{a_1}{\delta^2} - \frac{a_2}{\delta - \delta_0} \right) g_0(z) - \left( \frac{b_2}{\delta - \delta_0} - \frac{b_1}{\delta^2} \right) u_0(z) - \left( \frac{d_2}{\delta - \delta_0} - \frac{d_1}{\delta^2} \right). \quad (1.44)$$

The following expression is obtained from equations (1.44) and (1.43) pasted into boundary condition (1.7)

$$\begin{aligned} \left( r \frac{\partial U_0}{\partial r} + \beta_0 U_0 \right) \Big|_{\delta} = \\ -\delta \left( \left( \frac{a_1}{\delta^2} - \frac{a_2}{\delta - \delta_0} \right) g_0(z) + \left( \frac{b_2}{\delta - \delta_0} - \frac{b_1}{\delta^2} \right) u_0(z) + \left( \frac{d_2}{\delta - \delta_0} - \frac{d_1}{\delta^2} \right) \right) + \beta_0 g_0(z) = 0 \\ g_0(z) = b_0 u_0(z) + d_0, \end{aligned} \quad (1.45)$$

where  $b_0 = \frac{B_0}{K_0}$ ,  $d_0 = \frac{D_0}{K_0}$ ,  $B_0 = \frac{b_2 \delta}{\delta - \delta_0} - \frac{b_1}{\delta}$ ,  $D_0 = \frac{d_2 \delta}{\delta - \delta_0} - \frac{d_1}{\delta}$ ,  $K_0 = \frac{a_2 \delta}{\delta - \delta_0} - \frac{a_1}{\delta} + \beta_0$ .

The next equation is obtained by putting equation (1.45) into equation (1.43):

$$\begin{aligned} U_0(r, z) = \left( 1 + \frac{\delta - r}{\delta r} a_1 - \frac{\delta - r}{\delta - \delta_0} a_2 \right) (b_0 u_0(z) + d_0) + \\ + (\delta - r) \left( \frac{b_2}{\delta - \delta_0} - \frac{b_1}{\delta r} \right) u_0(z) + (\delta - r) \left( \frac{d_2}{\delta - \delta_0} - \frac{d_1}{\delta r} \right) \\ U_0(r, z) = \Phi_0(r) u_0(z) + \psi_0(r), \end{aligned} \quad (1.46)$$

where

$$\begin{aligned} \Phi_0(r) = (\delta - r) \left( \frac{b_0 a_1 - b_1}{\delta r} + \frac{b_2 - b_0 a_2}{\delta - \delta_0} \right) + b_0 \text{ and} \\ \psi_0(r) = (\delta - r) \left( \frac{d_0 a_1 - d_1}{\delta r} + \frac{d_2 - d_0 a_2}{\delta - \delta_0} \right) + d_0. \end{aligned}$$

Now integrating partial differential equation (1.5):

$$\left( r \frac{\partial U_0(r, z)}{\partial r} \right) \Big|_{\delta_0}^{\delta} + \frac{\partial^2 \int_{\delta_0}^{\delta} r U_0(r, z) dr}{\partial z^2} = 0 \quad (1.47)$$

and taking into account equation (1.34), the following expression is gotten

$$\frac{2}{\delta^2 - \delta_0^2} \left( r \frac{\partial U_0}{\partial r} \right) \Big|_{\delta_0}^{\delta} + \frac{d^2 u_0}{dz^2} = 0. \quad (1.48)$$

Using boundary conditions (1.6 (③)), (1.7 (④)):

$$r \frac{\partial U_0}{\partial r} \Big|_{\delta_0} = -\beta_0^0 (1 - U_0) \Big|_{\delta_0} \text{ and } r \frac{\partial U_0}{\partial r} \Big|_{\delta} = -\beta_0 U_0 \Big|_{\delta}, \quad (1.49)$$



and putting them into (1.48) we can get:

$$\frac{2}{\delta^2 - \delta_0^2} \left( -\beta_0 U_0(\delta, z) + \beta_0^0 (1 - U_0(\delta_0, z)) \right) + \frac{d^2 u_0}{dz^2} = 0. \quad (1.50)$$

Using equation (1.46) the equation (1.50) can be rewritten in the following form:

$$\begin{aligned} \frac{2}{\delta^2 - \delta_0^2} \left[ -\beta_0 (\Phi_0(\delta) u_0(z) + \psi_0(\delta)) + \beta_0^0 (1 - \Phi_0(\delta_0) u_0(z) - \psi_0(\delta_0)) \right] + \frac{d^2 u_0}{dz^2} = 0 \\ \frac{d^2 u_0}{dz^2} - k_3^2 u_0 = -Q_2, \end{aligned} \quad (1.51)$$

where

$$k_3^2 = \frac{2(\beta_0^0 \Phi_0(\delta_0) + \beta_0 \Phi_0(\delta))}{\delta^2 - \delta_0^2}, \quad Q_2 = \frac{2(\beta_0^0 (1 - \psi_0(\delta_0)) - \beta_0 \psi_0(\delta))}{\delta^2 - \delta_0^2}.$$

Equation (1.51) is the second order linear non homogeneous differential equation. It can be solve with a common approach. At first, the homogeneous equation should be solved and then one particular solution must be found [33]. The solution of the homogeneous equation of (1.51) is:

$$\frac{d^2 \tilde{u}_0}{dz^2} - k_3^2 \tilde{u}_0 = 0 \quad (1.52)$$

$$\lambda^2 - k_3^2 = 0$$

$$\lambda_{1,2} = \pm k_3$$

$$\tilde{u}_0(z) = \tilde{C}_1 e^{\lambda_1 z} + \tilde{C}_2 e^{\lambda_2 z} = \tilde{C}_1 e^{k_3 z} + \tilde{C}_2 e^{-k_3 z}. \quad (1.53)$$

The particular solution in form  $\tilde{u}_0(z) = A$  is searched, therefore:

$$\tilde{u}_0'(z) = 0, \tilde{u}_0''(z) = 0. \quad (1.54)$$

Putting (1.54) into (1.51) we get  $-k_3^2 A = -Q_2$ ,  $A = \frac{Q_2}{k_3^2}$  then the solution for differential equation (1.51) is:

$$u_0(z) = \tilde{u}_0(z) + \tilde{u}_0(z) = \tilde{C}_1 e^{k_3 z} + \tilde{C}_2 e^{-k_3 z} + \frac{Q_2}{k_3^2}. \quad (1.55)$$

Still, two unknown constants are represented. The boundary condition (1.8 (⊙)), equations (1.51) and (1.46) are used to express solution through one constant:

$$\left. \frac{\partial U_0}{\partial z} \right|_{z=1} = \Phi_0(r) u_0'(1) = 0$$

$$u_0'(1) = \tilde{C}_1 k_3 e^{k_3} - \tilde{C}_2 k_3 e^{-k_3} = 0$$

$$\tilde{C}_1 = \tilde{C}_2 e^{-2k_3}. \quad (1.56)$$

Now equation (1.55) through (1.56) can be written in the following way:

$$u_0(z) = \tilde{C}_2 e^{-2k_3} e^{k_3 z} + \tilde{C}_2 e^{-k_3 z} + \frac{Q_2}{k_3^2}$$

$$u_0(z) = C_2 \cosh(k_3(z-1)) + \frac{Q_2}{k_3^2}, \quad (1.57)$$

where  $C_2 = 2\tilde{C}_2 e^{-k_3}$ .

Function  $u_0(z)$  is solved and now problem for upper wall reduces from (1.46) and (1.57) to the following form

$$U_0(r, z) = \left( (\delta - r) \left( \frac{b_0 a_1 - b_1}{\delta r} + \frac{b_2 - b_0 a_2}{\delta - \delta_0} \right) + b_0 \right) \cdot \left( C_2 \cosh(k_3(z-1)) + \frac{Q_2}{k_3^2} \right) + (\delta - r) \left( \frac{d_0 a_1 - d_1}{\delta r} + \frac{d_2 - d_0 a_2}{\delta - \delta_0} \right) + d_0. \quad (1.58)$$

Problem for upper wall (1.58) reduces to the problem of finding constant  $C_2$ .

#### 1.4.2.2 Solution for Lower Wall

From (1.43) we get:

$$U_0(\delta, z) = g_0(z). \quad (1.59)$$

From (1.32) we get:

$$U(\delta, z) = C_1 (\mu_1 I_0(\mu\delta) + K_0(\mu\delta)) \Phi(z), \quad (1.60)$$

$$\left. \frac{\partial U(r, z)}{\partial r} \right|_{\delta} = C_1 \mu (\mu_1 I_1(\mu\delta) - K_1(\mu\delta)) \Phi(z). \quad (1.61)$$

Using equation (1.59), (1.60) and (1.61), and putting them into conjugation condition (1.9 (⊙)) at value  $r = \delta$  following equations are obtained:

$$g_0(z) = C_1 F_2 \Phi(z), \quad (1.62)$$

where  $F_2 = \mu_1 I_0(\mu\delta) + K_0(\mu\delta) - \mu\alpha (\mu_1 I_1(\mu\delta) - K_1(\mu\delta))$ .

Equation (1.23) is rewritten as:

$$\Phi(z) = F_0 - F_1 \cosh(\rho z), \quad (1.63)$$

where  $F_1 = \frac{1}{\frac{\sinh(1)}{\beta b} + \cosh(1) - \sinh(1)}$  and  $F_0 = \left[ \frac{\sinh(1)}{\beta b} + \cosh(1) \right] F_1$ .

Let's continue with equation (1.62) and (1.63) and get:

$$g_0(z) = C_1 F_2 (F_0 - F_1 \cosh(\rho z))$$

$$g_0(z) = C_1 (\tilde{F}_0 - \tilde{F}_1 \cosh(\rho z)), \quad (1.64)$$

where  $\tilde{F}_i = F_2 F_i$ ,  $i = 0, 1$ .

From the second conjugation condition (1.9) and (1.61), (1.63) follows:

$$\begin{aligned}\delta \frac{\partial U_0}{\partial r} \Big|_{r=\delta} &= \delta \frac{\beta_0}{\beta} C_1 \mu (\mu_1 I_1(\mu\delta) - K_1(\mu\delta)) (F_0 - F_1 \cosh(\rho z)) \text{ or} \\ \delta \frac{\partial U_0}{\partial r} \Big|_{r=\delta} &= C_1 (\tilde{F}_0 - \tilde{F}_1 \cosh(\rho z)),\end{aligned}\quad (1.65)$$

where  $\tilde{F}_i = \delta \frac{\beta_0}{\beta} \mu (\mu_1 I_1(\mu\delta) - K_1(\mu\delta)) F_i$ ,  $i = 0, 1$ .

From equation (1.64) and the derivation of equation (1.43) at value  $r = \delta_0$  follows:

$$\begin{aligned}\frac{\partial U_0}{\partial r} \Big|_{r=\delta_0} &= C_1 (\tilde{F}_0 - \tilde{F}_1 \cosh(\rho z)) \left( \frac{a_2}{\delta - \delta_0} - \frac{a_1}{\delta_0^2} \right) + \\ &+ \left( \frac{b_1}{\delta_0^2} - \frac{b_2}{\delta - \delta_0} \right) u_0(z) - \left( \frac{d_2}{\delta - \delta_0} - \frac{d_1}{\delta_0^2} \right).\end{aligned}\quad (1.66)$$

Now submitting equations (1.65) and (1.66) in equation (1.48) the following differential equation is obtained

$$\begin{aligned}\frac{d^2 u_0}{dz^2} + \frac{2C_1}{\delta^2 - \delta_0^2} (\tilde{F}_0 - \tilde{F}_1 \cosh(\rho z)) - \\ - \frac{2\delta_0}{\delta^2 - \delta_0^2} \left( C_1 (\tilde{F}_0 - \tilde{F}_1 \cosh(\rho z)) \left( \frac{a_2}{\delta - \delta_0} - \frac{a_1}{\delta_0^2} \right) + \right. \\ \left. + \left( \frac{b_1}{\delta_0^2} - \frac{b_2}{\delta - \delta_0} \right) u_0(z) - \left( \frac{d_2}{\delta - \delta_0} - \frac{d_1}{\delta_0^2} \right) \right) = 0\end{aligned}$$

$$\frac{d^2 u_0}{dz^2} - u_0 M = FC_1 - H + EC_1 \cosh(\rho z), \quad (1.67)$$

where

$$\begin{aligned}M &= \frac{2\delta_0}{\delta^2 - \delta_0^2} \left( \frac{b_1}{\delta_0^2} - \frac{b_2}{\delta - \delta_0} \right), \quad E = \frac{2}{\delta^2 - \delta_0^2} \left( \tilde{F}_1 + \delta_0 \tilde{F}_1 \left( \frac{a_2}{\delta - \delta_0} - \frac{a_1}{\delta_0^2} \right) \right), \\ F &= \frac{2}{\delta^2 - \delta_0^2} \left( \delta_0 \tilde{F}_0 \left( \frac{a_2}{\delta - \delta_0} - \frac{a_1}{\delta_0^2} \right) + \tilde{F}_0 \right), \quad H = \frac{2\delta_0}{\delta^2 - \delta_0^2} \left( \frac{d_2}{\delta - \delta_0} - \frac{d_1}{\delta_0^2} \right).\end{aligned}$$

Equation (1.67) is a second order linear non homogeneous differential equation. Similar as equation (1.51) it can be solved with a common approach. The solution of the homogeneous equation of (1.67) is:

$$\begin{aligned}\frac{d^2 u_0}{dz^2} - u_0 M &= 0 \\ \lambda^2 - M &= 0 \\ \lambda_{1,2} &= \pm \sqrt{M} \\ \tilde{u}_0(z) &= \tilde{C}_1 e^{\lambda_1 z} + \tilde{C}_2 e^{\lambda_2 z} = \tilde{C}_1 e^{\sqrt{M}z} + \tilde{C}_2 e^{-\sqrt{M}z}.\end{aligned}\quad (1.68)$$

Now particular solution for equation(1.67) should be found in following form:

$$\tilde{u}_0(z) = B \cosh(\rho z) - A. \quad (1.69)$$

From solution (1.69) pasted into equation (1.67) follows:

$$\tilde{u}_0'(z) = B\rho \sinh(\rho z)$$

$$\tilde{u}_0''(z) = B\rho^2 \cosh(\rho z)$$

$$B\rho^2 \cosh(\rho z) - (B \cosh(\rho z) + A)M = FC_1 - H + EC_1 \cosh(\rho z)$$

$$B = \frac{EC_1}{\rho^2 - M}, \quad A = \frac{H - FC_1}{M}. \quad (1.70)$$

The solution of differential equation (1.67) looks like the sum of particular solution (1.69) and homogeneous equation solution (1.68) in the following way:

$$u_0(z) = \tilde{C}_3 e^{\sqrt{M}z} + \tilde{C}_4 e^{-\sqrt{M}z} + \frac{EC_1}{\rho^2 - M} \cosh(\rho z) - \frac{FC_1 - H}{M}. \quad (1.71)$$

From condition conditions (1.8 (⊙)), (1.46) and (1.71) it follows that:

$$\left. \frac{\partial U_0}{\partial z} \right|_{z=0} = \Phi_0(r) u_0'(0) = 0$$

$$u_0'(0) = \tilde{C}_3 \sqrt{M} e^0 - \tilde{C}_4 \sqrt{M} e^0 + \frac{C_1 E \rho}{\rho^2 - M} \sinh(0) = 0 \quad (1.72)$$

From (1.72) it follows that, constants  $\tilde{C}_3 = \tilde{C}_4$ , therefore  $u_0(z)$  equation (1.71) can rewrite in the following way:

$$u_0(z) = C_3 \cosh(\sqrt{M}z) + \frac{EC_1}{\rho^2 - M} \cosh(\rho z) - \frac{FC_1 - H}{M}, \quad (1.73)$$

where  $C_3 = 2\tilde{C}_3$ .

Putting together equations (1.43), (1.64) and (1.73) follows:

$$\begin{aligned} U_0(r, z) = & C_1 \left( \tilde{F}_0 - \tilde{F}_1 \cosh(\rho z) \right) \left( 1 + (\delta - r) \left( \frac{a_1}{\delta r} - \frac{a_2}{\delta - \delta_0} \right) \right) + \\ & + (\delta - r) \left( \frac{b_2}{\delta - \delta_0} - \frac{b_1}{\delta r} \right). \\ & \left( C_3 \cosh(\sqrt{M}z) + \frac{EC_1}{\rho^2 - M} \cosh(\rho z) - \frac{FC_1 - H}{M} \right) + \\ & + (\delta - r) \left( \frac{d_2}{\delta - \delta_0} - \frac{d_1}{\delta r} \right). \end{aligned} \quad (1.74)$$

Now the solution for lower wall contains only two unknown constants,  $C_1$  and  $C_3$ .

### 1.4.3 Solution

We have two additional conditions on function  $u_0(z)$ , respectively:

$$u_0(b)_{b-0} = u_0(b)_{b+0} \quad (1.75)$$

and

$$\left. \frac{\partial u_0}{\partial z} \right|_{b-0} = \left. \frac{\partial u_0}{\partial z} \right|_{b+0}. \quad (1.76)$$

Also, in point  $(\delta, b)$ , the values of functions  $U_0(r, z)$  and  $U(r, z)$  must satisfy contact resistance condition (1.9 (⊙)) at point  $(\delta, b)$ :

$$U_0(\delta, b) = U(\delta, b) - \alpha \left. \frac{\partial U(r, z)}{\partial r} \right|_{(\delta, b)} \quad (1.77)$$

Equations (1.73) and (1.57) are used to satisfy equation (1.75)

$$\begin{aligned} C_2 \cosh(k_3(b-1)) + \frac{Q_2}{k_3^2} - \frac{H}{M} = \\ = C_3 \cosh(\sqrt{M}b) + C_1 \left( \frac{E}{\rho^2 - M} \cosh(1) - \frac{F}{M} \right), \end{aligned} \quad (1.78)$$

but the derivative of equation (1.73) and (1.57) is used to satisfy equation (1.76)

$$\begin{aligned} \sqrt{M}C_3 \sinh(\sqrt{M}b) + \frac{E\rho}{\rho^2 - M} C_1 \sinh(1) = \\ = C_2 k_3 \sinh(k_3(b-1)). \end{aligned} \quad (1.79)$$

From (1.60) and (1.61) follows:

$$U(\delta, b) = C_1 (\mu_1 I_0(\mu\delta) + K_0(\mu\delta)) \Phi(b) \quad (1.80)$$

$$\left. \frac{\partial U(r, z)}{\partial r} \right|_{(\delta, b)} = C_1 \mu (\mu_1 I_1(\mu\delta) - K_1(\mu\delta)) \Phi(b). \quad (1.81)$$

But from (1.46) and (1.73) we get:

$$U_0(\delta, b) = b_0 \left( C_3 \cosh(\sqrt{M}b) + \frac{EC_1}{\rho^2 - M} \cosh(1) - \frac{FC_1 - H}{M} \right) + d_0. \quad (1.82)$$

Equations (1.80), (1.81) and (1.82) are used to satisfy equation (1.77)

$$\begin{aligned} C_1 (\mu_1 I_0(\mu\delta) + K_0(\mu\delta) - \alpha \mu (\mu_1 I_1(\mu\delta) - K_1(\mu\delta))) \Phi(b) = \\ = b_0 \left( C_3 \cosh(\sqrt{M}b) + \frac{EC_1}{\rho^2 - M} \cosh(1) - \frac{FC_1 - H}{M} \right) + d_0. \end{aligned} \quad (1.83)$$

And now, the problem is reduced to the solution of three linear equations (1.78), (1.79) and (1.83), for three unknown constants  $C_i, i = 1, 2, 3$

$$\begin{cases}
C_2 \cosh(k_3(b-1)) + \frac{Q_2}{k_3^2} - \frac{H}{M} = \\
= C_3 \cosh(\sqrt{M}b) + C_1 \left( \frac{E}{\rho^2 - M} \cosh(1) - \frac{F}{M} \right) \\
C_1 (\mu_1 I_0(\mu\delta) + K_0(\mu\delta) - \alpha\mu (\mu_1 I_1(\mu\delta) - K_1(\mu\delta))) \Phi(b) = \\
= b_0 \left( C_3 \cosh(\sqrt{M}b) + \frac{EC_1}{\rho^2 - M} \cosh(1) - \frac{FC_1 - H}{M} \right) + d_0 \\
\sqrt{M} C_3 \sinh(\sqrt{M}b) + \frac{E\rho}{\rho^2 - M} C_1 \sinh(1) = \\
= C_2 k_3 \sinh(k_3(b-1)).
\end{cases} \quad (1.84)$$

Constants  $C_i, i = 1, 2, 3$  can be determined from system (1.84). The value of the temperature at any point of 2D domain can be calculated using equations (1.32), (1.58) and (1.74) and values of  $C_i, i = 1, 2, 3$ .

### 1.5 1D Solution as the Simple Case of the 3D Solution

Similar to papers [24], [25], a 1D solution is obtained. In this chapter, a 1D solution by conservative averaging method from 2D solution is discussed. Semi-analytical 1D solution is gained.

Define the following integral values for equations (1.5):

$$v_0(r) = \int_0^1 U_0(r, z) dz \quad (1.85)$$

and for equation (1.10):

$$v(r) = \frac{1}{b} \int_0^b U(r, z) dz. \quad (1.86)$$

The assumption that the temperature field does not depend on z direction is considered:

$$U_0(r, z) \equiv v_0(r) \text{ and } U(r, z) \equiv v(r). \quad (1.87)$$

Let's integrate equation (1.5) and use boundary conditions (1.8), assumptions (1.85) and (1.87):

$$\begin{aligned}
\int_0^1 \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial U_0}{\partial r} \right) dz + \int_0^1 \frac{\partial^2 U_0}{\partial z^2} dz &= 0 \\
\int_0^1 \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial U_0}{\partial r} \right) dz + \frac{\partial U_0}{\partial z} \Big|_{z=1} - \frac{\partial U_0}{\partial z} \Big|_{z=0} &= 0
\end{aligned}$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \int_0^1 U_0 dz}{\partial r} \right) = 0$$

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{v_0}{dr} \right) = 0, \quad r \in (\delta_0, \delta). \quad (1.88)$$

Similarly equation (1.10) is integrated:

$$\int_0^b \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial U}{\partial r} \right) dz + \int_0^b \frac{\partial^2 U}{\partial z^2} dz = 0$$

$$\int_0^b \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial U}{\partial r} \right) dz + \frac{\partial U}{\partial z} \Big|_{z=b} - \frac{\partial U}{\partial z} \Big|_{z=0} = 0. \quad (1.89)$$

From boundary conditions (1.11) and (1.13) we get:

$$\frac{\partial U}{\partial z} \Big|_{z=b} = -\beta U \quad \text{and} \quad \frac{\partial U}{\partial z} \Big|_{z=0} = 0. \quad (1.90)$$

And together form equations (1.86), (1.87), (1.89) and (1.90) follows:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \int_0^b U dz}{\partial r} \right) - \beta U \Big|_{z=b} = 0$$

$$\frac{b}{r} \frac{d}{dr} \left( r \frac{dv}{dr} \right) - \beta v = 0, \quad r \in (\delta, \delta + l). \quad (1.91)$$

Boundary condition (1.6) is integrated and assumptions (1.85) and (1.87) used:

$$\int_0^1 r \frac{\partial U_0}{\partial r} dz + \beta_0^0 \left( 1 - \int_0^1 U_0 dz \right) = 0$$

$$r \frac{dv_0}{dr} + \beta_0^0 (1 - v_0) = 0, \quad r = \delta_0. \quad (1.92)$$

From boundary condition (1.12), (1.86) and (1.87) follows:

$$r \frac{dv}{dr} + \beta v = 0, \quad r = \delta + l. \quad (1.93)$$

Conjugation condition (1.9) together with equations (1.85), (1.86) and (1.87) can be rewritten:

$$v_0 \Big|_{r=\delta-0} = \left( v - \alpha \frac{dv}{dr} \right) \Big|_{r=\delta+0}, \quad \beta \frac{dv_0}{dr} \Big|_{r=\delta-0} = \beta_0 \frac{dv}{dr} \Big|_{r=\delta+0}. \quad (1.94)$$

Boundary condition (1.7) is used as the final step for problem formulation in following way:

$$r \frac{dv_0}{dr} + \beta_0 v_0 = 0, \quad r = \delta, \quad (1.95)$$

but from equations (1.94) and (1.95) follows:

$$\begin{aligned} rb\beta \frac{dv_0}{dr} \Big|_{r=\delta-0} &= rb\beta_0 \frac{dv}{dr} \Big|_{r=\delta+0} \\ br\beta \frac{dv_0}{dr} + \beta \left( r \frac{dv_0}{dr} + \beta_0 v_0 \right) &= rb\beta_0 \frac{dv}{dr} \\ \beta \left( br \frac{dv_0}{dr} + r \frac{dv_0}{dr} + \beta_0 v_0 \right) &= rb\beta_0 \frac{dv}{dr} \\ \beta \left( -b\beta_0 v_0 + r \frac{dv_0}{dr} + \beta_0 v_0 \right) &= rb\beta_0 \frac{dv}{dr} \\ \beta \left( r \frac{dv_0}{dr} + \beta_0 v_0 (1-b) \right) &= rb\beta_0 \frac{dv}{dr}, \quad r = \delta. \end{aligned} \quad (1.96)$$

Now the 1D problem is formulated with equations (1.88) and (1.91) also conditions (1.92), (1.93) and (1.96).

The equation (1.88) can be solved in the following form:

$$\begin{aligned} \frac{d}{dr} \left( r \frac{dv_0}{dr} \right) &= 0 \\ r \frac{dv_0}{dr} &= C_1 \\ dv_0 &= \frac{C_1}{r} dr \\ v_0 &= C_1 \ln r + C_2. \end{aligned} \quad (1.97)$$

The solution of equation (1.91) can be rewritten in a well-known form in the following way:

$$\begin{aligned} \frac{b}{r} \frac{d}{dr} \left( r \frac{dv}{dr} \right) - \beta v &= 0 \\ r^2 v'' + r v' - \frac{\beta}{b} r^2 v &= 0. \end{aligned} \quad (1.98)$$

The following replacement  $r_1 = r \sqrt{\frac{\beta}{b}}$  is applied and the well-known form of a modified Bessel's differential equation [32] obtained:

$$r_1^2 v'' + r_1 v' - (r^2 + 0^2) v = 0. \quad (1.99)$$

with the following solution:

$$v(r) = C_3 I_0 \left( \sqrt{\frac{\beta}{b}} r \right) + C_4 K_0 \left( \sqrt{\frac{\beta}{b}} r \right), \quad (1.100)$$

where  $I_0, K_0$  are Bessel's modified functions.



Together, equations (1.97) and (1.100) give the solution for the 1D problem:

$$\begin{cases} v_0(r) = C_1 \ln r + C_2 \\ v(r) = C_3 I_0 \left( \sqrt{\frac{\beta}{b}} r \right) + C_4 K_0 \left( \sqrt{\frac{\beta}{b}} r \right). \end{cases} \quad (1.101)$$

Using boundary (1.92), (1.93) and conjugation conditions (1.96), constants  $C_1 \div C_4$  from (1.101) can be excluded.

From (1.100) follows:

$$v'(r) = C_3 \sqrt{\frac{\beta}{b}} I_1 \left( \sqrt{\frac{\beta}{b}} r \right) - C_4 \sqrt{\frac{\beta}{b}} K_1 \left( \sqrt{\frac{\beta}{b}} r \right). \quad (1.102)$$

Equations (1.100) and (1.102) placed in equation (1.93) give:

$$\begin{aligned} & r \sqrt{\frac{\beta}{b}} \left( C_3 I_1 \left( \sqrt{\frac{\beta}{b}} r \right) - C_4 K_1 \left( \sqrt{\frac{\beta}{b}} r \right) \right) + \\ & + \beta \left( C_3 I_0 \left( \sqrt{\frac{\beta}{b}} r \right) + C_4 K_0 \left( \sqrt{\frac{\beta}{b}} r \right) \right) = 0 \quad , \quad r = \delta + l \\ & C_3 = C_4 \frac{(\delta + l) \sqrt{\frac{\beta}{b}} K_1 \left( \sqrt{\frac{\beta}{b}} (\delta + l) \right) - \beta K_0 \left( \sqrt{\frac{\beta}{b}} (\delta + l) \right)}{(\delta + l) \sqrt{\frac{\beta}{b}} I_1 \left( \sqrt{\frac{\beta}{b}} (\delta + l) \right) + \beta I_0 \left( \sqrt{\frac{\beta}{b}} (\delta + l) \right)}. \end{aligned} \quad (1.103)$$

From (1.97) follows:

$$v_0' = \frac{C_1}{r}. \quad (1.104)$$

Equations (1.92), (1.97) and (1.104), where  $r = \delta_0$ , give:

$$\begin{aligned} & r \frac{C_1}{r} + \beta_0^0 (1 - C_1 \ln r - C_2) = 0 \\ & C_1 + \beta_0^0 - \beta_0^0 \ln r C_1 - \beta_0^0 C_2 = 0 \\ & C_1 = \frac{\beta_0^0 (C_2 - 1)}{1 - \beta_0^0 \ln \delta_0}. \end{aligned} \quad (1.105)$$

Putting expressions (1.97), (1.100) and (1.102) into equation (1.94), where  $r = \delta$ , the following connection is obtained:

$$\begin{aligned} & C_1 \ln \delta + C_2 = C_3 I_0 \left( \sqrt{\frac{\beta}{b}} \delta \right) + C_4 K_0 \left( \sqrt{\frac{\beta}{b}} \delta \right) - \\ & - \alpha \sqrt{\frac{\beta}{b}} \left( C_3 I_1 \left( \sqrt{\frac{\beta}{b}} \delta \right) - C_4 K_1 \left( \sqrt{\frac{\beta}{b}} \delta \right) \right). \end{aligned} \quad (1.106)$$

Whereas putting expressions (1.97), (1.99), (1.102) and (1.104) into equation (1.96), where  $r = \delta$ , the next connection follows:

$$\begin{aligned}
\beta \left( r \frac{C_1}{r} + \beta_0 (C_1 \ln r + C_2)(1-b) \right) &= rb\beta_0 \sqrt{\frac{\beta}{b}} \left( C_3 I_1 \left( \sqrt{\frac{\beta}{b}} \delta \right) - C_4 K_1 \left( \sqrt{\frac{\beta}{b}} \delta \right) \right) \\
C_1 \beta (1 + \beta_0 (1-b) \ln \delta) + C_2 \beta \beta_0 (1-b) &= \\
= C_3 \delta \beta_0 \sqrt{\beta b} I_1 \left( \sqrt{\frac{\beta}{b}} \delta \right) - C_4 \delta \beta_0 \sqrt{\beta b} K_1 \left( \sqrt{\frac{\beta}{b}} \delta \right). & \quad (1.107)
\end{aligned}$$

As we can see from equations (1.103), (1.105), (1.106) and (1.107), the four unknown constants can be easily determined from the linear equation system:

$$\left\{ \begin{aligned}
C_1 &= \frac{\beta_0^0 (C_2 - 1)}{1 - \beta_0^0 \ln \delta_0} \\
C_3 &= C_4 \frac{(\delta + l) \sqrt{\frac{\beta}{b}} K_1 \left( \sqrt{\frac{\beta}{b}} (\delta + l) \right) - \beta K_0 \left( \sqrt{\frac{\beta}{b}} (\delta + l) \right)}{(\delta + l) \sqrt{\frac{\beta}{b}} I_1 \left( \sqrt{\frac{\beta}{b}} (\delta + l) \right) + \beta I_0 \left( \sqrt{\frac{\beta}{b}} (\delta + l) \right)} \\
C_1 \beta (1 + \beta_0 (1-b) \ln \delta) + C_2 \beta \beta_0 (1-b) &= \\
= C_3 \delta \beta_0 \sqrt{\beta b} I_1 \left( \sqrt{\frac{\beta}{b}} \delta \right) - C_4 \delta \beta_0 \sqrt{\beta b} K_1 \left( \sqrt{\frac{\beta}{b}} \delta \right) \\
C_1 \ln \delta + C_2 &= C_3 I_0 \left( \sqrt{\frac{\beta}{b}} \delta \right) + C_4 K_0 \left( \sqrt{\frac{\beta}{b}} \delta \right) - \\
-\alpha \sqrt{\frac{\beta}{b}} \left( C_3 I_1 \left( \sqrt{\frac{\beta}{b}} \delta \right) - C_4 K_1 \left( \sqrt{\frac{\beta}{b}} \delta \right) \right). & \quad (1.108)
\end{aligned} \right.$$

Equations (1.101) together with constants from system (1.108) give the solution of the 1D heat transfer problem for the cylindrical system with the fin.

## 1.6 Calculations and Results

2D unstructured mesh [34] was constructed for numerical calculation of problems solved with the conservative averaging method (see chapter 1.3.) The node count varies between 500 ÷ 700 , but triangle count - from 800 ÷ 1200 (see Fig. 1.5).

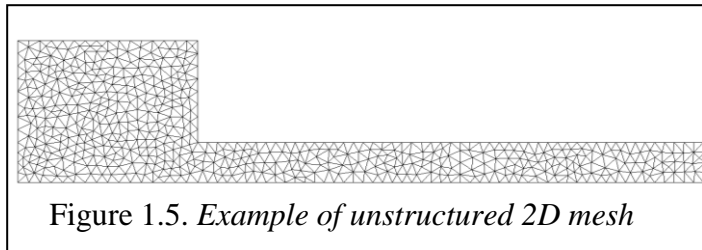


Figure 1.5. Example of unstructured 2D mesh

Software in Delphi environment for numerical solutions of 1D and 2D problems were developed. Developed software depending on the given parameters calculates the solution of systems of linear equations for 2D (1.84) and 1D (1.108) problems. The software calculates values of temperature fields depending on the values of  $r, z$  at any point of generated mesh for a 2D problem using formulae (1.74), (1.58) or

(1.32) and for 1D problem using formula (1.101). The temperature value on each mesh node  $(r, z)$  is displayed as a field for 2D problem and 1D as well. Extracting of representation of the solution for 1D to 2D is done, respectively; all values for 1D solution of temperature do not depend on the value of  $z$ . At constant value of  $r = const$  the solution reaches the same temperature value for all values of  $z$ , but generally  $\forall r$  reaches different temperature value. Several sections at constant value of  $z$  through the 2D domain are constructed, thus 2D and 1D solutions can be easily compared and temperature profile displayed.

Obviously the 1D solution for the wall has a logarithmic nature (1.101), however 2D solution for the wall contains exponents (1.58) and (1.74), and do not have linear nature in  $z$  direction. The question is how much do these solutions differ one from another? Since both solutions contain quite complicated analytical expressions, it is more affordable to compare some particular calculations using the same input data.

Calculations were done at different input data. Most interesting of them will be discussed (see table 1.1., 1.2. and 1.3.). Input data are summarised in Table 1.1 [35], other dimensionless parameters which are common for all numerical calculations are presented in Table 1.2. Table 1.3 contains data which differs for each version of the calculation. All calculations are done for problems with dimensionless parameters; real temperatures obtained using formulae (A.13) and (A.14).

Table 1.1. *Input data common of all calculations.*

Parameter	$T_1,$ [°C]	$T_0,$ [°C]	$\Delta,$ [m]	$k_0,$ [ $\frac{W}{mK}$ ]	$k,$ [ $\frac{W}{mK}$ ]	$k',$ [ $\frac{W}{mK}$ ]	$h_0,$ [ $\frac{W}{m^2K}$ ]	$h,$ [ $\frac{W}{m^2K}$ ]
Value	121	26	0.00368	45	100	0.025	1000	100

Table 1.2. *Input data with common dimensionless parameters*

Parameter	$\delta_0$	$\delta$	$\beta_0^0$	$\beta_0$	$\beta$
Value	0.2	1.17197	0.03140	0.00314	0.00698

Table 1.3. *Different input data for calculations*

Parameter	1.	2.	3.	4.	5.
$B, [m]$	0.000445	0.000445	0.000445	0.00089	0.002225
$R, [m]$	0.002695	0.002695	0.002695	0.002250	0.000470
$L, [m]$	0.012699	0.006350	0.025398	0.012699	0.012699
$b$	0.141719	0.141719	0.141719	0.283439	0.708598
$l$	4.044585	2.022292	8.0891714	4.044585	4.044585
$\alpha$	0	0	0	0	0

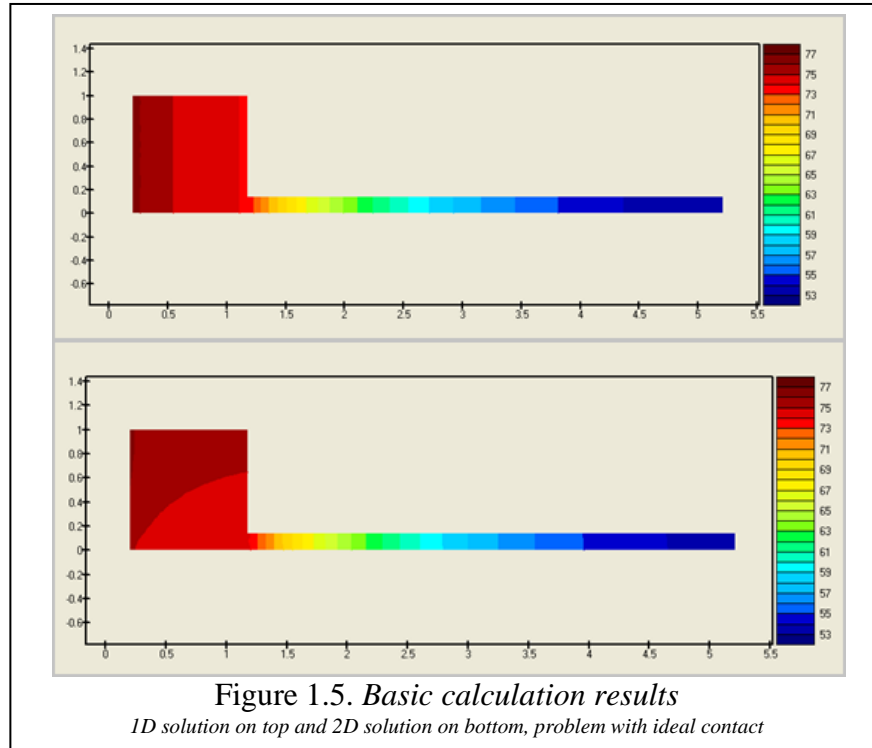
Table 1.3. *Different input data for calculations (continue)*

Parameter	6.	7.	8.
$B$ , [m]	0.000445	0.000445	0.000445
$R$ , [m]	0.002695	0.002695	0.002695
$L$ , [m]	0.012699	0.012699	0.012699
$b$	0.141719	0.141719	0.141719
$l$	4.044585	4.044585	4.044585
$\alpha$	0.01333	0.1333	1.333

The solution of problems can be shown by using a table or figure. 2D solution for the first calculation (further in text - basic calculation) is also shown in numerical format in Table 1.4. Basic calculation of the 2D and 1D solution is shown in Figure 1.5. It is obviously that the isolines of the 1D solution are perpendicular to the  $r$  axis, as it was mentioned previously. However, the isolines of the 2D solution are curves, which, in general case, are not perpendicular to the  $r$  axis. In the 1D solution, we do not see the influence of extended surface to the solution and see that the temperature is constant on line  $r = \delta_0$ . The 2D solution confirms the temperature difference on the internal surface of the wall (see Fig. 1.5 – 1.7). Tangents of isolines at the upper and lower edges of the wall are perpendicular to the  $r$  axis because of symmetry conditions for 2D problem.

Table 1.4. *Numerical results of calculation for basic case.*

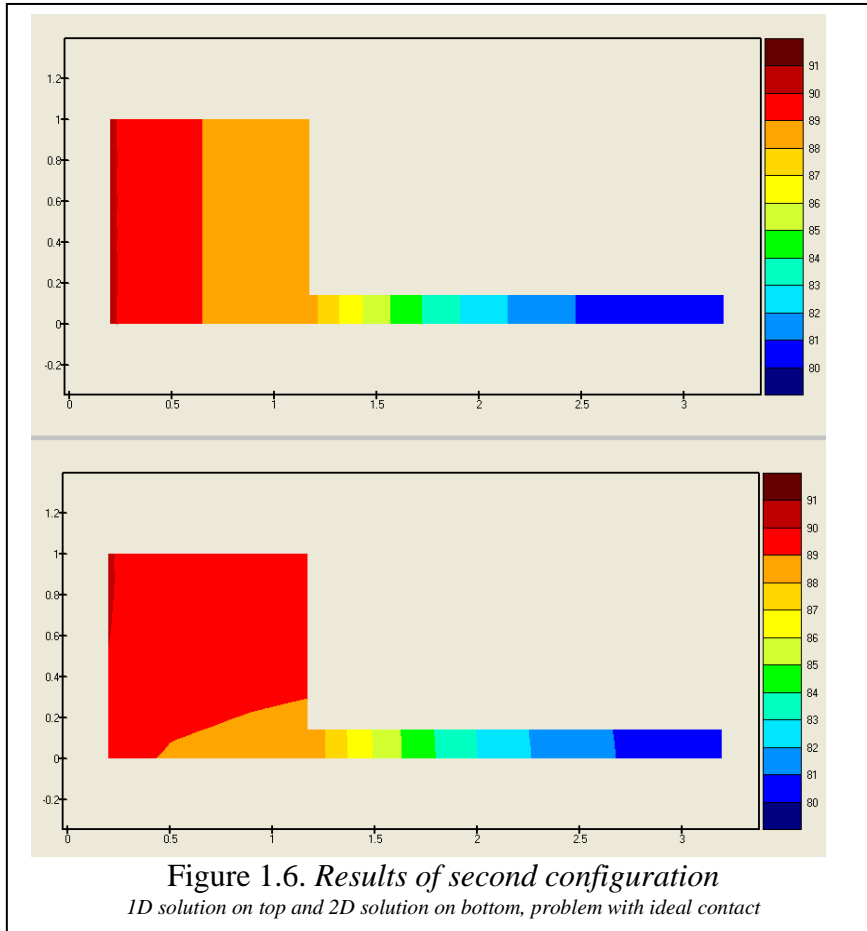
$z r$	0.2	0.52	0.84	1.17	1.74	2.32	2.90	3.48	4.06	4.63	5.21
1.00	76.22	75.36	75.18	75.11							
0.90	76.21	75.35	75.17	75.10							
0.80	76.18	75.33	75.14	75.07							
0.70	76.13	75.28	75.10	75.03							
0.60	76.07	75.21	75.03	74.96							
0.50	75.99	75.13	74.95	74.88							
0.40	75.89	75.03	74.84	74.77							
0.30	75.77	74.90	74.72	74.65							
0.20	75.63	74.76	74.58	74.51							
0.14	75.54	74.67	74.49	74.44	66.67	61.73	58.42	56.20	54.78	54.00	53.74
0.10	75.39	74.56	74.46	74.43	66.68	61.74	58.43	56.21	54.79	54.00	53.74
0.00	75.25	74.45	74.44	74.42	66.69	61.75	58.44	56.22	54.80	54.01	53.75



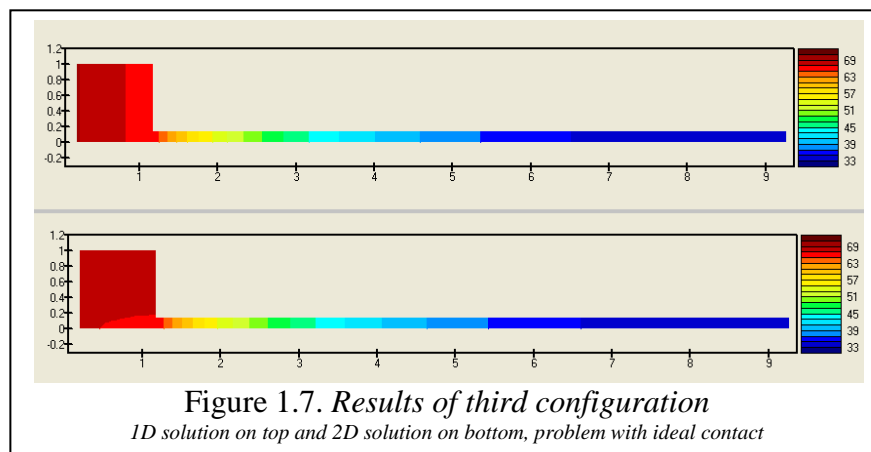
Let's shortly consider the solution for the fin. It is obviously that the isolines for the fin barely decline to the left. It is caused by heat exchange between the surrounding environment on the upper edge of the fin and symmetry conditions on lower edge of the fin. This is similar to the case of the wall tangent of isolines at the lower edge of the fin is perpendicular to the  $r$  axis.

Since a general view of both 1D and 2D solution method is obtained, further calculation results at different input data and geometry will be discussed and compared using basic calculation.

The fin is doubled by length in the third calculation compared to basic calculation (Figure 1.7). It is obvious from Figure 1.7 that the prolongation of the fin gives an effective cooling of the wall. There is an approximately  $8^{\circ}\text{C}$  difference of the average temperature in the wall compared to the basic calculation. In case of two times shorter fin comparing to basic calculation (Figure 1.6) temperature in the wall is higher for approximately  $14^{\circ}\text{C}$ .



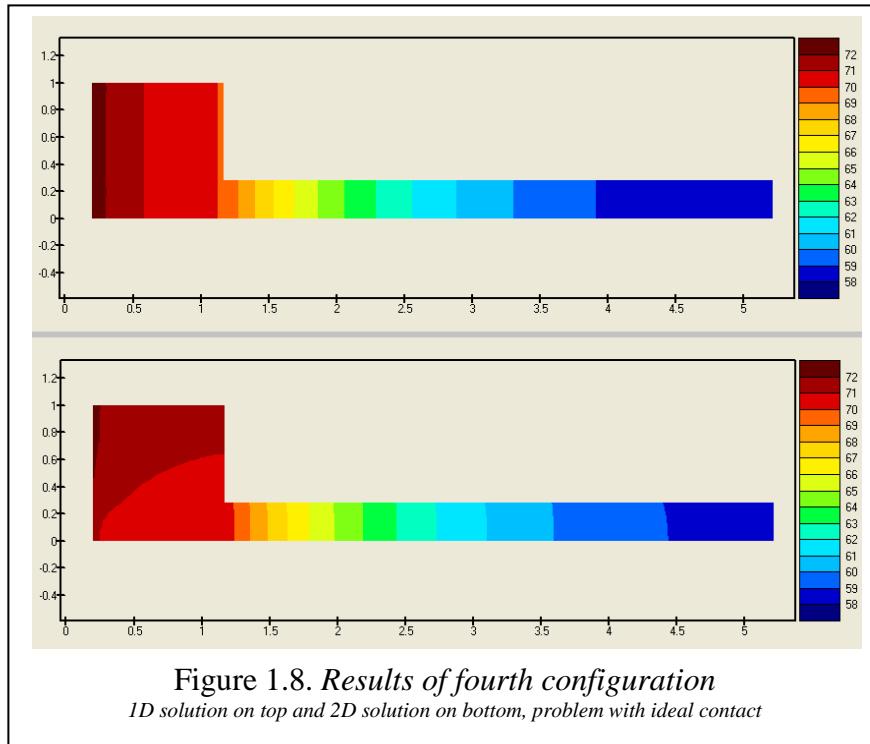
Comparing temperature value in Figures 1.5, 1.6, and 1.7 at length of the fin  $\sim 1.5$ ,  $\sim 3$ ,  $\sim 5$  and  $\sim 9$ , it is obvious that a longer fin better cools itself and the wall as well.



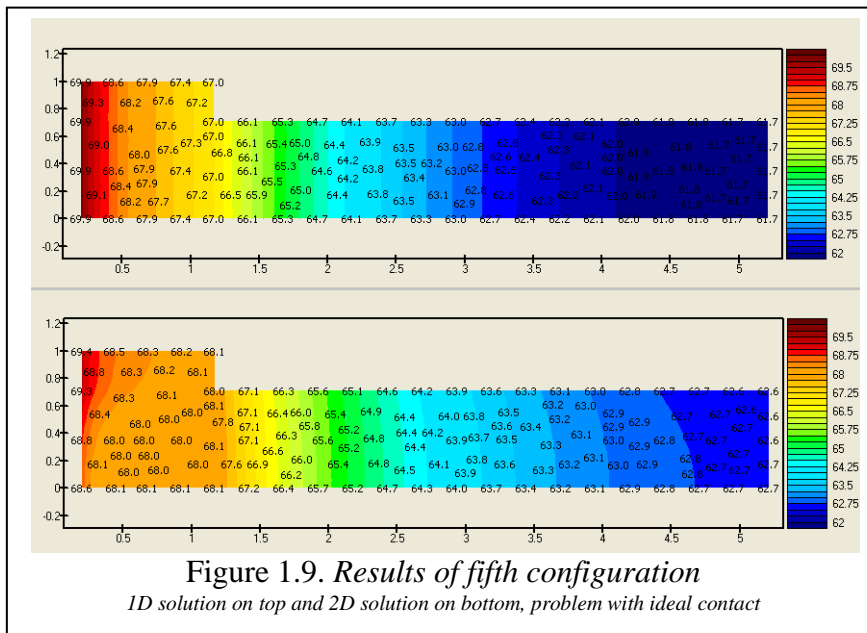
Influence of length of the fin is clear, still, the question of influence of fin height to temperature distribution is open.

The height of the fin in the fourth configuration is doubled compared to the basic configuration (Fig. 1.8). Comparing both solutions in Figures 1.5 and 1.8, is

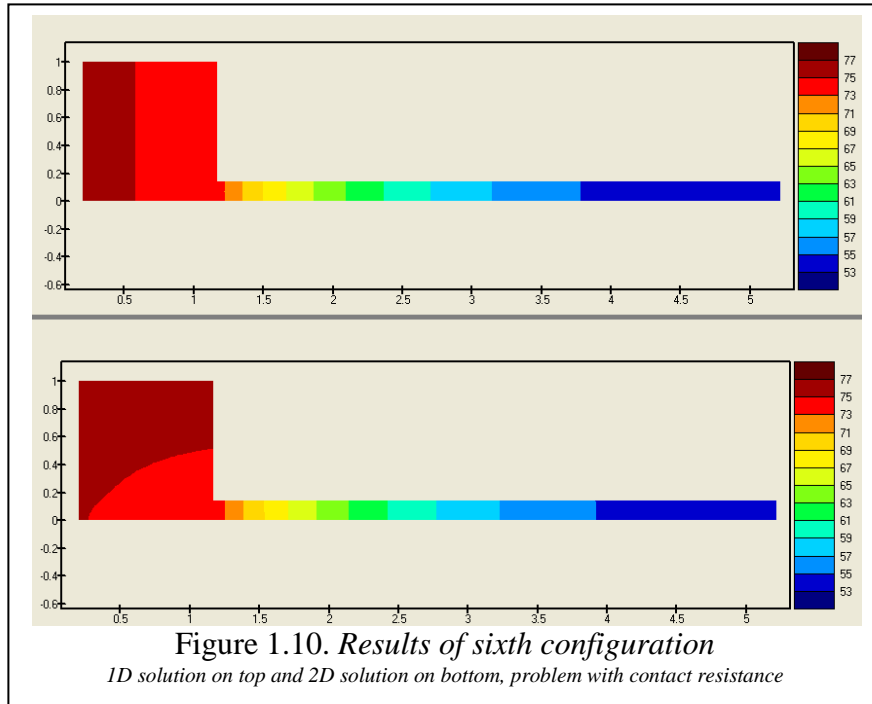
clear that the configuration with a higher fin has lower temperature in the wall, but still has a higher temperature inside the fin.



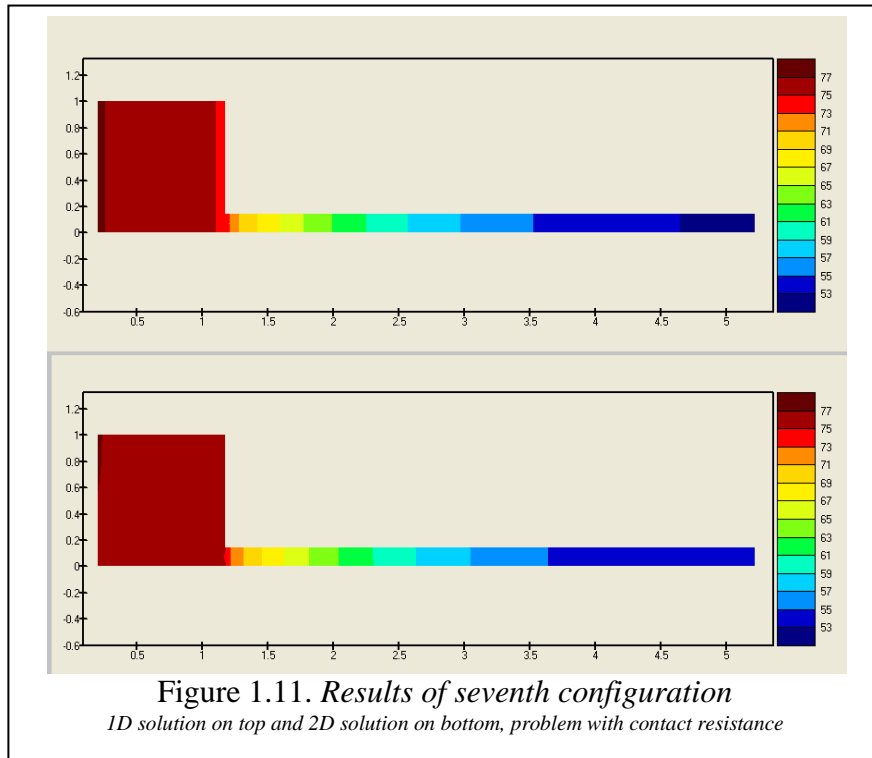
Fifth configuration consists of fin five times higher than in the basic configuration. And Figure 1.9 once more shows and approves already known fact from theory [2], [23], that the higher fin will better cool the wall, and temperature gradient in the fin is less than in the case of a lower fin with same length.



Since the fin has better heat transfer properties than the surrounding environment, it better conducts heat from the wall and fin contact surface as well. Temperature values at different points of 2D and 1D solution in Figure 1.9 are shown.

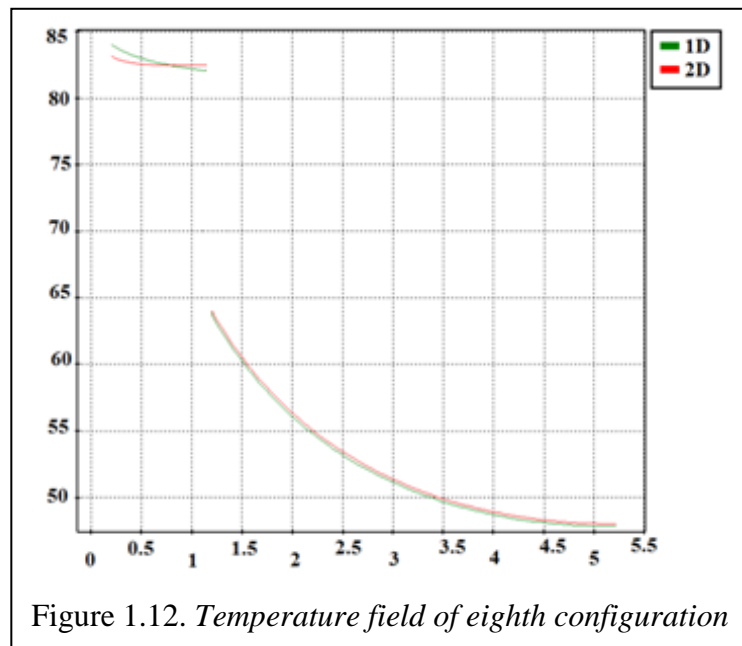


All six configurations have fine compatibility between 1D and 2D solutions in the mean of average temperatures (see, for example Fig. 1.9). Since a 1D solution cannot show the dependence of temperature distribution on argument  $z$ , still gives

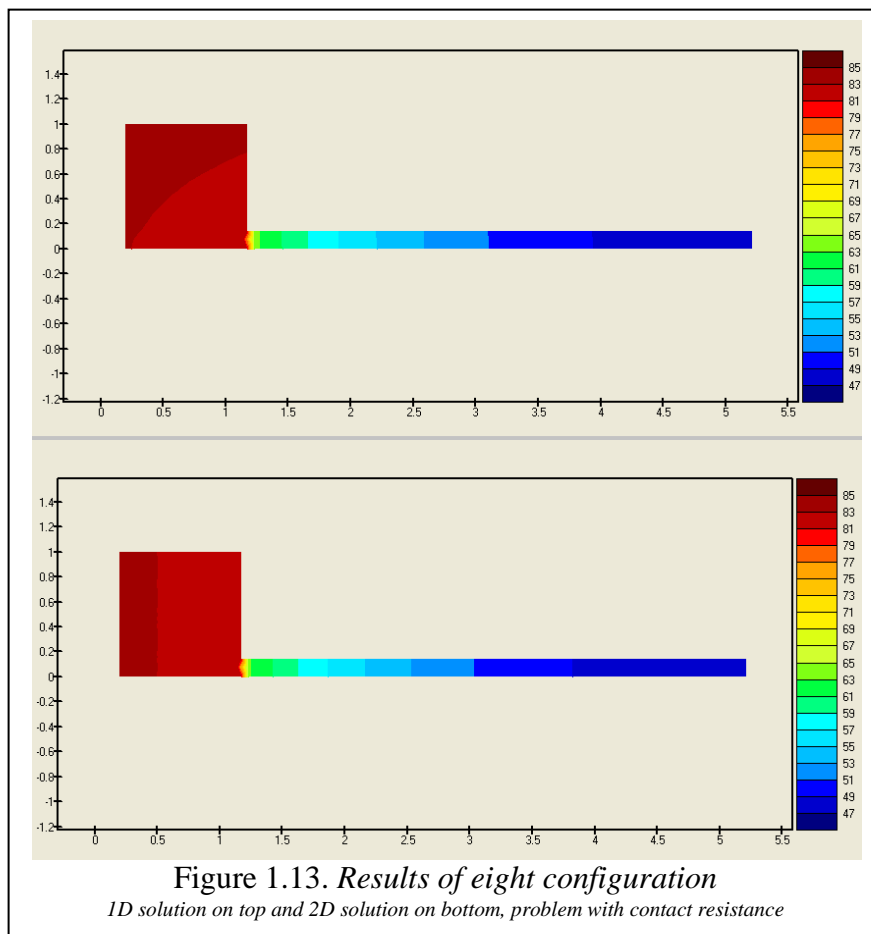




quite a good evaluation of the particular case of this steady-state process.



The next three calculations for the case of contact resistance are done. The rugged contact between the wall and fin is, that there is a bigger temperature difference between both elements [23].



The assumption that the gap between the fin and wall is filled with air is done. It is considered that the air convection is not present since the gap is less than  $<0.5\text{mm}$  [35]. Therefore, the coefficient, well-known in literature, of air heat conduction  $0.025\text{ W/mK}$  is used [35].

Several examples of contact resistance are displayed in Figures 1.10 – 1.12. The solution of the case when contact resistance parameter  $\alpha$  is equal to 0.01333 (see Table 1.3, sixth configuration) in Figure 1.10 is showed. It can also be interpreted as contact resistance with an air gap approximately  $10^{-4}\text{mm}$  in width. Obviously, temperature changes compared to the basic configuration are relatively small, within  $1^\circ\text{C}$ .

Increasing contact resistance difference between solutions of basic and other configurations appears. Insignificant differences in the wall can be observed. The seventh configuration has a gap  $\sim 10^{-3}\text{mm}$  in width (see Fig. 1.11).

Whereas increasing the gap by a jump of ten times, the 1D and 2D solution at the contact surface of the fin and wall can be observed (Figure 1. 12). The solution for the seventh configuration comparing with the solution of the basic configuration shows a temperature difference in the wall for  $\sim 8^\circ\text{C}$ , and a temperature difference at the left side of the fin approx.  $10^\circ\text{C}$ , but approx.  $6^\circ\text{C}$  at the right side of the fin.

More details on constructed and applied numerical models and analysis of their advantages and disadvantages are discussed in Chapter 2.6.

## 1.7 Conclusion

Some approximate three dimensional analytical solutions for a periodical system with a cylindrical fin when the wall and the fin consist of materials which have different thermal properties are constructed. Semi-analytical mathematical models of the 2D and 1D solution with a conservative averaging method are constructed. Some examples are analysed. The influence of model geometry and contact resistance at the contact between the wall and fin to the solution is shown.

## 2. ANALYTICAL 2D STEADY-STATE SOLUTION FOR CYLINDRICAL FIN

### 2.1 Problem Statement

In this chapter, the steady-state heat transfer problem in the cylindrical wall with fin is discussed. In Chapter 2.4, a similar problem as in Chapter 1.1 is described. Also, a non-homogeneous problem with a non-homogeneous environment is discussed. Similar methodology for a full 3D problem can be applied. It is considered that, fluxes in  $\varphi$  direction are comparable and do not affect the solution essentially. The correct reduction of the problem from 3D to 2D at its solution is shown. A similar approach of the analytical solution for a rectangular fin in the Cartesian coordinate system is discussed in the Master theses of author [9] and paper [27]. A short description of the generalised method of the Green function is described in Appendix A, Chapter A.3. The solution of discussed problems can be used in the solving of more complicated problems of mathematical physics, for example, solutions of a cylindrical fin with an outer hydrodynamic field [20], [21].

### 2.2 Mathematical Formulation of 3D Problem and Exact its Reduction to Non-homogeneous 2D Problem

At first, the accurate three-dimensional formulation of the steady-state problem for a system of cylindrical wall and fin (Figure 2.1) is considered. The one element of the wall is placed in the domain  $\{\tilde{r} \in [R_0, R_1], \tilde{z} \in [0, H], \varphi \in [0, \Phi]\}$  and temperature field  $\tilde{V}_0(\tilde{r}, \tilde{z}, \varphi)$  in the wall is described with the Laplace equation:

$$\frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left( \tilde{r} \frac{\partial \tilde{V}_0}{\partial \tilde{r}} \right) + \frac{\partial^2 \tilde{V}_0}{\partial \tilde{z}^2} + \frac{1}{\tilde{r}^2} \frac{\partial^2 \tilde{V}_0}{\partial \varphi^2} = 0. \quad (2.1)$$

The cylindrical fin of length  $L$  occupies the domain  $\{\tilde{r} \in [R_1, R_2], \tilde{z} \in [0, H_0], \varphi \in [0, \Phi]\}$  and the temperature field  $\tilde{V}(\tilde{r}, \tilde{z}, \varphi)$  fulfils the equation:

$$\frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left( \tilde{r} \frac{\partial \tilde{V}}{\partial \tilde{r}} \right) + \frac{\partial^2 \tilde{V}}{\partial \tilde{z}^2} + \frac{1}{\tilde{r}^2} \frac{\partial^2 \tilde{V}}{\partial \varphi^2} = 0. \quad (2.2)$$

The following boundary conditions in  $\varphi$  direction is considered (other boundary conditions will be added in non-dimensional form in chapter 2.2.1 and 2.2.2):

$$\left. \frac{\partial \tilde{V}_0}{\partial \varphi} \right|_{\varphi=0} = \tilde{q}_0^0(\tilde{r}, \tilde{z}),$$

$$\begin{aligned}
\left. \frac{\partial \tilde{V}_0}{\partial \varphi} \right|_{\varphi=\Phi} &= \tilde{q}_1^0(\tilde{r}, \tilde{z}), \\
\left. \frac{\partial \tilde{V}}{\partial \varphi} \right|_{\varphi=0} &= \tilde{q}_0(\tilde{r}, \tilde{z}), \\
\left. \frac{\partial \tilde{V}}{\partial \varphi} \right|_{\varphi=\Phi} &= \tilde{q}_1(\tilde{r}, \tilde{z}).
\end{aligned} \tag{2.3}$$

Equations (2.1) and (2.2) from 3D to 2D can be reduced by introducing the following average integral values for argument  $\varphi$ :

$$\begin{aligned}
\tilde{U}_0(\tilde{r}, \tilde{z}) &= \\
&= \frac{1}{\Phi} \int_0^\Phi \tilde{V}_0(\tilde{r}, \tilde{z}, \varphi) d\varphi, \\
\tilde{U}(\tilde{r}, \tilde{z}) &= \\
&= \frac{1}{\Phi} \int_0^\Phi \tilde{V}(\tilde{r}, \tilde{z}, \varphi) d\varphi.
\end{aligned} \tag{2.4}$$

Integration of equation (2.1) for the wall over  $\varphi \in [0, \Phi]$  gives the following equation (exact consequence of 3D partial differential equation (2.1)):

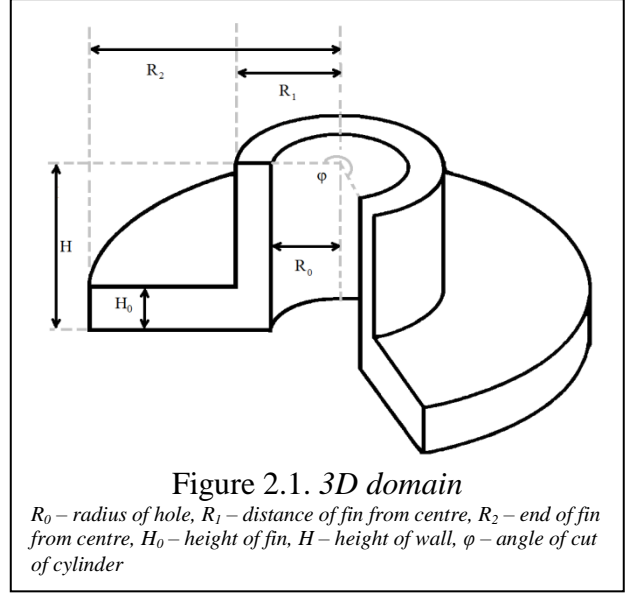
$$\begin{aligned}
\frac{1}{\Phi} \int_0^\Phi \frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left( \tilde{r} \frac{\partial \tilde{V}_0}{\partial \tilde{r}} \right) d\varphi + \frac{1}{\Phi} \int_0^\Phi \frac{\partial^2 \tilde{V}_0}{\partial \tilde{z}^2} d\varphi + \frac{1}{\tilde{r}^2 \Phi} \left( \left. \frac{\partial \tilde{V}_0}{\partial \varphi} \right|_{\varphi=\Phi} - \left. \frac{\partial \tilde{V}_0}{\partial \varphi} \right|_{\varphi=0} \right) &= 0 \\
\frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left( \tilde{r} \frac{\partial \tilde{U}_0}{\partial \tilde{r}} \right) + \frac{\partial^2 \tilde{U}_0}{\partial \tilde{z}^2} + \frac{1}{\tilde{r}^2 \Phi} \left( \left. \frac{\partial \tilde{V}_0}{\partial \varphi} \right|_{\varphi=\Phi} - \left. \frac{\partial \tilde{V}_0}{\partial \varphi} \right|_{\varphi=0} \right) &= 0.
\end{aligned} \tag{2.5}$$

The first pair of boundary conditions (2.3) allows rewriting the last equality (2.5) in the form of two-dimensional non-homogeneous equation as follows:

$$\frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left( \tilde{r} \frac{\partial \tilde{U}_0}{\partial \tilde{r}} \right) + \frac{\partial^2 \tilde{U}_0}{\partial \tilde{z}^2} + \tilde{Q}_0(\tilde{r}, \tilde{z}) = 0, \text{ where } \tilde{Q}_0(\tilde{r}, \tilde{z}) = \frac{\tilde{q}_1^0(\tilde{r}, \tilde{z}) - \tilde{q}_0^0(\tilde{r}, \tilde{z})}{\tilde{r}^2 \Phi}. \tag{2.6}$$

A similar equation (2.2) together with boundary condition (2.3), the second pair can be rewritten in the following form:

$$\frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left( \tilde{r} \frac{\partial \tilde{U}}{\partial \tilde{r}} \right) + \frac{\partial^2 \tilde{U}}{\partial \tilde{z}^2} + \tilde{Q}(\tilde{r}, \tilde{z}) = 0, \text{ where } \tilde{Q}(\tilde{r}, \tilde{z}) = \frac{\tilde{q}_1(\tilde{r}, \tilde{z}) - \tilde{q}_0(\tilde{r}, \tilde{z})}{\tilde{r}^2 \Phi}. \tag{2.7}$$



### 2.2.1 Problem Formulation in Dimensionless Arguments

The following dimensionless arguments and parameters to transform problem (2.5) to dimensionless problem are used:

$$r = \frac{\tilde{r}}{H}, \quad z = \frac{\tilde{z}}{H}, \quad \rho_0 = \frac{R_0}{H},$$

$$\rho_1 = \frac{R_1}{H}, \quad \rho_2 = \frac{R_2}{H}, \quad b = \frac{H_0}{H},$$

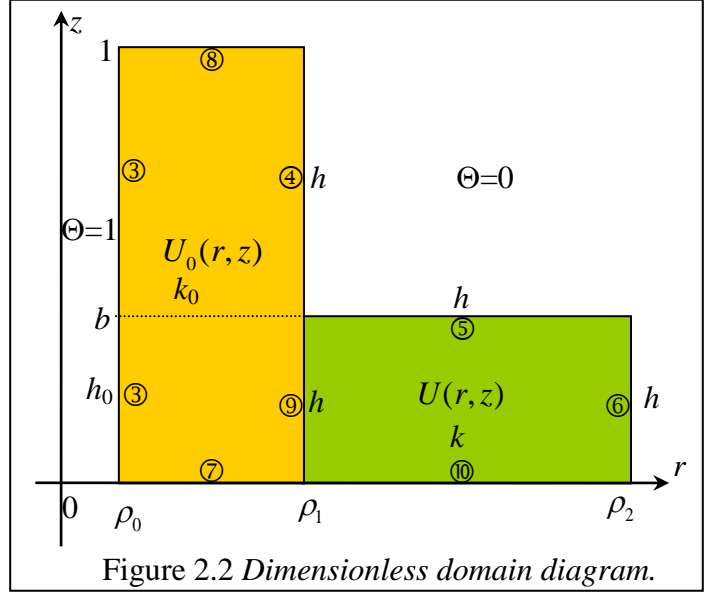
$$\beta_0 = \frac{hH}{k_0}, \quad \beta = \frac{hH}{k}, \quad \beta_0^0 = \frac{h_0H}{k_0},$$

$$\gamma_0 = \frac{\beta_0}{\rho_1}, \quad \gamma = \frac{\beta}{\rho_1}, \quad \gamma_0^0 = \frac{\beta_0^0}{\rho_0} \quad \text{and}$$

temperatures:

$$U(r, z) = \frac{\tilde{U}(r, z) - T_a}{T_b - T_a},$$

$$U_0(r, z) = \frac{\tilde{U}_0(r, z) - T_a}{T_b - T_a}.$$



Here,  $k$  and  $k_0$  - heat

conductivity coefficient for the fin and wall,  $h$  and  $h_0$  - heat exchange coefficient for the fin and wall,  $H_0$  - height of the fin,  $L$  - length of the fin,  $H$  - height of the wall,  $T_b$  - the surrounding temperature on the left (hot) side (the heat source side) of the wall,  $T_a$  - the surrounding temperature on the right (cold) side (the heat sink side) of the wall and the fin.

### 2.2.2 Description of Temperature Field in the Wall

One element of the wall placed in the dimensionless domain (Fig. 2.2) is now  $\{r \in [\rho_0, \rho_1], z \in [0, 1]\}$  and the dimensionless temperature field  $U_0(r, z)$  in the wall is described with following equation:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial U_0}{\partial r} \right) + \frac{\partial^2 U_0}{\partial z^2} + Q_0(r, z) = 0. \quad (2.8)$$

Additional boundary condition ③ and ④ are defined similarly as described in Chapter 1.3.2:

$$\frac{\partial U_0}{\partial r} + \gamma_0^0 (1 - U_0) = 0, \quad r = \rho_0, \quad z \in [0, 1], \quad (2.9)$$

$$\frac{\partial U_0}{\partial r} + \gamma_0 U_0 = 0, \quad r = \rho_1, \quad z \in [b, 1]. \quad (2.10)$$

And, in contradiction to the statements of problems in paper [4] and Chapter 1.2.2, general non-homogeneous boundary conditions on the bottom ⑦ and the top ⑧ of the wall are considered:

$$\left. \frac{\partial U_0}{\partial z} \right|_{z=0} = q_0(r), r \in [\rho_0, \rho_1], \quad (2.11)$$

$$\left( \frac{\partial U_0}{\partial z} + \alpha \beta_0 U_0 \right) \Big|_{z=1} = q_1(r), \quad (2.12)$$

where  $\alpha$  is a parameter with two possible values 0 or 1. This kind of construction of boundary condition allows discussing several different problems: the second and third kind non homogeneous BC; only the second kind non homogeneous BC. The statement where  $q_0(r) = q_1(r) \equiv 0$  and  $\alpha \equiv 0$  match with the problem described in Chapter 1.

Conjugation conditions ⑨ on the surface between the wall and the fin are assumed as ideal thermal contact - there is no contact resistance:

$$U_0 \Big|_{r=\rho_1-0} = U \Big|_{r=\rho_1+0}, \gamma \left. \frac{\partial U_0}{\partial r} \right|_{r=\rho_1-0} = \gamma_0 \left. \frac{\partial U}{\partial r} \right|_{r=\rho_1+0}. \quad (2.13)$$

### 2.2.3 Description of Temperature Field in the Fin

The cylindrical fin of length  $l$  in dimensionless problem (Fig. 2.2) occupies the domain  $\{r \in [\rho_1, \rho_2], z \in [0, b]\}$  and the temperature field  $U(r, z)$  fulfils the equation:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial U}{\partial r} \right) + \frac{\partial^2 U}{\partial z^2} + Q(r, z) = 0. \quad (2.14)$$

The following boundary conditions on top ⑤ and right side ⑥ of the fin are considered:

$$\frac{\partial U}{\partial z} + \beta U = 0, r \in [\rho_1, \rho_2], z = b, \quad (2.15)$$

$$\frac{\partial U}{\partial r} + \gamma U = 0, r = \rho_2, z \in [0, b]. \quad (2.16)$$

And non-homogeneous boundary conditions on the bottom ⑩ of fin (similar as for wall) are defined:

$$\left. \frac{\partial U}{\partial z} \right|_{z=0} = q(r), r \in [\rho_1, \rho_2]. \quad (2.17)$$

### 2.3 Solution of Non Steady-State Problem for 2D Sub-Domain

The solution of the non steady-state problem with axial symmetric rectangular domain known in literature [11] is discussed in this chapter. Also main ideas on how to use this solution for the steady-state problem with cylindrical wall and fin are described.

Axial symmetric non steady-state heat exchange problem with positive or negative heat source in cylindrical system of coordinates is formulated in the following way, where  $\tilde{U} = \tilde{U}(r, z, t)$ :

$$\frac{\partial \tilde{U}}{\partial t} = a \left( \frac{\partial^2 \tilde{U}}{\partial r^2} + \frac{1}{r} \frac{\partial \tilde{U}}{\partial r} + \frac{\partial^2 \tilde{U}}{\partial z^2} \right) + \Phi(r, z, t). \quad (2.18)$$

Problem (2.18) is formulated in domain  $\tilde{\rho}_1 \leq r \leq \tilde{\rho}_2$ ,  $0 \leq z \leq \tilde{l}$  with the following initial condition and boundary conditions (①, ②, ③, ④) of 3<sup>rd</sup> kind (Fig. 2.3):

$$\begin{aligned} \tilde{U}(r, z, 0) &= f(r, z), \\ \frac{\partial \tilde{U}}{\partial r} - k_1 \tilde{U} &= g_1(z, t), \quad r = \tilde{\rho}_1, \\ \frac{\partial \tilde{U}}{\partial r} + k_2 \tilde{U} &= g_2(z, t), \quad r = \tilde{\rho}_2, \\ \frac{\partial \tilde{U}}{\partial z} - k_3 \tilde{U} &= g_3(r, t), \quad z = 0, \\ \frac{\partial \tilde{U}}{\partial z} + k_4 \tilde{U} &= g_4(r, t), \quad z = \tilde{l}. \end{aligned} \quad (2.19)$$

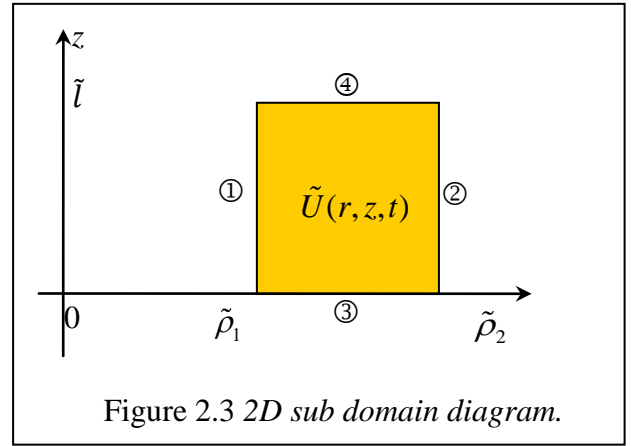


Figure 2.3 2D sub domain diagram.

The solution of problem (2.18) – (2.19) with mean of Green's function is the following:

$$\begin{aligned} \tilde{U}(r, z, t) &= 2\pi \int_0^{\tilde{l}} \int_{\tilde{\rho}_1}^{\tilde{\rho}_2} \xi f(\xi, \eta) G(r, z, \xi, \eta, t) d\xi d\eta - \\ &\quad - 2\pi \tilde{\rho}_1 a \int_0^t \int_0^{\tilde{l}} g_1(\eta, \tau) G(r, z, \tilde{\rho}_1, \eta, t - \tau) d\eta d\tau + \\ &\quad + 2\pi \tilde{\rho}_2 a \int_0^t \int_0^{\tilde{l}} g_2(\eta, \tau) G(r, z, \tilde{\rho}_2, \eta, t - \tau) d\eta d\tau - \\ &\quad - 2\pi a \int_0^t \int_{\tilde{\rho}_1}^{\tilde{\rho}_2} \xi g_3(\xi, \tau) G(r, z, \xi, 0, t - \tau) d\xi d\tau + \\ &\quad + 2\pi a \int_0^t \int_{\tilde{\rho}_1}^{\tilde{\rho}_2} \xi g_4(\xi, \tau) G(r, z, \xi, \tilde{l}, t - \tau) d\xi d\tau + \\ &\quad + 2\pi \int_0^t \int_0^{\tilde{l}} \int_{\tilde{\rho}_1}^{\tilde{\rho}_2} \xi \Phi(\xi, \eta, \tau) G(r, \xi, z, \eta, t - \tau) d\xi d\eta d\tau. \end{aligned} \quad (2.20)$$

Where Green's functions are:

$$G(r, z, \xi, \eta, \tau) = G_1(r, \xi, \tau) \cdot G_2(z, \eta, \tau), \quad (2.21)$$

$$G_1(r, \xi, t) = \frac{\pi}{4} \sum_{n=1}^{\infty} \frac{\lambda_n^2}{B_n} [k_2 J_0(\lambda_n \tilde{\rho}_2) - \lambda_n J_1(\lambda_n \tilde{\rho}_2)]^2 H_n(r) H_n(\xi) e^{-\lambda_n^2 at}, \quad (2.22)$$

$$G_2(z, \eta, t) = \sum_{m=1}^{\infty} \frac{\varphi_m(z) \varphi_m(\eta)}{\|\varphi_m\|^2} e^{-\mu_m^2 at}, \quad (2.23)$$

and:

$$\begin{aligned} B_n &= (\lambda_n^2 + k_2^2) [k_1 J_0(\lambda_n \tilde{\rho}_1) + \lambda_n J_1(\lambda_n \tilde{\rho}_1)]^2 - (\lambda_n^2 + k_1^2) [k_2 J_0(\lambda_n \tilde{\rho}_2) - \lambda_n J_1(\lambda_n \tilde{\rho}_2)]^2, \\ H_n(r) &= [k_1 Y_0(\lambda_n \tilde{\rho}_1) + \lambda_n Y_1(\lambda_n \tilde{\rho}_1)] J_0(\lambda_n r) - [k_1 J_0(\lambda_n \tilde{\rho}_1) + \lambda_n J_1(\lambda_n \tilde{\rho}_1)] Y_0(\lambda_n r), \\ \varphi_m(z) &= \mu_m \cos(\mu_m z) + k_3 \sin(\mu_m z), \\ \|\varphi_m\|^2 &= \frac{k_4}{2} \frac{\mu_m^2 + k_3^2}{\mu_m^2 + k_4^2} + \frac{k_3}{2} + \frac{\tilde{l}}{2} (\mu_m^2 + k_3^2), \\ J_0(\lambda), J_1(\lambda), Y_0(\lambda), Y_1(\lambda) &- \text{Bessels' functions.} \end{aligned} \quad (2.24)$$

But  $\lambda_n$  are positive roots of the following transcendent equation:

$$\begin{aligned} & [k_1 J_0(\lambda_n \tilde{\rho}_1) + \lambda_n J_1(\lambda_n \tilde{\rho}_1)] [k_2 Y_0(\lambda_n \tilde{\rho}_2) - \lambda_n Y_1(\lambda_n \tilde{\rho}_2)] - \\ & - [k_2 J_0(\lambda_n \tilde{\rho}_2) - \lambda_n J_1(\lambda_n \tilde{\rho}_2)] [k_1 Y_0(\lambda_n \tilde{\rho}_1) + \lambda_n Y_1(\lambda_n \tilde{\rho}_1)] = 0. \end{aligned} \quad (2.25)$$

But  $\mu_m$  are positive roots of the following transcendent equation:

$$\frac{tg \mu \tilde{l}}{\mu} = \frac{k_3 + k_4}{\mu^2 - k_3 k_4}. \quad (2.26)$$

To adapt solution (2.20) for the steady state problem (2.1) – (2.17), we can use the general properties of Green's function and their solutions [11]. The main idea is presumed that the steady-state case is the boundary case of the non steady-state problem when  $t \rightarrow +\infty$ .

Since the solution of problem (2.18) – (2.19) is formulated, we can adapt solution (2.20) for the 2D problem for a cylindrical system with fin - separately for the wall and fin.

For simple cases with the first or second kind of boundary conditions, the method of separation of variables can be used [36]. Based on the theorem of Steklov [37] and property of superposition of solutions, a separate solution for particular sub-domains can be found [38].

## 2.4 Solution of Homogeneous Steady State 2D Problem

In this chapter, the case of homogeneous equations for the cylindrical wall (2.8) and fin (2.14) are analysed and, for simplicity, additionally homogeneous boundary conditions (2.11), (2.12), (2.17) are assumed, id est.:

$$Q(r, z) = Q_0(r, z) \equiv 0, \quad q(r) = q_0(r) = q_1(r) \equiv 0. \quad (2.27)$$



The general case (with non-homogeneous differential equations and non-homogeneous boundary conditions) will be considered in Chapter 2.5.

#### 2.4.1 The Separate 2D Problem for the Wall

BC (2.10) together with the conjugation conditions (2.13) can be written in the following common form:

$$\left. \left( \frac{\partial U_0}{\partial r} + \gamma_0 U_0 \right) \right|_{r=\rho_1-0} = \begin{cases} \gamma_0 F_0(z), & 0 < z < b \\ 0 & , b < z < 1 \end{cases}, \text{ where} \quad (2.28)$$

$$F_0(z) = \left. \left( \frac{1}{\gamma} \frac{\partial U}{\partial r} + U \right) \right|_{r=\rho_1+0}.$$

Values and conditions harmonised with the nomenclature of the problem in chapter 2.3 are described in Table 2.1.

Table 2.1. Formulation of problem for wall.

$g_i$	$k_i$	Other values
$g_1(z, t) = -\gamma_0^0$	$k_1 = \gamma_0^0$	$a = 1$
$g_2(z, t) = \begin{cases} \gamma_0 F_0(z), & 0 < z < b \\ 0 & , b < z < 1 \end{cases}$	$k_2 = \gamma_0$	$\tilde{\rho}_1 = \rho_0$ $\tilde{\rho}_2 = \rho_1$
$g_3(r, t) = 0$	$k_3 = 0$	$\tilde{l} = 1$
$g_4(r, t) = 0$	$k_4 = \alpha\beta_0$	$\Phi(r, z, t) = 0$ $U(r, z, t) = U_0(r, z)$

Solution (2.20) can be written in the following form because of the homogeneity of the equation and boundary conditions for problem (2.8) – (2.13):

$$U_0(r, z) = \lim_{t \rightarrow \infty} 2\pi \int_0^1 \int_{\rho_0}^{\rho_1} \xi f(\xi, \eta) G(r, z, \xi, \eta, t) d\xi d\eta +$$

$$+ 2\pi \rho_0 \lim_{t \rightarrow \infty} \int_0^t \int_0^1 \gamma_0^0 G(r, z, \rho_0, \eta, t - \tau) d\eta d\tau +$$

$$+ 2\pi \rho_1 \lim_{t \rightarrow \infty} \int_0^t \int_0^b \gamma_0 F_0(\eta) G(r, z, \rho_1, \eta, t - \tau) d\eta d\tau. \quad (2.29)$$

Obviously all the rest of the addends of equation (2.20) for the wall are equal to zero due to the homogeneity of the problem or boundary conditions (see Table 4.1). It can easily be shown that the first addend of formula (2.29) also is zero:

$$\lim_{t \rightarrow \infty} 2\pi \int_0^1 \int_{\rho_0}^{\rho_1} \xi f(\xi, \eta) G(r, z, \xi, \eta, t) d\xi d\eta =$$

$$= 2\pi \int_0^1 \int_{\rho_0}^{\rho_1} \xi f(\xi, \eta) \lim_{t \rightarrow \infty} (G_1(r, \xi, t) G_2(z, \eta, t)) d\xi d\eta. \quad (2.30)$$

Let's consider  $\lim_{t \rightarrow \infty} (G_1(r, \xi, t)G_2(z, \eta, t))$ . From equations (2.22) and (2.23) it is obvious that both Greens' functions can be rewritten in following way:

$$G_1(r, \xi, t) = \frac{\pi}{4} \sum_{n=1}^{\infty} A_n(r, \xi) e^{-\lambda_n^2 t} \quad \text{and} \quad G_2(z, \eta, t) = \sum_{m=1}^{\infty} X_m(z, \eta) e^{-\mu_m^2 t}, \quad (2.31)$$

where  $A_n$  and  $X_m$  are:

$$\frac{\lambda_n^2}{B_n} [\gamma_0 J_0(\lambda_n \rho_1) - \lambda_n J_1(\lambda_n \rho_1)]^2 H_n(r) H_n(\xi) \stackrel{\text{denoted}}{=} A_n(r, \xi) \quad \text{and}$$

$$\frac{\varphi_m(z) \varphi_m(\eta)}{\|\varphi_m\|^2} \stackrel{\text{denoted}}{=} X_m(z, \eta),$$

$$\varphi_m(z) = \mu_m \cos(\mu_m z).$$

Now easy we can see that

$$\begin{aligned} \lim_{t \rightarrow \infty} (G_1(r, \xi, t)G_2(z, \eta, t)) &= \\ &= \frac{\pi}{4} \lim_{t \rightarrow \infty} \left( \sum_{n=1}^{\infty} A_n(r, \xi) e^{-\lambda_n^2 t} \sum_{m=1}^{\infty} X_m(z, \eta) e^{-\mu_m^2 t} \right) \rightarrow 0. \end{aligned} \quad (2.32)$$

Formula (2.29) taking into account (2.32) looks in the following way:

$$\begin{aligned} U_0(r, z) &= 2\pi\beta_0^0 \lim_{t \rightarrow \infty} \int_0^t \int_0^1 G(r, z, \rho_0, \eta, t - \tau) d\eta d\tau + \\ &+ 2\pi\beta_0 \lim_{t \rightarrow \infty} \int_0^t \int_0^b F_0(\eta) G(r, z, \rho_1, \eta, t - \tau) d\eta d\tau. \end{aligned} \quad (2.33)$$

Further, the first part of formula (2.33) will be discussed. From equation (2.21) and properties of Greens' function [11] order of integration can be changed:

$$\begin{aligned} \lim_{t \rightarrow \infty} \int_0^t \int_0^1 G(r, z, \rho_0, \eta, t - \tau) d\eta d\tau &= \\ &= \int_0^1 \lim_{t \rightarrow \infty} \int_0^t G_1(r, \rho_0, t - \tau) G_2(z, \eta, t - \tau) d\tau d\eta. \end{aligned} \quad (2.34)$$

Unifying equation (2.33) with the nomenclature of (2.31), the following connection is obtained:

$$\begin{aligned} \lim_{t \rightarrow \infty} \int_0^t \int_0^1 G(r, z, \rho_0, \eta, t - \tau) d\eta d\tau &= \\ &= \frac{\pi}{4} \int_0^1 \lim_{t \rightarrow \infty} \int_0^t \sum_{n=1}^{\infty} A_n(r, \rho_0) e^{-\lambda_n^2 (t-\tau)} \sum_{m=1}^{\infty} X_m(z, \eta) e^{-\mu_m^2 (t-\tau)} d\tau d\eta = \\ &= \frac{\pi}{4} \int_0^1 \lim_{t \rightarrow \infty} \int_0^t \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} X_m(z, \eta) A_n(r, \rho_0) e^{-\lambda_n^2 (t-\tau)} e^{-\mu_m^2 (t-\tau)} d\tau d\eta = \\ &= \frac{\pi}{4} \int_0^1 \lim_{t \rightarrow \infty} \int_0^t \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Q_{m,n}(r, z, \rho_0, \eta) e^{-(\lambda_n^2 + \mu_m^2)(t-\tau)} d\tau d\eta = \\ &= \frac{\pi}{4} \int_0^1 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Q_{m,n}(r, z, \rho_0, \eta) \lim_{t \rightarrow \infty} e^{-(\lambda_n^2 + \mu_m^2)t} \int_0^t e^{(\lambda_n^2 + \mu_m^2)\tau} d\tau d\eta = \end{aligned}$$

$$\begin{aligned}
&= \frac{\pi}{4} \int_0^1 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Q_{m,n}(r, z, \rho_0, \eta) \lim_{t \rightarrow \infty} e^{-(\lambda_n^2 + \mu_m^2)t} \frac{e^{(\lambda_n^2 + \mu_m^2)\tau}}{(\lambda_n^2 + \mu_m^2)} \Big|_0^t d\eta = \\
&= \frac{\pi}{4} \int_0^1 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{Q_{m,n}(r, z, \rho_0, \eta)}{\lambda_n^2 + \mu_m^2} \lim_{t \rightarrow \infty} \left( 1 - e^{-t(\lambda_n^2 + \mu_m^2)} \right) d\eta = \\
&= \frac{\pi}{4} \int_0^1 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{Q_{m,n}(r, z, \rho_0, \eta)}{\lambda_n^2 + \mu_m^2} d\eta, \tag{2.35}
\end{aligned}$$

where  $Q_{m,n}(r, z, \xi, \eta) = A_n(r, \xi) \cdot X_m(z, \eta)$ .

Further, equation (2.35) is integrated taking into account values of Table 4.1 and connections (2.31):

$$\begin{aligned}
&\frac{\pi}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{A_n(r, \rho_0) \varphi_m(z)}{(\lambda_n^2 + \mu_m^2) \|\varphi_m\|^2} \int_0^1 \varphi_m(\eta) d\eta = \\
&= \frac{\pi}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{A_n(r, \rho_0) \varphi_m(z)}{(\lambda_n^2 + \mu_m^2) \|\varphi_m\|^2} \mu_m \int_0^1 \cos(\mu_m z) d\eta = \\
&= \frac{\pi}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{A_n(r, \rho_0) \varphi_m(z)}{(\lambda_n^2 + \mu_m^2) \|\varphi_m\|^2} \sin \mu_m. \tag{2.36}
\end{aligned}$$

Equation (2.36) pasted into formulas (2.34) and (2.33), and obtained:

$$\begin{aligned}
U_0(r, z) &= \frac{\pi^2 \beta_0^0}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{A_n(r, \rho_0) \varphi_m(z) \sin \mu_m}{(\lambda_n^2 + \mu_m^2) \|\varphi_m\|^2} + \\
&\quad + 2\pi \beta_0 \lim_{t \rightarrow \infty} \int_0^t \int_0^b F_0(z) G(r, z, \rho_1, \eta, t - \tau) d\eta d\tau. \tag{2.37}
\end{aligned}$$

Further, the second part of equality (2.33) is discussed. Again, the order of integration is changed and obtained the following:

$$\begin{aligned}
&\lim_{t \rightarrow \infty} \int_0^t \int_0^b F_0(\eta) G(r, z, \rho_1, \eta, t - \tau) d\eta d\tau = \\
&= \int_0^b F_0(\eta) \lim_{t \rightarrow \infty} \int_0^t G(r, z, \rho_1, \eta, t - \tau) d\tau d\eta. \tag{2.38}
\end{aligned}$$

Using similar methods as (2.35), equation (2.38) can partly submit in the following way:

$$\begin{aligned}
&\lim_{t \rightarrow \infty} \int_0^t \int_0^b F_0(\eta) G(r, z, \rho_1, \eta, t - \tau) d\eta d\tau = \\
&= \int_0^b F_0(\eta) \frac{\pi}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{Q_{m,n}(r, z, \rho_1, \eta)}{\lambda_n^2 + \mu_m^2} d\eta. \tag{2.39}
\end{aligned}$$

Joining equations (2.39) and (2.37) the solution for the wall in the following form is gained:

$$\begin{aligned}
U_0(r, z) = & \frac{\pi^2 \beta_0^0}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{A_n(r, \rho_0) \varphi_m(z) \sin \mu_m}{(\lambda_n^2 + \mu_m^2) \|\varphi_m\|^2} + \\
& + \frac{\pi^2 \beta_0}{2} \int_0^b F_0(\eta) \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{Q_{m,n}(r, z, \rho_1, \eta)}{\lambda_n^2 + \mu_m^2} d\eta.
\end{aligned} \tag{2.40}$$

Using Table 4.1 and formulas (2.24) – (2.26) are obtained:

$$\begin{aligned}
B_n = & (\lambda_n^2 + \gamma_0^2) [\gamma_0^0 J_0(\lambda_n \rho_0) + \lambda_n J_1(\lambda_n \rho_0)]^2 - \\
& - (\lambda_n^2 + (\gamma_0^0)^2) [\gamma_0 J_0(\lambda_n \rho_1) - \lambda_n J_1(\lambda_n \rho_1)]^2, \\
H_n(r) = & [\gamma_0^0 Y_0(\lambda_n \rho_0) + \lambda_n Y_1(\lambda_n \rho_0)] J_0(\lambda_n r) - \\
& - [\gamma_0^0 J_0(\lambda_n \rho_0) + \lambda_n J_1(\lambda_n \rho_0)] Y_0(\lambda_n r), \\
\|\varphi_m\|^2 = & \frac{1}{2} \mu_m^2 \left( 1 + \frac{\alpha \beta_0}{\mu_m^2 + (\alpha \beta_0)^2} \right).
\end{aligned} \tag{2.41}$$

But  $\lambda_n$  are the roots of the following transcendent equation:

$$\begin{aligned}
& [\gamma_0^0 J_0(\lambda_n \rho_0) + \lambda_n J_1(\lambda_n \rho_0)] [\gamma_0 Y_0(\lambda_n \rho_1) - \lambda_n Y_1(\lambda_n \rho_1)] - \\
& - [\gamma_0 J_0(\lambda_n \rho_1) - \lambda_n J_1(\lambda_n \rho_1)] [\gamma_0^0 Y_0(\lambda_n \rho_0) + \lambda_n Y_1(\lambda_n \rho_0)] = 0.
\end{aligned} \tag{2.42}$$

But  $\mu_m$  are positive roots of the following transcendent equation:

$$\mu_m = \beta_0 \operatorname{ctg} \mu_m, \text{ if } \alpha = 1, \text{ and } \operatorname{tg} \mu_m = 0, \text{ if } \alpha = 0. \tag{2.43}$$

Representation (2.40) can be rewritten in a shorter form (with known function  $\Phi_0(r, z)$ ):

$$U_0(r, z) = \Phi_0(r, z) + \frac{\pi^2 \beta_0}{2} \int_0^b F_0(\eta) \tilde{G}_0(r, z, \rho_1, \eta) d\eta, \tag{2.44}$$

where  $\Phi_0(r, z)$  and  $\tilde{G}_0(r, z, \rho_1, \eta)$  are given by expressions:

$$\Phi_0(r, z) = \frac{\pi^2 \beta_0^0}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{A_n(r, \rho_0) \varphi_m(z) \sin \mu_m}{(\lambda_n^2 + \mu_m^2) \|\varphi_m\|^2}, \tag{2.45}$$

$$\tilde{G}_0(r, z, \rho_1, \eta) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{Q_{m,n}(r, z, \rho_1, \eta)}{\lambda_n^2 + \mu_m^2}. \tag{2.46}$$

The representation (2.44)-(2.46) is not the solution because of unknown function  $F_0(\eta)$ .

#### 2.4.2 The Separate 2D Problem for the Fin

Conjugation conditions (2.13) are written in the following form:

$$\left. \left( \frac{\partial U}{\partial r} - \gamma U \right) \right|_{r=\rho_1+0} = \gamma F(z), F(z) = \left. \left( \frac{1}{\gamma_0} \frac{\partial U_0}{\partial r} - U_0 \right) \right|_{r=\rho_1-0}. \quad (2.47)$$

Values and conditions unified with the nomenclature of the problem described in chapter 2.3 represented in Table 2.2.

Table 2.2. Formulation of problem for fin.

$g_i$	$k_i$	Other values
$g_1(z, t) = \gamma F(z), z \in [0, b]$	$k_1 = \gamma$	$a = 1$
$g_2(z, t) = 0$	$k_2 = \gamma$	$\tilde{\rho}_1 = \rho_1$ $\tilde{\rho}_2 = \rho_2$
$g_3(r, t) = 0$	$k_3 = 0$	$\tilde{l} = b$
$g_4(r, t) = 0$	$k_4 = \beta$	$\Phi(r, z, t) = 0$ $U(r, z, t) = U(r, z)$

In the equation of the fin, similar to the case of the wall, some of addends are zero because of the homogeneity of the boundary conditions and the problem. Using (2.20) solution for the fin can be represented in the following form:

$$U(r, z) = -2\pi\rho_1 \int_0^t \int_0^b \gamma F(\eta) G(r, z, \rho_1, \eta, t - \tau) d\eta d\tau. \quad (2.48)$$

Changing order of integration and using same idea as in (2.35) we obtain:

$$U(r, z) = -\frac{\pi^2 \beta}{2} \int_0^b F(\eta) \tilde{G}(r, z, \rho_1, \eta) d\eta, \quad (2.49)$$

$$\text{where } \tilde{G}(r, z, \rho_1, \eta) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{Q_{k,l}^*(r, z, \rho_1, \eta)}{(\lambda_k^*)^2 + (\mu_l^*)^2}, \quad (2.50)$$

$$Q_{k,l}^*(r, z, \xi, \eta) = A_k^*(r, \xi) \cdot X_l^*(z, \eta),$$

$$\frac{(\lambda_k^*)^2}{B_k^*} \left[ \gamma J_0(\lambda_k^* \rho_2) - \lambda_k^* J_1(\lambda_k^* \rho_2) \right]^2 H_k^*(r) H_k^*(\xi) \stackrel{\text{denoted}}{=} A_k^*(r, \xi),$$

$$\frac{\varphi_l^*(z) \varphi_l^*(\eta)}{\|\varphi_l^*\|^2} \stackrel{\text{denoted}}{=} X_l^*(z, \eta) \text{ and } \varphi_l^*(z) = \mu_l^* \cos(\mu_l^* z).$$

Using Table 4.2 and expressions (2.24) – (2.26) following equations obtained:

$$\begin{aligned} B_k^* &= \left( (\lambda_k^*)^2 + \gamma^2 \right) \left( \left[ \gamma J_0(\lambda_k^* \rho_1) + \lambda_k^* J_1(\lambda_k^* \rho_1) \right]^2 - \left[ \gamma J_0(\lambda_k^* \rho_2) - \lambda_k^* J_1(\lambda_k^* \rho_2) \right]^2 \right), \\ H_k^*(r) &= \left[ \gamma Y_0(\lambda_k^* \rho_1) + \lambda_k^* Y_1(\lambda_k^* \rho_1) \right] J_0(\lambda_k^* r) - \\ &\quad - \left[ \gamma J_0(\lambda_k^* \rho_1) + \lambda_k^* J_1(\lambda_k^* \rho_1) \right] Y_0(\lambda_k^* r), \end{aligned} \quad (2.51)$$

$$\|\varphi_l^*\|^2 = \frac{(\mu_l^*)^2}{2} \left( b + \frac{\beta}{(\mu_l^*)^2 + \beta^2} \right),$$

here  $\lambda_k^*$  and  $\mu_l^*$  are positive roots of the following transcendental equations:

$$\begin{aligned} & \left[ \gamma J_0(\lambda_k^* \rho_1) + \lambda_k^* J_1(\lambda_k^* \rho_1) \right] \left[ \gamma Y_0(\lambda_k^* \rho_2) - \lambda_k^* Y_1(\lambda_k^* \rho_2) \right] - \\ & - \left[ \gamma J_0(\lambda_k^* \rho_2) - \lambda_k^* J_1(\lambda_k^* \rho_2) \right] \left[ \gamma Y_0(\lambda_k^* \rho_1) + \lambda_k^* Y_1(\lambda_k^* \rho_1) \right] = 0, \\ & \mu_l^* = \beta \operatorname{ctg}(\mu_l^* b). \end{aligned} \quad (2.52)$$

### 2.4.3 The Conjugation of Two Separate Problems

Easily from representations (2.47) together with (2.44) and (2.28) together with (2.49), the two following equalities are obtained:

$$F(z) = \tilde{\Phi}(\rho_1, z) - \frac{\pi^2 \rho_1}{2} \int_0^b F_0(\eta_0) \tilde{G}_0(\rho_1, z, \rho_1, \eta_0) d\eta_0, \quad (2.53)$$

$$F_0(\eta_0) = -\frac{\pi^2 \rho_1}{2} \int_0^b F(\eta) \tilde{G}(\rho_1, \eta_0, \rho_1, \eta) d\eta, \quad (2.54)$$

here:

$$\begin{aligned} \tilde{\Phi}(r, z) &= \frac{1}{\gamma_0} \frac{\partial \Phi_0}{\partial r} - \Phi_0, \\ \tilde{G}_0(r, z, \rho_1, \eta_0) &= \gamma_0 \tilde{G}_0 - \frac{\partial \tilde{G}_0}{\partial r}, \\ \tilde{G}(r, z, \rho_1, \eta) &= \frac{\partial \tilde{G}}{\partial r} + \gamma \tilde{G}. \end{aligned} \quad (2.55)$$

Equation (2.54) substituted into (2.53) allows writing out the following second kind of Fredholm's integral equation:

$$F(z) = \tilde{\Phi}(\rho_1, z) + \int_0^b F(\eta) \cdot \Gamma(z, \eta) d\eta, \quad (2.56)$$

here:

$$\Gamma(z, \eta) = \left( \frac{\pi^2 \rho_1}{2} \right)^2 \int_0^b \tilde{G}(\rho_1, \eta_0, \rho_1, \eta) \cdot \tilde{G}_0(\rho_1, z, \rho_1, \eta_0) d\eta_0. \quad (2.57)$$

This is an inhomogeneous integral Fredholm equation of the second kind (2.56) by the given kernel  $\Gamma(z, \eta)$ . Equation (2.57) has exactly one solution [39]. Knowing  $F(z)$  we can find from representation (2.49) the solution for the fin.

Fredholm's method to obtain the solution of (2.56) is used. At first, interval  $z \in (0, b)$  into  $N$  exact parts  $z_i$ ,  $i = \overline{0, N}$  are divided, thus  $z_0 = 0$ , but  $z_N = b$ . Now

equation (2.56) considered in separate points  $z_i$ ,  $i = \overline{0, N}$  can be rewritten in the form of a system of linear equations:

$$F_i = \tilde{\Phi}_i + \sum_{j=0}^N \Gamma_{ij} F_j h, \quad (2.58)$$

where:

$$F(z_i) = F_i, \quad \tilde{\Phi}(z_i) = \tilde{\Phi}_i, \quad \Gamma(z_i, z_j) = \Gamma_{ij} \text{ un } z_{i+1} - z_i = h = \frac{b-0}{N}. \quad (2.59)$$

In this method we replace the integral with rectangular squaring formula, but it can be replaced by a trapezium or any other squaring formula. It is necessary to express  $\tilde{\Phi}(z)$  and  $\Gamma(z, \eta)$  to solve the system of linear equations (2.58). Let's express function  $\tilde{\Phi}(z_i)$  using (2.55) and (2.45):

$$\begin{aligned} \tilde{\Phi}_i &= \frac{\pi^2 \beta_0^0}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{\gamma_0} \frac{\partial A_n(\rho_1, \rho_0)}{\partial r} \frac{\varphi_m(z_i) \sin \mu_m}{(\lambda_n^2 + \mu_m^2) \|\varphi_m\|^2} - \\ &- \frac{\pi^2 \beta_0^0}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{A_n(\rho_1, \rho_0) \varphi_m(z_i) \sin \mu_m}{(\lambda_n^2 + \mu_m^2) \|\varphi_m\|^2} = \\ &= \frac{\pi^2 \beta_0^0}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left( \frac{1}{\gamma_0} \frac{\partial A_n(\rho_1, \rho_0)}{\partial r} - A_n(\rho_1, \rho_0) \right) \frac{\varphi_m(z_i) \sin \mu_m}{(\lambda_n^2 + \mu_m^2) \|\varphi_m\|^2}, \end{aligned} \quad (2.60)$$

where:

$$\begin{aligned} \frac{\partial A_n(r, \xi)}{\partial r} &= \frac{\lambda_n^2}{B_n} [\gamma_0 J_0(\lambda_n \rho_1) - \lambda_n J_1(\lambda_n \rho_1)]^2 H'_n(r) H_n(\xi) \text{ and} \\ H'_n(r) &= [\gamma_0^0 J_0(\lambda_n \rho_0) + \lambda_n J_1(\lambda_n \rho_0)] \lambda_n Y_1(\lambda_n r) - \\ &- [\gamma_0^0 Y_0(\lambda_n \rho_0) + \lambda_n Y_1(\lambda_n \rho_0)] \lambda_n J_1(\lambda_n r). \end{aligned} \quad (2.61)$$

It is necessary to express  $\tilde{G}_0(\rho_1, z_i, \rho_1, \eta_0)$  and  $\tilde{G}(\rho_1, \eta_0, \rho_1, z_j)$  for  $\Gamma_{ij}$ .

From equations (2.55) and (2.46) follows:

$$\tilde{G}_0(\rho_1, z_i, \rho_1, \eta_0) = \gamma_0 \tilde{G}_0(\rho_1, z_i, \rho_1, \eta_0) - \frac{\partial \tilde{G}_0(\rho_1, z_i, \rho_1, \eta_0)}{\partial r}. \quad (2.62)$$

Equations (2.31), (2.35) and (2.46) inserted into (2.62) can be written in the following form:

$$\begin{aligned} \tilde{G}_0(\rho_1, z_i, \rho_1, \eta_0) &= \gamma_0 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_n(\rho_1, \rho_1) \frac{X_m(z_i, \eta_0)}{\lambda_n^2 + \mu_m^2} - \\ &- \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\partial A_n(\rho_1, \rho_1)}{\partial r} \frac{X_m(z_i, \eta_0)}{\lambda_n^2 + \mu_m^2}, \\ \tilde{G}_0(\rho_1, z_i, \rho_1, \eta_0) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \tilde{A}_n \frac{X_m(z_i, \eta_0)}{\lambda_n^2 + \mu_m^2}, \end{aligned} \quad (2.63)$$

where:

$$\tilde{A}_n^{\text{denoted}} = \gamma_0 A_n(\rho_1, \rho_1) - \frac{\partial A_n(\rho_1, \rho_1)}{\partial r}. \quad (2.64)$$

Now, only left to find is the expression of  $\tilde{G}(\rho_1, \eta_0, \rho_1, z_j)$ . From formula (2.55) is obtained:

$$\tilde{G}(\rho_1, \eta_0, \rho_1, z_j) = \frac{\partial \tilde{G}(\rho_1, \eta_0, \rho_1, z_j)}{\partial r} + \gamma \tilde{G}(\rho_1, \eta_0, \rho_1, z_j). \quad (2.65)$$

From equations (2.65) and (2.50) follow:

$$\begin{aligned} \tilde{G}(\rho_1, \eta_0, \rho_1, z_j) &= \gamma \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} A_k^*(\rho_1, \rho_1) \frac{X_l^*(\eta_0, z_j)}{(\lambda_k^*)^2 + (\mu_l^*)^2} + \\ &\quad + \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{X_l^*(\eta_0, z_j)}{(\lambda_k^*)^2 + (\mu_l^*)^2} \frac{\partial A_k^*(\rho_1, \rho_1)}{\partial r} \\ \tilde{G}(\rho_1, \eta_0, \rho_1, z_j) &= \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \tilde{A}_k^* \frac{X_l^*(\eta_0, z_j)}{(\lambda_k^*)^2 + (\mu_l^*)^2}, \end{aligned} \quad (2.66)$$

where:

$$\tilde{A}_k^{\text{denoted}} = \gamma A_k^*(\rho_1, \rho_1) + \frac{\partial A_k^*(\rho_1, \rho_1)}{\partial r}. \quad (2.67)$$

Now from expressions (2.57), (2.63) and (2.66) follow:

$$\Gamma(z_i, z_j) = \left( \frac{\pi^2 \rho_1}{2} \right)^2 \int_0^b \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \tilde{A}_n \frac{X_m(z_i, \eta_0)}{\lambda_n^2 + \mu_m^2} \cdot \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \tilde{A}_k^* \frac{X_k^*(\eta_0, z_j)}{(\lambda_k^*)^2 + (\mu_l^*)^2} d\eta_0. \quad (2.68)$$

With the help of equations (2.31) and (2.50) expression (2.68) is written in the next form:

$$\begin{aligned} \Gamma(z_i, z_j) &= \left( \frac{\pi^2 \rho_1}{2} \right)^2 \int_0^b \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{\tilde{A}_k^* \cdot \tilde{A}_n \cdot \varphi_l^*(\eta_0) \cdot \varphi_l^*(z_j) \cdot \varphi_m(z_i) \cdot \varphi_m(\eta_0)}{\left( (\lambda_k^*)^2 + (\mu_l^*)^2 \right) \cdot (\lambda_n^2 + \mu_m^2) \cdot \|\varphi_l^*\|^2 \cdot \|\varphi_m\|^2} d\eta_0 = \\ &= \left( \frac{\pi^2 \rho_1}{2} \right)^2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{\tilde{A}_k^* \cdot \tilde{A}_n \cdot \varphi_l^*(z_j) \cdot \varphi_m(z_i) \cdot \int_0^b \varphi_l^*(\eta_0) \cdot \varphi_m(\eta_0) d\eta_0}{\left( (\lambda_k^*)^2 + (\mu_l^*)^2 \right) \cdot (\lambda_n^2 + \mu_m^2) \cdot \|\varphi_l^*\|^2 \cdot \|\varphi_m\|^2} = \\ &= \left( \frac{\pi^2 \rho_1}{2} \right)^2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{\tilde{A}_k^* \cdot \tilde{A}_n \cdot \varphi_l^*(z_j) \cdot \varphi_m(z_i) \cdot \Psi_{m,l}}{\left( (\lambda_k^*)^2 + (\mu_l^*)^2 \right) \cdot (\lambda_n^2 + \mu_m^2) \cdot \|\varphi_l^*\|^2 \cdot \|\varphi_m\|^2}, \end{aligned} \quad (2.69)$$

where:

$$\Psi_{k,m} = \frac{1}{2} \frac{\mu_m \mu_l^*}{\mu_l^{*2} - \mu_m^2} \begin{pmatrix} \sin[b\mu_m(\mu_l^* - \mu_m)] + \sin[b\mu_l^*(\mu_l^* - \mu_m)] \\ -\sin[b\mu_m(\mu_l^* + \mu_m)] + \sin[b\mu_l^*(\mu_l^* + \mu_m)] \end{pmatrix}. \quad (2.70)$$



The system of linear equation (2.58) contains unknown values of function  $F(z)$  at fixed  $z_i$ ,  $i = \overline{0, N}$ . Function  $\Gamma(z_i, z_j)$  (2.70) and  $\tilde{\Phi}(z_i)$  (2.60) are stated with analytical expressions and can be calculated at any value of  $z_i$  and  $z_j$ . The further set of values of  $F(z_i)$  is obtained. Now, again integral in expression (2.49) with the rectangular squaring formula is replaced in following way:

$$U(r, z) = -\frac{\pi^2 \beta}{2} \sum_{j=0}^N F(z_j) \tilde{G}(r, z, \rho_1, z_j) h. \quad (2.71)$$

Heat transfer at any point of the fin using expression (2.71) can be calculated. Now, easily the values of second unknown function  $F_0(z)$  can also be calculated. Equations (2.50) and (2.71) pasted in formula (2.28):

$$\begin{aligned} F_0(z) &= \left( \frac{1}{\gamma} \frac{\partial U}{\partial r} + U \right) \Big|_{r=\rho_1+0} = \\ &= -\frac{\pi^2 \beta}{2} \sum_{j=0}^N F(z_j) \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{X_l^*(z, z_j) A_k^*(\rho_1, \rho_1)}{(\lambda_k^*)^2 + (\mu_l^*)^2} h - \\ &= -\frac{\pi^2 \beta}{2 \gamma} \sum_{j=0}^N F(z_j) \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{X_l^*(z, z_j)}{(\lambda_k^*)^2 + (\mu_l^*)^2} \frac{\partial A_k^*(\rho_1, \rho_1)}{\partial r} h = \\ &= -\frac{\pi^2 \rho_1 h}{2} \sum_{j=0}^N F(z_j) \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{X_l^*(z, z_j)}{\lambda_k^{*2} + \mu_l^{*2}} \tilde{A}_k^* = \\ &= -\frac{\pi^2 \rho_1 h}{2} \sum_{j=0}^N F(z_j) \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{\varphi_l^*(z) \varphi_l^*(z_j)}{\|\varphi_l^*\|^2} \frac{\tilde{A}_k^*}{\lambda_k^{*2} + \mu_l^{*2}}. \end{aligned} \quad (2.72)$$

And one more time, the rectangular squaring formula for the integral of formula (2.44) is applied, and by means of formula (2.46) and (2.72), the value of heat transfer at any point of the wall can be calculated:

$$U_0(r, z) = \Phi_0(r, z) + \frac{\pi^2 \beta_0}{2} \sum_{i=0}^N F_0(z_i) \tilde{G}_0(r, z, \rho_1, z_i) h. \quad (2.73)$$

Equations (2.71) and (2.73) are the solutions of problem (2.8) – (2.17) when  $q(r) = q_1(r) = q_0(r) \equiv 0$  and  $Q(r, z) \equiv 0$  (2.27).

## 2.5 Solution by Non-homogeneous Environment Temperature

In this chapter, the case of non-homogeneous equations and non-homogeneous boundary conditions is considered.

### 2.5.1 The Statement of the Full Mathematical Problem

As the main equations for the wall and the fin we take differential equations (2.8), (2.14):

$$\begin{aligned} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial U_0}{\partial r} \right) + \frac{\partial^2 U_0}{\partial z^2} + Q_0(r, z) &= 0, \\ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial U}{\partial r} \right) + \frac{\partial^2 U}{\partial z^2} + Q(r, z) &= 0. \end{aligned} \quad (2.74)$$

The non homogenous BC for the wall are as follows:

$$\begin{aligned} \frac{\partial U_0}{\partial r} - \gamma_0^0 U_0 &= -\gamma_0^0 g_0(z), \quad z \in [0, 1], \quad r = \rho_0, \\ \frac{\partial U_0}{\partial r} + \gamma_0 U_0 &= \gamma_0 g(\rho_1, z), \quad z \in [b, 1], \quad r = \rho_1, \\ \frac{\partial U_0}{\partial z} &= q_0(r), \quad r \in [\rho_0, \rho_1], \quad z = 0, \\ \frac{\partial U_0}{\partial z} + \beta_0 U_0 &= q_1(r), \quad r \in [\rho_0, \rho_1], \quad z = 1. \end{aligned} \quad (2.75)$$

Similar are the non homogeneous boundary conditions for the fin:

$$\begin{aligned} \frac{\partial U}{\partial z} + \beta U &= \beta g(r, b), \quad r \in [\rho_1, \rho_2], \quad z = b, \\ \frac{\partial U}{\partial r} + \gamma U &= \gamma g(\rho_2, z), \quad z \in [0, b], \quad r = \rho_2, \\ \frac{\partial U}{\partial z} &= q(r), \quad r \in [\rho_1, \rho_2], \quad z = 0. \end{aligned} \quad (2.76)$$

To complete the full statement of generalised problem (2.74)-(2.76) the solution must fulfil the conjugations conditions (2.13):

$$U_0|_{\rho=\rho_1-0} = U|_{\rho=\rho_1+0}, \quad \gamma \frac{\partial U_0}{\partial r} \Big|_{\rho=\rho_1-0} = \gamma_0 \frac{\partial U}{\partial r} \Big|_{\rho=\rho_1+0}. \quad (2.77)$$

### 2.5.2 The Separate Solutions for the Wall and the Fin

In the same way as in the sub-section 2.4.1, the notation (2.28) is introduced. Then the solution in the wall can again be presented in the same form as (2.44):

$$U_0(r, z) = \Upsilon_0(r, z) + \frac{\pi^2 \beta_0}{2} \int_0^b F_0(\eta) \tilde{G}_0(r, z, \rho_1, \eta) d\eta. \quad (2.78)$$

Now the expression for the first term of the right hand side has a significantly more complicated form:

$$\begin{aligned} \Upsilon_0(r, z) = & 2\pi\beta_0 \int_b^1 \mathcal{G}(\rho_1, \eta_0) \tilde{G}_0(r, z, \rho_1, \eta_0) d\eta_0 + \\ & + 2\pi\beta_0^0 \int_0^1 \mathcal{G}_0(\eta_0) \tilde{G}_0(r, z, \rho_0, \eta_0) d\eta_0 + \\ & + 2\pi \int_{\rho_0}^{\rho_1} \xi_0 q_1(\xi_0) \tilde{G}_0(r, z, \xi_0, 1) d\xi_0 - \\ & - 2\pi \int_{\rho_0}^{\rho_1} \xi_0 q_0(\xi_0) \tilde{G}_0(r, z, \xi_0, 0) d\xi_0 + \\ & + 2\pi \int_{\rho_0}^{\rho_1} \xi_0 d\xi_0 \int_0^1 Q_0(\xi_0, \eta_0) \tilde{G}_0(r, z, \rho_0, \eta_0) d\eta_0. \end{aligned} \quad (2.79)$$

The solution for the fin with boundary conditions (2.76) can be presented in a similar way as for the wall:

$$U(r, z) = \Upsilon(r, z) - \frac{\pi^2 \beta}{2} \int_0^b F(\eta) \tilde{G}(r, z, \rho_1, \eta) d\eta. \quad (2.80)$$

The expression for the first term of right hand side has the form:

$$\begin{aligned} \Upsilon(r, z) = & 2\pi\gamma\rho_2 \int_0^b \mathcal{G}(\rho_2, \eta) \tilde{G}(r, z, \rho_2, \eta) d\eta + \\ & + 2\pi\beta \int_{\rho_1}^{\rho_2} \xi \mathcal{G}(\xi, b) \tilde{G}(r, z, \xi, b) d\xi - \\ & - 2\pi \int_{\rho_1}^{\rho_2} \xi q(\xi) \tilde{G}(r, z, \xi, 0) d\xi + \\ & + 2\pi \int_{\rho_1}^{\rho_2} \xi d\xi \int_0^b Q(\xi, \eta) \tilde{G}(r, z, \xi, \eta) d\eta. \end{aligned} \quad (2.81)$$

### 2.5.3 The Junction of Solutions for the Wall and the Fin

The following notations are introduced:

$$\tilde{Y}_0(r, z) = \frac{1}{\gamma_0} \frac{\partial Y_0}{\partial r} - Y_0, \quad \tilde{Y}(r, z) = \frac{1}{\gamma} \frac{\partial Y}{\partial r} + Y, \quad (2.82)$$

$$L_0(r, z, \xi, \eta) = \gamma_0 \tilde{G}_0 - \frac{\partial \tilde{G}_0}{\partial r}, \quad L(r, z, \rho_1, \eta) = \frac{\partial \tilde{G}}{\partial r} + \gamma \tilde{G}.$$

The representations (2.79) and (2.81) allow to easily obtain the following two equations:

$$F(z) = \tilde{Y}_0(\rho_1, z) - 2\pi\rho_1 \int_0^b F_0(\eta_0) L_0(\rho_1, z, \rho_1, \eta_0) d\eta_0, \quad (2.83)$$

$$F_0(z) = \tilde{Y}(\rho_1, z) - 2\pi\gamma\rho_1 \int_0^b F(\eta) L(\rho_1, z, \rho_1, \eta) d\eta.$$

From this system (2.83) the following second kind of Fredholm integral equation is established:

$$F(z) = \Psi_0(\rho_1, z) + \int_0^b F(\eta) \Gamma(z, \eta) d\eta. \quad (2.84)$$

Here the kernel of the Fredholm integral equation is given by the same formula as (2.57):

$$\Gamma(z, \eta) = \left( \frac{\pi\rho_1}{2} \right)^2 \int_0^b L_0(\rho_1, z, \rho_1, \eta_0) \cdot L(\rho_1, z, \rho_1, \eta) d\eta_0. \quad (2.85)$$

In its turn, the first term in the right hand side has a more complicated expression:

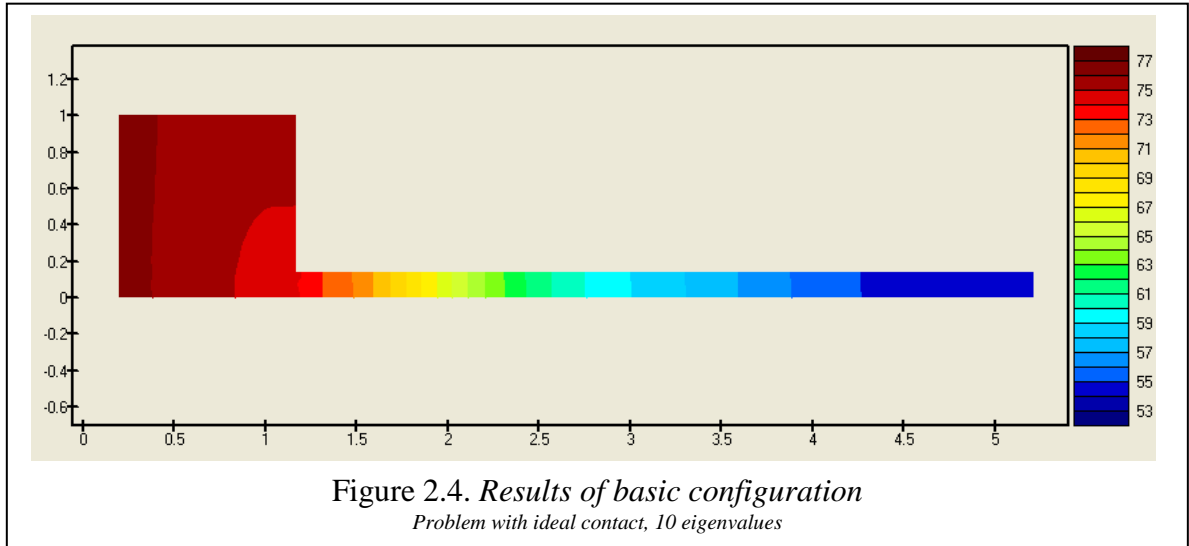
$$\Psi_0(\rho_1, z) = \tilde{Y}_0(\rho_1, z) - 2\pi\rho_1 \int_0^b \tilde{Y}(\rho_1, \eta_0) L_0(\rho_1, z, \rho_1, \eta_0) d\eta_0. \quad (2.86)$$

Evidently this second kind of Fredholm integral equation (2.84) has exact one solution [39]. Again, by the known  $F(z)$ , the representation (4.80) allows to find the solution for the fin. In a similar way, the integral equation for the function  $F_0(z)$  can be constructed and the solution for the wall obtained.

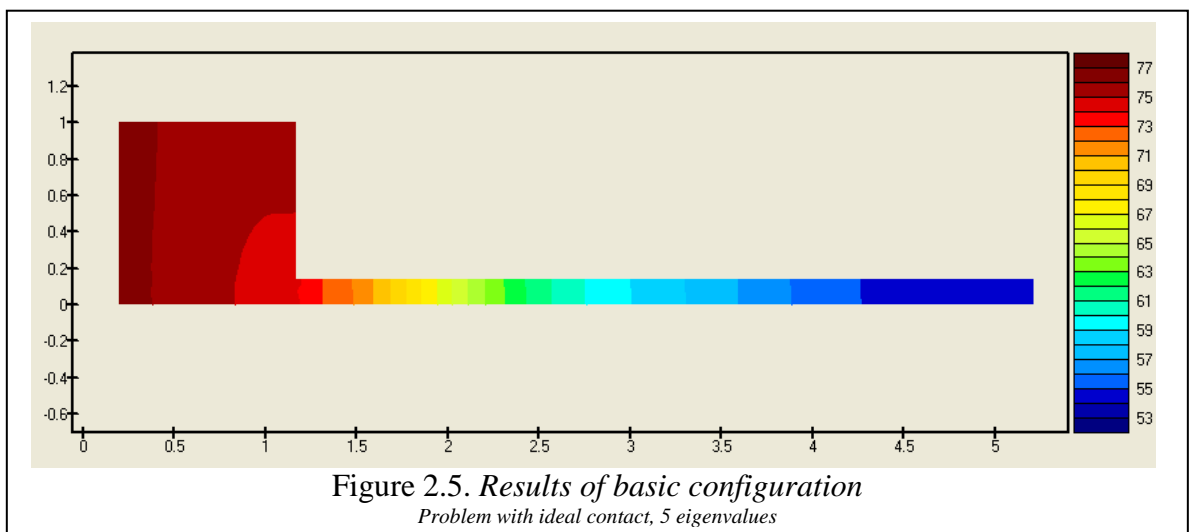
## 2.6 Calculations and Results

A mathematical model for the problem described in chapter 2.4 was developed. The visualisation of the results and principles applied are the same as in section 1.6. The parameters used in calculations are shown in tables 1.2 and 1.3 and correspondingly three cases (1; 2 and 5) were considered. The cases are shown in table 1.4.

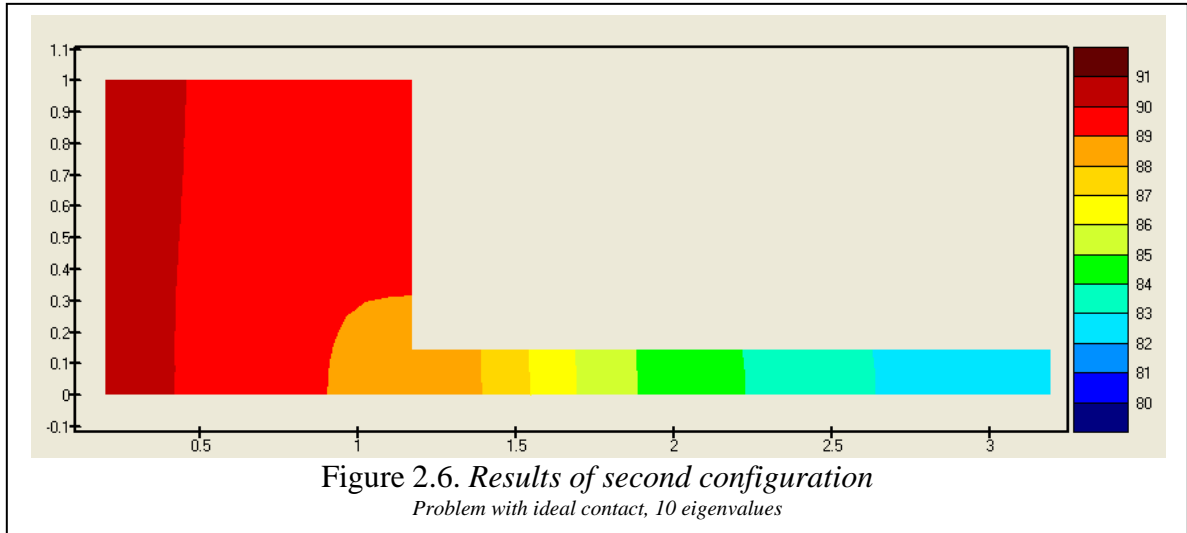
The basic case of configuration where the Green's function method is used for the solution is shown in Fig. 2.4. The analytical result by method of Green's function can be easily compared with the solution shown in Fig. 1.5. Evidently, both the 1D and 2D semi-analytical results obtained using the method of conservative averaging and the analytical 2D solution using the Green's function method have no significant differences.



When the calculations are performed using the Green's function method, it is necessary to take into account the time needed for computation. In Fig.2.4 the first 10 eigenvalues are used and the computation time was approx. 5 minutes. However, when the number of eigenvalues is increased the computational time increases significantly. In Figure 2.5, the result using the same input dataset as in Fig. 2.4 is shown, however, this time only the first five eigenvalues are used. It is evident that the use of only the first five eigenvalues is sufficient to produce a representative result.



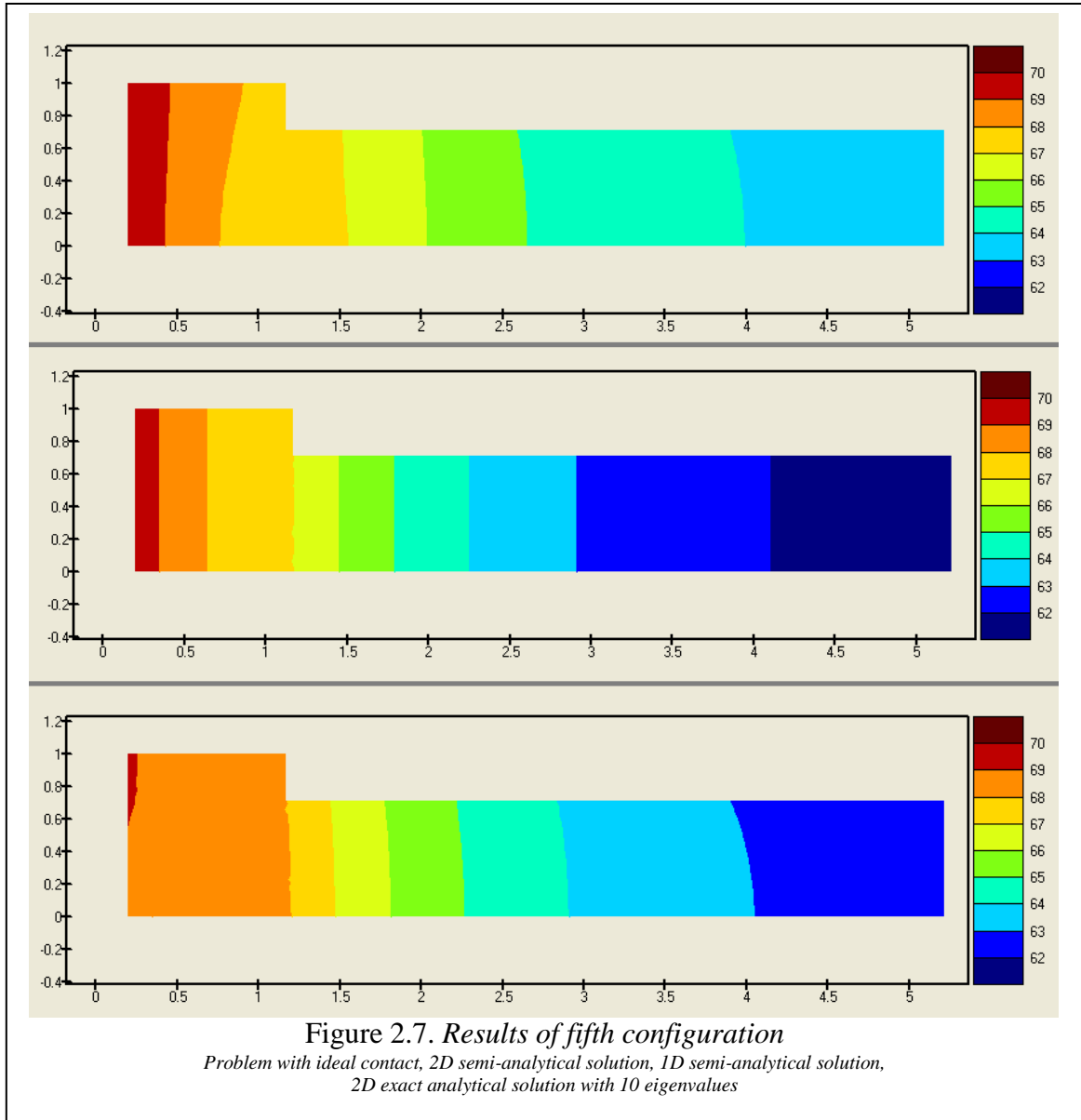
In Figure 2.6, the result using the second parameter data set is shown. A comparison with the semi-analytical solution shows that the most pronounced differences are in the fin; in general the average temperature difference is approx. 2°C.



In Figure 2.7, the semi-analytical 2D, 1D and analytical 2D results are shown. It is evident, that the features of the solution are similar; however there are small differences in numerical values.

Depending on the nature of the physical problem in question and the precision constraints of the results, it is possible to choose any of the models developed in Chapter 1 and 2. Each one of them has its advantages and disadvantages:

- The derivation of the 1D semi-analytical solution and the amount of code that needs to be written are comparatively small. When the calculations are carried out using the model, the result is acquired instantaneously. A significant flaw is the fact that only one dimension is considered and no understanding about the temperature distribution in the 2D domain is gained.
- The mathematical derivation of the 2D semi-analytical solution requires careful work and the amount of code increases when compared to the 1D model. From a practical point of view, the possibility of errors increases when the amount of derivation and code increases. However, it is possible to relatively easily identify and rectify the errors because the solution is reduced to the analytical solution of a system of linear equations. The result can be obtained in a few milliseconds.



- The analytical solution in 2D is the most time consuming and most voluminous of the models considered in this work. The numerical solution reduces to a number of different sub-problems – finding eigenvalues, solving a Fredholm integral equation and a number of multiple sums. Some of the sub-steps in the solution were compared with the results from Maple software to ascertain the congruence and the correctness of the results. This model is especially non-transparent because of the number of multiple sums, and even if the smallest mistakes are made it is hard to identify their cause. However, the advantage is that it is very clear that the result of the computation is the solution of the problem. The disadvantage of this model is the time needed for computation that, depending on the precision needed requires more and more resources. The increase of time needed is not linear but exponential. Most of the resources are used for finding the

eigenvalues (formulas (1.42), (1.43) and (1.52)) and the computation of values for different terms that consists of the construction of multiple sums.

## **2.7 Conclusions**

Two 2-dimensional analytical solutions (in both cases: homogeneous chapter (2.4) and non-homogeneous chapter (2.5) environment) for a system with cylindrical fin when the wall and the fin consist of materials with different thermal properties is constructed.

Numerical mathematical model of homogeneous solution is produced. Some examples of calculation and main benefits and disadvantages of technical performance are represented.



### 3. ANALYTICAL SOLUTIONS FOR NON STEADY-STATE HYPERBOLIC HEAT EXCHANGE EQUATION

#### 3.1 Problem Introduction

The intensive quenching method was patented several decades ago [15], [40], [14]. This method of intensive quenching is offered in water in contrast to other hardening methods in oil. To describe the standard hardening process, the classical equation is used [41], [42]. Dr. N. Kobasko has kindly provided the experimental result shown in Figure 3.1. Curve Nr. 2 on this figure shows that the solution has the character of an equation of hyperbolic type. In [16] it was proposed to describe the quenching process by the hyperbolic heat equation. In the paper it was concluded that hyperbolic type heat exchange equation better describes steel quenching physical processes.

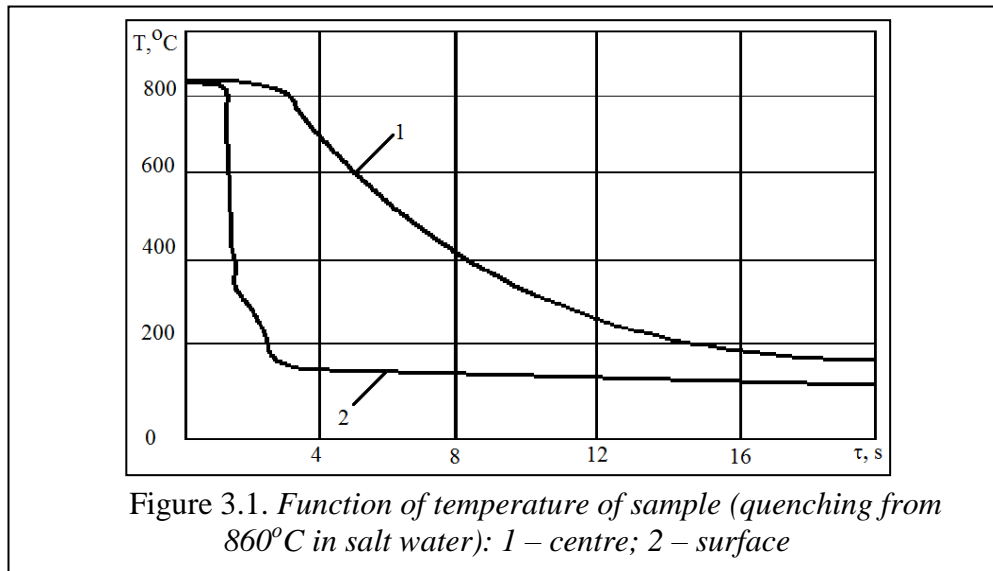


Figure 3.1. Function of temperature of sample (quenching from 860°C in salt water): 1 – centre; 2 – surface

Intensive quenching technical processes as well as mathematical models meet with several difficulties caused by a number of reasons. Firstly, if cooling process of part exceeds some critical value thin layer of water steam can form on the surface of part and so called film boiling can begin [2]. During film boiling process component is often fractured [13], [16], [43], [44], [45]. Then nucleate boiling process caused by fractures begins, which means that, non linear boundary conditions are required for mathematical process description. Secondly, there are no exact mathematical verified solutions for parts with more complicated geometry. Usually Kondratjev number (form factor) is used to reduce a complex problem to a problem with simpler domain. Thirdly, solution of hyperbolic heat exchange equation requires additional initial conditions (heat flux), that are not known; therefore issue is reduced to inverse non steady-state problem.

Understanding of intensive quenching process requires lot of experimental work including mathematical modelling as well. Because of the unknown values of

initial heat flux, there are two additional conditions at the moment  $t = T$  of process are given for further discussed problems.

Time inverted hyperbolic heat exchange problem for cylinder is discussed in chapter 3.2. Closed form 1D solution for a thin cylinder is obtained.

In chapter 3.3, a non steady-state solution for the hyperbolic heat equation for a cylindrical sample with fin is constructed. Usually mathematical modelling of systems with extended surfaces is realised by one dimensional steady-state assumptions [1], [3]. In [4], [5], [26], [28] two and three dimensional semi-analytical and analytical solutions for the steady-state process are constructed. This technique gives a more suitable form of the solution in the form of the Fredholm integral equation. In this chapter, the exact 3D problem is reduced to 2D and analytical solutions by the Green function method are obtained.

### 3.2 Mathematical Formulation of 3D Non Steady-State Hyperbolic Heat Exchange Problem for Cylinder

Definition of problem in domain  $r \in (0, R)$ ,  $z \in (0, H)$ ,  $t \in (0, T]$ ,  $\varphi \in [0, 2\pi]$  is as follows:

$$\tau_r \frac{\partial^2 U}{\partial t^2} + \frac{\partial U}{\partial t} = a^2 \left[ r^{-1} \frac{\partial}{\partial r} \left( r \frac{\partial U}{\partial r} \right) + \frac{\partial^2 U}{\partial z^2} + r^{-2} \frac{\partial^2 U}{\partial \varphi^2} \right], \quad (3.1)$$

where  $\tau$  is a known relaxation parameter (constant) from experimental results.

The additional following boundary conditions when  $t \in [0, T]$ ,  $\varphi \in [0, 2\pi]$  are formulated:

$$\frac{\partial U}{\partial r} + k_1 U = \gamma_1(z, \varphi, t), \quad r = R, \quad z \in [0, H], \quad (3.2)$$

$$\frac{\partial U}{\partial z} + k_2 U = \gamma_2(r, \varphi, t), \quad r \in [0, R], \quad z = H, \quad (3.3)$$

$$\frac{\partial U}{\partial z} - k_3 U = \gamma_3(r, \varphi, t), \quad r \in [0, R], \quad z = 0. \quad (3.4)$$

Periodical (continuity) boundary conditions when  $r \in (0, R)$ ,  $z \in (0, H)$ ,  $t \in (0, T]$  are the following:

$$U(r, z, 0, t) = U(r, z, 2\pi, t), \quad \frac{\partial U(r, z, 0, t)}{\partial r} = \frac{\partial U(r, z, 2\pi, t)}{\partial r}. \quad (3.5)$$

Additional condition in the centre is the following:

$$r \frac{\partial U}{\partial r} \rightarrow 0, \quad r \rightarrow 0, \quad z \in [0, Z], \quad t \in [0, T]. \quad (3.6)$$

Usually for hyperbolic heat equation, the two following initial conditions are required:

$$U = u^0(r, z, \varphi), \quad t = 0, \quad r \in [0, R], \quad z \in [0, H], \quad (3.7)$$

$$\frac{\partial U}{\partial t} = V_0(r, z, \varphi), \quad t = 0, \quad r \in [0, R], \quad z \in [0, H]. \quad (3.8)$$

Value  $V_0(r, z, \varphi)$  of initial condition (3.8) is usually unknown. Therefore two additional conditions at the moment  $t = T$  of process are given instead of initial condition (3.8):

$$U(r, z, \varphi, T) = U_T(r, z, \varphi), \quad \frac{\partial U}{\partial t}(r, z, \varphi, T) = V_T(r, z, \varphi) \quad (3.9)$$

### 3.2.1 Solution for Cylinder

The simplest case when BC (3.2)-(3.4), initial conditions (3.7), (3.8) and additional conditions (3.9) on the right hand side depend on two arguments  $r, z$  (it means practically the thin cylinder or cylinder with rotation symmetry):

$$\frac{\partial U}{\partial r} + k_1 U = \gamma_1(z, t), \quad r = R, \quad z \in [0, H], \quad (3.10)$$

$$\frac{\partial U}{\partial z} + k_2 U = \gamma_2(r, t), \quad r \in [0, R], \quad z = H, \quad (3.11)$$

$$\frac{\partial U}{\partial z} - k_3 U = \gamma_3(r, t), \quad r \in [0, R], \quad z = 0. \quad (3.12)$$

We introduce on the basis of conservative averaging method [46], [47] the 2D model:

$$V(r, z, t) = \frac{1}{2\pi} \int_0^{2\pi} U(r, z, \varphi, t) d\varphi. \quad (3.13)$$

Integration of the main differential equation (3.1) gives main 2D differential equation:

$$\begin{aligned} \tau_r \frac{\partial^2 V}{\partial t^2} + \frac{\partial V}{\partial t} &= \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V}{\partial r} \right) + \frac{\partial^2 V}{\partial z^2} + \frac{1}{r^2 2\pi} \left( \frac{\partial U}{\partial \varphi} \Big|_{\varphi=2\pi} - \frac{\partial U}{\partial \varphi} \Big|_{\varphi=0} \right), \\ \tau_r \frac{\partial^2 V}{\partial t^2} + \frac{\partial V}{\partial t} &= \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V}{\partial r} \right) + \frac{\partial^2 V}{\partial z^2}. \end{aligned} \quad (3.14)$$

The problem for this 2D cylinder is investigated in paper [43]. In this paper only one additional condition for classical heat equation was used:

$$V(r, z, T) = U_T(r, z). \quad (3.15)$$

The main equation (3.14) can be written in the form of the heat equation with unknown source term:

$$\frac{\partial V}{\partial t} = a^2 \left[ r^{-1} \frac{\partial}{\partial r} \left( r \frac{\partial V}{\partial r} \right) + \frac{\partial^2 V}{\partial z^2} \right] + F(r, z, t, \tau, V), \quad (3.16)$$

here:

$$F(r, z, t, \tau, V) = f(r, z, t) - \tau_r W(r, z, t), \quad W(r, z, t) = \frac{\partial^2 V}{\partial t^2}. \quad (3.17)$$

The solution of the problem with known source function  $F$  for the classical heat conduction equation has the well-known form by employing the Green function  $G(r, \rho, z, \zeta, t)$  for the finite cylinder with mixed type (second and third type) boundary conditions [11]:

$$G(r, \rho, z, \zeta, t) = G_1(r, \rho, t) \cdot G_2(z, \zeta, t), \quad (3.18)$$

here:

$$G_1(r, \rho, t) = \frac{1}{\pi R^2} \sum_{n=1}^{\infty} \frac{\mu_n^2 J_0\left(\frac{\mu_n r}{R}\right) J_0\left(\frac{\mu_n \rho}{R}\right)}{\left(k_1^2 R^2 + \mu_n^2\right) J_0^2(\mu_n)} e^{-\frac{\mu_n^2}{R^2} a^2 t},$$

$$G_2(z, \zeta, t) = \sum_{m=1}^{\infty} \frac{\left[ \cos(\lambda_m z) + \frac{k_2}{\lambda_m} \sin(\lambda_m z) \right] \left[ \cos(\lambda_m \zeta) + \frac{k_2}{\lambda_m} \sin(\lambda_m \zeta) \right]}{\frac{k_3}{2\lambda_m^2} \frac{\lambda_m^2 + k_2^2}{\lambda_m^2 + k_3^2} + \frac{k_2}{2\lambda_m^2} + \frac{H}{2} \left( 1 + \frac{k_2^2}{\lambda_m^2} \right)} e^{-a^2 \lambda_m^2 t}, \quad (3.19)$$

where  $\mu_n > 0$  and  $\lambda_m > 0$  are the roots of the following transcendental equations:

$$\mu J_1(\mu) - k_1 R J_0(\mu) = 0, \quad \left( \lambda - \frac{k_2 k_3}{\lambda} \right) \operatorname{tg}(\lambda H) = \frac{k_2 + k_3}{\lambda^2 - k_2 k_3}. \quad (3.20)$$

This fact allows us to write the solution in short form as the following expression:

$$V(r, z, t) = \Gamma(r, z, t) - 2\pi \tau_r \int_0^t d\tau \int_0^H d\zeta \int_0^R \rho G(r, \rho, z, \zeta, t - \tau) W(\rho, \zeta, \tau) d\rho. \quad (3.21)$$

Here the function  $\Gamma(r, z, t)$  contains all the other integrals where the Green function is multiplied by known boundary and initial conditions:

$$\begin{aligned} \Gamma(r, z, t) = & 2\pi \int_0^H d\zeta \int_0^R \rho u^0(\rho, \zeta) G(r, \rho, z, \zeta, t) d\rho + \\ & + 2R \int_0^t d\tau \int_0^H \gamma_1(\zeta, \tau) G(r, R, z, \zeta, t - \tau) d\zeta - \\ & - 2\pi a^2 \int_0^t d\tau \int_0^R \rho \gamma_2(\rho, \tau) G(r, \rho, z, 0, t - \tau) d\rho - \\ & - 2\pi a^2 \int_0^t d\tau \int_0^R \rho \gamma_3(\rho, \tau) G(r, z, \rho, H, t - \tau) d\rho + \\ & + 2\pi \int_0^t d\tau \int_0^H d\zeta \int_0^R \rho f(\rho, \zeta, \tau) G(r, z, \rho, H, t - \tau) d\rho. \end{aligned} \quad (3.22)$$

Now it is the right time to use the additional condition (3.9) in formula (3.21) and reduce equality to the first kind of Fredholm type integral equation for function  $W(r, z, t)$  :

$$\int_0^T d\tau \int_0^H d\zeta \int_0^R \rho G(r, z, \rho, \zeta, T - \tau) \tilde{W}(\rho, \zeta, \tau) d\rho = \frac{\Gamma(r, z, T) - U_T(r, z)}{2\pi\tau_r}. \quad (3.23)$$

### 3.2.1.1 2D solution by Tihonov's Regularization Method

The solving of the integral equation (3.23) is an ill-posed problem, but regardless of this, it can be solved, e.g. by Tihonov regularisation method [48]. Let's denote the regularized solution with  $\tilde{W}(r, z, t)$ . Then, the approximate (regularised) solution  $\tilde{V}(r, z, t)$  follows from the formula (3.21):

$$\begin{aligned} \tilde{V}(r, z, t) = & \Gamma(r, z, t) - \\ & - 2\pi\tau_r \int_0^t d\tau \int_0^H d\zeta \int_0^R \rho G(r, \rho, z, \zeta, t - \tau) \tilde{W}(\rho, \zeta, \tau) d\rho. \end{aligned} \quad (3.24)$$

From here the following expression can be written for the first time derivative of the solution:

$$\begin{aligned} \frac{\partial \tilde{V}(r, z, t)}{\partial t} = & \frac{\partial \Gamma(r, z, t)}{\partial t} - \\ & - 2\pi\tau_r \int_0^H d\zeta \int_0^R \rho G(r, \rho, z, \zeta, +0) \tilde{W}(\rho, \zeta, t) d\rho - \\ & - 2\pi\tau_r \int_0^t d\tau \int_0^H d\zeta \int_0^R \rho \frac{\partial G(r, \rho, z, \zeta, t - \tau)}{\partial t} \tilde{W}(\rho, \zeta, \tau) d\rho. \end{aligned} \quad (3.25)$$

The well-known filtration property of the Green function [49] allows rewriting the last equation in the following form:

$$\begin{aligned} \frac{\partial \tilde{V}(r, z, t)}{\partial t} = & \frac{\partial \Gamma(r, z, t)}{\partial t} - 2\pi\tau_r \tilde{W}(r, z, t) - \\ & 2\pi\tau_r \int_0^t d\tau \int_0^H d\zeta \int_0^R \rho \frac{\partial G(r, \rho, z, \zeta, t - \tau)}{\partial t} \tilde{W}(\rho, \zeta, \tau) d\rho. \end{aligned} \quad (3.26)$$

The unknown function  $V_0(r, z)$  can now be obtained by passage to the limit  $t \rightarrow +0$  in the equation (3.27). So the solution of the time inverse problem looks like the following:

$$V_0(r, z) = \frac{\partial \Gamma(r, z, +0)}{\partial t} - \tau_r \tilde{W}(r, z, +0). \quad (3.27)$$

### 3.2.1.2 1D solution by Green Function Method for Thin Cylinder

The main idea of the solution will be shown for the 1D case hyperbolic heat exchange equation. For cases of thin cylinder, it is assumed that the right side of BC (3.11) and (3.12) do not depend on  $r$ :

$$\frac{\partial U}{\partial z} + k_2 U = \gamma_2(t), \quad r \in [0, R], \quad z = H, \quad (3.28)$$

$$\frac{\partial U}{\partial z} - k_3 U = \gamma_3(t), \quad r \in [0, R], \quad z = 0. \quad (3.29)$$

As the next step by conservative averaging, the 1D model is introduced [7]:

$$v(r, t) = \frac{1}{H} \int_0^H V(r, z, t) dz. \quad (3.30)$$

Integration of the differential equation (3.1) and using BC (3.28) and (3.29) gives the main 1D differential equation:

$$\begin{aligned} \tau_r \frac{\partial^2 v}{\partial t^2} + \frac{\partial v}{\partial t} &= \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) + \frac{1}{H} \left( \frac{\partial V}{\partial z} \Big|_{z=H} - \frac{\partial V}{\partial z} \Big|_{z=0} \right), \\ \tau_r \frac{\partial^2 v}{\partial t^2} + \frac{\partial v}{\partial t} &= \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) - \frac{k_2 + k_3}{H} v + \frac{\gamma_3(t) - \gamma_2(t)}{H}, \end{aligned} \quad (3.31)$$

To exclude the first derivation by time, following unknown function is substituted:

$$v(r, t) = e^{-\frac{t}{2\tau_r}} u(r, t). \quad (3.32)$$

As a result from (3.31) and (3.32), the following equation is obtained:

$$\tau_r \frac{\partial^2 u}{\partial t^2} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) - \left( \frac{k_2 + k_3}{H} - \frac{1}{4\tau_r} \right) u + \frac{\gamma_3(t) - \gamma_2(t)}{H} e^{\frac{t}{2\tau_r}}. \quad (3.33)$$

Equation (3.33) is the Klein –Gordon equation [50] with a new BC instead of (3.10):

$$\frac{\partial u}{\partial r} + k_1 u = \tilde{\gamma}_1(t) e^{\frac{t}{2\tau_r}}, \quad r = R, \quad \tilde{\gamma}_1(t) = \frac{1}{H} \int_0^H \gamma_1(z, t) dz \quad (3.34)$$

and updated initial conditions (3.7) – (3.8):

$$\begin{aligned} u &= \tilde{u}^0(r), \quad t = 0, \quad r \in [0, R], \\ \frac{\partial u}{\partial t} &= \tilde{U}_0(r), \quad t = 0, \quad r \in [0, R]. \end{aligned} \quad (3.35)$$

The additional conditions (3.9) at final moment are the following:

$$u(r, T) = e^{\frac{T}{2\tau_r}} \tilde{U}_T(r), \quad \frac{\partial u}{\partial t}(r, T) = e^{\frac{T}{2\tau_r}} \tilde{V}_T(r). \quad (3.36)$$

The solution of direct problem [11] is:

$$\begin{aligned} u(r, t) &= \int_0^R \tilde{u}^0(\xi) \frac{\partial}{\partial t} G(r, \xi, t) d\xi + \int_0^R \tilde{U}_0(\xi) G(r, \xi, t) d\xi + \\ &+ a^2 \int_0^t \gamma_1(\tau) e^{\frac{\tau}{2\tau_r}} G(r, R, t - \tau) d\tau + \int_0^t d\tau \int_0^R \frac{\tilde{\gamma}_1(\tau) - \tilde{\gamma}_2(\tau)}{H} e^{\frac{\tau}{2\tau_r}} G(r, \xi, t - \tau) d\xi, \end{aligned} \quad (3.37)$$

here:

$$G(r, \xi, t) = \frac{2}{R^2} \sum_{n=1}^{\infty} \frac{\mu_n^2 \xi}{(k_1^2 R^2 + \mu_n^2) J_0^2(\mu_n)} J_0\left(\frac{\mu_n r}{R}\right) J_0\left(\frac{\mu_n \xi}{R}\right) \frac{\sin(t\sqrt{\lambda_n})}{\sqrt{\lambda_n}}, \quad (3.38)$$

$$\lambda_n = \left(\frac{a\mu_n}{R}\right)^2 + \frac{k_2 + k_3}{H},$$

where eigenvalues of the equation can be found as positive roots of the following transcendent equation:

$$\mu J(\mu) = \frac{k_2 + k_3}{H} R J_0(\mu). \quad (3.39)$$

With the first end condition (3.36), it leads to the first kind of Fredholm integral equation:

$$\begin{aligned} v(r, T) = \tilde{U}_T(r) = & \int_0^R u^0(\xi) \frac{\partial}{\partial t} G(r, \xi, T) d\xi + \int_0^R U^0(\xi) G(r, \xi, T) d\xi + \\ & + a^2 \int_0^T \gamma_1(\tau) G(r, R, T - \tau) d\tau + \int_0^T d\tau \int_0^R \frac{\gamma_1(\tau) - \gamma_2(\tau)}{H} G(r, \xi, T - \tau) d\xi. \end{aligned} \quad (3.40)$$

For solution of function  $U^0(x)$  we obtain the following first kind of Fredholm integral equation:

$$\begin{aligned} \int_0^R U^0(\xi) G(r, \xi, T) d\xi = F(r, T), \\ F(r, T) = \tilde{U}_T(r) - \int_0^R u^0(\xi) \frac{\partial}{\partial t} G(r, \xi, T) d\xi - \\ - a^2 \int_0^T \gamma_1(\tau) G(r, R, T - \tau) d\tau - \int_0^T d\tau \int_0^R \frac{\gamma_1(\tau) - \gamma_2(\tau)}{H} G(r, \xi, T - \tau) d\xi. \end{aligned} \quad (3.41)$$

If it is assumed that  $\tilde{\gamma}_1(\tau) = \tilde{\gamma}_2(\tau)$  then time depending heat flux, consider the following:

$$\begin{aligned} \frac{\partial}{\partial t} u(r, t) = & \int_0^R \tilde{u}^0(\xi) \frac{\partial^2}{\partial t^2} G(r, \xi, t) d\xi + \int_0^R \tilde{U}^0(\xi) \frac{\partial}{\partial t} G(r, \xi, t) d\xi + \\ & + a^2 \int_0^t \gamma_1(\tau) e^{\frac{\tau}{2\tau_r}} \frac{\partial}{\partial t} G(r, R, t - \tau) d\tau + a^2 \gamma_1(t) e^{\frac{t}{2\tau_r}}. \end{aligned} \quad (3.42)$$

Using this equation, start condition (3.8) can be found. In this case, the first kind of Fredholm's integral equation must be solved. If both additional conditions at the end of process (3.9) are used, then coordinates by time in equation (3.33) can be transformed:

$$\tilde{t} = T - t, \quad \tilde{u}(x, y, \tilde{t}) = u(x, y, T - \tilde{t}). \quad (3.43)$$

Equation (3.33) has a similar look:

$$\tau_r \frac{\partial^2 \tilde{u}}{\partial \tilde{t}^2} + \frac{\partial \tilde{u}}{\partial \tilde{t}} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \tilde{u}}{\partial r} \right) - \frac{k_2 + k_3}{H} \tilde{u} + \frac{\gamma_3(\tilde{t}) - \gamma_2(\tilde{t})}{H} e^{\frac{T-\tilde{t}}{2\tau_r}}. \quad (3.44)$$

Initial conditions look like the following:

$$\begin{aligned}\tilde{u}|_{\tilde{t}=0} &= u|_{t=T} = \tilde{U}_T(r) e^{\frac{T}{2\tau_r}}, \\ \frac{\partial \tilde{u}}{\partial \tilde{t}} \Big|_{\tilde{t}=0} &= -\frac{\partial u}{\partial t} \Big|_{t=T} = -\tilde{V}_T(r) e^{\frac{T}{2\tau_r}}.\end{aligned}\tag{3.45}$$

The derivation of equation (3.42) by time is:

$$\begin{aligned}\frac{\partial}{\partial \tilde{t}} \tilde{u}(r, \tilde{t}) &= \int_0^R \tilde{U}_T(\xi) e^{\frac{T}{2\tau_r}} \frac{\partial^2}{\partial \tilde{t}^2} G(r, \xi, \tilde{t}) d\xi - \\ &- \int_0^R \tilde{V}_T(\xi) e^{\frac{T}{2\tau_r}} \frac{\partial}{\partial \tilde{t}} G(r, \xi, \tilde{t}) d\xi \\ &+ a^2 \int_0^{\tilde{t}} \gamma_1(\tau) e^{\frac{\tau}{2\tau_r}} \frac{\partial}{\partial \tilde{t}} G(r, R, \tilde{t} - \tau) d\tau + a^2 \gamma_1(\tilde{t}) e^{\frac{\tilde{t}}{2\tau_r}}.\end{aligned}\tag{3.46}$$

By putting  $\tilde{t} = T \Rightarrow t = 0$  into (3.46), the following connection is obtained. This is initial condition (3.8), and is also direct problem:

$$\begin{aligned}\frac{\partial}{\partial \tilde{t}} \tilde{u}(r, T) &= \int_0^R \tilde{U}_T(\xi) \exp\left(\frac{T}{2\tau_r}\right) \frac{\partial^2}{\partial \tilde{t}^2} G(r, \xi, \tilde{t}) \Big|_{\tilde{t}=T} d\xi - \\ &- \int_0^R \tilde{V}_T(\xi) \exp\left(\frac{T}{2\tau_r}\right) \frac{\partial}{\partial \tilde{t}} G(r, \xi, \tilde{t}) \Big|_{\tilde{t}=T} d\xi + \\ &+ a^2 \int_0^T \gamma_1(\tau) \exp\left(\frac{\tau}{2\tau_r}\right) \frac{\partial}{\partial \tilde{t}} G(r, R, \tilde{t} - \tau) \Big|_{\tilde{t}=T} d\tau + \\ &+ a^2 \gamma_1(T) \exp\left(\frac{T}{2\tau_r}\right).\end{aligned}\tag{3.47}$$

### 3.2.2 Solution for Thin Disk

In case of thin disk BC (3.10) can be rewritten in the following way:

$$\frac{\partial V}{\partial r} + k_1 V = \gamma_1(t), \quad r = R, \quad z \in [0, H].\tag{3.48}$$

1D model introduced by averaging:

$$u(z, t) = \frac{2}{R^2} \int_0^R r V(r, z, t) dr.\tag{3.49}$$

Integration of the main differential equation (3.1) and using BC (3.11) and (3.12) give the main 2D differential equation:

$$\begin{aligned}\tau_r \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial z^2} + \frac{2}{R} \frac{\partial V}{\partial r} \Big|_{r=R}, \\ \tau_r \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial z^2} - \frac{2k_1}{R} u + \frac{2\gamma_1(t)}{R}.\end{aligned}\tag{3.50}$$



Further construction of solution is similar as in case of thin cylinder (see subchapter 3.2.1.2).

Equation (3.33) and (3.50) can be transformed to an ordinary differential equation by time, by means of the conservative averaging method. Thus a semi-analytical solution can be found. Monographs [51], [52] are useful for the solution in more complicated cases of domain described in Chapter 3.3.

### 3.3 Mathematical Formulation of 3D Problem for Cylindrical Fin

Further the cylindrical wall with fin is discussed. Let's assume that surface  $z=0$  and  $\varphi=0$  is middle surfaces with second type homogeneous boundary conditions. In other words, the angle of the sample is  $2\Phi$  and the height in  $z$  direction is  $2H$ . All other surfaces come into contact with continuously flowing water at constant temperature  $\Theta_0$ . It is assumed that the heat exchange processes on surfaces can be described by linear third type boundary conditions.

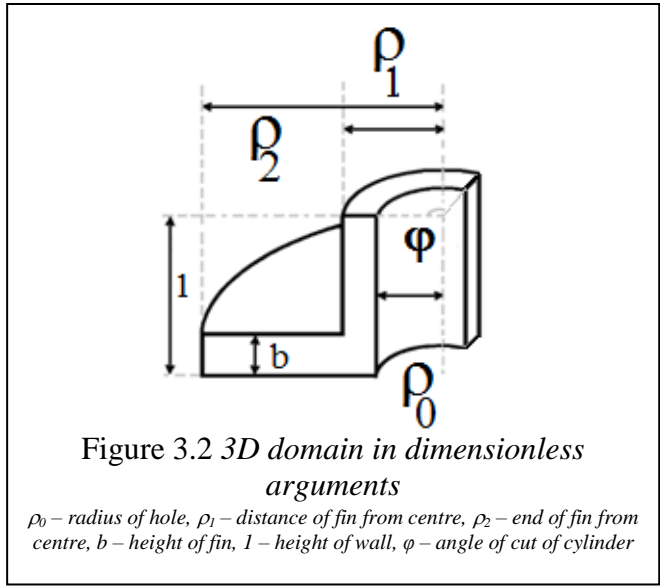


Figure 3.2 3D domain in dimensionless arguments  
 $\rho_0$  – radius of hole,  $\rho_1$  – distance of fin from centre,  $\rho_2$  – end of fin from centre,  $b$  – height of fin,  $1$  – height of wall,  $\varphi$  – angle of cut of cylinder

The steel sample is heated up to an initial temperature  $V_{in}$  and placed in the facility for quenching. This physical system can be described with the following mathematical model.

Let's start with an accurate 3D formulation of the transient problem for a system of a cylindrical wall and fin. One element of the wall (base) is placed in the dimensionless domain  $\{r \in [\rho_0, \rho_1], z \in [0, 1], \varphi \in [0, \Phi]\}$ . The cylindrical fin occupies the domain  $\{r \in [\rho_1, \rho_2], z \in [0, b], \varphi \in [0, \Phi]\}$ .

The following dimensionless arguments, parameters to transform the problem to a dimensionless problem are used:  $r = \frac{\tilde{r}}{H}$ ,  $z = \frac{\tilde{z}}{H}$ ,  $\rho_0 = \frac{r_0}{H}$ ,  $\rho_1 = \frac{r_1}{H}$ ,  $\rho_2 = \frac{r_2}{H}$ ,  $b = \frac{H_0}{H}$ ,  $\beta = \frac{hH}{k}$ ,  $\gamma = \frac{\beta}{\rho_1}$ ,  $a^2 = \frac{k}{c\rho}$ . Here the  $k$  - heat conductivity coefficient for the fin and wall,  $c$  - is specific heat capacity,  $\rho$  - density,  $h$  - heat exchange coefficient for the system,  $H_0$  - height of the fin,  $L$  - length of the fin,  $H$  - height of the wall, parameter  $\tau$  is so called relaxation time.

The most interesting shapes of this system are the following. The case when  $r_0 = \rho_0 = 0$  there are two shape options:

- a)  $\Phi < 2\pi$  partial cylinder (cross section in  $z$  direction is sector) with the fin;  
b)  $\Phi = 2\pi$  complete cylinder with fin. Case when  $r_0 > 0$  and  $\Phi = 2\pi$  we get a tube with the fin.

The dimensional temperature field of the steel sample (wall) is described by function  $\bar{V}_0(r, z, \varphi, t)$  and the following dimensionless temperature field is introduced  $V_0(r, z, \varphi, t) = \frac{\bar{V}_0(r, z, \varphi, t) - \Theta_0}{V_{in} - \Theta_0}$ , here  $V_{in}$  is some characteristic value. It means, that dimensionless temperature field  $V_0(r, z, \varphi, t)$  in the wall is described with the equation:

$$\tau \frac{\partial^2 V_0}{\partial t^2} + \frac{\partial V_0}{\partial t} = \frac{a^2}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V_0}{\partial r} \right) + a^2 \frac{\partial^2 V_0}{\partial z^2} + \frac{a^2}{r^2} \frac{\partial V_0}{\partial \varphi^2}. \quad (3.51)$$

Similarly, the temperature field  $V(r, z, \varphi, t)$  in the fin fulfils the equation:

$$\tau \frac{\partial^2 V}{\partial t^2} + \frac{\partial V}{\partial t} = \frac{a^2}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V}{\partial r} \right) + a^2 \frac{\partial^2 V}{\partial z^2} + \frac{a^2}{r^2} \frac{\partial V}{\partial \varphi^2}. \quad (3.52)$$

And the following boundary conditions in  $\varphi$  and  $z$  directions are formulated:

$$\begin{aligned} \left. \frac{\partial V_0}{\partial \varphi} \right|_{\varphi=0} = 0, \left( \frac{\partial V_0}{\partial \varphi} + \beta V_0 \right) \Big|_{\varphi=\Phi} = 0, \left. \frac{\partial V}{\partial \varphi} \right|_{\varphi=0} = 0, \left( \frac{\partial V}{\partial \varphi} + \beta V \right) \Big|_{\varphi=\Phi} = 0, \\ \left. \frac{\partial V_0}{\partial z} \right|_{z=0} = 0, \left( \frac{\partial V_0}{\partial z} + \beta V_0 \right) \Big|_{z=1} = 0, \left. \frac{\partial V}{\partial z} \right|_{z=0} = 0, \left( \frac{\partial V}{\partial z} + \beta V \right) \Big|_{z=b} = 0. \end{aligned} \quad (3.53)$$

Analogously the boundary conditions in  $r$  direction (one symmetry condition and three heat exchange conditions) are formulated:

$$\begin{aligned} \left( \frac{\partial V_0}{\partial r} - \beta V_0 \right) \Big|_{r=\rho_0} = 0, \left( \frac{\partial V_0}{\partial r} + \beta V_0 \right) \Big|_{r=\rho_1} = 0, \\ \left. \frac{\partial V}{\partial r} \right|_{r=0} = 0, \left( \frac{\partial V}{\partial r} + \beta V \right) \Big|_{r=\rho_2} = 0. \end{aligned} \quad (3.54)$$

The conjugations conditions on the surface between the wall and the fin are assumed as ideal thermal contact - there is no contact resistance (it is the continuity of temperature and the heat flux):

$$U_0 \Big|_{\rho_1-0} = U \Big|_{\rho_1+0}, \left. \frac{\partial U_0}{\partial r} \right|_{r=\rho_1-0} = \left. \frac{\partial U}{\partial r} \right|_{r=\rho_1+0}, z \in [0, b]. \quad (3.55)$$

The initial conditions are assumed in the following form:

$$V_0 \Big|_{t=0} = V_0^0(r, z, \varphi), V \Big|_{t=0} = V_0(r, z, \varphi), \quad (3.56)$$

$$\left. \frac{\partial V_0}{\partial t} \right|_{t=0} = \tilde{W}_0^0(r, z, \varphi), \left. \frac{\partial V}{\partial t} \right|_{t=0} = \tilde{W}_0(r, z, \varphi). \quad (3.57)$$

From the practical point of view, both conditions (3.57) are unrealistic. For this case the initial heat flux must be determined theoretically. Additional conditions

are assumed that the temperature distribution and the heat fluxes distribution at the end of process are given (known):

$$\begin{aligned} V_0|_{t=T} &= V_T^0(r, z, \varphi), V|_{t=T} = V_T(r, z, \varphi), \\ \frac{\partial V_0}{\partial t}\Big|_{t=T} &= \tilde{W}_T^0(r, z, \varphi), \frac{\partial V}{\partial t}\Big|_{t=T} = \tilde{W}_T(r, z, \varphi). \end{aligned} \quad (3.58)$$

In case of  $\Phi = 2\pi$ , boundary condition (3.53) is in the following form:

$$\frac{\partial V_0}{\partial \varphi}\Big|_{\varphi=0} = \frac{\partial V_0}{\partial \varphi}\Big|_{\varphi=\Phi}, \frac{\partial V}{\partial \varphi}\Big|_{\varphi=0} = \frac{\partial V}{\partial \varphi}\Big|_{\varphi=\Phi}. \quad (3.59)$$

### 3.3.1 Exact 3D Reduction to 2D Problem

Equations (3.51) and (3.54) can be reduced from 3D to 2D problem by introducing the following average integral values for argument  $\varphi$ :

$$U(r, z, t) = \frac{1}{\Phi} \int_0^\Phi V(r, z, \varphi, t) d\varphi, U_0(r, z, t) = \frac{1}{\Phi} \int_0^\Phi V_0(r, z, \varphi, t) d\varphi. \quad (3.60)$$

Integration of the equation (3.51) for the wall over  $\varphi \in [0, \Phi]$  gives following equation (exact consequence of 3D partial differential equation (3.51)):

$$\tau \frac{\partial^2 U_0}{\partial t^2} + \frac{\partial U_0}{\partial t} = \frac{a^2}{r} \frac{\partial}{\partial r} \left( r \frac{\partial U_0}{\partial r} \right) + a^2 \frac{\partial^2 U_0}{\partial z^2} + \frac{a^2}{r^2 \Phi} \left( \frac{\partial V_0}{\partial \varphi}\Big|_{\varphi=\Phi} - \frac{\partial V_0}{\partial \varphi}\Big|_{\varphi=0} \right). \quad (3.61)$$

The first pair of boundary conditions (3.53) allows rewriting the last equality in the form of two dimensional equations (assuming  $U_0$  is constant regarding argument  $\varphi$ ):

$$\tau \frac{\partial^2 U_0}{\partial t^2} + \frac{\partial U_0}{\partial t} = \frac{a^2}{r} \frac{\partial}{\partial r} \left( r \frac{\partial U_0}{\partial r} \right) + a^2 \frac{\partial^2 U_0}{\partial z^2} - d(r)U_0, d(r) = \frac{\beta a^2}{r^2 \Phi}. \quad (3.62)$$

Similarly, the dimensionless temperature field  $U(r, z, t)$  in the fin can be described with the equation:

$$\tau \frac{\partial^2 U}{\partial t^2} + \frac{\partial U}{\partial t} = \frac{a^2}{r} \frac{\partial}{\partial r} \left( r \frac{\partial U}{\partial r} \right) + a^2 \frac{\partial^2 U}{\partial z^2} - d(r)U. \quad (3.63)$$

The needed boundary conditions and conjugations conditions are defined as follow:

$$\frac{\partial U_0}{\partial r} - \beta U_0 = 0, r = \rho_0, z \in [0, 1], \quad (3.64)$$

$$\frac{\partial U_0}{\partial r} + \beta U_0 = 0, r = \rho_1, z \in [b, 1], \quad (3.65)$$

$$\frac{\partial U_0}{\partial z} + \beta U_0 = 0, r \in [\rho_0, \rho_1], z = 1 \quad (3.66)$$

$$\frac{\partial U_0}{\partial z} = 0, r = [\rho_0, \rho_1], z = 0, \quad (3.67)$$

$$\frac{\partial U}{\partial r} + \beta U = 0, r = \rho_2, z \in [0, b], \quad (3.68)$$

$$\frac{\partial U}{\partial z} = 0, r = [\rho_1, \rho_2], z = 0, \quad (3.69)$$

$$\frac{\partial U}{\partial z} + \beta U = 0, r = [\rho_1, \rho_2], z = b, \quad (3.70)$$

$$U_0|_{\rho_1-0} = U|_{\rho_1+0}, \frac{\partial U_0}{\partial r}|_{\rho_1-0} = \frac{\partial U}{\partial r}|_{\rho_1+0}. \quad (3.71)$$

The initial conditions transform into the form:

$$U_0|_{t=0} = U_0^0(r, z), U|_{t=0} = U_0(r, z), \quad (3.72)$$

$$\frac{\partial U_0}{\partial t}|_{t=0} = W_0^0(r, z), \frac{\partial U}{\partial t}|_{t=0} = W_0(r, z). \quad (3.73)$$

Additional conditions transform into following:

$$U_0|_{t=T} = U_T^0(r, z), U|_{t=T} = U_T(r, z), \quad (3.74)$$

$$\frac{\partial V_0}{\partial t}|_{t=T} = W_T^0(r, z), \frac{\partial V}{\partial t}|_{t=T} = W_T(r, z).$$

All initial and additional conditions (3.72) – (3.74) are obtained by the integration of conditions (3.56) – (3.57) by direction  $\varphi$  regarding equation (3.60). Equations (3.51), (3.52) in case of  $\Phi = 2\pi$  taking into account the boundary condition (3.59) are rewritten in a more simple form:

$$\tau \frac{\partial^2 U_0}{\partial t^2} + \frac{\partial U_0}{\partial t} = \frac{a^2}{r} \frac{\partial}{\partial r} \left( r \frac{\partial U_0}{\partial r} \right) + a^2 \frac{\partial^2 U_0}{\partial z^2}, \quad (3.75)$$

$$\tau \frac{\partial^2 U}{\partial t^2} + \frac{\partial U}{\partial t} = \frac{a^2}{r} \frac{\partial}{\partial r} \left( r \frac{\partial U}{\partial r} \right) + a^2 \frac{\partial^2 U}{\partial z^2}.$$

Let's make the following transformation to functions  $v(r, z, t)$ ,  $v_0(r, z, t)$ :

$$U(r, z, t) = e^{-\frac{t}{2\tau}} v(r, z, t), U_0(r, z, t) = e^{-\frac{t}{2\tau}} v_0(r, z, t). \quad (3.76)$$

The main equations (3.61) and (3.62) transform in following form:

$$\frac{\partial^2 v_0}{\partial t^2} = \frac{a_\tau^2}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_0}{\partial r} \right) + a_\tau^2 \frac{\partial^2 v_0}{\partial z^2} - c v_0, a_\tau^2 = \frac{a^2}{\tau}, \quad (3.77)$$

$$\frac{\partial^2 v}{\partial t^2} = \frac{a_\tau^2}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) + a_\tau^2 \frac{\partial^2 v}{\partial z^2} - c v, c = -\frac{1}{4\tau^2}.$$

Boundary conditions (3.67) - (3.74) for functions  $v(r, z, t)$ ,  $v_0(r, z, t)$  stay in the same form, only initial conditions (3.75), (3.76) are the following:

$$v_0|_{t=0} = U_0^0(r, z), v|_{t=0} = U_0(r, z), \quad (3.78)$$

$$\left. \frac{\partial v_0}{\partial t} \right|_{t=0} = W_0^0(r, z) + \frac{U_0^0(r, z)}{2\tau}, \quad \left. \frac{\partial v}{\partial t} \right|_{t=0} = W_0(r, z) + \frac{U_0(r, z)}{2\tau}. \quad (3.79)$$

Additional conditions (3.77) are the following:

$$\begin{aligned} v_0|_{t=T} &= e^{\frac{T}{2\tau}} U_T^0(r, z), \quad \left. \frac{\partial v_0}{\partial t} \right|_{t=T} = e^{\frac{T}{2\tau}} \left[ W_T^0(r, z) + \frac{U_T^0(r, z)}{2\tau} \right], \\ v|_{t=T} &= e^{\frac{T}{2\tau}} U_T(r, z), \quad \left. \frac{\partial v}{\partial t} \right|_{t=T} = e^{\frac{T}{2\tau}} \left[ W_T(r, z) + \frac{U_T(r, z)}{2\tau} \right]. \end{aligned} \quad (3.80)$$

### 3.3.2 Solution of 2D Problem

There, one shape of sample will be discussed. All other mentioned cases have a similar methodology of research as papers [45], [53]. Let's consider a complete cylinder with the fin. In this case, to split up the sample into two complete cylinders connected with surface  $z = b$  is preferred:

$$v_0|_{z=b-0} = v|_{z=b+0}, \quad \left. \frac{\partial v_0}{\partial z} \right|_{z=b-0} = \left. \frac{\partial v}{\partial z} \right|_{z=b+0}, \quad r \in [0, \rho_2]. \quad (3.81)$$

The boundary condition for a cylinder on the right hand side (3.68) together with the conjugation conditions (3.81) can be rewritten in following common form:

$$\left. \left( \frac{\partial v_0}{\partial z} + \beta v_0 \right) \right|_{z=b-0} = \begin{cases} F_0(r, t), & 0 < r < \rho_2, \\ 0, & \rho_2 < r < \rho_1, \end{cases} \quad F_0(r, t) = \left. \left( \frac{\partial v}{\partial z} + \beta v \right) \right|_{z=b+0}. \quad (3.82)$$

The solution for the complete cylinder can be written in the well-known form by means of the Green function, see, [11]:

$$\begin{aligned} v_0(r, z, t) &= \Phi_0(r, z, t) + a_\tau^2 \int_0^t d\tau_0 \int_0^{\rho_2} F_0(\rho_0, \tau_0) G_0(r, z, \rho_0, b, t - \tau_0) d\rho_0, \\ \Phi_0(r, z, t) &= \int_0^{\rho_1} d\xi_0 \int_0^1 \left[ \frac{1+\tau}{\tau} U_0^0(\xi_0, \eta_0) + W_0^0(\xi_0, \eta_0) \right] G_0(r, z, \xi_0, \eta_0, t) d\eta_0. \end{aligned} \quad (3.83)$$

The Green function has the form:

$$\begin{aligned} G_0(r, z, \xi, \eta, t) &= \\ \frac{2\xi}{\rho_1^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\mu_n^2}{(\beta^2 \rho_1^2 + \mu_n^2)} J_0\left(\frac{\mu_n r}{\rho_1}\right) J_0\left(\frac{\mu_n \xi}{\rho_1}\right) \frac{\varphi_m(z) \varphi_m(\eta) \sin(t\sqrt{\lambda_{nm}})}{\|\varphi_m\|^2 \sqrt{\lambda_{nm}}}, \\ \varphi_m(z) &= \cos(\beta_m z), \\ \lambda_{nm} &= a_\tau^2 \left( \frac{\mu_n^2}{\rho_1^2} + \beta_m^2 \right) - \frac{1}{4\tau^2}, \\ \|\varphi_m\|^2 &= \frac{b}{2} + \frac{\beta}{2} \left( 1 + \frac{\beta_m^2}{\beta^2 + \beta^2} \right). \end{aligned} \quad (3.84)$$

There  $\beta_m$  and  $\mu_n$  are positive roots of following transcendental equations:

$$\beta_m = \frac{(m-1)\pi + \pi/4}{b}, m \in N$$

$$\frac{J_1(\mu_n)}{J_0(\mu_n)} = \frac{\beta\rho_1}{\mu_n}. \quad (3.85)$$

The representation (3.83) is not the solution because of the unknown function  $F_0(r, t)$ . As the next step some conditions for the fin are modified. The conjugations conditions are rewritten in following form:

$$\left( \frac{\partial v}{\partial z} - \beta v \right) \Big|_{z=b+0} = \begin{cases} F(r, t), & 0 < r < \rho_2, \\ 0, & \rho_2 < r < \rho_1, \end{cases} F(r, t) = \left( \frac{\partial v_0}{\partial z} - \beta v_0 \right) \Big|_{z=b-0}. \quad (3.86)$$

Similar to the case of formula (3.83) the solution for the fin can be represented in following form:

$$v(r, z, t) = \Phi(r, z, t) + a_\tau^2 \int_0^t d\tau \int_0^{\rho_2} F(\rho, \tau) G_0(r, z, \rho, b, t - \tau) d\rho. \quad (3.87)$$

Formulae (3.83), (3.87) can be rewritten for  $F_0(r, t)$ ,  $F(r, t)$  similar as in Chapter 2.4.3 and Fredholm's integral equations of the second kind obtained. Similar to papers [45], [53], the system can be reduced to one integral equation.

### 3.4 Conclusions

The solution of the time inverse problem in closed form for a thin cylinder is obtained. The methodology of the solution of inverse problems in non canonical area is described. An two-dimensional analytical solution for a system with a cylindrical fin (when the wall and the fin consist of the same material) is obtained.

## CONCLUSIONS

The ideas concerning steady-state and non-steady-state semi-analytical and analytical solutions for a number of systems with a different number of dimensions are examined in this work and can be used as an additional learning tool when studying methods of mathematical physics for problems of heat transfer. A survey of the available literature shows that 2D problems in cylindrical coordinates for non-canonical domains using analytical or semi-analytical solutions have not been considered before; therefore it stands to reason that this Thesis adds to the educational and scientific literature.

Numerical models developed in this Thesis give a better insight into the advantages of analytical and semi-analytical solutions. Despite the fact that many scientists regard the analytical approach as archaic; in my opinion this work shows the solutions in a different light and highlights the contemporary flexibility and the wide scale of their applications. Evolving technologies change the requirements for applied methods. Therefore, in my opinion there is reason to think that, if advanced in this direction, analytical solutions can experience their renaissance especially concerning engineering applications in different fields.

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$$k_0 \frac{\partial \bar{U}_0}{\partial r} = -h \left( \bar{U}_0 - T_0 \right). \quad (\text{A.5})$$

A similar connection for edges ⑥ and ⑤ can be obtained (taking into account flux direction):

$$k \frac{\partial \bar{U}}{\partial r} = -h \left( \bar{U} - T_0 \right), \quad (\text{A.6})$$

$$k \frac{\partial \bar{U}}{\partial z} = -h \left( \bar{U} - T_0 \right). \quad (\text{A.7})$$

Taking into account heat balance, continuity of heat flux and connection (A.2) the following connection at the fin and wall contact surface  $r = \Delta, z \in (0, B)$  (⑨), is obtained:

$$k_0 \frac{\partial \bar{U}_0}{\partial r} \Big|_{r=\Delta-0} = k \frac{\partial \bar{U}}{\partial r} \Big|_{r=\Delta+0}. \quad (\text{A.8})$$

For case of ideal thermal contact [4] on line ⑨, the following connection is valid:

$$\bar{U}_0 \Big|_{x=\Delta-0} = \bar{U} \Big|_{x=\Delta+0}. \quad (\text{A.9})$$

In case of a periodical system, id est., system symmetric regarding lines  $z = 0$  and  $z = B + R$ , there is no heat flux on edges  $r \in (0, \Delta), z = 0$  and  $r \in (0, \Delta), z = B + R$  (⑩) and  $r \in (\Delta, \Delta + L), z = 0$  (⑪), respectively:

$$\frac{\partial \bar{U}_0}{\partial z} \Big|_{z=0} = \frac{\partial \bar{U}_0}{\partial z} \Big|_{z=1} = 0 \quad \text{and} \quad \frac{\partial \bar{U}}{\partial z} \Big|_{z=0} = 0. \quad (\text{A.10})$$

Now the reduction to dimensionless arguments can be done. That means new a dimensionless problem must be constructed within temperature range  $T \in [0, 1]$ , and the new domain always reduces that wall to one unit high:

$$\bar{r} = \frac{r}{B + R} \quad \text{and} \quad \bar{z} = \frac{z}{B + R}. \quad (\text{A.11})$$

New values of wall width, fin length and fin height in dimensionless domain are:

$$\delta = \frac{\Delta}{B + R}, \quad l = \frac{L}{B + R} \quad \text{and} \quad b = \frac{B}{B + R}. \quad (\text{A.12})$$

Heat transfer in the wall is:

$$U_0 = \frac{\bar{U}_0 - T_0}{T_1 - T_0}$$

$$\bar{U}_0 = U_0 (T_1 - T_0) + T_0, \quad (\text{A.13})$$

but heat transfer in the fin is:

$$U = \frac{\bar{U} - T_0}{T_1 - T_0}$$

$$\bar{U} = U(T_1 - T_0) + T_0. \quad (\text{A.14})$$

Now equations (A.11) and ((A.13) or (A.14)) pasted into the equation (A.4) gives the following new BC for edge ③:

$$\frac{\partial U_0}{\partial r} + \frac{h_0}{k_0}(B + R)(1 - U_0) = 0, \quad (\text{A.15})$$

BC for edge ④:

$$\frac{\partial U_0}{\partial r} + \frac{h_0}{k_0}(B + R)U_0 = 0, \quad (\text{A.16})$$

BC for edge ⑤:

$$\frac{\partial U}{\partial r} + \frac{h}{k}(B + R)U = 0 \quad (\text{A.17})$$

and BC for edge ⑥:

$$\frac{\partial U}{\partial z} + \frac{h}{k}(B + R)U = 0. \quad (\text{A.18})$$

Since the coefficients in equations (A.7), (A.15) – (A.18) are constants, the following denotations are done:

$$\beta_0^0 = \frac{h_0(B + R)}{k_0}, \quad \beta_0 = \frac{h(B + R)}{k_0}, \quad \beta = \frac{h(B + R)}{k}. \quad (\text{A.19})$$

Similarly, problem reduction to the problem of dimensionless arguments can be done for other discussed issues in the Thesis.

## ***A.2 Method of Conservative Averaging***

The main idea of the conservative averaging method is assuming that the solution has a known form of functional connection of one direction. This assumption is done to simplify the original problem, but still conserving by means of the average value original formulation of heat balance. The method of conservative averaging is openly discussed in papers [7], [19]. In short, the main ideas and steps for applied conservative averaging method for the problem described in the first chapter are the following:

1. The solution for the fin in  $z$  direction assumed to be a combination of exponential functions and written in the form which contains three additional unknown functions;
2. The average temperature connection in  $z$  direction for the fin is defined;
3. The solution for the wall in  $r$  direction is assumed to be a combination of polynomial functions and written in the form which contains three additional unknown functions;
4. The average temperature connection in  $r$  direction for the wall is defined;

5. Consequently, BC on the fin and wall is applied to exclude unknown functions;
6. The solution for the wall is divided into two solutions:
  - a. solution for the lower wall, respectively, sub-domain which has boundary only with the fin
  - b. solution for the upper wall, id est., sub-domain which has boundary only with the outer environment
7. A junction condition for wall lower part and fin is applied.
8. Additional conditions for the unknown function are defined: Continuity of function and its derivative at the contact of the wall and fin is requested.
9. At last, completing of the junction condition at the common point of all three sub-domains  $(\delta, b)$  is requested.
10. After completing points 1 – 8, a linear system of equations is obtained. And the solution to the original problem of the solution for the system of linear equations is reduced.

### ***A.3 Generalised Method of the Green Function***

The main idea of the generalised method of the Green function for non canonical domains is (1) applying of the Green function method for each canonical sub-domain and (2) merging both analytical solutions through junction conditions. The generalised method of the Green function is applied for problems in Chapter 2 and 3.3. Following the main steps of the method, the following may be pointed out:

1. Non-canonical domain is divided in two canonical sub-domains.
2. Junction conditions are combined together with boundary conditions. Non-homogeneous boundary condition of third kind with unknown functions on right side for each canonical sub-domain is written.
3. Analytical solution for each sub-domain is constructed where unknown function in boundary condition like known function is treated. But still it is not solution of basic problem, since it contains unknown function.
4. Obtained solutions with unknown function are pasted in boundary conditions and second kind of Fredholm's integral equation is written.
5. The solution of Fredholm integral equation for one sub-domain is obtained. Thus unknown function for each sub-domain is gained.

The existence and unity of the described solution is guaranteed by the unity and existence of the solution of Fredholm's integral equation and the continuity of its kernel [39].