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Fuzzy matrices and generalized aggregation operators: theoretical foundations and possible applications

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Annotation

According to the author's observations two mainstreams in the development of the theory of fuzzy sets can be isolated: the fuzzification of already known notions and the development of notions which either were originated in the frame of the theory of fuzzy sets or are tightly related to the theory. Many approaches and notions in topology, algebra, financial calculus and other fields were generalized by using fuzzy sets. Under the second mainstream we can mention such notions as the extension principle, a t-norm, a possibility distribution and others.

The goal of the thesis is to contribute to the both mainstreams. The following task is completed in the thesis: the theory of fuzzy matrices and the theory of generalized aggregation operators are developed and possible practical applications of the obtained results are outlined.

Years over years fuzzy sets community comes with a plenty of new and interesting results in the theory of fuzzy sets. Introduction of new and bright results is the complimentary but not easy task. This contribution has already interested at least one scientist from the community, i.e. the scientific supervisor of the thesis, thus the author considers that its development was not useless.

Contents

1 Introduction

Processing of inexact data is one of the challenging problems for modern engineers and practitioners in different areas. Since Kolmogorov had introduced the axiomatic of the modern probability theory in 1933 and even before it was the main tool used by scientists. But the apparatus of probability theory can not treat all kinds of vagueness, thus the necessity of a new approach was obvious.

At the beginning of the 20th century philosophers actively discussed impossibility of putting real processes or objects into the strict frames based on the principles of bivalent logic. Provided intensive and fruitful work in the 20th century in the circles of philosophers and mathematicians the prompt emergency of the theory of fuzzy sets was obvious. And it was crystallized in Zadeh's pioneering paper [40] in 1965 where foundations of the theory of fuzzy sets were presented. Two years later in 1967 Goguen ([8]) put the basis for the theory of L-fuzzy sets thus extending the theory proposed by Zadeh. Since these historical publications the era of the theory of fuzzy sets started ([6]). In the last forty five years many contributions have developed the theory of fuzzy sets to the impressive theory with the extensive list of notions, tools, theoretical results and the broad area of applications. The author in the thesis had contributed to the development of the theory of fuzzy sets. In order to reach the goal the following **task** was defined in the thesis: **to develop the theory of fuzzy matrices and the theory of generalized aggregation operators and to outline possible applications of the obtained results.**

Results provided in the thesis can be divided into two mainstreams: generalization of classical mathematical notions by means of fuzzy sets and contribution to the integral part of the fuzzy sets theory.

This work provides the main results presented in the thesis. The structure is the following: we recall the basic notions and results of the fuzzy set theory in the next chapter; the third chapter provides the main results on theory of fuzzy matrices; and the last chapter is devoted to the development of the theory of generalized aggregation operators.

2   Preliminaries

We flash results of the theory of fuzzy set in this chapter. Mainly the source [35] from the bibliography list is cited, but other sources, e.g. [6, 19] can also be used.

Definition 1 ([35]) *A mapping $M : X \to [0,1]$ is called a fuzzy subset of the set X or simply a fuzzy set.*

The set of all fuzzy subsets of X will be denoted $F(X)$.
Let a fuzzy set M be given, if we fix $\alpha \in [0,1]$ then:

Definition 2 ([35]) $M^{\alpha} = \{x : M(x) \geq \alpha\}$ *is called an α-cut of the fuzzy set M.*

Definition 3 ([35]) $M_{\alpha} = \{x : M(x) > \alpha\}$ *is called a strict α-cut of the fuzzy set M.*

Extension principle is one of the ways how to extend known results to more general cases, and in particular to fuzzy sets.

Definition 4 ([35]) *Let a mapping $\varphi : X \times Y \to Z$ be given, then the mapping $\tilde{\varphi} : F(X) \times F(Y) \to F(Z)$ defined by the formula*

$$\tilde{\varphi}(M, N)(z) = \sup\{\min(M(x), N(y)) | x \in X, y \in Y, \varphi(x, y) = z\},$$

where $M \in F(X)$, $N \in F(Y)$,
is called the extension of the function $\varphi(x, y)$ to the sets $F(X), F(Y)$.

Any arithmetic operation can be extended to the operation on fuzzy sets of real numbers.
Further we observe different properties of fuzzy sets, which play an important role in the work:

Definition 5 ([35]) *A mapping $f : X \to \mathbb{R}$ is upper semicontinuous, if for all $t \in \mathbb{R}$ the set $\{x | f(x) \geq t\}$ is closed.*

Fuzzy quantities form a special class of fuzzy sets:

Definition 6 ([35]) *A convex, upper semicontinuous fuzzy set $M : \mathbb{R} \to [0,1]$ with bounded α-cuts for all $\alpha > 0$ is called a fuzzy quantity.*

The class of all fuzzy quantities will be denoted $FQ(\mathbb{R})$.
Fuzzy intervals $(FI(\mathbb{R}))$ and fuzzy numbers $(FN(\mathbb{R}))$ are subclasses of fuzzy quantities:

Definition 7 ([35]) *A fuzzy quantity P is called a fuzzy interval if $\exists I = [a, b] \subseteq (-\infty, +\infty) : P(x) = 1 \Leftrightarrow x \in I$. Interval I is called the vertex of P.*

Definition 8 ([35]) *Fuzzy quantity P is called a fuzzy number if $\exists! x \in \mathbb{R} : P(x) = 1$. Point x is called the vertex of P.*

The notion of a t-norm is fundamental in different areas of fuzzy sets theory, and it plays an important role in our study. Detailed information on t-norms can be found e.g. in [15, 35]:

Definition 9 ([35]) *A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a t-norm provided that:*
(1) $T(x, y) = T(y, x)$ - symmetry
(2) $T(T(x, y), z) = T(x, T(y, z))$ - associativity
(3) $x_1 \leq x_2 \Rightarrow T(x_1, y) \leq T(x_2, y))$ - monotonicity
(4) $T(x, 1) = x$.

Examples of t-norms:
Min t-norm:
$$T_M(x, y) = \min(x, y)$$

Drastic t-norm:
$$T_W(x, y) = \begin{cases} \min(x, y), & \text{if } \max(x, y) = 1 \\ 0, & \text{otherwise} \end{cases}$$

Product t-norm:
$$T_P(x, y) = x \cdot y$$

Lukasiewicz t-norm:

$$T_L(x, y) = \max\{x + y - 1, 0\}.$$

3 The theory of fuzzy matrices: theoretical foundations and practical applications

We set out the theory of fuzzy matrices in this chapter. First we define a fuzzy matrix and operations with fuzzy matrices. Later we introduce the notion of the fuzzy inverse, which is based on the notion of the inverse of an interval matrix ([26, 27, 30]). Afterwards we focus on calculation of the fuzzy inverse in special cases and its estimation. We conclude the chapter by practical applications.

3.1 Fuzzy matrix: basic notions

We define a fuzzy matrix and operations with fuzzy matrices in this section.

Definition 10 *A matrix $\boldsymbol{A}_F = (A_{ij})_{m \times n}$, where A_{ij} $\forall i, j$ is a fuzzy number is called a fuzzy matrix.*

The set of all fuzzy matrices is denoted $\mathbb{M}$.
We introduce definitions of lower and upper dominants of a fuzzy matrix:

Definition 11 *$\boldsymbol{A}_F^U$ is an upper dominant of $\boldsymbol{A}_F$, if $A_{ij}(x) \leq A_{ij}^U(x)$, $\forall x \in \mathbb{R}, \forall i, j \in \overline{1, n}$.*

Definition 12 *$\boldsymbol{A}_F^L$ is a lower dominant of $\boldsymbol{A}_F$, if $A_{ij}(x) \geq A_{ij}^L(x)$, $\forall x \in \mathbb{R}, \forall i, j \in \overline{1, n}$.*

Operations with fuzzy matrices are defined similarly like in the crisp case. Operations with fuzzy numbers (elements of fuzzy matrices) are extension of classical operations via T_M.
The sum of fuzzy matrices $\mathbf{A}_F = (A_{ij})_{m \times n}$, $\mathbf{B}_F = (B_{ij})_{m \times n}$ is a fuzzy matrix

$$\mathbf{C}_F = \mathbf{A}_F + \mathbf{B}_F,$$

where $\mathbf{C}_F = (C_{ij})_{m \times n} = (A_{ij} + B_{ij})_{m \times n}$.
Multiplication of a fuzzy matrix $\mathbf{A}_F = (A_{ij})_{m \times n}$ with a fuzzy number C is a fuzzy matrix

$$\mathbf{B}_F = C\mathbf{A}_F,$$

where $\mathbf{B}_F = (B_{ij})_{m \times n} = (CA_{ij})_{m \times n}$.
Multiplication of two fuzzy matrices $\mathbf{A}_F = (A_{ij})_{m \times n}$, $\mathbf{B}_F = (B_{ij})_{n \times l}$ is a fuzzy matrix:

$$\mathbf{C}_F = \mathbf{A}_F \mathbf{B}_F,$$

where $\mathbf{C}_F = (C_{ij})_{m \times l} = (\sum_{k=1}^{n} A_{ik} B_{kj})_{m \times l}$.

Proposition 3.1 *The set $\mathbb{M}$ is closed w.r.t. fuzzy matrices addition, multiplication and multiplication with $C \in FN(\mathbb{R})$.*

3.2 Fuzzy inverse matrix

3.2.1 The inverse matrix of an interval matrix

We build the fuzzy inverse from special interval matrices. Necessary notions and results on the inverse of an interval matrix can be found in [18], [26], [27] and [30]. Work on interval arithmetic [23] will make reading easier. Matrix A_I is called an interval matrix if its elements are closed intervals:

$$A_I = [\underline{A}, \overline{A}],$$

where $\underline{A}, \overline{A}$ are correspondingly the matrix of lower bounds and the matrix of upper bounds.

A_I is called regular if all $A \in A_I$ are non-singular. For each regular interval matrix we define the inverse matrix as the narrowest interval matrix, which contains all inverse matrices: $(A_I)^{-1} = \{A^{-1} : A \in A_I\}$.

We calculate the inverse of an interval matrix by means of algorithm proposed by J.Rohn ([27], [30]). The essence of the algorithm is the following: calculate inverse matrices of all special type 2^{2n-1} vertex matrices. Then take minimum and maximum of each element, thus constructing intervals. Elements of these special type vertex matrices are elements of $\underline{A}$ and $\overline{A}$.

3.2.2 The inverse matrix of a fuzzy matrix

Let's consider a square fuzzy matrix $A_F = (A_{ij})_{n \times n}$. If one fixes an arbitrary $\alpha \in (0; 1]$ then $A_F^\alpha = \left(\left[(\underline{a})_{ij}^\alpha, (\overline{a})_{ij}^\alpha \right] \right)_{n \times n}$, is an interval matrix, whose elements are the corresponding α-cuts of the elements A_{ij}. When $\alpha = 0$ we take the closure of the strict α-cut of $A_{ij}, \forall i, j$ in the role of elements of A_F^0, thus obtaining an interval matrix.

We calculate inverse matrices of interval matrices A_F^α and thus we get the spectrum of interval matrices:

$$B_F^0, B_F^{\alpha_1}, ..., B_F^{\alpha_n}, ... B_F^1,$$

where $B_F^\alpha = (A_F^\alpha)^{-1}$ and its ij element is $[(\underline{b})_{ij}^\alpha; (\overline{b})_{ij}^\alpha]$.

We say that fuzzy matrix A_F is regular, if for all $\alpha \in [0, 1]$ interval matrix A_F^α is regular.

Inverse matrix of a regular fuzzy matrix is defined in the following way:

Definition 13 *A matrix $B_F = (B_{ij})_{n \times n}$ with its elements $B_{ij} : \mathbb{R} \to [0,1]$ defined in the following way:*

$$B_{ij}(x) = \max\{\alpha : x \in [(\underline{b})_{ij}^{\alpha}; (\overline{b})_{ij}^{\alpha}]\} \ \forall x \in \mathbb{R}$$

is called the inverse matrix of a regular fuzzy matrix $A_F = (A_{ij})_{n \times n}$.

B_F is fuzzy matrix and further we observe its calculation, estimation and practical applications.

3.2.3 Calculation of the fuzzy inverse in special cases and estimation of the fuzzy inverse

It is shown in the work that in the case of 2×2 fuzzy matrix A_F with the same sign pattern elements (all positive or all negative) only 2 inverse matrices need to be calculated on each α-cut.

Calculation of the fuzzy inverse is simplified also in the case when A_F is an M-fuzzy matrix. A fuzzy matrix A_F is called an M-fuzzy matrix if $\forall A \in A_F^0$ is an M-matrix. Characterization and properties of M-matrices can be found e.g. in [1]. Similarly like previously only 2 inverse matrices need to be calculated (in this case the inverses of the matrix of lower and upper bonds).

Estimation of the fuzzy inverse is reasonable when calculation of the fuzzy inverse is time consuming and the estimation is sufficient for practical purposes.

We conduct the construction in the following way: first we build estimation (upper and lower) for the inverse of interval matrices $A_F^{\alpha}, \alpha \in [0,1)$; after that by means of special construction we build the fuzzy inverse.

The following lemma is proven in the work:

Lemma 3.2 *Let $\boldsymbol{A}_I = ([\underline{a}_{ij}, \overline{a}_{ij}])_{n \times n}$ be an arbitrary regular interval matrix and $\boldsymbol{B}_I = ([\underline{b}_{ij}, \overline{b}_{ij}])_{n \times n}$ be its inverse matrix, then for $[\underline{b}_{ij}, \overline{b}_{ij}]$ the following estimations hold:*

$$[\underline{b}_{ij}, \overline{b}_{ij}] \subseteq \left[\underline{a}_{ij}^{-1} - 2\frac{\Delta_2^{n-1}}{\Delta}, \underline{a}_{ij}^{-1} + 2\frac{\Delta_2^{n-1}}{\Delta}\right] \bigcap \left[\overline{a}_{ij}^{-1} - 2\frac{\Delta_2^{n-1}}{\Delta}, \overline{a}_{ij}^{-1} + 2\frac{\Delta_2^{n-1}}{\Delta}\right],$$

where
$\underline{A}^{-1} = (\underline{a}_{ij}^{-1})_{n \times n}, \overline{A}^{-1} = (\overline{a}_{ij}^{-1})_{n \times n}$,
Δ_2^{n-1} *is estimation from lemma 2.11 (proven in the work), which is based on $(n-1)$-dimensional matrix, $\Delta = \min_{k=1,\dots,2^{2n-1}}(|\det A^k|)$.*

We use the fact that $\underline{A}^{-1}, \overline{A}^{-1} \in B_I$ and we get the following estimation:

$$[\underline{b}_{ij}, \overline{b}_{ij}] \supseteq [\min\{(\underline{a}_{ij})^{-1}, (\overline{a}_{ij})^{-1}\}, \max\{(\underline{a}_{ij})^{-1}, (\overline{a}_{ij})^{-1}\}]. \tag{1}$$

We provide the construction here:

Construction 1 *We have an arbitrary fuzzy matrix $\boldsymbol{A}_F$ and corresponding set of interval matrices:*

$$\boldsymbol{A}_F^0, ..., \boldsymbol{A}_F^\alpha = [\underline{A}^\alpha, \overline{A}^\alpha] = ([\underline{a}_{ij}^\alpha, \overline{a}_{ij}^\alpha])_{n\times n}, ..., \boldsymbol{A}_F^1.$$

According to lemma 3.2 for an arbitrary element of interval matrix $(\boldsymbol{A}_F^\alpha)^{-1} = \boldsymbol{B}_F^\alpha = ([\underline{b}_{ij}^\alpha, \overline{b}_{ij}^\alpha])_{n\times n} \ \alpha \in [0,1)$ the following estimation holds:

$$[\underline{b}_{ij}^\alpha, \overline{b}_{ij}^\alpha] \subseteq \left[(\underline{a}_{ij}^\alpha)^{-1} - 2\frac{\Delta_2^{n-1}}{\Delta}, (\underline{a}_{ij}^\alpha)^{-1} + 2\frac{\Delta_2^{n-1}}{\Delta}\right] \bigcap$$

$$\bigcap \left[(\overline{a}_{ij}^\alpha)^{-1} - 2\frac{\Delta_2^{n-1}}{\Delta}, (\overline{a}_{ij}^\alpha)^{-1} + 2\frac{\Delta_2^{n-1}}{\Delta}\right], \tag{2}$$

where
$(\underline{A}^\alpha)^{-1} = ((\underline{a}_{ij}^\alpha)^{-1})_{n\times n}, \ (\overline{A}^\alpha)^{-1} = ((\overline{a}_{ij}^\alpha)^{-1})_{n\times n},$
Δ_2^{n-1} *is estimation from lemma 2.11, which is based on $n-1$-dimensional matrix, $\Delta = \min_{k=1,...,2^{2n-1}}(|\det A^k|)$ and A^k are vertex matrices of interval matrix $\boldsymbol{A}_F^\alpha$.*
When $\alpha = 1$ elements of the crisp matrix $\boldsymbol{B}_F^1 = (b_{ij}^1)_{n\times n}$ are evaluated by means of corresponding elements of the crisp matrix:

$$(\boldsymbol{A}_F^1)^{-1} = ((a_{ij}^1)^{-1})_{n\times n} \tag{3}$$

We denote $I_\alpha, \ \alpha \in [0,1]$ interval, which includes $[\underline{b}_{ij}^\alpha, \overline{b}_{ij}^\alpha]$ according to formulas (2) and (3).
For all $x \in \mathbb{R}$ we assign the set of indices N_x in the following way:

$$\alpha \in N_x \Leftrightarrow x \in I_\alpha \tag{4}$$

The upper dominant $\boldsymbol{B}_F^U = (B_{ij}^U)_{n\times n}$ of the fuzzy inverse matrix $\boldsymbol{B}_F = (B_{ij})_{n\times n}$ is defined in the following way:

$$B_{ij}^U(x) = \max_{\alpha \in N_x} \alpha, \ \forall i,j = 1, ..., n. \tag{5}$$

Obviously $B_{ij}(x) \leq B_{ij}^U(x) \ \forall x \ \forall i,j = 1, ..., n$ and we finish construction here.

We build the lower dominant of the fuzzy inverse using formula (1) and the above construction.

3.3 Applications of the fuzzy inverse

We show practical applications of fuzzy inverse in this section. First we observe the estimation of the solution of the system of fuzzy linear equations (hereinafter SFLE), later we show some economical applications. More on SFLE and its solutions can be found e.g. in [7], [25], [33],[38].

The system of equations

$$A_F x = c_F, \tag{6}$$

where A_F is an $n \times n$ a fuzzy matrix and c_F is a one-column fuzzy matrix we call SFLE.

We introduce the notion of fuzzy approximate solution (AFS):

Definition 14 *A fuzzy vector*

$$x = A_F^{-1} c_F = B_F c_F$$

is AFS of (6).

Elements of x are fuzzy numbers and it can be shown using definition of A_F^{-1}. If A_F^{-1} is not given, we can use the upper dominant of it.

It is shown in the work that AFS is the upper dominant of the fuzzy solutions which coincide with the special type interval solutions. More on coincidence of fuzzy and interval solutions can be found in [20], [27],[28], [29].

Using AFS we can find approximate solutions for some economical problems. Let's the input-output model ([22]) in the fuzzy environment be given. Input-output analysis in the uncertain environment([13], [14], [31] un [2]):

$$(E - A_F)X_F = Y_F, \tag{7}$$

where E is the identity matrix.

The model (7) is used to solve the two main problems in planning:
(P_1) to find a gross output x which yields a given net output y
(P_2) to find a net output y corresponding to a given gross output x.

According to the definition of AFS it and its upper dominant are the estimations of the solution of (P_1).

Similar approach we use for the estimation of fuzzy economic multipliers ([22],[24]). The mathematical model is the following:

$$M = (L^{-1})^{'}(1, 1, ..., 1), \tag{8}$$

where L is the matrix of interrelations, which characterize the economical environment. When L is a fuzzy matrix, the calculation of fuzzy multiplier is related to calculation of the fuzzy inverse.

3.4   Concluding remarks on fuzzy matrices

The notion of the fuzzy inverse is central in this chapter because it has the largest practical value. As the first priority directions for further study we outline the following:

- development of simpler algorithms for calculation of the fuzzy inverse

- development of algorithms for more accurate estimation of the fuzzy inverse.

Also it is interesting to know what happens with the fuzzy inverse if operations with fuzzy matrices are defined by means of some other t-norm, not only T_M.

4 Generalized aggregation: theoretical foundations and practical applications

The second chapter of the work is dealing with generalized aggregation. We briefly provide important results here.

We develop further the notion introduced by Takači in [36], although other interesting concepts (e.g. [21, 32, 39]) can be found in literature.

Term "generalized" refers to the generalized input and output of aggregation operators (hereinafter agops), i.e. generalized aggregation operators (hereinafter gagops) aggregate fuzzy sets. We consider two construction methods of gagop and namely pointwise extension and T-extension.

At first we provide fundamental results on agops, then we consider γ-agops and later we focus on generalized aggregation. We conclude the chapter outlining practical applications of gagops and showing directions for future research.

4.1 Fundamentals on aggregation operators

We provide the basics on aggregation operators ([3, 4, 15]).

Definition 15 ([3]) *A mapping* $A : \cup_{n \in \mathbb{N}}[0, 1]^n \to [0, 1]$ *is an agop on the unit interval if for every* $n \in \mathbb{N}$ *the following conditions hold:*
(A1) $A(0, ..., 0) = 0$
(A2) $A(1, ..., 1) = 1$
(A3) $(\forall i = \overline{1, n})$ $(x_i \leq y_i) \Rightarrow A(x_1, x_2, ..., x_n) \leq A(y_1, y_2, ..., y_n)$

Conditions (A1) and (A2) are called boundary conditions, condition (A3) resembles the monotonicity property of A.

In general, the number of the input values to be aggregated is unknown, and therefore an agop can be presented as a family $A = (A_{(n)})_{n \in \mathbb{N}}$, where $A_{(n)} = A|_{[0,1]^n}$. Operators $A_{(n)}$ and $A_{(m)}$ for different n and m need not be related. We use convention $A_{(1)}(x) = x \ \forall x \in [0, 1]$.

Problem of aggregation is very broad in general, and we use the following two restrictions in the work: the number of input values is finite and $I = [0, 1]$ is the set of inputs and outputs. If the second restriction is a matter of rescaling then the first divides the global aggregation into two parts, i.e aggregation of finite number of inputs and aggregation of infinite number of inputs. But even with a such restriction the problem of aggregation is still very general. The following mappings are agops in the sense of definition 15:

$$\Pi(x_1, ..., x_n) = \prod_{i=1}^{n} x_i,$$

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$$M(x_1, ..., x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i,$$

$$max(x_1, ..., x_n) = \max(x_1, ..., x_n),$$

$$min(x_1, ..., x_n) = \min(x_1, ..., x_n),$$

$$P_F(x_1, ..., x_n) = x_1, \ P_L(x_1, ..., x_n) = x_n.$$

Further we consider properties of agops.

Definition 16 ([3]) *An element $x \in [0, 1]$ is called an A-idempotent element whenever $A_{(n)}(x, ..., x) = x, \forall n \in \mathbb{N}$. A is called an idempotent agop if each $x \in [0, 1]$ is an idempotent element of A.*

Definition 17 ([3]) *An agop $A : \cup_{n \in \mathbb{N}}[0, 1]^n \to [0, 1]$ is called a continuous agop if for all $n \in \mathbb{N}$ the operators $A_{(n)} : [0, 1]^n \to [0, 1]$ are continuous, that is, if*

$$\forall x_1, ..., x_n \in [0, 1], \forall (x_{1j})_{j \in \mathbb{N}}, ..., (x_{nj})_{j \in \mathbb{N}} \in [0, 1]^{\mathbb{N}} : \lim_{j \to \infty} x_{ij} = x_i$$

for $i = 1, ..., n$ then

$$\lim_{j \to \infty} A_{(n)}(x_{1j}, ..., x_{nj}) = A_{(n)}(x_1, ..., x_n).$$

Definition 18 ([3]) *An agop $A : \cup_{n \in \mathbb{N}}[0, 1]^n \to [0, 1]$ is called a symmetric agop if*

$$\forall n \in \mathbb{N}, \forall x_1, ..., x_n \in [0, 1] : A(x_1, ..., x_n) = A(x_{\pi(1)}, ..., x_{\pi(n)})$$

for all permutations $\pi = (\pi(1), ..., \pi(n))$ of $(1, ..., n)$.

Definition 19 ([3]) *An agop $A : \cup_{n \in \mathbb{N}}[0, 1]^n \to [0, 1]$ is associative if*

$$\forall n, m \in \mathbb{N}, \forall x_1, ..., x_n, y_1, ..., y_m \in [0, 1] :$$

$$A(x_1, ..., x_n, y_1, ..., y_m) = A(A(x_1, ..., x_n), A(y_1, ..., y_m))$$

Definition 20 ([3]) *An agop $A : \cup_{n \in \mathbb{N}}[0, 1]^n \to [0, 1]$ is bisymmetric if*

$$\forall n, m \in \mathbb{N}, \forall x_{11}, ..., x_{mn} \in [0, 1] :$$

$$A_{(mn)}(x_{11}, ..., x_{mn}) = A_{(m)}(A_{(n)}(x_{11}, ..., x_{1n}), ..., A_{(n)}(x_{m1}, ..., x_{mn})) =$$

$$= A_{(n)}(A_{(m)}(x_{11}, ..., x_{m1}), ..., A_{(m)}(x_{1n}, ..., x_{mn}))$$

Definition 21 ([3]) *An element $e \in [0,1]$ is called a neutral element of A if $\forall n \in \mathbb{N}, \forall x_1, ..., x_n, \in [0,1]$ if $x_i = e$ for some $i \in \{1,...,n\}$ then*

$$A(x_1, ..., x_n) = A(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$$

Definition 22 ([3]) *An element $a \in [0,1]$ is called an absorbing element of A if*

$$\forall n \in \mathbb{N}, \forall x_1, ..., x_n, \in [0,1] : a \in \{x_1, ..., x_n\} \Rightarrow A(x_1, ..., x_n) = a$$

Definition 23 ([3]) *An agop $A : \cup_{n \in \mathbb{N}}[0,1]^n \to [0,1]$ is said to be:*
(1) shift-invariant if

$$\forall n \in \mathbb{N}, \forall b \in (0,1), \forall x_1, ..., x_n \in [0, 1-b] :$$

$$A(x_1 + b, ..., x_n + b) = A(x_1, ..., x_n) + b$$

(2) homogeneous if

$$\forall n \in \mathbb{N}, \forall b \in (0,1), \forall x_1, ..., x_n \in [0,1] :$$

$$A(bx_1, ..., bx_n) = bA(x_1, ..., x_n)$$

(3) linear if it homogeneous and shift-invariant
(4) additive if

$$\forall n \in \mathbb{N}, \forall x_1, ..., x_n, y_1, ..., y_n \in [0,1] \text{ such that } x_1 + y_1, ..., x_n + y_n \in [0,1] :$$

$$A(x_1 + y_1, ..., x_n + y_n) = A(x_1, ..., x_n) + A(y_1, ..., y_n)$$

More on agops and their definitions can be found e.g. in [3].

4.2 New class of aggregation operators: γ-agops

This section is devoted to γ-agops, which are a generalization of the class of agops in some sense. Let $\gamma \in [0,1]$ and $\varphi_\gamma : [0,1] \to \{0\} \cup [\gamma, 1]$ be defined in the following way:

$$\varphi_\gamma(x) = \begin{cases} 0, & \text{if } x < \gamma, \\ x, & \text{if } x \geq \gamma \end{cases}$$

Definition 24 *$A : \cup_{n \in \mathbb{N}}[0,1]^n \to [0,1]$ is an γ-agop on the unit interval if the following conditions hold:*
(A1) $A(0, ..., 0) = 0$
(A2) $A(1, ..., 1) = 1$
(A_γ) $(\forall i = \overline{1, n}, \gamma \in [0,1]) (\varphi_\gamma(x_i) \leq \varphi_\gamma(y_i)) \Rightarrow A(x_1, ...x_n) \leq A(y_1, ..., y_n)$

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If $\gamma = 0$ then $\varphi_0(x) = x$ and (A_γ) is equal to (A3) (definition 15).
We show in the work that each γ-agop is an agop as well.

Example 4.1 *An agop*

$$A_2(x_1, ..., x_n) = \min(w_1 x_1, ..., w_n x_n),$$

where
$$w_i = \begin{cases} 0, & \text{if } x_i < \gamma, \\ 1, & \text{if } x_i \geq \gamma \end{cases} \quad \text{is } \gamma\text{-agop as well.}$$

Let's introduce relation $\equiv_{\varphi_\gamma}$ on $[0,1]^n$ in the following way:

$$(x_1, ..., x_n) \equiv_{\varphi_\gamma} (y_1, ..., y_n) \Leftrightarrow$$

$$\Leftrightarrow (\varphi_\gamma(x_1), ..., \varphi_\gamma(x_n)) = (\varphi_\gamma(y_1), ..., \varphi_\gamma(y_n)). \qquad (9)$$

Relation $\equiv_{\varphi_\gamma}$ is reflexive, symmetric, transitive and thus it is an equivalence relation.
We will denote equivalence classes X_k, $k = 1, 2, ...$

Proposition 4.2 *If* $(x_1, ..., x_n), (y_1, ..., y_n) \in X_k$, *A is* γ*-agop then* $A(x_1, ..., x_n) = A(y_1, .., y_n)$

γ - agops have the following properties:

- γ-agops $\forall \gamma > 0$ are not idempotent;

- γ-agops $\forall \gamma > 0$ are not shift-invariant, are not homogeneous and thus are not linear;

- if A_γ, $\gamma \in (0,1]$ is a γ-agop and a is its absorbing element then $a = 0$ or $a > \gamma$.

Regardless lack of important properties γ-agops are usefull in the frame of a generalized aggregation, what we show further.

4.3   Order relations

Generalized aggregation is considered in the frame of an order relation. In this section we summarize order relations considered further in the work.
Vertical order relation $\subseteq_{F_1}^\alpha$:

Definition 25 *Let* $\alpha \in [0,1]$, $P, Q \in F(\mathbb{R})$

$$P \subseteq_{F_1}^\alpha Q \Leftrightarrow (\forall x \in \mathbb{R})(P(x) \geq \alpha \Rightarrow P(x) \leq Q(x)).$$

The greatest element w.r.t. $\subseteq^{\alpha}_{F_1}$ is defined in the following way:

$$\tilde{1}(x) = 1, \forall x \in \mathbb{R}. \tag{10}$$

Let

$$\Theta = \{\tilde{0}(x) | \tilde{0}(x) \le \alpha, \forall x \in \mathbb{R}\}. \tag{11}$$

Capital Θ denotes the class of elements, where $\tilde{0}(x) = 0, \forall x \in \mathbb{R}$ is the least. Provided the essence of the parameter α (it "ignores" value if it is less than α) we consider all elements of Θ to be equivalent. Further speaking about boundary condition of a generalized agop (w.r.t.$\subseteq^{\alpha}_{F_1}$) we require that $\forall n \in \mathbb{N}$ n-ary aggregation of arbitrary elements from this class should be equal to an element from this class, then we say that the boundary condition is satisfied. Further we call Θ the class of minimal elements.

Horizontal order relation $\subseteq^{\alpha}_{F_2}$:

Definition 26 *Let $\alpha \in (0, 1]$, $P, Q \in F([a, b])$*

$$P \subseteq^{\alpha}_{F_2} Q \Leftrightarrow \overline{P}^{\alpha} \le \underline{Q}^{\alpha},$$

where

$$P^{\alpha} = \{x : P(x) \ge \alpha\}, \qquad \min P^{\alpha} = \underline{P}^{\alpha}, \qquad \max P^{\alpha} = \overline{P}^{\alpha}$$
$$Q^{\alpha} = \{x : Q(x) \ge \alpha\}, \qquad \min Q^{\alpha} = \underline{Q}^{\alpha}, \qquad \max Q^{\alpha} = \overline{Q}^{\alpha}.$$

The classes

$$\Theta = \{\tilde{0}(x) | \tilde{0}(x) = 1, \text{ if } x = a \text{ and } \tilde{0}(x) < \alpha \text{ if } x \in (a, b]\},$$

$$\Sigma = \{\tilde{1}(x) | \tilde{1}(x) = 1, \text{ if } x = b \text{ and } \tilde{1}(x) < \alpha \text{ if } x \in [a, b)\}$$

we will call correspondingly the class of minimal and maximal elements. The least element is defined in the following way:

$$\tilde{0}(x) = \begin{cases} 1, & \text{if } x = a \\ 0, & \text{otherwise} \end{cases}$$

but the greatest element does not exist.

The necessity of the whole class instead of just one the least (or the greatest) element is motivated by the essence of parameter α. Further in the context of generalized aggregation (w.r.t.$\subseteq^{\alpha}_{F_2}$) we require that $\forall n \in \mathbb{N}$ n-ary aggregation of an arbitrary element from the class of minimal (maximal) elements is equal to an arbitrary element from the same class, if this holds we say that boundary condition is satisfied.

Remark 1 *Relations $\subseteq^{\alpha}_{F_1}$ and $\subseteq^{\alpha}_{F_2}$ are transitive and asymmetric.*

4.4 Auxiliary results

Continous t-norms play an important role in the work. Following theorems 4.3, 4.4 and 4.5 are proven in the work for an arbitrary continous t-norm, and they are generalization of the results in [35] pp. 75-77.

Theorem 4.3 *If $\circ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous operation, T is a continuous t-norm and $P, Q \in F(\mathbb{R})$ are upper semicontinuous fuzzy sets with bounded α-cuts $\forall \alpha > 0$ then for all $z \in \mathbb{R}$, $z = x \circ y$ $\exists x_0, y_0 \in \mathbb{R}$ such that $z = x_0 \circ y_0$ and $(P \circ Q)(z) = T(P(x_0), Q(y_0))$.*

Theorem 4.4 *If $\circ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous operation, T is a continuous t-norm and $P, Q \in F(\mathbb{R})$ are upper semicontinuous fuzzy sets with bounded α-cuts $\forall \alpha > 0$ then*
$$(P \circ Q)^{T(\alpha, \beta)} = P^\alpha \circ Q^\beta.$$

Theorem 4.5 *If $\circ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous operation, T is a continuous t-norm and $P, Q \in FQ(\mathbb{R})$ then $P \circ Q \in FQ(\mathbb{R})$.*

4.5 Generalized aggregation: introduction

Let $\prec$ be some order relation on $F(\mathbb{R})$ with the least element $\tilde{0} \in F(\mathbb{R})$ and the greatest element $\tilde{1} \in F(\mathbb{R})$.

Definition 27 *[36] A mapping $\tilde{A} : \cup_{n \in \mathbb{N}} F(\mathbb{R})^n \to F(\mathbb{R})$ is called a generalized aggregation operator w.r.t. the order relation $\prec$, if for every $n \in \mathbb{N}$ the following conditions hold:*
($\tilde{A}1$) $\tilde{A}(\tilde{0}, ..., \tilde{0}) = \tilde{0}$
($\tilde{A}2$) $\tilde{A}(\tilde{1}, ..., \tilde{1}) = \tilde{1}$
($\tilde{A}3$) $(\forall i = \overline{1, n}) \ (P_i \prec Q_i) \Rightarrow \tilde{A}(P_1, ..., P_n) \prec \tilde{A}(Q_1, ..., Q_n)$,
where $P_1, ..., P_n, Q_1, ..., Q_n \in F(\mathbb{R})$.

We use convention $\tilde{A}_{(1)}(P(x)) = P(x)$ for all $P(x) \in F(\mathbb{R})$.
Inputs of gagops are upper semicontinuous fuzzy sets with bounded α-cuts, $\alpha > 0$, therefore further in the work $F(\mathbb{R})$ denotes the class of fuzzy sets with the mentioned characteristics. Sets $FQ(\mathbb{R}), FI(\mathbb{R}), FN(\mathbb{R}), FTI(\mathbb{R})$ and $FTN(\mathbb{R})$ denote correspondingly the set of fuzzy quantities, the set of fuzzy intervals, the set of fuzzy numbers, the set of fuzzy trapezoidal intervals and the set of fuzzy triangular numbers.
Properties of a gagop are defined in the same way like in the case of agop, but with the adoption to the fuzzy inputs. Operations with fuzzy sets (e.g. in the case of shift-invariance property) are performed via extension principle

and using an arbitrary continuous t-norm.

We have shown in the paper that if a gagop has the neutral (or absorbing) element then it is unique.

4.6   Pointwise extension

We start study on a pointwise extension.

$$\tilde{A} : \cup_{n \in \mathbb{N}} F(\mathbb{R})^n \to F(\mathbb{R})$$

is a pointwise extension of an arbitrary agop A provided that:

$$\forall x \in \mathbb{R} \ \tilde{A}(P_1, ..., P_n)(x) = A(P_1(x), ..., P_n(x)). \tag{12}$$

Output is a fuzzy set with a bounded α-cuts, $\alpha > 0$, because we have finite number of input values. If we lose upper semicontinuity of an output value (e.g. in the case of not continuous agop A) we use the special construction, which helps us to restore upper semicontinuity. Thus we say that the result of aggregation will always belong to $F(\mathbb{R})$.

Pointwise extension does not preserve the convexity of input values. Thus taking $FQ(\mathbb{R})$, $FI(\mathbb{R})$ or $FN(\mathbb{R})$ in the role of the set of input values we can not expect that in all cases the output value belongs to the same class.

Pointwise extension is a gagop w.r.t. $\subseteq_{F_1}^{\alpha}$. Also pointwise extension of a γ-agop respects $\subseteq_{F_1}^{\alpha}$:

Theorem 4.6 *If $\tilde{A}$ is a pointwise extension of a γ-agop A, and $\gamma > \alpha$, then it is a gagop w.r.t. order relation $\subseteq_{F1}^{\alpha}$.*

Speaking about an arbitrary agop (not γ-agop) idempotence is essential:

Theorem 4.7 *If $\tilde{A}$ is a pointwise extension of an idempotent agop A, then it is a gagop w.r.t. order relation $\subseteq_{F1}^{\alpha}$.*

γ-agops are not idempotent, but property (A_γ) substitutes the idempotence. A pointwise extension does not respect $\subseteq_{F2}^{\alpha}$ (for γ-agops we have the same result). Inconsistency arises from the fact that $\tilde{A}$ is defined on y axis, but $\subseteq_{F2}^{\alpha}$ acts on x axis.

We have shown the following properties of a pointwise extension in the work:

Proposition 4.8 *Let $\tilde{A}$ be a pointwise extension of A, then the following assertions hold:*
(1) if A is symmetric then $\tilde{A}$ is symmetric,
(2) if A is associative then $\tilde{A}$ is associative,
(3) if A is bisymmetric then $\tilde{A}$ is bisymmetric,
(4) if A is idempotent then $\tilde{A}$ is idempotent.

Proposition 4.9 *If $\tilde{A}$ is a pointwise extension of A, a and e are correspondingly absorbing and neutral elements of A, then the following assertions hold:*
(1) $R(x) = a$, $\forall x \in \mathbb{R}$ is the absorbing element of $\tilde{A}$,
(2) $E(x) = e$, $\forall x \in \mathbb{R}$ is the neutral element of $\tilde{A}$.

Proposition 4.10 *Let $\tilde{A} : \cup_{n \in \mathbb{N}} F(\mathbb{R})^n \to F(\mathbb{R})$ be the pointwise extension of $A = max$ and T is an arbitrary continuous t-norm, then $\tilde{A}$ is a shift-invariant gagop.*

The assertion of proposition 4.10 holds for an arbitrary continuous operation, thus $\tilde{A}$ ir homogeneous and linear in the case $A = max$.

4.7 T- extension

We study a T-extension of an arbitrary agop in this section. We will denote a T-extension $\widehat{A}$. Let T be an arbitrary continuous t-norm, then

$$\widehat{A}(P_1, ..., P_n)(x) = \sup\{T(P_1(x_1), ..., P_n(x_n))|$$

$$(x_1, ..., x_n) \in \mathbb{R}^n : A(x_1, ..., x_n) = x\} \tag{13}$$

is called a T-extension of an agop A.
We consider agops defined on the unit interval, and thus further $F([0,1])$ (the set of input values of $\widehat{A}$) contains upper semicontinuous fuzzy sets defined on $[0,1]$.
T-extension preserves convexity of input values, uniqueness of vertex and straight lines in special cases. Therefore $\widehat{A}$ can be defined on $FQ([0,1])$, $FI([0,1])$, $FN([0,1])$ and in special cases on $FTI([0,1])$ and $FTN([0,1])$.
T-extension is a gagop w.r.t. order relations defined previously:

Theorem 4.11 *An arbitrary T-extension $\widehat{A} : \cup_{n \in \mathbb{N}} F([0,1])^n \to F([0,1])$ of an arbitrary continuous agop A is a gagop w.r.t. $\subseteq_{F_1}^{\alpha}$.*

Theorem 4.12 *An arbitrary T-extension $\widehat{A} : \cup_{n \in \mathbb{N}} F([0,1])^n \to F([0,1])$ of an arbitrary continuous agop A is a gagop w.r.t. $\subseteq_{F_2}^{\alpha}$.*

Now we consider properties of a T-extension. If A is a continuous agop, then $\widehat{A}$ preserves symmetry, associativity and bisymmetry of A.

Proposition 4.13 *If A is a continuous and symmetric agop then*

$$\widehat{A} : \cup_{n \in \mathbb{N}} F([0,1])^n \to F([0,1])$$

is a symmetric gagop.

Proposition 4.14 *If A is a continuous and associative agop then*

$$\widehat{A} : \cup_{n\in\mathbb{N}}F([0,1])^n \to F([0,1])$$

is an associative gagop.

Proposition 4.15 *If A is a continuous and bisymmetric agop then*

$$\widehat{A} : \cup_{n\in\mathbb{N}}F([0,1])^n \to F([0,1])$$

is a bisymmetric gagop.

In a general case $\widehat{A}$ is not idempotent, and convexity of the input values is crucial:

Proposition 4.16 *If $\widehat{A} : \cup_{n\in\mathbb{N}}FQ([0,1])^n \to FQ([0,1])$ is T_M-extension of an arbitrary continuous, idempotent agop A then it is an idempotent gagop.*

T_M is the only idempotent t-norm, and it can be shown that assertion of proposition 4.16 does not hold for other t-norms.
Neutral and absorbing elements are special type fuzzy sets:

Proposition 4.17 *Let $\widehat{A} : \cup_{n\in\mathbb{N}}F([0,1])^n \to F([0,1])$ be an arbitrary T-extension of a continuous agop A, and e be the neutral element of A, then*

$$E(x) = \begin{cases} 1, & \text{if } x = e \\ 0, & \text{if } x \neq e \end{cases}$$

is the neutral element of $\widehat{A}$.

Proposition 4.18 *Let $\widehat{A} : \cup_{n\in\mathbb{N}}F([0,1])^n \to F([0,1])$ be an arbitrary T-extension of a continuous agop A, then*

$$R(x) = 0 \ \forall x \in [0,1]$$

is the absorbing element of $\widehat{A}$.

Neutral element of $\widehat{A}$ is a real number, and absorbing element is equal to 0 in each point. Both elements are not classical representative of the set of fuzzy elements.
When we speak about shift-invariance of $\widehat{A}$ we assume that all necessary operations are defined (see definition 23). We denote T_1 the t-norm which is used to extend the addition operation, and correspondingly T_2 - the t-norm used to extend the agop A. Both T_1 and T_2 are continuous. In general T-extension is not shift-invariant, but the following results characterize $\widehat{A}$ from the prospective of this property.

Proposition 4.19 *If $T_1 = T_2 = T_M$, A is a continuous shift-invariant agop defined by means of operations of addition and multiplication with $c \in \mathbb{R}$, then*

$$\widehat{A} : \cup_{n \in \mathbb{N}} FTN([0,1])^n \to FTN([0,1])$$

is a shift-invariant gagop.

Proposition 4.20 *If $T_1 = T_2 = T$ is an arbitrary t-norm, A is a continuous, additive agop, B is a crisp interval and $\widehat{A} : \cup_{n \in \mathbb{N}} F([0,1])^n \to F([0,1])$, then*

$$\widehat{A}(P_1, ..., P_n) + B = \widehat{A}(P_1 + B, ..., P_n + B).$$

Proposition 4.21 *If $T_1 = T_2 = T$ is an arbitrary t-norm, A is a continuous and additive agop, $\widehat{A} : \cup_{n \in \mathbb{N}} FQ([0,1])^n \to FQ([0,1])$ is an idempotent gagop then $\widehat{A}$ is a shift-invariant gagop.*

We have noted before that only T_M ensures idempotence of $\widehat{A}$, thus the assertion of proposition 4.21 holds only in the case when $T_1 = T_2 = T_M$.

4.8 Outline of practical applications of gagops

Areas of practical applications of gagops can be the same like of agops: decision making and multi-attributes classification ([21]), classification problems based on interacting criteria ([9, 10]), application in intelligent systems ([32]) and other.
If properties of objects or criteria are represented in the form of fuzzy sets in the above mention areas, then gagops can be used.
Aggregation of fuzzy relations ([5, 11]) and fuzzy cognitive maps ([16, 17, 37]) are another areas, where gagops can be applied.

4.9 Concluding remarks on generalized aggregation

Opulence of generalized aggregation with new results and urgency of practical applications make this theory charming for us for further development. The following directions for the further research can be outlined:

- other properties of generalized agops (apart from those considered in the work),

- other construction methods of generalized agops.

The notion of T-extension can be generalized if we substitute the t-norm with an arbitrary agop A^*:

$$\widehat{A}(P_1, ..., P_n)(x) = \sup\{A^*(P_1(x_1), ..., P_n(x_n)) : A(x_1, ..., x_n) = x\}.$$

A^*-extension and γ-agop (which is not studied in great details in the work) are another areas for further study.

5 References

References

[1] A. Berman, R. J. Plemmons. Nonnegative Matrices in the Mathematical Sciences (Classics in applied mathematics). SIAM, the Society for Industrial and Applied mathematics, USA, 1994

[2] J.J.Buckley. *Fuzzy eigenvalues and input-output analysis.* Fuzzy sets and systems, **34**: 187–195, 1990

[3] T. Calvo, A. Kolesarova, M. Komornikova and R. Mesiar. *Aggregation Operators: Properties, Classes and Construction methods.* In: Aggregation Operators: New Trends and Applications. Studies in Fuzziness and Soft Computing, T.Calvo, G. Mayor, R.Mesiar (eds.), pp. 3 – 104, Physica - Verlag, New York, 2002

[4] M. Detyniecki. Fundamentals on aggregation operators. [online] http://www.spatial.maine.edu/ worboys/SIE565/papers/ aggregation%20operators.pdf , 2001

[5] U. Dudziak. *Weak properties of fuzzy relations in the context of aggregation process.* In: Abstracts of Ninth International Conference on Fuzzy Set Theory and Applications, FSTA, p. 38, 2008

[6] D.Dubois, W.Ostasiewicz and H.Prade. Fuzzy Sets: History and Basic Notions: Fundamentals of Fuzzy Sets. Kluwer Academic Publ., Boston, Dodrecht, London, 1999

[7] M. Friedman, Ma. Ming and A. Kandel. *Fuzzy Linear System.* Fuzzy Sets and Systems, **96**:201–209, 1998

[8] J.A.Goguen. *L-Fuzzy Sets.* Journal of Mathematical Analysis and Applications, **18**:145–174, 1967

[9] M. Grabisch. *Fuzzy integral for classification and feature extraction.* In: Fuzzy Measures and Integrals. Theory and Applications, M. Grabisch, T. Murofushi, M. Sugeno, and J. Kacprzyk (eds), pp. 415–434, Physica - Verlag, Berlin, 2000

[10] M. Grabisch. *The application of fuzzy integrals in multicriteria decision making.* Journal of Operational Research, **89**:445–456, 1996

[11] O. Grigorenko. *Fuzzy POS Category and aggregation of fuzzy order relations.* In Abstracts of 14th International Conference on Mathematical Modeling and Analysis, p. 30, 2009

[12] R.Horn and Ch.Dzonson. Matrichnij analiz (in Russian). "Mir", Moscow, 1989

[13] Max E. Jerrell. *Interval Arithmetic for Input-output Models with Inexact Data.* Computational Economics **10**:89–100, 1997

[14] Max E. Jerrell. *Economic Input-Output Models.* In: Applications of interval computations, R.Baker Kearfott, V. Kreinovich (eds.) pp. 133–144, Kluwer Academic Publishers, Netherlands, 1996

[15] E. Klement, R. Mesiar and E. Pap. Triangular Norms. Series: Trends in Logic, Vol. 8, Dordrecth: Kluwer Academic Publishers, 2000

[16] B. Kosko. *Fuzzy cognitive maps.* International Journal Man-Machine Studies, **24**:65–75, 1986

[17] B. Kosko. Fuzzy thinking: the new science of fuzzy logic. Hyperion, New York, 1993

[18] V. Kreinovich. *Optimal Finite Characterization of Linear Problems with Inexact Data.* Reliable Computing, **11** (6):479–489, 2005

[19] R. Kruse, J.Gebhardt and F.Klawon. Foundations of Fuzzy Systems. John Wiley and Sons, Chichester, New York, Birsbane, Toronto, Singapore, 1998

[20] A.V. Lakeyev. *Computational complexity of estimation of generalized solution sets for interval linear systems.* Vicisliteljnije tehnologiji, **8** (1), 2003

[21] J.M. Merigo and M.C. Ramon. *The induced generalized hybrid averaging operator and its application in financial decision making.* International Journal of Business, Economics, Finance and Management Sciences, **1**(2): 95–101, 2009

[22] W. H. Miernyk. Input-Output Analysis. Random House, New York, 1965

[23] R.E. Moore. *Interval arithmetic and automatic error analysis in digital computing.* Stanford University, Technical report No. 25, Stanford, CA, 1962

[24] W.L.Peterson. Principles of Economics. Macro. 8th Edition, Richard D. Irwin, Inc, Boston, 1991

[25] H. Reynaerts and S. Muzzioli. *Solving fuzzy systems of linear equations by a nonlinear programming method.*

[26] J.Rohn. *Regularity of Interval Matrices and Theorems of the Alternatives.* Reliable computing, **12**:99–105, 2006

[27] J. Rohn. A Handbook of Results on Interval Linear Problems. [online] http://www.cs.cas.cz/rohn/handbook, 2005

[28] J. Rohn. *Solvability of systems of linear interval equations.* SIAM Journal on Matrix Analysis and Applications, **25**(1): 237–245, 2003

[29] J. Rohn. *Enclosing Solutions of Overdetermined Systems of Linear Interval Equations.* Reliable Computing, **2**:167–171, 1996

[30] J.Rohn. *Inverse interval matrix.* SIAM Journal on Numerical Analysis, **30**:864–870, 1993

[31] J. Rohn. *Input-Output Model with Interval Data.* Econometrica, **48**(3):767–769, 1980

[32] I. J. Rudas and J. Fodor. *Information Aggregation in Intelligent Systems Using Generalized Operators.* International Journal of Computers, Communications and Control, **1**(1):47–57, 2006

[33] P. Senthilkumar and G. Rajendran. *Solution of Fuzzy Linear Systems by Using Fuzzy Centre.* Applied Mathematical Sciences, **3**(49):2411 – 2419, 2009

[34] I. Skalna, M.V. Rama Rao and A. Pownuk. *Systems of fuzzy equations in structural mechanics.* Research Report No. 2007-01, 2007

[35] A.Šostaks. *L*-kopas un *L*-vērtīgas struktūras (in Latvian). Rīga, Latvijas Universitāte, Mācību grāmata, 2003

[36] A. Takači. *General aggregation operators acting on fuzzy numbers induced by ordinary aggregation operators.* Novi Sad J. Math., **33**(2): 67–76, 2003

[37] W.B. Vasantha Kandasamy and F. Smarandache. Fuzzy cognitive maps and neutrosophic cognitive maps. [online] http://fs.gallup.unm.edu//NCMs.pdf, 2003

[38] A. Vroman G. Deschrijver and E. E. Kerrel. *Using Parametric Functions to Solve Systems of Linear Fuzzy Equations with a Symmetric Matrix.* International Journal of Computational Intelligence Systems **1**(3):248–261, 2008

[39] R. Yager. *Generalized OWA Aggregation Operators.* Fuzzy Optimization and Decision Making, **3**(1):93–107, 2004

[40] L.A.Zadeh. *Fuzzy Sets.* Information and Control, **8**:338–353, 1965

Author's publications

[41] J.Lebedinska. *γ-aggregation operators and some aspects of generalized aggregation problem.* Math. Model. Anal., **15**(1) :83-96, 2010

[42] J.Lebedinska. *On another view of an inverse of an interval matrix.* Soft Computing, 2009, DOI 10.1007/s00500-009-0482-5

[43] J.Lebedinska. *On the inverse matrix of a fuzzy matrix.* ICTAA 2008 Proceedings, pp. 60–64, 2008