REDUCTION PRINCIPLE FOR DYNAMICAL SYSTEMS

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Summary of habilitation work

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Reviewers:

Prof., Dr. hab. math. U. Raitums (University of Latvia)

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CHARACTER OF THE WORK

1. CHARACTER OF THE WORK. The collection of papers.
2. CONTENTS OF THE WORK.
   (a) SUBJECT. Discrete dynamical and semidynamical systems and their dynamical extensions in the complete metric space. The systems of impulsive differential equations at the Banach space.
   (b) PURPOSE. Reduction of discrete dynamical and semidynamical systems and impulsive differential equations to a simpler form, including partial decoupling and linearization.
   (c) MAIN RESULTS. The conditions of the reduction for dynamical and semidynamical systems as well as for systems of impulsive differential equations.
   (d) THEORETICAL SIGNIFICANCE. A new original technique and its modifications for proving dynamical equivalence are developed.
   (e) PRACTICAL SIGNIFICANCE. The work possesses theoretical character, however the developed technique can be applied for solving practical problems, such as investigation into the stability of solutions and their behavior for large values of variables.
   (f) APPROVAL OF THE WORK. The results were represented at the following seminars and international conferences:
      i. 3rd International Colloquium on Differential Equations. Plovdiv, Bulgaria, August 18–22, 1992.
      ii. 8th UIC conference "Qualitative Theory of Differential Equations". Samarkand, Uzbekistan, September 5–10, 1992.
      iii. 4th Colloquium on the Qualitative Theory of Differential Equations. Szeged, Hungary, August 18–21, 1993.
      iv. Equadiff 8. Czecho–Slovak Conference on Differential Equations and their Applications. Bratislava, Slovakia, August 24–28, 1993.

v. Workshop "Dynamical Systems". Augsburg, Germany, June 27 – July 2, 1994.

vi. International Congress of Mathematicians. Zürich, Switzerland, August 3–11, 1994.

vii. 3rd SIAM Conference on Applications of Dynamical Systems. Snowbird, UT, USA, May 21–24, 1995.

viii. 2nd International Conference on Dynamic Systems and Applications. Atlanta, GA, USA, May 24–27, 1995.

ix. NSF–CBMS Regional Conference on Approximation Dynamics with Applications to Numerical Analysis. Columbia, MO, USA, June 1–5, 1995.

x. 2nd International Conference on Difference Equations and Applications. Veszprém, Hungary, August 7–11, 1995.

xi. 6th International Colloquium on Differential Equations. Plovdiv, Bulgaria, August 18–23, 1995.

xii. 5th International Conference on Differential Equations and Applications. Rousse, Bulgaria, August 24–29, 1995.

xiii. Conference "Problems of Pure and Applied Mathematics". Tallinn, Estonia, October 13–14, 1995.

xiv. 1st Latvian Mathematical Conference. Rīga, October 20–21, 1995.

xv. Seminar on Dynamical systems (Prof. F. Dumortier). Limburgs Universitair Centrum, Diepenbeek, Belgium, March 29, 1996.

xvi. 4th International Conference on Integral Methods in Science and Engineering. Oulu, Finland, June 17–20, 1996.

xvii. 2nd World Congress of Nonlinear Analysts. Athens, Greece, July 10–17, 1996.

xviii. Conference "Topological Methods in Differential Equations and Dynamical Systems". Krakóv–Przegorzaly, Poland, July 17–20, 1996.

xix. 2nd European Congress of Mathematics. Budapest, Hungary, July 22–26, 1996.

xx. 5th Colloquium on the Qualitative Theory of Differential Equations. Szeged, Hungary, July 29 – August 2, 1996.

xxi. 2nd International Conference "Mathematical Modelling and Complex Analysis". Vilnius, Lithuania, June 3–4, 1997.

xxii. International Conference on Differential Equations and Dynamical Systems. Waterloo, Ontario, Canada, August 1–4, 1997.

xxiii. Equadiff 9. Conference on Differential Equations and their Applications. Brno, Czech Republic, August 25–29, 1997.

xxiv. Conference "Topological, Variational & Singularities Methods in Nonlinear Analysis". Gdansk–Jurata, Poland, September 15–19, 1997.

xxv. 2nd Latvian Mathematical Conference. Rīga, October 31 – November 1, 1997.

(g) PUBLICATIONS. The main results are published in the following papers:

i. A. Reinfelds, *Global topological equivalence of nonlinear flows*, Differential Equations **8** (1974), 1474–1476. MR 47 # 9592, Zbl 244.54026, 288.54043.

ii. A. Reinfelds, *A reduction theorem*, Differential Equations **10** (1975), no. 5, 645–649. MR 58 # 1397, Zbl 286.34054, 315.34046.

iii. A. Reinfelds, *A reduction theorem for closed trajectories*, Differential Equations **11** (1976), 1353–1358. MR 52 # 11202, Zbl 318.34058, 345.34036.

iv. A. Reinfelds, *A generalized Grobman–Hartman theorem*, Latv. Mat. Ezhegodnik **29** (1985), 84–88 (Russian). MR 87a:34074, Zbl 582.34057.

v. A. Reinfelds, *Invariant sets in a metric space*, Latv. Mat. Ezhegodnik **30** (1986), 83–91 (Russian). MR 88d:54056, Zbl 634. 58028.

vi. A. Reinfelds, *Conjugation of homeomorphisms in a metric space*, Latv. Mat. Ezhegodnik **31** (1988), 236 (Russian).

vii. A. Reinfelds, *A reduction theorem for extensions of dynamical systems*, Latv. Mat. Ezhegodnik **33** (1989), 67–75 (Russian). MR 90m:34096, Zbl 695.34047.

viii. A. Reinfelds, *Dynamical equivalence of dynamical extensions*, Reports of the extended sessions of the seminar of the I. N. Vekua Institute of Applied Mathematics **5** (1990), no. 3, 164–166 (Russian).

ix. A. Reinfelds and L. Sermone, *Equivalence of differential equations with impulse action*, Latv. Univ. Zināt. Raksti **553** (1990), 124–130 (Russian). MR 92j:34021.

x. A. Reinfelds and L. Sermone, *Equivalence of nonlinear differential equations with impulse effect in Banach space*, Latv. Univ. Zināt. Raksti **577** (1992), 68–73. MR 95b:34014.

xi. A. Reinfelds, *Existence of central manifold for differential equations with impulses in a Banach space*, Latv. Univ. Zināt. Raksti **577** (1992), 81–88. MR 95b:34015.

xii. A. Reinfelds, *Invariant sets for splitting mapping in metric space*, Latv. Univ. Zināt. Raksti **588** (1993), 35–44. MR 96j:54041.

xiii. A. Reinfelds, *Decoupling of mappings in a metric space*, Proc. Latv. Acad. Sci. Sect. **B 1994**, no. 2(559), 67–75. Zbl 865.54038.

xiv. A. Reinfelds, *The reduction principle for discrete dynamical and semidynamical systems in metric spaces*, Z. Angew. Math. Phys. **45** (1994), no. 6, 933–955. MR 95m:54039, Zbl 824.34049.

xv. A. Reinfelds, *Invariant sets for noninvertible mapping*, Latv. Univ. Zināt. Raksti **592** (1994), 115–124. MR 96m:54079, Zbl 852.39011.

xvi. A. Reinfelds, *Partial decoupling for semidynamical system*, Latv. Univ. Zināt. Raksti **593** (1994), 54–61. MR 96j:54042, Zbl 854.34045.

xvii. A. Reinfelds, *Partial decoupling for noninvertible mappings*, Differential Equations Dynam. Systems **2** (1994), no. 3, 205–215. MR 97c:39007, Zbl 869.39009.

xviii. A. Reinfelds, *The stability of semidynamical system in metric space*, Latv. Univ. Zināt. Raksti **599** (1995), 140–145. MR 97a:54045, Zbl 854.34046.

xix. A. Reinfelds, *The reduction principle for discrete dynamical and semidynamical systems in metric spaces*, in S. Bilchev and S. Tersian (eds.), *Differential equations and applications*. Proceedings of the fifth international conference on differential equations and applications, Rousse, Bulgaria, August 24–29, 1995. Union of Bulgarian Mathematicians, Rousse, 1995, pp. 94–102. MR 97d:54067, Zbl 857.34047.

xx. A. Reinfelds, *Reduction theorem for differential equations with impulse effect in a Banach space*, J. Math. Anal. Appl. **203** (1996), no. 1, 187–210. MR 97h:34010, Zbl 860.34027.

xxi. A. Reinfelds, *Invariant sets and dynamical equivalence*, Proc. Est. Acad. Sci. Phys. Math. **45** (1996), no. 2–3, 216–225. MR 97g:54057, Zbl 862.34039.

xxii. A. Reinfelds, *The reduction of discrete dynamical and semidynamical systems in metric spaces*, in B. Aulbach and F. Colonius (eds.), *Six lectures on dynamical systems*, World Sci. Publishing, River Edge, NJ, 1996, pp. 267–312. MR 98d:58138.

xxiii. A. Reinfelds, *The shadowing lemma in metric space*, Univ. Iagel. Acta Math. **35** (1997), 205–210. MR 98d:58139.

xxiv. A. Reinfelds, *The reduction of discrete dynamical systems in metric space*, in S. Elaydi, I. Győri and G. Ladas (eds.), *Advances in difference equations*. Proceedings of the second international conference on difference equations, Veszprém, Hungary, August 7–11, 1995. Gordon and Breach, Yverdon, 1997, pp. 525–536. Zbl 980.22938.

xxv. A. Reinfelds, *Grobman's–Hartman's theorem for time–dependent difference equations*, Latv. Univ. Zinātn. Raksti **605** (1997), 9–13.

xxvi. A. Reinfelds, *Decoupling of impulsive differential equations in a Banach space*, in C. Constanda, J. Saranen and S. Seikkala (eds.), *Integral methods in science and engineering. Volume one: analytic methods*, Pitman Res. Notes Math. Ser., **374**, Longman, Harlow, 1997, pp. 144–148. MR 98i:00021, Zbl 980.34434.

xxvii. A. Reinfelds, *Decoupling of impulsive differential equations*, in R. Čiegis (ed.), *Mathematical modelling and complex analysis*. Proceedings of the second international conference "Mathematical modelling and complex analysis", Vilnius, Lithuania, June 3–4, 1997. "Technika", Vilnius, 1997, pp. 130–137.

xxviii. A. Reinfelds, *Dynamical equivalence of impulsive differential equations*, Nonlinear Anal. **30** (1997), no. 5, 2743–2752. MR 98m:34024, Zbl 980.11174.

xxix. A. Reinfelds, *Partial decoupling of semidynamical system in metric space*, J. Tech. Univ. Plovdiv Fundam. Sci. Appl. Ser. A Pure Appl. Math. **5** (1997), 33–40. CMP 98:12.

xxx. A. Reinfelds, *Dynamical equivalence of dynamical systems*, Univ. Iagel. Acta Math. **36** (1998), 149–155.

xxxi. O. Dumbrajs, R. Meyer–Spasche and A. Reinfelds, *Analysis of electron trajectories in a gyrrotron resonator*, IEEE Trans. Plasma Science **26** (1998), no. 3, 846–853.

xxxii. A. Reinfelds, *Partial decoupling of impulsive differential equations*, Latv. Univ. Zinātn. Raksti **612** (1998), 107–114.

Table of Contents

1. Introduction

A basic problem of the qualitative theory of differential equations is to classify the systems of differential equations with respect to some characteristic properties of solutions. Such a classification allows the investigation of complicated system of differential equations to be replaced with a simpler system of differential equations from the same class. In a sufficiently small neighborhood of invariant set satisfactorily classification gives the concept of topological (dynamical) equivalence.

Two systems of autonomous differential equations are topologically equivalent if there is a homeomorphism of phase space mapping trajectories of the first system of differential equations onto trajectories of the second system of differential equations preserving the orientation.

If, in addition, the corresponding homeomorphism maps solutions of the first system of differential equations into solutions of the second system of differential equations, then the considered systems of differential equations are dynamically equivalent. If we examine contraction in a small neighborhood of invariant set, then we have a local topological (dynamical) equivalence.

The source of the concept of topological equivalence of the system of differential equations may be found in the papers by H. Poincaré [104]. He considered the problem of existence of such mapping of phase space that maps an autonomous system of nonlinear differential equations onto that of linear, or in modern interpretation, he sought an analytical diffeomorphism which realized dynamical equivalence of a nonlinear and a linear system of differential equations. In 50–ties S. Sternberg [161, 162] weakened the hypothesis of the analyticity of diffeomorphism and substituted it for the existence of a sufficiently smooth diffeomorphism. A strong concept of topological equivalence was introduced by A. A. Andronov and L. S. Pontryagin [2] in 1937, in the paper on the structural stable systems of differential equations.

The problem of finding criteria for topological equivalence of differential equations systems in the neighborhood of stationary points was set by V. V. Nemitskiĭ [73]. The solution of problem in case of the system of linear autonomous differential equations having only elementary stationary points was found by E. M. Vaisbord [163], A. Reiziņa and L. Reiziņš [149] and V. I. Arnol'd [5]. In the general case, the systems of linear autonomous differential equations were classified topologically by N. N. Ladis [59].

D. M. Grobman [24, 35, 36, 37, 38, 39] un P. Hartman [43, 44, 45, 46] proved that the system of autonomous differential equations

$$dx/dt = Ax + f(x, y), \quad x \in \mathbb{R}^n$$

is dynamically equivalent to the linear system of differential equations

$$dx/dt = Ax, \quad x \in \mathbb{R}^n,$$

if matrix A has no eigenvalues with zero real parts, f is a Lipschitz mapping with sufficiently small Lipschitz constant and such that cancels at the origin. To prove his theorem, D. M. Grobman constructed some mapping, with the help of a formula variation for constants, and then proved that this mapping is a homeomorphism which realizes dynamical equivalence of nonlinear and linear system of differential equations. According to P. Hartman's technique, the given system of differential equations is reduced to diffeomorphisms and a condition of their conjugacy is found. Note that in the proof of the equivalence is important a circumstance that $\mathbb{R}^n$ is a local compact space.

In 1962, L. Reiziņš [150, 151, 153] generalized the theorem of Grobman–Hartman to the neighborhoods of elementary cycles. For this purpose, he introduced into the neighborhoods of cycle pseudolocal coordinates and reduced the investigation into the topological structure of dynamical system in the cycle neighborhood to studies of a halfperiodic system of differential equations in the vicinity of origin. Analogous results were later obtained by M. Irvin [50] with the help of Hartman's technique for the Poincaré mappings. Then K. Palmer [85, 90] generalized the theorems of Vaisbord and Grobman–Hartman for systems of nonautonomous differential equations, the linear parts of which satisfy the conditions of exponential dichotomy.

The analog of the Vaisbord theorem in a Banach space was proved by A. Reinfelds [108]. To prove the Grobman–Hartman theorem in a Banach space there was needed an essentially fundamentally new proof. By using the ideas of J. Moser [70] it was done by C. Pugh [102] and J. Palis [83]. A short proof of the Grobman–Hartman theorem for extensions of dynamical systems in the Banach space based on Green's type mappings was given by A. Reinfelds [117]. The corresponding homeomorphism that realizes the dynamical equivalence is presented as a solution of some functional integral equation. Note that such an approach was very successful, so later A. Reinfelds developed it and used for proving the theorems of reduction type for systems of impulsive differential equations in a Banach space. The Grobman–Hartman theorem and its modifications were proved using different technique by M. A. Boudourides [20, 21],

I. U. Bronshtein and V. A. Glavan [23], Nguyen Van Minh [65], A. Reinfelds [105, 110, 139].

L. Reiziņš [153, 154, 155], R. M. Mints [68], N. N. Ladis [56, 57, 58], C. Coleman [31, 32] and A. Reinfelds [106, 111] investigated dynamical equivalence of systems of differential equations in the neighborhoods of compound stationary points and cycles that have decoupled truncation.

In studying topological equivalence for systems of differential equations the reduction theorem occupies a significant part. According to this theorem there is a Lipschitz mapping v such that the nonlinear system of differential equations

$$\begin{cases} dx/dt &= Ax + f(x,y), \\ dy/dt &= By + g(x,y) \end{cases}$$

is dynamically equivalent to the partially linearized system of differential equations

$$\begin{cases} dx/dt &= Ax \\ dy/dt &= By + g(v(y),y), \end{cases}$$

if matrix A has no eigenvalues with zero real part, while all real parts of matrix B eigenvalues are equal to zero; f and g are Lipschitz mappings with a sufficiently small Lipschitz constant and such that vanish at the origin. The proof of the theorem in case y is onedimensional vector and under additional constraints was given by L. Reiziņš [152]. In general case, the theorem for systems of differential equations with C^2 smooth right–hand side was announced by A. N. Shoshitaishvili [6, 159] (the proof was published only in 1975 [160]). A. Reinfelds [107, 109], using a different method, proved the reduction theorem for the case when mappings f and g are Lipschitzian with a sufficiently small Lipschitz constant. K. Palmer [55, 86, 87, 88, 89, 91] applied slightly varying method and proved this theorem and its modifications in space $\mathbb{R}^n$. For nonautonomous systems of differential equations the reduction theorem was proved also by Nguen Van Minh [66].

Various criteria of topological equivalence in the neighborhood of a normal hyperbolic set, including those in the neighborhood of an invariant torus were given by G. S. Osipenko [76, 77, 78, 79, 80, 81, 82], J. Palis and F. Takens [84], M. Hirsh, C. Pugh and M. Shub [49], C. Pugh and M. Shub [103] and A. Reinfelds both in $\mathbb{R}^n$ [112, 113, 114, 115, 116] and in the Banach space [118, 119].

In Banach space the dynamical equivalence was studied by K. Lu [63] and P. W. Bates with K. Lu [16] for systems of differential equations whose linear part is an unbounded closed operator.

Investigations into the topological equivalence for discrete dynamical systems in $\mathbb{R}^n$ begin with papers by P. Hartman [43, 44, 45, 46] and M. Irvin [50].

U. Kirchgraber [53, 54, 55] proved the reduction theorem for discrete dynamical systems in $\mathbb{R}^n$. G. Papaschinopoulos [95] proved this theorem for difference equations.

In 1991–1993, B. Aulbach and B. M. Garay [9, 10, 11] published the first papers on the equivalence of noninvertible mappings in a Banach space. Such mappings arise in case we investigate solutions of evolutionary partial equations continuing in one direction. B. Aulbach and B. M. Garay introduced a hypothesis about reduction of noninvertible mappings and proved it for a special case. A. Reinfelds [129, 132, 137] did this for the general case.

In the second summary chapter of the habilitation work the main concepts are defined including the topological equivalence of dynamical (semidynamical) systems. It is considered a discrete dynamical system generated by homeomorphism in the Decart product of two complete metric spaces. The corresponding homeomorphism satisfies the given metric inequalities. Such inequalities are valid for mappings satisfying the conditions of the Grobman–Hartman theorem or the reduction theorem in the space $\mathbb{R}^n$. Analogous inequalities were used by Yu. I. Neĭmark [71, 72] and V. A. Pliss [100, 101] for proving the existence of invariant manifold. In the reduction theorem, important place are taken by global Lipschitz mappings whose graphs are invariant sets. For such a type of Lipschitz mappings the properties of uniqueness are fulfilled. In mathematical literature there are many papers devoted to existence of invariant sets for mappings and for systems of differential equations, both in $\mathbb{R}^n$ and Banach space. Given in the summary necessary and sufficient conditions generalize and specify the results of J. Hadamard [40] and other mathematicians [3, 17, 18, 19, 25, 28, 30, 34, 41, 42, 47, 48, 49, 51, 52, 61, 62, 69, 92, 94, 97, 164, 165]. The obtained results allow one to specify the statements of the reduction theorem. The new original technique makes it possible to prove the reduction theorem and its different modifications under various conditions both in the complete metric space and in the Banach space. Note that quite often the conditions of the theorem cannot be improved. Besides, the reduction theorems for semidynamical systems generated by noninvertible mappings has been proved. Therefore the hypothesis of B. Aulbach and B. M. Garay is valid.

In the third chapter the results of the previous chapter are generalized for dynamical extensions. They are natural generalization of systems of nonautonomous differential equations.

In the fourth chapter we studied the dynamical equivalence of systems of impulsive differential equations in Banach space. On the one hand they cover the systems of nonautonomous differential equations, on the other – those systems of differential equations with solutions which are continuable only in one direction. The systems of impulsive differential equations provide an adequate mathematical model of evolutionary processes that suddenly change their state at certain moments. The first investigators of impulsive differential equations were A. D. Mishkis and V. D. Mil'man [67]. In monographs by V. Lakshmikantham, D. D. Bainov and P. S. Simeonov [60] and A. M. Samoĭlenko and N. A. Perestyuk [156] there is systematic presentation of the theory for systems of impulsive differential equations.

The dynamical equivalence for systems of impulsive differential equations were considered first by the author [124, 125, 135, 141, 142, 143, 146, 148] and L. Sermone [124, 125, 157, 158] and D. D. Bainov, S. I. Kostadinov and Nguyen Van Minh [14, 15]. In the given chapter, the different modifications for systems of impulsive differential equations in a Banach space are proved (including those for noninvertible systems), assuming that the system splits into two parts. In proving the reduction theorem for systems of impulsive differential equations of significance are the global Lipschitz mappings whose graphs are invariant sets [12, 13, 126, 135, 143]. Often it is possible to use the reduction theorems many times, which allows further simplifications of the given system. By using standard technique local variants of the reduction theorem are also obtainable.

The sufficient conditions for dynamical equivalence are given using inequalities containing integrals from corresponding evolutionary operators. The obtained results on the one hand precise the known results for systems of ordinary differential equations in $\mathbb{R}^n$, and on the other they give a technique to solve analogue problems in functional spaces.

In the last chapter, we consider applications of the technique developed in the previous chapters. The reduction principle in the theory of stability for systems of autonomous differential equations was proved by V. A. Pliss [98, 99, 101]. For systems of nonautonomous differential equations it was generalized by B. Aulbach [7, 8]. To the various modifications of the reduction principle in the theory of stability papers [14, 15, 26, 64, 69, 96, 156] are devoted. In the given summary there is given a short proof for semidynamical systems in metric space by using conjugacy of mappings.

In implicit form the "shadowing" lemma occurs in connection with diffeomorphisms of D. V. Anosov [4]. There is wide mathematical literature on different modifications of "shadowing" lemma, both in local compact spaces [22, 29, 74, 75, 93] and in Banach space [1, 27]. We give a short proof of "shadowing" lemma in metric space using functional equations similar to those we used for proving dynamical equivalence.

At the end of chapter, we prove the asymptotic equivalence of two nonlinear differential equations that describe the electron trajectories in a gyrotron resonator [33].

2. Topological equivalence of discrete dynamical and semidynamical systems

2.1 Introduction

Consider a discrete dynamical (semidynamical) system generated by a homeomorphism (continuous mapping) T

$$T(x,y) = (f(x,y), g(x,y))$$

in an arbitrarily complete metric space. We will get the necessary and sufficient conditions for the existence of global Lipschitz mappings whose graphics are invariant sets of a dynamical (semidynamical) system. The obtained intermediate results allow one to get the sufficient conditions for decoupling and simplifying a dynamical (semidynamical) system and thus for reducing investigation of the given system to that of a simpler system. The resultant theorems are generalizations of the classical Grobman–Hartman theorem and of the reduction principle in the complete metric space.

2.2 Preliminaries

In this section we set out some basic facts needed for later sections and specify the form of mapping T.

Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be complete metric spaces with metrics ρ_1 and ρ_2, respectively.

Definition 2.1 A mapping $T\colon \mathbf{X}_1 \to \mathbf{X}_2$ is *Lipschitzian* (with constant k) if, for all $x, x' \in \mathbf{X}_1$,

$$\rho_2(T(x), T(x')) \leq k\rho_1(x, x').$$

Definition 2.2 A *fixed point* of T is any $x \in \mathbf{X}$ such that $T(x) = x$.

Theorem 2.3 Contraction mapping theorem. *Let* $\mathbf{M}$ *be a closed subset of the complete metric space* $\mathbf{X}$ *and let* $T\colon \mathbf{M} \to \mathbf{X}$ *be a Lipschitz mapping with constant* $k < 1$*. If* $T(\mathbf{M}) \subset \mathbf{M}$ *then mapping* T *has a unique fixed point in* $\mathbf{M}$*.*

Definition 2.4 A *homeomorphism* is a continuous mapping $H\colon \mathbf{X} \to \mathbf{X}$ which is bijective and its inverse mapping is continuous.

Definition 2.5 A one–parameter family $\{T^n\}$, $n \in \mathbf{Z}$ of continuous mappings with $T^1 = T\colon \mathbf{X} \to \mathbf{X}$ is a *discrete dynamical system* if:

(*i*) $T^0 = id$, where id is identity mapping.

(*ii*) $T^n \circ T^k = T^{n+k}$.

If the one–parameter family of mappings is defined only for nonnegative integers, we have a *discrete semidynamical system.*

Note that in the case of discrete dynamical systems the mapping T is a homeomorphism.

Definition 2.6 Two discrete dynamical (semidynamical) systems $T_1^n, T_2^n\colon \mathbf{X} \to \mathbf{X}$ are *topologically equivalent* if there exists a homeomorphism $H\colon \mathbf{X} \to \mathbf{X}$ such that the diagram

$$\begin{CD} \mathbf{X} @>{T_1^n}>> \mathbf{X} \\ @V{H}VV @VV{H}V \\ \mathbf{X} @>{T_2^n}>> \mathbf{X} \end{CD}$$

commutes.

Definition 2.7 Two mappings $T_1, T_2\colon \mathbf{X} \to \mathbf{X}$ are *topologically conjugate* if there exists a homeomorphism $H\colon \mathbf{X} \to \mathbf{X}$ such that the diagram

$$\begin{CD} \mathbf{X} @>{T_1}>> \mathbf{X} \\ @V{H}VV @VV{H}V \\ \mathbf{X} @>{T_2}>> \mathbf{X} \end{CD}$$

commutes.

It is easily verified that two discrete dynamical (semidynamical) systems T_1^n and T_2^n, generated by mappings T_1 and T_2, are topologically equivalent if and only if mappings T_1 and T_2 are topologically conjugate.

Let $\mathbf{X}$ and $\mathbf{Y}$ be complete metric spaces with metrics ρ_1 and ρ_2, respectively. The object of this chapter is to study continuous mappings $T\colon \mathbf{X}\times\mathbf{Y}\to\mathbf{X}\times\mathbf{Y}$ of the type

$$T(x,y) = (f(x,y), g(x,y)).$$

We will make the following hypotheses:

(**H1**) $\rho_1(x,x') \le \alpha\rho_1(f(x,y), f(x',y))$, $\alpha > 0$.

(**H2**) $\rho_1(f(x,y), f(x,y')) \le \beta\rho_2(y,y')$.

(**H3**) $\rho_2(g(x,y), g(x',y')) \le \gamma\rho_1(x,x') + \delta\rho_2(y,y')$, where $\alpha(\delta + 2\sqrt{\beta\gamma}) < 1$.

(**H4**) Mapping $f(\cdot,y)\colon \mathbf{X}\to\mathbf{X}$ is surjective.

Our aim is to decouple and simplify the given mapping T by means of a topological transformation.

Example 2.8 Let us consider the following mapping in Banach space

$$\begin{aligned} x^1 &= Ax + F(x,y),\\ y^1 &= By + G(x,y), \end{aligned} \tag{2.1}$$

where $x\in\mathbf{X}$, $y\in\mathbf{Y}$, A and B are bounded linear mappings, A is invertible, $\|B\| < \|A^{-1}\|^{-1}$, mappings $F\colon \mathbf{X}\times\mathbf{Y}\to\mathbf{X}$, $G\colon \mathbf{X}\times\mathbf{Y}\to\mathbf{Y}$ satisfy the Lipschitz conditions

$$\begin{aligned} |F(x,y) - F(x',y')| &\le \varepsilon(|x-x'| + |y-y'|),\\ |G(x,y) - G(x',y')| &\le \varepsilon(|x-x'| + |y-y'|). \end{aligned}$$

It is easy to verify that this mapping satisfies the hypotheses (**H1**) – (**H4**), where $\alpha = (\|A^{-1}\|^{-1} - \varepsilon)^{-1}$, $\beta = \gamma = \varepsilon$, $\delta = \|B\| + \varepsilon$. The condition $\alpha(\delta + 2\sqrt{\beta\gamma}) < 1$ reduces to the inequality

$$\varepsilon < \frac{\|A^{-1}\|^{-1} - \|B\|}{4}.$$

The mapping given by formula $x^1 = Ax + F(x,y)$ for a fixed y is surjective if $\varepsilon\|A^{-1}\| < 1$. Note that $\varepsilon\|A^{-1}\| < 1/4$.

Remark. Consider a mapping which shows that in the general case the inequality $\alpha(\delta + 2\sqrt{\beta\gamma}) < 1$ is impossible to replace with an equality. It is easy to verify that the linear mapping

$$x^1 = \alpha^{-1}x - \beta y,$$
$$y^1 = \gamma x + \delta y,$$

where $x \in \mathbb{R}^1$, $y \in \mathbb{R}^1$ and $\alpha, \beta, \gamma, \delta > 0$, satisfies the hypotheses (**H1**)–(**H4**).

If $\alpha(\delta + 2\sqrt{\beta\gamma}) < 1$, then the given mapping has a fixed point at the origin and two invariant straight lines – the graphics of Lipschitz mappings. If, on the contrary, $\alpha(\delta + 2\sqrt{\beta\gamma}) = 1$, then the given mapping has only one invariant straight line going through the origin. The characteristic equation of the corresponding linear mapping has a double root, the degree of the elementary divisor being two. Theorem 2.22 is not valid.

2.3 Auxiliary lemmas

To prove the main results, we use three lemmas. Consider the set of mappings

$$\mathbf{Lip}(k) = \{u \mid u\colon \mathbf{X} \to \mathbf{Y} \text{ and } \rho_2(u(x), u(x')) \le k\rho_1(x, x')\}.$$

Lemma 2.9 *Let $\alpha\beta k < 1$ and $u \in \mathbf{Lip}(k)$. Then the mapping $\varphi\colon \mathbf{X} \to \mathbf{X}$ defined by $\varphi(x) = f(x, u(x))$ is a homeomorphism.*

Next, introduce operator $\mathcal{L}$ acting on $\mathbf{Lip}(k)$ defined by the equality

$$(\mathcal{L}u)(f(x, u(x))) = g(x, u(x)).$$

Lemma 2.10 *There exists $k \ge 0$ such that $\mathcal{L}(\mathbf{Lip}(k)) \subset \mathbf{Lip}(k)$.*

Next, let us consider the set of mappings

$$\mathbf{Lip}(l) = \{v \mid v\colon \mathbf{Y} \to \mathbf{X} \text{ and } \rho_1(v(y), v(y')) \le l\rho_2(y, y')\}$$

and let us introduce the operator $\mathcal{K}$ acting on $\mathbf{Lip}(l)$ by the equality

$$f(\mathcal{K}v(y), y) = v(g(v(y), y)).$$

The operator $\mathcal{K}$ is well defined, because the mapping $f(\cdot, y)\colon \mathbf{X} \to \mathbf{X}$ is surjective and hypothesis (**H1**) is fulfilled.

Lemma 2.11 *There exists an $l \ge 0$ such that $\mathcal{K}(\mathbf{Lip}(l)) \subset \mathbf{Lip}(l)$.*

Later, in Chapters 1 and 2 we assume that

$$k = \frac{2\alpha\gamma}{1 - \alpha\delta + \sqrt{(1 - \alpha\delta)^2 - 4\alpha^2\beta\gamma}}$$

and

$$l = \frac{2\alpha\beta}{1 - \alpha\delta + \sqrt{(1-\alpha\delta)^2 - 4\alpha^2\beta\gamma}}.$$

It should be noted that $\beta k = \gamma l$, $\alpha(\gamma + \delta k)(1-\alpha\beta k)^{-1} = k$, $\alpha l(\gamma l + \delta) + \alpha\beta = l$. $\alpha\beta k = \alpha\gamma l < 1/2$ and $kl < 1$.

Example 2.12 Let us consider the mapping (2.1). We derive

$$k = l = \frac{2\varepsilon}{\|A^{-1}\|^{-1} - \|B\| - 2\varepsilon + \sqrt{(\|A^{-1}\|^{-1} - \|B\|)(\|A^{-1}\|^{-1} - \|B\| - 4\varepsilon)}}$$

$$= \frac{\|A^{-1}\|^{-1} - \|B\| - 2\varepsilon - \sqrt{(\|A^{-1}\|^{-1} - \|B\|)(\|A^{-1}\|^{-1} - \|B\| - 4\varepsilon)}}{2\varepsilon} < 1.$$

2.4 Fixed point

We will give the sufficient conditions for the existence of a fixed point.

Theorem 2.13 *If* $(1-\alpha)(1-\delta) - \alpha\beta\gamma > 0$, *then the mapping* T *has a unique fixed point* $T(x_0, y_0) = (x_0, y_0)$.

Example 2.14 Let us consider a mapping of form (2.1). The condition $(1 - \alpha)(1-\delta) > \alpha\beta\gamma$ reduces to the inequality

$$\varepsilon < \frac{(\|A^{-1}\|^{-1} - 1)(1 - \|B\|)}{\|A^{-1}\|^{-1} - \|B\|}.$$

Using the relation between geometric and arithmetic means, we obtain

$$\frac{(\|A^{-1}\|^{-1} - 1)(1 - \|B\|)}{\|A^{-1}\|^{-1} - \|B\|} \le \frac{\|A^{-1}\|^{-1} - \|B\|}{4}.$$

2.5 Invariant sets

We will give the necessary and sufficient conditions for the existence of mappings $u\colon \mathbf{X} \to \mathbf{Y}$ and $v\colon \mathbf{Y} \to \mathbf{X}$, whose graphs are invariant sets.

Theorem 2.15 *Let the hypotheses* (**H1**)–(**H4**) *hold. For the existence of mappings* $u\colon \mathbf{X} \to \mathbf{Y}$ *and* $v\colon \mathbf{Y} \to \mathbf{X}$ *that satisfy the functional equations*

$$u(f(x, u(x))) = g(x, u(x)), \tag{2.2}$$

$$f(v(y), y) = v(g(v(y), y)) \tag{2.3}$$

and the Lipschitz conditions

$$\rho_2(u(x), u(x')) \le k\rho_1(x, x'), \tag{2.4}$$

$$\rho_1(v(y), v(y')) \le l\rho_2(y, y'), \tag{2.5}$$

it is necessary and sufficient that the mapping T has a fixed point $T(x_0, y_0) = (x_0, y_0)$.

Let us note that if $\alpha\delta + 1 \le 2\alpha$, then

$$\beta k + \delta = \frac{1 - \alpha\delta - \sqrt{(1-\alpha\delta)^2 - 4\alpha^2\beta\gamma}}{2\alpha} + \delta < \frac{1-\alpha\delta}{2\alpha} + \delta \le 1.$$

In the case $\alpha\delta + 1 \ge 2\alpha$ we get

$$\alpha(1+\gamma l) = \alpha + \frac{1 - \alpha\delta - \sqrt{(1-\alpha\delta)^2 - 4\alpha^2\beta\gamma}}{2} < \alpha + \frac{1-\alpha\delta}{2} \le 1.$$

Lemma 2.16 *If $\beta k + \delta < 1$ and $\alpha(1+\gamma l) < 1$, then $(1-\alpha)(1-\delta) > \alpha\beta\gamma$, and conversely, if $(1-\alpha)(1-\delta) > \alpha\beta\gamma$, then $\beta k + \delta < 1$ and $\alpha(1+\gamma l) < 1$.*

Theorem 2.17 *Let the hypotheses* (**H1**)–(**H4**) *hold, and let there be $\beta k+\delta < 1$. For the existence of a mapping $u\colon \mathbf{X} \to \mathbf{Y}$ that satisfies the functional equation* (2.2) *and the Lipschitz condition* (2.4) *it is necessary and sufficient that there exists a mapping $u_0 \in \mathbf{Lip}(k)$ such that*

$$\sup_x \rho_2(u_0(f(x, u_0(x))), g(x, u_0(x))) < +\infty. \tag{2.6}$$

Theorem 2.18 *Let the hypotheses* (**H1**)–(**H4**) *hold, and let there be $\alpha(1+\gamma l) < 1$. For the existence of a mapping $v\colon \mathbf{Y} \to \mathbf{X}$ that satisfies the functional equation* (2.3) *and the Lipschitz condition* (2.5) *it is necessary and sufficient that there exists a mapping $v_0 \in \mathbf{Lip}(l)$ such that*

$$\sup_y \rho_1(v_0(g(v_0(y), y)), f(v_0(y), y)) < +\infty. \tag{2.7}$$

Remark. It is easy to verify the following estimates

$$\rho_2(u(f(x,y)),g(x,y)) \leq \rho_2(u(f(x,y)),u(f(x,u(x))))$$
$$+\rho_2(g(x,u(x)),g(x,y)) \leq (\beta k+\delta)\rho_2(u(x),y)$$

and

$$\rho_1(v(y),x) \leq \alpha\rho_1(f(v(y),y),f(x,y))$$
$$=\alpha\rho_1(v(g(v(y),y)),f(x,y)) \leq \alpha\rho_1(f(x,y),v(g(x,y)))+\alpha\gamma l\rho_1(v(y),x).$$

It follows that

$$\rho_1(v(y),x) \leq \alpha(1-\alpha\gamma l)^{-1}\rho_1(v(g(x,y)),f(x,y)).$$

Example 2.19 Let us consider the mapping (2.1). The condition (2.6) is fulfilled if

$$\sup_x |G(x,0)| < +\infty,$$

and (2.7) is fulfilled if

$$\sup_y |F(0,y)| < +\infty.$$

Lemma 2.20 *Let T be a homeomorphism and let there be a mapping $v\colon \mathbf{Y} \to \mathbf{X}$ satisfying* (2.3) *and* (2.5)*. Then the mapping $\psi\colon \mathbf{Y} \to \mathbf{Y}$, defined by $\psi(y) = g(v(y),y)$, is a homeomorphism.*

Corollary 2.21 *Let T be a homeomorphism and let the mappings $u\colon \mathbf{X} \to \mathbf{Y}$ and $v\colon \mathbf{Y} \to \mathbf{X}$ satisfy* (2.2)–(2.5)*. Then the mapping $S\colon \mathbf{X}\times\mathbf{Y} \to \mathbf{X}\times\mathbf{Y}$ defined by the equality $S(x,y) = (f(x,u(x)), g(v(y),y))$ is a homeomorphism.*

2.6 Conjugacy of homeomorphisms. 1

We now consider the case when the mapping T is a homeomorphism having a fixed point.

Theorem 2.22 *Let the hypotheses* (**H1**)–(**H4**) *hold and let T be a homeomorphism with a fixed point. Then there exists a homeomorphism $H\colon \mathbf{X}\times\mathbf{Y} \to \mathbf{X}\times\mathbf{Y}$ such that the diagram*

$$\begin{array}{ccc}
\mathbf{X}\times\mathbf{Y} & \xrightarrow{\;T\;} & \mathbf{X}\times\mathbf{Y} \\
H\downarrow & & \downarrow H \\
\mathbf{X}\times\mathbf{Y} & \xrightarrow{\;S\;} & \mathbf{X}\times\mathbf{Y}
\end{array}$$

commutes, where $S(x,y) = (f(x,u(x)), g(v(y),y))$.

Proof. The proof of the theorem consists of several steps.

Step 1. Mapping p: The functional equation

$$f(p(x,y), u(p(x,y))) = p(T(x,y))$$

has a unique solution $p \in \mathbf{M}_1$, where

$$\mathbf{M}_1 = \left\{ p \;\middle|\; p\colon \mathbf{X} \times \mathbf{Y} \to \mathbf{X} \text{ is continuous and } \sup_{x,y} \frac{\rho_1(p(x,y),x)}{\rho_2(u(x),y)} < +\infty \right\}$$

is the complete metric space.

Step 2. Mapping π: The functional equation

$$g(v(\pi(x,y)), \pi(x,y)) = \pi(T(x,y))$$

has a unique solution $\pi \in \mathbf{M}_2$, where

$$\mathbf{M}_2 = \left\{ \pi \;\middle|\; \pi\colon \mathbf{X} \times \mathbf{Y} \to \mathbf{Y} \text{ is continuous and } \sup_{x,y} \frac{\rho_2(\pi(x,y),y)}{\rho_1(v(y),x)} < +\infty \right\}$$

is the complete metric space.

Step 3. Mapping q: The functional equation

$$f(q(x,z),z) = q(f(x,u(x)), g(q(x,z),z))$$

has a unique solution $q \in \mathbf{M}_1(l)$, where

$$\mathbf{M}_1(l) = \left\{ q \in \mathbf{M}_1 \;\middle|\; \sup_{x,y} \frac{\rho_1(q(x,y),x)}{\rho_2(u(x),y)} \le l \text{ and } \rho_1(q(x,z), q(x,z')) \le l\rho_2(z,z') \right\}$$

is the complete metric space.

Step 4. Mapping θ: The functional equation

$$\theta(S(x,y)) = g(q(x,\theta(x,y)), \theta(x,y))$$

has a unique solution $\theta \in \mathbf{M}_3$, where

$$\mathbf{M}_3 = \left\{ \theta \;\middle|\; \theta\colon \mathbf{X} \times \mathbf{Y} \to \mathbf{Y} \text{ is continuous and} \sup_{x,y} \frac{\rho_2(\theta(x,y),y)}{\rho_1(q(x,y),v(y))} < +\infty \right\}$$

is the complete metric space.

Step 5. Mapping P: The functional equation

$$P(S(x,y)) = f(P(x,y), u(P(x,y)))$$

has a unique solution $P \in \mathbf{M}_4$, where

$$\mathbf{M}_4 = \left\{ P \,\middle|\, P\colon \mathbf{X} \times \mathbf{Y} \to \mathbf{X} \text{ is continuous and} \sup_{x,y} \frac{\rho_1(P(x,y),x)}{\rho_2(\theta(x,y),u(x))} < +\infty \right\}$$

is the complete metric space and

$$P(x,y) = p(q(x,\theta(x,y)),\theta(x,y)) = x.$$

Step 6. Mapping Π: The functional equation

$$\Pi(S(x,y)) = g(v(\Pi(x,y)), \Pi(x,y))$$

has a unique solution $\Pi \in \mathbf{M}_3$, where

$$\Pi(x,y) = \pi(q(x,\theta(x,y)),\theta(x,y)) = y.$$

Step 7. Mapping Q: The functional equation

$$Q(T(x,y), g(Q(x,y,z),z)) = f(Q(x,y,z),z)$$

has a unique solution $Q \in \mathbf{M}_5$, where

$$\mathbf{M}_5 = \left\{ Q \,\middle|\, Q\colon \mathbf{X} \times \mathbf{Y} \times \mathbf{Y} \to \mathbf{X} \text{ is continuous,} \right.$$

$$\left. \rho_1(Q(x,y,z), Q(x,y,z')) \le l\rho_2(z,z') \text{ and } \sup_{x,y,z} \frac{\rho_1(Q(x,y,z),x)}{\max\left(\rho_2(u(x),y), \rho_2(z,y)\right)} < \infty \right\}$$

is the complete metric space. We have $Q(x,y,z) = q(p(x,y),z)$. It is easily verified that $Q(x,y,y) = x$. Therefore $q(p(x,y),y) = x$.

Step 8. Mapping Θ: The functional equation

$$\Theta(T(x,y)) = g(Q(x,y,\Theta(x,y)), \Theta(x,y))$$

has a unique solution $\Theta \in \mathbf{M}_2$, where

$$\Theta(x,y) = \theta(p(x,y), \pi(x,y)) = y.$$

We obtain that the mappings H, $\Gamma\colon \mathbf{X} \times \mathbf{Y} \to \mathbf{X} \times \mathbf{Y}$ defined by

$$H(x,y) = (p(x,y), \pi(x,y))$$

and

$$\Gamma(x, y) = (q(x, \theta(x, y)), \theta(x, y))$$

are inverse to each other and H is a homeomorphism establishing conjugacy of the mappings T and S. The theorem is proven.

Example 2.23 Assume in addition that the mapping (2.1) is a homeomorphism having a fixed point. Using Theorem 2.22 we obtain that the homeomorphism (2.1) is topologically conjugate to

$$\begin{aligned} x^1 &= Ax + F(x, u(x)), \\ y^1 &= By + G(v(y), y). \end{aligned}$$

2.7 Conjugacy of noninvertible mappings

We consider the case when the mapping T has an invariant set.

Theorem 2.24 *Let the hypotheses* (**H1**)–(**H4**) *hold and let there be a mapping* $u: \mathbf{X} \to \mathbf{Y}$ *that satisfies* (2.2) *and* (2.4). *Then there exists a continuous mapping* $q: \mathbf{X} \times \mathbf{Y} \to \mathbf{X}$, *which is Lipschitzian with respect to the second variable, and a homeomorphism* $H: \mathbf{X} \times \mathbf{Y} \to \mathbf{X} \times \mathbf{Y}$ *such that the diagram*

$$\begin{array}{ccc} \mathbf{X} \times \mathbf{Y} & \xrightarrow{\ T\ } & \mathbf{X} \times \mathbf{Y} \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle H} \\ \mathbf{X} \times \mathbf{Y} & \xrightarrow{\ R\ } & \mathbf{X} \times \mathbf{Y} \end{array}$$

commutes, where $R(x, y) = (f(x, u(x)), g(q(x, y), y))$.

Theorem 2.25 *Let the hypotheses* (**H1**)–(**H4**) *hold, and let there be a homeomorphism* $f_0: \mathbf{X} \to \mathbf{X}$ *such that* f_0^{-1} *satisfies the Lipschitz conditions with a constant less than* 1. *If* $\alpha(1 + \gamma l) < 1$ *and*

$$\sup_{x,y} \rho_1(f(x, y), f_0(x)) < +\infty,$$

then there exists a continuous mapping $q: \mathbf{X} \times \mathbf{Y} \to \mathbf{X}$, *which is Lipschitzian with respect to the second variable, and a homeomorphism* $H: \mathbf{X} \times \mathbf{Y} \to \mathbf{X} \times \mathbf{Y}$ *such that the diagram*

$$\begin{array}{ccc} \mathbf{X}\times\mathbf{Y} & \xrightarrow{\quad T \quad} & \mathbf{X}\times\mathbf{Y} \\ H \downarrow & & \downarrow H \\ \mathbf{X}\times\mathbf{Y} & \xrightarrow{\quad N_1 \quad} & \mathbf{X}\times\mathbf{Y} \end{array}$$

commutes, where $N_1(x, y) = (f_0(x), g(q(x, y), y))$.

Example 2.26 In the general case of noninvertible mappings using Theorem 2.24 we have that (2.1) is topologically conjugate to

$$\begin{aligned} x^1 &= Ax + F(x, u(x)), \\ y^1 &= By + G(q(x, y), y). \end{aligned}$$

2.8 Conjugacy of homeomorphisms. 2

We consider the case, when T is a homeomorphism without fixed points.

Theorem 2.27 *Let the hypotheses* (**H1**)–(**H4**) *hold, and let there be a mapping* $v: \mathbf{Y} \to \mathbf{X}$ *that satisfies* (2.3) *and* (2.5). *If* T *is a homeomorphism, then there exists a continuous mapping* $\theta: \mathbf{X} \times \mathbf{Y} \to \mathbf{Y}$, *which is Lipschitzian with respect to the first variable, and a homeomorphism* $H: \mathbf{X} \times \mathbf{Y} \to \mathbf{X} \times \mathbf{Y}$ *such that the diagram*

$$\begin{array}{ccc} \mathbf{X}\times\mathbf{Y} & \xrightarrow{\quad T \quad} & \mathbf{X}\times\mathbf{Y} \\ H \downarrow & & \downarrow H \\ \mathbf{X}\times\mathbf{Y} & \xrightarrow{\quad N \quad} & \mathbf{X}\times\mathbf{Y} \end{array}$$

commutes, where $N(x, y) = (f(x, \theta(x, y)), g(v(y), y))$.

Theorem 2.28 *Let the hypotheses* (**H1**)–(**H4**) *hold, and let there be a homeomorphism* $g_0: \mathbf{Y} \to \mathbf{Y}$ *such that satisfies the Lipschitz condition with a constant less than* 1. *If* T *is a homeomorphism,* $\beta k + \delta < 1$ *and*

$$\sup_{x,y} \rho_2(g(x, y), g_0(y)) < +\infty, \tag{2.8}$$

then there exists a homeomorphism $H: \mathbf{X} \times \mathbf{Y} \to \mathbf{X} \times \mathbf{Y}$ *such that the diagram*

$$\begin{array}{ccc} \mathbf{X}\times\mathbf{Y} & \xrightarrow{\quad T \quad} & \mathbf{X}\times\mathbf{Y} \\ H\downarrow & & \downarrow H \\ \mathbf{X}\times\mathbf{Y} & \xrightarrow{\quad R_0 \quad} & \mathbf{X}\times\mathbf{Y} \end{array}$$

commutes, where $R_0(x,y) = (f(x,u(x)), g_0(y))$.

Theorem 2.29 *Let the hypotheses* (**H1**)–(**H4**) *hold, and let there be a homeomorphism* $f_0 \colon \mathbf{X} \to \mathbf{X}$ *such that* f_0^{-1} *satisfies the Lipschitz conditions with a constant less than* 1*. If* T *is a homeomorphism,* $\alpha(1+\gamma l) < 1$ *and*

$$\sup_{x,y} \rho_1(f(x,y), f_0(x)) < +\infty, \tag{2.9}$$

then there exists a homeomorphism $H \colon \mathbf{X}\times\mathbf{Y} \to \mathbf{X}\times\mathbf{Y}$ *such that the diagram*

$$\begin{array}{ccc} \mathbf{X}\times\mathbf{Y} & \xrightarrow{\quad T \quad} & \mathbf{X}\times\mathbf{Y} \\ H\downarrow & & \downarrow H \\ \mathbf{X}\times\mathbf{Y} & \xrightarrow{\quad N_0 \quad} & \mathbf{X}\times\mathbf{Y} \end{array}$$

commutes, where $N_0(x,y) = (f_0(x), g(v(y), y))$.

Example 2.30 Let us consider the mapping (2.1). The condition (2.8) is fulfilled if

$$\sup_{x,y} |G(x,y) - G(0,y)| < +\infty$$

and (2.9) is fulfilled if

$$\sup_{x,y} |F(x,y) - F(x,0)| < +\infty.$$

Let the mapping (2.1) be a homeomorphism and let B be invertible, $\|B\| < 1$, $\sup_{x,y} |G(x,y)| < +\infty$ and

$$\varepsilon < \begin{cases} \dfrac{\|A^{-1}\|^{-1} - \|B\|}{4} & \text{if } \|A^{-1}\|^{-1} + \|B\| \le 2 \\[2ex] \dfrac{(\|A^{-1}\|^{-1} - 1)(1 - \|B\|)}{\|A^{-1}\|^{-1} - \|B\|} & \text{if } \|A^{-1}\|^{-1} + \|B\| > 2. \end{cases}$$

Using Theorem 2.28 we obtain that homeomorphism (2.1) is topologically conjugate to

$$\begin{aligned} x^1 &= Ax + F(x, u(x)), \\ y^1 &= By. \end{aligned}$$

Now suppose that $\|A^{-1}\| < 1$, $\sup_{x,y} |F(x, y)| < +\infty$ and

$$\varepsilon < \begin{cases} \dfrac{(\|A^{-1}\|^{-1} - 1)(1 - \|B\|)}{\|A^{-1}\|^{-1} - \|B\|} & \text{if } \|A^{-1}\|^{-1} + \|B\| \le 2 \\[2ex] \dfrac{\|A^{-1}\|^{-1} - \|B\|}{4} & \text{if } \|A^{-1}\|^{-1} + \|B\| > 2. \end{cases}$$

By Theorem 2.29 homeomorphism (2.1) is topologically conjugate to

$$\begin{aligned} x^1 &= Ax, \\ y^1 &= By + G(v(y), y). \end{aligned}$$

2.9 Notes

The results of this section are based on [105, 120, 121, 128, 129, 137].

3. Topological equivalence of discrete dynamical extensions

3.1 Preliminaries

In this chapter we set out some basic facts needed for later sections and specify the form of the mapping T.

Let $\mathbf{X}$ and $\mathbf{Y}$ be complete metric spaces with metrics ρ_1 and ρ_2, respectively, and let $\boldsymbol{\Lambda}$ be a topological space. The object of this chapter is to extend the reduction theorem for discrete dynamical (semidynamical) extensions generated by a homeomorphism (continuous mapping) in a complete metric space. We consider the continuous mapping T defined by

$$(x, y, \lambda) \mapsto (f(x, y, \lambda), g(x, y, \lambda), \sigma(\lambda)).$$

We will propose the following hypotheses:

(H1) $\rho_1(x, x') \leq \alpha\rho_1(f(x, y, \lambda), f(x', y, \lambda))$, $\alpha > 0$.

(H2) $\rho_1(f(x, y, \lambda), f(x, y', \lambda)) \leq \beta\rho_2(y, y')$.

(H3) $\rho_2(g(x, y, \lambda), g(x', y', \lambda)) \leq \gamma\rho_1(x, x') + \delta\rho_2(y, y')$, where $\alpha(\delta + 2\sqrt{\beta\gamma}) < 1$.

(H4) Mapping $f(\cdot, y, \lambda) \colon \mathbf{X} \to \mathbf{X}$ is surjective.

(H5) Mapping $\sigma \colon \boldsymbol{\Lambda} \to \boldsymbol{\Lambda}$ is a homeomorphism.

Our aim is to decouple and simplify the given mapping T by means of a topological transformation.

Example 3.1 Consider a nonautonomous system of difference equations on $\mathbb{Z}$ of the form

$$\begin{aligned} x(n+1) &= A(n)x(n) + F(x(n), y(n), n), \\ y(n+1) &= B(n)y(n) + G(x(n), y(n), n), \end{aligned}$$

where $x \in \mathbf{X}$, $y \in \mathbf{Y}$, $\mathbf{X}$ and $\mathbf{Y}$ are Banach spaces, $A(n)$ and $B(n)$ are bounded linear maps, $A(n)$ is invertible, $\|B(n)\| < \|A^{-1}(n)\|^{-1}$ and the maps $F\colon \mathbf{X} \times \mathbf{Y} \times \mathbb{Z} \to \mathbf{X}$, $G\colon \mathbf{X} \times \mathbf{Y} \times \mathbb{Z} \to \mathbf{Y}$ satisfy the Lipschitz conditions

$$|F(x, y, n) - F(x', y', n)| \le \varepsilon(|x - x'| + |y - y'|),$$
$$|G(x, y, n) - G(x', y', n)| \le \varepsilon(|x - x'| + |y - y'|).$$

It is easy to verify that this mapping satisfies the hypotheses (**H1**) – (**H5**), where $\alpha = ((\sup_n \|A^{-1}(n)\|)^{-1} - \varepsilon)^{-1}$, $\beta = \gamma = \varepsilon$, $\delta = \sup_n \|B(n)\| + \varepsilon$ and $\sigma(n) = n + 1$. The condition $\alpha(\delta + 2\sqrt{\beta\gamma}) < 1$ reduces to the inequality

$$\varepsilon < \frac{(\sup_n \|A^{-1}(n)\|)^{-1} - \sup_n \|B(n)\|}{4}.$$

The mapping given by formula $x_1 = A(n)x + F(x, y, n)$ for fixed n and y is surjective if $\varepsilon \sup_n \|A^{-1}(n)\| < 1$. Let us note that $\varepsilon \sup_n \|A^{-1}(n)\| < 1/4$.

3.2 Auxiliary lemmas

In order to prove the main results we use three lemmas. Let us consider the set of mappings

$$\mathbf{Lip}(k) = \{u \mid u\colon \mathbf{X} \times \boldsymbol{\Lambda} \to \mathbf{Y} \text{ and } \rho_2(u(x, \lambda), u(x', \lambda)) \le k\rho_1(x, x')\}.$$

Lemma 3.2 *Let $\alpha\beta k < 1$ and $u \in \mathbf{Lip}(k)$. Then the mapping $\varphi\colon \mathbf{X}\times\boldsymbol{\Lambda} \to \mathbf{X}\times\boldsymbol{\Lambda}$, defined by $\varphi(x, \lambda) = (f(x, u(x, \lambda), \lambda), \sigma(\lambda))$, is a homeomorphism.*

Next, introduce the operator $\mathcal{L}$ acting on $\mathbf{Lip}(k)$ defined by the equality

$$(\mathcal{L}u)(f(x, u(x, \lambda), \lambda), \sigma(\lambda)) = g(x, u(x, \lambda), \lambda).$$

Lemma 3.3 *There exists $k \ge 0$ such that $\mathcal{L}(\mathbf{Lip}(k)) \subset \mathbf{Lip}(k)$.*

Next, let us consider the set of mappings

$$\mathbf{Lip}(l) = \{v \mid v\colon \mathbf{Y} \times \boldsymbol{\Lambda} \to \mathbf{X} \text{ and } \rho_1(v(y, \lambda), v(y', \lambda)) \le l\rho_2(y, y')\}$$

and let us introduce the operator $\mathcal{K}$ acting on $\mathbf{Lip}(l)$ by the equality

$$f(\mathcal{K}v(y, \lambda), y, \lambda) = v(g(v(y, \lambda), y, \lambda), \sigma(\lambda)).$$

The operator $\mathcal{K}$ is well defined, because the mapping $f(\cdot, y, \lambda)\colon \mathbf{X} \to \mathbf{X}$ is surjective and hypothesis (**H1**) is fulfilled.

Lemma 3.4 *There exists an $l \ge 0$ such that $\mathcal{K}(\mathbf{Lip}(l)) \subset \mathbf{Lip}(l)$.*

3.3 Invariant sets

We will give the necessary and sufficient conditions for the existence of mappings $u: \mathbf{X} \times \boldsymbol{\Lambda} \to \mathbf{Y}$ and $v: \mathbf{Y} \times \boldsymbol{\Lambda} \to \mathbf{X}$, whose graphs are invariant sets.

Theorem 3.5 *Let the hypotheses* (**H1**) – (**H5**) *hold. For the existence of mappings* $u: \mathbf{X} \times \boldsymbol{\Lambda} \to \mathbf{Y}$ *and* $v: \mathbf{Y} \times \boldsymbol{\Lambda} \to \mathbf{X}$ *that satisfy the functional equations*

$$u(f(x, u(x, \lambda), \lambda), \sigma(\lambda)) = g(x, u(x, \lambda), \lambda), \tag{3.1}$$

$$f(v(y, \lambda), y, \lambda) = v(g(v(y, \lambda), y, \lambda), \sigma(\lambda)) \tag{3.2}$$

and the Lipschitz conditions

$$\rho_2(u(x, \lambda), u(x', \lambda)) \leq k\rho_1(x, x'), \tag{3.3}$$

$$\rho_1(v(y, \lambda), v(y', \lambda)) \leq l\rho_2(y, y') \tag{3.4}$$

it is necessary and sufficient that there exist continuous mappings $x_0: \boldsymbol{\Lambda} \to \mathbf{X}$ *and* $y_0: \boldsymbol{\Lambda} \to \mathbf{Y}$ *such that*

$$f(x_0(\lambda), y_0(\lambda), \lambda) = x_0(\sigma(\lambda)) \text{ and } g(x_0(\lambda), y_0(\lambda), \lambda) = y_0(\sigma(\lambda)).$$

Theorem 3.6 *Let the hypotheses* (**H1**) – (**H5**) *hold, and let* $\beta k + \delta < 1$. *For the existence of a mapping* $u: \mathbf{X} \times \boldsymbol{\Lambda} \to \mathbf{Y}$ *that satisfies the functional equation* (3.1) *and the Lipschitz condition* (3.3) *it is necessary and sufficient that there exists a mapping* $u_0 \in \mathbf{Lip}(k)$ *such that*

$$\sup_{x,\lambda} \rho_2(u_0(f(x, u_0(x, \lambda), \lambda), \sigma(\lambda)), g(x, u_0(x, \lambda), \lambda)) < +\infty. \tag{3.5}$$

Theorem 3.7 *Let the hypotheses* (**H1**) – (**H5**) *hold, and let* $\alpha(1 + \gamma l) < 1$. *For the existence of a mapping* $v: \mathbf{Y} \times \boldsymbol{\Lambda} \to \mathbf{X}$ *that satisfies the functional equation* (3.2) *and the Lipschitz condition* (3.4) *it is necessary and sufficient that there exists a mapping* $v_0 \in \mathbf{Lip}(l)$ *such that*

$$\sup_{y,\lambda} \rho_1(v_0(g(v_0(y, \lambda), y, \lambda), \sigma(\lambda)), f(v_0(y, \lambda), y, \lambda)) < +\infty. \tag{3.6}$$

Lemma 3.8 *Let* T *be a homeomorphism and let there be a continuous mapping* $v: \mathbf{Y} \times \boldsymbol{\Lambda} \to \mathbf{X}$ *satisfying* (3.2) *and* (3.4). *Then the mapping* $\psi: \mathbf{Y} \times \boldsymbol{\Lambda} \to \mathbf{Y} \times \boldsymbol{\Lambda}$, *defined by* $\psi(y, \lambda) = (g(v(y, \lambda), y, \lambda), \sigma(\lambda))$, *is a homeomorphism.*

Corollary 3.9 *Let* T *be a homeomorphism and let the mappings* $u: \mathbf{X} \times \boldsymbol{\Lambda} \to \mathbf{Y}$ *and* $v: \mathbf{Y} \times \boldsymbol{\Lambda} \to \mathbf{X}$ *satisfy* (3.1)–(3.4). *Then the mapping* $S: \mathbf{X} \times \mathbf{Y} \times \boldsymbol{\Lambda} \to \mathbf{X} \times \mathbf{Y} \times \boldsymbol{\Lambda}$ *defined by the equality* $S(x, y, \lambda) = (f(x, u(x, \lambda), \lambda), g(v(y, \lambda), y, \lambda), \sigma(\lambda))$ *is a homeomorphism.*

3.4 Conjugacy of homeomorphisms. 1

We now consider the case when mapping T is a homeomorphism.

Theorem 3.10 *Let the hypotheses* (**H1**)–(**H5**) *hold and let there exist continuous mappings* $x_0\colon \boldsymbol{\Lambda} \to \mathbf{X}$ *and* $y_0\colon \boldsymbol{\Lambda} \to \mathbf{Y}$ *such that*

$$f(x_0(\lambda), y_0(\lambda), \lambda) = x_0(\sigma(\lambda)) \text{ and } g(x_0(\lambda), y_0(\lambda), \lambda) = y_0(\sigma(\lambda)).$$

Then there exists a homeomorphism $H\colon \mathbf{X} \times \mathbf{Y} \times \boldsymbol{\Lambda} \to \mathbf{X} \times \mathbf{Y} \times \boldsymbol{\Lambda}$ *such that the diagram*

$$\begin{array}{ccc} \mathbf{X} \times \mathbf{Y} \times \boldsymbol{\Lambda} & \xrightarrow{\quad T \quad} & \mathbf{X} \times \mathbf{Y} \times \boldsymbol{\Lambda} \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle H} \\ \mathbf{X} \times \mathbf{Y} \times \boldsymbol{\Lambda} & \xrightarrow{\quad S \quad} & \mathbf{X} \times \mathbf{Y} \times \boldsymbol{\Lambda} \end{array}$$

commutes, where $S(x, y, \lambda) = (f(x, u(x, \lambda), \lambda), g(v(y, \lambda), y, \lambda), \sigma(\lambda))$.

3.5 Conjugacy of noninvertible mappings

Theorem 3.11 *Let the hypotheses* (**H1**)–(**H4**) *hold and let there be a mapping* $u\colon \mathbf{X} \times \boldsymbol{\Lambda} \to \mathbf{Y}$ *that satisfies* (3.1) *and* (3.3)*. Then there exists a continuous mapping* $q\colon \mathbf{X} \times \mathbf{Y} \times \boldsymbol{\Lambda} \to \mathbf{X}$*, which is Lipschitzian with respect to the second variable, and a homeomorphism* $H\colon \mathbf{X} \times \mathbf{Y} \to \mathbf{X} \times \mathbf{Y}$ *such that the diagram*

$$\begin{array}{ccc} \mathbf{X} \times \mathbf{Y} \times \boldsymbol{\Lambda} & \xrightarrow{\quad T \quad} & \mathbf{X} \times \mathbf{Y} \times \boldsymbol{\Lambda} \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle H} \\ \mathbf{X} \times \mathbf{Y} \times \boldsymbol{\Lambda} & \xrightarrow{\quad R \quad} & \mathbf{X} \times \mathbf{Y} \times \boldsymbol{\Lambda} \end{array}$$

commutes, where $R(x, y, \lambda) = (f(x, u(x, \lambda), \lambda), g(q(x, y, \lambda), y, \lambda), \sigma(\lambda))$.

Let continuous mapping $f_0\colon \mathbf{X} \times \boldsymbol{\Lambda} \to \mathbf{X}$ satisfy conditions:

(i) $\rho_1(x, x') \leq c_1 \rho_1(f_0(x, \lambda), f_0(x', \lambda))$ and $c_1 < 1$.

(ii) Mapping $f_0(\cdot, \lambda)\colon \mathbf{X} \to \mathbf{X}$ is surjective.

Theorem 3.12 *Let the hypotheses* (**H1**)–(**H4**) *hold. If* $\alpha(1+\gamma l) < 1$ *and*

$$\sup_{x,y,\lambda} \rho_1(f(x,y,\lambda), f_0(x,\lambda)) < +\infty,$$

then there exists a continuous mapping $q\colon \mathbf{X}\times\mathbf{Y}\times\boldsymbol{\Lambda} \to \mathbf{X}$, *which is Lipschitzian with respect to the second variable, and a homeomorphism* $H\colon \mathbf{X}\times\mathbf{Y}\times\boldsymbol{\Lambda} \to \mathbf{X}\times\mathbf{Y}\times\boldsymbol{\Lambda}$ *such that the diagram*

$$\begin{array}{ccc} \mathbf{X}\times\mathbf{Y}\times\boldsymbol{\Lambda} & \xrightarrow{\quad T \quad} & \mathbf{X}\times\mathbf{Y}\times\boldsymbol{\Lambda} \\ H \big\downarrow & & \big\downarrow H \\ \mathbf{X}\times\mathbf{Y}\times\boldsymbol{\Lambda} & \xrightarrow{\quad N_1 \quad} & \mathbf{X}\times\mathbf{Y}\times\boldsymbol{\Lambda} \end{array}$$

commutes, where $N_1(x,y,\lambda) = (f_0(x,\lambda), g(q(x,y,\lambda),y,\lambda), \sigma(\lambda))$.

3.6 Conjugacy of homeomorphisms. 2

Theorem 3.13 *Let the hypotheses* (**H1**)–(**H5**) *hold, and let there be a mapping* $v\colon \mathbf{Y}\times\boldsymbol{\Lambda} \to \mathbf{X}$ *that satisfies* (3.2) *and* (3.4). *If* T *is a homeomorphism, then there exists a continuous mapping* $\theta\colon \mathbf{X}\times\mathbf{Y}\times\boldsymbol{\Lambda} \to \mathbf{Y}$, *which is Lipschitzian with respect to the first variable, and a homeomorphism* $H\colon \mathbf{X}\times\mathbf{Y}\times\boldsymbol{\Lambda} \to \mathbf{X}\times\mathbf{Y}\times\boldsymbol{\Lambda}$ *such that the diagram*

$$\begin{array}{ccc} \mathbf{X}\times\mathbf{Y}\times\boldsymbol{\Lambda} & \xrightarrow{\quad T \quad} & \mathbf{X}\times\mathbf{Y}\times\boldsymbol{\Lambda} \\ H \big\downarrow & & \big\downarrow H \\ \mathbf{X}\times\mathbf{Y}\times\boldsymbol{\Lambda} & \xrightarrow{\quad N \quad} & \mathbf{X}\times\mathbf{Y}\times\boldsymbol{\Lambda} \end{array}$$

commutes, where $N(x,y,\lambda) = (f(x,\theta(x,y,\lambda)), g(v(y,\lambda),y,\lambda), \sigma(\lambda))$.

Let mapping $g_0\colon \mathbf{Y}\times\boldsymbol{\Lambda} \to \mathbf{Y}$ satisfy conditions:

(i) $\rho_2(g_0(y,\lambda), g_0(y',\lambda)) \le c_2\rho_2(y,y')$ and $c_2 < 1$.

(ii) Mapping defined by $(y,\lambda) \mapsto (g_0(y,\lambda), \sigma(\lambda))$ is a homeomorphism.

Theorem 3.14 *Let the hypotheses* (**H1**)–(**H5**) *hold, and let there be a mapping* $u\colon \mathbf{X}\times\boldsymbol{\Lambda} \to \mathbf{Y}$ *that satisfies* (3.1) *and* (3.3). *If* T *is a homeomorphism,* $\beta k+\delta < 1$ *and*

$$\sup_{x,y,\lambda} \rho_2(g(x,y,\lambda), g_0(y,\lambda)) < +\infty, \tag{3.7}$$

then there exists a homeomorphism $H\colon \mathbf{X} \times \mathbf{Y} \times \boldsymbol{\Lambda} \to \mathbf{X} \times \mathbf{Y} \times \boldsymbol{\Lambda}$ *such that the diagram*

$$\begin{array}{ccc} \mathbf{X} \times \mathbf{Y} \times \boldsymbol{\Lambda} & \xrightarrow{\quad T \quad} & \mathbf{X} \times \mathbf{Y} \times \boldsymbol{\Lambda} \\ H \big\downarrow & & \big\downarrow H \\ \mathbf{X} \times \mathbf{Y} \times \boldsymbol{\Lambda} & \xrightarrow{\quad R_0 \quad} & \mathbf{X} \times \mathbf{Y} \times \boldsymbol{\Lambda} \end{array}$$

commutes, where $R_0(x,y,\lambda) = (f(x, u(x,\lambda), \lambda), g_0(y,\lambda), \sigma(\lambda))$.

Let mapping $f_0\colon \mathbf{X} \times \boldsymbol{\Lambda} \to \mathbf{X}$ satisfy conditions:

(i) $\rho_1(x,x') \le c_1\rho_1(f_0(x,\lambda), f_0(x',\lambda))$ and $c_1 < 1$.

(ii) Mapping defined by $(x,\lambda) \mapsto (f_0(x,\lambda), \sigma(\lambda))$ is a homeomorphism.

Theorem 3.15 *Let the hypotheses* (**H1**)–(**H5**) *hold, and let there be a mapping* $v\colon \mathbf{Y} \times \boldsymbol{\Lambda} \to \mathbf{X}$ *that satisfies* (3.2) *and* (3.4). *If* T *is a homeomorphism,* $\alpha(1 + \gamma l) < 1$ *and*

$$\sup_{x,y,\lambda} \rho_1(f(x,y,\lambda), f_0(x,\lambda)) < +\infty, \tag{3.8}$$

then there exists a homeomorphism $H\colon \mathbf{X} \times \mathbf{Y} \times \boldsymbol{\Lambda} \to \mathbf{X} \times \mathbf{Y} \times \boldsymbol{\Lambda}$ *such that the diagram*

$$\begin{array}{ccc} \mathbf{X} \times \mathbf{Y} \times \boldsymbol{\Lambda} & \xrightarrow{\quad T \quad} & \mathbf{X} \times \mathbf{Y} \times \boldsymbol{\Lambda} \\ H \big\downarrow & & \big\downarrow H \\ \mathbf{X} \times \mathbf{Y} \times \boldsymbol{\Lambda} & \xrightarrow{\quad N_0 \quad} & \mathbf{X} \times \mathbf{Y} \times \boldsymbol{\Lambda} \end{array}$$

commutes, where $N_0(x,y,\lambda) = (f_0(x,\lambda), g(v(y,\lambda), y, \lambda), \sigma(\lambda))$.

3.7 Notes

The results of this section are based on [127, 130, 131, 132, 133, 136, 139, 140, 144, 145, 147].

4. Equivalence of impulsive differential equations

4.1 Introduction

The dynamical equivalence for systems of impulsive differential equations was first considered by the author and L. Sermone, and later D. D. Bainov, S. I. Kostadinov and Nguyen Van Minh. In the present chapter, different modifications of the reduction theorem for systems of impulsive differential equations in a Banach space (also for noninvertible systems) are proved assuming that the systems split into two parts. Often it is possible to use the reduction theorems many times, which allows further simplification of the given system. Using the standard technique, local modifications of the reduction theorem are obtainable.

4.2 Preliminaries

Let $\mathbf{X}$ and $\mathbf{Y}$ be Banach spaces. By $\mathcal{L}(\mathbf{X})$ and $\mathcal{L}(\mathbf{Y})$ we mean the Banach spaces of bounded linear operators. Consider the following system of impulsive differential equations:

$$\begin{cases} dx/dt &= A(t)x + f(t,x,y), \\ dy/dt &= B(t)y + g(t,x,y), \\ \Delta x|_{t=\tau_i} &= x(\tau_i+0) - x(\tau_i-0) \\ &= C_i x(\tau_i-0) + p_i(x(\tau_i-0), y(\tau_i-0)), \\ \Delta y|_{t=\tau_i} &= y(\tau_i+0) - y(\tau_i-0) \\ &= D_i y(\tau_i-0) + q_i(x(\tau_i-0), y(\tau_i-0)), \end{cases} \tag{4.1}$$

where

(i) the mappings $A\colon \mathbb{R} \to \mathcal{L}(\mathbf{X})$ and $B\colon \mathbb{R} \to \mathcal{L}(\mathbf{Y})$ are locally integrable in the Bochner sense;

(ii) the mappings $f: \mathbb{R} \times \mathbf{X} \times \mathbf{Y} \to \mathbf{X}$ and $g: \mathbb{R} \times \mathbf{X} \times \mathbf{Y} \to \mathbf{Y}$ are locally integrable in the Bochner sense with respect to t for fixed x and y, and, in addition, they satisfy the Lipschitz conditions

$$|f(t,x,y) - f(t,x',y')| \le \varepsilon(|x-x'| + |y-y'|),$$

$$|g(t,x,y) - g(t,x',y')| \le \varepsilon(|x-x'| + |y-y'|);$$

(iii) for $i \in \mathbf{Z}$, $C_i \in \mathcal{L}(\mathbf{X})$, $D_i \in \mathcal{L}(\mathbf{Y})$, the mappings $p_i: \mathbf{X} \times \mathbf{Y} \to \mathbf{X}$, $q_i: \mathbf{X} \times \mathbf{Y} \to \mathbf{Y}$ satisfy the Lipschitz conditions

$$|p_i(x,y) - p_i(x',y')| \le \varepsilon(|x-x'| + |y-y'|),$$

$$|q_i(x,y) - q_i(x',y')| \le \varepsilon(|x-x'| + |y-y'|);$$

(iv) the mappings $(x,y) \mapsto (x + C_i x + p_i(x,y), y + D_i y + q_i(x,y))$, $x \mapsto x + C_i x$ are homeomorphisms;

(v) the moments τ_i of impulse form a strictly increasing sequence

$$\ldots < \tau_{-2} < \tau_{-1} < \tau_0 < \tau_1 < \tau_2 < \ldots,$$

where the limit points may be only $\mp\infty$.

We now proceed with definitions of the solution for a systems of impulsive differential equations and dynamically equivalence in the large.

Definition 4.1 By the *solution* to an impulsive system we mean a piecewise absolutely continuous mapping with discontinuities of the first kind at the points $t = \tau_i$ which for almost all t satisfies system (4.1) and for $t = \tau_i$, satisfies the conditions of a "jump".

Note that condition (iv) implies continuability of solutions of (4.1) in the negative direction. Furthermore, condition (v), together with the Lipschitz property with respect to x and y of the right–hand side, ensures that there is a unique solution defined on $\mathbb{R}$.

Let $\Phi(\cdot, s, x, y) = (x(\cdot, s, x, y), y(\cdot, s, x, y)): \mathbb{R} \to \mathbf{X} \times \mathbf{Y}$ be the solution of system (4.1), where $\Phi(s+0, s, x, y) = (x(s+0, s, x, y), y(s+0, s, x, y)) = (x, y)$. At the break points τ_i the values for all solutions are taken at $\tau_i + 0$ unless otherwise indicated. For short, we will use the notation $\Phi(t) = (x(t), y(t))$.

Let $\mathbf{U}$ be a Banach space. Consider two impulsive differential equations

$$du/dt = P(t,u), \;\; \Delta u|_{t=\tau_i} = S_i(u(\tau_i - 0)) \tag{4.2}$$

and

$$du/dt = Q(t,u), \;\; \Delta u|_{t=\tau_i} = T_i(u(\tau_i - 0)) \tag{4.3}$$

that satisfy the conditions of the existence and uniqueness theorem. We assume that the maximum interval of the existence of the solutions is $\mathbb{R}$. Let $\phi(\cdot, s, u)\colon \mathbb{R} \to \mathbf{U}$ and $\psi(\cdot, s, u)\colon \mathbb{R} \to \mathbf{U}$ be the solutions of the above equations, respectively. Suppose that there is a function $e\colon \mathbf{U} \to \mathbb{R}_+$ such that

$$\max\{|P(t,u) - Q(t,u)|, \sup_i |S_i(u) - T_i(u)|\} \le e(u).$$

Definition 4.2 The two impulsive differential equations (4.2) and (4.3) are *dynamically equivalent in the large* if there exists a mapping $H\colon \mathbb{R} \times \mathbf{U} \to \mathbf{U}$ and a positive constant c such that:

(i) $H(t,\cdot)\colon \mathbf{U} \to \mathbf{U}$ is a homeomorphism;

(ii) $H(t, \phi(t,s,u)) = \psi(t, s, H(s,u))$ for all $t \in \mathbb{R}$;

(iii) $\max\{|H(t,u) - u|, |H^{-1}(t,u) - u|\} \le ce(u)$;

(iv) in case the differential equations are autonomous and have no impulse effect, then the mapping H does not depend on t.

Note that without (iii) and (iv) the concept of dynamical equivalence would be trivial, since in this case the equality $H(s,u) = \psi(s, 0, \phi(0, s, u))$ gives a dynamical equivalence. It is significant that in the case of the classical global Grobman–Hartman theorem for autonomous differential equations, the corresponding function $e(x) = a > 0$ and appropriate constant c depend on the linear truncation only.

4.3 Auxiliary lemmas

Let $X(t,\tau)$ and $Y(t,\tau)$ be the evolutionary operators of the impulsive linear differential equations

$$\begin{cases} dx/dt & = & A(t)x, \\ \Delta x|_{t=\tau_i} & = & x(\tau_i + 0) - x(\tau_i - 0) = C_i x(\tau_i - 0) \end{cases}$$

and, respectively,

$$\begin{cases} dy/dt & = & B(t)y, \\ \Delta y|_{t=\tau_i} & = & y(\tau_i + 0) - y(\tau_i - 0) = D_i y(\tau_i - 0). \end{cases}$$

Assume that

$$\nu_1 = \sup_s \int_{-\infty}^{s} |Y(s,t)||X(t,s)|\,dt + \sup_s \sum_{\tau_i \le s} |Y(s,\tau_i)||X(\tau_i - 0, s)|,$$

$$\nu_2 = \sup_s \int_{s}^{+\infty} |X(s,t)||Y(t,s)|\,dt + \sup_s \sum_{s < \tau_i} |X(s,\tau_i)||Y(\tau_i - 0, s)|$$

and

$$\nu = \max\{\nu_1, \nu_2\}.$$

Let $\mathbf{PC}(\mathbb{R}\times\mathbf{X},\mathbf{Y})$ be a set of mappings $u\colon \mathbb{R}\times\mathbf{X}\to\mathbf{Y}$ that are continuous for $(t,x)\in[\tau_i,\tau_{i+1})\times\mathbf{X}$ and have discontinuities of the first kind for $t=\tau_i$. The set

$$\mathbf{B}_1 = \left\{ u \in \mathbf{PC}(\mathbb{R}\times\mathbf{X},\mathbf{Y}) \;\middle|\; \sup_{s,x} |u(s,x)| < +\infty \right\}$$

becomes a Banach space if we use the norm

$$\|u\| = \sup_{s,x} |u(s,x)|.$$

For $k>0$ the set

$$\mathbf{B}_1(k) = \{u \in \mathbf{B}_1 \mid |u(s,x) - u(s,x')| \le k|x-x'|\}$$

is a closed subset of $\mathbf{B}_1$. Assume that

$$\mu_1 = \sup_s \left(\int_{-\infty}^{s} |Y(s,t)|\,dt + \sum_{\tau_i \le s} |Y(s,\tau_i)| \right) < +\infty.$$

Lemma 4.3 *Let $u, u' \in \mathbf{B}_1(k)$ and let $\varepsilon(1+k)\nu_1 < 1$. Then the following estimate is valid:*

$$\int_{-\infty}^{s} |Y(s,t)||z(t) - z'(t)|\,dt + \sum_{\tau_i \le s} |Y(s,\tau_i)||z(\tau_i - 0) - z'(\tau_i - 0)|$$

$$\le \nu_1 \left(1 - \varepsilon\nu_1(1+k)\right)^{-1} \left(|x-x'| + \varepsilon\mu_1\|u-u'\|\right),$$

where $z\colon(-\infty,s]\to\mathbf{X}$ is the solution of the impulsive differential equation

$$\begin{cases} dz/dt &= A(t)z + f(t,z,u(t,z)), \;\; z(s) = x, \\ \Delta z|_{t=\tau_i} &= C_i z(\tau_i - 0) + p_i(z(\tau_i - 0), u(\tau_i - 0, z(\tau_i - 0))). \end{cases}$$

The set

$$\mathbf{N}_1 = \left\{ u \in \mathbf{PC}(\mathbb{R} \times \mathbf{X}, \mathbf{Y}) \;\middle|\; \sup_{s,x \neq 0} \frac{|u(s,x)|}{|x|} < +\infty \right\}$$

becomes a Banach space if we use the norm

$$\|u\| = \sup_{s,x \neq 0} \frac{|u(s,x)|}{|x|}.$$

For $k > 0$ the set

$$\mathbf{N}_1(k) = \{u \in \mathbf{N}_1 \mid |u(s,x) - u(s,x')| \leq k|x - x'|\}$$

is a closed subset of $\mathbf{N}_1$.

Lemma 4.4 *Let $u, u' \in \mathbf{N}_1(k)$, $f(t,0,0) = 0$, $p_i(0,0) = 0$ and $\varepsilon(1+k)\nu_1 < 1$. Then the following estimate is valid:*

$$\int_{-\infty}^{s} |Y(s,t)||z(t) - z'(t)|\, dt + \sum_{\tau_i \leq s} |Y(s,\tau_i)||z(\tau_i - 0) - z'(\tau_i - 0)|$$

$$\leq \nu_1 \left(1 - \varepsilon\nu_1(1+k)\right)^{-1} \left(|x - x'| + \varepsilon\nu_1(1 - \varepsilon\nu_1(1+k))^{-1}|x|\|u - u'\|\right),$$

where $z\colon (-\infty, s] \to \mathbf{X}$ is the solution of the impulsive differential equation

$$\begin{cases} dz/dt &= A(t)z + f(t,z,u(t,z)), \;\; z(s) = x, \\ \Delta z|_{t=\tau_i} &= C_i z(\tau_i - 0) + p_i(z(\tau_i - 0), u(\tau_i - 0, z(\tau_i - 0))). \end{cases}$$

Remark. Lemmas 4.3 and 4.4 are also valid for noninvertible impulsive systems if the condition $\varepsilon(1+k)\nu_1 < 1$ is replaced by the stronger condition

$$\varepsilon(1+k)\max\{\nu_1, \sup_i |(id_x + C_i)^{-1}|\} < 1.$$

The set

$$\mathbf{B}_2 = \left\{ v \in \mathbf{PC}(\mathbb{R} \times \mathbf{Y}, \mathbf{X}) \;\middle|\; \sup_{s,y} |v(s,y)| < +\infty \right\}$$

becomes a Banach space if we use the norm

$$\|v\| = \sup_{s,y} |v(s,y)|.$$

For $l > 0$ the set

$$\mathbf{B}_2(l) = \{v \in \mathbf{B}_2 \mid |v(s,y) - v(s,y')| \le l|y-y'|\}$$

is a closed subset of $\mathbf{B}_2$. Assume that

$$\mu_2 = \sup_s \left(\int_s^{+\infty} |X(s,t)|\, dt + \sum_{s<\tau_i} |X(s,\tau_i)| \right) < +\infty.$$

Lemma 4.5 *Let $v, v' \in \mathbf{B}_2(l)$ and let $\varepsilon(1+l)\nu_2 < 1$. Then the following estimate is valid:*

$$\int_s^{+\infty} |X(s,t)||w(t) - w'(t)|\, dt + \sum_{s<\tau_i} |X(s,\tau_i)||w(\tau_i - 0) - w'(\tau_i - 0)|$$

$$\le \nu_2 \left(1 - \varepsilon\nu_2(1+l)\right)^{-1} \left(|y-y'| + \varepsilon\mu_2\|v - v'\|\right),$$

where $w\colon [s,+\infty) \to \mathbf{Y}$ is the solution of the impulsive differential equation

$$\begin{cases} dw/dt &= B(t)w + g(t, v(t,w), w), \;\; w(s) = y, \\ \Delta w|_{t=\tau_i} &= D_i w(\tau_i - 0) + q_i(v(\tau_i - 0, w(\tau_i - 0)), w(\tau_i - 0)). \end{cases}$$

Analogously, the set

$$\mathbf{N}_2 = \left\{ v \in \mathbf{PC}(\mathbb{R} \times \mathbf{Y}, \mathbf{X}) \,\middle|\, \sup_{s, y \ne 0} \frac{|v(s,y)|}{|y|} < +\infty \right\}$$

becomes a Banach space if we use the norm

$$\|v\| = \sup_{s,y\ne 0} \frac{|v(s,y)|}{|y|}.$$

For $l > 0$ the set

$$\mathbf{N}_2(l) = \{v \in \mathbf{N}_2 \mid |v(s,y) - v(s,y')| \le l|y-y'|\}$$

is a closed subset of $\mathbf{N}_2$.

Lemma 4.6 *Let $v, v' \in \mathbf{N}_2(l)$, $g(t,0,0) = 0$, $q_i(0,0) = 0$ and $\varepsilon(1+l)\nu_2 < 1$. Then the following estimate is valid:*

$$\int_s^{+\infty} |X(s,t)||w(t) - w'(t)|\, dt + \sum_{s<\tau_i} |X(s,\tau_i)||w(\tau_i - 0) - w'(\tau_i - 0)|$$

$$\le \nu_2 \left(1 - \varepsilon\nu_2(1+l)\right)^{-1} \left(|y-y'| + \varepsilon\nu_2(1-\varepsilon\nu_2(1+l))^{-1}|y|\|v - v'\| \right),$$

where $w\colon [s,+\infty) \to \mathbf{Y}$ is the solution of the impulsive differential equation

$$\begin{cases} dw/dt &= B(t)w + g(t, v(t,w), w), \;\; w(s) = y, \\ \Delta w|_{t=\tau_i} &= D_i w(\tau_i - 0) + q_i(v(\tau_i - 0, w(\tau_i - 0)), w(\tau_i - 0)). \end{cases}$$

Later in this chapter we assume that

$$k = (2\varepsilon\nu_1)^{-1}(1 - 2\varepsilon\nu_1 - \sqrt{1 - 4\varepsilon\nu_1})$$

and

$$l = (2\varepsilon\nu_2)^{-1}(1 - 2\varepsilon\nu_2 - \sqrt{1 - 4\varepsilon\nu_2}).$$

4.4 Invariant sets

Invariant sets play a significant role in the equivalence theory.

Theorem 4.7 *Let* $4\varepsilon\nu < 1$, $f(t,0,0) = 0$, $g(t,0,0) = 0$, $p_i(0,0) = 0$ *and* $q_i(0,0) = 0$. *Then there exists a unique pair of mappings* $u \in \mathbf{N}_1(k)$ *and* $v \in \mathbf{N}_2(l)$ *with the following properties:*

(i) $u(t, x(t,s,x,u(s,x))) = y(t,s,x,u(s,x))$ *for all* $t \in \mathbb{R}$;

(ii) $|u(s,x) - u(s,x')| \le k|x - x'|$;

(iii) $$\int_s^{+\infty} |X(s,t)||y(t,s,x,y) - u(t, x(t,s,x,y))|\,dt + \sum_{s<\tau_i} |X(s,\tau_i)||y(\tau_i - 0, s, x, y) - u(\tau_i - 0, x(\tau_i - 0, s, x, y))| \le \nu_2(1 - \varepsilon(1+k)\nu_2)^{-1}|y - u(s,x)|;$$

(iv) $v(t, y(t,s,v(s,y),y)) = x(t,s,v(s,y),y)$ *for all* $t \in \mathbb{R}$;

(v) $|v(s,y) - v(s,y')| \le l|y - y'|$;

(vi) $$\int_{-\infty}^{s} |Y(s,t)||x(t,s,x,y) - v(t, y(t,s,x,y))|\,dt + \sum_{\tau_i \le s} |Y(s,\tau_i)||x(\tau_i - 0, s, x, y) - v(\tau_i - 0, y(\tau_i - 0, s, x, y))| \le \nu_1(1 - \varepsilon(1+l)\nu_1)^{-1}|x - v(s,y)|.$$

Proof. For the functional equations

$$u(s,x) = \int_{-\infty}^{s} Y(s,\tau) g(\tau, z(\tau), u(\tau, z(\tau)))\,d\tau$$

$$+\sum_{\tau_i \le s} Y(s,\tau_i) q_i(z(\tau_i - 0), u(\tau_i - 0, z(\tau_i - 0)))$$

and

$$v(s,y) = \int_s^{+\infty} X(s,\tau) f(\tau, v(\tau, w(\tau)), w(\tau))\, d\tau$$
$$+\sum_{s<\tau_i} X(s,\tau_i) p_i(v(\tau_i - 0, w(\tau_i - 0)), w(\tau_i - 0))$$

in $\mathbf{N}_1(k)$ (respectively in $\mathbf{N}_2(l)$) there exists a unique solution, where $z\colon (-\infty, s] \to \mathbf{X}$ is the solution of the impulsive differential equations

$$\begin{cases} dz/dt &= A(t)z + f(t,z,u(t,z)),\ z(s) = x, \\ \Delta z|_{t=\tau_i} &= C_i z(\tau_i - 0) + p_i(z(\tau_i - 0), u(\tau_i - 0, z(\tau_i - 0))) \end{cases}$$

and $w\colon [s, +\infty) \to \mathbf{Y}$ is the solution of the impulsive differential equations

$$\begin{cases} dw/dt &= B(t)w + g(t, v(t,w), w),\ w(s) = y, \\ \Delta z|_{t=\tau_i} &= D_i w(\tau_i - 0) + q_i(v(\tau_i - 0, w(\tau_i - 0)), w(\tau_i)). \end{cases}$$

Remark. The conditions (i)–(v) of Theorem 4.7 are also valid for noninvertible impulse systems if we add the condition:

$$2\varepsilon \sup_i |(id_x + C_i)^{-1}| < 1 + \sqrt{1 - 4\varepsilon\nu_1}.$$

Theorem 4.8 *Let* $4\varepsilon\nu \le 1$, $\sup_{t,x} |g(t,x,0)| < +\infty$, $\sup_{i,x} |q_i(x,0)| < +\infty$ *and* $2\varepsilon\mu_1 < 1 + \sqrt{1 - 4\varepsilon\nu_1}$. *Then there exists a unique mapping* $u \in \mathbf{B}_1(k)$ *with the following properties:*

(i) $u(t, x(t,s,x,u(s,x))) = y(t,s,x,u(s,x))$ *for all* $t \in \mathbb{R}$;

(ii) $|u(s,x) - u(s,x')| \le k|x - x'|$;

(iii) $$\int_s^{+\infty} |X(s,t)||y(t,s,x,y) - u(t, x(t,s,x,y))|\, dt$$
$$+ \sum_{s<\tau_i} |X(s,\tau_i)||y(\tau_i - 0, s, x, y) - u(\tau_i - 0, x(\tau_i - 0, s, x, y))|$$
$$\le \nu_2 (1 - \varepsilon(1+k)\nu_2)^{-1} |y - u(s,x)|.$$

Remark. Theorem 4.8 is also valid for noninvertible impulse systems if we add the condition

$$2\varepsilon \sup_i |(id_x + C_i)^{-1}| < 1 + \sqrt{1 - 4\varepsilon\nu_1}.$$

Theorem 4.9 *Let* $4\varepsilon\nu \le 1$, $\sup_{t,y} |f(t,0,y)| < +\infty$, $\sup_{i,y} |p_i(0,y)| < +\infty$ *and* $2\varepsilon\mu_2 < 1 + \sqrt{1-4\varepsilon\nu_2}$. *Then there exists a unique mapping* $v \in \mathbf{B}_2(l)$ *with the following properties:*

(iv) $v(t, y(t,s,v(s,y),y)) = x(t,s,v(s,y),y)$ *for all* $t \in \mathbb{R}$*;*

(v) $|v(s,y) - v(s,y')| \le l|y-y'|$*;*

(vi)
$$\int_{-\infty}^{s} |Y(s,t)||x(t,s,x,y) - v(t,y(t,s,x,y))|\,dt + \sum_{\tau_i \le s} |Y(s,\tau_i)||x(\tau_i - 0, s,x,y) - v(\tau_i - 0, y(\tau_i - 0, s,x,y))| \le \nu_1(1-\varepsilon(1+l)\nu_1)^{-1}|x - v(s,y)|.$$

4.5 Dynamical equivalence of invertible systems. 1

Consider now a system of reduced impulsive differential equations

$$\begin{cases} dx/dt &= A(t)x + f(t,x,u(t,x)), \\ dy/dt &= B(t)y + g(t,v(t,y),y), \\ \Delta x|_{t=\tau_i} &= C_i x(\tau_i - 0) + p_i(x(\tau_i - 0), u(\tau_i - 0, x(\tau_i - 0))), \\ \Delta y|_{t=\tau_i} &= D_i y(\tau_i - 0) + q_i(v(\tau_i - 0, y(\tau_i - 0)), y(\tau_i - 0)). \end{cases} \tag{4.4}$$

This system splits into two parts. The first of them does not contain the variable y, while the second one is independent of x. Let $\Psi(\cdot,s,x,y) = (x_0(\cdot,s,x), y_0(\cdot,s,y))\colon \mathbb{R} \to \mathbf{X}\times\mathbf{Y}$ be a solution of system (25), where $\Psi(s+0,s,x,y) = (x,y)$. For short, we will use the notation $\Psi(t) = (x_0(t), y_0(t))$.

Theorem 4.10 *Let* $4\varepsilon\nu < 1$ *and let there exist mappings* $u\colon \mathbb{R}\times\mathbf{X} \to \mathbf{Y}$ *and* $v\colon \mathbb{R}\times\mathbf{Y} \to \mathbf{X}$ *satisfying* (i) – (vi). *Then systems* (4.1) *and* (4.4) *are dynamically equivalent in the large.*

Proof. The proof of the theorem consists of several steps.

Step 1. The space

$$\mathbf{N}_3 = \left\{ \kappa \in \mathbf{PC}(\mathbb{R}\times\mathbf{X}\times\mathbf{Y}, \mathbf{X}) \,\middle|\, \sup_{s,x,y} \frac{|\kappa(s,x,y)|}{|y - u(s,x)|} < +\infty \right\}$$

equipped with the norm

$$\|\kappa\| = \sup_{s,x,y} \frac{|\kappa(s,x,y)|}{|y-u(s,x)|}$$

is a Banach space. In $\mathbf{N}_3$, there exists a unique solution of the functional equation

$$\kappa_1(s,x,y)$$
$$= \int_s^{+\infty} X(s,\tau)(f(\tau,\Phi(\tau)) - f(\tau, x(\tau)+\kappa_1(\tau,\Phi(\tau)), u(\tau, x(\tau)+\kappa_1(\tau,\Phi(\tau))))) \, d\tau$$
$$+ \sum_{s<\tau_i} X(s,\tau_i)(p_i(\Phi(\tau_i - 0))$$
$$-p_i(x(\tau_i-0)+\kappa_1(\tau_i-0,\Phi(\tau_i-0)), u(\tau_i-0, x(\tau_i-0)+\kappa_1(\tau_i-0,\Phi(\tau_i-0))))).$$

Step 2. The space

$$\mathbf{N}_4 = \left\{ \lambda \in \mathbf{PC}(\mathbb{R}\times\mathbf{X}\times\mathbf{Y},\mathbf{Y}) \;\middle|\; \sup_{s,x,y} \frac{|\lambda(s,x,y)|}{|x - v(s,y)|} < +\infty \right\}$$

equipped with the norm

$$\|\lambda\| = \sup_{s,x,y} \frac{|\lambda(s,x,y)|}{|x - v(s,y)|}$$

is a Banach space. In $\mathbf{N}_4$, there exists a unique solution of the functional equation

$$\lambda_1(s,x,y)$$
$$= \int_{-\infty}^{s} Y(s,\tau)(g(\tau, v(\tau, y(\tau)+\lambda_1(\tau,\Phi(\tau))), y(\tau)+\lambda_1(\tau,\Phi(\tau))) - g(\tau,\Phi(\tau))) \, d\tau$$
$$+ \sum_{\tau_i \le s} Y(s,\tau_i)(q_i(v(\tau_i-0, y(\tau_i-0)+\lambda_1(\tau_i-0,\Phi(\tau_i-0))), y(\tau_i-0)$$
$$+\lambda_1(\tau_i-0,\Phi(\tau_i-0))) - q_i(\Phi(\tau_i-0))).$$

Define the mapping H_1 by the equality

$$H_1(s,x,y) = (x+\kappa_1(s,x,y), y+\lambda_1(s,x,y)).$$

From the uniqueness of the solution we get for all $t \in \mathbb{R}$ that

$$H_1(t,\Phi(t,s,x,y)) = \Psi(t,s,H_1(s,x,y)).$$

Step 3. The set

$$\mathbf{N}_3(l) = \{\kappa \in \mathbf{N}_3 \mid |\kappa(s,x,y) - \kappa(s,x,y')| \le l|y-y'|\}$$

is a closed subset of the Banach space $\mathbf{N}_3$. In $\mathbf{N}_3(l)$, there exists a unique solution of the functional equation

$$\kappa_2(s,x,w)$$

$$= \int_s^{+\infty} X(s,\tau)(f(\tau,x_0(\tau),u(\tau,x_0(\tau))) - f(\tau,x_0(\tau)+\kappa_2(\tau,x_0(\tau),\eta(\tau)),\eta(\tau)))\,d\tau$$

$$+ \sum_{s<\tau_i} X(s,\tau_i)(p_i(x_0(\tau_i-0),u(\tau_i-0,x_0(\tau_i-0)))$$

$$-p_i(x_0(\tau_i-0)+\kappa_2(\tau_i-0,x_0(\tau_i-0),\eta(\tau_i-0)),\eta(\tau_i-0))),$$

$$\eta(t) = Y(t,s)w + \int_s^t Y(t,\tau)g(\tau,x_0(\tau)+\kappa_2(\tau,x_0(\tau),\eta(\tau)),\eta(\tau))\,d\tau$$

$$+ \sum_{s<\tau_i\le t} Y(t,\tau_i)q_i(x_0(\tau_i-0)+\kappa_2(\tau_i-0,x_0(\tau_i-0),\eta(\tau_i-0)),\eta(\tau_i-0)).$$

Step 4. The space

$$\mathbf{N}_5 = \left\{\lambda \in \mathbf{PC}(\mathrm{I\!R}\times\mathbf{X}\times\mathbf{Y},\mathbf{Y}) \;\middle|\; \sup_{s,x,y} \frac{|\lambda(s,x,y)|}{|x+\kappa_2(s,x,y)-v(s,y)|} < +\infty\right\}$$

equipped with the norm

$$\|\lambda\| = \sup_{s,x,y} \frac{|\lambda(s,x,y)|}{|x+\kappa_2(s,x,y)-v(s,y)|}$$

is a Banach space. In $\mathbf{N}_5$, there exists a unique solution of the functional equation

$$\lambda_2(s,x,y) = \int_{-\infty}^s Y(s,\tau)(g(\tau,x_0(\tau)+\kappa_2(\tau,x_0(\tau),y_0(\tau)$$

$$+\lambda_2(\tau,\Psi(\tau))),y_0(\tau)+\lambda_2(\tau,\Psi(\tau))) - g(\tau,v(\tau,y_0(\tau)),y_0(\tau)))\,d\tau$$

$$+ \sum_{\tau_i\le s} Y(s,\tau_i)(q_i(x_0(\tau_i-0)+\kappa_2(\tau_i-0,x_0(\tau_i-0),y_0(\tau_i-0)$$

$$+\lambda_2(\tau_i-0,\Psi(\tau_i-0))),y_0(\tau_i-0)+\lambda_2(\tau_i-0,\Psi(\tau_i-0)))$$

$$-q_i(v(\tau_i-0,y_0(\tau_i-0)),y_0(\tau_i-0))).$$

The next step is to define the mapping H_2 by the equality

$$H_2(s,x,y) = (x+\kappa_2(s,x,y+\lambda_2(s,x,y)),y+\lambda_2(s,x,y)).$$

Then the mapping H_2 satisfies the functional equation

$$H_2(t,\Psi(t,s,x,y)) = \Phi(t,s,H_2(s,x,y))$$

for all $t\in\mathrm{I\!R}$.

Step 5. The space

$$\mathbf{N}_6 = \left\{\kappa \in \mathbf{PC}(\mathrm{I\!R}\times\mathbf{X}\times\mathbf{Y},\mathbf{X}) \;\middle|\; \sup_{s,x,y} \frac{|\kappa(s,x,y)|}{|y+\lambda_2(s,x,y)-u(s,x)|} < +\infty\right\}$$

equipped with the norm

$$\|\kappa\| = \sup_{s,x,y} \frac{|\kappa(s,x,y)|}{|y + \lambda_2(s,x,y) - u(s,x)|}$$

is a Banach space. In $\mathbf{N}_6$, there exists a unique trivial solution of the functional equations

$$\kappa_3(s,x,y) = \int_s^{+\infty} X(s,\tau)(f(\tau, x_0(\tau), u(\tau, x_0(\tau)))$$
$$-f(\tau, x_0(\tau) + \kappa_3(\tau, \Psi(\tau)), u(\tau, x_0(\tau) + \kappa_3(\tau, \Psi(\tau))))) \, d\tau$$
$$+ \sum_{s<\tau_i} X(s,\tau_i)(p_i(x_0(\tau_i - 0), u(\tau_i - 0, x_0(\tau_i - 0))) - p_i(x_0(\tau_i - 0)$$
$$+\kappa_3(\tau_i - 0, \Psi(\tau_i - 0)), u(\tau_i - 0, x_0(\tau_i - 0) + \kappa_3(\tau_i - 0, \Psi(\tau_i - 0))))).$$

Step 6. In $\mathbf{N}_5$, there exists a unique trivial solution of the functional equations

$$\lambda_3(s,x,y) = -\int_{-\infty}^{s} Y(s,\tau)(g(\tau, v(\tau, y_0(\tau)), y_0(\tau))$$
$$-g(\tau, v(\tau, y_0(\tau) + \lambda_3(\tau, \Psi(\tau))), y_0(\tau) + \lambda_3(\tau, \Psi(\tau)))) \, d\tau$$
$$+ \sum_{\tau_i \le s} Y(s,\tau_i)(q_i(v(\tau_i - 0, y_0(\tau_i - 0) + \lambda_3(\tau_i - 0, \Psi(\tau_i - 0))), y_0(\tau_i - 0))$$
$$+\lambda_3(\tau_i - 0, \Psi(\tau_i - 0))) - q_i(v(\tau_i - 0, y_0(\tau_i - 0)), y_0(\tau_i - 0))).$$

Step 7. Notice that the mappings $\alpha_1 \colon \mathbb{R}\times\mathbf{X}\times\mathbf{Y} \to \mathbf{X}$ and $\beta_1 \colon \mathbb{R}\times\mathbf{X}\times\mathbf{Y} \to \mathbf{Y}$ defined by equalities

$$\alpha_1(s,x,y)$$
$$= \kappa_2(s,x,y+\lambda_2(s,x,y)) + \kappa_1(s, x + \kappa_2(s,x,y+\lambda_2(s,x,y)), y + \lambda_2(s,x,y))$$

and

$$\beta_1(s,x,y) = \lambda_2(s,x,y) + \lambda_1(s, x + \kappa_2(s,x,y+\lambda_2(s,x,y)), y + \lambda_2(s,x,y))$$

also satisfy the functional equations of Steps 5 and 6, respectively. Besides, $\alpha_1 \in \mathbf{N}_6$ and $\beta_1 \in \mathbf{N}_5$. Hence $\alpha_1(s,x,y) = 0$ and $\beta_1(s,x,y) = 0$. It follows that the identity

$$H_1(s, H_2(s,x,y)) = (x,y)$$

holds true.

Step 8. The space

$$\mathbf{N}_7 = \Big\{ \kappa \in \mathbf{PC}(\mathbb{R} \times \mathbf{X} \times \mathbf{Y} \times \mathbf{Y}, \mathbf{X}) \,\Big|$$

$$\sup_{s,x,y,w} \frac{|\kappa(s,x,y,w)|}{\max\{|y-u(s,x)|, |y-w|\}} < +\infty \Big\}$$

equipped with the norm

$$\|\kappa\| = \sup_{s,x,y,w} \frac{|\kappa(s,x,y,w)|}{\max\{|y-u(s,x)|, |y-w|\}}$$

is a Banach space. The set

$$\mathbf{N}_7(l) = \{\kappa \in \mathbf{N}_7 \mid |\kappa(s,x,y,w) - \kappa(s,x,y,w')| \le l|w-w'|\}$$

is a closed subset of $\mathbf{N}_7$. In $\mathbf{N}_7(l)$, there exists a unique solution of the functional equations

$$\kappa_4(s,x,y,w)$$

$$= \int_s^{+\infty} X(s,\tau)(f(\tau,\Phi(\tau)) - f(\tau, x(\tau) + \kappa_4(\tau,\Phi(\tau),\eta(\tau)),\eta(\tau)))\,d\tau$$

$$+ \sum_{s<\tau_i} X(s,\tau_i)(p_i(\Phi(\tau_i-0)) - p_i(x(\tau_i-0)$$

$$+\kappa_4(\tau_i-0,\Phi(\tau_i-0),\eta(\tau_i-0)),\eta(\tau_i-0))),$$

$$\eta(t) = Y(t,s)w + \int_s^t Y(t,\tau)g(\tau,x(\tau)+\kappa_4(\tau,\Phi(\tau),\eta(\tau)),\eta(\tau))\,d\tau$$

$$+ \sum_{s<\tau_i\le t} Y(t,\tau_i)q_i(x(\tau_i-0)+\kappa_4(\tau_i-0,\Phi(\tau_i-0),\eta(\tau_i-0)),\eta(\tau_i-0)).$$

Step 9. In $\mathbf{N}_4$, there exists a unique solution of the functional equations

$$\lambda_4(s,x,y) = -\int_{-\infty}^s Y(s,\tau)(g(\tau,\Phi(\tau))$$

$$-g(\tau,x(\tau)+\kappa_4(\tau,\Phi(\tau),y(\tau)+\lambda_4(\tau,\Phi(\tau))),y(\tau)+\lambda_4(\tau,\Phi(\tau))))\,d\tau$$

$$+\sum_{\tau_i\le s} Y(s,\tau_i)(q_i(x(\tau_i-0)+\kappa_4(\tau_i-0,\Phi(\tau_i-0),y(\tau_i-0)$$

$$+\lambda_4(\tau_i-0,\Phi(\tau_i-0))),y(\tau_i-0)+\lambda_4(\tau_i-0,\Phi(\tau_i-0)))-q_i(\Phi(\tau_i-0))).$$

Step 10. The mappings $\alpha_2\colon \mathbb{R}\times\mathbf{X}\times\mathbf{Y}\times\mathbf{Y}\to\mathbf{X}$ and $\beta_2\colon \mathbb{R}\times\mathbf{X}\times\mathbf{Y}\to\mathbf{Y}$ defined by equalities

$$\alpha_2(s,x,y,w) = \kappa_1(s,x,y) + \kappa_2(s,x+\kappa_1(s,x,y),w)$$

and

$$\beta_2(s,x,y) = \lambda_1(s,x,y) + \lambda_2(s,x+\kappa_1(s,x,y),y+\lambda_1(s,x,y))$$

also satisfy the functional equations of Steps 8 and 9, respectively. Besides, $\alpha_2 \in \mathbf{N}_7(l)$ and $\beta_2 \in \mathbf{N}_4$. Hence $\alpha_2(s,x,y,y) = 0$ and $\beta_2(s,x,y) = 0$. We get that the following identity:

$$H_2(s, H_1(s,x,y)) = (x,y)$$

holds true.

Taking into account Steps 1, 2, 7 and 10, we get that $H_1(s,\cdot)$ is a homeomorphism establishing the dynamical equivalence of systems (4.1) and (4.4) in the large. It is easy to verify that if system (4.1) of differential equations is autonomous and has no impulse effect, then the mappings u, v, H_1 and H_2 are independent of $s \in \mathbb{R}$. Note that in our case $e(x,y) = \varepsilon(|x| + |y|)$. Thus, the proof of the theorem is complete.

4.6 Dynamical equivalence of noninvertible systems

For noninvertible impulse systems the condition (iv) is replaced by another one,

(iv') : the mappings $x \mapsto x + C_i x$ are homeomorphisms.

Theorem 4.11 *Let $4\varepsilon\nu < 1$ and let there exists a mapping $u\colon \mathbb{R} \times \mathbf{X} \to \mathbf{Y}$ satisfying* (i)–(iii). *Then there is a mapping $q\colon \mathbb{R} \times \mathbf{X} \times \mathbf{Y} \to \mathbf{X}$, which is Lipschitzian with respect to the third variable, such that systems* (4.1) *and*

$$\begin{cases} dx/dt &= A(t)x + f(t,x,u(t,x)), \\ dy/dt &= B(t)y + g(t,q(t,x,y),y), \\ \Delta x|_{t=\tau_i} &= C_i x(\tau_i - 0) + p_i(x(\tau_i - 0), u(\tau_i - 0, x(\tau_i - 0))), \\ \Delta y|_{t=\tau_i} &= D_i y(\tau_i - 0) + q_i(q(\tau_i - 0, x(\tau_i - 0), y(\tau_i - 0)), y(\tau_i - 0)) \end{cases} \tag{4.5}$$

are dynamically equivalent for $t \geq s$.

Let us give another sufficient condition for the dynamical equivalence of two impulsive systems. Suppose there exist mappings $f_0\colon \mathbb{R} \times \mathbf{X} \to \mathbf{X}$ and $p_{i0}\colon \mathbf{X} \to \mathbf{X}$ locally integrable in the Bochner sense with respect to t for fixed x and such that

$$\sup_{t,x,y} |f(t,x,y) - f_0(t,x)| < +\infty;$$

$$\sup_{i,x,y} |p_i(x,y) - p_{i0}(x)| < +\infty;$$

$$|f_0(t,x) - f_0(t,x')| \leq \varepsilon |x - x'|;$$

$$|p_{i0}(x) - p_{i0}(x')| \le \varepsilon |x - x'|.$$

Theorem 4.12 *Let* $4\varepsilon\nu < 1$ *and* $2\varepsilon\mu_2 < 1+\sqrt{1-4\varepsilon\nu_2}$. *Then there is a mapping* $q\colon \mathbb{R}\times\mathbf{X}\times\mathbf{Y}\to\mathbf{X}$, *which is Lipschitzian with respect to the third variable, such that systems* (4.1) *and*

$$\begin{cases} dx/dt &= A(t)x + f_0(t,x), \\ dy/dt &= B(t)y + g(t, q(t,x,y), y), \\ \Delta x|_{t=\tau_i} &= C_i x(\tau_i - 0) + p_{i0}(x(\tau_i - 0)), \\ \Delta y|_{t=\tau_i} &= D_i y(\tau_i - 0) + q_i(q(\tau_i - 0, x(\tau_i - 0), y(\tau_i - 0)), y(\tau_i - 0)) \end{cases} \tag{4.6}$$

are dynamically equivalent for $t \ge s$.

4.7 Dynamical equivalence of invertible systems. 2

Theorem 4.13 *Let* $4\varepsilon\nu < 1$ *and let there exists a mapping* $v\colon \mathbb{R}\times\mathbf{Y}\to\mathbf{X}$ *satisfying* (iv)–(vi). *Then there is a mapping* $\theta\colon \mathbb{R}\times\mathbf{X}\times\mathbf{Y}\to\mathbf{Y}$, *which is Lipschitzian with respect to the second variable, such that systems* (4.1) *and*

$$\begin{cases} dx/dt &= A(t)x + f(t,x,\theta(t,x,y)), \\ dy/dt &= B(t)y + g(t, v(t,y), y), \\ \Delta x|_{t=\tau_i} &= C_i x(\tau_i - 0) + p_i(x(\tau_i - 0), \theta(x(\tau_i - 0), y(\tau_i - 0))), \\ \Delta y|_{t=\tau_i} &= D_i y(\tau_i - 0) + q_i(v(\tau_i - 0, y(\tau_i - 0)), y(\tau_i - 0)) \end{cases} \tag{4.7}$$

are dynamically equivalent in the large.

Suppose there exist mappings $g_0\colon \mathbb{R}\times\mathbf{Y}\to\mathbf{Y}$ and $q_{i0}\colon \mathbf{Y}\to\mathbf{Y}$ locally integrable in the Bochner sense with respect to t for fixed y and, in addition, they satisfy the estimates:

$$\sup_{t,x,y} |g(t,x,y) - g_0(t,y)| < +\infty;$$

$$\sup_{i,x,y} |q_i(x,y) - q_{i0}(y)| < +\infty;$$

$$|g_0(t,y) - g_0(t,y')| \le \varepsilon |y - y'|;$$

$$|q_{i0}(y) - q_{i0}(y')| \le \varepsilon |y - y'|.$$

Theorem 4.14 *Let* $4\varepsilon\nu < 1$ *and* $2\varepsilon\mu_1 < 1+\sqrt{1-4\varepsilon\nu_1}$. *If there exists a mapping* $u\colon \mathbb{R}\times\mathbf{X}\to\mathbf{Y}$ *satisfying* (i)–(iii) *and the mappings* $y \mapsto y + D_i y + q_{i0}(y)$ *are homeomorphisms, then systems* (4.1) *and*

$$\begin{cases} dx/dt &= A(t)x + f(t,x,u(t,x)), \\ dy/dt &= B(t)y + g_0(t,y), \\ \Delta x|_{t=\tau_i} &= C_i x(\tau_i - 0) + p_i(x(\tau_i - 0), u(\tau_i - 0, x(\tau_i - 0))), \\ \Delta y|_{t=\tau_i} &= D_i y(\tau_i - 0) + q_{i0}(y(\tau_i - 0)). \end{cases} \tag{4.8}$$

are dynamically equivalent in the large.

Theorem 4.15 *Let* $4\varepsilon\nu < 1$ *and* $2\varepsilon\mu_2 < 1+\sqrt{1-4\varepsilon\nu_2}$. *If there exists a mapping* $v\colon \mathbb{R} \times \mathbf{Y} \to \mathbf{X}$ *satisfying* (iv)–(vi) *and the mappings* $x \mapsto x + C_i x + p_{i0}(y)$ *are homeomorphisms, then systems* (4.1) *and*

$$\begin{cases} dx/dt &= A(t)x + f_0(t,x), \\ dy/dt &= B(t)y + g(t,v(t,y),y), \\ \Delta x|_{t=\tau_i} &= C_i x(\tau_i - 0) + p_{i0}(x(\tau_i - 0)), \\ \Delta y|_{t=\tau_i} &= D_i y(\tau_i - 0) + q_i(v(\tau_i - 0, y(\tau_i - 0)), y(\tau_i - 0)) \end{cases} \tag{4.9}$$

are dynamically equivalent in the large.

4.8 Notes

The results of this section are based on [107, 109, 117, 122, 123, 124, 125, 126, 135, 141, 142, 143, 146, 148].

5. Applications

5.1 Applications to the stability theory

We will prove that the asymptotic behavior of a semidynamical system generated by a continuous mapping $T\colon \mathbf{X}\times\mathbf{Y}\to\mathbf{X}\times\mathbf{Y}$, where

$$T(x,y) = (f(x,y), g(x,y))$$

is determined by a reduced semidynamical system generated by a continuous mapping $\varphi\colon \mathbf{X}\to\mathbf{X}$, where

$$\varphi(x) = f(x, u(x)).$$

Theorem 5.1 *Let the hypotheses* (**H1**)–(**H4**) *hold and let the mapping* T *have a fixed point* $T(x_0,y_0) = (x_0,y_0)$. *Then for any* $(x,y)\in\mathbf{X}\times\mathbf{Y}$ *there exists a* $\xi\in\mathbf{X}$ *such that*

$$\rho_1(x^n,x_0) \le l_1(k\beta+\delta)^n(\rho_2(y,y_0)+k\rho_1(x,x_0)) + \rho_1(\xi^n,x_0),$$

$$\rho_2(y^n,y_0) \le (1+kl_1)(k\beta+\delta)^n(\rho_2(y,y_0)+k\rho_1(x,x_0)) + k\rho_1(\xi^n,x_0)$$

and

$$\rho_1(\xi,x_0) \le l_1\rho_2(y,y_0) + (1+kl_1)\rho_1(x,x_0),$$

where

$$T^n(x,y) = (f^n(x,y), g^n(x,y)) = (x^n,y^n)$$

is the n*–th iterate of* T *and*

$$l_1 = \frac{\alpha\beta}{\sqrt{(1-\alpha\delta)^2 - 4\alpha^2\beta\gamma}}.$$

Corollary 5.2 *If* $\beta k+\delta\le 1$ *and* x_0 *is a stable fixed point of* φ, *then* (x_0,y_0) *is a stable fixed point of* T. *If* $\beta k+\delta<1$ *and* x_0 *is an asymptotically stable fixed point of* φ, *then* (x_0,y_0) *is an asymptotically stable fixed point of* T.

Example 5.3 Consider the mapping (2.1) and let there be $F(0,0) = 0$, $G(0,0) = 0$. According to Theorem 5.1 we get the estimates

$$|x^n| \leq l_1(k\beta + \delta)^n(|y| + k|x|) + |\xi^n|,$$

$$|y^n| \leq (1 + kl_1)(k\beta + \delta)^n(|y| + k|x|) + k|\xi^n|,$$

where

$$\xi^{n+1} = A\xi^n + F(\xi^n, u(\xi^n))$$

and

$$|\xi| \leq l_1|y| + (1 + kl_1)|x|.$$

5.2 Shadowing lemma

We shall prove that the mapping T has a shadowing property.

Definition 5.4 A sequence $\{x^n, y^n\}_{n\in\mathbb{Z}}$ is an *orbit* of T if

$$(x^{n+1}, y^{n+1}) = T(x^n, y^n)$$

for all $n \in \mathbb{Z}$.

Definition 5.5 A sequence $\{\zeta^n, \eta^n\}_{n\in\mathbb{Z}}$ is a Δ*–pseudo–orbit* of T if

$$\max\{\rho_1(f(\zeta^n, \eta^n), \zeta^{n+1}), \rho_2(g(\zeta^n, \eta^n), \eta^{n+1})\} \leq \Delta$$

for all $n \in \mathbb{Z}$.

Definition 5.6 A mapping T *is said to have the shadowing property* if for every $e > 0$ there exists $\Delta > 0$ such that any Δ–pseudo–orbit $\{\zeta^n, \eta^n\}_{n\in\mathbb{Z}}$ is e–traced by some genuine orbit $\{x^n, y^n\}_{n\in\mathbb{Z}}$, i.e.

$$\max\{\rho_1(x^n, \zeta^n), \rho_2(y^n, \eta^n)\} \leq e$$

for all $n \in \mathbb{Z}$.

Theorem 5.7 *Let the hypotheses* (**H1**)–(**H4**) *hold and let there fulfil* $(1-\alpha)(1-\delta) > \alpha\beta\gamma$*. Then the mapping* T *has the shadowing property.*

Remark. Theorem 5.7 remains valid in the case of a *positive orbit* and a *positive* Δ*–pseudo–orbit.*

Example 5.8 Consider the mapping (2.1). Using Theorem 5.7 we obtain that the mapping (2.1) has the shadowing property if

$$\varepsilon < \frac{(\|A^{-1}\|^{-1} - 1)(1 - \|B\|)}{\|A^{-1}\|^{-1} - \|B\|}$$

and

$$e = \frac{\max\{(\|A^{-1}\|^{-1} - 1), (1 - \|B\|)\}}{(\|A^{-1}\|^{-1} - 1)(1 - \|B\|) - \varepsilon(\|A^{-1}\|^{-1} - \|B\|)} \Delta.$$

5.3 Equation of the gyrotron resonator

The equation which describes the electron motion in a gyrotron resonator has its standard form

$$p' + i(\Delta + |p|^2 - 1)p = iFf(t). \tag{5.1}$$

Consider the nonperturbed equation ($f(t) \equiv 0$)

$$q' + i(\Delta + |q|^2 - 1)q = 0. \tag{5.2}$$

Definition 5.9 Two differential equations (5.1) and (5.2) are *asymtotically equivalent* if there exists a map $H\colon [t_0, +\infty) \times \mathbb{C} \to \mathbb{C}$ such that:

(i) $H(t, \cdot)\colon \mathbb{C} \to \mathbb{C}$ is a homeomorphism;

(ii) $H(t, p(t, t_0, p_0)) = q(t, t_0, H(t_0, p_0))$ for all $t \in [t_0, +\infty)$;

(iii) $\lim_{t\to+\infty} |p(t, t_0, p_0) - q(t, t_0, H(t_0, p_0))| = 0$.

Theorem 5.10 *Suppose that integrals* $\int_{t_0}^{+\infty} f(s)\,ds$ *and* $\int_{t_0}^{+\infty} (s - t_0) f(s)\,ds$ *converge absolutely. Then differential equations* (5.1) *and* (5.2) *are asymptotically equivalent.*

Corollary 5.11 *There exists the asymptotic expression*

$$p(t, t_0, p_0) = H(t_0, p_0) \exp\Big(i(1 - \Delta - |H(t_0, p_0)|^2)(t - t_0)\Big) + \delta(t, t_0, p_0),$$

where

$$\lim_{t\to+\infty} \delta(t, t_0, p_0) = 0.$$

5.4 Notes

The 5th chapter has been written after works [134, 137, 138, 33].

References

[1] E. Akin, *The general topology of dynamical systems*, Grad. Stud. Math., **1**, Amer. Math. Soc., Providence, RI, 1993.

[2] A. A. Andronov and L. S. Pontryagin, *Systemes grosiers*, Dokl. Akad. Nauk SSSR **14** (1937), no. 5, 247–250.

[3] D. V. Anosov, *Many–dimensional analog of Hadamard's theorem*, Nauchn. Dokl. Vyssh. Shkoly. Ser. Fiz.–Mat. Nauk **1959**, no. 1, 3–12 (Russian).

[4] D. V. Anosov, *Geodesic flows on closed Riemannian manifolds with negative curvature*, Proc. Steklov Inst. Math. **90** (1967).

[5] V. I. Arnol'd, *Ordinary differential equations*, "Nauka", Moscow, 1971 (Russian).

[6] V. I. Arnol'd, *Lectures on bifurcations in versal families*, Russian Math. Surveys **27** (1972), 54–123.

[7] B. Aulbach, *A reduction principle for nonautonomous differential equations*, Arch. Mat. **39** (1982), no. 3, 217–232.

[8] B. Aulbach, *Continuous and discrete dynamics near manifolds of equilibria*, Lecture Notes in Math., **1058**, Springer, Berlin, 1984.

[9] B. Aulbach and B. M. Garay, *Linearization and decoupling of dynamical and semidynamical systems*, in D. Bainov and V. Covachev (eds.), *The second colloquium on differential equations*, World Sci. Publishing, Singapore, 1992, pp. 15–27.

[10] B. Aulbach and B. M. Garay, *Linearizing the expanding part of noninvertible mappings*, Z. Angew. Math. Phys. **44** (1993), no. 3, 469–494.

[11] B. Aulbach and B. M. Garay, *Partial linearization for noninvertible mappings*, Z. Angew. Math. Phys. **45** (1994), no. 4, 505–542.

[12] D. D. Bainov, S. I. Kostadinov, Nguyen Hong Thai and P. P. Zabreiko, *Existence of integral manifolds for impulsive differential equations in a Banach space*, Internat. J. Theoret. Phys. **28** (1989), no. 7, 815–833.

[13] D. D. Bainov, S. I. Kostadinov, Nguyen Van Minh, Nguyen Hong Thai and P. P. Zabreiko, *Integral manifolds of impulsive differential equations*, J. Appl. Math. Stochastic Anal. **5** (1992), no. 2, 99–110.

[14] D. D. Bainov, S. I. Kostadinov and Nguyen Van Minh, *Dichotomies and integral manifolds of impulsive differential equations*, Science Culture Techn. Publishing, Singapore, 1994.

[15] D. D. Bainov and S. I. Kostadinov, *Abstract impulsive differential equations*, Descartes Press, Koriyama, 1996.

[16] P. W. Bates and K. Lu, *A Hartman–Grobman theorem for Cahn–Hillard equations and phases–field equations*, J. Dynam. Differential Equations **6** (1994), no. 1, 101–145.

[17] N. N. Bogolyubov, *On some statistical methods in mathematical physics*, AN USSR, L'vov, 1945 (Russian).

[18] N. N. Bogolyubov and Ju. A. Mitropol'skiĭ, *Asymptotic methods in the theory of nonlinear oscillations*, Gordon and Breach, New York, 1962.

[19] P. Bohl, *Über die Bewegung eines mechanischen Systems in der Nähe einer Gleichgewichtslage*, J. Reine Angew. Math. **127** (1904), no. 3–4, 179–276.

[20] M. A. Boudourides, *Topological equivalence of monotone nonlinear nonautonomous differential equations*, Portugal. Math. **41** (1982), no. 1–4, 287–294.

[21] M. A. Boudourides, *Hyperbolic Lipschitz homeomorphisms and flows*, in *Equadiff* 82, *Proceedings* 82, Lecture Notes in Math., **1017**, 1983, pp. 101–106.

[22] R. Bowen, *ω–limit sets for Axiom A diffeomorphisms*, J. Differential Equations, **18**, (1975), no. 2, 333–339.

[23] I. U. Bronshtein and V. A. Glavan, *The Grobman–Hartman theorem for extensions of dynamical systems*, Differential Equations **14** (1978), no. 8, 1071–1073.

[24] B. F. Bylov, R. E. Vinograd, D. M. Grobman and V. V. Nemytskiĭ, *Theory of Lyapunov exponents*, "Nauka", Moscow, 1966 (Russian).

[25] J. Carr, *Applications of centre manifold theory*, Springer, Berlin, 1981.

[26] O. O. Chernikova, *A reduction principle for systems of differential equations that have impulses effect*, Ukrainian Math. J. **34** (1983), 487–492.

[27] S. N. Chow, X. B. Lin and K. J. Palmer, *A shadowing lemma with applications in semi–linear parabolic equations*, SIAM J. Math. Anal. **20** (1989), no. 3, 547–557.

[28] E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw–Hill, New–York, 1955.

[29] B. A. Coomes, H. Koçak and K. J. Palmer, *A shadowing theorem for ordinary differential equations*, Z. Angew. Math. Phys. **46** (1995), no. 1, 85–106.

[30] E. Cotton, *Sur les solutions asymptotiques des équations différentielles*, Ann. Sci. École Norm. Sup. (**3**) **28** (1911), 473–521.

[31] C. Coleman, *Local trajectory equivalence of differential systems*, Proc. Amer. Math. Soc. **16** (1965), no. 5, 890–892.

[32] C. Coleman, *Addendum Ibid*, Proc. Amer. Math. Soc. **17** (1966), 770.

[33] O. Dumbrajs, R. Meyer–Spasche and A. Reinfelds, *Analysis of electron trajectories in a gyrotron resonator*, IEEE Trans. Plasma Science **26** (1998), no. 3, 846–853.

[34] N. Fenishel, *Persistence and smoothness of invariant manifolds and flows*, Indiana Univ. Math. J. **21** (1971/72), no. 3, 193–226.

[35] D. M. Grobman, *Homeomorphisms of systems of differential equations*, Dokl. Akad. Nauk SSSR **128** (1959), no. 5, 880–881 (Russian).

[36] D. M. Grobman, *Topological classification of neighborhoods of a singularity in n-space*, Mat. Sb. (N.S.) **56**(**98**) (1962), no. 1, 77–94 (Russian).

[37]D. M. Grobman, *Topological equivalence of dynamical systems*, Dokl. Akad. Nauk SSSR **175** (1967), no. 6, 1211–1212 (Russian).

[38]D. M. Grobman, *Topological equivalence in the large for systems of differential equations*, Math. USSR, Sb. **2** (1967), 535–542.

[39]D. M. Grobman, *Homeomorphism of dynamical systems*, Differential Equations **5** (1972), 995–1001.

[40]J. Hadamard, *Sur l'itération et les solutions asymptotiques des équations différentielles*, Bull. Soc. Math. France **29** (1901), 224–228.

[41]J. K. Hale, *Integral manifolds of perturbed differential systems*, Ann. of Math. (**2**) **73** (1961), no. 3, 496–531.

[42]J. K. Hale, *Oscillations in nonlinear systems*, McGraw–Hill, New York, 1963.

[43]P. Hartman, *On local homeomorphisms of Euclidean spaces*, Bol. Soc. Mat. Mexicana (**2**) **5** (1960), no. 2, 220–241.

[44]P. Hartman, *A lemma in the theory of structural stability of differential equations*, Proc. Amer. Math. Soc. **11** (1960), no. 4, 610–620.

[45]P. Hartman, *On the local linearization of differential equations*, Proc. Amer. Math. Soc. **14** (1963), no. 4, 568–573.

[46]P. Hartman, *Ordinary differential equations*, John Willey & Sons, New York, 1964.

[47]D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Math., **840**, Springer, Berlin, 1981.

[48]M. W. Hirsch and C. C. Pugh, *Stable manifolds and hyperbolic sets*, in *Global analysis*. Proc. Sympos. Pure Math., **14**, Amer. Math. Soc., Providence, RI, 1970, pp. 133–163.

[49]M. W. Hirsch, C. C. Pugh and M. Shub, *Invariant manifolds*, Lecture Notes in Math., **583**, Springer, Berlin, 1977.

[50]M. Irwin, *A classification of elementary cycles*, Topology **9**, (1970), no. 1, 35–47.

[51]A. Kelley, *The stable, center–stable, center, center–unstable, unstable manifolds*, J. Differential Equations **3** (1967), no. 4, 546-570.

[52]A. Kelley, *Stability of the center–stable manifold*, J. Math. Anal. Appl. **18** (1967), no. 2, 336–344.

[53]U. Kirchgraber, *Sur les propriétés géométriques au voisinage d'une variété invariante*, C. R. Acad. Sci. Paris Ser. **A 288** (1979), no. 9, 511–514.

[54]U. Kirchgraber, *The geometry in the neighbourhood of an invariant manifold and topological conjugacy*, Research Report 83–01, ETH Zürich, 1983.

[55]U. Kirchgraber and K. J. Palmer, *Geometry in the neighborhood of invariant manifolds of maps and flows and linearization*, Pitman Res. Notes Math. Ser. **233**, Longman, Harlow, 1990.

[56]N. N. Ladis, *The structural stability of certain systems of differential equations*, Differential Equations **7** (1971), 322–325.

[57]N. N. Ladis, *Topological equivalence of certain differential systems*, Differential Equations **8** (1972), 856–859.

[58] N. N. Ladis, *Conditions for the structural stability of differential systems that splits*, Vestnik Beloruss. Gos. Univ. Ser. I Fiz. Mat. Mekh. **2** (1972), 18–19 (Russian).

[59] N. N. Ladis, *Topological equivalence of linear flows*, Differential Equations **9** (1975), 938–947.

[60] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, *Theory of impulsive differential equations*, World Sci. Publishing, River Edge, NJ, 1989.

[61] S. Lattes, *Sur les équations fonctionelles qui définissent une courbe ou une surface invariante par une transformation*, Ann. Mat. Pura Appl. (**3**) **13** (1906), 1–137.

[62] D. C. Lewis, *Invariant manifolds near an invariant point of unstable type*, Amer. J. Math. **60** (1938), no. 3, 577–587.

[63] K. Lu, *A Hartman–Grobman theorem for scalar reaction–diffusion equations*, J. Differential Equations **93** (1991), no. 2, 364–394.

[64] O. B. Lykova, *The reduction principle in Banach space*, Ukrainian Math. J. **23** (1971), 391–397.

[65] Nguyen Van Minh, *On a topological classification of non–autonomous differential equations*, in B. Sz.–Nagy and L. Hatvani (eds.), *Qualitative theort of differential equations*, pp. 421–426, Colloq. Math. Soc. János Bolyai, **53**, North-Holland, Amsterdam, 1990.

[66] Nguyen Van Minh, *A reduction principle for topological classification of nonautonomous differential equations*, Proc. Roy. Soc. Edinburgh Sect. **A 123** (1993), no. 4, 621–632.

[67] V. D. Mil'man and A. D. Myshkis, *On the stability of motion in the presence of impulses*, Sibirsk. Mat. Zh. **1** (1960), no. 2, 233-237 (Russian).

[68] R. M. Mints, *Toplogical equivalence of singular points for system of three differential equations*, Nauchn. Dokl. Vyssh. Shkoly Fiz.–Mat. Nauk **1958**, no. 1, 19–24 (Russian).

[69] Ju. A. Mitropol'skiĭ and O. B. Lykova, *Integral manifolds in nonlinear mechanics*, "Nauka", Moscow, 1973 (Russian).

[70] J. Moser, *On a theorem of Anosov*, J. Differential Equations **5** (1969), no. 3, 411–440.

[71] Ju. I. Neĭmark, *Existence and structural stability of invariant manifolds of pointwise mappings*, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **10** (1967), no. 3, 311–320 (Russian).

[72] Ju. I. Neĭmark, *The method of point transformations in the theory of nonlinear oscillations*, "Nauka", Moscow, 1972 (Russian).

[73] V. V. Nemytskiĭ, *Topological classification of singular points and generalized Lyapunov functions*, Differential Equations **3** (1967), 179–185.

[74] S. E. Newhouse, *Lectures on dynamical systems*, in *Dynamical Systems*, Progr. Math., **8**, Birkhäuser, Boston, 1980, pp. 1–114.

[75] J. Ombach, *The simplest shadowing*, Ann. Polon. Math. **58** (1993), no. 3, 253–258.

[76] G. S. Osipenko, *Topological equivalence of differential equations*, Differential Equations **11** (1976), 1023–1029.

[77] G. S. Osipenko, *Local dynamical equivalence of differential equations*, Differential Equations **12** (1977), no. 12, 1530–1535.

[78] G. S. Osipenko, *Dynamical equivalence near invariant manifolds*, Differential Equations **14** (1978), no. 9, 1214–1216.

[79] G. S. Osipenko, *The behavior of solutions of differential equations near invariant manifolds*, Differential Equations **15** (1979), no. 2, 179–185.

[80] G. S. Osipenko, *Perturbation of dynamic systems near invariant manifolds. I*, Differential Equations **15** (1979), no. 11, 1403–1412.

[81] G. S. Osipenko, *Perturbation of dynamic systems near invariant manifolds. II*, Differential Equations **16** (1980), no. 4, 381–387.

[82] G. S. Osipenko, *On the problem of linearization in the neighborhood of invariant manifolds*, Differential Equations **17** (1981), no. 4, 433–436.

[83] J. Palis, *On the local structure of hyperbolic fixed points in Banach spaces*, An. Acad. Brasil. Ci. **40** (1968), no. 3, 263–266.

[84] J. Palis and F. Takens, *Topological equivalence of normally hyperbolic dynamical systems*, Topology **16** (1977), no. 4, 335–345.

[85] K. J. Palmer, *A generalization of Hartman's linearization theorem*, J. Math. Anal. Appl. **41** (1973), no. 3, 753–758.

[86] K. J. Palmer, *Linearization near an integral manifold*, J. Math. Anal. Appl. **51** (1975), no. 1, 243–255.

[87] K. J. Palmer, *Linearization of systems with an integral*, J. Math. Anal. Appl. **60** (1977), no. 3, 781–793.

[88] K. J. Palmer, *Linearization of reversible systems*, J. Math. Anal. Appl. **60** (1977), no. 3, 794–808.

[89] K. J. Palmer, *Topological equivalence and the Hopf bifurcation*, J. Math. Anal. Appl. **66** (1978), no. 3, 586–598.

[90] K. J. Palmer, *A characterization of exponential dichotomy in terms of topological equivalence*, J. Math. Anal. Appl. **69** (1979), no. 1, 8–16.

[91] K. J. Palmer, *Qualitative behavior of a system of ODE near an equilibrium point – a generalization of the Hartman–Grobman theorem*, Preprint No. 372, Universität Bonn, 1980.

[92] K. J. Palmer, *On the stability of the center manifold*, Z. Angew. Math. Phys. **38** (1987), no. 2, 273–278.

[93] K. J. Palmer, *Exponential dichotomies, the shadowing lemma and transversal homoclinic points*, Dynamics Reported, vol. 1, Teubner, Stuttgart, 1988, pp. 265–306.

[94] G. Papaschinopoulos, *On the summable manifold for discrete systems*, Math. Japon. **33** (1988), no. 3, 457–468.

[95] G. Papaschinopoulos, *Linearization near the summable manifold for discrete systems*, Studia Sci. Math. Hungar. **25** (1990), no. 3, 275–289.

[96] N. A. Perestyuk, *Invariant sets of a class of discontinuous dynamical systems*, Ukrainian Math. J. **36** (1984), no. 1, 58–62.

[97] O. Perron, *Über Stabilität and asymptotisches Verhalten der Integrale von Differentialgleichungssystemen*, Math. Z. **29** (1928), no. 1, 129–169.

[98] V. A. Pliss, *The reduction principle in the theory of stability of motion*, Soviet Math. Dokl. **5** (1964), 247–250.

[99] V. A. Pliss, *A reduction principle in the theory of stability of motion*, Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), no. 6, 1297–1324 (Russian).

[100] V. A. Pliss, *On the theory of invariant surfaces*, Differentsial'nye Uravneniya **2** (1966), no. 9, 1139–1150 (Russian).

[101] V. A. Pliss, *Integral sets of periodic systems of differential equations*, "Nauka", Moscow, 1977 (Russian).

[102] C. C. Pugh, *On a theorem of P. Hartman*, Amer. J. Math. **91** (1969), no. 2, 363–367.

[103] C. C. Pugh and M. Shub, *Linearization of normally hyperbolic diffeomorphisms and flows*, Invent. Math. **10** (1970), no. 3, 187–198.

[104] H. Poincaré, *Sur le properties des fonctions définies par les équations aux différences partielles*, Thése, Gautier–Villars, Paris, 1879.

[105] A. Reinfelds, *Global topological equivalence of nonlinear flows*, Differential Equations **8** (1974), 1474–1476.

[106] A. Reinfelds, *Topological equivalence of differential equations with delayed truncation*, Differential Equations **9** (1975), 356–358.

[107] A. Reinfelds, *A reduction theorem*, Differential Equations **10** (1975), no. 5, 645–649.

[108] A. Reinfelds, *Topological equivalence of linear differential equations in a Banach space*, Latv. Mat. Ezhegodnik **15** (1974), 97–99 (Russian).

[109] A. Reinfelds, *A reduction theorem for closed trajectories*, Differential Equations **11** (1976), 1353–1358.

[110] A. Reinfelds, *Conjugacy of flows in a Banach space*, Latv. Mat. Ezhegodnik **16** (1975), 233–240 (Russian).

[111] A. Reinfelds, *Dynamical equivalence of the full and truncated equation*, Latv. Mat. Ezhegodnik **19** (1976), 222–232 (Russian).

[112] A. Reinfelds, *Dynamical equivalence of differential equations in the neighborhood of a torus*, Latv. Mat. Ezhegodnik **21** (1977), 90–93 (Russian).

[113] A. Reinfelds, *Dynamic equivalence of differential equations in the neighborhood of an invariant manifold*, Latv. Mat. Ezhegodnik **24** (1980), 156–171 (Russian).

[114] A. Reinfelds, *Dynamical equivalence of dynamical systems in a neighbourhood of a torus*, in M. Farkas (ed.), *Qualitative theory of differential equations*, Vol. II, pp. 857–864, Colloq. Math. Soc. János Bolyai, **30**, North-Holland, Amsterdam, 1981.

[115] A. Reinfelds, *The homeomorphism of dynamical systems in the neighborhood of a stable invariant manifold*, Differential Equations **19** (1983), no. 12, 1491–1499.

[116] A. Reinfelds, *Dynamic equivalence near a conditionally stable invariant manifold*, Latv. Mat. Ezhegodnik **27** (1983), 119–121 (Russian).

[117] A. Reinfelds, *A generalized Grobman-Hartman theorem*, Latv. Mat. Ezhegodnik **29** (1985), 84–88 (Russian).

[118] A. Reinfelds, *Differential equations in the neighborhood of a stable invariant manifold in Banach space*, Differential Equations **21** (1985), no. 12, 1387–1390.

[119] A. Reinfelds, *Dynamic equivalence in a neighborhood of an asymptotically stable manifold in a Banach space*, Latv. Mat. Ezhegodnik **30** (1986), 76–82 (Russian).

[120] A. Reinfelds, *Invariant sets in a metric space*, Latv. Mat. Ezhegodnik **30** (1986), 83–91 (Russian).

[121] A. Reinfelds, *Conjugation of homeomorphisms in a metric space*, Latv. Mat. Ezhegodnik **31** (1988), 236 (Russian).

[122] A. Reinfelds, *A reduction theorem for extensions of dynamical systems*, Latv. Mat. Ezhegodnik **33** (1989), 67–75 (Russian).

[123] A. Reinfelds, *Dynamical equivalence of dynamical extensions*, Reports of the extended sessions of the seminar of the I. N. Vekua Institute of Applied Mathematics **5** (1990), no. 3, 164–166 (Russian).

[124] A. Reinfelds and L. Sermone, *Equivalence of differential equations with impulse action*, Latv. Univ. Zināt. Raksti **553** (1990), 124–130 (Russian).

[125] A. Reinfelds and L. Sermone, *Equivalence of nonlinear differential equations with impulse effect in Banach space*, Latv. Univ. Zināt. Raksti **577** (1992), 68–73.

[126] A. Reinfelds, *Existence of central manifold for differential equations with impulses in a Banach space*, Latv. Univ. Zināt. Raksti **577** (1992), 81–88.

[127] A. Reinfelds, *Invariant sets for splitting mapping in metric space*, Latv. Univ. Zināt. Raksti **588** (1993), 35–44.

[128] A. Reinfelds, *Decoupling of mappings in a metric space*, Proc. Latv. Acad. Sci. Sect. **B 1994**, no. 2(559), 67–75.

[129] A. Reinfelds, *The reduction principle for discrete dynamical and semidynamical systems in metric spaces*, Z. Angew. Math. Phys. **45** (1994), no. 6, 933–955.

[130] A. Reinfelds, *Invariant sets for noninvertible mapping*, Latv. Univ. Zināt. Raksti **592** (1994), 115–124.

[131] A. Reinfelds, *Partial decoupling of semidynamical system*, Latv. Univ. Zināt. Raksti **593** (1994), 54–61.

[132] A. Reinfelds, *Partial decoupling for noninvertible mappings*, Differential Equations Dynam. Systems **2** (1994), no. 3, 205–215.

[133] A. Reinfelds, *The reduction principle for discrete dynamical and semidynamical systems in metric spaces*, in S. Bilchev and S. Tersian (eds.), *Differential equations and applications*. Proceedings of the fifth international conference on differential equations and applications, Rousse, Bulgaria, August 24–29, 1995. Union of Bulgarian Mathematicians, Rousse, 1995, pp. 94–102.

[134] A. Reinfelds, *The stability of semidynamical system in metric space*, Latv. Univ. Zināt. Raksti **599** (1995), 140–145.

[135] A. Reinfelds, *A reduction theorem for systems of differential equations with impulse effect in a Banach space*, J. Math. Anal. Appl. **203** (1996), no. 1, 187–210.

[136] A. Reinfelds, *Invariant sets and dynamical equivalence*, Proc. Est. Acad. Sci., Phys. Math. **45** (1996), no. 2–3, 216–225.

[137] A. Reinfelds, *The reduction of discrete dynamical and semidynamical systems in metric spaces*, in B. Aulbach and F. Colonius (eds.), *Six lectures on dynamical systems*, World Sci. Publishing, River Edge, NJ, 1996, pp. 267–312.

[138] A. Reinfelds, *The shadowing lemma in a metric space*, Univ. Iagel. Acta Math. **35** (1997), 205–210.

[139] A. Reinfelds, *Grobman's–Hartman's theorem for time–dependent difference equations*, Latv. Univ. Zinār. Raksti **605** (1997), 9–13.

[140] A. Reinfelds, *The reduction of discrete dynamical systems in metric space*, in S. Elaydi, I. Győri and G. Ladas (eds.), *Advances in Difference Equations.* Proceedings of the second international conference on difference equations, Veszprém, Hungary, August 7–11, 1995. Gordon and Breach, Yverdon, 1997, pp. 525–536.

[141] A. Reinfelds, *Decoupling of impulsive differential equations*, in R. Čiegis (ed.), *Mathematical Modelling and Complex Analysis.* Proceedings of the second international conference "Mathematical modelling and complex analysis", Vilnius, Lithuania, June 3–4, 1997. "Technika", Vilnius, 1997, pp. 130–137.

[142] A. Reinfelds, *Decoupling of impulsive differential equations in a Banach space*, in C. Constanda, J. Saranen and S. Seikkala (eds.), *Integral methods in science and engineering. Volume one: analytic methods*, Pitman Res. Notes Math. Ser. **374**, Longman, Harlow, 1997, pp. 144–148.

[143] A. Reinfelds, *Dynamical equivalence of impulsive differential equations*, Nonlinear Anal. **30** (1997), no. 5, 2743–2752.

[144] A. Reinfelds, *Partial decoupling of semidynamical system in metric space*, J. Tech. Univ. Plovdiv Fundam. Sci. Appl. Ser. A Pure Appl. Math. **5** (1997), 33–40.

[145] A. Reinfelds, *Dynamical equivalence of dynamical systems*, Univ. Iagel. Acta Math. **36** (1998), 149–155.

[146] A. Reinfelds, *Partial decoupling of impulsive differential equations*, Latv. Univ. Zinār. Raksti **612** (1998), 107–114.

[147] A. Reinfelds, *The reduction principle for discrete dynamical systems in metric space*, J. Difference Equations Appl. (submitted).

[148] A. Reinfelds, *Partial decoupling of noninvertible impulsive differential equations*, Funct. Differ. Equ. (in print).

[149] A. Reiziņa and L. Reiziņš, *Topological classification of linear systems of differential equations*, Latv. Mat. Ezhegodnik **2** (1967), 261–264.

[150] L. Reiziņš, *A homeomorphism of a system of differential equations in neighbourhoods of closed trajectories*, Soviet Math. Dokl. **5** (1964), 290–293.

[151] L. Reiziņš, *A homeomorphism of systems of differential equations in neighbourhoods of closed trajectories*, Mat. Sb. **63(105)** (1964), no. 3, 392–408. (Russian)

[152] L. Reiziņš, *Local topological equivalence of systems of differential equations*, Differential Equations **4** (1968), 99-107.

[153] L. Reiziņš, *Local equivalence of differential equations*, "Zinātne", Rīga, 1971 (Russian).

[154] L. Reiziņš, *Local equivalence of certain flows*, VII Internationale Konferenz über Nichtlineare Schwingungen (Berlin, 1975), vol. 1, no. 2, Akademie–Verlag, Berlin, 1977, pp. 163–169. (Russian).

[155] L. Reiziņš, *Local topological equivalence of differential equations*, in M. Farkas (ed.), *Differential equations*, pp. 355–363, Colloq. Math. Soc. János Bolyai, **15**, North–Holland, Amsterdam, 1977.

[156] A. M. Samoĭlenko and N. A. Perestyuk, *Impulsive differential equations*, World Sci. Publishing, River Edge, NJ, 1995.

[157] L. Sermone, *Equivalence of linear differential equations with impulse effect*, Proc. Latv. Acad. Sci. Sect. **B 1994**, no. 2(559), 78–80.

[158] L. Sermone, *Reduction of differential equations with impulse effect*, J. Appl. Math. Stochastic. Anal. **10** (1997), no. 1, 79–87.

[159] A. N. Shoshitaĭshvili, *Bifurcations of topological type at singular points of parametrized vector fields*, Funct. Anal. Appl. **6** (1972), 169–170.

[160] A. N. Shoshitaĭshvili, *The bifurcation of the topological type of the singular points of vector fields that depend on parameters*, Trudy Sem. Petrovsk. **1** (1975), 279–309 (Russian).

[161] S. Sternberg, *On the behavior of invariant curves near a hyperbolic point of a surface transformation*, Amer. J. Math. **77** (1955), no. 3, 526–534.

[162] S. Sternberg, *Local contractions and a theorem of Poincaré*, Amer. J. Math. **79** (1957), no. 4, 809–824.

[163] E. M. Vaisbord, *About equivalence of systems of differential equations in the neighbourhood of singular point*, Nauchn. Dokl. Vyssh. Shkoly Fiz.–Mat. Nauk **1958**, no. 1, 37–42 (Russian).

[164] M. Yorinaga, *Invariant manifolds under a certain transformation*, J. Sci. Hiroshima Univ. Ser. A–I Math. **25** (1961), no. 2, 117–125.

[165] J. C. Wells, *Invariant manifolds of non–linear operators*, Pacific. J. Math. **62** (1976), no. 1, 285–293.