RIGA TECHNICAL UNIVERSITY

PH.D. THESIS

2007

RIGA TECHNICAL UNIVERSITY

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SOLUTIONS TO SOME PROBLEMS ON THE INFLUENCE OF A CONDUCTING INHOMOGENEOUS MEDIUM ON A SOURCE OF CURRENT

Ph.D. Thesis

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Riga 2007

This work has been partly supported by the European Social Fund within the National Program "Support for the carrying out doctoral study program's and post-doctoral researches" project "Support for the development of doctoral studies at Riga Technical University".

ABSTRACT

The main topic of this PhD thesis is the mathematical analysis of problems in non-destructive testing by eddy currents. Non-destructive testing methods are widely used in the industry for quality control of products and materials. The term “non-destructive testing” usually refers to inspection methods for testing properties of materials and the quality of products without damaging or impairing the test objects.

Since many high-technology devices operate under extreme temperature and pressure conditions, or come into prolonged contact with chemically active materials, etc., it is important to develop non-destructive testing methods which ensure their safety and reliability. In fact, the main objective of non-destructive testing is to decide whether a device (or material) can successfully perform specified functions. Perfect devices or materials are rare. From a practical point of view, a material is considered to be of good quality if its parameters lie within specified tolerances. Therefore, the purpose of non-destructive testing can be formulated in the following way: to determine whether the relevant parameters of a material (or characteristics of a device) lie within prescribed limits.

Among the non-destructive testing methods in use today, are the following: X-rays, Messbauer analysis, neuron activation, ultrasound, acoustic emission, microwaves, dielectric spectroscopy, and eddy currents.

Eddy current testing has its origins with Michael Faraday’s discovery of electromagnetic induction in 1831. Nowadays eddy current testing devices are widely used for quality control of electrically conducting objects, such as metals, alloys and semiconductors. As early as 1879, Hughes recorded changes in the properties of an exciting-sensing coil placed in contact with metals of different conductivity and permeability. However, practical use of these effects in testing quality of materials started only after the Second World War. In the industry eddy current devices are used, for example, to control the size of products, to measure the diameter of wires and tubes, the thickness of walls and metal sheets. They are widely used to control the thickness of metal covering and the thickness of layers in multiplayer products. They can also be used to estimate the rate of destructive corrosion.

Another widespread field of applications of eddy current methods is the detection of flaws in materials: cracks, fiberings and non-metallic inclusions. Flaw detection is very important in the transport industry, including aircraft, ship and automobile. Another important application is the quality control of spot welding. Eddy current methods are widely used in the

nuclear industry and, in particular, to determine flaws and the thickness of walls in heat exchanger tubes.

Devices for non-destructive inspection in engineering are based on either theoretical or experimental research. Experiments are often used to develop the theory for particular cases. However, in many cases the problem of interest contains many parameters and it may be difficult to “connect” the output signal of a specific coil to the parameters of the flaw in a conducting medium.

Moreover, the necessity to have a theoretical generalization of experimental data, on the one hand, and analytical difficulties, on the other hand, incites one to consider particular cases for only some range of the parameters. In this case, the obtained results can have only a limited domain of applicability. In addition, it is often extremely difficult to interpret, numerically or graphically, the obtained solution of the problem. This happens if the solution contains integrals that cannot be expressed in terms of elementary or special, tabulated functions. However, these difficulties are overcome by approximating the solution.

Analytical solutions for the simplest cases have been derived and associated computer programs have been developed. But in order to solve more complicated problems one uses different approximation methods. In practice, one needs only know some integral characteristics of the solution, which, in some cases, can be expressed in terms of computationally suitable formulas.

In general, eddy current testing problems depend on several parameters. The use of mathematical models, along with the exact form of the solution, facilitates the study of the influence of these parameters on the output characteristics of the signal and on the testing process. Such a task may be costly and difficult, if at all possible, to achieve experimentally.

The present thesis represents a theoretical study on non-destructive inspection problems. In particular, summarizing all the above-mentioned problems, the thesis is devoted to the methods of solving eddy current testing problems and getting ways to simplify the obtained solutions in order to make them more adaptable to computer calculations and engineering practice. Mathematical methods are used in the thesis to solve several direct problems related to eddy current testing of conducting materials. One of the obstacles in using mathematical models of eddy current testing in engineering practice is the complexity of solutions. The approximate solutions developed in this thesis allow one to implement the results in eddy current testing. In addition, the simplified forms of the obtained solutions can be successfully used to solve important practical inverse problems in eddy current testing.

This thesis is divided into four parts. The first part describes the physics of eddy current testing method, the detailed derivation of the basic equations for the vector potential and its boundary conditions, and introduces the meaning of the induced change in impedance. The basic equations are Maxwell's equations for a linear, isotropic and homogeneous medium. By introducing the vector-potential, Maxwell's equations lead to Helmholtz' equation for the vector potential. Generally, in the literature on formulating the vector potential problem, the form of the vector potential (i.e. the number of non-zero components and dependent variables) and, consequently, its boundary conditions are taken into account only by considering the symmetry of the problem (for example, the geometry of the source of current) or even without any derivation. Thus, the detailed derivation of the structure of the vector potential seems to have been done only in the present thesis (see also the author's paper [36]) for the few mostly cases used in the literature. The form and the number of non-zero components of the vector potential are determined for the following cases: a double conductor line, a wire of finite length, and a single-turn coil above a uniform conducting half-space. As a generalization of these cases, the vector potential problem is formulated for a wire of an arbitrary form located in a vertical plane above a uniform conducting half-space.

The second part of this thesis considers exact analytical solutions to problems on electromagnetic waves spreading from emitters of different forms. It includes the integral representation of the solution for Helmholtz' vector equation in arbitrary orthogonal curvilinear coordinates; an exact analytical solution to the vector potential problem of a rectangular frame with current, and an exact solution to the vector potential problem of a wire of an arbitrary form with given current. Since Helmholtz' vector equation describes the eddy current problems, the integral representation of the solution to this equation is very important. In the integral representation of the solution to Helmholtz' vector equation known in the literature (see [59]), the vector potential of the electromagnetic field is expressed in terms of a triple integral of the product of the external current vector and the fundamental solution of Helmholtz' scalar equation. This representation has its simplest form in rectangular coordinates where the unit vectors $\vec{e}_x, \vec{e}_y, \vec{e}_z$ do not depend on the spatial coordinates. In the applications, other coordinate systems are also used. In the present thesis (see also the author's papers [17], [18]), the integral representation of the solution to Helmholtz' vector equation is obtained for a system of arbitrary orthogonal curvilinear coordinates where the unit vectors $\vec{e}_{q1}, \vec{e}_{q2}, \vec{e}_{q3}$ are prescribed functions of the spatial coordinates. As particular cases of the obtained representation, the integral representations of the solution to Helmholtz' vector equation are found for cylindrical and spherical coordinates.

The obtained representation of the solution to Helmholtz' vector equation is used in this thesis for the vector potential problems of a rectangular frame with current and of a wire of arbitrary form with given current. The reaction of a conducting half-space to a rectangular frame with current has been studied theoretically only in the case where the ratio of the frame's sides is 1:4 or smaller. In this case, a double conductor line is considered as a convenient and sufficiently accurate model of the rectangular frame. In this thesis (see also the author's paper [13]), an exact solution to the vector potential problem of the electromagnetic field induced by a rectangular frame with current is obtained without using any double conductor line approximation. Due to the linearity of the problem it is sufficient to find the vector potential of the electromagnetic field created by one side of the frame having the form of a straight segment and by the other side having the form of a circular arc. Different formulas known in the literature can be obtained from particular cases of the presented solutions. Note that in the literature the problem of electromagnetic waves spreading from a linear harmonic emitter is only solved in the so-called dipole approximation. The dipole approximation is used for the analysis of electromagnetic waves spreading under the assumption that the waves' length is much greater than the emitter's length. In non-destructive testing problems, however, the size of a defect situated in a conducting medium may be compared with the emitter's length, l, or it may be even larger than l. Hence, non-destructive testing methods require solutions to the problem of electromagnetic waves spreading from a linear harmonic emitter without using the dipole approximation. The solution is presented in this thesis (see also the author's paper [13]).

In the same section (see also the author's paper [15]), an exact solution is obtained for the problem of electromagnetic waves spreading from a wire of finite length having an arbitrary form. Writing the equation for the curve describing the wire in cylindrical polar and Cartesian coordinates, and using Helmholtz' equation and the integral representation of its solution, we obtain the solution in the form of a single definite integral of an elementary function. Moreover, using the obtained solution, some new formulas for electromagnetic waves spreading are also found for the particular cases of a wire in the form of Archimedes's spiral, of an elliptical or circular helix and in the form of fractals. The case of a fractal wire is interesting for antenna analysis in radio engineering.

The third part of this thesis is devoted to the analysis of the impedance change in homogeneous conducting media. Analyzing the influence of a uniform or non-uniform conducting media with different geometries on a source of current, that has the form of a double conductor line or a single-turn coil, one obtains the solution for the induced impedance change in terms of improper integrals (see [6]), whose integrands are either combinations of irrational and trigonometric functions, or combinations of irrational and Bessel functions, so that the two classes of definite integrals can be separated. In this thesis (see also the author’s papers [12], [8]), these new classes of definite integrals are evaluated in closed form by means of divergent integrals that converge in the sense of Abel [4]). Only one particular case of these integrals is known in the literature (see [48]), where the method used is also appropriate only for that case.

Particular cases of these integrals have been used for the evaluation of impedance change in the cases of a double conductor line and a single-turn coil located on the surface of a conducting half-space. In the case of a double conductor line, the expression for the impedance change has been evaluated in closed form, but in the case of a single-turn coil, the expression for the impedance has been transformed into the simpler form of a fast-convergent series. Furthermore, simple asymptotic formulas for the impedance of arbitrarily located double lines and coils are obtained in the limit as the frequency tends to infinity. These results are presented in the thesis (see also the author’s papers [9], [7]).

The change in impedance of a rectangular frame with current inside a conducting cylindrical tube has been studied theoretically only in the double conductor line approximation (see [6]). In this thesis (see also the authors’ papers [14], [35]), the exact analytical solution of a similar problem is obtained without using any approximation. As a possible application, the obtained solution can be used to determine the wall thickness of a tube directly under the frame for the case of non-concentric wall’s surfaces.

Finally, the fourth part of this thesis is devoted to some problems on the impedance change of media containing flaws (or defects). The exact analytical solution for the problem of the influence of a conducting medium with an arbitrary flaw on a source of current is not known. Therefore, since 1960 different approximate analytical and numerical methods for such problems have been used. Different approximate methods – two methods of additional currents – are developed in this thesis (see also the author’s papers [10], [11], [34]). The presence of additional currents in the region of a flaw is assumed by the first method. The direction of this current is opposite to the direction of the eddy current that flows in the same region in the absence of the flaw. The additional currents used in the second method are chosen so that the differential equation for a uniform conducting medium is transformed into a differential equation for the flaw. This problem has an exact analytical solution that allows one to estimate the error of both methods.

The next two sections are devoted to the proof of a new exact analytical formula for the impedance change and a well-known formula in the literature. The derivation of the new formula is based on Green’s formula while the Lorentz theorem is used for obtaining the other formula.

In the literature, the formula for the induced change in impedance, describing the influence of a conducting medium with a flaw of arbitrary shape on a source of current, seems to have been used before without a rigorous proof. In the present thesis (see also the author’s papers [19], [37], [38]), this formula is analytically proved and its correctness is analyzed. The formula has the form of a triple integral of a scalar product of amplitude electric field vectors.

The new formula for the impedance change has been found in the present thesis (see also the author’s papers [16], [19], [34], [37]). This new formula has the form of a triple integral of a scalar product of two vector potentials: the vector potential in the flaw and the vector potential in the same region in the absence of the flaw over the region containing the flaw.

The equivalence of these two formulas was proved (see the author’s papers [19], [37]). However, since analytical solutions of many problems contain the expressions for the vector potential, then the newly obtained formula is more convenient for calculations than the well-known formula.

ANOTĀCIJA

Promocijas darba galvenā tēma ir nesagraujošās kontroles problēmu matemātiskā analīze ar virpuļstrāvu metodes izmantošanu. Nesagraujošās kontroles metodes plaši izmanto industrijā, lai kontrolētu produkcijas un materiālu kvalitāti. "Nesagraujošās kontroles" termins attiecas uz inspekcijas metodēm, lai testētu materiālu īpašības un produkcijas kvalitāti bez testējamā objekta salaušanas vai sabojāšanas.

Tā kā daudzas augstas tehnoloģijas ierīces strādā ekstremālos temperatūras un spiediena apstākļos vai atrodas ilgstošā kontaktā ar ķīmiski aktīviem materiāliem, ir svarīgi attīstīt nesagraujošās kontroles metodes, kuras nodrošina ierīču darbību un drošību. Faktiski, nesagraujošās kontroles galvenais mērķis ir noteikt, vai ierīce (jeb materiāls) var sekmīgi veikt specifiskas funkcijas. Ideālas ierīces vai materiāli ir reti sastopami. No praktiskā viedokļa skatoties, materiālu uzskata par labas kvalitātes, ja tā parametri atrodas noteiktā pieļaujamā diapazonā. Tāpēc nesagraujošās kontroles galveno mērķi var formulēt sekojošā veidā: noteikt, vai materiāla attiecīgie parametri (jeb ierīces raksturojums) atrodas uzdotās robežās.

Šodienas nesagraujošās kontroles metožu teorijā atšķir sekojošas metodes: rentgenoskopijas analīze, Mesbauera analīze, neirona aktivizācijas, ultraskaņa, akustiska emisija, mikroviļņi, dielektriskā spektroskopija, un virpuļstrāva.

Virpuļstrāvu kontroles pamati nāk no Maikla Faradeja laikiem, kad viņš atklāja elektromagnētisko indukciju 1831. gadā. Mūsdienās virpuļstrāvu ierīces plaši izmanto, lai kontrolētu kvalitāti tādiem elektrību vadošiem objektiem kā metāli, sakausējumi un pusvadītāji. Jau 1879. gadā Hugs reģistrējis īpašību izmaiņas ierosmes-zondēšanas spolē, kura tika novietota kontaktā ar metāliem ar dažādu vadāmību un caurlaidību. Bet, lai testētu materiālus, šos efektus sāka izmantot praktiski tikai pēc otrā pasaules kara. Industrijā virpuļstrāvu ierīces izmanto, piemēram, lai kontrolētu produktu izmēru, lai mērītu vadu un cauruļu diametru, lai noteiktu biezumu sienām un metāla plātnēm. Tās arī plaši izmanto, lai kontrolētu metāla segumu biezumu un slāņu biezumu daudzslāņu vidē. Tādas ierīces var arī izmantot, lai novērtētu postošas korozijas ātrumu.

Cita plaša sfēra virpuļstrāvu metožu pielietošanai ir defektu atrašana metālos, t.i. plīsumi, noslāņošanās un nemetāla defekti. Defektu atrašana ir ļoti svarīga transporta industrijā, ieskaitot lidmašīnas, kuģus un automašīnas. Ļoti svarīgs pielietojums ir punktu metināšanas kvalitātes kontrole. Virpuļstrāvu metodes plaši izmanto kodolindustrijā un, piemēram, lai atrastu defektus un noteiktu sienu biezumu siltumapmaiņu caurulēm.

Inženierzinātnēs nesagraujošās kontroles ierīču darbība balstās vai nu uz teorētiskiem pētījumiem, vai arī uz eksperimentiem. Eksperimentus bieži izmanto, lai attīstītu teoriju atsevišķiem gadījumiem. Tomēr daudzos gadījumos problēma satur vairākus parametrus un ir grūti saistīt no speciālas spoles izejošo signālu ar defekta parametriem, ja defekts atrodas elektrību vadošā vidē.

Turklāt, eksperimentālo datu teorētiskā vispārinājuma nepieciešamība no vienas puses un analītiskā risinājuma grūtības no otras puses piespiež aplūkot atsevišķus gadījumus ar dažu parametru diapazonu. Šajā gadījumā iegūtiem rezultātiem būs pielietojums tikai ierobežotā sfērā. Pie tam, bieži ir ļoti grūti interpretēt, skaitliski vai grafiski, problēmas iegūto atrisinājumu. Tas notiek, ja atrisinājums satur integrāļus, kurus nav iespējams izteikt ar elementāro vai speciālo tabulas funkciju integrāļiem. Bet šīs grūtības ir pārvaramas ar atrisinājuma aproksimāciju.

Ir iegūti analītiski atrisinājumi vienkāršākajiem gadījumiem un izstrādāta atbilstoša programmatūra. Bet, lai risinātu sarežģītas problēmas, izmanto dažādas aproksimācijas metodes. Taču praktiski nepieciešams zināt tikai dažas īpašības atrisinājumam, kuru atsevišķos gadījumos var izteikt datoram piemērotu formulu veidā.

Vispār, virpuļstrāvu kontroles problēmas ir atkarīgas no dažiem parametriem. Matemātisko modeļu izmantošana kopā ar precīzo atrisinājumu atvieglo pētījumu par šo parametru ietekmi uz signāla izejošajām īpašībām un uz testēšanas procesu. Tāds uzdevums var būt dārgs un sarežģīts vai vispār nav sasniedzams eksperimentāli.

Dotais promocijas darbs ir teorētisks pētījums par nesagraujošās kontroles problēmām. Konkrēti, apkopojot visas iepriekš minētās problēmas, promocijas darbs ir veltīts virpuļstrāvu kontroles problēmu risināšanas metodēm, kā arī metodēm, kas palīdz vienkāršot iegūto problēmu atrisinājumus, lai adaptētu pēdējos skaitliskiem aprēķiniem datorā un praktiskiem pielietojumiem inženierzinātnēs. Matemātiskās metodes šajā promocijas darbā ir izmantotas, lai risinātu dažas tiešas problēmas, kas saistītas ar virpuļstrāvu kontroli materiāliem ar vadāmību. Viens no virpuļstrāvu kontroles matemātisko modeļu izmantošanas iemesliem inženieru praksē ir atrisinājumu sarežģītība. Tuvināti atrisinājumi, kas iegūti šajā darbā, dod iespēju iegūt rezultātus virpuļstrāvu kontrolē. Pie tam, iegūto atrisinājumu vienkāršotas formas var tikt sekmīgi izmantotas, lai risinātu svarīgas praktiskas inversās problēmas vilpuļstrāvu kontrolē.

Promocijas darbs ir sadalīts 4 daļās. Pirmajā daļā aprakstīta virpuļstrāvu metodes fizika, vektora potenciāla pamatvienādojumu un robežnosacījumu detalizēta iegūšana, un inducētas izmaiņas impedancē ieviešana. Par pamatvienādojumiem uzskata Maksvela vienādojumu sistēmu lineārai, izotropai un homogēnai videi. Vektora potenciāla ieviešana Maksvela vienādojumus reducē uz Helmholca vienādojumu vektora potenciālam. Literatūrā, formulējot problēmu vektora potenciālam, vektora potenciāla forma (t.i. nenulles komponentes un atkarīgie mainīgie) un, sekojoši, tās robežnosacījumi ir ņemti bez strāvas avota ģeometrijas apskata vai vispār bez kāda pierādījuma. Šķiet, ka detalizēts pierādījums ir izdarīts tikai šajā promocijas darbā (sk. arī autora rakstu [36]), tas veikts četrām literatūrā visbiežāk sastopamajām problēmām par vektora potenciālu. Vektora potenciāla forma un nenulles komponentu skaits ir definēti gadījumiem ar divvadu līniju, ar galīga garuma vadu un ar viena vijuma spoli ar strāvu virs homogēnas elektrību vadošas pustelpas. Kā šo gadījumu vispārinājums, vektora potenciāla problēma ir noformulēta patvaļīgas formas vadam, kurš novietots vertikālajā plaknē virs homogēnas elektrību vadošas pustelpas.

Promocijas darba otrajā daļā tiek apskatīts analītisks atrisinājums problēmai par elektromagnētisko viļņu izplatību no dažādu formu izstarotājiem. Šī daļa satur atrisinājumu vektoriālam Helmholca vienādojumam integrālā formā patvaļīgā ortogonālā līklīniju koordinātu sistēmā; precīzu analītisku atrisinājumu problēmai par vektora potenciālu taisnstūrveida rāmītim ar strāvu, un precīzu analītisku atrisinājumu problēmai par vektora potenciālu patvaļīgas formas vadam ar uzdoto strāvu vektora potenciālu. Sakarā ar to, ka vektoriālais Helmholca vienādojums apraksta virpuļstrāvu problēmas, atrisinājums šim vienādojumam integrālā formā ir ļoti svarīgs. Literatūrā pazīstams atrisinājums vektoriālam Helmholca vienādojumam ir dots integrālā formā, elektromagnētiskā lauka vektora potenciāls ir izteikts ar trīskāršo integrāli no strāvas blīvuma vektora un Helmholca vienādojuma fundamentālā atrisinājuma skalārā reizinājuma. Tādu atrisinājumu visvieglāk izmantot taisnleņķa koordinātu sistēmā, kurā vektori $\vec{e}_x, \vec{e}_y, \vec{e}_z$ nav atkarīgi no koordinātēm. Bet, izstrādājot šo promocijas darbu, parādījās nepieciešamība izmantot citas koordinātu sistēmas. Sakarā ar to, ka literatūrā atrisinājums integrālā formā nav atrodams, šajā promocijas darbā (sk. arī autora rakstu [17], [18]), iegūts atrisinājums vektoriālam Helmholca vienādojumam integrālā formā patvaļīgā ortogonālā līklīniju koordinātu sistēmā, kurā vienības vektori $\vec{e}_{q1}, \vec{e}_{q2}, \vec{e}_{q3}$ ir dotas koordinātu funkcijas. Kā partikulārie gadījumi no šī atrisinājuma iegūti atrisinājumi vektoriālam Helmholca vienādojumam integrālā formā cilindriskā un sfēriskā koordinātu sistēmā.

Iegūtais atrisinājums vektoriālam Helmholca vienādojumam tiek izmantots problēmai par vektora potenciālu taisnstūrveida rāmītim ar strāvu, kā arī problēmai par vektora potenciālu patvaļīgas formas vadam ar uzdoto strāvu, kas aplūkoti tajā pašā nodaļā. Elektrību vadošas pustelpas reakcija uz taisnstūrveida rāmīti ar strāvu tika pētīta teorētiski tikai gadījumā, kad rāmīša malu proporcija ir 1:4 vai mazāka. Šajā gadījumā, divvadu līniju uzskata par ērtu un pietiekoši precīzu taisnstūrveida rāmīša modeli. Šajā promocijas darbā (sk. arī autora rakstu [13]), precīzs atrisinājums problēmai par elektromagnētiskā lauka vektora potenciālu, kas inducēts taisnstūrveida rāmītim ar strāvu, iegūts bez divvadu aproksimācijas. Problēmas linearitātes dēļ, pietiek atrast vektora potenciālu elektromagnētiskajam laukam, kas izveidots ar vienu rāmīša malu, kurai ir taisnas līnijas forma, un otru malu, kurai ir riņķa līnijas loka forma. Atsevišķos gadījumos no atrastā atrisinājuma var iegūt dažādas literatūrā pazīstamas formulas. Bet literatūrā problēma par elektromagnētisko izstarojumu no lineārā harmoniskā izstarotāja risināta tikai tā saucamajā dipolu aproksimācijā. Tāda veida aproksimāciju izmanto elektromagnētisko viļņu izplatīšanas analīzei, pieņemot, ka viļņu garums ir daudzreiz lielāks nekā izstarotāja garums. Tomēr nesagraujošās kontroles problēmās, defekta izmērs, kas atrodas vidē ar vadāmību, mēdz būt samērojams ar izstarotāja garumu l vai arī lielāks nekā l. Tāpēc problēmu par elektromagnētisko viļņu izplatību no lineāra harmoniska izstarotāja vajag risināt bez dipolu aproksimācijas. Tas tika izdarīts šajā promocijas darbā (sk. arī autora rakstu [13]).

Vienā no otrās daļas paragrāfiem (sk. arī autora rakstu [15]), precīzs atrisinājums iegūts problēmai par elektromagnētisko viļņu izplatību no galīga garuma patvaļīgas formas vada. Sastādot vienādojumu līnijai, kas apraksta vadu cilindriskajā un Dekarta koordinātu sistēmā, un izmantojot Helmholca vienādojumu un tā atrisinājuma integrālo formu, iegūstam atrisinājumu, kuram ir vienkārša noteiktā integrāļa forma no elementāras funkcijas. Pie tam, izmantojot iegūto atrisinājumu, atrod dažas jaunas formulas par elektromagnētisko viļņu izplatību gadījumiem ar vadu, kuram ir Arhimēda spirāles forma, eliptiskas vai riņķveida spirāles forma, kā arī fraktālam vadam. Gadījums ar fraktālo vadu ir interesants telekomunikācijas nozarei antenu analīzei.

Trešajā daļā tiek analizētas izmaiņas impedancē homogēnas elektrību vadošas pustelpas dēļ. Analizējot homogēnas vai nehomogēnas elektrību vadošas vides ar dažādu ģeometriju ietekmi uz strāvas avotu, kuram ir divvadu līnijas vai viena vijuma spoles forma, atrisinājumu izmaiņām impedancē ieguvām neīsto integrāļu formā (sk. [5]). Šo integrāļu zemintegrāļa funkcija ir vai nu iracionālo un trigonometrisko funkciju kombinācija, vai arī iracionālo un Beseļa funkciju kombinācija. Tas ļauj atdalīt divas noteikto integrāļu klases. Šajā promocijas darbā (sk. arī autora rakstu [12], [8]) šīs noteikto integrāļu jaunās klases novērtē slēgtā formā, izmantojot diverģējošos integrāļus, kas konverģē Ābeļa nozīmē. Literatūrā zināms tikai viens šo integrāļu atsevišķs gadījums (sk. [49]), bet tur izmantotā metode ir pielietojama tikai aprakstītajam gadījumam.

Šo integrāļu atsevišķus gadījumus izmanto impedances novērtēšanai gadījumos ar divvadu līniju un viena vijuma spoli, novietotiem uz elektrību vadošas pustelpas virsmas. Divvadu līnijas gadījumā impedances izteiksmi novērtē slēgtā formā, bet gadījumā ar viena vijuma spoli, impedances izteiksmi var transformēt vienkāršākā formā, kas satur ātri konverģējošas rindas. Turklāt, vienkārša asimptotiska formula impedancei iegūta gadījumā ar patvaļīgi novietotu divvadu līniju un viena vijuma spoli, kad frekvence tiecas uz bezgalību. Iegūtais rezultāts arī pretendē uz oriģinalitāti un piedāvāts šajā promocijas darbā (sk. arī autora rakstu [9], [7]).

Izmaiņas impedancē taisnstūrveida rāmītim ar strāvu, kas novietots cilindriskas elektrību vadošas caurules iekšpusē, teorētiski tika pētītas tikai izmantojot divvadu aproksimāciju (sk. [5]). Šajā promocijas darbā (sk. arī autora rakstu [14], [35]) līdzīgai problēmai atrasts precīzs analītisks atrisinājums, bet bez aproksimācijas izmantošanas. Viens no atrisinājuma iespējamiem pielietojumiem var būt caurules sienas biezuma noteikšana gadījumā, ja caurule ar nekoncentriskām sienām atrodas tieši zem rāmīša.

Beidzot, promocijas darba ceturtā daļa veltīta dažādām problēmām par izmaiņām impedancē videi, kas satur defektu. Literatūrā nav zināms precīzs analītisks atrisinājums problēmai par vides ar vadāmību, kura satur patvaļīgas formas defektu, ietekmi uz strāvas avotu. Tāpēc, sākot no 1960. gada, izmanto dažādas analītiskas un skaitliskas aproksimācijas metodes, lai risinātu šāda veida problēmas. Citas aproksimācijas metodes – divas metodes, kas izmanto papildstrāvu – tika izstrādātas šajā promocijas darbā (sk. arī autora rakstu [10], [11], [34]). Saskaņā ar pirmo metodi pieņemts, ka defekta apgabalā eksistē papildstrāva. Šīs strāvas virziens ir pretēji vērsts tās strāvas virzienam, kura tek tajā pašā apgabalā, tikai gadījumā, kad defekta tur nav. Papildstrāvu, kuru izmanto otrajā metodē, izvēlās tā, lai diferenciālvienādojumus homogēnai elektrību vadošai videi varētu pārveidot par diferenciālvienādojumiem defektam. Šai problēmai eksistē precīzs analītisks atrisinājums un tas dod iespēju novērtēt kļūdu abām metodēm.

Nākamie divi paragrāfi apraksta pierādījumu jaunai precīzai analītiskai formulai par izmaiņām impedancē, kā arī literatūrā pazīstamai formulai. Pierādījums jaunai formulai balstās uz Grīna formulas, bet otras formulas pierādījumam izmanto Lorenca teorēmu. Šķiet, ka literatūrā formula izmaiņām impedancē, kas raksturo elektrību vadošas pustelpas, kas satur patvaļīgas formas defektu, ietekmi uz strāvas avotu, tika izmantota bez stingra pierādījuma. Dotajā promocijas darbā (sk. arī autora rakstu [19], [37], [38]), šī literatūrā pazīstamā formula tika analītiski pierādīta un analizēts tās patiesums. Formulai ir trīskāršā integrāļa forma pa apgabalu, kas satur defektu, no divu elektriskā lauka vektoru skalārā reizinājuma.

Arī šajā promocijas darbā tika iegūta jauna formula izmaiņām impedancē (sk. arī autora rakstu [16], [19], [34], [37]). Formula iegūta trīskāršā integrāļa formā pa apgabalu, kas satur defektu, no divu vektoru potenciālu skalārā reizinājuma: vektora potenciālu defektā un vektora potenciālu tajā pašā apgabalā, gadījumā, kad defekta nav.

Stingri pierādīts, ka jaunā vienkāršākā formula ir ekvivalenta formulai, kas izmantota literatūrā (sk. arī autora rakstu [19], [37]). Bet sakarā ar to, ka analītiski atrisinājumi daudzām problēmām satur izteiksmes vektora potenciālam, jaunā formula izmaiņām impedancē ir ērtāka skaitļošanai nekā literatūrā zināmā formula.

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INTRODUCTION

The introductory part of the thesis is devoted to a review of the literature. A full review of theoretical papers on eddy current methods for non-destructive testing published up to 1997 is given in monograph [6]. Therefore, only the most important theoretical papers on eddy current testing, which are close to the topic of the thesis and published up to 2004, are considered below. The review is divided into two parts, where the first part deals with papers on the influence of a homogeneous conducting medium on an emitter, while the second part analyzes problems on inhomogeneous conducting medium containing a defect (or a flaw).

1. One of the first papers dealing with problems on homogeneous conducting medium is the paper [62], published in 1963. In this paper, an exact formula is obtained for the part of vector potential, induced into a single-turn coil (and also a superposed coil) due to the presence of a conducting half-space. Besides, the full complex resistance (so-called impedance), $Z = Z_c(\alpha, \beta)$, is calculated by means of the formula

$$Z_c = j\beta \int_0^\infty \frac{s - \sqrt{s^2 + j}}{s + \sqrt{s^2 + j}} J_1^2(\beta s) e^{-2\hat{\alpha}\beta s} ds := X + jY \,, \qquad (0.1)$$

where $X = \operatorname{Re} Z_c$, $Y = \operatorname{Im} Z_c$, $j = \sqrt{-1}$, $J_1(z)$ is the Bessel function of the first kind of order 1, $\hat{\alpha} = 2h / r_c$ and $\beta = r_c \sqrt{\omega\sigma\mu_0}$ are dimensionless variables, r_c is the radius of the coil, h is the height of the coil above the conducting medium, ω is the frequency, σ is the conductivity of the medium, μ_0 is the magnetic constant. Unfortunately, integral (0.1) cannot be expressed in terms of known elementary or special functions. In [49] integral (0.1) is calculated approximately assuming that the parameter β is small (i.e. assuming that either the coil radius is small, or the frequency or the conductivity of the medium is small). In this case, Z_c is expressed in terms of an elliptic integral. In turn, in Section 3.2.2 of the thesis, integral (0.1) is represented in the form of a fast-convergent series for all values of the parameter β (see also the author's papers [9], [7]):

$$Z_c(\beta) = 8j \sum_{n=1}^{\infty} \frac{(-1)^n (\sqrt{j}\beta)^{n+1}}{n(n+1)(n+2)(n+3)} \frac{1}{\Gamma^2(n/2)} \,, \qquad (0.2)$$

where $\Gamma(z)$ is the gamma function. The solution corresponds to the case $\hat{\alpha} = 0$, i.e. where the single-turn coil is located on the surface $z = 0$. Series (0.2) converges very rapidly so that if

$\beta \le 3$, then it is sufficient to take the first five terms of the series for having a computational error less than 3% compared with the exact solution. Moreover, in this thesis, an exact asymptotic of integral (0.1) as $\beta \to \infty$ and $\hat{\alpha} \ne 0$ is found (see Section 3.2.3 and the author's papers [9], [7]):

$$\lim_{\beta\to\infty} Z_c = -j\frac{1}{\pi\theta}\left[(2-\theta^2)K(\theta) - 2E(\theta)\right], \qquad \theta = \frac{1}{\sqrt{1+\hat{\alpha}^2}}, \tag{0.3}$$

where $K(\theta)$ and $E(\theta)$ are the full elliptic integrals of the first and second kind, respectively, tabulated, for instance, in [53].

After the first paper by Sobolev [62], different aspects of this problem were considered, for instance, in [23], [26], [30], [64] and [49]. As a generalization of the problem considered in [62], the problem on the influence of a flat conducting layer on a single-turn coil is considered in [46], [63], [52], and the problem on the influence of a multi-layer conducting medium on a single-turn coil or double conductor line is considered in [6], [29], [51], [24] and [40].

Eddy current testing problems can be reduced to a two-dimensional problem if the flaw in the inhomogeneous half-space is a cylindrical body coaxial with a single-turn coil carrying the external current (see [66], [67]), or if the flaw is an infinitely long cylinder parallel to the double conductor line carrying the external current (see [1]). Similar reduction takes place in the case of a homogeneous conducting half-space where the external current is generated either by a single-turn coil or double conductor line in the plane parallel to the half-space.

In some applications, the source of current has the form of a rectangular frame located above the conducting half-space in the plane parallel to the plane $z = 0$. It is shown in [49] that if the ratio of the frame's sides is 1:4 or smaller, then the so-called double conductor line (see Fig.1.2) may be considered as a convenient and sufficiently accurate model of a rectangular probe. In this case, the formula for impedance has the form

$$Z_l = j\int_0^\infty \frac{s-\sqrt{s^2+j}}{s+\sqrt{s^2+j}} e^{-2\hat{\alpha}\beta s}(1-\cos\beta s)\frac{ds}{s} := X + jY, \tag{0.4}$$

where $\hat{\alpha} = h/d$, $\beta = d\sqrt{\omega\sigma_1\mu_0}$, h is the height of the double conductor line above the surface of the conducting half-space, $d = y_1 - y_0$ is the distance between the wires. In this thesis (see Section 3.2.1 and also the author's papers [9], [7]), integral (0.4) has been calculated in the exact analytical form for $\hat{\alpha} = 0$ (i.e. the case when the double line is located on the surface $z = 0$):

$$Z_l\big|_{\hat{\alpha}=0} = \frac{\pi}{4} + j\left[\frac{1}{2} - C - \ln\frac{\beta}{2}\right] - 2\left[\frac{1}{\beta^2} - \frac{1+j}{\beta\sqrt{2}} K_1(\beta\sqrt{j})\right], \quad (0.5)$$

where $K_1(z)$ is the modified Bessel function of the second kind of order 1 and $C = 0{,}577215\ldots$ is the Euler constant. The Bessel functions in (0.5) can be expressed in terms of the Kelvin functions, $\mathrm{ker}_1(\beta)$ and $\mathrm{kei}_1(\beta)$, tabulated, for instance, in [53]:

$$K_1(\beta\sqrt{j}) = j\,[\mathrm{ker}_1(\beta) + j\,\mathrm{kei}_1(\beta)]. \quad (0.6)$$

Asymptotic of integral (0.4) as $\beta \to \infty$ and $\hat{\alpha} \neq 0$ has also been obtained in the present thesis (see Section 3.2.3 and also the author's paper [9]), and it has the form

$$\lim_{\beta\to\infty} Z_l = -j\ln\frac{\sqrt{4\hat{\alpha}^2 + 1}}{2\hat{\alpha}}. \quad (0.7)$$

Computational results of hodograph (i.e. curves $X = X(\beta)$ and $Y = Y(\beta)$ for different fixed values of $\hat{\alpha}$), obtained by means of the exact analytical formula (0.4) practically coincide with the simple asymptotic formula (0.7) if the parameter β satisfies the inequality $\beta \geq 10$. The asymptotic formula (0.7) is also valid in the case the coil carrying an alternating current is located above a plate of finite thickness d. An exact analytical solution of this problem is found in [46] and [63]. The results for a half-space and a plate practically coincide due to the so-called skin effect as $\beta \to \infty$ (i.e. as $\omega \to \infty$), when induced currents in the plate are concentrated near the plate's surface $z = 0$, so that the plate's thickness d does not influence the asymptotic value of the impedance Z as $\beta \to \infty$.

Another form of a conducting medium that has many applications is a conducting tube of cylindrical shape. In this case, the emitter having the form of a single-turn coil or a rectangular frame carrying the current is located either inside or outside the tube (see [2], [25], [33], [43], [45], [30], [31], [32], [39]). In many cases the excitation coil is located in a concentric position with respect to the sample to be tested. If the axis of the coil does not coincide with the tube's axis, the solution of the problem becomes more complicated. In [1] the coil is displaced so that the tube's axis is situated on the coil. In Section 3.3 of this thesis (see also the author's papers [14], [35]), an analytical solution is obtained for the following two problems on the influence of the tube's wall on the emitters located inside the tube.

(1) The emitter has the form of a linear segment of finite length. The emitter is located parallel to the tube's walls, but not on the tube's axis;

(2) The emitter has the form of a circular arc of finite length with the center situated on the tube's axis.

The solution of these two problems allows one to obtain an analytical solution on the influence of circular tube's walls on a rectangular frame with current. The frame consists of two parallel linear segments and two parallel circular arcs. The location of such a frame in the vicinity of the tube's walls gives information about the thickness of the walls situated under the emitter, i.e. the information about the tube walls' deterioration.

2. The second group of problems, important for applications, is on the influence of a conducting medium, containing flaws of different conductivities, on an emitter. These problems have approximate analytical solutions in the following two cases: (1) the case of a homogeneous half-space as a conducting region, when the source of the external current is located in the plane parallel to the half-space and it has the form either of a single-turn coil or a double conductor line; (2) the case where the flaw of the inhomogeneous half-space has the form of a cylindrical body coaxial with a single-turn coil carrying the external current (see [66], [67]), or if the flaw is an infinitely long cylinder parallel to the double conductor line carrying the external current (see [1]).

One of these assumptions corresponds to the case where the conductivities, σ_1 and σ_2, of a conducting medium and a flaw, respectively, do not differ by much. In this case one can define a small parameter $\varepsilon = 1 - \sigma_2/\sigma_1$ ($\sigma_2 < \sigma_1$) and consider a perturbation series with respect to ε. Using the perturbation series in ε, one can solve a wide range of problems in eddy current testing. This is the so-called perturbation method (see [3]). Such situation often occurs in practice, for instance, in eddy current testing of weld joints. Assuming that σ_1 is the conductivity of the welding region of an aircraft body and σ_2 is the conductivity of the aircraft body, then the parameter is varying in the range $0.05 \le \varepsilon \le 0.3$ (see [66]). This method was used in [66] to find an approximate analytical solution for the problem on the influence of a cylindrical flaw on a single-turn coil carrying the current. Later this result was obtained in [42] and [44]. In the second part of monograph [6], the application of this method is considered for conducting media and flaws of different shapes.

A different approximate method - the method of additional currents – is developed in this thesis in Section 4.1 (see also the author's papers [10], [11], [34]). The main idea of the method is as follows. Instead of the region containing a flaw, the problem is solved for a uniform medium containing a source of additional current located in the region of the flaw. For example, if $\sigma_F = 0$, then in region V_F the additional source of current is taken to be opposite compared with the current that exists in the uniform medium containing the flaw. For small values of the parameter $\varepsilon = 1 - \sigma_2/\sigma_1$, this method coincides with the perturbation method. But the advantage of the method of additional currents is that it can be applied to the whole range of values of the parameter ε: $0 < \varepsilon \le 1$.

Another approximate method used for such problems is the layer approximation. The method was suggested in [47]. Using this method, the region of the flaw, V_F, is replaced by a region $\tilde{V}_F$ which has the form of a flat horizontal layer with the same conductivity σ_F as the region V_F. The layer is located in region $z_1 \le z \le z_2$. The planes $z = z_1$ and $z = z_2$ are tangential planes to region V_F. For such region $\tilde{V}_F$, the problem can be solved analytically.

In order to calculate the impedance change induced by the presence of the flaw, we substitute the obtained solution for $\vec{E}_F$ into the formula for the impedance change (see [20], [21]):

$$Z^{\text{ind}} = -\frac{(\sigma_F - \sigma)}{I^2} \iiint_{V_F} \vec{E} \cdot \vec{E}_F \, dV, \tag{0.8}$$

where V_F is the region of the flaw, σ_F and σ are the conductivities of the flawed and flawless regions, respectively, $\vec{E}_F$ is the complex-valued amplitude electric field vector in the flawed region, $\vec{E}$ is the complex-valued amplitude electric field vector in the same region in the absence of the flaw, I is the amplitude of the current density. Equation (0.8) is also used in [28], [50], [27], [47], [22].

A more convenient formula is obtained in this thesis (see Section 4.2.2 and also the author's papers [19], [16], [34], [37]) for calculating the impedance change:

$$Z^{\text{ind}} = \frac{\omega^2(\sigma_F - \sigma)}{I^2} \iiint_{V_F} \vec{A} \cdot \vec{A}_F \, dV, \tag{0.9}$$

where $\vec{A}_F$ is the complex-valued amplitude vector potential in the flawed region, $\vec{A}$ is the complex-valued amplitude vector potential in the same region in the absence of the flaw.

Note that the relationship between the vectors $\vec{E}$ and $\vec{A}$ in the case of harmonic oscillations of the external current with frequency ω is given by (see [6]):

$$\vec{E} = -j\omega\vec{A} + \frac{1}{\tilde{k}_1^2} \operatorname{grad} \operatorname{div} \vec{A}, \tag{0.10}$$

where $\tilde{k}_1^2 = \mu_0\mu(\sigma + j\varepsilon_0\hat{\varepsilon}\omega)$ if the displacement current is taken into account and $\tilde{k}_1^2 = \mu_0\mu\sigma$, if the displacement current is neglected, ε_0 and μ_0 are the electric and magnetic constants, respectively; $\hat{\varepsilon}$ and μ are the relative permittivity and relative magnetic permeability of the medium, respectively.

In Section 4.2 of this thesis (see also the author's papers [19], [37]), the equivalence of Eqs. (0.8) and (0.9) is proved. Since analytical solutions of many problems contain expressions for the vector potential $\vec{A}$, then formula (0.9) is more convenient for calculations than formula (0.8). Besides, in order to calculate the impedance change, the authors in [50] and [47] performed an incorrect substitution of the vector $\vec{E}$ by the vector $\vec{A}$ using Eq. (0.9). The point is that, in the simplest case, Eqs. (0.8) and (0.9) coinside if $\operatorname{div}\vec{A}=0$, although in general case, $\operatorname{div}\vec{A}\neq 0$. Thus, in the previous studies (see [50], [47]), it is assumed that $\operatorname{div}\vec{A}=0$ in Eq. (0.10). Besides, in [50] it is assumed that the scalar potential gives the change in the static field only. That statement is not true. In [47] it is suggested to use the Coulomb's gauge, i.e. $\operatorname{div}\vec{A}=0$, but the authors use the following equation for the vector potential $\vec{A}$:

$$\Delta\vec{A}+k^2\vec{A}=\mu_0\mu\vec{I}^{\,\mathrm{ext}},\ k^2=-j\omega\sigma\mu_0\mu\,. \tag{0.11}$$

It is well known that Eq. (0.11) is not correct in this case. In fact, in the case of Coulomb's gauge the equation for the vector potential is more complicated (see [6], p.10), and it has the form

$$\Delta\vec{A}=\mu_0\mu\sigma\left(\nabla\varphi+\frac{\partial\vec{A}}{\partial t}\right)+\mu_0\varepsilon_0\mu\hat{\varepsilon}\frac{\partial}{\partial t}\left(\nabla\varphi+\frac{\partial\vec{A}}{\partial t}\right)-\mu_0\mu\vec{I}^{\,e}, \tag{0.12}$$

where φ is the scalar potential. That fact made us to prove and verify the correctness of Eq. (0.8) in Section 4.2.3 of the present thesis (see also the author's papers [19], [37] and [38]).

1. FORMULATION OF THE PROBLEM AND BASIC EQUATIONS

1.1. Physics of eddy current method

The main topic of this thesis is the analysis of non-destructive testing problems by the eddy current methods. The eddy current method is based on the law of electromagnetic induction, i.e. on inducing electrical currents in the material being inspected and observing the interaction between these currents and the material. The idea of the method is as follows (see Fig. 1.1). Suppose that a coil carrying an alternating current, $\vec{I}^e$, is situated in the vicinity of a conducting medium to be tested. The current, passing through the coil (so-called excitation coil), generates a varying magnetic field, Φ_p. This magnetic field (so-called primary field) induces varying currents (known as eddy currents because of their circulatory paths) in the electrically conducting medium according to the principle of electromagnetic induction. These currents, in turn, produce a varying magnetic field, Φ_s, (so-called secondary field). The effects of the secondary field can be seen from the variation of the output signal of the excitation coil or from the output signal of a second coil (so-called detector coil) situated nearby. In general, the output signal represents the resultant field, that is, the sum of the primary and secondary fields.

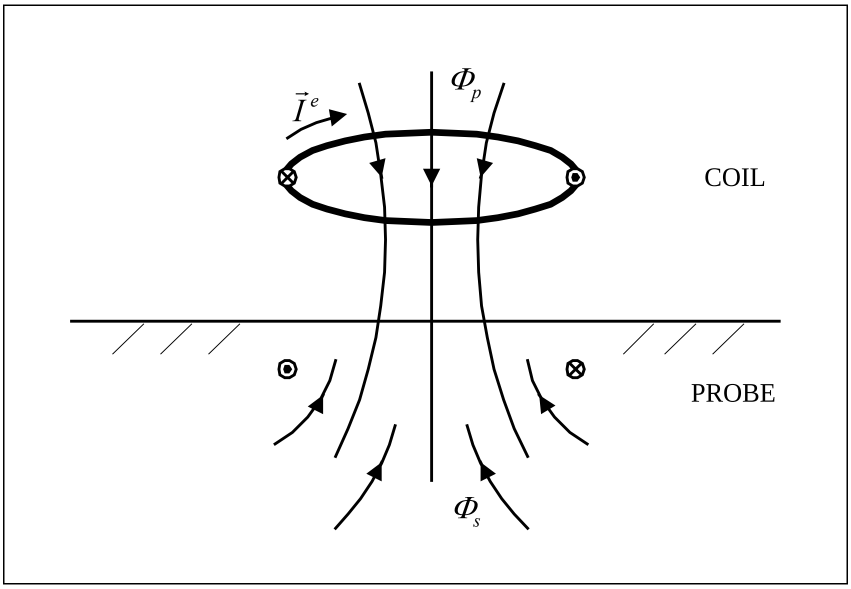

Fig.1.1. A coil carrying alternating current above a conducting medium

The output signal of the detector coil depends on several parameters, such as the magnitude and frequency of the alternating current, the electrical conductivity and magnetic permeability of the medium, as well as the relative position of the coil and the medium. It also reflects the presence of inhomogeneities (so-called flaws) in the medium. The eddy current flow changes the coil impedance. Hence, it is important to study the basic principles of the interaction of eddy currents in non-uniform conducting media. By monitoring the coil impedance, the electrical, magnetic and geometric properties of the probe can be measured. Thereby, the most general formulation of the main problem of eddy current flaw inspection and control can be stated as a problem of finding a dependence of a detector coil's parameters on a flaw's shape, its size, properties of the medium in the flawed region, and its location with respect to the detector coil.

Eddy current responses are conveniently described by reference to the "impedance plane". This is a graphical representation of the complex probe impedance where the abscissa (X value) represents the resistance and the ordinate (Y value) represents the inductive reactance. Note that, while the general form of the impedance plane remains the same, the details are unique for a particular probe and frequency.

1.2. Basic equations

Maxwell's equations for a homogeneous isotropic medium are

$$\operatorname{curl} \vec{\tilde{E}} = -\frac{\partial \vec{\tilde{B}}}{\partial t}, \tag{1.1}$$

$$\operatorname{curl} \vec{\tilde{B}} = \mu \left(\vec{\tilde{I}} + \vec{\tilde{I}}^{e} + \frac{\partial \vec{\tilde{D}}}{\partial t} \right), \tag{1.2}$$

$$\operatorname{div} \vec{\tilde{B}} = 0, \tag{1.3}$$

$$\operatorname{div} \vec{\tilde{D}} = \tilde{\rho}, \tag{1.4}$$

$$\vec{\tilde{I}} = \sigma \vec{\tilde{E}}, \tag{1.5}$$

$$\vec{\tilde{B}} = \mu_0 \mu \vec{\tilde{H}}, \tag{1.6}$$

$$\vec{\tilde{D}} = \varepsilon_0 \hat{\varepsilon} \vec{\tilde{E}}, \tag{1.7}$$

where $\vec{\tilde{E}}$ and $\vec{\tilde{H}}$ are the electric and magnetic field vectors, respectively; $\vec{\tilde{D}}$ and $\vec{\tilde{B}}$ are the electric and magnetic induction vectors, respectively; $\vec{\tilde{I}}$ is the current vector density; $\vec{\tilde{I}}^{e}$ is the external current vector density; σ is the conductivity; ε_0 and μ_0 are the electric and magnetic constants, respectively; $\hat{\varepsilon}$ and μ are the relative permittivity and relative magnetic permeability of the medium, respectively; and $\tilde{\rho}$ is the charge density.

Equations (1.1)-(1.7) can be rewritten in another convenient form in terms of the vector potential $\vec{\tilde{A}}$ of the magnetic field. This magnetic vector potential is introduced by the relation

$$\vec{\tilde{B}} = \operatorname{curl} \vec{\tilde{A}}. \tag{1.8}$$

Using Eq. (1.3) one can rewrite Eq. (1.1) in the form:

$$\operatorname{curl}\left(\vec{\tilde{E}} + \frac{\partial \vec{\tilde{A}}}{\partial t} \right) = 0, \tag{1.9}$$

where the vector $\vec{\tilde{E}} + \frac{\partial \vec{\tilde{A}}}{\partial t}$ is the potential. The scalar electric potential $\tilde{\varphi}$ is defined by the relation:

$$\vec{\tilde{E}} + \frac{\partial \vec{\tilde{A}}}{\partial t} = -\operatorname{grad} \tilde{\varphi}, \tag{1.10}$$

where

$$\operatorname{grad} = \nabla = \frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z. \tag{1.11}$$

Substituting Eq. (1.10) into Eqs. (1.5) and (1.7), we have

$$\vec{\tilde{I}} = -\sigma\left(\nabla\tilde{\varphi} + \frac{\partial \vec{\tilde{A}}}{\partial t} \right), \tag{1.12}$$

$$\vec{\tilde{D}} = -\varepsilon_0\hat{\varepsilon}\left(\nabla\tilde{\varphi} + \frac{\partial \vec{\tilde{A}}}{\partial t} \right). \tag{1.13}$$

Substituting Eqs. (1.8), (1.12) and (1.13) into Eq. (1.2), and using the formula $\operatorname{curl}\operatorname{curl}\vec{\tilde{A}} = \nabla \operatorname{div}\vec{\tilde{A}} - \Delta\vec{\tilde{A}}$, yields

$$\nabla \operatorname{div}\vec{\tilde{A}} - \Delta\vec{\tilde{A}} = -\mu_0\mu\sigma\left(\nabla\tilde{\varphi} + \frac{\partial \vec{\tilde{A}}}{\partial t} \right) - \mu_0\varepsilon_0\mu\hat{\varepsilon}\frac{\partial}{\partial t}\left(\nabla\tilde{\varphi} + \frac{\partial \vec{\tilde{A}}}{\partial t} \right) + \mu_0\mu\vec{\tilde{I}}^e, \tag{1.14}$$

where $\Delta = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian in three dimensions. Eq. (1.14) can be rewritten as:

$$\nabla\left[\operatorname{div}\vec{\tilde{A}} + \mu_0\mu\sigma\tilde{\varphi} + \mu_0\mu\varepsilon_0\hat{\varepsilon}\frac{\partial\tilde{\varphi}}{\partial t} \right] - \Delta\vec{\tilde{A}} = -\mu_0\mu\sigma\frac{\partial\vec{\tilde{A}}}{\partial t} - -\mu_0\varepsilon_0\mu\hat{\varepsilon}\frac{\partial^2\vec{\tilde{A}}}{\partial t^2} + \mu_0\mu\vec{\tilde{I}}^e. \tag{1.15}$$

For given $\vec{\tilde{B}}$ and $\vec{\tilde{E}}$, the vector potential $\vec{\tilde{A}}$ is not uniquely defined by Eq. (1.8). Therefore, to determine $\vec{\tilde{A}}$ uniquely an additional condition, *Lorentz' gauge*, can be used:

$$\operatorname{div}\vec{\tilde{A}} + \mu_0\mu\sigma\tilde{\varphi} + \mu_0\varepsilon_0\mu\hat{\varepsilon}\frac{\partial\tilde{\varphi}}{\partial t} = 0\,. \tag{1.16}$$

In this case, Eq. (1.15) has the form:

$$\Delta\vec{\tilde{A}} = \mu_0\varepsilon_0\mu\hat{\varepsilon}\frac{\partial^2\vec{\tilde{A}}}{\partial t^2} + \mu_0\mu\sigma\frac{\partial\vec{\tilde{A}}}{\partial t} - \mu_0\mu\vec{\tilde{I}}^{\,e}\,, \tag{1.17}$$

and a similar equation holds for the scalar potential $\tilde{\varphi}$:

$$\Delta\tilde{\varphi} = \mu_0\varepsilon_0\mu\hat{\varepsilon}\frac{\partial^2\tilde{\varphi}}{\partial t^2} + \mu_0\mu\sigma\frac{\partial\tilde{\varphi}}{\partial t} - \frac{\tilde{\rho}}{\varepsilon_0\hat{\varepsilon}}\,. \tag{1.18}$$

The first term on the right-hand side of Eq. (1.17) represents the displacement current, while the second term represents the current due to conductivity. Since, for metals, the displacement current is much smaller than the current due to conductivity, the first term may be neglected. Then Eq. (1.17) takes the form

$$\Delta\vec{\tilde{A}} = \mu_0\mu\sigma\frac{\partial\vec{\tilde{A}}}{\partial t} - \mu_0\mu\vec{\tilde{I}}^{\,e}\,. \tag{1.19}$$

The displacement current can also be neglected in free space due to the sufficiently low frequencies used in eddy current testing. Moreover, in free space the current due to conductivity is equal to zero. Then Eq. (1.19) can be rewritten as

$$\Delta\vec{\tilde{A}} = -\mu_0\mu\vec{\tilde{I}}^{\,e}\,. \tag{1.20}$$

Another useful gauge is *Coulomb's gauge* which requires that

$$div\,\vec{\tilde{A}} = 0\,. \tag{1.21}$$

In the case of Coulomb's gauge, it follows from Eqs. (1.4), (1.7), (1.10) and (1.21) that

$$\tilde{\rho} = \operatorname{div}\vec{\tilde{D}} = \operatorname{div}(\varepsilon_0\hat{\varepsilon}\,\vec{\tilde{E}}) = \varepsilon_0\hat{\varepsilon}\operatorname{div}\left(-\operatorname{grad}\tilde{\varphi} - \frac{\partial\vec{\tilde{A}}}{\partial t}\right) = -\varepsilon_0\hat{\varepsilon}\left(\Delta\tilde{\varphi} + \frac{\partial(\operatorname{div}\vec{\tilde{A}})}{\partial t}\right) = -\varepsilon_0\hat{\varepsilon}\,\Delta\tilde{\varphi}\,.$$

Thus, the equation for the scalar potential $\tilde{\varphi}$ is

$$\Delta\tilde{\varphi} = -\frac{\tilde{\rho}}{\varepsilon_0\hat{\varepsilon}}\,. \tag{1.22}$$

Using Coulomb's gauge, the vector potential satisfies the equation

$$\Delta \tilde{\vec{A}} = \mu_0 \mu \sigma \left(\nabla \tilde{\varphi} + \frac{\partial \tilde{\vec{A}}}{\partial t} \right) + \mu_0 \varepsilon_0 \mu \hat{\varepsilon} \frac{\partial}{\partial t} \left(\nabla \tilde{\varphi} + \frac{\partial \tilde{\vec{A}}}{\partial t} \right) - \mu_0 \mu \tilde{\vec{I}}^{\,e}, \qquad (1.23)$$

Lorentz' gauge presents the advantage of decoupling the equations for the scalar and the vector potentials so that, in some cases, these equations can be solved independently.

Assume that all vectors and scalars that describe the electromagnetic field are periodic in time, t, with the same frequency ω. For example,

$$\tilde{\vec{E}} = \vec{E}\cos(\omega t + \varphi) = \mathrm{Re}[\vec{E}e^{j(\omega t+\varphi)}] = \mathrm{Re}[\dot{\vec{E}}e^{j\omega t}]\ , \ \text{where } \dot{\vec{E}} = \vec{E}e^{j\varphi}, \qquad (1.24)$$

$$\tilde{\vec{I}} = \vec{I}\cos(\omega t + \varphi) = \mathrm{Re}[\vec{I}e^{j(\omega t+\varphi)}] = \mathrm{Re}[\dot{\vec{I}}e^{j\omega t}]\ , \ \text{where } \dot{\vec{I}} = \vec{I}e^{j\varphi},$$

and so on. Thus, the vectors and scalars can be written as

$$\tilde{\vec{E}} = \dot{\vec{E}}e^{j\omega t}, \qquad \tilde{\vec{B}} = \dot{\vec{B}}e^{j\omega t}, \qquad \tilde{\vec{H}} = \dot{\vec{H}}e^{j\omega t}, \qquad \tilde{\vec{D}} = \dot{\vec{D}}e^{j\omega t},$$

$$\tilde{\vec{I}} = \dot{\vec{I}}e^{j\omega t}, \qquad \tilde{\vec{I}}^{\,e} = \dot{\vec{I}}^{\,e}e^{j\omega t}, \qquad \tilde{\vec{A}} = \dot{\vec{A}}e^{j\omega t}, \qquad \tilde{\varphi} = \dot{\varphi}e^{j\omega t}, \qquad (1.25)$$

where the factors multiplied with $e^{j\omega t}$ are complex-valued amplitude vectors and functions of the spatial coordinates. It is to be noted that everywhere below instead of $\dot{\vec{E}}$, $\dot{\vec{I}}$, $\dot{\vec{B}}$, $\dot{\vec{I}}^{\,e}$, $\dot{\vec{H}}$, $\dot{\vec{A}}$, $\dot{\vec{D}}$, $\dot{\varphi}$ we use $\vec{E}$, $\vec{I}$, $\vec{B}$, $\vec{I}^{\,e}$, $\vec{H}$, $\vec{A}$, $\vec{D}$, φ, respectively, without the dot.

Under assumptions (1.25), Eq. (1.17) for the complex-valued amplitude magnetic vector potential (everywhere below it is called just vector potential) takes the form

$$\Delta \vec{A} + k^2 \vec{A} = -\mu_0 \mu \vec{I}^{\,e}, \qquad (1.26)$$

where $k^2 = -j\omega\mu_0\mu(\sigma + j\varepsilon_0\hat{\varepsilon}\omega)$. Equation (1.26) is known in the literature (see [59]) as Helmholtz' equation. If the displacement current is neglected, then Eq. (1.19) takes the form

$$\Delta \vec{A} + \hat{k}^2 \vec{A} = -\mu_0 \mu \vec{I}^{\,e}, \qquad (1.27)$$

where $\hat{k}^2 = -j\omega\mu_0\mu\sigma$. Besides, in free space, due to the absence of conductivity, the equation for the vector potential has the form

$$\Delta \vec{A} = -\mu_0 \mu \vec{I}^{\,e}. \qquad (1.28)$$

It also follows from Eqs. (1.25) and (1.16) that

$$\varphi = -\frac{1}{\tilde{k}^2}\,\mathrm{div}\,\vec{A}, \quad \tilde{k}^2 = \mu_0\mu(\sigma + j\varepsilon_0\hat{\varepsilon}\omega). \qquad (1.29)$$

By means of Eqs. (1.10) and (1.29) the vector potential $\vec{A}$ completely determines the electric field vector $\vec{E}$,

$$\vec{E} = -j\omega\vec{A} - \text{grad}\,\varphi = -j\omega\vec{A} + \frac{1}{\tilde{k}^2}\text{grad div}\,\vec{A}. \tag{1.30}$$

Applying Coulomb's gauge (1.21) to Eq. (1.30), we obtain

$$\vec{E} = -j\omega\vec{A}. \tag{1.31}$$

To guarantee the uniqueness of the electromagnetic potentials in the case of a non-uniform medium, it is necessary to use one of the above-mentioned gauges together with some boundary conditions for $\vec{A}$ at the interface between the media and at infinity.

1.3. Boundary conditions

It is shown in [6] that the solution of Maxwell's equations is unique if the tangent components of the vectors $\vec{E}$ and $\vec{H}$ are continuous at the interface between the two media, and conditions at infinity are prescribed. The boundary conditions for the tangential components $E_{1,\tau}$ and $E_{2,\tau}$ of the vector $\vec{E}$, and for $H_{1,\tau}$ and $H_{2,\tau}$ of the vector $\vec{H}$ are, respectively

$$E_{1\tau} = E_{2\tau}, \quad H_{1\tau} = H_{2\tau}. \tag{1.32}$$

In view of Eqs. (1.8) and (1.30), $\vec{A}$ completely determines $\vec{E}$ and $\vec{H}$, despite the fact that the problem for $\vec{A}$ is completely decoupled from the problem for φ.

In the literature on formulating the vector potential problem, the form of the vector potential (i.e. the number of nonzero components and arguments) and, consequently, the boundary conditions are determined only by considering the geometry of the source of current or even without any proof. The detailed proof seems to be given only in the present thesis (see also the author's paper [36]). The proof begins with the assumption that all three components of the vector potential are nonzero. Then it is proved that some components in particular cases are equal to zero. The form and the number of nonzero components of the vector potential are proved for the case of a double conductor line, of a wire of finite length, and of a single-turn coil above a uniform conducting half-space. As a generalization of these cases, the vector potential problem is formulated for a wire of an arbitrary form located in a vertical plane above a uniform conducting half-space.

Let us consider boundary conditions for these four mostly used cases of the vector potential problem formulated in the literature (see [6]).

Moreover, despite the fact that in eddy current testing problems the displacement current is often neglected, it is convenient to consider the influence of that current on the following boundary conditions. The displacement current can be neglected in the final expressions of a problem.

Boundary value problem 1: A double conductor line above a uniform conducting half-space. Consider two infinitely long wires located in free space in region $R_0 = \{z > 0\}$ at height h (see Fig.1.2). The wires are parallel to the x-axis and are passing through the points $(0, y_0, h)$ and $(0, y_1, h)$. The conducting half-space is located in region $R_1 = \{z < 0\}$.

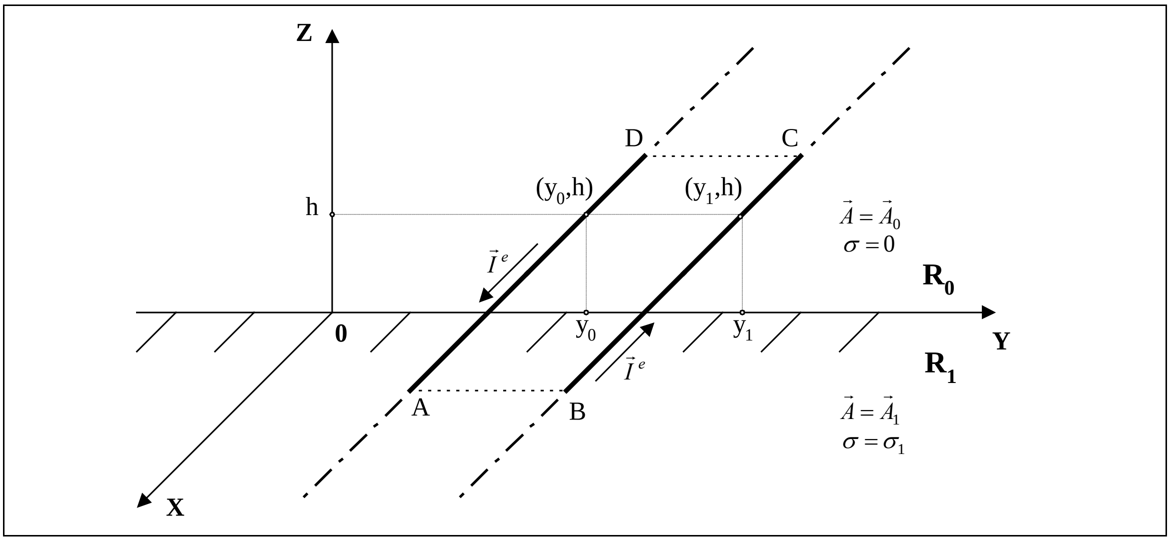

Fig.1.2. A double conductor line in free space R_0 over a uniform conducting half-space R_1

Assume that in region R_0: $\vec{A} = \vec{A}_0$, $\vec{E} = \vec{E}_0$, $\vec{H} = \vec{H}_0$, and in region R_1: $\vec{A} = \vec{A}_1$, $\vec{E} = \vec{E}_1$, $\vec{H} = \vec{H}_1$. The wires carry the alternating current $\pm \vec{I} e^{j\omega t}$, where $j = \sqrt{-1}$, ω is the frequency, $\vec{I}$ is the complex-valued amplitude current vector density (everywhere below it is called just current vector as it is used in the English scientific literature). In practice, two infinitely long wires are used as a model of a rectangular frame with current with sides' ratio $1:4$ or smaller, i.e. $AD : DC \geq 4$ (see Fig.1.2).

According to Eq. (1.26), the equation for the vector potential $\vec{A}$ in region R_0 (free space with conductivity $\sigma = 0$), and in region R_1 (absence of external currents) is, respectively,

$$\Delta \vec{A}_0 + k_0^2 \vec{A}_0 = -\mu_0 \tilde{\mu}_0 \vec{I}^e, \quad \vec{I}^e = I\delta(z-h)[\delta(y-y_0) - \delta(y-y_1)]\vec{e}_x, \quad \text{in } R_0, \tag{1.33}$$

$$\Delta \vec{A}_1 + k_1^2 \vec{A}_1 = 0\,, \quad \text{in } R_1\,, \tag{1.34}$$

where

$$\Delta \vec{A} = \vec{e}_x\left[\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\right] + \vec{e}_y\left[\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2}\right] + \vec{e}_z\left[\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2}\right] \tag{1.35}$$

is the Laplacian of the vector function $\vec{A}$, $k_0^2 = \mu_0 \varepsilon_0 \tilde{\mu}_0 \hat{\varepsilon}_0 \omega^2$, $k_1^2 = -j\omega\mu_0\mu_1(\sigma_1 + j\varepsilon_0\hat{\varepsilon}_1\omega)$, $\tilde{\mu}_0$ and μ_1 are the relative magnetic permeabilities in regions R_0 and R_1, respectively, $\hat{\varepsilon}_0$ and $\hat{\varepsilon}_1$ are the relative permittivities in regions R_0 and R_1, respectively, $\vec{I}^{\,e}$ is the complex-valued amplitude external current vector density (everywhere below it is called external current vector), σ_1 is the conductivity of region R_1, and $\delta(x)$ is Dirac's delta function.

Since the wires are infinitely long along the x-axis, the right-hand side of Eq. (1.33) does not depend on the variable x, and it is natural to suppose that the left-hand side does not depend on x either. Thus, the vector potential depends only on y and z. Besides, we suppose that the vector potential has the full form (i.e. it has all three components):

$$\vec{A} = A_x(y,z)\vec{e}_x + A_y(y,z)\vec{e}_y + A_z(y,z)\vec{e}_z \tag{1.36}$$

The expression for the electric field vector is

$$\vec{E} = -j\omega\vec{A} + \frac{1}{\tilde{k}^2}\,\text{grad div}\,\vec{A} = -j\omega(A_x\vec{e}_x + A_y\vec{e}_y + A_z\vec{e}_z) + \tag{1.37}$$

$$+\frac{1}{\tilde{k}^2}\left[\frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\vec{e}_x + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\vec{e}_y + \frac{\partial}{\partial z}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\vec{e}_z\right],$$

where $\tilde{k}^2 = \mu_0\mu(\sigma + j\varepsilon_0\hat{\varepsilon}\omega)$, and in the present case, $\frac{\partial A_x}{\partial x} = 0$, $\frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial y} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) = 0$.

Thus,

$$\vec{E} = -j\omega(A_x\vec{e}_x + A_y\vec{e}_y + A_z\vec{e}_z) + \frac{1}{\tilde{k}^2}\left[\frac{\partial}{\partial y}\left(\frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\vec{e}_y + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\vec{e}_z\right]. \tag{1.38}$$

It follows from the equations $\vec{B} = \text{curl}\,\vec{A}$ and $\vec{B} = \mu_0\mu\vec{H}$ that the magnetic field vector takes the form

$$\vec{H} = \frac{1}{\mu_0\mu}\text{curl}\,\vec{A} = \frac{1}{\mu_0\mu}\begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix} \tag{1.39}$$

or

$$\vec{H}=\frac{1}{\mu_0\mu}\left[\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\vec{e}_x-\left(\frac{\partial A_z}{\partial x}-\frac{\partial A_x}{\partial z}\right)\vec{e}_y+\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\vec{e}_z\right]. \tag{1.40}$$

Hence, in the present case, the magnetic field vector has the form

$$\vec{H}=\frac{1}{\mu_0\mu}\left[\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\vec{e}_x+\frac{\partial A_x}{\partial z}\vec{e}_y-\frac{\partial A_x}{\partial y}\vec{e}_z\right]. \tag{1.41}$$

In view of Eqs. (1.38) and (1.41), the vectors $\vec{E}$ and $\vec{H}$ have two tangent components, and the boundary conditions (1.32) for these components are rewritten as

$$z=0: \quad E_{0x}=E_{1x}, \quad E_{0y}=E_{1y}; \tag{1.42}$$

$$z=0: \quad H_{0x}=H_{1x}, \quad H_{0y}=H_{1y}. \tag{1.43}$$

Using Eqs. (1.38) and (1.41), the boundary conditions (1.42) and (1.43) at $z=0$ take the form

$$E_{0x}=E_{1x}: \quad A_{0x}=A_{1x}; \tag{1.44}$$

$$E_{0y}=E_{1y}: \quad A_{0y}+\frac{1}{\tilde{k}_0^2}\frac{\partial}{\partial y}\left(\frac{\partial A_{0y}}{\partial y}+\frac{\partial A_{0z}}{\partial z}\right)=A_{1y}+\frac{1}{\tilde{k}_1^2}\frac{\partial}{\partial y}\left(\frac{\partial A_{1y}}{\partial y}+\frac{\partial A_{1z}}{\partial z}\right); \tag{1.45}$$

$$H_{0x}=H_{1x}: \quad \frac{1}{\tilde{\mu}_0}\left(\frac{\partial A_{0z}}{\partial y}-\frac{\partial A_{0y}}{\partial z}\right)=\frac{1}{\mu_1}\left(\frac{\partial A_{1z}}{\partial y}-\frac{\partial A_{1y}}{\partial z}\right); \tag{1.46}$$

$$H_{0y}=H_{1y}: \quad \frac{1}{\tilde{\mu}_0}\frac{\partial A_{0x}}{\partial z}=\frac{1}{\mu_1}\frac{\partial A_{1x}}{\partial z}, \tag{1.47}$$

where $\tilde{k}_0^2=\mu_0\tilde{\mu}_0(\sigma_0+j\varepsilon_0\hat{\varepsilon}_0\omega)$, $\tilde{k}_1^2=\mu_0\mu_1(\sigma_1+j\varepsilon_0\hat{\varepsilon}_1\omega)$, σ_0 and σ_1 is the conductivity of regions R_0 and R_1, respectively, but $\sigma_0=0$ in free space.

It follows from the boundary conditions (1.44) and (1.47), and the vector equations (1.33) and (1.34) that the problem for the x-component, A_x, of the vector potential $\vec{A}$, is decoupled from the equations for the other components. Hence, the problem for A_x has the form

$$\begin{cases}\Delta A_{0x}+k_0^2A_{0x}=-\mu_0\tilde{\mu}_0I\delta(z-h)[\delta(y-y_0)-\delta(y-y_1)], & \text{in } R_0,\\ \Delta A_{1x}+k_1^2A_{1x}=0, & \text{in } R_{1.}\end{cases} \tag{1.48}$$

with the boundary conditions

$$z=0: \quad A_{0x}=A_{1x}, \quad \frac{1}{\tilde{\mu}_0}\frac{\partial A_{0x}}{\partial z}=\frac{1}{\mu_1}\frac{\partial A_{1x}}{\partial z}. \tag{1.49}$$

The problems for the components A_y and A_z are homogeneous:

$$\begin{cases} \Delta A_{0y} + k_0^2 A_{0y} = 0, & \text{in } R_0, \\ \Delta A_{1y} + k_1^2 A_{1y} = 0, & \text{in } R_{1.} \end{cases} \quad \text{and} \quad \begin{cases} \Delta A_{0z} + k_0^2 A_{0z} = 0, & \text{in } R_0, \\ \Delta A_{1z} + k_1^2 A_{1z} = 0, & \text{in } R_{1.} \end{cases} \tag{1.50}$$

with boundary conditions (1.45) and (1.46). The problems for these components cannot be separated, but their zero-solutions, $A_y = 0$ and $A_z = 0$, satisfy Eq. (1.48) and the boundary conditions (1.45) and (1.46). Due to the uniqueness theorem, these solutions are unique.

Consequently, in the case of a double conductor line above a uniform conducting half-space, the vector potential has only the x-component:

$$\vec{A} = A_x(y,z)\vec{e}_x\,. \tag{1.51}$$

Boundary value problem 2: A horizontal emitter over a uniform conducting half-space. Consider a horizontal emitter of finite length, parallel to the x-axis and passing through the point $(0, y_0, h)$ (see Fig.1.3).

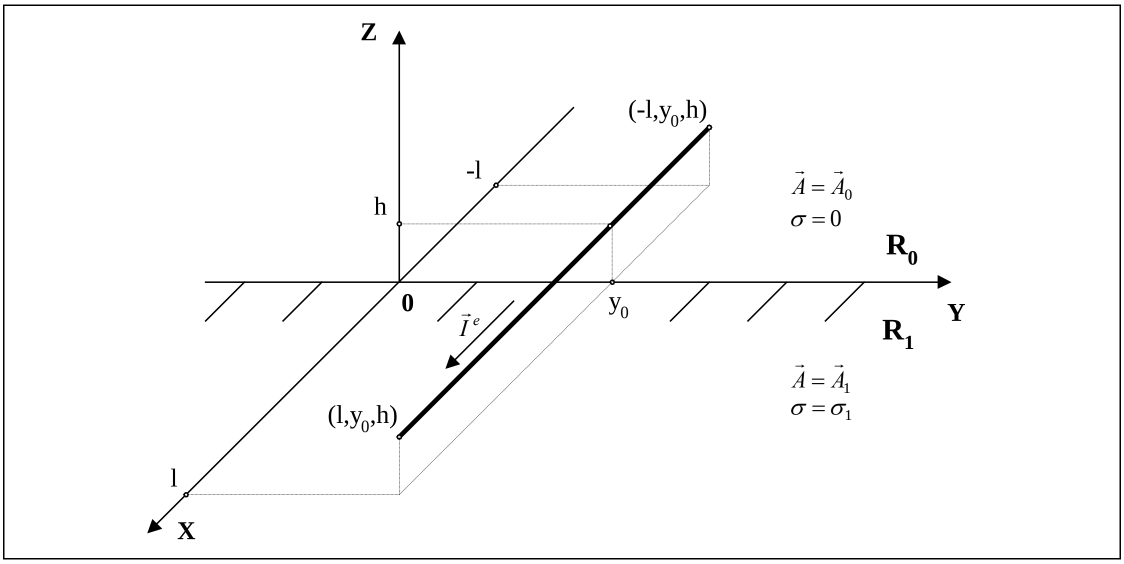

Fig.1.3. A horizontal emitter in free space R_0 over a uniform conducting half-space R_1

In free space R_0, the equation for the vector potential $\vec{A}$ is

$$\Delta \vec{A}_0 + k_0^2 \vec{A}_0 = -\mu_0 \tilde{\mu}_0 \vec{I}^e, \qquad \vec{I}^e = \begin{cases} I\delta(y-y_0)\delta(z-h)\vec{e}_x, & x \in (-l,l), \\ 0, & x \notin (-l,l). \end{cases} \tag{1.52}$$

In this case, the external current vector, $\vec{I}^e$, has only the x-component, I_x^e. But this does not mean that the vector potential $\vec{A}$ has only the x-component, A_x.

Initially we suppose that all three components of the vector potential are not equal to zero:

$$\vec{A} = A_x(x,y,z)\vec{e}_x + A_y(x,y,z)\vec{e}_y + A_z(x,y,z)\vec{e}_z \,. \tag{1.53}$$

We will prove that $A_y = 0$, but $A_z \neq 0$, because in this case the vector $\vec{I}^e$ depends on all three variables, x, y, z. Hence, the left-hand side of Eq. (1.52) depends also on the variables x, y, z. In the present case, the electric and magnetic field vectors are given by Eqs. (1.37) and (1.40), respectively, and the boundary conditions (1.42) and (1.43) for the tangent components of these vectors at $z = 0$, become

$E_{0x} = E_{1x}$:

$$-j\omega A_{0x} + \frac{1}{\tilde{k}_0^2}\frac{\partial}{\partial x}\left(\frac{\partial A_{0x}}{\partial x} + \frac{\partial A_{0y}}{\partial y} + \frac{\partial A_{0z}}{\partial z}\right) = -j\omega A_{1x} + \frac{1}{\tilde{k}_1^2}\frac{\partial}{\partial x}\left(\frac{\partial A_{1x}}{\partial x} + \frac{\partial A_{1y}}{\partial y} + \frac{\partial A_{1z}}{\partial z}\right); \tag{1.54}$$

$E_{0y} = E_{1y}$:

$$-j\omega A_{0y} + \frac{1}{\tilde{k}_0^2}\frac{\partial}{\partial y}\left(\frac{\partial A_{0x}}{\partial x} + \frac{\partial A_{0y}}{\partial y} + \frac{\partial A_{0z}}{\partial z}\right) = -j\omega A_{1y} + \frac{1}{\tilde{k}_1^2}\frac{\partial}{\partial y}\left(\frac{\partial A_{1x}}{\partial x} + \frac{\partial A_{1y}}{\partial y} + \frac{\partial A_{1z}}{\partial z}\right); \tag{1.55}$$

$$H_{0x} = H_{1x}: \quad \frac{1}{\tilde{\mu}_0}\left(\frac{\partial A_{0z}}{\partial y} - \frac{\partial A_{0y}}{\partial z}\right) = \frac{1}{\mu_1}\left(\frac{\partial A_{1z}}{\partial y} - \frac{\partial A_{1y}}{\partial z}\right); \tag{1.56}$$

$$H_{0y} = H_{1y}: \quad \frac{1}{\tilde{\mu}_0}\left(\frac{\partial A_{0x}}{\partial z} - \frac{\partial A_{0z}}{\partial x}\right) = \frac{1}{\mu_1}\left(\frac{\partial A_{1x}}{\partial z} - \frac{\partial A_{1z}}{\partial x}\right). \tag{1.57}$$

It is easy to see that if there exists a zero solution $A_y = 0$, then the equations for A_y are satisfied so that the problem for A_x is decoupled. Indeed, by substituting $A_{0y} = A_{1y} = 0$ into the boundary conditions (1.54)-(1.57), the boundary conditions become

$$z = 0: \quad -j\omega A_{0x} + \frac{1}{\tilde{k}_0^2}\frac{\partial}{\partial x}\left(\frac{\partial A_{0x}}{\partial x} + \frac{\partial A_{0z}}{\partial z}\right) = -j\omega A_{1x} + \frac{1}{\tilde{k}_1^2}\frac{\partial}{\partial x}\left(\frac{\partial A_{1x}}{\partial x} + \frac{\partial A_{1z}}{\partial z}\right); \tag{1.58}$$

$$z = 0: \quad \frac{1}{\tilde{k}_0^2}\frac{\partial}{\partial y}\left(\frac{\partial A_{0x}}{\partial x} + \frac{\partial A_{0z}}{\partial z}\right) = \frac{1}{\tilde{k}_1^2}\frac{\partial}{\partial y}\left(\frac{\partial A_{1x}}{\partial x} + \frac{\partial A_{1z}}{\partial z}\right); \tag{1.59}$$

$$z = 0: \quad \frac{1}{\tilde{\mu}_0}\frac{\partial A_{0z}}{\partial y} = \frac{1}{\mu_1}\frac{\partial A_{1z}}{\partial y}; \tag{1.60}$$

$$z = 0: \quad \frac{1}{\tilde{\mu}_0}\left(\frac{\partial A_{0x}}{\partial z} - \frac{\partial A_{0z}}{\partial x}\right) = \frac{1}{\mu_1}\left(\frac{\partial A_{1x}}{\partial z} - \frac{\partial A_{1z}}{\partial x}\right). \tag{1.61}$$

In Eq. (1.59) the differentiation $\partial/\partial y$ can be neglected, since $z = 0$ is fixed, but the variables x, y are not fixed, and, moreover, Eq. (1.59) can be easily obtained by differentiating Eq.

(1.62) with respect to y, so that we use

$$\frac{1}{\widetilde{k}_0^2}\left(\frac{\partial A_{0x}}{\partial x}+\frac{\partial A_{0z}}{\partial z}\right)=\frac{1}{\widetilde{k}_1^2}\left(\frac{\partial A_{1x}}{\partial x}+\frac{\partial A_{1z}}{\partial z}\right). \tag{1.62}$$

Differentiating Eq. (1.62) with respect to x,

$$\frac{1}{\widetilde{k}_0^2}\frac{\partial}{\partial x}\left(\frac{\partial A_{0x}}{\partial x}+\frac{\partial A_{0z}}{\partial z}\right)=\frac{1}{\widetilde{k}_1^2}\frac{\partial}{\partial x}\left(\frac{\partial A_{1x}}{\partial x}+\frac{\partial A_{1z}}{\partial z}\right), \tag{1.63}$$

and comparing the result with Eq. (1.58) obviously yields

$$z=0: \quad A_{0x}=A_{1x}. \tag{1.64}$$

Neglecting the differentiation with respect to y in Eq. (1.60) yields

$$\frac{1}{\widetilde{\mu}_0}A_{0z}=\frac{1}{\mu_1}A_{1z}. \tag{1.65}$$

Differentiating Eq. (1.65) with respect to x,

$$\frac{1}{\widetilde{\mu}_0}\frac{\partial A_{0z}}{\partial x}=\frac{1}{\mu_1}\frac{\partial A_{1z}}{\partial x}, \tag{1.66}$$

and comparing the result with Eq. (1.61) we obtain that at $z=0$:

$$\frac{1}{\widetilde{\mu}_0}\frac{\partial A_{0x}}{\partial z}=\frac{1}{\mu_1}\frac{\partial A_{1x}}{\partial z}. \tag{1.67}$$

Thus, the problem for the x-component of the vector potential is decoupled and has the form

$$\Delta A_{0x}+k_0^2A_{0x}=\begin{cases}-\mu_0\widetilde{\mu}_0 I\delta(y-y_0)\delta(z-h), & x\in(-l,l),\\ 0, & x\notin(-l,l),\end{cases} \qquad \text{in } R_0, \tag{1.68}$$

$$\Delta A_{1x}+k_1^2A_{1x}=0, \qquad \text{in } R_1,$$

with the boundary conditions (see Eqs. (1.64) and (1.67))

$$z=0: \quad A_{0x}=A_{1x}, \quad \frac{1}{\widetilde{\mu}_0}\frac{\partial A_{0x}}{\partial z}=\frac{1}{\mu_1}\frac{\partial A_{1x}}{\partial z}. \tag{1.69}$$

The formulation of the problem for the z-component of the vector potential is

$$\begin{cases}\Delta A_{0z}+k_0^2A_{0z}=0, & \text{in } R_0,\\ \Delta A_{1z}+k_1^2A_{1z}=0, & \text{in } R_1.\end{cases} \tag{1.70}$$

The problem for A_z does not have a zero-solution, because the boundary condition (1.62) for A_z is not homogeneous. This problem is solved after obtaining the solution of the problem for A_x by using the boundary conditions (1.59) and (1.65):

$$z=0: \quad \frac{1}{\tilde{k}_0^2}\frac{\partial}{\partial y}\left(\frac{\partial A_{0x}}{\partial x}+\frac{\partial A_{0z}}{\partial z}\right)=\frac{1}{\tilde{k}_1^2}\frac{\partial}{\partial y}\left(\frac{\partial A_{1x}}{\partial x}+\frac{\partial A_{1z}}{\partial z}\right); \tag{1.71}$$

$$z=0: \quad \frac{1}{\tilde{\mu}_0}A_{0z}=\frac{1}{\mu_1}A_{1z}\,. \tag{1.72}$$

Consequently, in the case of a horizontal emitter over a uniform conducting half-space, the vector potential has only the following two components:

$$\vec{A}=A_x(x,y,z)\vec{e}_x+A_z(x,y,z)\vec{e}_z\,. \tag{1.73}$$

Besides, if $A_y=0$ and $A_z=0$, then it follows from Eq. (1.62) that A_x must satisfy the boundary condition

$$z=0: \frac{1}{k_0^2}A_{0x}=\frac{1}{k_1^2}A_{1x}\,. \tag{1.74}$$

Note that if $k_0^2 \neq k_1^2$, then condition (1.74) contradicts Eq. (1.69), i.e. the condition $A_{0x}=A_{1x}$. Consequently, this condition cannot be satisfied. To satisfy the boundary condition (1.62), A_z must not be equal to zero. In that case, condition (1.62) is a boundary condition for A_z and it provides the uniqueness of solution $A_z \neq 0$.

Boundary value problem 3: A circular single-turn coil above a uniform conducting half-space. Consider a circular single-turn coil of radius r_c located at height h in free space $R_0=\{z>0\}$ above a uniform conducting half-space in region $R_1=\{z<0\}$ (see Fig.1.4).

In free space R_0, the equation for the vector potential is

$$\Delta\vec{A}_0+k_0^2\vec{A}_0=-\mu_0\tilde{\mu}_0\vec{I}^{\,e}, \qquad \vec{I}^{\,e}=I\delta(r-r_c)\delta(z-h)\vec{e}_\varphi\,. \tag{1.75}$$

where

$$\Delta\vec{A}=\vec{e}_r\left(\Delta A_r-\frac{A_r}{r^2}-\frac{2}{r^2}\frac{\partial A_\varphi}{\partial\varphi}\right)+\vec{e}_\varphi\left(\Delta A_\varphi-\frac{A_\varphi}{r^2}+\frac{2}{r^2}\frac{\partial A_r}{\partial\varphi}\right)+\vec{e}_z\Delta A_z\,, \tag{1.76}$$

$$\Delta f(r,\varphi,z)=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 f}{\partial\varphi^2}+\frac{\partial^2 f}{\partial z^2}\,, \tag{1.77}$$

$\Delta f(r,\varphi,z)$ is the Laplacian of a scalar function $f(r,\varphi,z)$ in cylindrical polar coordinates.

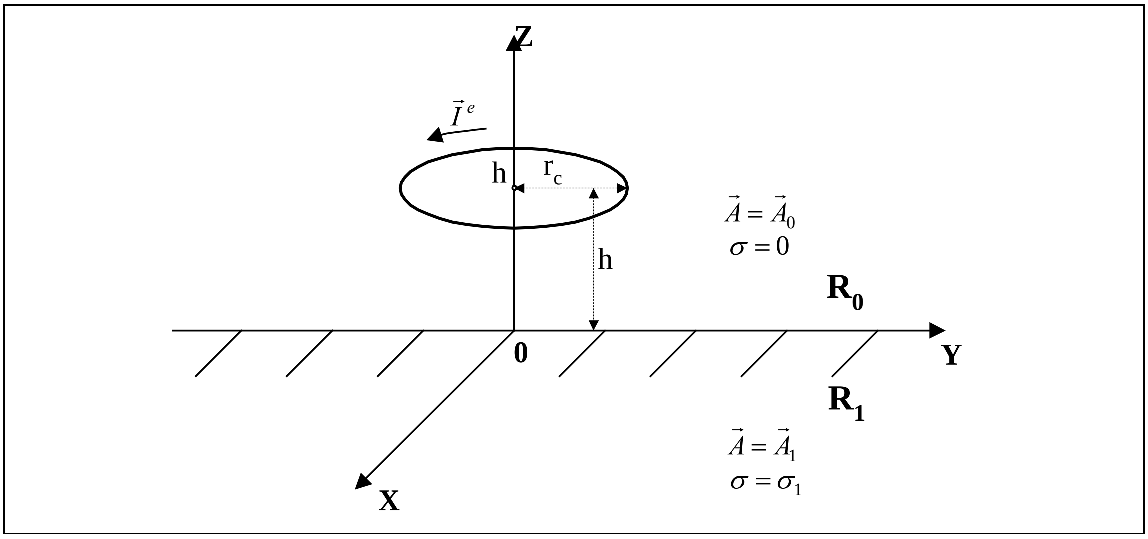

Fig.1.4. A circular single-turn coil in free space R_0 over a uniform conducting half-space R_1

In the case of a circular coil, due to axial symmetry, the vector potential does not depend on the variable φ and it may have the full form

$$\vec{A} = A_r(r,z)\vec{e}_r + A_\varphi(r,z)\vec{e}_\varphi + A_z(r,z)\vec{e}_z\,. \tag{1.78}$$

But in Eq. (1.76) the Laplacian takes the form

$$\Delta\vec{A} = \vec{e}_r\left(\Delta A_r - \frac{A_r}{r^2}\right) + \vec{e}_\varphi\left(\Delta A_\varphi - \frac{A_\varphi}{r^2}\right) + \vec{e}_z\Delta A_z\,, \tag{1.79}$$

where the scalar function becomes

$$\Delta f(r,z) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{\partial^2 f}{\partial z^2}\,. \tag{1.80}$$

The expression for the electric field vector in cylindrical polar coordinates is

$$\vec{E} = -j\omega\vec{A} + \frac{1}{\tilde{k}^2}\text{grad div}\,\vec{A} = -j\omega(A_r\vec{e}_r + A_\varphi\vec{e}_\varphi + A_z\vec{e}_z) +$$

$$+\frac{1}{\tilde{k}^2}\left[\vec{e}_r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z}\right) + \vec{e}_\varphi\frac{\partial}{\partial\varphi}\left(\frac{1}{r}\frac{\partial(rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z}\right) +\right.$$

$$\left. +\vec{e}_z\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial(rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z}\right)\right], \tag{1.81}$$

where, in the present case, $\partial/\partial\varphi = 0$ (since the functions A_r, A_φ, A_z do not depend on φ). Thus,

$$\vec{E} = -j\omega(A_r\vec{e}_r + A_\varphi\vec{e}_\varphi + A_z\vec{e}_z) +$$

$$+\frac{1}{\tilde{k}^2}\left[\vec{e}_r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rA_r)}{\partial r}+\frac{\partial A_z}{\partial z}\right)+\vec{e}_z\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial(rA_r)}{\partial r}+\frac{\partial A_z}{\partial z}\right)\right]. \tag{1.82}$$

Using the expression for $\text{curl}\,\vec{A}$ in cylindrical polar coordinates

$$\text{curl}\,\vec{A}=\vec{e}_r\left(\frac{1}{r}\frac{\partial A_z}{\partial\varphi}-\frac{\partial A_\varphi}{\partial z}\right)+\vec{e}_\varphi\left(\frac{\partial A_r}{\partial z}-\frac{\partial A_z}{\partial r}\right)+\vec{e}_z\frac{1}{r}\left(\frac{\partial(rA_\varphi)}{\partial r}-\frac{\partial A_r}{\partial\varphi}\right), \tag{1.83}$$

and taking into account that, in this case, $\partial/\partial\varphi=0$, one finds that

$$\vec{H}=\frac{1}{\mu_0\mu}curl\,\vec{A}=\frac{1}{\mu_0\mu}\left[\vec{e}_r\left(-\frac{\partial A_\varphi}{\partial z}\right)+\vec{e}_\varphi\left(\frac{\partial A_r}{\partial z}-\frac{\partial A_z}{\partial r}\right)+\vec{e}_z\frac{1}{r}\frac{\partial(rA_\varphi)}{\partial r}\right]. \tag{1.84}$$

In this case, at $z=0$ the boundary conditions are

$$E_{0r}=E_{1r}:$$
$$-j\omega A_{0r}+\frac{1}{\tilde{k}_0^2}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rA_{0r})}{\partial r}+\frac{\partial A_{0z}}{\partial z}\right)=-j\omega A_{1r}+\frac{1}{\tilde{k}_1^2}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rA_{0r})}{\partial r}+\frac{\partial A_{1z}}{\partial z}\right); \tag{1.85}$$

$$E_{0\varphi}=E_{1\varphi}:\quad A_{0\varphi}=A_{1\varphi}; \tag{1.86}$$

$$H_{0r}=H_{1r}:\quad \frac{1}{\tilde{\mu}_0}\frac{\partial A_{0\varphi}}{\partial z}=\frac{1}{\mu_1}\frac{\partial A_{1\varphi}}{\partial z}; \tag{1.87}$$

$$H_{0\varphi}=H_{1\varphi}:\quad \frac{1}{\tilde{\mu}_0}\left(\frac{\partial A_{0r}}{\partial z}-\frac{\partial A_{0z}}{\partial r}\right)=\frac{1}{\mu_1}\left(\frac{\partial A_{1r}}{\partial z}-\frac{\partial A_{1z}}{\partial r}\right). \tag{1.88}$$

It is obvious from the boundary conditions (1.86), (1.87) and Eq. (1.79) that the problem for the φ-component, A_φ, is decoupled. The problems for the other two components, A_r and A_z, cannot be decoupled. But each of these problems is a system of homogeneous equations with homogeneous boundary conditions, whose zero-solutions, $A_r=0$ and $A_z=0$, satisfy the boundary conditions (1.85) and (1.88). Due to the uniqueness theorem, there are no other solutions of these problems.

Consequently, in the case of a circular single-turn coil above a uniform conducting half-space, the vector potential has only the φ-component:

$$\vec{A}=A_\varphi(r,z)\vec{e}_\varphi. \tag{1.89}$$

Then the problem for the vector potential has the form

$$\begin{cases}\Delta_\varphi A_{0\varphi}+k_0^2A_{0\varphi}=-\mu_0\tilde{\mu}_0I\delta(r-r_c)\delta(z-h), & \text{in } R_0,\\ \Delta_\varphi A_{1\varphi}+k_1^2A_{1\varphi}=0, & \text{in } R_{1.}\end{cases} \tag{1.90}$$

where Δ_φ is the Laplacian given by Eqs. (1.79) and (1.80):

$$\Delta_\varphi f(r,z) = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial z^2} - \frac{1}{r^2} f \,, \tag{1.91}$$

with the boundary conditions (1.86) and (2.88):

$$z = 0: \quad A_{0\varphi} = A_{1\varphi}\,, \quad \frac{1}{\tilde{\mu}_0}\frac{\partial A_{0\varphi}}{\partial z} = \frac{1}{\mu_1}\frac{\partial A_{1\varphi}}{\partial z}\,. \tag{1.92}$$

Boundary value problem 4: A contour of arbitrary form in a vertical plane above a uniform conducting half-space. Without loss of generality, we assume that the vertical plane is the plane $y = 0$, since the choice of the coordinate system is free. Let us consider a contour L located in the vertical plane $y = 0$ in free space $R_0 = \{z > 0\}$ above a uniform conducting half-space in region $R_1 = \{z < 0\}$ (see Fig.1.5). It is to be noted that in the case of a closed contour, the contour is to be divided into two parts: the upper and the lower contours. The problem for each contour is solved separately and the results are added together.

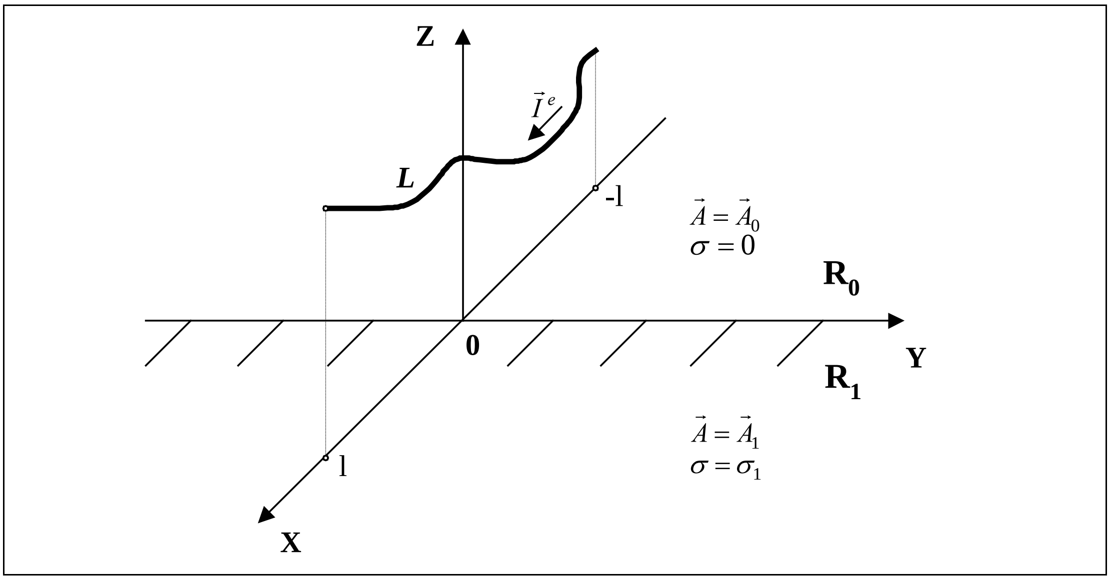

Fig.1.5. Contour L in plane $y = 0$ in free space R_0 over a uniform conducting half-space R_1

Let the contour L be described by the equations

$$\{x = t,\ y = 0,\ z = z(t)\}\,. \tag{1.93}$$

On the other hand, the line L can be defined in vector form as

$$\vec{\mathbf{r}}(x) = x \cdot \vec{i} + 0 \cdot \vec{j} + z(x) \cdot \vec{k}\,. \tag{1.94}$$

The unit tangent vector to the line $\vec{\mathbf{r}} = \vec{\mathbf{r}}(x)$ is

$$\vec{e}_\tau = \frac{\vec{\mathbf{r}}'(x)}{|\vec{\mathbf{r}}'(x)|} = \frac{\vec{i} + z'(x)\vec{k}}{\sqrt{1+z'^2(x)}}. \tag{1.95}$$

In this case, the external current vector is

$$\vec{I}^e = \begin{cases} I\delta(y)\delta[z - z(x)]\vec{e}_\tau, & x \in (-l,l), \\ 0, & x \notin (-l,l), \end{cases} \tag{1.96}$$

or

$$\vec{I}^e = \begin{cases} I\delta(y)\delta[z - z(x)]\dfrac{\vec{i} + z'(x)\vec{k}}{\sqrt{1+z'^2(x)}}, & x \in (-l,l), \\ 0, & x \notin (-l,l). \end{cases} \tag{1.97}$$

In free space R_0 the equation for the vector potential has the form

$$\Delta \vec{A}_0 + k_0^2 \vec{A}_0 = -\mu_0 \tilde{\mu}_0 \vec{I}^e, \tag{1.98}$$

where $\vec{I}^e$ is defined by Eq. (1.97).

The vector potential depends on all variables and is assumed to have the form

$$\vec{A} = A_x(x,y,z)\vec{e}_x + A_y(x,y,z)\vec{e}_y + A_z(x,y,z)\vec{e}_z. \tag{1.99}$$

In the present case, the boundary conditions are the same as for the case of a finite length horizontal emitter (see the boundary value problem 2). Besides, similarly to the 2nd boundary value problem, the problem for A_y is homogeneous and has a zero-solution, $A_y = 0$, which is unique due to the uniqueness theorem. Thus, if we substitute $A_y = 0$ into the boundary conditions (1.58)-(1.61) and take the vector equation (1.98) with Eq. (1.97) into account, the problem for A_x is decoupled:

$$\Delta A_{0x} + k_0^2 A_{0x} = \begin{cases} -\mu_0 \tilde{\mu}_0 I\delta(y)\delta[z - z(x)]\dfrac{1}{\sqrt{1+z'^2(x)}}, & x \in (-l,l), \\ 0, & x \notin (-l,l), \end{cases} \quad \text{in } R_0, \tag{1.100}$$

$$\Delta A_{1x} + k_1^2 A_{1x} = 0, \quad \text{in } R_1, \tag{1.101}$$

with the boundary conditions

$$z = 0: \quad A_{0x} = A_{1x}, \quad \frac{1}{\tilde{\mu}_0}\frac{\partial A_{0x}}{\partial z} = \frac{1}{\mu_1}\frac{\partial A_{1x}}{\partial z}. \tag{1.102}$$

The problem for A_z is solved after obtaining the solution of the problem for A_x. But unlike the 2nd boundary value problem, the projection of Eq. (1.98) for the vector potential onto the

z -axis, gives a nonhomogeneous problem for A_z in the form

$$\Delta A_{0z} + k_0^2 A_{0z} = \begin{cases} -\mu_0 \tilde{\mu}_0 I \delta(y) \delta[z - z(x)] \dfrac{z'(x)}{\sqrt{1 + z'^2(x)}}, & x \in (-l, l), \\ 0, & x \notin (-l, l), \end{cases} \quad \text{in } R_0, \tag{1.103}$$

$$\Delta A_{1z} + k_1^2 A_{1z} = 0, \quad \text{in } R_1, \tag{1.104}$$

with the boundary conditions (1.71) and (1.72)

$$z = 0: \quad \frac{1}{\tilde{\mu}_0} A_{0z} = \frac{1}{\mu_1} A_{1z}, \tag{1.105}$$

$$z = 0: \quad \frac{1}{\tilde{k}_0^2} \frac{\partial}{\partial y} \left(\frac{\partial A_{0x}}{\partial x} + \frac{\partial A_{0z}}{\partial z} \right) = \frac{1}{\tilde{k}_1^2} \frac{\partial}{\partial y} \left(\frac{\partial A_{1x}}{\partial x} + \frac{\partial A_{1z}}{\partial z} \right), \tag{1.106}$$

Consequently, in the case of an emitter of arbitrary form in a vertical plane above a uniform conducting half-space, the vector potential has only the following two components:

$$\vec{A} = A_x(x, y, z)\vec{e}_x + A_z(x, y, z)\vec{e}_z. \tag{1.107}$$

It is to be noted that the above-mentioned problems can be solved by integral transform methods (see [4]). In this thesis all the problems are formulated and solved in terms of a vector potential and the excitation is considered as a time-harmonic.

1.4. Impedance change

Once the vector potential is determined, one can calculate the main characteristic used in eddy current testing — the change in impedance of a detector coil — affected by the presence of a conducting medium.

The induced change in impedance of a closed contour C of arbitrary form is defined by the relation

$$Z^{\text{ind}} = -\frac{\tilde{E}(t)}{I}, \tag{1.108}$$

where $\tilde{E}(t)$ is the electromotive force and I is the amplitude of the current.

The electromotive force is the work needed to move a positive unit charge over the closed contour C:

$$\tilde{E}(t) = \oint_C E_l(M) dl, \tag{1.109}$$

where $\vec{E} = -j\omega\vec{A}^{\text{ind}} + \frac{1}{\tilde{k}^2}\text{grad}\,\text{div}\,\vec{A}^{\text{ind}}$, $\tilde{k}^2 = \mu_0\mu(\sigma + j\varepsilon_0\hat{\varepsilon}\omega)$, $\vec{A}^{\text{ind}}$ is the vector potential induced by the presence of the external current. Thus,

$$\tilde{E}(t) = \oint_C [-j\omega A_l^{\text{ind}} + \frac{1}{\tilde{k}^2}(\text{grad}\,\text{div}\,\vec{A}^{\text{ind}})_l]dl = -j\omega\oint_C A_l^{\text{ind}} dl\,, \tag{1.110}$$

since $\oint_C \text{grad}\,\varphi = 0$ for any function φ and for any closed contour C. Consequently, Eq. (1.108) for the induced change in impedance takes the form

$$Z^{\text{ind}} = \frac{j\omega}{I}\oint_C A_l^{\text{ind}} dl\,, \tag{1.111}$$

where C is the closed contour of a source of current.

An impedance diagram represents the variations of amplitude and phase of the coil impedance, $Z = X + jY$, which can be resolved into its real and imaginary components, X and $Y = \omega L$, called the resistive and the reactive components, respectively; and L is the coil inductance.

2. EXACT ANALYTICAL SOLUTIONS TO PROBLEMS OF ELECTROMAGNETIC WAVES SPREADING FROM EMITTERS OF DIFFERENT FORMS

2.1. Integral representation of the solution to Helmholtz' vector equation in arbitrary orthogonal curvilinear coordinates

Since Helmholtz' vector equation describes eddy current problems, the integral representation of the solution to this equation is very important. The integral representation of this solution known in the literature is based on the expression of the electromagnetic field vector potential in terms of a triple integral of the product of the external current vector and the fundamental solution of Helmholtz' scalar equation (see [59], [65]). This representation has its simplest form in rectangular coordinates in which the unit vectors $\vec{e}_x, \vec{e}_y, \vec{e}_z$ do not depend on the spatial coordinates. However, other coordinate systems are also widely used in applications. Since this integral representation of the solution is absent in the literature for other coordinate systems, it is obtained in the present section (see also [17], [18]) for a system of arbitrary orthogonal curvilinear coordinates in which the unit vectors $\vec{e}_{q1}, \vec{e}_{q2}, \vec{e}_{q3}$ are prescribed functions of the spatial coordinates. As particular cases of the representation obtained, the integral representations of the solution to Helmholtz' vector equation are found for cylindrical and spherical coordinates. The obtained representation of the solution to this equation is used for the vector potential problem of a rectangular frame with current considered below.

2.1.1. Formulation of the problem

Helmholtz' equation for the vector potential used in electrodynamics has the form

$$\Delta \vec{A} + k^2 \vec{A} = -\mu_0 \mu \vec{I}^{\,\mathrm{e}}, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \tag{2.1}$$

where $k^2 = \mu_0 \varepsilon_0 \mu \hat{\varepsilon} \omega^2$, $\vec{I}^{\,\mathrm{e}} = \vec{I}^{\,\mathrm{e}}(M)$ is the external current vector. The vectors $\vec{A}(M)$ and $\vec{I}^{\,e}(M)$ in Cartesian coordinates have the form

$$\vec{A}(M) = A_x(M)\vec{e}_x + A_y(M)\vec{e}_y + A_z(M)\vec{e}_z, \tag{2.2}$$

$$\vec{I}^{\,\mathrm{e}}(M) = I_x^{\mathrm{e}}(M)\vec{e}_x + I_y^{\mathrm{e}}(M)\vec{e}_y + I_z^{\mathrm{e}}(M)\vec{e}_z. \tag{2.3}$$

The integral representation of the solution to Helmholtz' equation in vector form for a point $M(x,y,z)$, situated in the region where the external current vector $\vec{I}^{\mathrm{e}}=0$ (see [59], p. 322), is

$$\vec{A}(M)=\frac{\mu_0\mu}{4\pi}\iiint_V \vec{I}^{\mathrm{e}}(\tilde{M})\frac{\exp(-jkr_{M\tilde{M}})}{r_{M\tilde{M}}}d\tilde{V}\,, \tag{2.4}$$

where the integration is performed over the points $\tilde{M}(\tilde{x},\tilde{y},\tilde{z})\in V$ where $\vec{I}^{\mathrm{e}}\neq 0$. Besides, $\hat{\varepsilon}=1$ since the wire is situated in free space, $r_{M\tilde{M}}$ is the distance between the points $M(x,y,z)$ and $M(\tilde{x},\tilde{y},\tilde{z})$ and is equal to

$$r_{M\tilde{M}}=\sqrt{(x-\tilde{x})^2+(y-\tilde{y})^2+(z-\tilde{z})^2}\,. \tag{2.5}$$

The conditions for all components, A_x, A_y, A_z, of the vector potential at infinity are called Sommerfeld's conditions of radiation (see [65], page 509):

$$R^2=r^2+z^2\to\infty: \qquad A=O\left(\frac{1}{R}\right),\quad \frac{\partial A}{\partial R}+jkA=o\left(\frac{1}{R}\right)\,, \tag{2.6}$$

where the symbol $O(1/R)$ means that A and $1/R$ are infinitesimals of the same order as $R\to\infty$, but the symbol $o(1/R)$ means that $\partial A/\partial R+jkA$ is an infinitesimal of higher order than $1/R$ as $R\to\infty$.

It can be easily verified that if the functions $I_x^{\mathrm{e}}(M)$, $I_y^{\mathrm{e}}(M)$, $I_z^{\mathrm{e}}(M)$ are continuous in some closed region V and, consequently, they are bounded in this region, then the vector function $\vec{A}(M)$ in Eq. (2.4) satisfies Sommerfeld's conditions (2.6). Consequently, in this case, Eq. (2.4) gives the solution to the problem (2.1), (2.6) providing that the vector function $\vec{I}^{\mathrm{e}}(M)$ is prescribed.

In Cartesian coordinates, the unit vectors $\vec{e}_x$, $\vec{e}_y$ and $\vec{e}_z$ are constant. Therefore, in this case, according to Eq. (2.4), each component of the vector $\vec{A}$ is expressed in terms of a triple integral of the corresponding component of the vector $\vec{I}^{\mathrm{e}}$ (i.e. A_x in terms of I_x^{e}, A_y in terms of I_y^{e} and A_z in terms of I_z^{e}). For example,

$$A_x(M)=\frac{\mu_0\mu}{4\pi}\iiint_V I_x^{\mathrm{e}}(\tilde{M})\frac{\exp(-jkr_{M\tilde{M}})}{r_{M\tilde{M}}}d\tilde{V}\,, \tag{2.7}$$

and so on. However, in all other orthogonal curvilinear coordinate systems, the unit vectors depend on the spatial coordinates. For example, in the system of cylindrical polar coordinates

(r,φ,z), the vector $\vec{I}^{\,\mathrm{e}}(\tilde{M})$ has the form

$$\vec{I}^{\,\mathrm{e}}(\tilde{M}) = I_r^{\mathrm{e}}(\tilde{M})\vec{e}_r + I_\varphi^{\mathrm{e}}(\tilde{M})\vec{e}_\varphi + I_z^{\mathrm{e}}(\tilde{M})\vec{e}_z\,, \tag{2.8}$$

and only the unit vector $\vec{e}_z$ is constant. Then, in this case, the equality

$$\iiint\limits_V I_r^{\mathrm{e}}(\tilde{M})\,\vec{e}_r(\tilde{M})\frac{\exp(-jkr_{M\tilde{M}})}{r_{M\tilde{M}}}d\tilde{V} = \iiint\limits_V I_r^{\mathrm{e}}(\tilde{M})\frac{\exp(-jkr_{M\tilde{M}})}{r_{M\tilde{M}}}d\tilde{V}\cdot\vec{e}_r(\tilde{M}) \tag{2.9}$$

and the similar equality for $\vec{I}_\varphi^{\mathrm{e}}(\tilde{M})\cdot\vec{e}_\varphi(\tilde{M})$ are wrong. Consequently, the components $A_r(M)$ and $A_\varphi(M)$ have the form of a triple integral of some linear combination of the components $I_r^{\mathrm{e}}(\tilde{M})$ and $I_\varphi^{\mathrm{e}}(\tilde{M})$. Problem (2.1), (2.6) is solved in the author's papers [13] and [15] in the cylindrical polar coordinates as follows. First, the triple integral (2.7) is transformed into a single integral in the Cartesian coordinates. Then the transformation to the cylindrical polar coordinates is performed in the obtained solution. This fact allows us to obtain the universal formula for the integral representation of the solution to the vector Helmholtz equation (2.4) in the systems of cylindrical polar, spherical and also arbitrary orthogonal curvilinear coordinates. The components of the vector $\vec{A}(M)$ are expressed in terms of a triple integral of the linear combinations of the components of the external current vector $\vec{I}^{\,\mathrm{e}}(\tilde{M})$.

2.1.2. Integral representation of the solution to Helmholtz' vector equation in cylindrical polar coordinates

In cylindrical polar coordinates (r,φ,z) the vectors $\vec{A}(M) = \vec{A}(r,\varphi,z)$ and $\vec{I}^{\,\mathrm{e}}(\tilde{M}) = \vec{I}^{\,\mathrm{e}}(\tilde{r},\tilde{\varphi},\tilde{z})$ have the form

$$\vec{A}(M) = A_r(M)\,\vec{e}_r(M) + A_\varphi(M)\,\vec{e}_\varphi(M) + A_z(M)\,\vec{e}_z\,, \tag{2.10}$$

$$\vec{I}^{\,\mathrm{e}}(\tilde{M}) = I_r^{\mathrm{e}}(\tilde{M})\,\vec{e}_r(\tilde{M}) + I_\varphi^{\mathrm{e}}(\tilde{M})\,\vec{e}_\varphi(\tilde{M}) + I_z^{\mathrm{e}}(\tilde{M})\,\vec{e}_z\,. \tag{2.11}$$

The components $A_r(M)$, $A_\varphi(M)$ can be expressed in terms of the components $A_x(M)$, $A_y(M)$ as

$$A_r = A_x\cos\varphi + A_y\sin\varphi\,, \tag{2.12}$$

$$A_\varphi = -A_x\sin\varphi + A_y\cos\varphi\,. \tag{2.13}$$

The components $I_x^{\mathrm{e}}(\tilde{M})$, $I_y^{\mathrm{e}}(\tilde{M})$ can be expressed in terms of the components $I_r^{\mathrm{e}}(\tilde{M})$ and $I_\varphi^{\mathrm{e}}(\tilde{M})$ as

$$I_x^{\mathrm{e}}(\tilde{M}) = I_r^{\mathrm{e}}(\tilde{M})\cos\tilde{\varphi} - I_\varphi^{\mathrm{e}}(\tilde{M})\sin\tilde{\varphi}, \quad (2.14)$$

$$I_y^{\mathrm{e}}(\tilde{M}) = I_r^{\mathrm{e}}(\tilde{M})\sin\tilde{\varphi} + I_\varphi^{\mathrm{e}}(\tilde{M})\cos\tilde{\varphi}. \quad (2.15)$$

It follows from Eq. (2.4) that

$$A_x(M) = \frac{\mu_0\mu}{4\pi}\iiint_V I_x^{\mathrm{e}}(\tilde{M})\Phi(M,\tilde{M})\,d\tilde{V}, \quad (2.16)$$

$$A_y(M) = \frac{\mu_0\mu}{4\pi}\iiint_V I_y^{\mathrm{e}}(\tilde{M})\Phi(M,\tilde{M})\,d\tilde{V}, \quad (2.17)$$

where

$$\Phi(M,\tilde{M}) = \frac{\exp(-jkr_{M\tilde{M}})}{r_{M\tilde{M}}}. \quad (2.18)$$

The distance, $r_{M\tilde{M}}$, between the points M and $\tilde{M}$ is defined by Eq. (2.5). Substituting

$$x = r\cos\varphi,\; y = r\sin\varphi,\; z = z; \quad \tilde{x} = \tilde{r}\cos\varphi,\; \tilde{y} = \tilde{r}\sin\tilde{\varphi},\; \tilde{z} = \tilde{z} \quad (2.19)$$

into Eq. (2.5), one obtains

$$r_{M\tilde{M}} = \sqrt{r^2 + \tilde{r}^2 - 2r\tilde{r}\cos(\varphi - \tilde{\varphi}) + (z - \tilde{z})^2}. \quad (2.20)$$

It follows from Eq. (2.12), by substituting Eqs. (2.16) and (2.17), that

$$A_r(M) = \frac{\mu_0\mu}{4\pi}\iiint_V I_x^{\mathrm{e}}(\tilde{M})\Phi(M,\tilde{M})\,d\tilde{V}\cdot\cos\varphi + \frac{\mu_0\mu}{4\pi}\iiint_V I_y^{\mathrm{e}}(\tilde{M})\Phi(M,\tilde{M})\,d\tilde{V}\cdot\sin\varphi. \quad (2.21)$$

Substituting Eqs. (2.14) and (2.15) into Eq. (2.21) yields

$$A_r(M) = \frac{\mu_0\mu}{4\pi}\iiint_V [I_r^{\mathrm{e}}(\tilde{M})\cos\tilde{\varphi} - I_\varphi^{\mathrm{e}}(\tilde{M})\sin\tilde{\varphi}]\Phi(M,\tilde{M})\,d\tilde{V}\cdot\cos\varphi +$$
$$+\frac{\mu_0\mu}{4\pi}\iiint_V [I_r^{\mathrm{e}}(\tilde{M})\sin\tilde{\varphi} + I_\varphi^{\mathrm{e}}(\tilde{M})\cos\tilde{\varphi}]\Phi(M,\tilde{M})\,d\tilde{V}\cdot\sin\varphi. \quad (2.22)$$

The final expression for the component $A_r(M)$ can be easily obtained from Eq. (2.22) by performing some elementary transformations, and it has the form

$$A_r(M) = \frac{\mu_0\mu}{4\pi}\iiint_V [I_r^{\mathrm{e}}(\tilde{M})\cos(\varphi - \tilde{\varphi}) + I_\varphi^{\mathrm{e}}(\tilde{M})\sin(\varphi - \tilde{\varphi})]\Phi(M,\tilde{M})\,d\tilde{V}, \quad (2.23)$$

where

$$d\tilde{V} = \tilde{r}\, d\tilde{r}\, d\tilde{\varphi}\, d\tilde{z} \; . \qquad (2.24)$$

The expression for the component $A_\varphi(M)$ is obtained by performing similar transformations and using Eq. (2.13) for $A_\varphi(M)$, Eq. (2.16) for $A_x(M)$ and Eq. (2.17) for $A_y(M)$, Eq. (2.14) for $I_x^{\mathrm{e}}(\tilde{M})$ and Eq. (2.15) for $I_y^{\mathrm{e}}(\tilde{M})$. It has the form

$$A_\varphi(M) = \frac{\mu_0\mu}{4\pi} \iiint_V [I_r^{\mathrm{e}}(\tilde{M})\sin(\varphi-\tilde{\varphi}) + I_\varphi^{\mathrm{e}}(\tilde{M})\cos(\varphi-\tilde{\varphi})]\Phi(M,\tilde{M})\, d\tilde{V} \; . \qquad (2.25)$$

The component $A_z(M)$ has the same form as in Cartesian coordinates:

$$A_z(M) = \frac{\mu_0\mu}{4\pi} \iiint_V I_z^{\mathrm{e}}(\tilde{M})\Phi(M,\tilde{M})\, d\tilde{V} \; . \qquad (2.26)$$

Thus, Eqs. (2.10), (2.23), (2.25) and (2.26) give the integral representation of the solution to Helmholtz' vector equation (2.4) in cylindrical polar coordinates.

2.1.3. Integral representation of the solution to Helmholtz' vector equation in spherical coordinates

In spherical coordinates (ρ, θ, φ) the vectors $\vec{A}(M) = \vec{A}(\rho,\theta,\varphi)$ and $\vec{I}^{\mathrm{e}}(\tilde{M}) = \vec{I}^{\mathrm{e}}(\tilde{\rho},\tilde{\theta},\tilde{\varphi})$ have the form

$$\vec{A}(M) = A_\rho(M)\vec{e}_\rho(M) + A_\theta(M)\vec{e}_\theta(M) + A_\varphi(M)\vec{e}_\varphi(M) , \qquad (2.27)$$

$$\vec{I}^{\mathrm{e}}(\tilde{M}) = I_\rho^{\mathrm{e}}(\tilde{M})\vec{e}_\rho(\tilde{M}) + I_\theta^{\mathrm{e}}(\tilde{M})\vec{e}_\theta(\tilde{M}) + I_\varphi^{\mathrm{e}}(\tilde{M})\vec{e}_\varphi(\tilde{M}) \; . \qquad (2.28)$$

The components $A_\rho(M)$, $A_\varphi(M)$, $A_\theta(M)$ can be expressed in terms of the components $A_x(M)$, $A_y(M)$, $A_z(M)$ as

$$A_\rho(M) = [A_x(M)\cos\varphi + A_y(M)\sin\varphi]\sin\theta + A_z(M)\cos\theta , \qquad (2.29)$$

$$A_\theta(M) = [A_x(M)\cos\varphi + A_y(M)\sin\varphi]\cos\theta - A_z(M)\sin\theta , \qquad (2.30)$$

$$A_\varphi(M) = -A_x(M)\sin\varphi + A_y(M)\cos\varphi \; . \qquad (2.31)$$

The components $I_x^{\mathrm{e}}(\tilde{M})$, $I_y^{\mathrm{e}}(\tilde{M})$, $I_z^{\mathrm{e}}(\tilde{M})$ can be expressed in terms of the components $I_\rho^{\mathrm{e}}(\tilde{M})$, $I_\varphi^{\mathrm{e}}(\tilde{M})$, $I_\theta^{\mathrm{e}}(\tilde{M})$ as (see [68], page 582)

$$I_x^{\mathrm{e}}(\tilde{M}) = I_\rho^{\mathrm{e}}(\tilde{M})\sin\tilde{\theta}\cos\tilde{\varphi} + I_\theta^{\mathrm{e}}(\tilde{M})\cos\tilde{\theta}\cos\tilde{\varphi} - I_\varphi^{\mathrm{e}}(\tilde{M})\sin\tilde{\varphi} , \qquad (2.32)$$

$$I_y^{\mathrm{e}}(\tilde{M}) = I_\rho^{\mathrm{e}}(\tilde{M})\sin\tilde{\theta}\sin\tilde{\varphi} + I_\theta^{\mathrm{e}}(\tilde{M})\cos\tilde{\theta}\sin\tilde{\varphi} + I_\varphi^{\mathrm{e}}(\tilde{M})\cos\tilde{\varphi} , \qquad (2.33)$$

$$I_z^{\mathrm{e}}(\tilde{M}) = I_\rho^{\mathrm{e}}(\tilde{M})\cos\tilde{\theta} - I_\theta^{\mathrm{e}}(\tilde{M})\sin\tilde{\theta}\,. \tag{2.34}$$

It follows from Eq. (2.4) that

$$\begin{pmatrix} A_x(M) \\ A_y(M) \\ A_z(M) \end{pmatrix} = \frac{\mu_0\mu}{4\pi}\iiint\limits_V \begin{pmatrix} I_x^{\mathrm{e}}(\tilde{M}) \\ I_y^{\mathrm{e}}(\tilde{M}) \\ I_z^{\mathrm{e}}(\tilde{M}) \end{pmatrix} F(M,\tilde{M})\,d\tilde{V}\,, \tag{2.35}$$

where

$$F(M,\tilde{M}) = \frac{\exp(-jkr_{M\tilde{M}})}{r_{M\tilde{M}}}\,. \tag{2.36}$$

Substituting

$$x = \rho\sin\theta\cos\varphi,\quad y = \rho\sin\theta\sin\varphi,\quad z = \rho\cos\theta\,,$$

$$\tilde{x} = \tilde{\rho}\sin\tilde{\theta}\cos\tilde{\varphi},\quad \tilde{y} = \tilde{\rho}\sin\tilde{\theta}\sin\tilde{\varphi},\quad \tilde{z} = \tilde{\rho}\cos\tilde{\theta} \tag{2.37}$$

into Eq. (2.5) for the distance $r_{M\tilde{M}}$ yields

$$r_{M\tilde{M}} = \sqrt{\rho^2 + \tilde{\rho}^2 - 2\rho\tilde{\rho}[\sin\theta\sin\tilde{\theta}\cos(\varphi-\tilde{\varphi}) + \cos\theta\cos\tilde{\theta}]}\,. \tag{2.38}$$

It follows from Eqs. (2.29) and (2.35) that

$$\begin{aligned} A_\rho(M) = {} & \frac{\mu_0\mu}{4\pi}\iiint\limits_V I_x^{\mathrm{e}}(\tilde{M})F(M,\tilde{M})\,d\tilde{V}\cdot\cos\varphi\sin\theta + \\ & + \frac{\mu_0\mu}{4\pi}\iiint\limits_V I_y^{\mathrm{e}}(\tilde{M})F(M,\tilde{M})\,d\tilde{V}\cdot\sin\varphi\sin\theta + \\ & + \frac{\mu_0\mu}{4\pi}\iiint\limits_V I_z^{\mathrm{e}}(\tilde{M})F(M,\tilde{M})\,d\tilde{V}\cdot\cos\theta\,. \end{aligned} \tag{2.39}$$

The final expression for the component $A_\rho(M)$ is obtained by substituting Eqs. (2.32)-(2.34) into Eq. (2.39) and by performing some elementary transformations. It has the form

$$\begin{aligned} A_\rho(M) = {} & \frac{\mu_0\mu}{4\pi}\iiint\limits_V \{I_\rho^{\mathrm{e}}(\tilde{M})[\sin\theta\sin\tilde{\theta}\cos(\varphi-\tilde{\varphi}) + \cos\theta\cos\tilde{\theta}] + \\ & + I_\theta^{\mathrm{e}}(\tilde{M})[\sin\theta\cos\tilde{\theta}\cos(\varphi-\tilde{\varphi}) - \cos\theta\sin\tilde{\theta}] + \\ & + I_\varphi^{\mathrm{e}}(\tilde{M})\sin\theta\sin(\varphi-\tilde{\varphi})\}\, F(M,\tilde{M})\,d\tilde{V}\,, \end{aligned} \tag{2.40}$$

where

$$d\tilde{V} = \tilde{\rho}^2\sin^2\tilde{\theta}\,d\tilde{\rho}\,d\tilde{\theta}\,d\tilde{\varphi}\,. \tag{2.41}$$

If we performe similar transformations for the components $A_\theta(M)$ and $A_\varphi(M)$, the final expression for these components is found in the form

$$A_\theta(M) = \frac{\mu_0\mu}{4\pi}\iiint\limits_V \{I_\rho^{\mathrm{e}}(\tilde{M})[\sin\tilde{\theta}\cos\theta\cos(\varphi-\tilde{\varphi}) - \cos\tilde{\theta}\sin\theta] +$$

$$+ I_\theta^{\mathrm{e}}(\tilde{M})[\cos\tilde{\theta}\cos\theta\cos(\varphi-\tilde{\varphi}) + \sin\tilde{\theta}\sin\theta] +$$

$$+ I_\varphi^{\mathrm{e}}(\tilde{M})\cos\theta\sin(\varphi-\tilde{\varphi})\}\, F(M,\tilde{M})\, d\tilde{V}\,, \tag{2.42}$$

$$A_\varphi(M) = -\frac{\mu_0\mu}{4\pi}\iiint\limits_V \{I_\rho^{\mathrm{e}}(\tilde{M})\sin\tilde{\theta}\sin(\varphi-\tilde{\varphi}) + I_\theta^{\mathrm{e}}(\tilde{M})\cos\tilde{\theta}\sin(\varphi-\tilde{\varphi}) -$$

$$- I_\varphi^{\mathrm{e}}(\tilde{M})\cos(\varphi-\tilde{\varphi})\}\, F(M,\tilde{M})\, d\tilde{V}\,. \tag{2.43}$$

Thus, Eqs. (2.27), (2.40), (2.42) and (2.43) give the integral representation of the solution to the Helmholtz' vector equation (2.4) in spherical coordinates.

2.1.4. Integral representation of the solution to Helmholtz' vector equation in arbitrary curvilinear coordinates

Let the arbitrary orthogonal curvilinear coordinates (q_1, q_2, q_3) be given by the functions

$$x = x(q_1,q_2,q_3)\,,\ y = y(q_1,q_2,q_3)\,,\ z = z(q_1,q_2,q_3) \tag{2.44}$$

and, respectively,

$$\tilde{x} = x(\tilde{q}_1,\tilde{q}_2,\tilde{q}_3)\,,\ \tilde{y} = y(\tilde{q}_1,\tilde{q}_2,\tilde{q}_3)\,,\ \tilde{z} = z(\tilde{q}_1,\tilde{q}_2,\tilde{q}_3)\,. \tag{2.45}$$

Let $\vec{e}_{q_1}$, $\vec{e}_{q_2}$, $\vec{e}_{q_3}$ be the unit vectors of this coordinate system. Then the vectors $\vec{A}(M)$ and $\vec{I}^{\,\mathrm{e}}(\tilde{M})$ have the form

$$\vec{A}(M) = A_{q_1}(M)\vec{e}_{q_1}(M) + A_{q_2}(M)\vec{e}_{q_2}(M) + A_{q_3}(M)\vec{e}_{q_3}(M)\,, \tag{2.46}$$

$$\vec{I}^{\,\mathrm{e}}(\tilde{M}) = I_{q_1}^{\mathrm{e}}(\tilde{M})\vec{e}_{q_1}(\tilde{M}) + I_{q_2}^{\mathrm{e}}(\tilde{M})\vec{e}_{q_2}(\tilde{M}) + I_{q_3}^{\mathrm{e}}(\tilde{M})\vec{e}_{q_3}(\tilde{M})\,. \tag{2.47}$$

The components $A_{q_1}(M)$, $A_{q_2}(M)$, $A_{q_3}(M)$ can be expressed in terms of the components $A_x(M)$, $A_y(M)$, $A_z(M)$ as

$$A_{q_j}(M) = \frac{1}{H_j(M)}\left[A_x(M)\frac{\partial x}{\partial q_j} + A_y(M)\frac{\partial y}{\partial q_j} + A_z(M)\frac{\partial z}{\partial q_j}\right], \quad j = 1,2,3\,. \tag{2.48}$$

The components $I_x^{\mathrm{e}}(\tilde{M})$, $I_y^{\mathrm{e}}(\tilde{M})$, $I_z^{\mathrm{e}}(\tilde{M})$ can be expressed in terms of the components $I_{q_1}^{\mathrm{e}}(\tilde{M})$, $I_{q_2}^{\mathrm{e}}(\tilde{M})$, $I_{q_3}^{\mathrm{e}}(\tilde{M})$ as (see [68], page 561, formulae (A.6.8), (A.6.9))

$$I_x^{\mathrm{e}}(\tilde{M}) = \sum_{k=1}^{3} I_{q_k}^{\mathrm{e}}(\tilde{M}) \frac{1}{\tilde{H}_k} \frac{\partial \tilde{x}}{\partial \tilde{q}_k}, \tag{2.49}$$

$$I_y^{\mathrm{e}}(\tilde{M}) = \sum_{k=1}^{3} I_{q_k}^{\mathrm{e}}(\tilde{M}) \frac{1}{\tilde{H}_k} \frac{\partial \tilde{y}}{\partial \tilde{q}_k}, \tag{2.50}$$

$$I_x^{\mathrm{e}}(\tilde{M}) = \sum_{k=1}^{3} I_{q_k}^{\mathrm{e}}(\tilde{M}) \frac{1}{\tilde{H}_k} \frac{\partial \tilde{x}}{\partial \tilde{q}_k}, \tag{2.51}$$

where $H_k = H_k(q_1, q_2, q_3)$, $\tilde{H}_k = H_k(\tilde{q}_1, \tilde{q}_2, \tilde{q}_3)$ are the Lame coefficients of the prescribed coordinate system (see [68] with the notation $H_k = h_k^{-1}$).

It follows from Eq. (2.4) that

$$\begin{pmatrix} A_x(M) \\ A_y(M) \\ A_z(M) \end{pmatrix} = \frac{\mu_0 \mu}{4\pi} \iiint_V \begin{pmatrix} I_x^{\mathrm{e}}(\tilde{M}) \\ I_y^{\mathrm{e}}(\tilde{M}) \\ I_z^{\mathrm{e}}(\tilde{M}) \end{pmatrix} G(M, \tilde{M})\, d\tilde{V}, \tag{2.52}$$

where

$$G(M, \tilde{M}) = \frac{\exp(-jkr_{M\tilde{M}})}{r_{M\tilde{M}}}. \tag{2.53}$$

The distance $r_{M\tilde{M}}$ is defined by Eq. (2.5), where x, y, z and $\tilde{x}, \tilde{y}, \tilde{z}$ are functions of q_1, q_2, q_3 and $\tilde{q}_1, \tilde{q}_2, \tilde{q}_3$, respectively, and they are given by Eqs. (2.44) and (2.45).

Substitution of Eqs. (2.49)-(2.51) into Eq. (2.52) followed by the substitution of Eq. (2.52) into Eq. (2.48) yields

$$A_{q_j}(M) = \frac{\mu_0 \mu}{4\pi} \frac{1}{H_j(M)} \iiint_V \sum_{k=1}^{3} \frac{1}{H_k(\tilde{M})} I_{q_k}^{\mathrm{e}}(\tilde{M}) \times$$

$$\times \left[\frac{\partial \tilde{x}}{\partial \tilde{q}_k} \frac{\partial x}{\partial q_j} + \frac{\partial \tilde{y}}{\partial \tilde{q}_k} \frac{\partial y}{\partial q_j} + \frac{\partial \tilde{z}}{\partial \tilde{q}_k} \frac{\partial z}{\partial q_j} \right] G(M, \tilde{M})\, d\tilde{V}, \quad j = 1, 2, 3, \tag{2.54}$$

where

$$d\tilde{V} = H_1(\tilde{M}) H_2(\tilde{M}) H_3(\tilde{M})\, d\tilde{q}_1 d\tilde{q}_2 d\tilde{q}_3. \tag{2.55}$$

Eqs. (2.46) and (2.54) give the integral representation of the solution to Helmholtz' vector equation (2.4) in arbitrary orthogonal curvilinear coordinates.

The integral representation of the solution to Helmholtz' vector equation (2.4) can be obtained for any orthogonal coordinate system by substituting the Lame coefficients of this coordinate system into Eqs. (2.54) and (2.46). For example, in cylindrical polar coordinates (r,φ,z):

$$q_1 = r\,,\ q_2 = \varphi\,,\ q_3 = z\,, \qquad H_1 = 1,\ H_2 = r\,,\ H_3 = 1\,,$$

$$\tilde{q}_1 = \tilde{r}\,,\ \tilde{q}_2 = \tilde{\varphi}\,,\ \tilde{q}_3 = \tilde{z}\,, \qquad \tilde{H}_1 = 1,\ \tilde{H}_2 = \tilde{r}\,,\ \tilde{H}_3 = 1\,,$$

$$G(M,\tilde{M}) = \Phi(M,\tilde{M})\,.$$

Substituting these expressions into Eq. (2.54) at $j = 1$ and using Eq. (2.19), we have

$$A_r(M) = \frac{\mu_0\mu}{4\pi}\iiint_V [I_r^{\mathrm{e}}(\tilde{M})(\cos\tilde{\varphi}\cos\varphi + \sin\tilde{\varphi}\sin\varphi) +$$

$$+I_\varphi^{\mathrm{e}}(\tilde{M})(-\sin\tilde{\varphi}\cos\varphi + \cos\tilde{\varphi}\sin\varphi)]\Phi(M,\tilde{M})\,d\tilde{V}\,. \qquad (2.56)$$

Eq. (2.56) completely coincides with the previously obtained Eq. (2.23). Similarly, Eqs. (2.25) for $A_\varphi(M)$ and (2.26) for $A_z(M)$ can be obtained from Eq. (2.54) by substituting $j = 2$ and $j = 3$, respectively.

In spherical coordinates (ρ,θ,φ), by Eqs. (2.40), (2.42) and (2.43) can be obtained from Eq. (2.54) by using the substitution

$$q_1 = \rho\,,\ q_2 = \theta\,,\ q_3 = \varphi\,, \qquad H_1 = 1,\ H_2 = \rho\,,\ H_3 = \rho\sin\theta\,,$$

$$\tilde{q}_1 = \tilde{\rho}\,,\ \tilde{q}_2 = \tilde{\theta}\,,\ \tilde{q}_3 = \tilde{\varphi}\,, \qquad \tilde{H}_1 = 1,\ \tilde{H}_2 = \tilde{\rho}\,,\ \tilde{H}_3 = \tilde{\rho}\sin\tilde{\theta}\,,$$

$$G(M,\tilde{M}) = F(M,\tilde{M})$$

and Eq. (2.37).

2.2. Exact analytical solution to the vector potential problem of a rectangular frame with current

The reaction of a conducting half-space on a rectangular frame with current has been studied theoretically only in the case where the ratio of the frame's sides is 1:4 or smaller. In this case, a double conductor line is considered as a convenient and sufficiently accurate model of the rectangular frame (see [6]). In this section (see also the author's paper [13]), an exact solution to the problem of the vector potential of the electromagnetic field induced by a rectangular frame with current is obtained without using the double conductor line approximation. Due to the linearity of the problem, it is sufficient to find the vector potential of the electromagnetic field created by one side of the frame having the form of a straight line and by the other side having the form of a circular arc. Similarly, the vector potential can be written for the other two other sides of the frame, and the results are added.

2.2.1. Solution to the problem of electromagnetic waves spreading from a harmonic emitter having the form of a straight line

In the literature the problem on electromagnetic waves spreading from a linear harmonic emitter is only solved in the so-called dipole approximation (see [57], p.666). The main idea is as follows. The emitter's length, l, tends to zero, but the current vector, I, in the emitter tends to infinity so that the product $I \cdot l = P$ (called the moment of dipole) stays constant. Such an approximation is used for the analysis of electromagnetic waves spreading under the assumption that the waves' length is much greater than the emitter's length. However, in non-destructive testing problems the size of a defect situated in a conducting medium may be compared with the emitter's length, l, or may even be larger than l. Therefore for problems of non-destructive testing, the problem on electromagnetic waves spreading from a linear harmonic emitter is to be solved without using the dipole approximation. This is done in the present thesis (see also the author's paper [13]).

Consider a vertical wire of length $2l$, located in the domain $\{ -l \le z \le l, \; r = 0 \}$ in free space (see Fig.2.1), where (r, φ, z) are cylindrical polar coordinates.

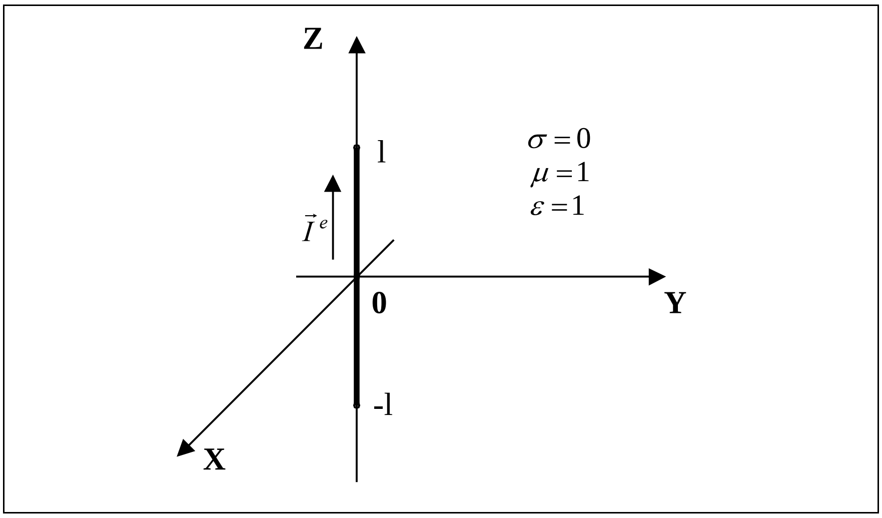

Fig.2.1. A linear harmonic emitter in free space

In the present case, due to axial symmetry, the magnetic vector potential does not depend on the coordinate φ, and it has only the z-component (see [57]):

$$\vec{A}(r,z) = A(r,z)\vec{e}_z \,. \tag{2.57}$$

The external current vector has the form (see [6]):

$$\vec{I}^{\,\mathrm{e}} = \begin{cases} I\dfrac{\delta(r)}{\pi r}\vec{e}_z, & -l < z < l, \\ 0, & z \notin (-l,l). \end{cases} \tag{2.58}$$

The use of the form $\dfrac{\delta(r)}{r}$ on the right-hand side of Eq. (2.58) was suggested in [13]. The right-hand side of Eq. (2.58) is chosen so that the full current vector in the wire is equal to I:

$$\iint\limits_D I^{\mathrm{e}} dxdy = \int\limits_0^{2\pi} d\varphi \int\limits_0^{\infty} I\frac{\delta(r)}{\pi r} rdr = 2\pi\, I \cdot \frac{1}{2\pi} = I. \tag{2.59}$$

In cylindrical polar coordinates, the mathematical formulation of the problem on electromagnetic waves spreading from a linear harmonic emitter has the form (see Eq. (1.26)):

$$\Delta\,\vec{A} + k^2\vec{A} = -\mu_0\mu\vec{I}^{\,\mathrm{e}}\,. \tag{2.60}$$

Since the vector potential has only the z-component, the problem for $A_z \equiv A$ has the form

$$\Delta A + k^2 A = \begin{cases} -\mu_0\mu I\dfrac{\delta(r)}{\pi r}, & -l < z < l, \\ 0, & z \notin (-l,l), \end{cases} \tag{2.61}$$

where $k^2 = \mu_0\varepsilon_0\mu\hat{\varepsilon}\omega^2$ (since $\sigma = 0$ in free space), $\Delta = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{\partial^2}{\partial z^2}$ is the Laplacian for the z-component of the vector potential in a system of cylindrical polar coordinates for the case of independence from the variable φ.

The conditions for the vector potential at infinity are Sommerfeld's radiation conditions (see [65], page 509):

$$R^2 = r^2 + z^2 \to \infty: \qquad A = O\left(\frac{1}{R}\right), \quad \frac{\partial A}{\partial R} + jkA = o\left(\frac{1}{R}\right) \,, \tag{2.62}$$

where $k = \omega\sqrt{\mu_0\varepsilon_0\mu\hat{\varepsilon}}$.

In order to solve problem (2.61)-(2.62), it is sufficient to use the integral representation of the solution to Helmholtz' equation in vector form in cylindrical polar coordinates (see Section 2.1.2):

$$\vec{A}(r,z)=\frac{\mu_0\mu}{4\pi}\iiint\limits_V \vec{I}^{e}(\tilde{M})\frac{\exp(-jkr_{M\tilde{M}})}{r_{M\tilde{M}}}\tilde{r}d\tilde{r}d\tilde{\varphi}d\tilde{z}\,, \tag{2.63}$$

where the integration is performed over the points $(\tilde{r},\tilde{\varphi},\tilde{z})\in V$ where $\vec{I}^{e}\neq 0$. Here $\mu=1,\ \hat{\varepsilon}=1,\ \sigma=0$, since the wire is situated in free space, and $r_{M\tilde{M}}$ is the distance given by the formula (see Eq.(2.20)):

$$r_{M\tilde{M}}=\sqrt{r^2+\tilde{r}^2-2r\tilde{r}\cos(\varphi-\tilde{\varphi})+(z-\tilde{z})^2}\,. \tag{2.64}$$

Substituting Eq. (2.58) for $\vec{I}^{e}$ into the projection of Helmholtz' equation (2.63) onto the z-axis, and taking into account that $\mu=1$, the solution for the z-component of the Helmholtz equation has the form

$$A(r,z)=\frac{\mu_0 I}{4\pi^2}\int_0^{2\pi}d\tilde{\varphi}\int_0^{\infty}\tilde{r}d\tilde{r}\int_{-l}^{l}\frac{\delta(\tilde{r})}{\tilde{r}}\frac{\exp(-jkr_{M\tilde{M}})}{r_{M\tilde{M}}}d\tilde{z}=$$

$$=\frac{\mu_0 I}{4\pi^2}\int_0^{2\pi}d\tilde{\varphi}\int_{-l}^{l}d\tilde{z}\int_0^{\infty}\delta(\tilde{r})\frac{\exp[-jk\sqrt{r^2+\tilde{r}^2-2r\tilde{r}\cos(\varphi-\tilde{\varphi})+(z-\tilde{z})^2}\,]}{\sqrt{r^2+\tilde{r}^2-2r\tilde{r}\cos(\varphi-\tilde{\varphi})+(z-\tilde{z})^2}}d\tilde{r}\,. \tag{2.65}$$

In order to calculate the integral of Eq. (2.65) with respect to $\tilde{r}$, the main property of the Dirac delta function is used. That is, for any continuous function $f(x)$ on $[a,b]$:

$$\int_a^b f(x)\delta(x-x_0)dx=\begin{cases} f(x_0), & a<x_0<b,\\ \frac{1}{2}f(x_0), & x_0=a\ \ or\ \ x_0=b,\\ 0, & x_0\notin[a,b]. \end{cases} \tag{2.66}$$

Using Eq. (2.66), it follows from Eq. (2.65) that

$$A(r,z)=\frac{\mu_0 I}{4\pi^2}\frac{1}{2}\int_0^{2\pi}d\tilde{\varphi}\int_{-l}^{l}\frac{\exp(-jk\sqrt{r^2+(z-\tilde{z})^2})}{\sqrt{r^2+(z-\tilde{z})^2}}d\tilde{z}=\frac{\mu_0 I}{4\pi}\int_{-l}^{l}\frac{\exp(-jk\sqrt{r^2+(z-\tilde{z})^2})}{\sqrt{r^2+(z-\tilde{z})^2}}d\tilde{z}\,. \tag{2.67}$$

Substituting $z-\tilde{z}=\xi$ with $d\tilde{z}=-d\xi$ into Eq. (2.67) yields

$$A(r,z)=\frac{\mu_0 I}{4\pi}\int_{z-l}^{z+l}\frac{\exp(-jk\sqrt{r^2+\xi^2})}{\sqrt{r^2+\xi^2}}d\xi\,. \tag{2.68}$$

It can be verified that $A(r,z)$ given by Eq. (2.68) satisfies Sommerfeld's conditions (2.62). Consequently, Eq. (2.68) is the solution of problem (2.61) - (2.62), and it gives the electromagnetic field's vector potential created by a linear harmonic emitter of length *2l* in an exact formulation without using the dipole approximation.

In the limit case as $k^2 \to 0$, Eq. (2.68) gives the result known in the literature. The term $k^2\vec{A}$ in Eq. (2.61) represents the displacement current. This term is neglected in free space and for sufficiently low frequencies. Consequently, substituting $k^2 = 0$ into Eq. (2.68), the vector potential can be written in the form

$$A(r,z)\big|_{k=0} = \frac{\mu_0 I}{4\pi}\int_{z-l}^{z+l}\frac{d\xi}{\sqrt{r^2+\xi^2}} = \frac{\mu_0 I}{4\pi}\ln\frac{z+l+\sqrt{(z+l)^2+r^2}}{z-l+\sqrt{(z-l)^2+r^2}}. \tag{2.69}$$

In particular, at $\mu = 1, \hat{\varepsilon} = 1$ and $\vec{A} = \vec{A}(r,z)$, it follows from Eq. (2.69) that the complex-valued amplitude magnetic induction vector $\vec{B}$ (everywhere below it is called the magnetic induction vector) has the form:

$$\vec{B} = \operatorname{curl}\vec{A} = -\frac{\partial A}{\partial r}\vec{e}_\varphi. \tag{2.70}$$

The known formula for the magnetic induction vector, produced by the current of an infinite length wire (see, for example, [61]), can be easily obtained from Eq. (2.70) as $l \to \infty$:

$$\vec{B} = \frac{\mu_0 I}{2\pi}\frac{1}{r}\vec{e}_\varphi. \tag{2.71}$$

Equations (2.68), (2.70) and (2.71) correspond to the quasi-steady case, because $\omega \neq 0$ in $\tilde{\vec{A}} = \vec{A}e^{j\omega t}$ and $\tilde{\vec{B}} = \vec{B}e^{j\omega t}$ (see Eq. (2.25)). Eq. (2.70) can also be obtained by using the Bio-Savare law.

It is to be noted that Eq. (2.68) gives the possibility to apply easily the dipole approximation as $l \to 0$, but $I \to \infty$ so that the product $I \cdot 2l = P = \text{const}$. Before passing to the limit, we transform the integral in Eq. (2.68) by using the mean value theorem for a definite integral. That is, if the function $f(\xi, r)$ (in the present case, the integrand in Eq. (2.68)) is continuous in the integration domain $\{z - l \le \xi \le z + l\}$, there exists a point $\bar{\xi}$ in this domain such that the integral in Eq. (2.68) is equal to the product of the integrand $f(\bar{\xi}, r)$ and the length of the integration interval, *2l*:

$$A(r,z) = \frac{\mu_0 I}{4\pi} f(\bar{\xi}, r)\cdot 2l = \frac{\mu_0 P}{4\pi} f(\bar{\xi}, r), \quad z - l \le \bar{\xi} \le z + l. \tag{2.72}$$

Now let $l \to 0$ and $I \to \infty$ so that $I \cdot 2l = P = \text{const}$, then $\bar{\xi} \to z$ and it follows from Eq. (2.68) that

$$A(r,z) = \mu_0 P f(z,r) = \frac{\mu_0 P}{4\pi}\frac{e^{-jk\sqrt{z^2+r^2}}}{\sqrt{z^2+r^2}}. \tag{2.73}$$

Equation (2.73) coincides with the known formula in the literature for the vector potential of a linear harmonic emitter (see for example, [65], [56]). But instead of SI measurement units the authors use the CGSE units in the above mentioned books; therefore Eq. (2.73) differs from the similar formula in [65] and [56] by the factor $1/4\pi$.

2.2.2. Solution to the problem of electromagnetic waves spreading from a harmonic emitter having the form of a circular arc

Consider a wire in the form of a circular arc of radius R situated in the domain $\{z=0, \quad r=R_a, \quad -\varphi_0 \le \varphi \le \varphi_0\}$ (see Fig.2.2). In this case, the vector potential depends on all variables, i.e. $\vec{A}=\vec{A}(r,z,\varphi)$.

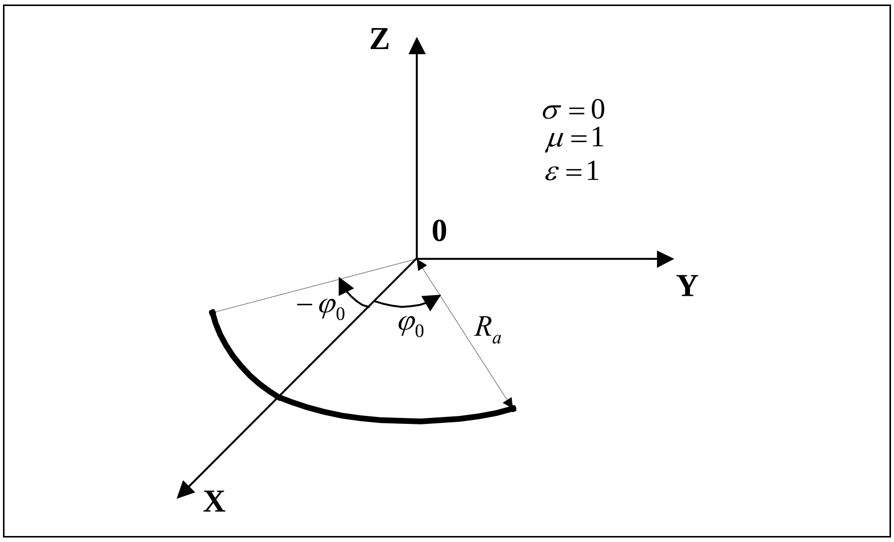

Fig.2.2. An emitter in the form of a circular arc in free space

The external current vector has the form (see the similar formula in [6], page 24):

$$\vec{I}^{\,\mathrm{e}} = \begin{cases} I\,\delta(r-R_a)\delta(z)\vec{e}_\varphi, & -\varphi_0 \le \varphi \le \varphi_0, \\ 0, & \varphi \notin [-\varphi_0, \varphi_0]. \end{cases} \tag{2.74}$$

The mathematical formulation of the problem has the form

$$\Delta\vec{A} + k^2\vec{A} = -\mu_0\mu\vec{I}^{\,e}, \tag{2.75}$$

with the Sommerfield's conditions of radiation at infinity:

$$R^2 = r^2 + z^2 \to \infty: \qquad A = O\left(\frac{1}{R}\right), \quad \frac{\partial A}{\partial R} + jkA = o\left(\frac{1}{R}\right). \tag{2.76}$$

In cylindrical polar coordinates, the Laplacian of the vector function has the form

$$\Delta\vec{A} = \vec{e}_r\left(\Delta A_r - \frac{A_r}{r^2} - \frac{2}{r^2}\frac{\partial A_\varphi}{\partial\varphi}\right) + \vec{e}_\varphi\left(\Delta A_\varphi - \frac{A_\varphi}{r^2} + \frac{2}{r^2}\frac{\partial A_r}{\partial\varphi}\right) + \vec{e}_z\Delta A_z\,, \tag{2.77}$$

where

$$\Delta f(r,\varphi,z) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\varphi^2} + \frac{\partial^2 f}{\partial z^2} \tag{2.78}$$

and $\Delta f(r,\varphi,z)$ is the Laplacian of a scalar function. It is easy to see from Eq. (2.77) that problem (2.74)-(2.78) for the z-component of the vector potential is decoupled. Besides, the problem for A_z is homogeneous and it has a unique zero solution, $A_z = 0$. Consequently, $A_z = 0$ can be substituted into Eq. (2.77). According to Eqs. (2.74) and (2.75), the problem for the r-component, A_r, is also homogeneous, but it does not have a zero solution, $A_r = 0$. Indeed, substituting $A_r = 0$ into Eq. (2.77), we find that $\partial A_\varphi / \partial\varphi = 0$, but this is a contradiction. The problem for A_φ is not homogeneous due to Eq. (2.74). Thus, the problems for A_r and A_φ cannot be decoupled.

Consequently, the vector potential $\vec{A}$ must have the form

$$\vec{A} = A_r(r,\varphi,z)\vec{e}_r + A_\varphi(r,\varphi,z)\vec{e}_\varphi\,. \tag{2.79}$$

Then the solution of problem (2.74)-(2.78) for the two non-zero components of the vector potential is obtained by the integral representation of the solution to Helmholtz' equation in vector form in cylindrical polar coordinates (see Section 2.1.2):

$$A_r(r,\varphi,z) = \frac{\mu_0\mu}{4\pi}\iiint_V I_\varphi^{\mathrm{e}}(\tilde{M})\sin(\varphi-\tilde{\varphi})\frac{\exp(-jkr_{M\tilde{M}})}{r_{M\tilde{M}}}\tilde{r}d\tilde{r}d\tilde{\varphi}d\tilde{z}\,, \tag{2.80}$$

$$A_\varphi(r,\varphi,z) = \frac{\mu_0\mu}{4\pi}\iiint_V I_\varphi^{\mathrm{e}}(\tilde{M})\cos(\varphi-\tilde{\varphi})\frac{\exp(-jkr_{M\tilde{M}})}{r_{M\tilde{M}}}\tilde{r}d\tilde{r}d\tilde{\varphi}d\tilde{z}\,, \tag{2.81}$$

where

$$r_{M\tilde{M}} = \sqrt{r^2 + \tilde{r}^2 - 2r\tilde{r}\cos(\varphi-\tilde{\varphi}) + (z-\tilde{z})^2}\,. \tag{2.82}$$

Substituting Eq. (2.74) for $\vec{I}^{\mathrm{e}}$ into the projection of Helmholtz' equation (2.80) on the r-axis, and taking into account that $\mu = 1$, Helmholtz equation for the r-component has the form

$$A_r(r,\varphi,z) = \frac{\mu_0 I}{4\pi}\int_{-\varphi_0}^{\varphi_0} d\tilde{\varphi}\int_0^{\infty}\tilde{r}d\tilde{r}\int_{-\infty}^{+\infty}\delta(\tilde{r}-R_a)\delta(\tilde{z})\sin(\varphi-\tilde{\varphi})\frac{\exp(-jkr_{M\tilde{M}})}{r_{M\tilde{M}}}d\tilde{z}\,. \tag{2.83}$$

Using the main property of Dirac's delta function, it follows from Eq. (2.83) that

$$A_r(r,\varphi,z) = \frac{\mu_0 I}{4\pi} \int_{-\varphi_0}^{\varphi_0} \sin(\varphi - \tilde{\varphi})\, \tilde{r} \left. \frac{\exp(-jkr_{M\tilde{M}})}{r_{M\tilde{M}}} \right|_{\substack{\tilde{z}=0 \\ \tilde{r}=R_a}} d\tilde{\varphi}$$

or

$$A_r(r,\varphi,z) = \frac{\mu_0 I R_a}{4\pi} \int_{-\varphi_0}^{\varphi_0} \sin(\varphi - \tilde{\varphi}) \frac{\exp(-jk\sqrt{r^2 + R_a^2 - 2rR_a\cos(\varphi - \tilde{\varphi}) + z^2})}{\sqrt{r^2 + R_a^2 - 2rR_a\cos(\varphi - \tilde{\varphi}) + z^2}} d\tilde{\varphi}. \tag{2.84}$$

Similarly, the solution for the φ-component can be obtained, and it has the form

$$A_\varphi(r,\varphi,z) = \frac{\mu_0 I R_a}{4\pi} \int_{-\varphi_0}^{\varphi_0} \cos(\varphi - \tilde{\varphi}) \frac{\exp(-jk\sqrt{r^2 + R_a^2 - 2rR_a\cos(\varphi - \tilde{\varphi}) + z^2})}{\sqrt{r^2 + R_a^2 - 2rR_a\cos(\varphi - \tilde{\varphi}) + z^2}} d\tilde{\varphi}. \tag{2.85}$$

It follows from Eqs. (2.84) and (2.85), by substituting $\varphi - \tilde{\varphi} = \psi, \quad -d\tilde{\varphi} = d\psi$, that

$$A_r(r,\varphi,z) = \frac{\mu_0 I R_a}{4\pi} \int_{\varphi-\varphi_0}^{\varphi+\varphi_0} \sin\psi \frac{\exp(-jk\sqrt{r^2 + R_a^2 - 2rR_a\cos\psi + z^2})}{\sqrt{r^2 + R_a^2 - 2rR_a\cos\psi + z^2}} d\psi, \tag{2.86}$$

$$A_\varphi(r,\varphi,z) = \frac{\mu_0 I R_a}{4\pi} \int_{\varphi-\varphi_0}^{\varphi+\varphi_0} \cos\psi \frac{\exp(-jk\sqrt{r^2 + R_a^2 - 2rR_a\cos\psi + z^2})}{\sqrt{r^2 + R_a^2 - 2rR_a\cos\psi + z^2}} d\psi. \tag{2.87}$$

It can easily be verified that $A_r(r,\varphi,z)$ and $A_\varphi(r,\varphi,z)$ in Eqs. (2.86) and (2.87) satisfy Sommerfeld's conditions (2.76). Consequently, Eqs. (2.86) and (2.87) give the solution of problem (2.74)-(2.78) with Sommerfeld's conditions (2.76) at infinity.

Remark. By substituting $\varphi_0 = \pi$ into Eqs. (2.86) and (2.87), one can obtain the solution for the problem on electromagnetic waves spreading from a coil carrying harmonic current. Then in Eqs. (2.86) and (2.87) the interval of integration is equal to the period of the functions $\sin\psi, \cos\psi$ and, consequently, the parameter φ can be deleted from the limits of integration. Besides, in this case the vector potential does not depend on the variable φ due to the axial symmetry. Then it follows from Eq. (2.86) that $A_r(r,z) = 0$, since the integrand in Eq. (2.86) is odd with respect to ψ. Thus we have

$$A_\varphi(r,z) = \frac{\mu_0 I R_a}{2\pi} \int_0^\pi \cos\psi \frac{\exp(-jk\sqrt{r^2 + R_a^2 - 2rR_a\cos\psi})}{\sqrt{r^2 + R_a^2 - 2rR_a\cos\psi}} d\psi. \tag{2.88}$$

For $k = 0$, solution (2.88) coincides with the one known in the literature (see for example, Eq. (2.1.15) in [6]) for the case of a bare single-turn coil located in free space:

$$A(r,z) = \frac{\mu_0 I R_a}{2} \int_0^\infty J_1(\lambda r) J_1(\lambda R_a) e^{-\lambda|z-h|} d\lambda\,, \tag{2.89}$$

where $J_1(s)$ is the Bessel function of the first kind of order 1 of a real argument. This can be easily verified by using the package "Mathematica".

One can see that if the multiplication of the Bessel functions in Eq. (2.89) is expressed in terms of the integral of trigonometric functions, by using formula (4) on page 426 in [55], i.e. by the formula

$$\int_0^\infty e^{-at} t^{\mu-\nu} J_\mu(bt) J_\nu(ct)\, dt = \frac{(b/2)^\mu (c/2)^\nu \Gamma(2\mu+1)}{\Gamma(\nu+1/2)\Gamma(1/2)} \int_0^\pi \frac{(\sin\varphi)^{2\nu}\, d\varphi}{(a^2 + 2iac\cos\varphi - c^2\cos^2\varphi + b^2)^{\mu+1/2}}\,,$$

$$\mathrm{Re}[a \pm bi \pm ci] > 0\,, \quad \mathrm{Re}[\mu] > -1/2\,. \tag{2.90}$$

then at $k = 0$ formula (2.89) may be expressed in terms of full elliptic integrals.

2.3. Exact analytical solution to the vector potential problem of a wire of arbitrary form with given current

In the previous section, an exact solution to the problem on electromagnetic waves spreading has been obtained for the case of a finite length wire in the form of a straight line and of a circular arc. In this section (see also the author's paper [15]), an exact solution to the similar problem is obtained for the case of a finite length wire of an arbitrary form. Writing the equation for the curve describing the wire in cylindrical polar and Cartesian coordinates and using Helmholtz' equation and the integral representation of its solution, the solution is obtained in the form of a single definite integral of an elementary function. Moreover, using the obtained solution, some new formulae for electromagnetic waves spreading are also found for the particular cases of a wire in the form of an Archimedes's spiral, of an elliptical or circular helix and in the form of a fractal wire. The case of the fractal wire is interesting for antenna analysis in radio engineering.

2.3.1. Solution to the problem of electromagnetic waves spreading from a harmonic emitter of an arbitrary form

Consider a wire located on a curve L (see Fig.2.3). In cylindrical polar coordinates (r, φ, z) centered at 0, with the z-axis directed upwards, the parametric equation describing the curve L is

$$\begin{cases} r = \hat{r}(\varphi) \\ z = \hat{z}(\varphi) \end{cases}, \quad \varphi_1 \le \varphi \le \varphi_2 . \tag{2.91}$$

where $\hat{r}(\varphi)$ and $\hat{z}(\varphi)$ are given functions of the angle φ. In Cartesian coordinates (x, y, z) the same curve is given by the parametric equations in the form

$$\begin{cases} x = \hat{r}(\varphi)\cos\varphi, \\ y = \hat{r}(\varphi)\sin\varphi, \\ z = \hat{z}(\varphi), \end{cases} \quad \varphi_1 \le \varphi \le \varphi_2 . \tag{2.92}$$

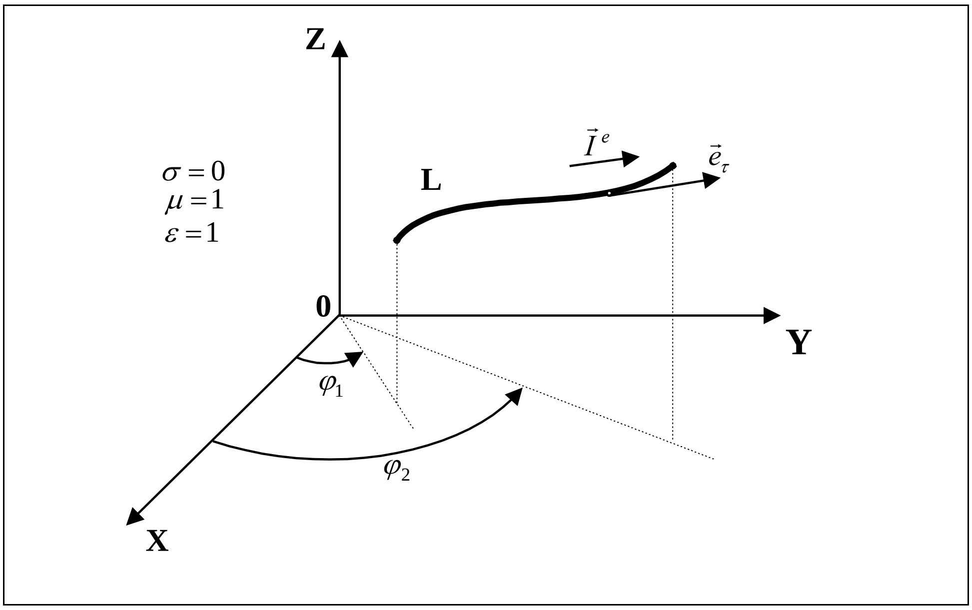

Fig.2.3. An arbitrary form wire in free space

The equation of the curve can be also written in the vector form

$$\vec{\mathbf{r}}(\varphi) = \hat{r}(\varphi)\cos\varphi \cdot \vec{i} + \hat{r}(\varphi)\sin\varphi \cdot \vec{j} + \hat{z}(\varphi)\cdot\vec{k} . \tag{2.93}$$

The unit tangent vector, $\vec{e}_\tau$, to the curve $\vec{\mathbf{r}} = \vec{\mathbf{r}}(\varphi)$ is

$$\vec{e}_\tau = \frac{\vec{\mathbf{r}}'(\varphi)}{|\vec{\mathbf{r}}'(\varphi)|} . \tag{2.94}$$

Taking into account Eq. (2.93), the unit tangent vector can be written in the form

$$\vec{e}_\tau = \frac{1}{|\vec{\mathbf{r}}'(\varphi)|}\left\{[\hat{r}'(\varphi)\cos\varphi - \hat{r}(\varphi)\sin\varphi]\vec{e}_x + [\hat{r}'(\varphi)\sin\varphi + \hat{r}(\varphi)\cos\varphi]\vec{e}_y + \hat{z}'(\varphi)\vec{e}_z\right\}, \qquad (2.95)$$

where $|\vec{\mathbf{r}}'(\varphi)| = \sqrt{\hat{r}^2(\varphi) + \hat{r}'^2(\varphi) + \hat{z}'^2(\varphi)}$, and the angle φ is

$$\varphi = \begin{cases} \arctan(y/x), & x > 0, \\ \arctan(y/x) + \pi, & x < 0,\ y > 0, \\ \arctan(y/x) - \pi, & x < 0,\ y < 0. \end{cases} \qquad (2.96)$$

The vector potential has the form

$$\vec{A} = A_x(x,y,z)\vec{e}_x + A_y(x,y,z)\vec{e}_y + A_z(x,y,z)\vec{e}_z, \qquad (2.97)$$

where the variables x, y, z are functions of the angle φ given by Eq. (2.92).

The mathematical formulation of the problem on electromagnetic waves spreading from the segment of the curve $\vec{\mathbf{r}} = \vec{\mathbf{r}}(\varphi)$ according to Eq. (1.2.26) has the form:

$$\Delta\vec{A} + k^2\vec{A} = -\mu_0\mu\vec{I}^{\,\mathrm{e}}, \qquad (2.98)$$

where $k^2 = \mu_0\varepsilon_0\mu\hat{\varepsilon}\omega^2$; $\mu = 1$, $\hat{\varepsilon} = 1$ and $\sigma = 0$, since the wire is located in free space.
The external current density is

$$\vec{I}^{\,\mathrm{e}} = \begin{cases} I\,\delta[r - \hat{r}(\varphi)]\delta[z - \hat{z}(\varphi)]\vec{e}_\tau, & \varphi_1 \le \varphi \le \varphi_2, \\ 0, & \varphi \notin [\varphi_1, \varphi_2], \end{cases} \qquad (2.99)$$

The conditions at infinity for all the components A_x, A_y, A_z of the vector potential are Sommerfeld's conditions of radiation (see Eq. (2.62)):

$$R^2 = r^2 + z^2 \to \infty: \qquad A = O(1/R), \quad \frac{\partial A}{\partial R} + jkA = o(1/R). \qquad (2.100)$$

Substituting Eq. (2.96) for $\vec{e}_\tau$ into the right-hand side of Eq. (2.99) for $\vec{I}^{\,\mathrm{e}}$, and projecting the vector equation (2.98) onto the x, y, z axes, one can obtain three scalar problems for the components A_x, A_y, A_z of the vector potential in the form:

$$\Delta A_x + k^2 A_x = \begin{cases} -\mu_0 I\,\delta[(r - \hat{r}(\varphi)]\delta[z - \hat{z}(\varphi)]\Phi_1(\varphi), & \varphi_1 < \varphi < \varphi_2, \\ 0, & \varphi \notin [\varphi_1, \varphi_2], \end{cases} \qquad (2.101)$$

$$\Delta A_y + k^2 A_y = \begin{cases} -\mu_0 I\,\delta[(r - \hat{r}(\varphi)]\delta[z - \hat{z}(\varphi)]\Phi_2(\varphi), & \varphi_1 < \varphi < \varphi_2, \\ 0, & \varphi \notin [\varphi_1, \varphi_2], \end{cases} \qquad (2.102)$$

$$\Delta A_z + k^2 A_z = \begin{cases} -\mu_0 I\, \delta[(r-\hat{r}(\varphi)]\delta[z-\hat{z}(\varphi)]\Phi_3(\varphi), & \varphi_1 < \varphi < \varphi_2, \\ 0, & \varphi \notin [\varphi_1, \varphi_2], \end{cases} \tag{2.103}$$

where

$$\Phi_1(\varphi) = \frac{1}{|\vec{\mathbf{r}}'(\varphi)|}[\hat{r}'(\varphi)\cos\varphi - \hat{r}(\varphi)\sin\varphi],$$

$$\Phi_2(\varphi) = \frac{1}{|\vec{\mathbf{r}}'(\varphi)|}[\hat{r}'(\varphi)\sin\varphi + \hat{r}(\varphi)\cos\varphi], \tag{2.104}$$

$$\Phi_3(\varphi) = \frac{\hat{z}'(\varphi)}{|\vec{\mathbf{r}}'(\varphi)|}.$$

In order to solve problems (2.101)-(2.103) with Sommerfeld's conditions (2.100), one uses the solution of the Helmholtz equation in the vector form (see Eqs. (2.4) - (2.5)):

$$\vec{A}(x,y,z) = \frac{\mu_0 \mu}{4\pi} \iiint_V \vec{I}^{\,e}(\tilde{x},\tilde{y},\tilde{z}) \frac{\exp(-jkr_{M\tilde{M}})}{r_{M\tilde{M}}} d\tilde{x}d\tilde{y}d\tilde{z}\,, \tag{2.105}$$

where $k = \omega\sqrt{\mu_0\varepsilon_0\mu\hat{\varepsilon}}$ and $r_{M\tilde{M}}$ is the distance between the points $M(x,y,z)$ and $M(\tilde{x},\tilde{y},\tilde{z})$:

$$r_{M\tilde{M}} = \sqrt{(x-\tilde{x})^2 + (y-\tilde{y})^2 + (z-\tilde{z})^2}\,. \tag{2.106}$$

It follows from Eq. (2.105) at $\mu = 1$ that the solution of Eq. (2.101) for the x-component has the form:

$$A_x(x,y,z) = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{+\infty} d\tilde{x} \int_{-\infty}^{+\infty} d\tilde{y} \int_{z(\varphi_1)}^{z(\varphi_2)} \delta[\tilde{r} - \hat{r}(\tilde{\varphi})]\delta[\tilde{z} - \hat{z}(\tilde{\varphi})]\Phi_1(\tilde{\varphi}) \frac{\exp(-jkr_{M\tilde{M}})}{r_{M\tilde{M}}} d\tilde{z}\,. \tag{2.107}$$

Using the main property of the delta function (see Eq. (2.66)) for the integration with respect to $\tilde{z}$ yields

$$A_x(x,y,z) = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{+\infty} d\tilde{x} \int_{-\infty}^{+\infty} \delta[\tilde{r} - \hat{r}(\tilde{\varphi})]\Phi_1(\tilde{\varphi}) \frac{\exp(-jkr_{M\tilde{M}})}{r_{M\tilde{M}}}\bigg|_{\tilde{z}=\hat{z}(\tilde{\varphi})} d\tilde{y}\,. \tag{2.108}$$

Substituting $\tilde{x} = \tilde{r}\cos\tilde{\varphi}$, $\tilde{y} = \tilde{r}\sin\tilde{\varphi}$, $x = r\cos\varphi$, $y = r\sin\varphi$ and $d\tilde{x}d\tilde{y} = \tilde{r}d\tilde{r}d\tilde{\varphi}$ into Eq. (2.108), one obtains

$$A_x(r,\varphi,z) = \frac{\mu_0 I}{4\pi} \int_{\varphi_1}^{\varphi_2} d\tilde{\varphi} \int_0^{\infty} \delta[\tilde{r} - \hat{r}(\tilde{\varphi})]\Phi_1(\tilde{\varphi}) \frac{\exp(-jkr_{M\tilde{M}})}{r_{M\tilde{M}}}\bigg|_{\tilde{z}=\hat{z}(\tilde{\varphi})} \tilde{r}d\tilde{r}\,, \tag{2.109}$$

where

$$r_{M\tilde{M}}\Big|_{\tilde{z}=z(\tilde{\varphi})} = \sqrt{(x-\tilde{x})^2 + (y-\tilde{y})^2 + [z-\hat{z}(\tilde{\varphi})]^2} =$$

$$= \sqrt{r^2 + \tilde{r}^2 - 2r\tilde{r}\cos(\varphi - \tilde{\varphi}) + [z - \hat{z}(\tilde{\varphi})]^2} \ . \quad (2.110)$$

It follows from Eq. (2.109) by using the main property of the delta function (see Eq. (2.66)) that

$$A_x(r,\varphi,z) = \frac{\mu_0 I}{4\pi} \int_{\varphi_1}^{\varphi_2} \Phi_1(\tilde{\varphi})\hat{r}(\tilde{\varphi}) \frac{\exp(-jkF)}{F} d\tilde{\varphi} \ , \quad (2.111)$$

where

$$F = \sqrt{r^2 + \hat{r}^2(\tilde{\varphi}) - 2r\hat{r}(\tilde{\varphi})\cos(\varphi - \tilde{\varphi}) + [z - \hat{z}(\tilde{\varphi})]^2} \ , \quad (2.112)$$

$$\Phi_1(\tilde{\varphi}) = \frac{1}{|\vec{\mathbf{r}}'(\tilde{\varphi})|}[\hat{r}'(\tilde{\varphi})\cos\tilde{\varphi} - \hat{r}(\tilde{\varphi})\sin\tilde{\varphi}] \ , \quad (2.113)$$

$$|\vec{\mathbf{r}}'(\tilde{\varphi})| = \sqrt{\hat{r}^2(\tilde{\varphi}) + \hat{r}'^2(\tilde{\varphi}) + \hat{z}'^2(\tilde{\varphi})} \ . \quad (2.114)$$

Similarly, the solution of Eqs. (2.102) and (2.103) has the form

$$A_y(r,\varphi,z) = \frac{\mu_0 I}{4\pi} \int_{\varphi_1}^{\varphi_2} \Phi_2(\tilde{\varphi})\hat{r}(\tilde{\varphi}) \frac{\exp(-jkF)}{F} d\tilde{\varphi} \ , \quad (2.115)$$

$$A_z(r,\varphi,z) = \frac{\mu_0 I}{4\pi} \int_{\varphi_1}^{\varphi_2} \Phi_3(\tilde{\varphi})\hat{r}(\tilde{\varphi}) \frac{\exp(-jkF)}{F} d\tilde{\varphi} \ , \quad (2.116)$$

where

$$\Phi_2(\tilde{\varphi}) = \frac{1}{|\vec{\mathbf{r}}'(\tilde{\varphi})|}[\hat{r}'(\tilde{\varphi})\sin\tilde{\varphi} + \hat{r}(\tilde{\varphi})\cos\tilde{\varphi}] \ , \quad (2.117)$$

$$\Phi_3(\tilde{\varphi}) = \frac{\hat{z}'(\tilde{\varphi})}{|\vec{\mathbf{r}}'(\tilde{\varphi})|} \ . \quad (2.118)$$

In order to get the solution of the same problem in terms of cylindrical polar coordinates, the components A_x and A_y are to be expressed in terms of the components A_r and A_φ as (see [68])

$$A_r = A_x\cos\varphi + A_y\sin\varphi \ , \quad A_\varphi = -A_x\sin\varphi + A_y\cos\varphi \ . \quad (2.119)$$

Substituting Eqs. (2.111) and (2.115) into Eq. (2.119), we can write the solution of the problem in terms of cylindrical polar coordinates as

$$A_r(r,\varphi,z) = \frac{\mu_0 I}{4\pi} \int_{\varphi_1}^{\varphi_2} \hat{r}(\tilde{\varphi}) \frac{1}{|\vec{\mathbf{r}}'(\tilde{\varphi})|} \left[\hat{r}'(\tilde{\varphi})\cos(\varphi - \tilde{\varphi}) + \hat{r}(\tilde{\varphi})\sin(\varphi - \tilde{\varphi})\right] \frac{\exp(-jkF)}{F} d\tilde{\varphi} \ , \quad (2.120)$$

$$A_\varphi(r,\varphi,z) = \frac{\mu_0 I}{4\pi}\int_{\varphi_1}^{\varphi_2} \hat{r}(\tilde{\varphi})\frac{1}{|\vec{\mathbf{r}}'(\tilde{\varphi})|}\left[\hat{r}(\tilde{\varphi})\cos(\varphi-\tilde{\varphi}) - \hat{r}'(\tilde{\varphi})\sin(\varphi-\tilde{\varphi})\right]\frac{\exp(-jkF)}{F}d\tilde{\varphi}\,, \quad (2.121)$$

$$A_z(r,\varphi,z) = \frac{\mu_0 I}{4\pi}\int_{\varphi_1}^{\varphi_2} \hat{r}(\tilde{\varphi})\frac{\hat{z}'(\tilde{\varphi})}{|\vec{\mathbf{r}}'(\tilde{\varphi})|}\frac{\exp(-jkF)}{F}d\tilde{\varphi}\,. \quad (2.122)$$

It can be easily verified that if the functions $\hat{r}(\varphi)$ and $\hat{r}'(\varphi)$ are continuous on the closed segment $\varphi_1 \le \varphi \le \varphi_2$ and, consequently, they are bounded on this segment, then the functions $A_r(r,\varphi,z)$, $A_\varphi(r,\varphi,z)$ and $A_z(r,\varphi,z)$ in Eqs. (2.120)-(2.122) satisfy Sommerfeld's conditions (2.100). Consequently, Eqs. (2.120)-(2.122) are the solutions to problem (2.98)-(2.100).

2.3.2. Some examples of the particular exact solutions obtained by using solutions of Section 2.3.1

Example 1. Consider a wire given in the form of a fragment of the Archimedes' spiral:

$$\hat{r}(\varphi) = a\varphi\,,\ \ a = \text{const}\,,\ \ \hat{z}(\varphi) = 0\,,\ \ \varphi_1 \le \varphi \le \varphi_2\,. \quad (2.123)$$

The solution to the problem for $A_r(r,\varphi,z)$, $A_\varphi(r,\varphi,z)$ and $A_z(r,\varphi,z)$ can be easily found from Eqs. (2.120)-(2.122) by substituting $\hat{r}(\tilde{\varphi}) = a\tilde{\varphi}$, $\hat{r}'(\tilde{\varphi}) = a$, $\hat{z}'(\tilde{\varphi}) = 0$, $|\vec{\mathbf{r}}'(\tilde{\varphi})| = a\sqrt{\tilde{\varphi}^2+1}$ and

$$F = \sqrt{r^2 + a^2\tilde{\varphi}^2 - 2ra\tilde{\varphi}\cos(\varphi-\tilde{\varphi}) + z^2}\,. \quad (2.124)$$

Then the solution to the problem has the form

$$A_r(r,\varphi,z) = \frac{\mu_0 I}{4\pi}a\int_{\varphi_1}^{\varphi_2} \frac{\tilde{\varphi}}{\sqrt{\tilde{\varphi}^2+1}}\left[\cos(\varphi-\tilde{\varphi}) + \tilde{\varphi}\sin(\varphi-\tilde{\varphi})\right]\frac{\exp(-jkF)}{F}d\tilde{\varphi}\,, \quad (2.125)$$

$$A_\varphi(r,\varphi,z) = \frac{\mu_0 I}{4\pi}a\int_{\varphi_1}^{\varphi_2} \frac{\tilde{\varphi}}{\sqrt{\tilde{\varphi}^2+1}}\left[\tilde{\varphi}\cos(\varphi-\tilde{\varphi}) - \sin(\varphi-\tilde{\varphi})\right]\frac{\exp(-jkF)}{F}d\tilde{\varphi}\,, \quad (2.126)$$

$$A_z(r,\varphi,z) = 0\,. \quad (2.127)$$

Example 2. Consider a wire given in the form of the elliptical helix:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,,\ \ \hat{z}(\varphi) = \hat{h}\varphi \quad (\ \hat{h} = h/2\pi\,), \quad \varphi_1 \le \varphi \le \varphi_2\,, \quad (2.128)$$

or by the substitution $x = \hat{r}(\varphi)\cos\varphi$ and $y = \hat{r}(\varphi)\sin\varphi$ into Eq. (2.128), the wire is described by

$$\hat{r}(\varphi)=\left(\frac{\cos^2\varphi}{a^2}+\frac{\sin^2\varphi}{b^2}\right)^{-1/2}, \quad z(\varphi)=\hat{h}\varphi, \quad \varphi_1\le\varphi\le\varphi_2. \tag{2.129}$$

In this case, the solutions for $A_r(r,\varphi,z)$, $A_\varphi(r,\varphi,z)$ and $A_z(r,\varphi,z)$ are obtained from Eqs. (2.120)-(2.122) by substituting

$$\hat{r}(\tilde{\varphi})=\left(\frac{\cos^2\tilde{\varphi}}{a^2}+\frac{\sin^2\tilde{\varphi}}{b^2}\right)^{-1/2}, \qquad \hat{r}'(\tilde{\varphi})=-\frac{1}{2}\left(\hat{r}(\tilde{\varphi})\right)^{-3/2}\sin 2\varphi\left(\frac{1}{b^2}-\frac{1}{a^2}\right),$$

$$\hat{z}(\tilde{\varphi})=\hat{h}\tilde{\varphi}, \quad \hat{z}'(\tilde{\varphi})=\hat{h}, \quad |\vec{\mathbf{r}}'(\tilde{\varphi})|=\sqrt{\hat{r}^2(\tilde{\varphi})+\hat{r}'^2(\tilde{\varphi})+\hat{h}^2},$$

$$F=\sqrt{r^2+\hat{r}^2(\tilde{\varphi})-2r\hat{r}(\tilde{\varphi})\cos(\varphi-\tilde{\varphi})+(z-\hat{h}\tilde{\varphi})^2}. \tag{2.130}$$

Example 3. Consider a wire given in the form of the circular helix (circular solenoid):

$$\hat{r}(\varphi)=R, \quad \hat{z}(\varphi)=\hat{h}\varphi \ \ (\hat{h}=h/2\pi), \quad \varphi_1\le\varphi\le\varphi_2. \tag{2.131}$$

In this case, the solutions for $A_r(r,\varphi,z)$, $A_\varphi(r,\varphi,z)$ and $A_z(r,\varphi,z)$ are obtained from Eqs. (2.120)-(2.122) by substituting

$$\hat{r}(\tilde{\varphi})=R, \quad \hat{r}'(\tilde{\varphi})=0, \quad \hat{z}(\tilde{\varphi})=\hat{h}\tilde{\varphi}, \quad \hat{z}'(\tilde{\varphi})=\hat{h}, \quad |\vec{\mathbf{r}}'(\tilde{\varphi})|=\sqrt{R^2+\hat{h}^2},$$

$$F=\sqrt{r^2+R^2-2rR\cos(\varphi-\tilde{\varphi})+[z-\hat{h}\tilde{\varphi}]^2}. \tag{2.132}$$

Thus, the solution to the problem has the form

$$A_r(r,\varphi,z)=\frac{\mu_0 I}{4\pi}\int_{\varphi_1}^{\varphi_2}\frac{R^2\sin(\varphi-\tilde{\varphi})}{\sqrt{R^2+\hat{h}^2}}\frac{\exp(-jkF)}{F}d\tilde{\varphi}, \tag{2.133}$$

$$A_\varphi(r,\varphi,z)=\frac{\mu_0 I}{4\pi}\int_{\varphi_1}^{\varphi_2}\frac{R^2\cos(\varphi-\tilde{\varphi})}{\sqrt{R^2+\hat{h}^2}}\frac{\exp(-jkF)}{F}d\tilde{\varphi}, \tag{2.134}$$

$$A_z(r,\varphi,z)=\frac{\mu_0 I}{4\pi}\int_{\varphi_1}^{\varphi_2}\frac{\hat{h}}{\sqrt{R^2+\hat{h}^2}}\frac{\exp(-jkF)}{F}d\tilde{\varphi}. \tag{2.135}$$

It is to be noted that in the limit case as $k^2\to 0$ and $r\to 0$ (i.e. on the axis of the solenoid), this example gives the result known in the literature (see [69]). The term $k^2\vec{A}$ in Eq. (2.98) for the vector potential represents the displacement current. For sufficiently low frequencies and at the absence of conductivity in free space, the multiplier $k^2=0$ can be directly substituted into Eq. (2.98) and, consequently, into solutions (2.120)-(2.122) for A_r, A_φ and A_z. In this case, integrals (2.120)-(2.122) and, consequently, the vectors $\vec{B}$ and $\vec{E}$ given by

Eqs. (1.8) and (1.30), respectively, can be expressed in terms of elliptical integrals. Besides, substituting $r=0$ into the same equations, integrals (2.133)-(2.103) can be expressed in terms of elementary functions.

It follows from Eqs. (2.133)-(2.135) as $k^2 \to 0$ that

$$A_r(r,\varphi,z)=\frac{\mu_0 I}{4\pi}\frac{R^2}{\sqrt{R^2+\hat{h}^2}}\int_{\varphi_1}^{\varphi_2}\sin(\varphi-\tilde{\varphi})\frac{d\tilde{\varphi}}{F}, \tag{2.136}$$

$$A_\varphi(r,\varphi,z)=\frac{\mu_0 I}{4\pi}\frac{R^2}{\sqrt{R^2+\hat{h}^2}}\int_{\varphi_1}^{\varphi_2}\cos(\varphi-\tilde{\varphi})\frac{d\tilde{\varphi}}{F}, \tag{2.137}$$

$$A_z(r,\varphi,z)=\frac{\mu_0 I}{4\pi}\frac{\hat{h}}{\sqrt{R^2+\hat{h}^2}}\int_{\varphi_1}^{\varphi_2}\frac{d\tilde{\varphi}}{F}. \tag{2.138}$$

In particular, in calculating the component of the magnetic induction vector $\vec{B}$ on the axis of the solenoid, i.e. the z-component, the formula for $\operatorname{curl}\vec{A}$ is used in polar cylindrical coordinates:

$$\vec{B}=\operatorname{curl}\vec{A}=(\operatorname{curl}\vec{A})_r\vec{e}_r+(\operatorname{curl}\vec{A})_\varphi\vec{e}_\varphi+(\operatorname{curl}\vec{A})_z\vec{e}_z, \tag{2.139}$$

where

$$(\operatorname{curl}\vec{A})_z=\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\varphi)-\frac{\partial A_r}{\partial\varphi}\right). \tag{2.140}$$

The magnetic induction vector $\vec{B}=B_z\vec{e}_z$ on the axis of the solenoid has the form

$$B_z\big|_{r=0}=(\operatorname{curl}\vec{A})_z\big|_{r=0}=\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\varphi)-\frac{\partial A_r}{\partial\varphi}\right)\Bigg|_{r=0}=\lim_{r\to0}\frac{\frac{\partial}{\partial r}(rA_\varphi)-\frac{\partial A_r}{\partial\varphi}}{r}=\left[\frac{0}{0}\right]=$$

$$=\lim_{r\to0}\frac{\partial}{\partial r}\left[\frac{\partial}{\partial r}(rA_\varphi)-\frac{\partial A_r}{\partial\varphi}\right]=\lim_{r\to0}\frac{\partial}{\partial r}\left[A_\varphi+r\frac{\partial A_\varphi}{\partial r}-\frac{\partial A_r}{\partial\varphi}\right]=$$

$$=\lim_{r\to0}\left[\frac{\partial A_\varphi}{\partial r}+\frac{\partial A_\varphi}{\partial r}-\frac{\partial^2 A_r}{\partial\varphi\partial r}\right]+\lim_{r\to0}r\frac{\partial^2 A_\varphi}{\partial r^2}. \tag{2.141}$$

Substituting Eq. (2.136) for A_r and Eq. (2.137) for A_φ into Eq. (2.141), one obtains

$$B_z\big|_{r=0}=\lim_{r\to0}\left\{\frac{\mu_0 I}{4\pi}\frac{2R^2}{\sqrt{R^2+\hat{h}^2}}\int_{\varphi_1}^{\varphi_2}\cos(\varphi-\tilde{\varphi})\left(-\frac{1}{F^2}\right)\left(\frac{1}{2F}(2r-2R\cos(\varphi-\tilde{\varphi}))\right)d\tilde{\varphi}-\right.$$

$$\left.-\frac{\mu_0 I}{4\pi}\frac{R^2}{\sqrt{R^2+\hat{h}^2}}\frac{\partial}{\partial r}\int_{\varphi_1}^{\varphi_2}\frac{\partial}{\partial\varphi}\left(\frac{\sin(\varphi-\tilde{\varphi})}{F}\right)d\tilde{\varphi}\right\}=$$

$$= \lim_{r\to 0}\left\{\frac{\mu_0 I}{4\pi}\frac{2R^2}{\sqrt{R^2+\hat{h}^2}}\int_{\varphi_1}^{\varphi_2}\cos(\varphi-\tilde{\varphi})\left(-\frac{1}{F^2}\right)\left(\frac{1}{2F}(2r-2R\cos(\varphi-\tilde{\varphi}))\right)d\tilde{\varphi}-\right.$$

$$\left.-\frac{\mu_0 I}{4\pi}\frac{R^2}{\sqrt{R^2+\hat{h}^2}}\frac{\partial}{\partial r}\int_{\varphi_1}^{\varphi_2}\left(\frac{\cos(\varphi-\tilde{\varphi})}{F}-\frac{\sin(\varphi-\tilde{\varphi})}{F^2}\frac{1}{2F}2Rr\sin(\varphi-\tilde{\varphi})\right)d\tilde{\varphi}\right\}=$$

$$= \lim_{r\to 0}\left\{\frac{\mu_0 I}{4\pi}\frac{2R^2}{\sqrt{R^2+\hat{h}^2}}\int_{\varphi_1}^{\varphi_2}\cos(\varphi-\tilde{\varphi})\left(-\frac{r-R\cos(\varphi-\tilde{\varphi})}{F^3}\right)d\tilde{\varphi}-\right.$$

$$-\frac{\mu_0 I}{4\pi}\frac{R^2}{\sqrt{R^2+\hat{h}^2}}\int_{\varphi_1}^{\varphi_2}\cos(\varphi-\tilde{\varphi})\left(-\frac{r-R\cos(\varphi-\tilde{\varphi})}{F^3}\right)d\tilde{\varphi}-$$

$$\left.-\frac{\mu_0 I}{4\pi}\frac{R^3}{\sqrt{R^2+\hat{h}^2}}\int_{\varphi_1}^{\varphi_2}\frac{\partial}{\partial r}\left(-\frac{r}{F^3}\sin^2(\varphi-\tilde{\varphi})\right)d\tilde{\varphi}\right\}=$$

$$= \frac{\mu_0 I}{4\pi}\frac{2R^3}{\sqrt{R^2+\hat{h}^2}}\int_{\varphi_1}^{\varphi_2}\left.\frac{\cos^2(\varphi-\tilde{\varphi})}{F^3}\right|_{r=0}d\tilde{\varphi}-\frac{\mu_0 I}{4\pi}\frac{R^3}{\sqrt{R^2+\hat{h}^2}}\int_{\varphi_1}^{\varphi_2}\left.\frac{\cos^2(\varphi-\tilde{\varphi})}{F^3}\right|_{r=0}d\tilde{\varphi}+$$

$$\left.+\frac{\mu_0 I}{4\pi}\frac{R^3}{\sqrt{R^2+\hat{h}^2}}\int_{\varphi_1}^{\varphi_2}\sin^2(\varphi-\tilde{\varphi})\left.\left(\frac{1}{F^3}+r\frac{\partial}{\partial r}\frac{1}{F^3}\right)\right|_{r=0}d\tilde{\varphi}\right\}=$$

$$= \frac{\mu_0 I}{4\pi}\frac{R^3}{\sqrt{R^2+\hat{h}^2}}\int_{\varphi_1}^{\varphi_2}\frac{\cos^2(\varphi-\tilde{\varphi})}{\left[R^2+(z-\hat{h}\tilde{\varphi})^2\right]^{3/2}}d\tilde{\varphi}+$$

$$+\frac{\mu_0 I}{4\pi}\frac{R^3}{\sqrt{R^2+\hat{h}^2}}\int_{\varphi_1}^{\varphi_2}\frac{\sin^2(\varphi-\tilde{\varphi})}{\left[R^2+(z-\hat{h}\tilde{\varphi})^2\right]^{3/2}}d\tilde{\varphi}=$$

$$= \frac{\mu_0 I}{4\pi}\frac{R^3}{\sqrt{R^2+\hat{h}^2}}\int_{\varphi_1}^{\varphi_2}\frac{d\tilde{\varphi}}{\left[R^2+(z-\hat{h}\tilde{\varphi})^2\right]^{3/2}}\,. \tag{2.142}$$

Substituting $\varphi_1 = -\hat{L}/\hat{h}$, $\varphi_2 = \hat{L}/\hat{h}$, $\hat{h}\tilde{\varphi} = \tilde{z}$, $\tilde{\varphi} = \tilde{z}/\hat{h}$ and $d\tilde{\varphi} = d\tilde{z}/\hat{h}$ into Eq. (2.142), where $\hat{L}$ is half the length of the solenoid, the magnetic induction vector on the axis of the solenoid takes the form

$$B_z\big|_{r=0} = \frac{\mu_0 I}{4\pi}\frac{R^3}{\sqrt{R^2+\hat{h}^2}}\frac{1}{\hat{h}}\int_{-\hat{L}}^{\hat{L}}\frac{d\tilde{z}}{\left[R^2+(z-\tilde{z})^2\right]^{3/2}}\,. \tag{2.143}$$

Substituting $z-\tilde{z} = R\tan t$ into Eq. (2.143) and calculating the integral, yields

$$B_z\big|_{r=0} = \frac{\mu_0 IR}{4\pi\hat{h}\sqrt{R^2+\hat{h}^2}}\left[\frac{\hat{L}+z}{\sqrt{R^2+(\hat{L}+z)^2}}+\frac{\hat{L}-z}{\sqrt{R^2+(\hat{L}-z)^2}}\right], \tag{2.144}$$

where $z = \hat{h}\varphi = h\varphi/2\pi$. It means that h is the pitch of a screw. Therefore, if there are $2N$

single-turn coils of the solenoid, then the length of the solenoid is equal to $2Nh$, but the number of the single-turn coils on one unit length of the solenoid is equal to $n = 2N/(2Nh) = 1/h$. Consequently, in the case $\hat{h}^2 << R^2$, Eq. (2.144) can be rewritten as

$$B_z\big|_{r=0} = \frac{\mu_0 nI}{2}\left[\frac{\hat{L}+z}{\sqrt{R^2+(\hat{L}+z)^2}} + \frac{\hat{L}-z}{\sqrt{R^2+(\hat{L}-z)^2}}\right]. \tag{2.145}$$

Eq. (2.145) coincides with a known formula in the literature (see [69]).

Example 4. Consider a fractal wire i.e. wire in the form of a broken curve. There exist linear branches on some links of this curve. Due to the linearity of the problem, it is sufficient to solve the problem for one linear segment of the wire that is given by the equation

$$\frac{x-x_{1k}}{l_k} = \frac{y-y_{1k}}{m_k} = \frac{z-z_{1k}}{n_k}, \quad x_{1k} \le x \le x_{2k}, \tag{2.146}$$

where

$$l_k = x_{2k} - x_{1k}, \;\; m_k = y_{2k} - y_{1k}, \;\; n_k = z_{2k} - z_{1k}, \; k = 1, 2, \ldots, N.$$

Then we must add the results of the solutions. The coordinate system XYZ may be chosen such that $l_k \ne 0$, $m_k \ne 0$, $n_k \ne 0$ for the each $k = 1, 2, \ldots, N$, and such that the each of segments (2.146) does not pass through the point (0,0,0). Substituting $x = \hat{r}(\varphi)\cos\varphi$, $y = \hat{r}(\varphi)\sin\varphi$ and $z = \hat{z}(\varphi)$ into Eq. (2.146), we can write the equation for the linear segment given by Eq. (2.146) in cylindrical polar coordinates as

$$\hat{r}(\varphi) = (m_k x_{1k} - l_k y_{1k})(m_k \cos\varphi - l_k \sin\varphi)^{-1},$$

$$\hat{z}(\varphi) = z_{1k} + (\hat{r}(\varphi)\cos\varphi - x_{1k})\, n_k l_k^{-1}. \tag{2.147}$$

The solutions for $A_r(r,\varphi,z)$, $A_\varphi(r,\varphi,z)$ and $A_z(r,\varphi,z)$ are obtained by substituting

$$\hat{r}(\tilde{\varphi}) = (m_k x_{1k} - l_k y_{1k})(m_k \cos\tilde{\varphi} - l_k \sin\tilde{\varphi})^{-1},$$

$$\hat{z}(\tilde{\varphi}) = z_{1k} + (\hat{r}(\tilde{\varphi})\cos\tilde{\varphi} - x_{1k})\, n_k l_k^{-1},$$

$$\hat{r}(\tilde{\varphi}) = (m_k x_{1k} - l_k y_{1k})(m_k \cos\tilde{\varphi} - l_k \sin\tilde{\varphi})^{-2}(m_k \sin\tilde{\varphi} - l_k \cos\tilde{\varphi}),$$

$$\hat{z}(\tilde{\varphi}) = z_{1k} + (\hat{r}(\tilde{\varphi})\cos\tilde{\varphi} - x_{1k})\, n_k l_k^{-1},$$

$$|\,\vec{\mathbf{r}}'(\tilde{\varphi})\,| = \sqrt{\hat{r}^2(\tilde{\varphi}) + \hat{r}'^2(\tilde{\varphi}) + \hat{z}'^2(\tilde{\varphi})}, \qquad \varphi_1 = \arctan y_{1k}/x_{1k}, \;\; \varphi_2 = \arctan y_{2k}/x_{2k} \;\text{ and}$$

$$F = \sqrt{r^2 + \hat{r}^2(\tilde{\varphi}) - 2r\hat{r}(\tilde{\varphi})\cos(\varphi-\tilde{\varphi}) + [z-\hat{z}(\tilde{\varphi})]^2} \tag{2.148}$$

into the same Eqs. (2.120)-(2.122). This case of a fractal wire can be applicable in radio engineering for the design of antennas.

3. SOME PROBLEMS ON IMPEDANCE CHANGE OF HOMOGENEOUS MEDIA

3.1. Evaluation of new classes of definite integrals

As mentioned in the Introduction, analytical solutions to eddy current testing problems are rare and can be found only for domains and emitters of simple geometry. However, even in these cases, the change in impedance is usually expressed in terms of improper integrals (see [6]).

The integrands of these integrals are either combinations of irrational and trigonometric functions, or combinations of irrational and Bessel functions, so that one can consider two classes of definite integrals. Thus, applications require the evaluation of different types of improper integrals. New classes of improper integrals which can be evaluated in closed analytical form and have direct applications to eddy current testing methods are considered in this section (see also the author's papers [12], [8]). The integrals are evaluated in closed form by means of divergent integrals that converge in the sense of Abel [4]).

Consider the following classes of definite integrals

$$A_n(\gamma) = \int_0^\infty \frac{\cos\gamma x\, dx}{(\sqrt{x^2 + a^2} + x)^{2n-1}}, \qquad B_{n,m}(b) = \int_0^\infty \frac{x^{m+1} J_m(bx)\, dx}{(\sqrt{x^2 + a^2} + x)^{2n-1}}, \tag{3.1}$$

where $n = 1, 2, 3, \ldots, m = 0, 1, 2, \ldots,$ and $J_m(z)$ is the Bessel function of the first kind of order m. Only the particular case $B_{1,0}(b)$ is evaluated in closed form in the literature (see [48]) and the used method is appropriate only for calculating $B_{1,0}(b)$. The general formula for $B_{n,m}(b)$ and even particular cases for $A_n(\gamma)$ seems to be absent in the literature.

3.1.1. Evaluation of the integral $A_n(\gamma)$

In order to evaluate $A_n(\gamma)$, one uses divergent integrals, which converge in the sense of Abel (see [4]). For example, for $\gamma > 0$ the integral

$$\int_0^\infty \sin\gamma x\, dx = \lim_{\delta \to +0} \int_0^\infty e^{-\delta x} \sin\gamma x dx = \lim_{\delta \to +0} \frac{\gamma}{\gamma^2 + \delta^2} = \frac{1}{\gamma}. \tag{3.2}$$

Differentiating both sides of Eq. (3.2) with respect to γ yields that, in the sense of Abel,

$$\int_0^\infty x\cos\gamma x\,dx = -\frac{1}{\gamma^2}\,, \quad \int_0^\infty x^2\sin\gamma x\,dx = -\frac{2}{\gamma^3}\,, \quad \int_0^\infty x^3\cos\gamma x\,dx = \frac{2\cdot 3}{\gamma^4}\,,\ \dots\ , \tag{3.3}$$

and, in general,

$$\int_0^\infty x^{2n+1}\cos\gamma x\,dx = (-1)^{n+1}\frac{(2n+1)!}{\gamma^{2n+2}}\,, \qquad n = 0,1,2,\dots . \tag{3.4}$$

Consider the integral (see [41])

$$\int_0^\infty \frac{e^{-\delta\sqrt{x^2+a^2}}}{\sqrt{x^2+a^2}}\cos\gamma x dx = K_0(a\sqrt{\delta^2+\gamma^2})\,, \tag{3.5}$$

where $K_0(z)$ is the modified Bessel function of the second kind of order 0 and $a = \text{const}$, $a > 0$. Differentiating both sides of Eq. (3.5) with respect to δ and using the formula $K_0'(z) = -K_1(z)$, one obtains

$$\int_0^\infty e^{-\delta\sqrt{x^2+a^2}}\cos\gamma x dx = \frac{a\delta K_1(a\sqrt{\delta^2+\gamma^2})}{\sqrt{\delta^2+\gamma^2}}\,, \quad \delta > 0\,, \tag{3.6}$$

where $K_1(z)$ is the modified Bessel function of the second kind of order 1. Differentiating ($2n+1$)-times both sides of Eq. (3.6) with respect to δ, one obtains the formulas

$$\int_0^\infty \sqrt{x^2+a^2}\,e^{-\delta\sqrt{x^2+a^2}}\cos\gamma x\,dx = -\frac{d}{d\delta}\left[\frac{a\delta K_1(a\sqrt{\delta^2+\gamma^2})}{\sqrt{\delta^2+\gamma^2}}\right], \tag{3.7}$$

$$\vdots$$

$$\int_0^\infty (\sqrt{x^2+a^2})^{2n+1}e^{-\delta\sqrt{x^2+a^2}}\cos\gamma x\,dx = -\frac{d^{2n+1}}{d\delta^{2n+1}}\left[\frac{a\delta K_1(a\sqrt{\delta^2+\gamma^2})}{\sqrt{\delta^2+\gamma^2}}\right]. \tag{3.8}$$

It follows from Eqs. (3.7), (3.8) and (3.4) that, in the sense of Abel,

$$\int_0^\infty \sqrt{x^2+a^2}\cos\gamma x\,dx = -\lim_{\delta\to+0}\frac{d}{d\delta}\left[\frac{a\delta K_1(a\sqrt{\delta^2+\gamma^2})}{\sqrt{\delta^2+\gamma^2}}\right], \tag{3.9}$$

$$\int_0^\infty (\sqrt{x^2+a^2})^{2n+1}\cos\gamma x\,dx = -\lim_{\delta\to+0}\frac{d^{2n+1}}{d\delta^{2n+1}}\left[\frac{a\delta K_1(a\sqrt{\delta^2+\gamma^2})}{\sqrt{\delta^2+\gamma^2}}\right], \tag{3.10}$$

$$\int_0^\infty x^{2m}(\sqrt{x^2+a^2})^{2n+1}\cos\gamma x\,dx = (-1)^{m+1}\frac{d^{2m}}{d\gamma^{2m}}\lim_{\delta\to+0}\frac{d^{2n+1}}{d\delta^{2n+1}}\left[\frac{a\delta\,K_1(a\sqrt{\delta^2+\gamma^2})}{\sqrt{\delta^2+\gamma^2}}\right]. \tag{3.11}$$

In view of the transformation

$$\frac{1}{(\kappa+x)^{2n-1}}=\frac{(\kappa-x)^{2n-1}}{(\kappa^2-x^2)^{2n-1}}=\frac{(\kappa-x)^{2n-1}}{(a^2)^{2n-1}}=\frac{1}{(a^2)^{2n-1}}\Big[\kappa^{2n-1}-p\kappa^{2n-2}x+\frac{p(p-1)}{2!}\kappa^{2n-3}x^2$$

$$-\frac{p(p-1)(p-2)}{3!}\kappa^{2n-4}x^3+\cdots-x^{2n-1}\Big],\quad a^2=\kappa^2-x^2,\; p=2n-1, \tag{3.12}$$

one can see that $A_n(\gamma)$ is a finite sum of integrals (3.4) and (3.11) for $\kappa=\sqrt{x^2+a^2}$. Consider the right-hand side of Eq. (3.12). The first term is

$$\kappa^{2n-1}=\left(\sqrt{x^2+a^2}\right)^{2n-1}=(x^2+a^2)^{n-1}\sqrt{x^2+a^2}\,. \tag{3.13}$$

Thus, the first term is a finite sum of the products of the term $\sqrt{x^2+a^2}$ and even powers of x (see the integrand in Eq. (3.11). The second term is

$$p\kappa^{2n-2}x=p(\sqrt{x^2+a^2})^{2n-2}x=p(x^2+a^2)^{n-1}x\,. \tag{3.14}$$

Thus, the second term is a finite sum of odd powers of x (see the integrand in Eq. (3.4)). The third term is

$$\frac{p(p-1)}{2!}\kappa^{2n-3}x^2=\frac{p(p-1)}{2!}(\sqrt{x^2+a^2})^{2n-3}x^2=\frac{p(p-1)}{2!}(x^2+a^2)^{n-1}x^2\sqrt{x^2+a^2}\,. \tag{3.15}$$

Thus, the third term has the same structure as the first term of Eq. (3.12). Similarly, the fourth term has the same structure as the second term of Eq. (3.11), and so on.

Hence, the integral $A_n(\gamma)$ indeed is the finite sum of integrals (3.4), (3.10) and (3.11). For example, it follows from Eq. (3.1) and decomposition (3.12) that

$$A_1(\gamma)=\int_0^\infty\frac{\cos\gamma x\,dx}{\sqrt{x^2+a^2}+x}=\frac{1}{a^2}\int_0^\infty(\sqrt{x^2+a^2}-x)\cos\gamma x\,dx$$

$$=\frac{1}{a^2}\int_0^\infty\sqrt{x^2+a^2}\cos\gamma x\,dx-\frac{1}{a^2}\int_0^\infty x\cos\gamma x\,dx\,, \tag{3.16}$$

$$A_2(\gamma)=\int_0^\infty\frac{\cos\gamma x\,dx}{(\sqrt{x^2+a^2}+x)^3}=\frac{1}{a^6}\int_0^\infty(\sqrt{x^2+a^2}-x)^3\cos\gamma x\,dx$$

$$=\frac{1}{a^6}\int_0^\infty\Big[(\sqrt{x^2+a^2})^3-3x(x^2+a^2)+3x^2\sqrt{x^2+a^2}-x^3\Big]\cos\gamma x\,dx$$

$$=\frac{1}{a^6}\int_0^\infty\Big[(\sqrt{x^2+a^2})^3-4x^3-3xa^2+3x^2\sqrt{x^2+a^2}\Big]\cos\gamma x\,dx\,. \tag{3.17}$$

Consider the expressions for $A_1(\gamma)$ in detail. The first integral on the right-hand side of Eq. (3.16) can be evaluated by using Eq. (3.9) and the formula

$$\frac{d}{dz}\left[\frac{K_\nu(z)}{z^\nu}\right] = -\frac{K_{\nu+1}(z)}{z^\nu}, \tag{3.18}$$

and it has the form

$$\int_0^\infty \sqrt{x^2+a^2}\cos\gamma x\,dx = -\lim_{\delta\to+0}\frac{d}{d\delta}\left[\frac{a\delta K_1(a\sqrt{\delta^2+\gamma^2})}{\sqrt{\delta^2+\gamma^2}}\right]$$

$$= -\lim_{\delta\to+0}\left[\frac{aK_1(a\sqrt{\delta^2+\gamma^2})}{\sqrt{\delta^2+\gamma^2}} - a\delta\frac{K_2(a\sqrt{\delta^2+\gamma^2})}{\sqrt{\delta^2+\gamma^2}}\frac{a\delta}{\sqrt{\delta^2+\gamma^2}}\right] = -\frac{a}{\gamma}K_1(a\gamma), \tag{3.19}$$

but the second integral on the right-hand side of Eq. (3.16) is found by Eq. (3.3). Similarly, in order to calculate $A_2(\gamma)$, four integrals on the right-hand side of Eq. (3.17) are evaluated by using Eq. (3.10), (3.4), (3.11) and (3.18). Thus,

$$A_1(\gamma) = \int_0^\infty \frac{\cos\gamma x\,dx}{\sqrt{x^2+a^2}+x} = \frac{1}{a^2}\left[\frac{1}{\gamma^2} - \frac{a}{\gamma}K_1(a\gamma)\right], \tag{3.20}$$

$$A_2(\gamma) = \int_0^\infty \frac{\cos\gamma x\,dx}{(\sqrt{x^2+a^2}+x)^3} = \frac{1}{a^6}\left[\frac{3a^3}{\gamma}K_3(a\gamma) + \frac{3a^2}{\gamma^2} - \frac{24}{\gamma^4}\right], \tag{3.21}$$

where $K_1(z)$ and $K_3(z)$ are the modified Bessel functions of the second kind of order 1 and 3, respectively. The correctness of Eqs. (3.20) and (3.21) has been verified numerically by using the package “Mathematica”.

Moreover, the correctness of Eq. (3.20) is rigorously proved by using the following analytical methods of Van der Pol [48]. Performing the transformation

$$\frac{1}{x+\sqrt{x^2+a^2}} = \frac{1}{a^2}\int_0^a \frac{v\,dv}{\sqrt{v^2+x^2}} = \frac{1}{a^2}\sqrt{v^2+x^2}\Big|_{v=0}^{v=a}$$

$$= \frac{1}{a^2}(\sqrt{x^2+a^2}-x) = \frac{1}{a^2}\frac{x^2+a^2-x^2}{\sqrt{x^2+a^2}+x} = \frac{1}{\sqrt{x^2+a^2}+x}, \tag{3.22}$$

we can rewrite the integral on the left-hand side of Eq. (3.20) in the form

$$\int_0^\infty \frac{\cos\gamma x\,dx}{\sqrt{x^2+a^2}+x} = \frac{1}{a^2}\int_0^\infty \cos\gamma x\,dx\int_0^a \frac{v\,dv}{\sqrt{v^2+x^2}} = \frac{1}{a^2}\int_0^\infty v\,dv\int_0^a \frac{\cos\gamma x\,dx}{\sqrt{v^2+x^2}}. \tag{3.23}$$

If one uses the integral

$$\int_0^\infty \frac{\cos\gamma x\,dx}{\sqrt{v^2+x^2}} = K_0(\gamma v)\,, \tag{3.24}$$

where $K_0(\gamma v)$ is the modified Bessel function of the second kind of order 0 (see [54], vol.1, page 18), then it follows from Eq. (3.23) that

$$\frac{1}{a^2}\int_0^\infty v dv \int_0^a \frac{\cos\gamma x dx}{\sqrt{v^2+x^2}} = \frac{1}{a^2}\int_0^a K_0(\gamma v) v dv = \left[\gamma v = z; v = \frac{1}{\gamma} z; dv = \frac{1}{\gamma} dz\right] = \frac{1}{a^2\gamma^2}\int_0^{\gamma a} K_0(z) z dz$$

or using the equation $zK_0(z) = -\frac{d}{dz}[zK_1(z)]$ and $K_1(z) = \frac{1}{z}$, as $z \to 0$, one obtains

$$\frac{1}{a^2\gamma^2}\int_0^{\gamma a} K_0(z) z dz = -\frac{1}{a^2\gamma^2} zK_1(z)\Big|_{z=0}^{z=\gamma a} = -\frac{1}{a^2\gamma^2}[\gamma a K_1(\gamma a) - 1] = \frac{1}{a^2\gamma^2} - \frac{1}{\gamma a} K_1(\gamma a)\,. \tag{3.25}$$

Thus, Eq. (3.20) is also proved analytically.

3.1.2. Evaluation of the integral $B_{n,m}(b)$

As in the previous evaluation of the integral $A_n(\gamma)$, divergent integrals that converge in the sense of Abel are used to evaluate the integral $B_{n,m}(b)$. Consider the integral (see [41], Eq. 6.623(2)):

$$\int_0^\infty e^{-\delta x} J_m(bx) x^{m+1} dx = \frac{2\delta}{\sqrt{\pi}} \frac{(2b)^m \Gamma(m+3/2)}{(\delta^2+b^2)^{m+3/2}}\,, \quad \delta > 0, \quad b > 0, \tag{3.26}$$

where $\Gamma(z)$ is Euler's gamma function, and consider the integral (see [58], p.171, Eq. (6.15.6)):

$$\int_0^\infty \frac{K_\mu(\delta\sqrt{x^2+a^2})}{(x^2+a^2)^{\mu/2}} J_\nu(bx) x^{\nu+1} dx = \frac{b^\nu}{\delta^\mu}\left(\frac{\sqrt{\delta^2+b^2}}{a}\right)^{\mu-\nu-1} K_{\mu-\nu-1}(a\sqrt{\delta^2+b^2}), \tag{3.27}$$

where $a > 0, b > 0, \delta > 0, \nu > -1$, and $K_\mu(z)$ is the modified Bessel function of the second kind of order μ. Substituting $\mu = 1/2$, $\nu = m$ into Eq. (3.27) yields

$$\int_0^\infty \frac{K_{1/2}(\delta\sqrt{x^2+a^2})}{(x^2+a^2)^{1/4}} J_m(bx) x^{m+1} dx = \frac{b^m}{\delta^{1/2}}\left(\frac{\sqrt{\delta^2+b^2}}{a}\right)^{-m-1/2} K_{-m-1/2}(a\sqrt{\delta^2+b^2})\,. \tag{3.28}$$

If we use the equations

$$K_{1/2}(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z} \quad \text{and} \quad K_{-\nu}(z) = K_{\nu}(z)\,, \tag{3.29}$$

then it follows from Eq. (3.28) that

$$\int_0^\infty \frac{e^{-\delta\sqrt{x^2+a^2}}}{\sqrt{x^2+a^2}} J_m(bx) x^{m+1} dx = \sqrt{\frac{2a}{\pi\kappa_2}} \left(\frac{ab}{\kappa_2}\right)^m K_{m+1/2}(a\kappa_2), \qquad \kappa_2 = \sqrt{\delta^2 + b^2}\,, \tag{3.30}$$

where

$$K_{m+1/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{k=0}^{m} \frac{(m+k)!}{k!(m-k)!(2z)^k}, \tag{3.31}$$

and, in the case $m = 0$, the sum in Eq. (3.31) is equal to 1.

Differentiating both sides of Eq. (3.26) ($2n+1$)-times with respect to δ, one obtains

$$-\int_0^\infty e^{-\delta x} x^{2n-1} J_m(bx)\, x^{m+1}\, dx = \frac{d^{2n-1}}{d\delta^{2n-1}} \left[\frac{2\delta}{\sqrt{\pi}} \frac{(2b)^m \Gamma(m+3/2)}{(\delta^2 + b^2)^{m+3/2}} \right], \tag{3.32}$$

or in the sense of Abel:

$$\int_0^\infty x^{2n-1} J_m(bx)\, x^{m+1}\, dx = -\lim_{\delta\to+0} \frac{d^{2n-1}}{d\delta^{2n-1}} \left[\frac{2\delta}{\sqrt{\pi}} \frac{(2b)^m \Gamma(m+3/2)}{(\delta^2 + b^2)^{m+3/2}} \right], \tag{3.33}$$

Differentiating both sides of Eq. (3.30) $2n$ -times with respect to δ yields

$$\int_0^\infty e^{-\delta\sqrt{x^2+a^2}} (\sqrt{x^2+a^2})^{2n-1} J_m(bx)\, x^{m+1}\, dx = \frac{d^{2n}}{d\delta^{2n}} \left[\sqrt{\frac{2a}{\pi}} (ab)^m \frac{K_{m+1/2}(a\kappa_2)}{\kappa_2^{m+1/2}} \right]. \tag{3.34}$$

We introduce the operator L_m such that

$$L_m = \frac{1}{b^{1-m}} \frac{d}{db} (b^{1-2m} \frac{d}{db}) b^m\,. \tag{3.35}$$

Then it follows from Eq. (3.34) that, in the sense of Abel,

$$\int_0^\infty x^{2r} (\sqrt{x^2+a^2})^{2n-1} J_m(bx)\, x^{m+1}\, dx = (-1)^r (L_m)^r \lim_{\delta\to+0} \frac{d^{2n}}{d\delta^{2n}} \left[\sqrt{\frac{2a}{\pi}} (ab)^m \frac{K_{m+1/2}(a\kappa_2)}{\kappa_2^{m+1/2}} \right], \tag{3.36}$$

where $r = 0,1,2,\ldots$, and the operator L_m is chosen such that

$$L_m[J_m(bx)] = -x^2 J_m(bx)\,, \tag{3.37}$$

and it gives the multiplier x^{2r} on the left-hand side of Eq. (3.36).

The next transformations are the same as for the integral $A_n(\gamma)$. The transformation given by Eq. (3.12) allows representing the integral $B_{n,m}(b)$ as a finite sum of integrals (3.33) and (3.36).

For example, for $m=0$ it follows from Eqs. (3.33), (3.36) and (3.37) that

$$\int_0^\infty x^{2n} J_0(bx)\,dx = -\lim_{\delta\to+0} \frac{d^{2n-1}}{d\delta^{2n-1}} \frac{\delta}{(\delta^2+b^2)^{3/2}} = (-1)^n (L_0)^n \frac{1}{b}, \qquad (3.38)$$

where

$$L_0 = \frac{1}{b}\frac{d}{db}\left(b\frac{d}{db}\right), \qquad (3.39)$$

$$\int_0^\infty x^{2r} (\sqrt{x^2+a^2})^{2n-1} J_0(bx)\,x\,dx = (-1)^r (L_0)^r \lim_{\delta\to+0} \frac{d^{2n}}{d\delta^{2n}} \frac{e^{-a\sqrt{\delta^2+b^2}}}{\sqrt{\delta^2+b^2}}, \quad r=0,1,2,\ldots \qquad (3.40)$$

It follows from Eq. (3.1), by using decomposition (3.12) and Eqs. (3.38) and (3.40) at $n=1$, $m=0$, $r=0$, that

$$B_{1,0}(b) = \int_0^\infty \frac{xJ_0(bx)dx}{\sqrt{x^2+a^2}+x} = \frac{1}{a^2}\int_0^\infty (\sqrt{x^2+a^2}-x)J_0(bx)\,x\,dx =$$

$$= \frac{1}{a^2}\left[\lim_{\delta\to+0}\frac{d^2}{d\delta^2}\frac{e^{-a\sqrt{\delta^2+b^2}}}{\sqrt{\delta^2+b^2}} + \frac{1}{b}\frac{d}{db}\left(b\frac{d}{db}\frac{1}{b}\right)\right] = \frac{1}{a^2}\left[\frac{1}{b^3} - \frac{1}{b^2}\left(a+\frac{1}{b}\right)e^{-ab}\right]; \qquad (3.41)$$

Similarly, at $n=1$, $m=0$, $r=0$, we have

$$B_{2,0}(b) = \int_0^\infty \frac{xJ_0(bx)dx}{(\sqrt{x^2+a^2}+x)^3} = \frac{1}{a^6}\int_0^\infty (\sqrt{x^2+a^2}-x)^3\,x\,J_0(bx)\,dx =$$

$$= \frac{1}{a^6}\int_0^\infty \left[(\sqrt{x^2+a^2})^3 - 4x^3 - 3xa^2 + 3x^2\sqrt{x^2+a^2}\right] x\,J_0(bx)\,dx =$$

$$= \frac{1}{a^6}\left[\frac{d^4}{d\delta^4} - 3L_0\frac{d^2}{d\delta^2}\right]\frac{e^{-a\sqrt{b^2+\delta^2}}}{\sqrt{b^2+\delta^2}}\Bigg|_{\delta=0} + \frac{3}{a^4b^3} - \frac{36}{a^6b^5}. \qquad (3.42)$$

The correctness of Eqs. (3.41)-(3.42) has been verified numerically by using the package “Mathematica”. However, there remains the question whether the following two limits are equal, i.e. the integrals

$$\lim_{\delta\to+0}\int_0^\infty e^{-\delta x}\sqrt{x^2+a^2}\cos\gamma x\,dx \qquad (3.43)$$

and

$$\lim_{\delta\to+0}\int_0^\infty e^{-\delta\sqrt{x^2+a^2}}\sqrt{x^2+a^2}\cos\gamma x\,dx\,. \qquad (3.44)$$

Equation (3.7) is used for evaluating the integral (3.44), but for the evaluation of integral (3.43), a similar formula does not exist. Integral (3.43) can be expressed only in terms of the Lommel functions, and for these functions, it is difficult to find the limit as $\delta\to+0$. It is rigorously proved that

$$\lim_{\delta\to+0}\int_0^\infty (e^{-\delta x}-e^{-\delta\sqrt{x^2+a^2}})\sqrt{x^2+a^2}\cos\gamma x\,dx=0\,, \qquad (3.45)$$

i.e. limits (3.43) and (3.44) are equal. The correctness of Eq. (3.45) has also been verified numerically by using the package “Mathematica”.

It is to be noted that the multiplier $\cos\gamma x$ in Eq. (3.45) is very important. It is proved that without this multiplier, the limit

$$\lim_{\delta\to+0}\int_0^\infty \left[e^{-\delta x}-e^{-\delta\sqrt{x^2+a^2}}\right]\sqrt{x^2+a^2}\,dx=\frac{a^2}{2}\neq 0\,, \qquad (3.46)$$

despite the fact that

$$\lim_{\delta\to+0}\left[e^{-\delta x}-e^{-\delta\sqrt{x^2+a^2}}\right]=0\,. \qquad (3.47)$$

3.2. Closed form solutions to some eddy current testing problems

In this section, we apply some of the integrals that have been evaluated in the previous section to some mathematical problems of eddy current testing. In particular, the integrals $A_1(\gamma)$ and $B_{1,0}(b)$ given by Eqs. (3.20) and (3.41), respectively, are used for evaluating impedance change in the cases of a double conductor line and a single-turn coil located on the surface of a conducting half-space. In the case of a double conductor line, the expression for the impedance change has been evaluated in closed form, but in the case of a single-turn coil, the impedance has been transformed into the simpler form of a fast-convergent series. Furthermore, we obtain the simple asymptotic formulae for the impedance of arbitrarily situated double lines and coils in the limit as the frequency tends to infinity. The obtained results are published in [9] and [7].

3.2.1. Double conductor line above a half-space

Consider two infinitely long wires carrying an alternating current and located in region $R_0 = \{z > 0\}$ at height h above a uniform conducting half-space (see Fig.3.1). The wires are parallel to the x-axis and pass through the points $(0, y_0, h)$ and $(0, y_1, h)$. The conducting half-space is located in region $R_1 = \{z < 0\}$.

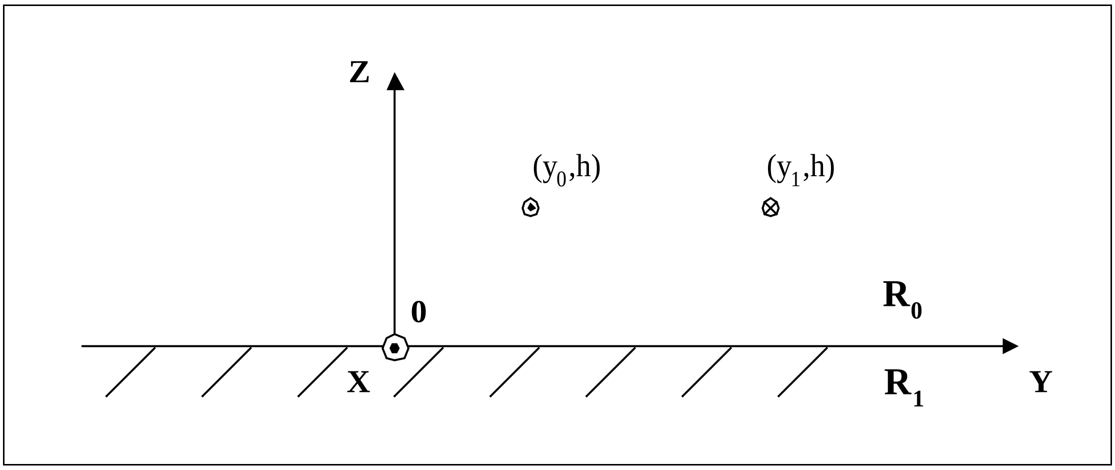

Fig.3.1. A double conductor line in free space R_0 at height h above a uniform conducting half-space R_1

The vector potential has only the x-component (see the 1st boundary value problem):

$$\vec{A} = A_x(y,z)\vec{e}_x \ . \tag{3.48}$$

The mathematical formulation of the problem has the form (see the 1st boundary value problem, Eqs. (1.48) and (1.49)):

$$\Delta A_0 = -\mu_0 I[\delta(y - y_0)\delta(z - h) - \delta(y - y_1)\delta(z - h)], \qquad z > 0, \tag{3.49}$$

$$\Delta A_1 + k_1^2 A_1 = 0, \qquad z < 0, \tag{3.50}$$

where $k_1^2 = -j\omega\mu_0\sigma_1$, and $\tilde{\mu}_0 = 1$, $\hat{\varepsilon}_0 = 1$ and $\sigma = 0$ in free space. The displacement current is assumed to be absent, but the relative magnetic permeability of region R_1 is equal to 1. The boundary conditions are

$$A_0\big|_{z=0+} = A_1\big|_{z=0-}, \qquad \left.\frac{\partial A_0}{\partial z}\right|_{z=0+} = \left.\frac{\partial A_1}{\partial z}\right|_{z=0-} \tag{3.51}$$

and the following conditions hold at infinity:

$$A_0, \quad A_1, \quad \frac{\partial A_0}{\partial y}, \quad \frac{\partial A_1}{\partial y}, \quad \frac{\partial A_0}{\partial z}, \quad \frac{\partial A_1}{\partial z} \to 0, \quad \text{as} \quad \sqrt{y^2+z^2} \to \infty . \tag{3.52}$$

The solution of problem (3.49)-(3.52) for region R_0 has the form (see [6])

$$A_0(y,z) = \frac{\mu_0 I}{4\pi} \ln \frac{(z-h)^2+(y-y_1)^2}{(z-h)^2+(y-y_0)^2} +$$

$$+ \frac{\mu_0 I}{2\pi} \int_0^\infty \frac{\lambda - q}{\lambda + q} e^{-(z+h)} [\cos\lambda(y-y_0) - \cos\lambda(y-y_1)] \frac{d\lambda}{\lambda}, \tag{3.53}$$

where $q = \sqrt{\lambda^2 - k_1^2}$.

Note that the first term on the right-hand side of Eq. (3.53) represents the vector potential due to the presence of the two solitary wires in unbounded free space and is associated with the primary field, while the second term on the same side gives the contribution $A_0^{ind}(y,z)$ to $A_0(y,z)$ due to the conducting medium and is associated with the secondary field. Thus, the induced vector potential is

$$A_0^{ind}(y,z) = \frac{\mu_0 I}{2\pi} \int_0^\infty \frac{\lambda - q}{\lambda + q} e^{-(z+h)} [\cos\lambda(y-y_0) - \cos\lambda(y-y_1)] \frac{d\lambda}{\lambda} . \tag{3.54}$$

The induced change in impedance, Z^{ind}, due to the conducting medium, is given by Eq. (1.111), i.e.

$$Z^{ind} = \frac{j\omega}{I} \oint_c A_{0l}^{ind} \, dl , \tag{3.55}$$

where C is the contour of the source of current. In the case of the double line, the equation of the contour C is given by $\{ z = h, \ y = y_0, \ 0 \le x \le 1$ and $z = h, \ y = y_1, \ 0 \le x \le 1 \}$. Introducing dimensionless variables, it follows from Eqs. (3.54) and (3.55), that the impedance change in the double conductor line per unit length has the form (see [6]):

$$Z^{ind} = \frac{\mu_0 \omega}{\pi} Z_l , \qquad Z_l = j \int_0^\infty \frac{s - \sqrt{s^2+j}}{s + \sqrt{s^2+j}} e^{-2\hat{\alpha}\beta s} (1 - \cos\beta s) \frac{ds}{s} := X + jY , \tag{3.56}$$

where $\hat{\alpha} = h/d$, $\beta = d\sqrt{\omega\sigma_1\mu_0}$ and $d = y_1 - y_0$ is the distance between the wires.

Consider the case $\hat{\alpha} = 0$, i.e. the current source is located on the surface $z = 0$. In order to evaluate integral (3.56) in closed form, we use Eq. (3.20) of the new classes of definite integrals, obtained in Section 3.1.1, i.e. the integral

$$A_1(\gamma)=\int_0^\infty \frac{\cos\gamma x\,dx}{\sqrt{x^2+a^2}+x}=\frac{1}{a^2}\left[\frac{1}{\gamma^2}-\frac{a}{\gamma}K_1(a\gamma)\right], \tag{3.57}$$

where $K_1(z)$ is the modified Bessel function of the second kind of order 1.

In order to transform integral (3.56) to the form of integral (3.57) at $\hat{\alpha}=0$, one uses the formula

$$\frac{x-\sqrt{x^2+a^2}}{x(x+\sqrt{x^2+a^2})}=\frac{2}{x+\sqrt{x^2+a^2}}-\frac{1}{x}. \tag{3.58}$$

To avoid the divergent integrals in Eq. (3.56) at $\hat{\alpha}=0$, we replace the multiplier $(1-\cos\beta x)$ with the multiplier $(\cos\hat{\alpha}x-\cos\beta x)$. Then by Eq. (3.58), the integral on the right-hand side of Eq. (3.56) at $\hat{\alpha}=0$ (and assuming $j:=a^2$) can be rewritten as

$$\int_0^\infty \frac{x-\sqrt{x^2+a^2}}{x+\sqrt{x^2+a^2}}(\cos\hat{\alpha}x-\cos\beta x)\frac{dx}{x}= 2\int_0^\infty \frac{\cos\hat{\alpha}x-\cos\beta x}{x+\sqrt{x^2+a^2}}\,dx-\int_0^\infty \frac{\cos\hat{\alpha}x-\cos\beta x}{x}\,dx\,. \tag{3.59}$$

The second integral on the right-hand side of Eq. (3.59) is known in the literature (see [41]):

$$\int_0^\infty \frac{\cos\hat{\alpha}x-\cos\beta x}{x}\,dx=\ln\frac{\beta}{\hat{\alpha}}. \tag{3.60}$$

The first integral on the right-hand side of Eq. (3.59) is evaluated by Eq. (3.57):

$$\int_0^\infty \frac{\cos\hat{\alpha}x-\cos\beta x}{x+\sqrt{x^2+a^2}}\,dx= A_1(\hat{\alpha})-A_1(\beta)=\frac{1}{a^2}\left[\frac{1}{\gamma^2}-\frac{a}{\gamma}K_1(a\gamma)\right]\Bigg|_{\gamma=\beta}^{\gamma=\hat{\alpha}}. \tag{3.61}$$

Substituting Eqs. (3.60) and (3.61) into Eq. (3.59) yields

$$\int_0^\infty \frac{x-\sqrt{x^2+a^2}}{x+\sqrt{x^2+a^2}}(\cos\hat{\alpha}x-\cos\beta x)\frac{dx}{x}=\frac{2}{a^2}\left[\frac{1}{\gamma^2}-\frac{a}{\gamma}K_1(a\gamma)\right]\Bigg|_{\gamma=\beta}^{\gamma=\hat{\alpha}}-\ln\frac{\beta}{\hat{\alpha}}$$

$$=\frac{2}{a^2}\left[F(\hat{\alpha})-F(\beta)\right]-\ln\frac{\beta}{\hat{\alpha}}, \tag{3.62}$$

where

$$F(\gamma)=\frac{1}{\gamma^2}-\frac{a}{\gamma}K_1(a\gamma)\,. \tag{3.63}$$

In order to take the limit as $\hat{\alpha}\to 0$ in Eq. (3.62), i.e. the limit of the part

$$\lim_{\hat{\alpha}\to+0}\left[\frac{2}{a^2}F(\hat{\alpha})-\ln\frac{\beta}{\hat{\alpha}}\right], \tag{3.64}$$

the asymptotic behaviour of $K_1(z)$ as $z\to 0$ is to be found, and not only by using the known formula (see [58], p.142, formula (5.7.12)):

$$K_n(z) \underset{z\to 0}{\approx} \frac{1}{2}(n-1)!\left(z/2\right)^{-n},\ n=1,2,\dots, \qquad \text{i.e.} \qquad K_1(z) \underset{z\to 0}{\approx} 1/z\,.$$

The general expression for $K_n(z)$ is

$$K_n(z) = \frac{1}{2}\sum_{k=0}^{n-1}\frac{(-1)^k (n-k-1)!}{k!}\left(z/2\right)^{2k-n}$$

$$+\frac{1}{2}(-1)^{n+1}\sum_{k=0}^{\infty}\frac{\left(z/2\right)^{2k+n}}{k!(k+n)!}\left[2\ln\frac{z}{2}-\psi(k+1)-\psi(k+n+1)\right],\quad n=0,1,2,\dots\,, \tag{3.65}$$

where $\psi(z)$ is the logarithmic derivative of Euler's gamma function, whose values can be calculated by means of the following formulae,

$$\psi(1)=-C\,,\ \ \psi(2)=-C+1,\ \ \psi(3)=-C+1+\frac{1}{2},\ \ \dots\ ,\ \ \psi(m+1)=-C+1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{m},$$

where $C = 0{,}577215\dots$ is Euler's constant. Besides, at $n=0$, the first sum in Eq. (3.65) for $K_0(z)$ is equal to zero.

It follows from Eq. (3.65) at $n=1$ that

$$K_1(z) = \frac{1}{z} + I_1(z)\ln\frac{z}{2} - \frac{1}{2}\sum_{k=0}^{\infty}\frac{\left(z/2\right)^{2k+1}}{k!(k+1)!}\left[\psi(k+1)+\psi(k+2)\right], \tag{3.66}$$

where

$$I_1(z) = \sum_{k=0}^{\infty}\frac{\left(z/2\right)^{2k+1}}{k!(k+1)!} \tag{3.67}$$

and $I_1(z)$ is the modified Bessel function of the first kind of order 1. It follows from Eq. (3.66) that

$$K_1(z) - \frac{1}{z} = I_1(z)\ln\frac{z}{2} - \frac{1}{2}\left\{\frac{1}{1!}\frac{z}{2}[\psi(1)+\psi(2)] + \frac{1}{1!2!}\left(\frac{z}{2}\right)^3[\psi(2)+\psi(3)]+\cdots\right\}, \tag{3.68}$$

where, in order to find the limit as $z\to 0$, the term $\frac{1}{2!}\left(\frac{z}{2}\right)^3[\psi(2)+\psi(3)]$ can be neglected, because this term gives zero contribution in the limit as $z\to 0$ and,

$$I_1(z) = \frac{z}{2} + \frac{1}{1!2!}\left(\frac{z}{2}\right)^3 + \cdots \tag{3.69}$$

The term $\frac{1}{1!2!}\left(\frac{z}{2}\right)^3$ in Eq. (3.69) does not give the contribution as $z\to 0$ either. Thus, as

$z \to 0$, Eq. (3.68) holds

$$K_1(z) - \frac{1}{z} = \frac{z}{2}\ln\frac{z}{2} - \frac{z}{4}(1-2C)\,. \tag{3.70}$$

Besides, it follows from Eqs. (3.63) and (3.69) that

$$F(\gamma) = \frac{1}{\gamma^2} - \frac{a}{\gamma}K_1(a\gamma) = \frac{1}{\gamma^2} - \frac{a}{\gamma}\left[\frac{1}{a\gamma} + \frac{a\gamma}{2}\ln\frac{a\gamma}{2} - \frac{a\gamma}{4}(1-2C)\right]$$

$$= -\frac{a^2}{2}\ln\frac{a\gamma}{2} + \frac{a^2}{4}(1-2C)\,, \quad \text{as } \gamma \to 0\,. \tag{3.71}$$

Substituting Eq. (3.71) at $\gamma = \hat{\alpha}$ into limit (3.64) yields

$$\lim_{\hat{\alpha}\to+0}\left[\frac{2}{a^2}F(\hat{\alpha}) - \ln\frac{\beta}{\hat{\alpha}}\right] = \lim_{\hat{\alpha}\to+0}\left[-\ln\frac{a\hat{\alpha}}{2} + \frac{1}{2}(1-2C) - \ln\frac{\beta}{\hat{\alpha}}\right]$$
$$= \lim_{\hat{\alpha}\to+0}\left[-\ln\frac{a}{2} - \ln\hat{\alpha} + \frac{1}{2}(1-2C) - \ln\beta + \ln\hat{\alpha}\right] = 0.5 - C - \ln\frac{a\beta}{2}\,.$$

Hence,

$$\lim_{\hat{\alpha}\to+0}\left[\frac{2}{a^2}F(\hat{\alpha}) - \ln\frac{\beta}{\hat{\alpha}}\right] = 0.5 - C - \ln\frac{a\beta}{2}\,. \tag{3.72}$$

At last, it follows from Eq. (3.62), by taking the limit as $\hat{\alpha} \to 0$ and using Eq. (3.72), that

$$\lim_{\hat{\alpha}\to0}\int_0^\infty \frac{x-\sqrt{x^2+a^2}}{x+\sqrt{x^2+a^2}}(\cos\hat{\alpha}x - \cos\beta x)\frac{dx}{x} = \lim_{\hat{\alpha}\to+0}\left[\frac{2}{a^2}F(\hat{\alpha}) - \ln\frac{\beta}{\hat{\alpha}}\right] - \frac{2}{a^2}F(\beta)$$

$$= 0.5 - C - \ln\frac{a\beta}{2} - \frac{2}{a^2}F(\beta)\,. \tag{3.73}$$

We substitute $F(\beta)$ given by Eq. (3.63) into Eq. (3.73) and if $\hat{\alpha} = 0$, then integral (3.59) takes the final form

$$\int_0^\infty \frac{x-\sqrt{x^2+a^2}}{x+\sqrt{x^2+a^2}}(1-\cos\beta x)\frac{dx}{x} = 0.5 - C - \ln\frac{a\beta}{2} - \frac{2}{a^2}\left[\frac{1}{\beta^2} - \frac{a}{\beta}K_1(a\beta)\right]. \tag{3.74}$$

Therefore, if we substitute Eq. (3.74) into Eq. (3.56) for Z_l and if $\hat{\alpha} = 0$ and $a = \sqrt{j}$, then the impedance change in the double conductor line has the form

$$Z_l\big|_{\hat{\alpha}=0} = j\left\{0.5 - C - \ln\frac{\beta\sqrt{j}}{2} - \frac{2}{j}\left[\frac{1}{\beta^2} - \frac{\sqrt{j}}{\beta}K_1(\beta\sqrt{j})\right]\right\}, \tag{3.75}$$

or if we take into account that

$$-\ln\frac{\beta\sqrt{j}}{2}=-\left[\ln\frac{\beta}{2}+\ln\sqrt{j}\right]=-\left[\ln\frac{\beta}{2}+\ln\sqrt{e^{j\frac{\pi}{2}}}\right]=-\left[\ln\frac{\beta}{2}+\ln e^{j\frac{\pi}{4}}\right]=-\left[\ln\frac{\beta}{2}+j\frac{\pi}{4}\right],$$

then the impedance change takes the final form

$$Z_l\Big|_{\hat{\alpha}=0}=\frac{\pi}{4}+j\left[\frac{1}{2}-C-\ln\frac{\beta}{2}\right]-2\left[\frac{1}{\beta^2}-\frac{1+j}{\beta\sqrt{2}}K_1(\beta\sqrt{j})\right]. \tag{3.76}$$

The Bessel function in Eq. (3.76) can be expressed in terms of the Kelvin functions, $\ker_1(\beta)$ and $\mathrm{kei}_1(\beta)$, which are tabulated in [53]:

$$K_1(\beta\sqrt{j})=j[\ker_1(\beta)+j\,\mathrm{kei}_1(\beta)]. \tag{3.77}$$

Eq. (3.77) allows separating the real and the imaginary parts in Eq. (3.76):

$$X(\beta)=\mathrm{Re}\,Z_l=\frac{\pi}{4}-\frac{2}{\beta^2}-\frac{\sqrt{2}}{\beta}\left[\ker_1(\beta)+\mathrm{kei}_1(\beta)\right], \tag{3.78}$$

$$Y(\beta)=\mathrm{Im}\,Z_l=\frac{1}{2}-C-\ln\frac{\beta}{2}+\frac{\sqrt{2}}{\beta}\left[\ker_1(\beta)-\mathrm{kei}_1(\beta)\right]. \tag{3.79}$$

Computational results obtained by means of Eqs. (3.78) and (3.79) are presented in Fig.3.2 for different values of the parameters $\hat{\alpha}$ and β (see curve $\hat{\alpha}=0$).

As can be seen from Fig.3.2, in the case $\hat{\alpha}=0$, i.e. when the current source is located on the surface of the conducting half-space, the curve becomes parallel to the imaginary axis as $\beta\to\infty$.

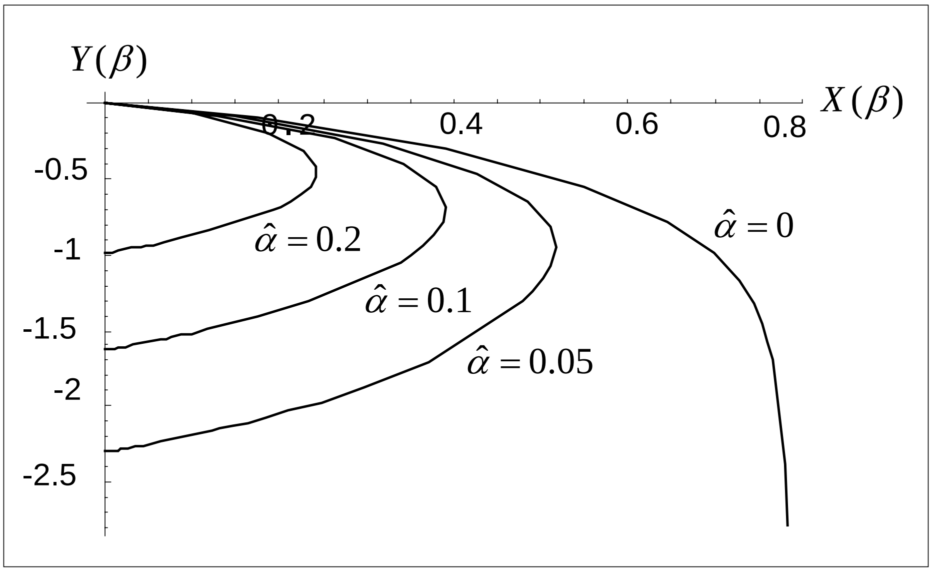

Fig.3.2. Curves describing the change in impedance of a double conductor line for different values of $\hat{\alpha}$ and β

Computational results obtained by means of Eqs. (3.78) and (3.79) at $\hat{\alpha}=0$ completely coincide with the computational results obtained by means of integral (3.56) with the use of the package "Mathematica".

3.2.2. Single-turn coil above a half-space

Consider a circular single-turn coil of radius r_c, carrying an alternating current and located at height h in free space, $R_0=\{z>0\}$, above a uniform conducting half-space in region $R_1=\{z<0\}$ (see Fig.3.3).

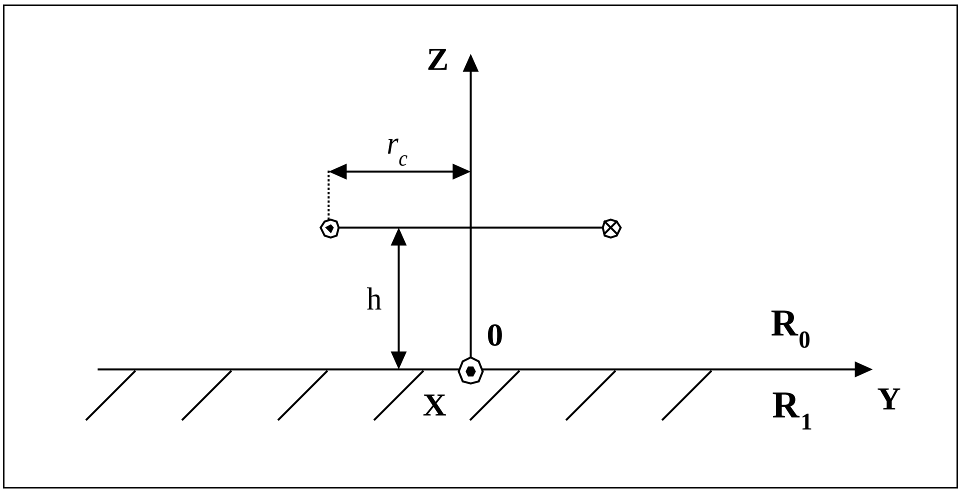

Fig.3.3. A single-turn coil of radius r_c in free space, R_0, at height h above the uniform conducting half-space, R_1

In this case, the vector potential has only the φ-component,

$$\vec{A}=A(r,z)\vec{e}_\varphi, \tag{3.80}$$

The mathematical formulation of the problem for the vector potential has the form (see the 3rd boundary value problem, Eqs. (1.89)-(1.92))

$$\begin{cases}\Delta_\varphi A_0=-\mu_0 I\delta(r-r_c)\delta(z-h), & \text{in } R_0,\\ \Delta_\varphi A_1+k_1^2A_1=0, & \text{in } R_{1.}\end{cases} \tag{3.81}$$

where

$$\Delta_\varphi f(r,z)=\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}+\frac{\partial^2 f}{\partial z^2}-\frac{1}{r^2}f, \tag{3.82}$$

with the boundary conditions

$$A_0\big|_{z=0+} = A_1\big|_{z=0-}, \qquad \left.\frac{\partial A_0}{\partial z}\right|_{z=0+} = \left.\frac{\partial A_1}{\partial z}\right|_{z=0-}, \tag{3.83}$$

and the conditions at infinity

$$A_0, \quad A_1, \quad \frac{\partial A_0}{\partial r}, \quad \frac{\partial A_1}{\partial r}, \quad \frac{\partial A_0}{\partial z}, \quad \frac{\partial A_1}{\partial z} \to 0, \quad \text{as} \quad \sqrt{r^2 + z^2} \to \infty. \tag{3.84}$$

The solution of problem (3.81)-(3.84) for region R_0 has the form (see [6])

$$A_0(r,z) = \frac{\mu_0 I r_c}{2}\int_0^\infty J_1(\lambda r_c) J_1(\lambda r) e^{-\lambda|z-h|}]d\lambda + \frac{\mu_0 I r_c}{2}\int_0^\infty \frac{\lambda - q}{\lambda + q} J_1(\lambda r_c) J_1(\lambda r) e^{-\lambda(z+h)} d\lambda, \tag{3.85}$$

where $J_1(z)$ is the Bessel function of the first kind of order 1. The first term on the right-hand side of Eq. (3.86) represents the vector potential of a solitary single-turn coil in unbounded free space, while the second term on the same side gives the contribution $A_0^{\text{ind}}(r,z)$ into $A_0(r,z)$ due to the conducting medium. Thus, the induced vector potential is

$$A_0^{\text{ind}}(r,z) = \frac{\mu_0 I r_c}{2}\int_0^\infty \frac{\lambda - q}{\lambda + q} J_1(\lambda r_c) J_1(\lambda r) e^{-\lambda(z+h)} d\lambda. \tag{3.86}$$

Substituting $A_0^{\text{ind}}(r,z)$ into Eq. (3.55) for Z^{ind}, the induced impedance change in the coil located above the conducting half-space (see [5]) has the form

$$Z^{\text{ind}} = \mu_0 \omega \pi r_c Z_c, \qquad Z_c = j\beta \int_0^\infty \frac{s - \sqrt{s^2 + j}}{s + \sqrt{s^2 + j}} J_1^2(\beta s) e^{-2\hat{\alpha}\beta s} ds := X + jY, \tag{3.87}$$

where $\hat{\alpha} = 2h/r_c$ and $\beta = r_c\sqrt{\omega\sigma\mu_0}$ are dimensionless variables. Formula (3.87) was obtained by [62] for the first time in 1960, although this problem was considered before in connection with expansion of electromagnetic waves in monograph [56] and textbook [57] under the magnetic dipole approximation assuming that the radius of the coil, r_c, tends to zero, but the amplitude of the current tends to infinity so that multiplier $r_c I = M$ remains constant. In this case, integral (3.87) is simplified and can be expressed in terms of elementary functions for the case $\hat{\alpha} = 0$ (see [48]). However, in the case of non-destructive testing, the magnetic dipole approximation is not appropriate, as the size of the defect can be commensurable with the diameter of the coil. Thus, one needs to simplify the integral (3.87).

Consider the case $\hat{\alpha} = 0$, i.e. when the current source is located on the surface $z = 0$. In order to evaluate integral (3.87) in closed form, divergent integrals convergent in the sense of Abel are used, i.e. the integral

$$B_{1,0}(b)=\int_0^{\infty}\frac{xJ_0(bx)dx}{\sqrt{x^2+a^2}+x}=\frac{1}{a^2}\left[\frac{1}{b^3}-\frac{1}{b^2}\left(a+\frac{1}{b}\right)e^{-ab}\right]. \tag{3.88}$$

Integral (3.87) at $\hat{\alpha}=0$ can be rewritten by using Eq. (3.58) as

$$\int_0^{\infty}\frac{x-\sqrt{x^2+a^2}}{x+\sqrt{x^2+a^2}}J_1^2(\beta x)dx=\lim_{b\to\infty}\left[2\int_0^{b}\frac{xJ_1^2(\beta x)}{x+\sqrt{x^2+a^2}}dx-\int_0^{b}J_1^2(\beta x)\,dx\right], \tag{3.89}$$

where $a=\sqrt{j}$. In Eq. (3.89) both integrals can be transformed by using the formula (see [55], page 166, formula (2)):

$$J_1^2(z)=-\frac{2}{\pi}\int_0^{\pi/2}J_0(2z\cos\theta)\cos 2\theta\,d\theta\,, \tag{3.90}$$

Then it follows from Eq. (3.90), by using formula

$$\int_0^{\infty}J_0(bz)\,dz=\frac{1}{b},\ b>0, \tag{3.91}$$

that the second integral on the right-hand side of Eq. (3.89) is

$$\lim_{b\to\infty}\int_0^{b}J_1^2(\beta x)\,dx=-\frac{2}{\pi}\int_0^{\pi/2}\cos 2\theta\,d\theta\,\lim_{b\to\infty}\int_0^{b}J_0(2\beta x\cos\theta)\,dx=-\frac{2}{\pi}\lim_{\varepsilon\to+0}\int_0^{\pi/2-\varepsilon}\frac{\cos 2\theta}{2\beta\cos\theta}d\theta\,. \tag{3.92}$$

Using Eqs. (3.90) and (3.88) for $B_{1,0}(b)$, the first integral on the right-hand side of Eq. (3.89) takes the form

$$\int_0^{\infty}\frac{xJ_1^2(\beta x)}{x+\sqrt{x^2+a^2}}dx=-\frac{2}{\pi}\int_0^{\pi/2}\cos 2\theta\,d\theta\int_0^{\infty}\frac{xJ_0(2\beta x\cos\theta)}{\sqrt{x^2+a^2}+x}dx=$$

$$=-\frac{2}{\pi}\int_0^{\pi/2}\cos 2\theta\cdot B_{1,0}(2\beta\cos\theta)\,d\theta\ =$$

$$=-\frac{2}{\pi}\int_0^{\pi/2}\cos 2\theta\cdot\frac{1}{a^2}\left[\frac{1}{(2\beta\cos\theta)^3}-\frac{1}{(2\beta\cos\theta)^2}\left(a+\frac{1}{2\beta\cos\theta}\right)e^{-2a\beta\cos\theta}\right]d\theta\ . \tag{3.93}$$

Substituting Eqs. (3.92) and (3.93) into Eq. (3.89), integral (3.89) is transformed to the form

$$\int_0^{\infty}\frac{x-\sqrt{x^2+a^2}}{x+\sqrt{x^2+a^2}}J_1^2(\beta x)dx=-\frac{2}{\pi}\int_0^{\pi/2}\frac{\cos 2\theta}{2\beta\cos\theta}d\theta-$$

$$-\frac{2}{\pi a^2}\int_0^{\pi/2}\frac{\cos 2\theta}{2\beta\cos\theta}\left[\frac{1}{4\beta^2\cos^2\theta}-e^{-2a\beta\cos\theta}\left(\frac{a}{2\beta\cos\theta}+\frac{1}{4\beta^2\cos^2\theta}\right)\right]d\theta\,. \tag{3.94}$$

Substituting Eq. (3.94) into Eq. (3.87) for Z_c when $\hat{\alpha}=0$, the impedance change in the coil is obtained in the form of a convergent integral that does not contain Bessel functions any longer, and it has the form

$$Z_c=\frac{j}{\pi}\int_0^{\pi/2}\frac{\cos 2\theta}{\cos\theta}\left\{1-\frac{1}{a^2}\left[\frac{1}{2\beta^2\cos^2\theta}\left(1-e^{-2a\beta\cos\theta}\right)-\frac{ae^{-2a\beta\cos\theta}}{\beta\cos\theta}\right]\right\}d\theta\,,\qquad a=\sqrt{j}\,. \qquad (3.95)$$

If θ tends to $\pi/2$, the leading terms of the expansion in power of $2a\beta\cos\theta$ for the expression in parentheses have the form:

$$\frac{4}{3}a\beta\cos\theta-a^2\beta^2\cos^2\theta+\ldots$$

Therefore, the integrand in Eq. (3.95) has a finite limit as $\theta\to\pi/2$. Computational results obtained by means of Eq. (3.95) are presented in Fig.3.4 for different values of the parameter β (see the curve $\hat{\alpha}=0$). These computational results completely coincide with results obtained by means of Eq. (3.87) at $\hat{\alpha}=0$.

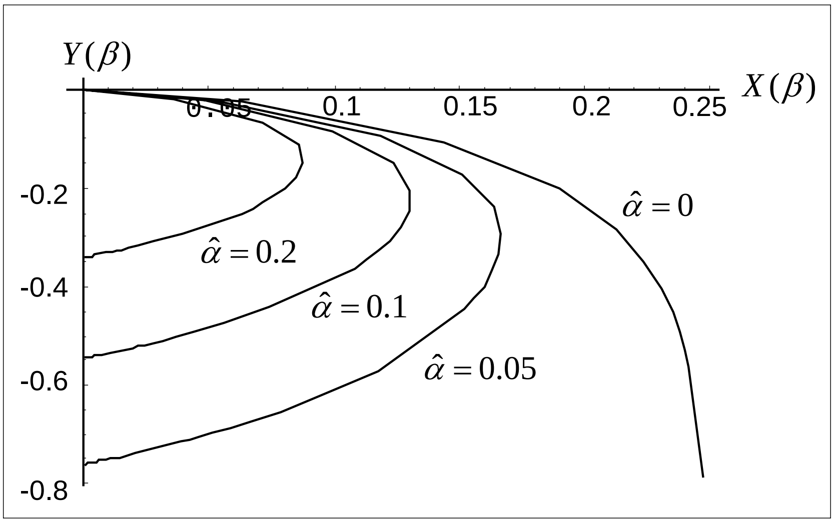

Fig.3.4. Curves describing the change of impedance in a coil for different values of $\hat{\alpha}$ and β

Besides, it follows from Eq. (3.95) that

$$\frac{d}{d\beta}\left[\frac{1}{\beta}\frac{d}{d\beta}(\beta^2 Z_c)\right]=\frac{4aj}{\pi}\int_0^{\frac{\pi}{2}}\cos 2\theta\, e^{-2a\beta\cos\theta}d\theta\,,\quad a=\sqrt{j}\,. \qquad (3.96)$$

Expanding the exponential function of integrand in Eq. (3.96) in series in powers of $\cos\theta$ and using the formula

$$\int_0^{\frac{\pi}{2}} \cos^n \theta \, d\theta = \sqrt{\frac{\pi}{2}} \frac{\Gamma\left((n+1)/2\right)}{\Gamma\left((n+2)/2\right)} := B_n , \tag{3.97}$$

where $\Gamma(z)$ is the Euler gamma function, it follows from Eq. (3.96) that

$$Z_c(\beta) = \frac{4aj}{\pi} \sum_{n=0}^{\infty} (-1)^n (2B_{n+2} - B_n) \frac{(2a)^n \beta^{n+1}}{(n+1)(n+3)} . \tag{3.98}$$

or

$$Z_c(\beta) = 8j \sum_{n=1}^{\infty} \frac{(-1)^n (a\beta)^{n+1}}{n(n+1)(n+2)(n+3)} \frac{1}{\Gamma^2\left(n/2\right)} , \qquad a = \sqrt{j} . \tag{3.99}$$

Series (3.99) converge very rapidly. If $\beta \le 3$, then the first five terms of the series are sufficient for error less than 3%, compared with the exact solution.

3.2.3. Asymptotic formula for the impedance as $\beta \to \infty$

In order to calculate the limit of the impedance, Z_l, in the case of a double line above a conducting half-space as $\beta \to \infty$ (skin effect), it is convenient to substitute $\beta s = \eta$ into Eq. (3.56) for Z_l,

$$Z_l = j \int_0^{\infty} \frac{\eta - \sqrt{\eta^2 + \beta^2 j}}{\eta + \sqrt{\eta^2 + \beta^2 j}} e^{-2\hat{\alpha}\eta} (1 - \cos\eta) \frac{d\eta}{\eta} . \tag{3.100}$$

Eq. (3.100) is more convenient for calculations than Eq. (3.57) for all $\beta \in (0, \infty)$. It follows from Eq. (3.100) by taking the limit as $\beta \to \infty$ that

$$Z_{l\infty} := \lim_{\beta \to \infty} Z_l , \qquad Z_{l\infty} = -j \int_0^{\infty} e^{-p\eta} \frac{1 - \cos\eta}{\eta} d\eta , \qquad p = 2\hat{\alpha} . \tag{3.101}$$

In order to calculate integral (3.101), the following property of the Laplace transform is used,

$$L\left[f(t)\right] = F(p) \qquad \Rightarrow \qquad L\left[\frac{f(t)}{t}\right] = \int_p^{\infty} F(q) dq , \tag{3.102}$$

that provide the convergence of the right-hand side of Eq. (3.102). We have

$$L[1 - \cos t] = \frac{1}{p} - \frac{p}{p^2 + 1} =: F(p) . \tag{3.103}$$

Hence, it follows from Eqs. (3.102) and (3.103) that

$$Z_{l\infty} = -j\int_p^\infty \left[\frac{1}{q} - \frac{q}{q^2+1}\right] dq = -j\ln\frac{\sqrt{p^2+1}}{p} = -j\ln\frac{\sqrt{4\hat{\alpha}^2+1}}{2\hat{\alpha}}. \qquad (3.104)$$

Eq. (3.104) gives the asymptotic value of the impedance for a double line above a conducting half-space as $\beta \to \infty$. Formula (3.104) seems to be absent in the literature. The computational results obtained by means of Eq. (3.100) for different values of the parameter β and for $\hat{\alpha} = 0.2, 0.1, 0.05, 0$ are shown in Fig. 3.2. As can be seen from this figure, all curves cross the imaginary axis at the points given by Eq. (3.104).

In order to calculate the limit of the impedance Z_c in the case of the coil located above a conducting half-space as $\beta \to \infty$ (skin effect), it is convenient to substitute $\beta s = \eta$ into Eq. (3.88):

$$Z_c = j\int_0^\infty \frac{\eta - \sqrt{\eta^2 + \beta^2 j}}{\eta + \sqrt{\eta^2 + \beta^2 j}} J_1^2(\eta) e^{-2\hat{\alpha}\eta} d\eta. \qquad (3.105)$$

Eq. (3.105) is more convenient for calculations than formula (3.88) for all $\beta \in (0, \infty)$. Taking the limit as $\beta \to \infty$, it follows from Eq. (3.105) that

$$\lim_{\beta\to\infty} Z_c =: Z_{c\infty} = -j\int_0^\infty e^{-p\eta} J_1^2(\eta) d\eta, \quad p = 2\hat{\alpha}. \qquad (3.106)$$

Integral (3.106) is evaluated in [55] and [60]:

$$Z_{c\infty} = -j\frac{1}{\pi\vartheta}\left[(2-\vartheta^2)K(\vartheta) - 2E(\vartheta)\right], \qquad \vartheta = \frac{1}{\sqrt{1+\hat{\alpha}^2}}, \qquad (3.107)$$

where $K(\vartheta)$ and $E(\vartheta)$ are the full elliptic integrals of the first and second kind, respectively, tabulated, for example, in [53]. The computational results obtained by means of Eq. (3.105) for different values of the parameter β and for $\hat{\alpha} = 0.2, 0.1, 0.05, 0$ are shown in Fig.3.4. As can be seen from this figure, all curves cross the imaginary axis exactly at the points given by Eq. (3.107).

3.3. Exact analytical solution to the problem on impedance change of a rectangular frame with current inside a cylindrical tube

3.3.1. Formulation of the problem

The change in impedance of a rectangular frame with current inside a conducting cylindrical tube has been studied theoretically only in the double conductor line approximation (see [6]). In this section, (see also the authors' paper [14], [35]), the exact analytical solution of the similar problem is obtained without using any approximation.

Consider a thin-wall tube located in the region $\{R \le r \le R_1,\ 0 \le \varphi \le 2\pi,\ -\infty < z < +\infty\}$, where r, φ, z are cylindrical polar coordinates. The rectangular frame $ABCD$ is located inside the tube. The sides AB and CD are linear segments located on the lines $\{r = r_0$, $\varphi = \pm\varphi_0,\ -l \le z \le l,\ (0 < r_0 < R)\}$, but BC and DA are arcs of a circle located on the lines $\{r = r_0,\ -\varphi_0 \le \varphi \le \varphi_0,\ z = \pm l\}$. Due to the linearity of the problem, it is sufficient to consider separately the vector potential problem on electromagnetic waves spreading from each side of the rectangle. The impedance change Z^{ind} of the whole frame is formed by the impedances of the sides AB, BC, CD and DA:

$$Z^{\text{ind}} = Z^{\text{ind}}_{AB} + Z^{\text{ind}}_{BC} + Z^{\text{ind}}_{CD} + Z^{\text{ind}}_{DA}. \qquad (3.108)$$

3.3.2. Emitter in the form of a linear segment

Consider an emitter located on the linear segment AB of the rectangular frame inside the tube (see Fig.3.5). It is known that in the case of a linear emitter, there are two components of the vector potential: $A_z(r,\varphi,z)$ and $A_r(r,\varphi,z)$ (see [6]). However, only A_z contributes to the impedance of frame ABCD. Besides, the problem for A_z is solved independently.

The mathematical formulation of the problem has the form (see [6]):

$$A_z = A_0(r,\varphi,z),\ 0 < r < R;$$
$$A_z = A_1(r,\varphi,z),\ R \le r \le R_1; \qquad (3.109)$$
$$A_z = A_2(r,\varphi,z),\ R_1 \le r < +\infty,$$

where the functions A_0, A_1, A_2 satisfy the following equations (assuming that the relative magnetic permeability of the wall is $\mu = 1$).

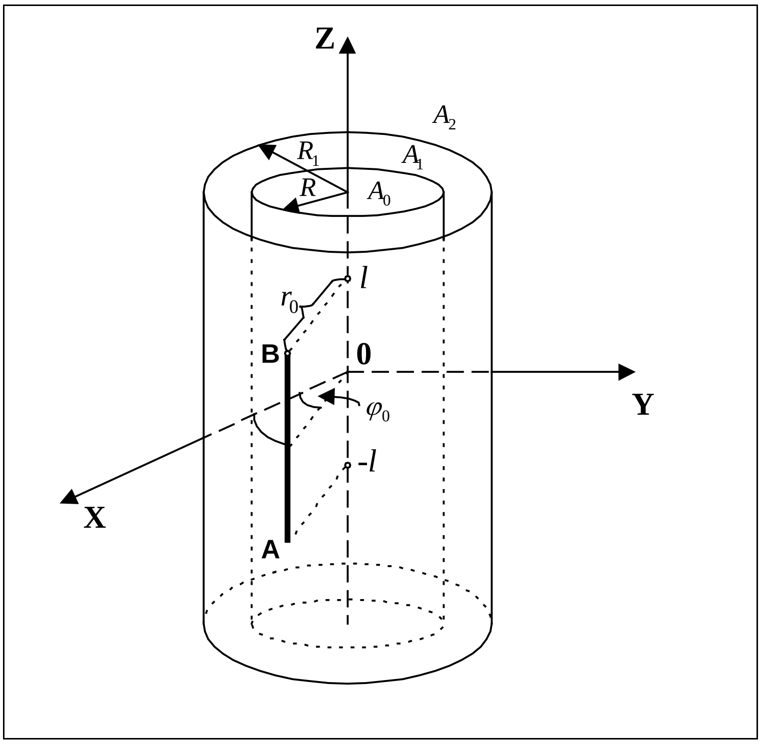

Fig.3.5. Emitter of the form of a linear segment located inside a tube

$$LA_0 = \begin{cases} -\mu_0 I \dfrac{\delta(r-r_0)}{r_0}\delta(\varphi-\varphi_0), & -l<z<l,\ 0<r<R, \\ 0, & z \notin (-l,l),\ 0<r<R, \end{cases} \tag{3.110}$$

$$LA_1 + k^2 A_1 = 0, \qquad R<r<R_1, \quad -\infty<z<+\infty, \tag{3.111}$$

$$LA_2 = 0, \qquad R_1<r<+\infty, \ -\infty<z<+\infty, \tag{3.112}$$

where

$$L = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2},$$

$k^2 = -j\omega\mu_0\sigma$ and σ is the conductivity of the tube. The right-hand side of Eq. (3.110) is selected so that the full current in each cross section $z = \text{const}$ $(-l \le z \le l)$ of the wire is equal to I:

$$\int_0^{2\pi}\int_0^{\infty} I\frac{\delta(r-r_0)}{r_0}\delta(\varphi-\varphi_0)rdrd\varphi \ = \frac{I}{r_0}\int_0^{2\pi}\delta(\varphi-\varphi_0)d\varphi\int_0^{\infty}\delta(r-r_0)rdr = I.$$

The boundary conditions are

$$r=R:\ A_0 = A_1,\ \frac{\partial A_0}{\partial r} = \frac{\partial A_1}{\partial r}; \qquad r=R_1:\ \ A_1 = A_2,\ \frac{\partial A_1}{\partial r} = \frac{\partial A_2}{\partial r}, \tag{3.113}$$

$$z \to \pm\infty:\ \ A_0, A_1, A_2 \to 0; \qquad r \to +\infty:\ \ A_2 \to 0. \tag{3.114}$$

Due to the symmetry of the problem, the functions $A_i(r,\varphi,z)$ ($i=0,1,2$) are even with respect to z, i.e. the following additional condition holds:

$$\left.\frac{\partial A_i}{\partial z}\right|_{z=0} = 0, \quad i = 0,1,2. \tag{3.115}$$

Applying the Fourier cosine transform with respect to z,

$$A_i^c(r,\varphi,\nu) = \sqrt{\frac{2}{\pi}}\int_0^\infty A_i(r,\varphi,z)\cos\nu z dz\,, \tag{3.116}$$

to problem (3.110)-(3.115), one obtains that

$$L_\nu A_0^c = -\mu_0 I\frac{\delta(r-r_0)}{r_0}\delta(\varphi-\varphi_0)\sqrt{\frac{2}{\pi}}\frac{\sin\nu l}{\nu}, \qquad 0<r<R, \tag{3.117}$$

$$L_\nu A_1^c + k^2 A_1^c = 0, \quad R<r<R_1, \tag{3.118}$$

$$L_\nu A_2^c = 0, \qquad R_1 < r < +\infty, \tag{3.119}$$

where

$$L_\nu = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} - \nu^2. \tag{3.120}$$

The boundary conditions are

$$r=R: \ A_0^c = A_1^c, \ \ \frac{\partial A_0^c}{\partial r} = \frac{\partial A_1^c}{\partial r}; \qquad r=R_1: \ A_1^c = A_2^c, \ \ \frac{\partial A_1^c}{\partial r} = \frac{\partial A_2^c}{\partial r}, \tag{3.121}$$

$$r\to+\infty: \ A_2^c \to 0. \tag{3.122}$$

The Fourier series of the function $\delta(\varphi-\varphi_0)$ has the form (see [6]):

$$\delta(\varphi-\varphi_0) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^\infty \cos n(\varphi-\varphi_0), \tag{3.123}$$

and series (3.123) converges conditionally in the sense of Abel (see [6]). The solution of problem (3.117)-(3.122) is sought in the form:

$$A_i^c(r,\varphi,\nu) = \frac{a_{i0}(r,\nu)}{2\pi} + \frac{1}{\pi}\sum_{n=1}^\infty a_{in}(r,\nu)\cos n(\varphi-\varphi_0), \ \ i=0,1,2, \tag{3.124}$$

where $a_{i0}(r,\nu)$ and $a_{in}(r,\nu)$ are unknown coefficients. Substituting Eqs. (3.123) and (3.124) into Eqs. (3.117)-(3.122), one obtains a boundary value problem for a system of ordinary differential equations with respect to the coefficients $a_{in}(r,\nu)$, $i=0,1,2$. Solving this system and substituting the coefficients $a_{in}(r,\nu)$ into Eq. (3.124), one obtains the solution of problem

(3.117)-(3.122). Applying the inverse Fourier cosine transform, one obtains the solution of problem (3.109)-(3.115) in the form

$$A_i(r,\varphi,z) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} A_i^c(r,\varphi,\nu)\cos\nu z d\nu\,,\quad i=0,1,2\,. \tag{3.125}$$

Since for calculating the impedance change we need only the value A_0, we consider the solution for this component in detail:

$$A_0(r,\varphi,z) = \frac{1}{\pi}\sqrt{\frac{2}{\pi}}\sum_{n=0}^{\infty}\gamma_n\int_0^{\infty} a_{0n}(r,\nu)\cos\nu z d\nu\cos n(\varphi-\varphi_0)\,, \tag{3.126}$$

where

$$\gamma_0 = \frac{1}{2}\,,\ \gamma_n = 1\,,\ (n = 1,2,3,...),$$

$$a_{0n}(r,\nu) = \begin{cases} C_{1n}I_n(\nu r), & 0 \le r \le r_0, \\ C_{2n}I_n(\nu r) + C_{3n}K_n(\nu r), & r_0 \le r \le R, \end{cases} \tag{3.127}$$

$$C_{1n} = \frac{E}{I_n(\nu R)}[r_0K_n(\nu r_0)I_n(\nu R) - r_0I_n(\nu r_0)K_n(\nu R) + f_n(\nu)]\,, \tag{3.128}$$

$$C_{2n} = C_{1n} - Er_0K_n(\nu r_0),\quad C_{3n} = r_0EI_n(\nu r_0),\quad E = \mu_0 I\sqrt{\frac{2}{\pi}}\frac{\sin\nu l}{\nu r_0}, \tag{3.129}$$

$$f_n(\nu) = \frac{r_0}{R}\frac{I_n(\nu r_0)}{I_n(\nu R)}\left[\frac{I_n'(\nu R)}{I_n(\nu R)} - q[C_4I_n'(qR) + C_5K_n'(qR)\right]^{-1}, \tag{3.130}$$

$$C_4 = R_1[\nu K_n'(\nu R_1)K_n(qR_1) - qK_n'(qR_1)K_n(\nu R_1)]C_6\,, \tag{3.131}$$

$$C_5 = R_1[qI_n'(qR_1)K_n(\nu R_1) - \nu K_n'(\nu R_1)I_n(qR_1)]C_6\,, \tag{3.132}$$

$$C_6 = \frac{1}{R_1}\{\nu K_n'(\nu R_1)[K_n(qR_1)I_n(qR) - I_n(qR_1)K_n(\nu R)]$$

$$+ qK_n(\nu R_1)[I_n'(qR_1)K_n(qR) + K_n'(qR_1)I_n(qR)]\}^{-1}, \tag{3.133}$$

$$q = \sqrt{\nu^2 - k^2}\,,\quad I_n'(\nu R) = \left.\frac{dI_n(z)}{dz}\right|_{z=\nu R},\quad K_n'(\nu R) = \left.\frac{dK_n(z)}{dz}\right|_{z=\nu R}, \tag{3.134}$$

$I_n(z)$ and $K_n(z)$ are the modified Bessel functions of the first and the second kinds, respectively, of order n. As $R_1 \to \infty$ (the thick-wall tube), the expression for $f_n(\nu)$ is much simpler, and it has the form

$$f_n(\nu) = \frac{r_0}{R}\frac{I_n(\nu r_0)K_n(qR)}{\nu K_n(qR)I_n'(\nu R) - qI_n(\nu R)K_n'(qR)}\,. \tag{3.135}$$

The vector potential $A_0(r,\varphi,z)$ for side CD is found from Eq. (3.126) by replacing φ_0 with $-\varphi_0$ and by changing the sign of the sum to its opposite. From the expression for $A_0(r,\varphi,z)$ in Eq. (3.126), we obtain $A_0^{\text{ind}}(r,\varphi,z)$, i.e. the part of the vector potential corresponding to the reaction of the tube's walls to the wire on side AB. Thus, it is sufficient to consider only the terms which depend on q, R and R_1 in Eq. (3.126):

$$A_0^{\text{ind}}(r,\varphi,z) = \frac{1}{\pi}\sqrt{\frac{2}{\pi}}\sum_{n=0}^{\infty}\gamma_n\int_0^{\infty} a_{0n}^{ind}(r,\nu)\cos\nu z d\nu \cos n(\varphi-\varphi_0)\,, \tag{3.136}$$

$$a_{0n}^{\text{ind}}(r,\nu) = \frac{E}{I_n(\nu R)}[f_n(\nu) - r_0 I_n(\nu r_0)K_n(\nu R)]I_n(\nu r) \tag{3.137}$$

(the exact solution of the problem on electromagnetic waves spreading of the isolated frame $ABCD$ with current has been obtained in [13]). The formula for calculating the impedance change, Z^{ind}, has the form (see Section 1.4 and [6]):

$$Z^{\text{ind}} = \frac{j\omega}{I}\oint_C A_{0l}^{\text{ind}} dl\,. \tag{3.138}$$

It follows from Eq. (3.138) that, in the case of two wires located on sides AB and CD, the impedance change has the form

$$Z_{AB}^{\text{ind}} + Z_{CD}^{\text{ind}} = 2\frac{j\omega}{I}\int_o^l [A_{0AB}^{\text{ind}}(r_0,\varphi_o,z) + A_{0CD}^{\text{ind}}(r_0,\varphi_0,z)]dz. \tag{3.139}$$

Substituting $A_{0AB}^{\text{ind}}(r_0,\varphi_0,z)$ taken from Eq. (3.136) and the corresponding expression for $A_{0CD}^{\text{ind}}(r_0,\varphi_0,z)$ into Eq. (3.139), one obtains

$$Z_{AB}^{\text{ind}} + Z_{CD}^{\text{ind}} = \frac{4j\omega\mu_0}{\pi^2 r_0}\sum_{n=0}^{\infty}\gamma_n\int_0^{\infty}\frac{\sin^2\nu l}{\nu^2}\frac{I_n(\nu r_0)}{I_n(\nu R)}[f_n(\nu) - r_0 I_n(\nu r_0)K_n(\nu R)]d\nu(1-\cos 2n\varphi_0)\,. \tag{3.140}$$

Calculations have been made only for the case where there exists only one emitter on the axis of the cylinder and $R_1 \to \infty$ (the thick-wall cylinder). In order to find A_0^{ind} and Z^{ind}, it is enough to take the limit in Eq. (3.136) as $r_0 \to 0$. Then all terms of series (3.136) tend to zero except the term with $n = 0$. The induced impedance change, Z^{ind}, has the form:

$$Z^{\text{ind}} = \frac{2}{\pi^2} j\omega\mu_0\int_0^{\infty}\left(\frac{\sin\nu l}{\nu}\right)^2 \frac{\hat{C}(\nu) - K_0(qR)}{I_0(\nu R)}d\nu\,, \tag{3.141}$$

$$\hat{C}(\nu) = \frac{1}{R}\frac{K_0(qR)}{qK_1(qR)I_0(\nu R) + \nu K_0(qR)I_1(\nu R)}, \qquad q = \sqrt{\nu^2 + j\omega\sigma\mu_0}\,. \tag{3.142}$$

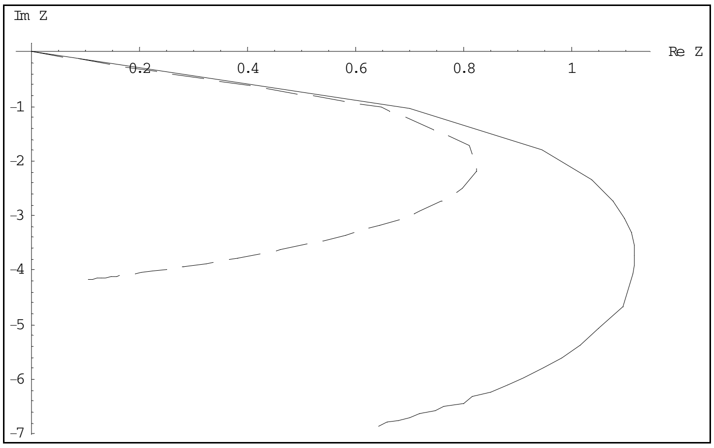

Fig.3.6. Curves describing the change in impedance for
$\alpha = 0.01$(—), $\alpha = 0.1$ (-- --), and for different values of β

Introducing the dimensionless variables $\alpha = R/l$, $\beta = \sqrt{\omega\sigma\mu_0}$ and $s = \nu\, l/\beta$, we obtain from Eqs. (3.141) and (3.142) that

$$Z^{\text{ind}} = \frac{2}{\pi^2}\omega\mu_0 l \cdot Z\,, \quad Z = j\beta\int_0^\infty \left(\frac{\sin\beta s}{\beta s}\right)^2 \frac{\hat{C}(s) - K_0(\alpha\beta s)}{I_0(\alpha\beta s)} ds\,, \tag{3.143}$$

where

$$\hat{C}(s) = \frac{1}{\alpha\beta}\frac{K_0(qR)}{\sqrt{s^2+j}\,K_1(\alpha\beta\sqrt{s^2+j})I_0(\alpha\beta s) + sK_0(\alpha\beta\sqrt{s^2+j})I_1(\alpha\beta s)}\,. \tag{3.144}$$

Calculation results of the impedance by means of Eq. (3.143) are presented in Fig.3.6. As can be seen from the figure, the influence of the conducting walls to the impedance increases as $\alpha = R/l$ decreases, i.e. as the emitter gets closer to the wall of the cylinder.

3.3.3. Circular arc emitter

The problem is solved for an emitter located on the arc $\tilde{B}\tilde{C} = \{\, r = r_0,\ -\varphi_0 \le \varphi \le \varphi_0,\ z = 0 \,\}$ (see Fig.3.7). However, one can easily obtain the solution of the prescribed problem, when the emitters are located on arcs BC and DA, from the obtained solution by replacing z with $z - L$ and $z + L$, respectively. In the case of emitter located on arc $\tilde{B}\tilde{C}$, the equations for the vector potential components A_φ and A_r, are not decoupled (see [6]) so that the solution of this problem is more complicated. Consider only the case $R_1 \to \infty$ (thick-wall tube), in order to explain the main idea of the method on deriving the solution. The solution for the case of finite R_1 is performed similarly, but it is more bulky.

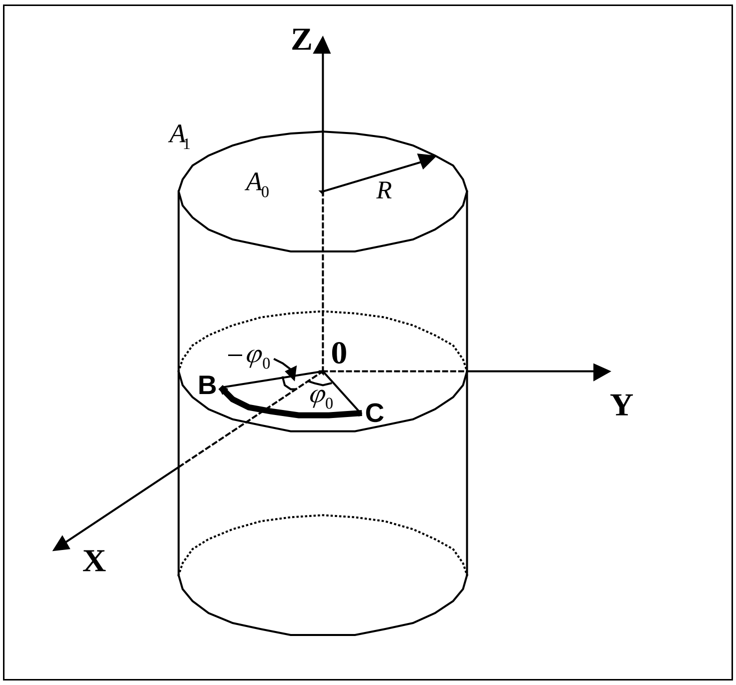

Fig.3.7. Circular arc emitter located inside a thick-wall tube

The formulation of the problem has the form:

$$\left.\begin{aligned} &\tilde{L}A_{0\varphi} + \frac{2}{r^2}\frac{\partial A_{0r}}{\partial \varphi} = -\mu_0 I^e \\ &\tilde{L}A_{0r} - \frac{2}{r^2}\frac{\partial A_{0\varphi}}{\partial \varphi} = 0 \end{aligned}\right\}, \quad 0 < r < R, \quad -\infty < z < +\infty, \qquad (3.145), (3.146)$$

$$\left.\begin{aligned} &\tilde{L}A_{1\varphi} + k^2 A_{1\varphi} + \frac{2}{r^2}\frac{\partial A_{1r}}{\partial \varphi} = 0 \\ &\tilde{L}A_{1r} + k^2 A_{1r} - \frac{2}{r^2}\frac{\partial A_{1\varphi}}{\partial \varphi} = 0 \end{aligned}\right\}, \quad R < r < +\infty, \quad -\infty < z < +\infty, \qquad (3.147), (3.148)$$

where

$$I^e = \begin{cases} I\delta(z)\delta(r - r_0), & -\varphi_0 \le \varphi \le \varphi_0, \\ 0, & \varphi \notin (-\varphi_0, \varphi_0), \end{cases} \qquad (3.149)$$

$$\tilde{L} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}, \quad \text{i.e. } \tilde{L} = \Delta^2 - \frac{1}{r^2} \ . \qquad (3.150)$$

Assuming that $\mu = 1$, the boundary conditions at $r = R$ for the components $A_{0\varphi}$, A_{0r}, $A_{1\varphi}$, A_{1r}, have the form (see similar boundary conditions on pp.19-20 in [6] for the case of a linear horizontal finite length emitter above a conducting half-space):

$$r = R: \quad A_{0\varphi} = A_{1\varphi}, \quad A_{0r} = A_{1r}, \quad \frac{\partial A_{0\varphi}}{\partial r} = \frac{\partial A_{1\varphi}}{\partial r}, \qquad (3.151)$$

$$r = R: \quad \frac{1}{\tilde{k}_0^2}\left[A_{0r} + R\frac{\partial A_{0r}}{\partial r} + \frac{\partial A_{0\varphi}}{\partial \varphi}\right] = \frac{1}{\tilde{k}^2}\left[A_{1r} + R\frac{\partial A_{1r}}{\partial r} + \frac{\partial A_{1\varphi}}{\partial \varphi}\right], \tag{3.152}$$

$$r \to \infty: \quad A_{1\varphi}, A_{1r} \to 0, \tag{3.153}$$

where $\tilde{k}_0^2 = j\omega\mu_0\mu\varepsilon_0\hat{\varepsilon}$, $\tilde{k}^2 = \mu_0\mu(\sigma + j\omega\varepsilon_0\hat{\varepsilon})$ and $\hat{\varepsilon}$ is the relative permittivity. Since the solution is an even function of z, the following additional boundary condition takes place,

$$z = 0: \quad \frac{\partial A_i}{\partial z} = 0, \quad i = 0,1. \tag{3.154}$$

Applying the Fourier cosine transform given by Eq. (3.116) to problem (3.145)-(3.154) and expanding the function I^{ec} (the Fourier cosine transform of the function I^e in Eq. (3.149)) and the unknown functions $A_{i\varphi}^c(r,\varphi,\nu)$, $A_{ir}^c(r,\varphi,\nu)$ into the Fourier series, we have:

$$I^{ec}(r,\varphi,\nu) = \frac{1}{2\pi}\sqrt{\frac{2}{\pi}}\delta(r - r_0)\left[\varphi_0 + 2\sum_{n=1}^{\infty}\frac{\sin n\varphi_0}{n}\cos n\varphi\right], \tag{3.155}$$

$$A_{i\varphi}^c(r,\varphi,\nu) = \frac{1}{\pi}a_{i0}^c(r,\nu) + \frac{2}{\pi}\sum_{n=1}^{\infty}a_{in}^c(r,\nu)\cos n\varphi, \quad i = 0,1, \tag{3.156}$$

$$A_{ir}^c(r,\varphi,\nu) = \frac{2}{\pi}\sum_{n=1}^{\infty}b_{in}^s(r,\nu)\sin n\varphi, \quad i = 0,1, \tag{3.157}$$

where $a_{i0}^c(r,\nu)$, $a_{in}^c(r,\nu)$, $b_{in}^s(r,\nu)$ are unknown coefficients. Substituting series (3.155)-(3.157) into the equations for the functions $A_{i\varphi}^c(r,\varphi,\nu)$, $A_{ir}^c(r,\varphi,\nu)$ and comparing the coefficients of $\cos n\varphi$ and $\sin n\varphi$, one can obtain the boundary value problem for the system of ordinary differential equations with respect to the coefficients $a_{in}^c(r,\nu)$, $b_{in}^s(r,\nu)$:

$$\left.\begin{aligned} &\tilde{L}_n a_{0n}^c + \frac{2n}{r^2}b_{0n}^s = -\frac{1}{2}\sqrt{\frac{2}{\pi}}\mu_0 I\frac{\sin n\varphi_0}{n}\delta(r - r_0) \\ &\tilde{L}_n b_{0n}^s + \frac{2n}{r^2}a_{0n}^c = 0 \end{aligned}\right\}, \quad 0 < r < R, \tag{3.158), (3.159}$$

$$\left.\begin{aligned} &\tilde{L}_n a_{1n}^c + k^2 a_{1n}^c + \frac{2n}{r^2}b_{1n}^s = 0 \\ &\tilde{L}_n b_{1n}^s + k^2 b_{1n}^s + \frac{2n}{r^2}a_{1n}^c = 0 \end{aligned}\right\}, \quad R < r < +\infty, \tag{3.160), (3.161}$$

where

$$\tilde{L}_n = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2+1}{r^2} - \nu^2, \quad n = 0,1,2,\ldots, \; b_{00}^s = b_{10}^s = 0. \tag{3.162}$$

The boundary conditions are

$$r = R: \quad a_{0n}^{c} = a_{1n}^{c}, \quad b_{0n}^{s} = b_{1n}^{s}, \qquad r \to \infty: \; a_{1n}^{c} \to 0, \; b_{1n}^{s} \to 0, \tag{3.163}$$

$$r = R: \quad \begin{cases} \dfrac{da_{0n}^{c}}{dr} = \dfrac{da_{1n}^{c}}{dr}, \\ \dfrac{1}{\tilde{k}_0^2}\left[b_{0n}^{s} + R\dfrac{db_{0n}^{s}}{dr} - na_{on}^{c}\right] = \dfrac{1}{\tilde{k}^2}\left[b_1^{s} + R\dfrac{db_1^{s}}{dr} - na_{1n}^{c}\right]. \end{cases} \tag{3.164), (3.165}$$

Since $b_{i0} = 0$ ($i = 0,1$), condition (3.165) at $n = 0$ is correct and the problem for a_{i0} ($i = 0,1$) is decoupled. Then the solution $a_{00}(r,z)$ has the form:

$$a_{00}(r,z) = \sqrt{\frac{2}{\pi}}\int_0^\infty a_{00}^{c}(r,\nu)\cos\nu z d\nu, \tag{3.166}$$

where

$$a_{00}^{c}(r,\nu) = C_{10}(\nu)I_1(\nu r), \qquad 0 \le r \le r_0;$$

$$a_{00}^{c}(r,\nu) = C_{20}(\nu)K_1(\nu r) + C_{30}(\nu)I_1(\nu r), \qquad r_0 \le r \le R;$$

$$C_{10}(\nu) = C_{30}(\nu) + \nu r_0 E_0 K_1(\nu r_0), \qquad C_{20}(\nu) = \nu r_0 E_0 I_1(\nu r_0), \qquad E_0 = \frac{\mu_0\varphi_0}{2\nu}\sqrt{\frac{2}{\pi}}I,$$

$$C_{30}(\nu) = I_1^{-1}(\nu R)[C_0(\nu) - C_{20}(\nu)K_1(\nu R)],$$

$$C_0(\nu) = C_{20}(\nu)K_1(\nu R)R^{-1}[\nu I_1'(\nu R)K_1(\nu R) - qK_1'(\nu R)I_1(\nu R)], \quad q = \sqrt{\nu^2 - k^2}. \tag{3.167}$$

The solution for $a_{00}(r,z)$ allows one to obtain the sum $Z_{BC}^{\text{ind}} + Z_{DA}^{\text{ind}}$ in the case the detector coil of radius $r_c \le R$ is concentric with respect to the tube (see [5]). Using boundary condition (3.163), $n = 1,2,3,\ldots$, the solution of the problem for the coefficients $a_{in}(r,z)$ and $b_{in}(r,z)$ has the form:

$$a_{in}(r,z) = \sqrt{\frac{2}{\pi}}\int_0^\infty a_{in}^{c}(r,\nu)\cos\nu z d\nu, \qquad b_{in}(r,z) = \sqrt{\frac{2}{\pi}}\int_0^\infty b_{in}^{s}(r,\nu)\cos\nu z d\nu, \tag{3.168}$$

where

$$a_{in}^{c}(r,\nu) = \frac{1}{2}[u_{in}^{c}(r,\nu) + v_{in}^{c}(r,\nu)], \qquad b_{in}^{s}(r,\nu) = \frac{1}{2}[u_{in}^{c}(r,\nu) - v_{in}^{c}(r,\nu)], \tag{3.169}$$

$$u_{0n}^{c}(r,\nu) = C_{1n}(\nu)I_m(\nu r), \qquad 0 \le r \le r_0;$$

$$u_{0n}^{c}(r,\nu) = C_{2n}(\nu)K_m(\nu r) + C_{3n}(\nu)I_m(\nu r), \quad r_0 \le r \le R;$$

$$u_{1n}^{c}(r,\nu) = \tilde{C}_n(\nu)K_m(qr)\big/K_m(qR), \qquad C_{1n}(\nu) = C_{3n}(\nu) + \nu r_0 E_n K_m(\nu r_0),$$

$$C_{2n}(\nu) = \nu r_0 E_n I_m(\nu r_0), \qquad C_{3n}(\nu) = I_m^{-1}(\nu R)[\tilde{C}_n(\nu) - C_{2n}(\nu)K_m(\nu R)],$$

$$E_n = \frac{\mu_0}{2\nu}\sqrt{\frac{2}{\pi}} I \frac{\sin n\varphi_0}{n}, \; m = n-1 \quad \text{for} \quad u_{in}^c(r,\nu)\,(i = 0{,}1). \tag{3.170}$$

The expressions for $v_{in}^c(r,\nu)$ are obtained from the formulas for $u_{in}^c(r,\nu)$ by replacing m with $n+1$ and $\tilde{C}_n(\nu)$ with $\hat{C}_n(\nu)$, where $\tilde{C}_n(\nu)$ and $\hat{C}_n(\nu)$ are unknown constants that are found by substituting Eq. (3.169) into the two boundary conditions (3.164) and (3.165).

The solution to problem (3.145)-(3.154) for $A_{0\varphi}(r,\varphi,z)$ has the form:

$$A_{0\varphi}(r,\varphi,z) = \frac{1}{\pi} a_{00}(r,z) + \frac{2}{\pi}\sum_{n=1}^{\infty} a_{0n}(r,z)\cos n\varphi, \tag{3.171}$$

where $a_{00}(r,z)$ and $a_{0n}(r,z)$ are given by Eqs. (3.166)-(3.170). In order to obtain the expression of the vector potential $A_{BC} + A_{DA}$, we replace $\cos\nu z$ with $\cos\nu(z-L) - \cos\nu(z+L)$.

It follows from Eq. (3.138) that in the case of two wires located on arcs BC and DA the impedance change is

$$Z_{BC}^{\text{ind}} + Z_{DA}^{\text{ind}} = 2\frac{j\omega r_0}{I}\int_0^l [A_{0\varphi}^{\text{ind}}(r_0, L, \varphi) + A_{0\varphi}^{\text{ind}}(r_0, -L, \varphi)]dz. \tag{3.172}$$

Choosing $A_{0\varphi}^{\text{ind}}$ in Eq. (3.171) and putting it into Eq. (3.172), we finally obtain:

$$Z_{BC}^{\text{ind}} + Z_{DA}^{\text{ind}} = 4j\omega\frac{r_0}{\pi I}\sqrt{\frac{2}{\pi}} \times$$

$$\times\int_0^\infty \left\{ C_{30}(\nu) I_\nu(r_0) + \sum_{n=1}^{\infty}[C_{3n}(\nu) I_{n-1}(\nu r_0) + D_{3n}(\nu) I_{n+1}(\nu r_0)]\frac{\sin n\varphi_0}{n}(1-\cos 2\nu L)\right\} d\nu,$$

$$D_{3n}(\nu) = I_{n+1}^{-1}(\nu R)[\hat{C}_n(\nu) - \nu r_0 E_n I_{n+1}(\nu r_0) K_{n+1}(\nu R)]. \tag{3.173}$$

Eqs. (3.140) and (3.173) give the solutions of the problem on the impedance of a frame with current inside a conducting tube. One possible application of the solution is the following. If the ratio of the frame's sides CD and AB, and the gap between the frame and tube's wall are sufficiently small, then the eddy currents in the tube's wall are mainly excited only under the frame. Therefore, the solution obtained can be used to determine the tube's wall thickness directly under the frame for the case of non-concentric wall's surfaces. The eccentricity in the tube's geometry arises in the exploitation process of heat exchanger tubes.

4. SOME PROBLEMS ON THE IMPEDANCE CHANGE OF MEDIA CONTAINING FLAWS

4.1. Calculation of impedance change by using the method of additional currents

The exact analytical solution for the problem of the influence of a conducting medium with an arbitrary flaw on a source of current is not known. Therefore, since 1960 different approximate analytical and numerical methods for that kind of problem have been used. The influence of a non-uniform conducting medium on a source of current has been investigated in monograph [6], by using the small parameter $\varepsilon = 1 - \sigma_2/\sigma_1$, where σ_1 and σ_2 are the conductivities of the conducting medium and the flaw, respectively. In order to obtain approximate and exact analytical solutions of that kind of problems two methods of additional current in a non-uniform conducting medium are developed in this thesis (see also the author's paper [10], [11], [34]). The presence of additional current in the region of a flaw is assumed by the first method. The direction of this current is opposite to the direction of the eddy current that flows in the same region when the flaw is absent. The additional current used in the second method is chosen so that the differential equation for the uniform conducting medium is transformed into a differential equation for the flaw. Both methods are illustrated for the problem of a double conductor line above a two-layer conducting half-space. This problem has an exact analytical solution that allows one to estimate the error for both methods.

4.1.1. Formulation of the problem

Consider two horizontal infinitely long parallel wires carrying an alternating current and placed in free space above a two-layer conducting medium. The wires are situated on the lines $\{y = y_0, \quad z = h, \ -\infty < x < +\infty\}$ and $\{y = y_1, \ z = h, \ -\infty < x < +\infty\}$. The upper layer of thickness d and conductivity σ_1 is situated in the region $\{-\infty < x, y < +\infty, \ -d \le z \le 0\}$; the lower layer situated in the region $\{-\infty < x, y < +\infty, \ -\infty < z \le -d\}$ is a half-space with conductivity σ_2 (see Fig.4.1).

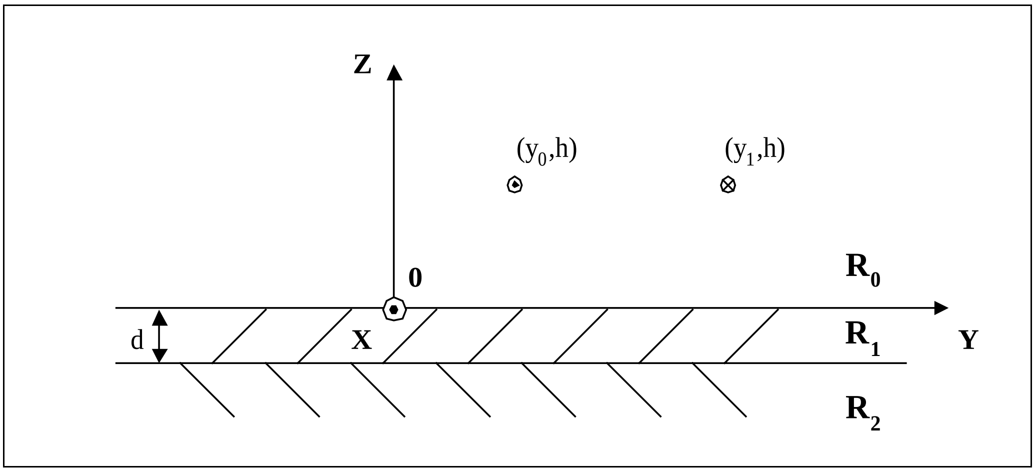

Fig.4.1. A double conductor line above a two-layer conducting medium

In this case the vector potential has only the *x*-component, i.e. $\vec{A} = A_x(y,z)$ (see the 1st boundary value problem) and the formulation of the vector potential problem for the *x*-component has the form (see [6]):

$$\Delta A_0 = -\mu_0 I^e, \quad I^e = I\,\delta(z-h)[\delta(y-y_0)-\delta(y-y_1)], \qquad z>0, \tag{4.1}$$

$$\Delta A_1 + k_1^2 A_1 = 0, \qquad -d<z<0, \tag{4.2}$$

$$\Delta A_2 + k_2^2 A_2 = 0, \qquad -\infty<z<-d, \tag{4.3}$$

where $k_i^2 = -j\omega\mu_0\sigma_i$, $i=1,2$ (i.e. the displacement current is absent) and σ_i is the conductivity of region R_i $(i=1,2)$. The boundary conditions are

$$z=0: \qquad A_0 = A_1, \quad \frac{\partial A_0}{\partial z} = \frac{\partial A_1}{\partial z}, \tag{4.4}$$

$$z=-d: \quad A_1 = A_2, \quad \frac{\partial A_1}{\partial z} = \frac{\partial A_2}{\partial z}. \tag{4.5}$$

Problem (4.1)-(4.5) has been solved in [6]. The reaction of the conducting plate on the double conductor line is the following

$$A_0^{\text{ind}}(y,z) = \frac{\mu_0 I}{2\pi}\int_0^\infty \frac{[(q_1+q_2)(\lambda-q_1)+(q_1-q_2)(\lambda+q_1)e^{-2q_1 d}]e^{-\lambda(z+h)}}{(\lambda+q_1)(q_1+q_2)-(\lambda-q_1)(q_2-q_1)e^{-2qd}}$$

$$\times[\cos\lambda(y-y_0)-\cos\lambda(y-y_1)]\frac{d\lambda}{\lambda}, \tag{4.6}$$

where $q_1 = \sqrt{\lambda^2 - k_1^2}$ and $q_2 = \sqrt{\lambda^2 - k_2^2}$. The induced change in impedance due to a conducting medium, per unit length of contour C of a double conductor line, is given in [6]:

$$Z^{\text{ind}} = \frac{j\omega}{I} A_0^{\text{ind}}(y,z)\bigg|_{\substack{y=y_0 \\ z=h}} = \frac{\mu_0\omega}{\pi} Z\,. \tag{4.7}$$

4.1.2. Approximate solution to the problem by the first method of additional currents

In order to solve problem (4.1)-(4.5) approximately, using the first method of additional current, the two-layer conducting medium should be replaced by the conducting half-space $\{-\infty < z < 0\}$ (this means that Eq. (4.2) takes place in the whole region $\{-\infty < z < 0\}$). In this case, the vector potential A_1 in the region $\{z < 0\}$ has the form (see [6]):

$$A_1(y,z) =: A_{1\infty} = \frac{\mu_0 I}{\pi} \int_0^\infty \frac{e^{q_1 z - \lambda h}}{\lambda + q_1} [\cos\lambda(y - y_0) - \cos\lambda(y - y_1)]\,d\lambda. \tag{4.8}$$

In the region $\{z < 0\}$ the eddy current $\tilde{I}^e$ is defined by the formula (see [6]):

$$\tilde{I}^e = -\sigma_1 j\omega A_{1\infty}. \tag{4.9}$$

First, consider the more explicit case $\sigma_2 = 0$ $(k_2^2 = 0)$: the region $\{-\infty < z < -d\}$ is free space and the conducting plate of thickness d is located in the region $\{-d < z < 0\}$. In order to pass from the conducting half-space to the conducting plate of finite thickness, it is assumed that there is an additional current, $\hat{I}^e$, in the region $\{-\infty < z < -d\}$ of the conducting half-space, and this current is opposite to the current given by Eq. (4.9):

$$\hat{I}^e = -\tilde{I}^e = \sigma_1 j\omega A_{1\infty}. \tag{4.10}$$

Then the equation for the vector potential in the region $\{-\infty < z < -d\}$ takes the form:

$$\Delta A_1 + k_1^2 A_1 = k_1^2 A_{1\infty}, \qquad -\infty < z < -d, \tag{4.11}$$

where $A_{1\infty}$ is given by Eq. (4.8).

In general, if $\sigma_2 \neq 0$ $(k_2^2 \neq 0)$, then, instead of Eq. (4.11), the following equation is obtained:

$$\Delta A_1 + k_1^2 A_1 = (k_1^2 - k_2^2) A_{1\infty}, \qquad -\infty < z < -d. \tag{4.12}$$

Then the problem can be formulated as follows:

$$\Delta A_0 = -\mu_0 I^e, \quad I^e = I\,\delta(z-h)[\delta(y-y_0) - \delta(y-y_1)], \quad z > 0, \tag{4.13}$$

$$\Delta A_1 + k_1^2 A_1 = \begin{cases} 0, & -d < z < 0, \\ (k_1^2 - k_2^2) A_{1\infty}, & -\infty < z < -d. \end{cases} \tag{4.14}$$

The boundary conditions are

$$z = 0: \quad A_0 = A_1, \quad \frac{\partial A_0}{\partial z} = \frac{\partial A_1}{\partial z}. \tag{4.15}$$

The right-hand side of Eq. (4.14) is transformed to the form $(k_1^2 - k_2^2)A_{1\infty} = \varepsilon\, k_1^2 A_{1\infty}$, where $\varepsilon = 1 - \sigma_2/\sigma_1 = 1 - k_2^2/k_1^2$. Therefore, one can prove that the solution of problem (4.13)-(4.15) completely coincides with the approximate solution of problem (4.1)-(4.5) obtained by the perturbation method (see [3]), if one assumes that

$$A_i(y,z) = A_i^{(0)}(y,z) + \varepsilon\, A_i^{(1)}(y,z), \qquad i = 1,2. \tag{4.16}$$

The function $A_i^{(0)}$ in Eq. (4.16) is the solution of problem (4.1)-(4.5) if $\varepsilon = 0$ (i.e. $k_2 = k_1$). The terms $A_i^{(1)}$ in Eq. (4.16) give a solution of problem (4.13)-(4.15), if we assume that the right-hand side of Eq. (4.13) is equal to zero, and substitute $\varepsilon = 1$ (i.e. $k_2 = 0$) into the right-hand side of Eq. (4.14). This method had been used first in [3] for the problem of a double conductor line above a flawed medium when the flaw with conductivity $\sigma_2 = \sigma_1(1-\varepsilon)$ was situated in the rectangular domain $\{-\infty < x < +\infty, -l \le y \le l, \quad -(a+b) \le z \le -a\}$. The change in impedance at $\sigma_2 = 0$ is obtained from the solution of problem (4.13)-(4.15) in the form

$$Z^{\text{ind}} = \frac{\mu_0 \omega}{\pi} Z_1, \qquad Z_1 = \beta^2 \int_0^\infty \frac{(1-\cos\xi)e^{-2\alpha\xi}\left[\dfrac{1}{\xi} - \dfrac{e^{-2\bar{d}\sqrt{\xi^2 + j\beta^2}}}{\sqrt{\xi^2 + j\beta^2}}\right]}{(\xi + \sqrt{\xi^2 + j\beta^2})^2} d\xi\,, \tag{4.17}$$

where

$$\beta = c\sqrt{\omega\sigma_1\mu_0}\,, \quad c = y_1 - y_0\,, \quad \alpha = h/c\,, \quad \bar{d} = d/c\,.$$

The difference between Z^{ind} obtained from the solution of problem (4.13)-(4.15) and the exact value of Z^{ind} by means of Eq. (4.7) does not exceed 2% if $0 \le \varepsilon \le 0.2$. However, if $\varepsilon = 1$, the difference tends to 20% (see Fig.4.2).

4.1.3. Approximate solution to the problem by the second method of additional currents

Using this method one should replace a two-layer conducting medium with a conducting half-space of conductivity σ_1. However, instead of equation (4.12) the following equation is considered:

$$\Delta A_1 + k_1^2 A_1 = (k_1^2 - k_2^2) I^e. \tag{4.18}$$

The additional current, I^e, is selected so that Eq. (4.18) is transformed into Eq. (4.3). For this purpose, we substitute $I^e = \tilde{A}_2$ into Eq. (4.18), where $\tilde{A}_2$ is the solution of Eq. (4.3) of the domain $\{-\infty < z < 0\}$, i.e. for the case where the layer $\{-d < z < 0\}$ of conductivity σ_1 is absent:

$$\tilde{A}_2(y,z) = \frac{\mu_0 I}{\pi}\int_0^\infty \frac{e^{q_2 z-\lambda h}}{\lambda + q_2}[\cos\lambda(y-y_0) - \cos\lambda(y-y_1)]d\lambda. \tag{4.19}$$

Then Eq. (4.18) can be written as

$$\Delta A_1 + k_1^2(A_1 - \tilde{A}_2) = -k_2^2\tilde{A}_2. \tag{4.20}$$

Substituting $A_1 = \tilde{A}_2$ into Eq. (4.20), we obtain

$$\Delta\tilde{A}_2 + k_2^2\tilde{A}_2 = 0, \tag{4.21}$$

i.e. Eq. (4.3) is obtained. In this case a complete formulation of the problem is given by Eqs. (4.13)-(4.15), where the value $A_{1\infty}$ is replaced by $\tilde{A}_2$ on the right-hand side of Eq. (4.14). If $\sigma_2 = 0$, the induced impedance change has the form

$$Z^{ind} = \frac{\mu_0\omega}{\pi}Z_2, \qquad Z_2 = \beta^2\int_0^\infty \frac{1-e^{-\bar{d}(\xi+\sqrt{\xi^2+j\beta^2})}}{(\xi+\sqrt{\xi^2+j\beta^2})^2}\, e^{-2\varepsilon\xi}(1-\cos\xi)\frac{d\xi}{\xi}. \tag{4.22}$$

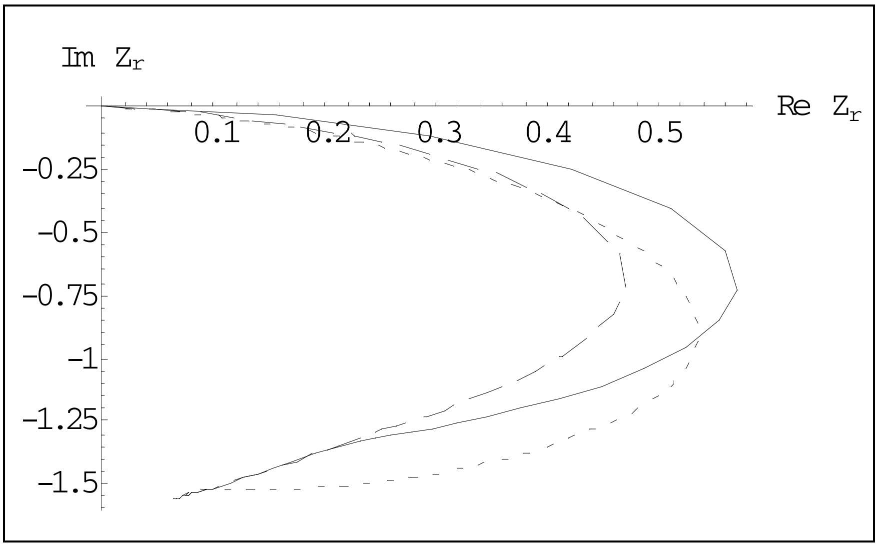

Fig.4.2. Curves describing the change in impedance for $\alpha = 0.1$ and for different values of β,

where the continuous line is the exact solution by means of formula (4.7), the dotted line is obtained by the second method by means of formula (4.22), the dashed line is obtained by the first method by means of formula (4.17)).

The real and imaginary parts of the impedance change as functions of the parameter β are plotted in Fig.4.2 by means of the exact formula (4.7), and by the approximate formulas (4.17) and (4.22) for $\alpha = 0.1$. Calculations show that the approximate solution (4.22) gives an error less than 6% for the value Z^{ind} given by Eq. (4.7), while the error in $|Z|$ by means of Eq. (4.17) reaches 20%.

Note that the second method gives a result that completely coincides with the exact solution if one uses the exact value of A_F.

4.2. Impedance change of a conducting medium with a flaw of an arbitrary shape

This section is devoted to the proof of a new exact analytical formula for the impedance change and the well-know formula in the literature (see [50] and [47]). Both formulas are proved in this thesis (see also the author's papers [16], [19], [34], [37]). The derivation of the new formula is based on Green's formula, since Lorentz' theorem is used for obtaining the other formula. The newly obtained formula for the impedance change has the form of a triple integral of a scalar product of two vector potentials: the vector potential in the flaw and the vector potential in the same region in the absence of the flaw over the region containing the flaw. A similar formula obtained earlier by previous authors has the form of a triple integral of a scalar product of amplitude electric field vectors.

4.2.1. Formulation of the problem

The formula for the change in impedance used in the literature (see [50], [47]) has the form

$$Z^{\text{ind}} = -\frac{(\sigma_F - \sigma)}{I^2} \iiint\limits_{V_F} \vec{E} \cdot \vec{E}_F \, dV, \tag{4.23}$$

where V_F is the region of the flaw, σ_F and σ are the conductivities of the flawed and flawless regions, respectively, $\vec{E}_F$ is the amplitude electric field vector in the flawed region, $\vec{E}$ is the amplitude electric field vector in the same region in the absence of the flaw and I is the amplitude of the current vector density.

The displacement current is neglected in Eq. (4.23) as it is used in the problems of eddy current testing and in the case of harmonic oscillations of the external current with frequency

ω (see [6]). In this section (see also the author's papers [16], [19], [34], [37]), a new formula for Z^{ind} is obtained in a form more suitable for computations:

$$Z^{\text{ind}} = \frac{\omega^2(\sigma_F - \sigma)}{I^2} \iiint_{V_F} \vec{A} \cdot \vec{A}_F \, dV, \tag{4.24}$$

where $\vec{A}_F$ is the amplitude vector potential in the flawed region, $\vec{A}$ is the amplitude vector potential in the same region in the absence of the flaw (i.e. the case when all physical properties of the region V_F are the same as the physical properties of the conducting region outside of region V_F) and ω is the frequency.

The aim is to prove that the right-hand sides of Eqs. (4.23) and (4.24) coincide (see also the author's papers [19] and [37]), i.e.

$$\iiint_{V_F} \vec{E} \cdot \vec{E}_F \, dV = -\omega^2 \iiint_{V_F} \vec{A} \cdot \vec{A}_F \, dV \,. \tag{4.25}$$

Note that the relationship between the vectors $\vec{E}$ and $\vec{A}$ in the case of harmonic oscillations of the external current with frequency ω is given by (see [6]):

$$\vec{E} = -j\omega\vec{A} + \frac{1}{\tilde{k}_1^2} \operatorname{grad} \operatorname{div} \vec{A}\,, \tag{4.26}$$

where $\tilde{k}_1^2 = \mu_0\mu(\sigma + j\varepsilon_0\hat{\varepsilon}\omega)$ if the displacement current is taken into account and $\tilde{k}_1^2 = \mu_0\mu\sigma$ if the displacement current is neglected, ε_0 and μ_0 are the electric and magnetic constants, respectively; $\hat{\varepsilon}$ and μ are the relative permittivity and relative magnetic permeability of the medium, respectively, and $j = \sqrt{-1}$ is the imaginary unit.

It follows from Eq. (4.26) that Eq. (4.25) is correct if

$$\operatorname{div} \vec{A} = 0\,, \quad \operatorname{div} \vec{A}_F = 0\,. \tag{4.27}$$

In fact Eq. (4.27) is only valid in the case of a homogeneous half-space as the conducting region and the external current located either on a single-turn coil or double conductor line in the plane parallel to the half-space. Eq. (4.27) is also valid if the flaw of the inhomogeneous half-space is a cylindrical body coaxial with a single-turn coil carrying the external current (see [66], [67]) or if the flaw is an infinitely long cylinder parallel to double conductor line carrying the external current (see [5]). In all other cases, $\operatorname{div} \vec{A} \neq 0$, $\operatorname{div} \vec{A}_F \neq 0$ in the region V_F. However, Eqs. (4.23), (4.24) and (4.25) are still true as it will be shown below.

It follows from Eqs. (4.25) and (4.26) that for a flaw situated in an arbitrary region V_F

$$\iiint_{V_F} [-j\omega\vec{A}\cdot\tilde{k}_1^2\,\mathrm{graddiv}\vec{A}_F - j\omega\vec{A}_F\cdot\tilde{k}_{1F}^2\,\mathrm{graddiv}\vec{A} + \mathrm{graddiv}\vec{A}\cdot\mathrm{graddiv}\vec{A}_F]\,dV = 0\,, \quad (4.28)$$

where $\tilde{k}_{1F}^2 = \mu_0\mu(\sigma_F + j\varepsilon_0\hat{\varepsilon}_F\omega)$.

At first sight, assuming the continuity of the functions $\vec{A}$, $\vec{A}_F$, $\mathrm{graddiv}\vec{A}$, $\mathrm{graddiv}\vec{A}_F$, one may conclude that $\mathrm{graddiv}\vec{A} = 0$, $\mathrm{graddiv}\vec{A}_F = 0$ (using the known theorem: if a function $f(M)$ is continuous in a closed region V_F and for any region $\tilde{V} \subset V_F$ the formula $\iiint_{\tilde{V}} f(M)\,dV = 0$ is valid, then $f(M) = 0$ for all $M \in V_F$). However, this is not true. In fact, by changing the region V_F, the functions $\vec{A}$ and $\vec{A}_F$ are changed too. Therefore, Eq. (4.28) is also valid if $\mathrm{div}\vec{A} \neq 0$, $\mathrm{div}\vec{A}_F \neq 0$ in the region V_F.

In the previous studies (see [50], [47]) in trying to prove Eq. (4.23) for impedance change, it was assumed that $\mathrm{div}\vec{A} = 0$ in Eq. (4.26). Besides, in [50] it was assumed that the scalar potential gives a change in the static field only. That statement is not true. It was suggested in [47] to use the Coulomb's gauge, i.e. $\mathrm{div}\vec{A} = 0$. At the same time, the authors use the following equation for the vector potential $\vec{A}$:

$$\Delta\vec{A} + k^2\vec{A} = \mu_0\mu\vec{I}^{ext},\ k^2 = -j\omega\sigma\mu_0\mu\,. \quad (4.29)$$

It is well known that Eq. (4.29) is not correct in this case. In fact, in the case of Coulomb's gauge the equation for the vector potential is more complicated (see [6], p.10), and has the form

$$\Delta\vec{A} = \mu_0\mu\sigma\left(\nabla\varphi + \frac{\partial\vec{A}}{\partial t}\right) + \mu_0\varepsilon_0\mu\hat{\varepsilon}\frac{\partial}{\partial t}\left(\nabla\varphi + \frac{\partial\vec{A}}{\partial t}\right) - \mu_0\mu\vec{I}^e\,, \quad (4.30)$$

where φ is the scalar potential.

Note also that in this problem by taking the displacement current into account, the coefficient $(\sigma_F - \sigma)/I^2$ in Eqs. (4.23) and (4.24) is transformed into the coefficient

$$\frac{\sigma_F - \sigma}{I^2} + \frac{j\omega\varepsilon_0(\hat{\varepsilon}_F - \hat{\varepsilon})}{I^2}\,, \quad (4.31)$$

where $\hat{\varepsilon}_F$ and $\hat{\varepsilon}$ are the relative electric permittivity in the flawed and flawless regions.

4.2.2. New convenient formula for the change in impedance

Let us prove the new formula (4.24) for the impedance change, which describes the influence of a conducting medium with a flaw of arbitrary shape on a source of current. Consider a conducting half-space situated in the region $V_1=\{ -\infty < x, y < +\infty,\ -\infty < z < 0 \}$ with a flaw in the region $V_F \in V_1$. The source of current is located in free space, V_0, on the closed curve described by the equation:

$$z = h\,,\quad \rho = \rho(\varphi)\,,\quad 0 \le \varphi \le 2\pi\,, \tag{4.32}$$

where ρ, φ, z are cylindrical polar coordinates. One also can use the Cartesian coordinates x, y, z (see Fig.4.3).

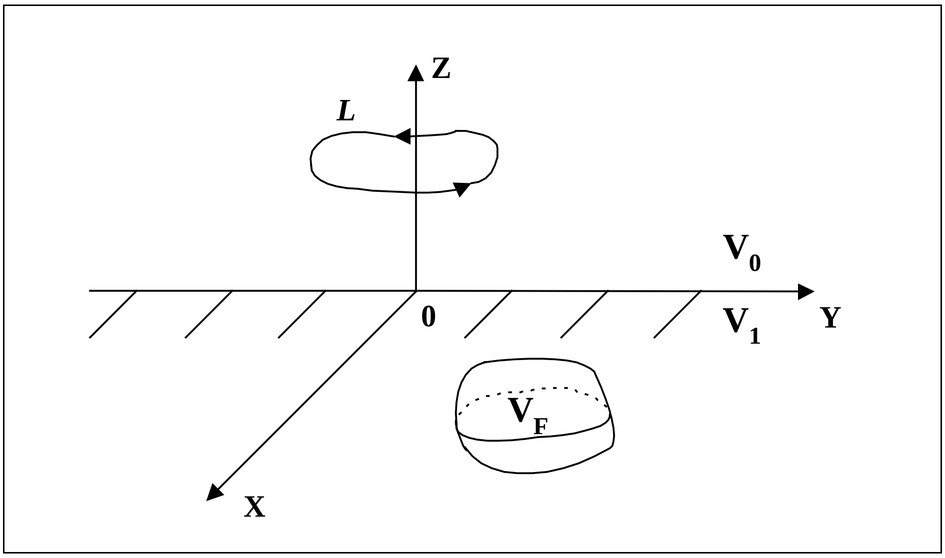

Fig.4.3. Contour L with current above a conducting half-space, V_1, containing a flaw of arbitrary form in region V_F

The current in the contour is given by

$$\vec{I}^{\,e} = I\delta[\rho - \rho(\varphi)]\delta(z-h)\vec{e}_\tau\,,\quad 0 \le \varphi \le 2\pi\,, \tag{4.33}$$

where I is the complex amplitude of the current's density, $\delta(x)$ is the Dirac delta function and $\vec{e}_\tau$ is the unit vector to the tangent of line (4.32). In this case the complex amplitude $\vec{A}(x, y, z)$ of the vector potential has three components A_x, A_y, A_z (see [6]):

$$\vec{A}(x, y, z) = \vec{A}_h(x, y, z) + A_z(x, y, z)\vec{e}_z\,, \tag{4.34}$$

$$\vec{A}_h(x, y, z) = A_x(x, y, z)\vec{e}_x + A_y(x, y, z)\vec{e}_y\,. \tag{4.35}$$

Since the source of current is situated in the horizontal plane, then only the horizontal component of the vector potential, $\vec{A}_h(x,y,z)$, contributes to the impedance. It is known that the problem for the horizontal component can be solved separately (see [6]). The solution of the problem to the vertical component needs only satisfy all boundary conditions for the components A_x, A_y, A_z in the plane $z=0$. That is why it is not necessary to solve the problem for A_z.

The mathematical formulation of the problem for the horizontal component has the form

$$\Delta\vec{A}_{0h} = -\mu_0 \vec{I}^{\,e}\;, \qquad -\infty < x,y < +\infty, \qquad 0 < z < +\infty, \tag{4.36}$$

$$\Delta\vec{A}_{1h} + k_1^2\vec{A}_{1h} = 0, \quad -\infty < x,y < +\infty, \quad -\infty < z < 0, \quad (x,y,z)\notin V_F, \tag{4.37}$$

$$\Delta\vec{A}_{Fh} + k_F^2\vec{A}_{Fh} = 0, \quad -\infty < x,y < +\infty, \quad -\infty < z < 0, \quad (x,y,z)\in V_F, \tag{4.38}$$

with the corresponding boundary conditions on the boundaries of regions V_0, V_1 and V_F. The problem for a non-uniform medium can be transformed into the problem for a uniform medium with non-uniform right-hand side so that system (4.36)-(4.38) can be rewritten in the form

$$\Delta\vec{A}_{0h} = -\mu_0 \vec{I}^{\,e}\;, \qquad -\infty < x,y < +\infty, \qquad 0 < z < +\infty, \tag{4.39}$$

$$\Delta\vec{A}_{1h} + k_1^2\vec{A}_{1h} = \begin{cases} 0, & (x,y,z)\notin V_F, \quad -\infty < z < 0, \\ (k_1^2 - k_F^2)\vec{A}_{Fh}, & (x,y,z)\in V_F, \quad -\infty < z < 0, \end{cases} \tag{4.40}$$

with the boundary conditions

$$z=0:\; \vec{A}_{0h} = \vec{A}_{1h}, \quad \frac{\partial\vec{A}_{0h}}{\partial z} = \frac{\partial\vec{A}_{1h}}{\partial z}; \tag{4.41}$$

$$x^2+y^2+z^2\to\infty\;\;(z>0):\;\; \vec{A}_{0h}\to 0; \qquad x^2+y^2+z^2\to\infty\;\;(z<0):\;\; \vec{A}_{1h}\to 0. \tag{4.42}$$

It is to be noted that $\vec{A}_{Fh}$ on the right-hand side of Eq. (4.40) is taken from the solution of problem (4.36)-(4.38) with the corresponding boundary conditions. Besides, the right-hand side of Eq. (4.40) is chosen such that substituting $\vec{A}_{1h} = \vec{A}_{Fh}$ into Eq. (4.40), Eq. (4.40) is transformed into Eq. (4.38).

In order to obtain Eq. (4.24) for Z^{ind}, and due to the linearity of the problem, the functions $\vec{A}_{0h}(x,y,z)$ and $\vec{A}_{1h}(x,y,z)$ can be written in the form

$$\vec{A}_{0h}(x,y,z) = \vec{A}_{0h}^{\text{absnt}}(x,y,z) + \vec{A}'_{0h,\text{ind}}(x,y,z) + \vec{A}''_{0h,\text{ind}}(x,y,z), \tag{4.43}$$

$$\vec{A}_{1h}(x,y,z) = \vec{A}'_{1h}(x,y,z) + \vec{A}''_{1h}(x,y,z), \tag{4.44}$$

where

$\vec{A}_{0h}^{\text{absnt}}$ is the solution of Eq. (4.39) in the absence of the conducting half-space in the region $z < 0$;

$\vec{A}'_{0h,\text{ind}}$ is the reaction of the conducting medium under the condition that the medium is uniform, i.e. $k_F = k_1$ (in other words, the right-hand side of Eq. (4.40) is equal to zero), but $I \neq 0$;

$\vec{A}''_{0h,\text{ind}}$ is a contribution to the reaction of the conducting medium under the condition that the medium is non-uniform , i.e. $k_F \neq k_1$, but $I = 0$ (in other words, $\vec{A}''_{0h,\text{ind}}$ is the solution of problem (4.39) - (4.42) when $I = 0$, but $k_F \neq k_1$).

Similarly,

$\vec{A}'_{1h}$ is the solution of Eq. (4.40) at $k_F = k_1$, but $I \neq 0$;

$\vec{A}''_{1h}$ is the solution of Eq. (4.40) at $k_F \neq k_1$, but $I = 0$.

It follows from the boundary conditions (4.41) and Eqs. (4.43), (4.44) that the following equalities are to be satisfied:

$$z = 0: \quad \vec{A}_{0h} = \vec{A}_{1h}, \qquad \Rightarrow \quad \vec{A}_{0h}^{\text{abst}} + \vec{A}'_{0h,\text{ind}} + \vec{A}''_{0h,\text{ind}} = \vec{A}'_{1h} + \vec{A}''_{1h}; \tag{4.45}$$

$$z = 0: \quad \frac{\partial \vec{A}_{0h}}{\partial z} = \frac{\partial \vec{A}_{1h}}{\partial z}, \quad \Rightarrow \quad \frac{\partial \vec{A}_{0h}^{\text{absnt}}}{\partial z} + \frac{\partial \vec{A}'_{0h,\text{ind}}}{\partial z} + \frac{\partial \vec{A}''_{0h,\text{ind}}}{\partial z} = \frac{\partial \vec{A}'_{1h}}{\partial z} + \frac{\partial \vec{A}''_{1h}}{\partial z}. \tag{4.46}$$

Thus, it follows from Eqs. (4.45) and (4.46) that

$$z = 0: \quad \begin{cases} \vec{A}'_{1h} = \vec{A}_{0h}^{\text{absnt}} + \vec{A}'_{0h,\text{ind}}; & \vec{A}''_{1h} = \vec{A}''_{0h,\text{ind}}; \\ \dfrac{\partial \vec{A}'_{1h}}{\partial z} = \dfrac{\partial \vec{A}_{0h}^{\text{absnt}}}{\partial z} + \dfrac{\partial \vec{A}'_{0h,\text{ind}}}{\partial z}; & \dfrac{\partial \vec{A}''_{1h}}{\partial z} = \dfrac{\partial \vec{A}''_{0h,\text{ind}}}{\partial z}. \end{cases} \tag{4.47}$$

Since $\vec{A}_{0h}^{\text{absnt}}$ is the solution of the inhomogeneous Eq. (4.39), then the functions $\vec{A}'_{0h,\text{ind}}$ and $\vec{A}''_{0h,\text{ind}}$ must be the solutions of the corresponding homogeneous equations

$$\Delta \vec{A}'_{0h,\text{ind}} = 0, \qquad \Delta \vec{A}''_{0h,\text{ind}} = 0. \tag{4.48), (4.49}$$

Since $\vec{A}'_{1h}$ is the solution of the homogeneous Eq. (4.40) when $k_F = k_1$, then the function $\vec{A}'_{1h}$ satisfies the equation

$$\Delta \vec{A}'_{1h} + k_1^2 \vec{A}'_{1h} = 0. \tag{4.50}$$

In the case of problem (4.36)-(4.38) for a non-uniform medium, the change in impedance due to a flaw in the conducting medium has the form (see [6]):

$$Z^{\text{ind}} = \frac{j\omega}{I}\oint_L \vec{A}''_{0h,\text{ind}}(x,y,z)\,d\vec{l}\ . \tag{4.51}$$

Consider Green's formula

$$\iiint_V (u\,\Delta v - v\,\Delta u)\,dV = \oiint_S \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS\,, \tag{4.52}$$

where S is the closed surface bounding the region V, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$, $\vec{n}$ is the outer normal to surface S, and the functions $u, v, \Delta u, \Delta v$ are continuous in the closed region. It is easy to prove that Green's formula (4.52) takes place for the two vector functions

$$\vec{A} = A_x(M)\vec{e}_x + A_y(M)\vec{e}_y + A_z(M)\vec{e}_z \quad \text{and} \quad \vec{B} = B_x(M)\vec{e}_x + B_y(M)\vec{e}_y + B_z(M)\vec{e}_z$$

in Cartesian coordinates, and has the form

$$\iiint_V (\vec{A}\,\Delta\vec{B} - \vec{B}\,\Delta\vec{A})\,dV = \oiint_S \left(\vec{A}\frac{\partial\vec{B}}{\partial n} - \vec{B}\frac{\partial\vec{A}}{\partial n}\right) dS\,. \tag{4.53}$$

In order to prove Eq. (4.53), it is enough to write Eq. (4.52) for the three pairs of projections (A_x, B_x), (A_y, B_y), (A_z, B_z) and to sum the written results.

Now in order to evaluate a formula for Z^{ind} of Eq. (4.51), it is necessary to consider the contour of integration in detail.

I. Consider the region $z < 0$. Taking the scalar product of Eq. (4.40) with $\vec{A}'_{1h}$ and Eq. (4.50) with $\vec{A}_{1h}$, and subtracting the first product from the second one, one obtains that

a) $\vec{A}'_{1h}$ times Eq. (4.40) equals

$$\vec{A}'_{1h}\Delta\vec{A}_{1h} + k_1^2\vec{A}'_{1h}\vec{A}_{1h} = \begin{cases} 0, & (x,y,z)\notin V_F, \quad -\infty < z < 0, \\ (k_1^2 - k_F^2)\vec{A}'_{1h}\vec{A}_{Fh}, & (x,y,z)\in V_F, \quad -\infty < z < 0, \end{cases} \tag{4.54}$$

b) $\vec{A}_{1h}$ times Eq. (4.50) equals

$$\vec{A}_{1h}\Delta\vec{A}'_{1h} + k_1^2\vec{A}_{1h}\vec{A}'_{1h} = 0, \tag{4.55}$$

c) Eq. (4.55) minus Eq. (4.54):

$$\vec{A}_{1h}\Delta\vec{A}'_{1h} - \vec{A}'_{1h}\Delta\vec{A}_{1h} = -(k_1^2 - k_F^2)\vec{A}'_{1h}\vec{A}_{Fh}\,. \tag{4.56}$$

It follows from Eq. (4.56) by integrating over the region $z < 0$ that

$$\iiint\limits_{z<0} (\vec{A}_{1h}\Delta\vec{A}'_{1h} - \vec{A}'_{1h}\Delta\vec{A}_{1h})dV = -(k_1^2 - k_F^2)\iiint\limits_{V_F} \vec{A}'_{1h}\vec{A}_{Fh}\,dV\,, \tag{4.57}$$

and it follows from Eq. (4.57) by using Green's formula (4.53) that

$$\oiint\limits_{S} \left(\vec{A}'_{1h}\frac{\partial\vec{A}_{1h}}{\partial n} - \vec{A}_{1h}\frac{\partial\vec{A}'_{1h}}{\partial n}\right)dS = (k_1^2 - k_F^2)\iiint\limits_{V_F} \vec{A}_{Fh}\vec{A}'_{1h}\,dV\,, \tag{4.58}$$

where S is the closed surface of the integration (see Fig.4.4).

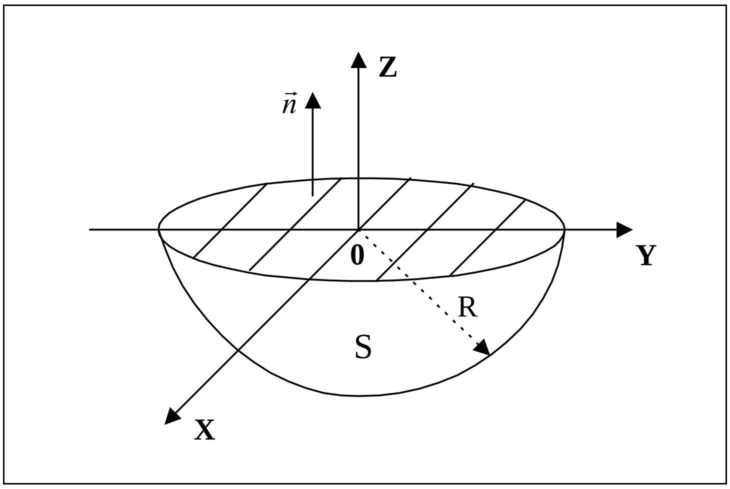

Fig. 4.4. Closed integration surface S ($z<0$)

Since $\vec{A}_{1h}$, $\vec{A}'_{1h} \to 0$ as $R^2 = x^2 + y^2 + z^2 \to \infty$, and $z<0$, then instead of the surface integral over the closed surface S (see Fig. 4.4.), only the double integral over the plane $z=0$ remains on the left-hand side of Eq. (4.58), and in the plane $z=0$:

$$\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}(\vec{A}'_{1h}\frac{\partial\vec{A}_{1h}}{\partial z} - \vec{A}_{1h}\frac{\partial\vec{A}'_{1h}}{\partial z})\bigg|_{z=0} dxdy = (k_1^2 - k_F^2)\iiint\limits_{V_F}\vec{A}_{Fh}\vec{A}'_{1h}\,dV\,. \tag{4.59}$$

Using the boundary conditions (4.47) and the decomposition (4.44), one can perform the transformations

$$z=0: \quad \vec{A}'_{1h}\frac{\partial\vec{A}_{1h}}{\partial z} - \vec{A}_{1h}\frac{\partial\vec{A}'_{1h}}{\partial z} = \vec{A}'_{1h}\frac{\partial}{\partial z}(\vec{A}'_{1h} + \vec{A}''_{1h}) - (\vec{A}'_{1h} + \vec{A}''_{1h})\frac{\partial\vec{A}'_{1h}}{\partial z}$$

$$= \vec{A}'_{1h}\frac{\partial\vec{A}''_{1h}}{\partial z} - \vec{A}''_{1h}\frac{\partial\vec{A}'_{1h}}{\partial z} = \vec{A}'_{0h}\frac{\partial\vec{A}''_{0h,\mathrm{ind}}}{\partial z} - \vec{A}''_{0h,\mathrm{ind}}\frac{\partial\vec{A}'_{0h}}{\partial z}\,, \tag{4.60}$$

where $\vec{A}'_{0h} = \vec{A}^{\mathrm{absnt}}_{0h} + \vec{A}'_{0h,\mathrm{ind}}$. Then it follows from Eqs. (4.59) and (4.60) that

$$\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} (\vec{A}'_{0h}\frac{\partial\vec{A}''_{0h,\mathrm{ind}}}{\partial z} - \vec{A}''_{0h}\frac{\partial\vec{A}'_{0h}}{\partial z})\bigg|_{z=0} dxdy = (k_1^2 - k_F^2)\iiint\limits_{V_F}\vec{A}_{Fh}\vec{A}'_{1h}\,dV\,. \tag{4.61}$$

II. Consider the region $z>0$. By a similar way, one can perform the same transformations with Eq. (4.39) for $\vec{A}_{0h}$ and with $\Delta\vec{A}''_{0h,\mathrm{ind}}=0$ in the region $z>0$. Taking the scalar product of Eq. (4.39) with $\vec{A}''_{0h,\mathrm{ind}}$ and $\Delta\vec{A}''_{0h,\mathrm{ind}}=0$ with $\vec{A}_{0h}$, and subtracting the first product from the second one, we obtain

$$\vec{A}_{0h}\Delta\vec{A}''_{0h,\mathrm{ind}}-\vec{A}''_{0h,\mathrm{ind}}\Delta\vec{A}_{0h}=\mu_0\vec{I}^{\,e}\vec{A}'_{0h,\mathrm{ind}}\,. \tag{4.62}$$

It follows from Eq. (4.62) by integrating over the region $z>0$ that

$$\iiint\limits_{z>0}(\vec{A}_{0h}\Delta\vec{A}''_{0h,\mathrm{ind}}-\vec{A}''_{0h,\mathrm{ind}}\Delta\vec{A}_{0h})dV=\mu_0\iiint\limits_{z>0}\vec{I}^{\,e}\vec{A}''_{0h,\mathrm{ind}}dV\,, \tag{4.63}$$

and it follows from Eq. (4.63) by using Green's formula (4.53) that

$$\oiint\limits_{S}(\vec{A}_{0h}\frac{\partial\vec{A}''_{0h,\mathrm{ind}}}{\partial n}-\vec{A}''_{0h,\mathrm{ind}}\frac{\partial\vec{A}_{0h}}{\partial n})dS=\mu_0\iiint\limits_{z>0}\vec{I}^{\,e}\vec{A}''_{0h,\mathrm{ind}}\,dV\,, \tag{4.64}$$

where S is the closed surface of integration (see Fig.4.5).

Since $\vec{A}_{0h}$, $\vec{A}''_{0h,\mathrm{ind}}\to 0$ as $R^2=x^2+y^2+z^2\to\infty$, and $z>0$, then instead of the surface integral over the closed surface S, only the double integral over the plane $z=0$ remains on the left-hand side of Eq. (4.64), and in the plane $z=0$:

$$-\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}(\vec{A}_{0h}\frac{\partial\vec{A}''_{0h,\mathrm{ind}}}{\partial z}-\vec{A}''_{0h,\mathrm{ind}}\frac{\partial\vec{A}_{0h}}{\partial z})\bigg|_{z=0}dxdy=\mu_0\iiint\limits_{z>0}\vec{I}^{\,e}\vec{A}''_{0h,\mathrm{ind}}\,dV\,. \tag{4.65}$$

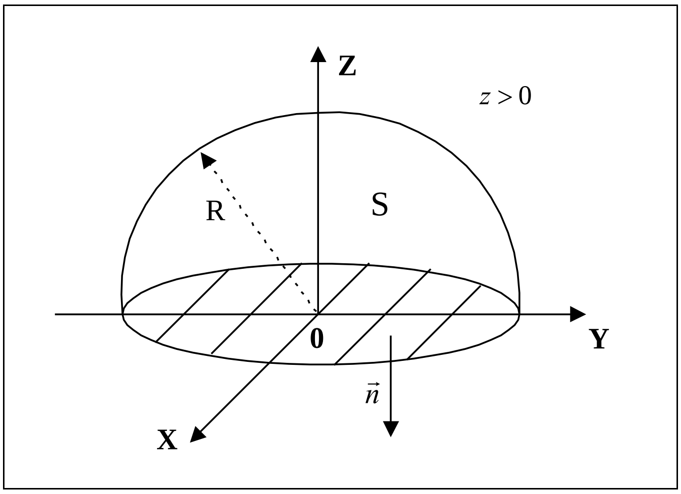

Fig. 4.5. Closed integration surface S ($z>0$)

The minus sign on the left-hand side of Eq. (4.65) comes from the fact that the outer normal to the surface $z=0$ in region $z>0$ is opposite to the direction of the z-axis.

Using the decomposition $\vec{A}_{0h} = \vec{A}_{0h}^{\text{absnt}} + \vec{A}'_{0h,\text{ind}} + \vec{A}''_{0h,\text{ind}}$, we perform the transformations

$$z=0:\ \vec{A}_{0h}\frac{\partial \vec{A}''_{0h,\text{ind}}}{\partial z} - \vec{A}''_{0h,\text{ind}}\frac{\partial \vec{A}_{0h}}{\partial z} = (\vec{A}_{0h}^{\text{absnt}} + \vec{A}'_{0h,\text{ind}} + \vec{A}''_{0h,\text{ind}})\frac{\partial \vec{A}''_{0h,\text{ind}}}{\partial z}$$

$$-\vec{A}''_{0h,\text{ind}}\frac{\partial}{\partial z}(\vec{A}_{0h}^{\text{absnt}} + \vec{A}'_{0h,\text{ind}} + \vec{A}''_{0h,\text{ind}}) = \vec{A}'_{0h}\frac{\partial \vec{A}''_{0h,\text{ind}}}{\partial z} - \vec{A}''_{0h,\text{ind}}\frac{\partial \vec{A}'_{0h}}{\partial z}, \tag{4.66}$$

where $\vec{A}'_{0h} = \vec{A}_{0h}^{\text{absnt}} + \vec{A}'_{0h,\text{ind}}$. Then it follows from Eq. (4.65), by using Eqs. (4.66) and (4.33), that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left.(\vec{A}'_{0h}\frac{\partial \vec{A}''_{0h,\text{ind}}}{\partial z} - \vec{A}''_{0h}\frac{\partial \vec{A}'_{0h}}{\partial z})\right|_{z=0} dxdy = -\mu_0 I \iiint_{z>0} \delta[\rho-\rho(\varphi)]\delta(z-h)\,\vec{e}_\tau \vec{A}''_{0h,\text{ind}}\,dV\,. \tag{4.67}$$

Transforming the right-hand side of Eq. (4.67) by using the main property of the delta function, we have

$$\iiint_{z>0} \delta[\rho-\rho(\varphi)]\delta(z-h)\,\vec{e}_\tau \vec{A}''_{0h,\text{ind}} dxdydz = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left.\delta[\rho-\rho(\varphi)]\vec{e}_\tau \vec{A}''_{0h,\text{ind}}\right|_{z=h} dxdy\,. \tag{4.68}$$

It follows from Eq. (4.68), passing to polar cylindrical coordinates by substituting $x=\rho\cos\varphi$, $y=\rho\sin\varphi$, $dxdy = \rho\, d\rho d\varphi$, that

$$F \equiv \int_0^{2\pi} d\varphi \int_0^{\infty} \left.\delta[\rho-\rho(\varphi)]\vec{e}_\tau \vec{A}''_{0h,\text{ind}}\right|_{z=h} \rho\, d\rho = \int_0^{2\pi} \vec{A}''_{0h,\text{ind}}(\rho(\varphi),h)\ \vec{e}_\tau\ \rho(\varphi) d\varphi\,. \tag{4.69}$$

But $\vec{e}_\tau \rho(\varphi)d\varphi = \vec{e}_\tau dl = d\vec{l}$. Consequently, it follows from Eqs. (4.69) and (4.51) that

$$F = \oint_L \vec{A}''_{0h,\text{ind}}(\rho(\varphi),h)\,d\vec{l} = \frac{I}{j\omega} Z^{\text{ind}}\,. \tag{4.70}$$

Since the left-hand sides of Eqs. (4.61) and (4.67) are equal, then the right-hand sides must be equal as well,

$$-\mu_0 I^2 Z^{\text{ind}} (j\omega)^{-1} = (k_1^2 - k_F^2)\iiint_{V_F} \vec{A}_{Fh}\vec{A}'_{1h}\,dV\,. \tag{4.71}$$

Since $k_1^2 = j\omega\sigma_1\mu_0$ and $k_F^2 = j\omega\sigma_F\mu_0$, it follows from Eq. (4.71) that

$$-I^2 \mu_0 Z^{\text{ind}} (j\omega)^{-1} = j\omega\mu_0(\sigma_1 - \sigma_F)\iiint_{V_F} \vec{A}_{Fh}\vec{A}'_{1h}\,dV\,. \tag{4.72}$$

It follows from Eq. (4.72) that

$$Z^{\text{ind}} = \frac{\omega^2(\sigma_F - \sigma_1)}{I^2}\iiint_{V_F} \vec{A}_{Fh}\vec{A}'_{1h}\,dV\,, \tag{4.73}$$

i.e. the obtained formula is similar to formula (4.24).

4.2.3. Formula for impedance change known in the literature

In the literature (see [50], [20]) formula (4.23) for induced changes in impedance, describing the influence of a conducting medium with a flaw of arbitrary shape on a source of current, seems to have been used without any strict proof before. In particular, it is usually obtained without describing the source of external current. This fact creates difficulties for estimating the degree of mathematical basis of the formula. In the present thesis (see also author's paper [19], [37]), this formula is analytically proved and its correctness is verified. The proof is performed taking into account the displacement current.

Let an emitter be located on a closed curve described in parametric form in polar cylindrical coordinates (ρ,φ,z) by the equation:

$$\begin{cases}\rho=\rho(\varphi),\\ z=z(\varphi),\end{cases} \quad 0\le\varphi\le 2\pi , \tag{4.74}$$

where $\rho(\varphi), z(\varphi)$ are prescribed functions and φ is a parameter. An equation describing any closed curve can be written by using Eq. (4.74) and choosing the appropriate system of rectangular coordinates (x,y,z).

Consider a sphere S_R of radius R with an interior closed surface S of arbitrary form (see Fig.4.6). The surface S covers a region V_{coil} containing a single-turn coil, and a region V of the conducting medium. A closed surface S_{coil} bounds the region V_{coil} containing only a single-turn coil. A closed surface S_V bounds the region V containing a conducting medium with conductivity $\sigma=\text{const}$ and the relative permittivity $\hat{\varepsilon}=\text{const}$, and a region V_F with conductivity $\sigma_F=\text{const}$ and the relative permittivity $\hat{\varepsilon}_F=\text{const}$. The region V_F is bounded by a closed surface S_F. Finally, $\tilde{V}$ is a region bounded by the surfaces S and S_R, and $\tilde{\tilde{V}}$ is a region bounded by the surfaces S, S_{coil} and S_V.

In the case of harmonic oscillations of the external current with frequency ω in the closed coil, Maxwell's equations for the complex-valued amplitude electric field vector $\vec{E}$ and the complex-valued amplitude magnetic field vector $\vec{H}$ have the form (see [6]):

$$\text{curl}\,\vec{E}=-j\omega\mu_0\mu\vec{H}\ , \tag{4.75}$$

$$\text{curl}\,\vec{H}=(\sigma+j\varepsilon_0\hat{\varepsilon}\omega)\vec{E}+\vec{I}^{\,e}\ . \tag{4.76}$$

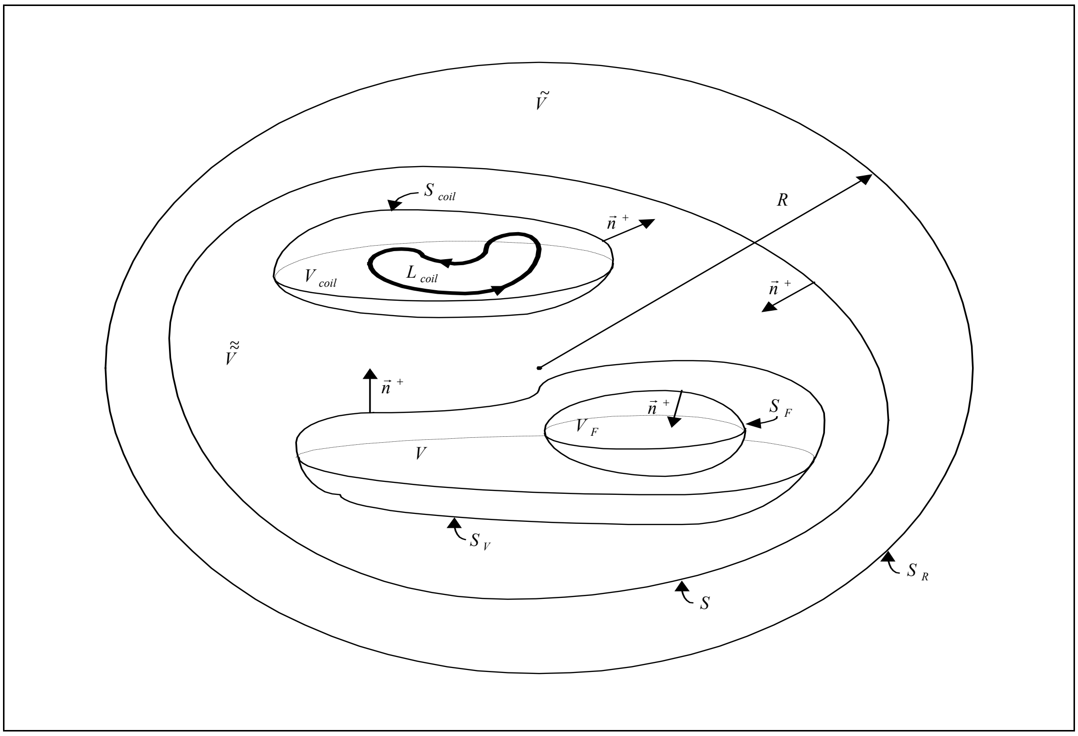

Fig.4.6. The disposition of the regions and closed surfaces

According to Eq. (4.74), one can write

$$\vec{I}^e = I h(\rho,\varphi)\delta[\rho-\rho(\varphi)]\delta[z-z(\varphi)]\vec{e}_\tau, \tag{4.77}$$

where $\vec{e}_\tau$ is a unit vector of the tangent to the curve given by Eq. (4.74), I is the complex-valued amplitude current vector density. The coefficient $h(\rho,\varphi)$ in Eq. (4.77) has the form:

$$h(\rho,\varphi) = \frac{1}{\rho}\sqrt{\rho^2 + [\rho'(\varphi)]^2 + [z'(\varphi)]^2}\,. \tag{4.78}$$

The coefficient $h(\rho,\varphi)$ is chosen so that the triple integral of $\left|\vec{I}^e\right|$ over the whole space is equal to the following constant:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left|\vec{I}^e\right| dV = I L_{\text{coil}} = \sigma_{\text{coil}}\text{E}_{\text{coil}}L_{\text{coil}}, \tag{4.79}$$

where σ_{coil} is the conductivity of the coil, L_{coil} is the length of the closed contour given by Eq. (4.74) with the current density $I = \text{const}$, $\text{E}_{\text{coil}}L_{\text{coil}}$ is the electromotive force that is necessary for supporting the current of density $I = \text{const}$ in this closed contour. It follows from Eq. (4.74) that the contour's length, L_{coil}, is equal to

$$L_{\text{coil}} = \int_0^{2\pi} \sqrt{[\rho(\varphi)]^2 + [\rho'(\varphi)]^2 + [z'(\varphi)]^2}\,d\varphi, \tag{4.80}$$

assuming that the origin of the coordinate system is located inside the coil. In order to prove Eq. (4.79), we substitute $\left|\vec{I}^e\right|$ given by Eq. (4.77) into the integral of Eq. (4.79). Using the main property of the delta function and Eq. (4.80), we obtain

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left|\vec{I}^e\right|dxdydz = I\int_0^{2\pi}d\varphi\int_0^{\infty}\frac{1}{\rho}\sqrt{\rho^2+[\rho'(\varphi)]^2+[z'(\varphi)]^2}\,\rho\,d\rho\times$$

$$\times\int_{-\infty}^{+\infty}\delta[\rho-\rho(\varphi)]\delta[z-z(\varphi)]dz = I\int_0^{2\pi}\sqrt{\rho^2+[\rho'(\varphi)]^2+[z'(\varphi)]^2}\,d\varphi =$$

$$= I L_{\text{coil}} = \sigma_{\text{coil}}\mathrm{E}_{\text{coil}}L_{\text{coil}}\,. \tag{4.81}$$

Hence, formula (4.79) is proved.

Consider the system of Eqs. (4.75)-(4.76) for the following two cases: for the case when the flaw is absent, i.e. $\sigma_F=\sigma$ in the region V_F (by substituting $\vec{E}=\vec{E}_{\text{abs}}$, $\vec{H}=\vec{H}_{\text{abs}}$), and in the presence of the flaw (by substituting $\vec{E}=\vec{E}_F$, $\vec{H}=\vec{H}_F$). Then assuming that the external current vector density $\vec{I}^e$ is the same in both cases and is defined by Eq. (4.77), one obtains

$$-\operatorname{curl}\vec{E}_{\text{abs}} = j\omega\mu_0\mu\vec{H}_{\text{abs}}\,, \tag{4.82}$$

$$\operatorname{curl}\vec{H}_{\text{abs}} = \tilde{k}^2_{\text{abs}}\vec{E}_{\text{abs}}+\vec{I}^e\,, \tag{4.83}$$

$$-\operatorname{curl}\vec{E}_F = j\omega\mu_0\mu\vec{H}_F\,, \tag{4.84}$$

$$\operatorname{curl}\vec{H}_F = \tilde{k}^2_F\vec{E}_F+\vec{I}^e\,, \tag{4.85}$$

where

$$\tilde{k}^2_{\text{abs}} = \begin{cases}\sigma + j\varepsilon_0\hat{\varepsilon}\omega, & M(x,y,z)\in V,\\ j\varepsilon_0\hat{\varepsilon}\omega, & M(x,y,z)\notin V,\end{cases} \tag{4.86}$$

$$\tilde{k}^2_F = \begin{cases}\sigma_F + j\varepsilon_0\hat{\varepsilon}_F\omega, & M(x,y,z)\in V_F,\\ \sigma + j\varepsilon_0\hat{\varepsilon}\omega, & M(x,y,z)\notin V_F.\end{cases} \tag{4.87}$$

In the above, $\vec{E}_{\text{abs}}$ and $\vec{H}_{\text{abs}}$ are the solutions of Eqs. (4.82)-(4.83) such that:

1) the tangent components of the vectors $\vec{E}_{\text{abs}}$ and $\vec{H}_{\text{abs}}$ are continuous on the surface S_V (see [6]);

2) the vectors $\vec{E}_{\text{abs}}$ and $\vec{H}_{\text{abs}}$ satisfy the radiation condition at infinity (see [65]).

Similarly, $\vec{E}_F$ and $\vec{H}_F$ are the solutions of Eqs. (4.84)-(4.85) such that:

1) the tangent components of the vectors $\vec{E}_F$ and $\vec{H}_F$ are continuous on the surfaces S_F and S_V;

2) the vectors $\vec{E}_F$ and $\vec{H}_F$ satisfy the radiation condition at infinity.

In order to prove Eq. (4.23) we use Lorentz' reciprocity theorem (see [6]). Taking the scalar product of Eq. (4.83) with $\vec{E}_F$ and of Eq. (4.84) with $\vec{H}_{\text{abs}}$, and summing both products, one obtains

$$\vec{E}_F \cdot \operatorname{curl} \vec{H}_{\text{abs}} - \vec{H}_{\text{abs}} \cdot \operatorname{curl} \vec{E}_F = \tilde{k}^2_{\text{abs}}\, \vec{E}_{\text{abs}} \cdot \vec{E}_F + \vec{I}^{\,e} \cdot \vec{E}_F + j\omega\mu_0\mu\, \vec{H}_{\text{abs}} \cdot \vec{H}_F\,. \tag{4.88}$$

From

$$\operatorname{div}(\vec{E}_F \times \vec{H}_{\text{abs}}) = \vec{H}_{\text{abs}} \cdot \operatorname{curl} \vec{E}_F - \vec{E}_F \cdot \operatorname{curl} \vec{H}_{\text{abs}}\,, \tag{4.89}$$

and Eq. (4.84), it follows

$$-\operatorname{div}(\vec{E}_F \times \vec{H}_{\text{abs}}) = \tilde{k}^2_{\text{abs}}\, \vec{E}_{\text{abs}} \cdot \vec{E}_F + \vec{I}^{\,e} \cdot \vec{E}_F + j\omega\mu_0\mu\, \vec{H}_{\text{abs}} \cdot \vec{H}_F\,. \tag{4.90}$$

Interchanging the subscripts abs and F in Eq. (4.90) (i.e. doing the similar operations with Eqs. (4.82) and (4.85)), we obtain

$$-\operatorname{div}(\vec{E}_{\text{abs}} \times \vec{H}_F) = \tilde{k}^2_F\, \vec{E}_F \cdot \vec{E}_{\text{abs}} + \vec{I}^{\,e} \cdot \vec{E}_{\text{abs}} + j\omega\mu_0\mu\, \vec{H}_F \cdot \vec{H}_{\text{abs}}\,. \tag{4.91}$$

Subtracting Eq. (4.90) from Eq. (4.91) yields

$$\operatorname{div}(\vec{E}_F \times \vec{H}_{\text{abs}} - \vec{E}_{\text{abs}} \times \vec{H}_F) = (\tilde{k}^2_F - \tilde{k}^2_{\text{abs}})\vec{E}_{\text{abs}} \cdot \vec{E}_F - \vec{I}^{\,e} \cdot (\vec{E}_F - \vec{E}_{\text{abs}})\,. \tag{4.92}$$

I. Integrating Eq. (4.92) over the region $\tilde{V}$ bounded by the closed surfaces S_R and S yields

$$\iiint\limits_{\tilde{V}} \operatorname{div}(\vec{E}_F \times \vec{H}_{\text{abs}} - \vec{E}_{\text{abs}} \times \vec{H}_F)\, dV = (\tilde{k}^2_F - \tilde{k}^2_{\text{abs}}) \iiint\limits_{\tilde{V}} \vec{E}_{\text{abs}} \cdot \vec{E}_F\, dV - \iiint\limits_{\tilde{V}} \vec{I}^{\,e} \cdot (\vec{E}_F - \vec{E}_{\text{abs}})\, dV\,. \tag{4.93}$$

Since $\tilde{k}^2_{\text{abs}} - \tilde{k}^2_F = 0$ and $\vec{I}^{\,e} = 0$ in the region $\tilde{V}$ (see Eqs. (4.77), (4.86), (4.87)), the right-hand side of Eq. (4.93) is equal to zero in the region $\tilde{V}$. The left-hand side is transformed using the Gauss' divergence theorem and taking into account that the boundary of the region $\tilde{V}$ consists of two closed surfaces S_R and S (see Fig.5.6). As a result, we obtain

$$\left[\oiint\limits_{S_R} + \oiint\limits_{S} \right] (\vec{E}_F \times \vec{H}_{\text{abs}} - \vec{E}_{\text{abs}} \times \vec{H}_F) \cdot \vec{n}^{+} dS = 0\,, \tag{4.94}$$

where $\vec{n}^{+}$ is a unit vector of the external normal to the boundary of region $\tilde{V}$. We assume that the integrand in Eq. (4.92) tends to zero faster than R^{-2} as $R \to \infty$. Since the surface S_R is a sphere of radius R, we have

$$\lim_{R\to\infty} \oiint\limits_{S_R} (\vec{E}_F \times \vec{H}_{\text{abs}} - \vec{E}_{\text{abs}} \times \vec{H}_F) \cdot \vec{n}^{+} dS = 0\,. \tag{4.95}$$

Thus, it follows from Eq. (4.95) that

$$\oiint_S \vec{R}\cdot\vec{n}^{+}dS = 0\,, \tag{4.96}$$

where

$$\vec{R} = \vec{E}_F \times \vec{H}_{\text{abs}} - \vec{E}_{\text{abs}} \times \vec{H}_F\,. \tag{4.97}$$

II. Integrating Eq. (4.92) over the region $\tilde{\tilde{V}}$ bounded by the three closed surfaces S, S_{coil} and S_V, using Gauss' divergence theorem and taking into account that in the region $\tilde{\tilde{V}}$ the right-hand side of Eq. (4.92) is equal to zero, we obtain

$$\Big(\oiint_S + \oiint_{S_{\text{coil}}} + \oiint_{S_V}\Big)\vec{R}\cdot\vec{n}^{-}dS = 0\,, \tag{4.98}$$

where $\vec{n}^{-} = -\vec{n}^{+}$ is a unit vector of the external normal to the boundary of region $\tilde{\tilde{V}}$. It follows from Eqs. (4.96) and (4.98) that

$$\oiint_{S_{\text{coil}}}\vec{R}\cdot\vec{n}^{-}dS = -\oiint_{S_V}\vec{R}\cdot\vec{n}^{-}dS = \oiint_{S_V}\vec{R}\cdot\vec{n}^{+}dS\,. \tag{4.99}$$

III. Integrating Eq. (4.92) over the region V_{coil} bounded by the closed surface S_{coil}, then using Gauss' divergence theorem and taking into account that in this region $\vec{I}^{e} \neq 0$ and $\vec{I}^{e}$ is defined by Eq. (4.77), one gets

$$\oiint_{S_{\text{coil}}}\vec{R}\cdot\vec{n}^{+}dS = -I\iiint_{V_{\text{coil}}} \delta[\rho-\rho(\varphi)]\delta[z-z(\varphi)]\; h(\rho,\varphi)\,\vec{e}_{\tau}\cdot\vec{E}^{\text{ind}}dV\,, \tag{4.100}$$

where

$$\vec{E}^{\text{ind}} = \vec{E}_F - \vec{E}_{\text{abs}}\,. \tag{4.101}$$

Using the main property of the delta function, the right-hand side of Eq. (4.100) is transformed as

$$\iiint_{V_{\text{coil}}} \delta[\rho-\rho(\varphi)]\delta[z-z(\varphi)]h(\rho,\varphi)\,\vec{e}_{\tau}\cdot\vec{E}^{\text{ind}}dxdydz$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \delta[\rho-\rho(\varphi)]\; h(\rho,\varphi)\,\vec{e}_{\tau}\cdot\vec{E}^{\text{ind}}\Big|_{z=z(\varphi)} dxdy\,, \tag{4.102}$$

where

$$h(\rho,\varphi) = \frac{1}{\rho}\sqrt{\rho^2 + [\rho'(\varphi)]^2 + [z'(\varphi)]^2}\,.$$

Introducing the polar cylindrical coordinates $x = \rho\cos\varphi$, $y = \rho\sin\varphi$, $dxdy = \rho\, d\rho d\varphi$ in Eq. (4.102) yields

$$F \equiv \int_0^{2\pi} d\varphi \int_0^{\infty} \delta[\rho - \rho(\varphi)]\, h(\rho,\varphi)\, \vec{e}_\tau \cdot \vec{E}^{\text{ind}}\Big|_{z=z(\varphi)} \rho\, d\rho$$

$$= \int_0^{2\pi} \vec{E}^{\text{ind}}(\rho(\varphi), z(\varphi))\, h(\rho(\varphi),\varphi)\, \vec{e}_\tau\, \rho(\varphi)\, d\varphi\,. \qquad (4.103)$$

However,

$$\vec{e}_\tau h(\rho(\varphi),\varphi)\rho(\varphi)d\varphi = \vec{e}_\tau dl = d\vec{l}\ , \qquad (4.104)$$

where $d\vec{l}$ is a vector such that its module is equal to the differential of the length of the line arc and it is directed along the tangent to this curve. Thus, it follows from Eq. (4.103) that

$$F = \oint_{L_{\text{coil}}} \vec{E}^{\text{ind}} \cdot d\vec{l} = -Z^{\text{ind}} I\,, \qquad (4.105)$$

where Z^{ind} is the change in impedance due to a flaw situated in the region V_F (see [6]). Consequently, Eq. (4.100) has the following form

$$\oiint_{S_{\text{coil}}} \vec{R} \cdot \vec{n}^+ dS = I^2 Z^{\text{ind}}\,. \qquad (4.106)$$

IV. Integrating Eq. (4.92) over the region V bounded by the two closed surfaces S_V and S_F, using Gauss' divergence theorem and taking into account that the right-hand side of Eq. (4.92) is equal to zero in this region, we obtain

$$\oiint_{S_V} \vec{R} \cdot \vec{n}^+ dS + \oiint_{S_F} \vec{R} \cdot \vec{n}^+ dS = 0\,. \qquad (4.107)$$

It follows from Eqs. (4.99), (4.106) and (4.107) that

$$-I^2 Z^{\text{ind}} = \oiint_{S_V} \vec{R} \cdot \vec{n}^+ dS = -\oiint_{S_F} \vec{R} \cdot \vec{n}^+ dS\,. \qquad (4.108)$$

V. Finally, integrating Eq. (4.92) over the region V_F, using Gauss' divergence theorem and taking into account that $\vec{I}^e = 0$ and $\tilde{k}_F^2 - \tilde{k}_{\text{abs}}^2 = \sigma_F - \sigma + j\omega\varepsilon_0(\hat{\varepsilon}_F - \hat{\varepsilon})$ in this region, we obtain

$$-\oiint_{S_F} \vec{R} \cdot \vec{n}^+ dS = [(\sigma_F - \sigma) + j\omega\varepsilon_0(\hat{\varepsilon}_F - \hat{\varepsilon})] \iiint_{V_F} \vec{E}_{\text{abs}} \cdot \vec{E}_F\, dV\,. \qquad (4.109)$$

The final formula follows from Eqs. (4.108) and (4.109):

$$Z^{\text{ind}} = -\frac{1}{I^2}[(\sigma_F - \sigma) + j\omega\varepsilon_0(\hat{\varepsilon}_F - \hat{\varepsilon})] \iiint_{V_F} \vec{E}_{\text{abs}} \cdot \vec{E}_F \, dV \,. \tag{4.110}$$

Eq. (4.110) gives the formula for calculating the induced impedance change in the case when the displacement current is taken into account. If the displacement current is neglected, one can obtain the corresponding Eq. (4.23) from Eq. (4.110) by the simple substitution ($\hat{\varepsilon}_F$ - $\hat{\varepsilon}$) equal to zero.

4.2.4. Proof of the equivalence of the two formulas.

Let us consider two arbitrary functions $u(M) = u(x, y, z)$ and $v(M) = v(x, y, z)$ that are continuous together with their second derivatives in the region V bounded by some closed surfaces $S_1, S_2, \ldots, S_m$. The Green's formula for these functions has the form

$$\iiint_V (u\Delta v - v\Delta u) dV = \left(\oiint_{S_1} + \oiint_{S_2} + \cdots + \oiint_{S_m} \right) \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) dS \,, \tag{4.111}$$

where $\vec{n}$ is the external normal vector to the region V. Formula (4.111) is also valid for the two vector functions $\vec{u}(M)$ and $\vec{v}(M)$.

Let $\vec{A}_{\text{abs}}(M)$ be the vector potential in the absence of the flaw, and $\vec{A}_F(M)$ be the vector potential in the presence of the flaw. The vectors $\vec{A}_{\text{abs}}$ and $\vec{A}_F$ satisfy the following equations (see Eq. (1.26) and [6]):

$$\Delta\vec{A}_{\text{abs}} + k_{\text{abs}}^2 \vec{A}_{\text{abs}} = -\mu_0 \mu \vec{I}^{\,e} \,, \tag{4.112}$$

$$\Delta\vec{A}_F + k_F^2 \vec{A}_F = -\mu_0 \mu \vec{I}^{\,e} \,, \tag{4.113}$$

where

$$k_{\text{abs}}^2 = \begin{cases} -j\omega\mu_0\mu(\sigma + j\omega\varepsilon_0\hat{\varepsilon}), & M(x, y, z) \in V, \\ \omega^2\mu_0\mu\varepsilon_0\hat{\varepsilon}, & M(x, y, z) \notin V, \end{cases} \tag{4.114}$$

$$k_F^2 = \begin{cases} -j\omega\mu_0\mu(\sigma_F + j\omega\varepsilon_0\hat{\varepsilon}_F), & M(x, y, z) \in V_F, \\ -j\omega\mu_0\mu(\sigma + j\omega\varepsilon_0\hat{\varepsilon}, & M(x, y, z) \notin V_F,\ M(x, y, z) \in V. \end{cases} \tag{4.115}$$

and $\vec{I}^{\,e}$ is defined by Eq. (4.77).

Green's formula (4.111) can be rewritten for the vectors $\vec{A}_{\text{abs}}$ and $\vec{A}_F$ in the region $\tilde{V}$ bounded by the closed surfaces S_R and S in the form (see Fig.4.6):

$$\iiint_{\tilde{V}} (\vec{A}_{\text{abs}} \Delta \vec{A}_F - \vec{A}_F \Delta \vec{A}_{\text{abs}})\, dV = \left(\oiint_{S_R} + \oiint_{S} \right) \left(\vec{A}_{\text{abs}} \frac{\partial \vec{A}_F}{\partial n} - \vec{A}_F \frac{\partial \vec{A}_{\text{abs}}}{\partial n} \right) dS \,. \tag{4.116}$$

Substituting $\Delta \vec{A}_{\text{abs}} = -k_{\text{abs}}^2 \vec{A}_{\text{abs}} - \mu_0 \mu \vec{I}^{\,e}$ of Eq. (4.112) and $\Delta \vec{A}_F = -k_F^2 \vec{A}_F - \mu_0 \mu \vec{I}^{\,e}$ of Eq. (4.113) into the left-hand side of Eq. (4.114), the integrand can rewritten as

$$\vec{A}_{\text{abs}} \Delta \vec{A}_F - \vec{A}_F \Delta \vec{A}_{\text{abs}} = (k_{\text{abs}}^2 - k_F^2) \vec{A}_{\text{abs}} \vec{A}_F + \mu_0 \mu \vec{I}^{\,e} (\vec{A}_F - \vec{A}_{\text{abs}}) \,. \tag{4.117}$$

Since the external current $\vec{I}^{\,e} = 0$ and $k_{\text{abs}}^2 = k_F^2$ in the region $\tilde{V}$, one can see that the right-hand side of Eq. (4.117) is equal to zero. Hence the left-hand side of Eq. (4.116) is also equal to zero. In the limit as $R \to \infty$, the integral over S_R tends to zero. Consequently, it follows from Eq. (4.116) that

$$\oiint_{S} \left(\vec{A}_{\text{abs}} \frac{\partial \vec{A}_F}{\partial n} - \vec{A}_F \frac{\partial \vec{A}_{\text{abs}}}{\partial n} \right) dS = 0 \text{ as } R \to \infty \,. \tag{4.118}$$

Formula (4.118) is completely equivalent to formula (4.96). Therefore, the rest of the proof of formula (4.24) is completely similar to the one of formula (4.23). Consequently,

$$Z^{\text{ind}} = \frac{\omega^2}{I^2} [\sigma_F - \sigma + j\omega\varepsilon_0 (\hat{\varepsilon}_F - \hat{\varepsilon})] \iiint_{V_F} \vec{A}_{\text{abs}} \cdot \vec{A}_F \, dV \,. \tag{4.119}$$

Eq. (4.119) gives the new formula for the induced change calculation when the displacement current is taken into account. Besides, it is to be noted that the vectors $\vec{E}_{\text{abs}}$, $\vec{H}_{\text{abs}}$, $\vec{E}_F$ and $\vec{H}_F$ are expressed in terms of the vectors $\vec{A}_{\text{abs}}$ and $\vec{A}_F$ by using the following expressions (see [6]):

$$\text{curl}\, \vec{A}_{\text{abs}} = \mu_0 \mu \vec{H}_{\text{abs}} \,, \qquad \vec{E}_{\text{abs}} = -j\omega \vec{A}_{\text{abs}} + \frac{1}{\mu_0 \mu} \frac{1}{\tilde{k}_{\text{abs}}^2} grad\, div \tilde{A}_{\text{abs}} \,, \tag{4.120}$$

$$\text{curl}\, \vec{A}_F = \mu_0 \mu \vec{H}_F \,, \qquad \vec{E}_F = -j\omega \vec{A}_F + \frac{1}{\mu_0 \mu} \frac{1}{\tilde{k}_F^2} \text{graddiv}\, \tilde{A}_F \,, \tag{4.121}$$

where the coefficients $\tilde{k}_{\text{abs}}^2$ and $\tilde{k}_F^2$ are given by Eqs. (4.86) and (4.87).

CONCLUSIONS

The present thesis is a theoretical work dealing with problems of non-destructive testing by eddy current methods. Experimental investigations in this field are costly and time-consuming so that theoretical studies can be an attractive alternative. This thesis presents methods to solve some eddy current testing problems and ways to simplify the obtained solutions and adapt them to computational calculations in engineering. The forward problems are solved: mathematical models of eddy current testing are constructed and the influence of the parameters of the flaw and the media on the input signal of eddy current probes is investigated.

In the present thesis the discussion on non-destructive testing problems starts from a detailed review of the literature devoted to this subject and mostly of recent papers.

The thesis is divided into four chapters. The first chapter is introductory. It describes the physics of the method, introduces the meaning of vector potential and impedance change, and gives the main equations and characteristics. Besides, the four mostly used boundary value problems, giving the form and non-zero components of vector potentials for separate cases, are strictly proved corresponding to the geometry of a source of current.

The second chapter is devoted to Helmholtz' vector equation and its solution, describing the problems of eddy current testing as an influence of a conducting medium on a source of current. In this thesis the integral representation of the solution to Helmholtz' vector equation is considered not only in the well-known form of the Cartesian coordinates, but it is also obtained for arbitrary orthogonal curvilinear coordinates. As particular cases, the integral representation of the solution to Helmholtz' vector equation is derived for cylindrical polar and spherical coordinates. Besides, in Chapter 2 the newly obtained representations of the solution to Helmholtz' vector equation are used for solving the problems of electromagnetic waves spreading from emitters of different forms. These are the vector potential problems of a rectangular frame with current and of a wire of arbitrary form with given current. In the present thesis, the problem of a rectangular frame with current is solved without using the dipole approximation, which is widely used for problems of electromagnetic waves spreading from linear emitters, but not suitable for problems of eddy current inspection. To solve the problem of a finite length wire of an arbitrary form, the integral representation of the solution to Helmholtz' vector equation is found in the form of a single definite integral of an elementary function. The obtained solution is used for deriving the solution for its particular cases of electromagnetic waves spreading from a wire in the form of an Archimedes's spiral, an elliptical or circular helix and, also, from a fractal form wire.

The third chapter is devoted to the problems of influence of the homogeneous conducting medium on a source of current of different geometries. Analytical solutions to eddy current testing problems even in simple geometries are expressed in terms of improper integrals containing special functions. Hence, the effective evaluation of such integrals is an important practical problem. In Chapter 3 several new improper integrals are evaluated in closed form. The results are used to calculate the change in impedance of a double conductor line and a single-turn coil above a conducting half-space. An asymptotic formula for large frequencies is also obtained in Chapter 3. Moreover, in this chapter the impedance change is obtained for the problem of a rectangular frame located inside a conducting cylindrical tube. Moreover, in this thesis the exact analytical solution of this problem is obtained without using either the double conductor line approximation or any other approximation.

The last chapter considers the problems of impedance change due to the influence of conducting media containing flaws (or defects) of arbitrary shapes. Since the exact analytical solution for the problem of the influence of a conducting medium with an arbitrary flaw on a source of current is not known, different approximate analytical and numerical methods have been developed and used. In the present thesis two methods, called methods of additional currents, are developed. One of these methods transforms the problem (differential equations) for a non-uniform conducting medium into a problem for a uniform conducting medium with a non-uniform right-hand side in the system of differential equations describing the problem. Besides, Chapter 4 is mostly devoted to the basic analytical formula of impedance change for non-uniform media. In the literature this well-known formula is based on Lorentz' theorem, but rigorous proof seems to be absent. In this thesis the formula is analytically proved and its correctness is analyzed. Moreover, the similar formula for impedance change, whose proof is based on Green's formula, is obtained in the present thesis. The last mentioned formula appears in the literature in some applied problems, but the connection between these two formulae and their equivalence, seems to be given only in the present thesis.

Future work can be devoted to the analysis of the method of additional currents applied to practical problems of non-destructive testing, for example, to problems with flaws in conducting media. In the applications, such problems presently require extensive computational resources and are seldom used in engineering practice because of the complexity of the analysis. The use of the method of additional currents may be useful for simplifying real world problems and obtaining practically useful engineering solutions for eddy current testing of conducting media with flaws. Moreover, the approximate solutions developed in this thesis and the simplified form of other solutions can be successfully used to solve important practical inverse problems in eddy current testing.

Appendix 1

NOMENCLATURE

List of Latin symbols

$\tilde{\vec{A}}$ magnetic vector potential, $\tilde{\vec{A}} = \vec{A}e^{j\omega t}$

$\vec{A}$ complex-valued amplitude magnetic vector potential

$\vec{A}_F$ complex-valued amplitude magnetic vector potential in a flawed region

A_i non-zero component of vector potential in region R_i, $i = 0,1$

A_i^{ind} induced vector potential intensity in R_i

$\tilde{\vec{B}}$ magnetic induction vector, $\tilde{\vec{B}} = \vec{B}e^{j\omega t}$

$\vec{B}$ complex-valued amplitude magnetic induction vector

C Euler constant, $C = 0.577215\ldots$

$\tilde{\vec{D}}$ electric induction vector, $\tilde{\vec{D}} = \vec{D}e^{j\omega t}$

$\vec{D}$ complex-valued amplitude electric induction vector

$\tilde{\vec{E}}$ electric field vector, $\tilde{\vec{E}} = \vec{E}\,e^{j\omega t}$

$\vec{E}$ complex-valued amplitude electric field vector

$\vec{E}_F$ complex-valued amplitude electric field vector in a flawed region

h height of emitter above conducting medium

$\tilde{\vec{H}}$ magnetic field vector, $\tilde{\vec{H}} = \vec{H}\,e^{j\omega t}$

$\vec{H}$ complex-valued amplitude magnetic field vector

$\tilde{\vec{I}}$ current vector density, $\tilde{\vec{I}} = \vec{I}\,e^{j\omega t}$

$\vec{I}$ complex-valued amplitude current vector density

$\tilde{\vec{I}}^{\,e}$ external current vector density, $\tilde{\vec{I}}^{\,e} = \vec{I}^{\,e}\,e^{j\omega t}$

$\vec{I}^{\,e}$ complex-valued amplitude external current vector density

$I_\nu(s)$ modified Bessel function of the first kind of order ν

j imaginary unit, $j = \sqrt{-1}$

$J_\nu(s)$ Bessel function of the first kind of order ν

$K_\nu(s)$ modified Bessel function of the second kind of order ν

$\mathrm{ker}_1(x)$
$\mathrm{kei}_1(x)$ Kelvin functions

L_m the operator $L_m = \frac{1}{b^{1-m}}\frac{d}{db}\left(b^{1-2m}\frac{d}{db}\right)b^m$

$\tilde{L}$ the operator $\tilde{L} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$, i.e. $\tilde{L} = \Delta^2 - \frac{1}{r^2}$

$\vec{n}$ normal to a surface

r_c coil radius

R_0 unbounded free space

R_1 conducting medium with or without a flaw (defect)

R_F flaw-region

X real part of Z

Y imaginary part of Z

$Y_\nu(s)$ Bessel function of the second kind of order ν

Z dimensionless induced change in impedance

Z_c dimensionless induced change in impedance (case of a coil)

$Z_{c\infty}$ asymptotic of impedance as $\beta \to \infty$ (case of a coil)

Z_l dimensionless induced change in impedance (case of a double line)

$Z_{l\infty}$ asymptotic of impedance as $\beta \to \infty$ (case of a double line)

Z^{ind} induced change in impedance

List of Greek symbols

$\Gamma(x)$ Euler gamma function

Δ Laplacian, $\Delta f(x,y,z) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

$$\Delta f(r,\varphi,z) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\Delta\vec{A}(r,\varphi,z) = \vec{e}_r\left(\Delta A_r - \frac{A_r}{r^2} - \frac{2}{r^2}\frac{\partial A_\varphi}{\partial \varphi}\right) + \vec{e}_\varphi\left(\Delta A_\varphi - \frac{A_\varphi}{r^2} + \frac{2}{r^2}\frac{\partial A_r}{\partial \varphi}\right) + \vec{e}_z\Delta A_z$$

Δ_φ the operator $\Delta_\varphi f(r,z) = \dfrac{\partial^2 f}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial f}{\partial r} + \dfrac{\partial^2 f}{\partial z^2} - \dfrac{1}{r^2} f$

$\delta(x)$ Dirac's delta function

ε_0 electric constant

$\hat{\varepsilon}$ relative permittivity

μ_0 magnetic constant

μ relative magnetic permeability

$\tilde{\rho}$ charge density

σ conductivity

Φ_p primary varying magnetic field

Φ_s secondary varying magnetic field

ψ scalar electric potential intensity

$\tilde{\psi}$ scalar electric potential, $\tilde{\psi} = \psi\, e^{j\omega t}$

ω frequency

Coefficients

$k^2 = -j\omega\mu_0\mu(\sigma + j\varepsilon_0\hat{\varepsilon}\omega)$

$\tilde{k}^2 = \mu_0\mu(\sigma + j\varepsilon_0\hat{\varepsilon}\omega)$

Coordinate systems

(x, y, z) Cartesian coordinates, $x, y, z \in \Re$

(r, φ, z) cylindrical polar coordinates, $r \geq 0,\ 0 \leq \varphi \leq 2\pi,\ z \in \Re$

(ρ, θ, φ) spherical coordinates, $\rho \geq 0,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \varphi \leq \pi$

Two classes of definite integrals

$$A_n(\gamma) = \int_0^\infty \frac{\cos\gamma x\, dx}{(\sqrt{x^2 + a^2} + x)^{2n-1}}, \qquad B_{n,m}(b) = \int_0^\infty \frac{x^{m+1} J_m(bx)\, dx}{(\sqrt{x^2 + a^2} + x)^{2n-1}}$$

Appendix 2

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