

RIGA TECHNICAL UNIVERSITY



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**Analytical and quasi-analytical solutions of direct
problems in eddy current testing**

Doctoral Thesis

In Partial Fulfilment of the Requirements of the Doctor Degree in
Mathematics

Subdiscipline of Mathematical modelling

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ABSTRACT

The thesis consists of two parts. Analytical solutions of direct problems in eddy current testing are constructed in the first part of the thesis. It is assumed that the properties of a conducting medium are functions of one spatial coordinate. Solutions are found for the following cases: (a) a planar conducting multilayer medium where the magnetic permeability and electrical conductivity are exponential functions of the vertical coordinate; (b) a cylindrical conducting multilayer medium where the magnetic permeability and electrical conductivity are power functions of the radial coordinate; (c) a double conductor line above a multilayer medium with varying properties for the cases of alternating current and step current; (d) a moving planar two-layer medium with varying properties. All solutions are found by the method of integral transforms where the corresponding system of ordinary differential equations is solved in terms of different special functions.

Quazi-analytical solutions are constructed in the second part of the thesis for the case of a conducting medium of finite size. It is assumed that all properties of the conducting medium are constant. Three cases of axisymmetric problems are considered: (a) a coil with alternating current above a conducting cylinder of finite size; (b) a coil with current above a conducting plate with bottom cylindrical hole; (c) a coil with current above a conducting half-space with a flaw in the form of a cylinder of finite size. The problems are solved by the method of separation of variables. The solution procedure includes two steps where numerical methods are used: (1) calculation of complex eigenvalues without good initial guess for the root and (2) solution of a system of linear algebraic equations. Results of numerical calculations are presented. Good comparison between theoretical calculations and experimental data is found.

Keywords: eddy currents, integral transform, special functions, variable conductivity and permeability, flaw detection, eigenvalues

ANOTĀCIJA

Promocijas darbs sastāv no divām daļām. Darba pirmajā daļā konstruēti virpuļstrāvu nesagraujošās kontroles tiešo problēmu analītiskie atrisinājumi. Tiek pieņemts, ka vadošas vides īpašības ir funkcijas, atkarīgas no vienas telpas koordinātas. Atrisinājums atrasts šādiem gadījumiem: (a) plakana vadošai daudzslāņu videi, kuras magnētiskā caurlaidība un elektriskā vadāmība ir eksponentfunkcijas no vertikālās koordinātas; (b) cilindriskai vadošai daudzslāņu videi, kuras magnētiskā caurlaidība un elektriskā vadāmība ir radiālās koordinātas pakāpes funkcijas; (c) divvadu līnijai virs daudzslāņu vadītāja ar mainīgām īpašībām gadījumos, kad divvadu līnijā plūst maiņstrāva un lēcienveida strāva; (d) plakana divslāņu vadošai videi ar mainīgām īpašībām, kad vide atrodas kustībā. Visi atrisinājumi atrasti pēc integrālo transformāciju metodes, kurā atbilstošo parasto diferenciālvienādojumu atrisinājumi izteikti ar dažādām speciālajām funkcijām.

Darba otrajā daļā konstruēti kvazianalītiskie atrisinājumi gadījumiem, kad vadošai videi ir galīgi izmēri. Tiek uzskatīts, ka visas vadošās vides īpašības ir konstantas. Aplūkotas trīs problēmas: (a) vijums ar maiņstrāvu virs vadoša galīga izmēra cilindra; (b) vijums ar strāvu virs vadošas plāksnes ar cilindriskā veida dobumu; (c) vijums ar strāvu virs vadošas pustelpas ar defektu galīga cilindra formā. Visos trijos gadījumos vijuma ass sakrīt ar atbilstošā cilindriskā ķermeņa asi. Minētās problēmas atrisinātas, izmantojot mainīgo atdalīšanas metodi. Risināšanas procedūra satur divus soļus, kuros izmantotas skaitliskās metodes: (1) komplekso īpašvērtību noteikšana, kad nav zināmi labi sākuma sakņu tuvinājumi; (2) lineāras algebrisko vienādojumu sistēmas atrisināšana. Darbā veikti skaitliski aprēķini. Ir novērota laba teorētisko aprēķinu un eksperimentālo datu sakrītība.

Atslēgas vārdi: virpuļstrāvas, integrālās transformācijas, speciālās funkcijas, mainīga magnētiskā caurlaidība un elektriskā vadāmība, defekta atrašana, īpašvērtības.

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NOMENCLATURE

List of Latin symbols

\bar{A}	amplitude of vector potential
\tilde{A}	vector potential, $\tilde{A} = \bar{A}e^{j\omega t}$
A_i	nonzero component of vector potential in region R_i
A_0^{ind}	induced vector potential intensity in free space
\bar{B}	amplitude of magnetic induction vector
\tilde{B}	magnetic induction vector, $\tilde{B} = \bar{B}e^{j\omega t}$
\bar{D}	amplitude of electric induction vector
\tilde{D}	electric induction vector, $\tilde{D} = \bar{D}e^{j\omega t}$
\bar{E}	amplitude of electric field vector
\tilde{E}	electric field vector, $\tilde{E} = \bar{E}e^{j\omega t}$
h	height of a single-turn coil above conducting medium
h_1	height of a coil above conducting medium
$h_2 - h_1$	height of a coil
\bar{H}	amplitude of magnetic field vector
\tilde{H}	magnetic field vector, $\tilde{H} = \bar{H}e^{j\omega t}$
\bar{I}	amplitude current vector
\bar{I}^e	amplitude external current vector
\tilde{I}	current vector, $\tilde{I} = \bar{I}e^{j\omega t}$
$I_\nu(s)$	modified Bessel function of the first kind of order ν
j	imaginary unit, $j = \sqrt{-1}$

$J_\nu(s)$	Bessel function of the first kind of order ν
$K_\nu(s)$	modified Bessel function of the second kind of order ν
r_0	radius of a single-turn coil
r_1	inner radius of a coil
r_2	outer radius of a coil
$Y_\nu(s)$	Bessel function of the second kind of order ν
Z	dimensionless induced change in impedance
Z^{ind}	induced change in impedance

List of Greek symbols

ε_0	electric constant
ε	relative permittivity
μ	relative magnetic permeability
μ_0	magnetic constant
$\tilde{\rho}$	charge density
σ	conductivity
ψ	scalar electric potential intensity
$\tilde{\psi}$	electric potential, $\tilde{\psi} = \psi e^{j\omega t}$
ω	frequency

INTRODUCTION

The structure of the thesis

The main objective of the PhD thesis is to develop mathematical models which can be used to describe nondestructive eddy current testing methods. The thesis consists of two parts. Analytical solutions of direct problems in eddy current testing are constructed in the first part of the thesis (**Chapters 2 – 5**). It is assumed that the properties of a conducting medium are functions of one spatial coordinate. Solutions are found for the following cases: (a) a planar conducting multilayer medium where the magnetic permeability and electrical conductivity are exponential functions of the vertical coordinate; (b) a cylindrical conducting multilayer medium where the magnetic permeability and electrical conductivity are power functions of the radial coordinate; (c) a double conductor line above a multilayer medium with exponentially varying properties for the cases of alternating current and step current; (d) a moving planar two-layer medium with varying properties. All solutions are found by the method of integral transforms where the corresponding system of ordinary differential equations is solved in terms of different special functions.

Chapter 1 (Introduction) is devoted to the derivation of the basic equations used in the study. In addition, literature survey is also presented.

In **Chapter 2** we construct analytical solutions of eddy current testing problems for the case of a planar conducting multilayer medium. The magnetic permeability and electrical conductivity of the layers are assumed to vary with respect to the vertical coordinate. Such a situation takes place in practice during special treatment of ferromagnetic metals (for example, surface hardening). Experimental data indicate that a relatively thin layer of reduced magnetic permeability is formed as a result of surface hardening. Analysis of experimental data shows that the magnetic permeability can be reasonably well approximated by an exponential function of the vertical coordinate. It is shown in **Chapter 2** that if the magnetic permeability and electrical conductivity are exponential functions of the vertical coordinate then the solution of the system of the Maxwell's equations can be found in closed form in terms of improper integrals containing Bessel functions of complex argument. The solution is obtained for the case of a multilayer medium. Two special cases (conducting half-space and two-layer medium) are considered in detail. The change in impedance is computed using software package „Mathematica”.

Chapter 3 is devoted to the analysis of a similar problem in cylindrical geometry where a coil with alternating current is located inside (or outside) a multilayer tube. It is assumed that the axis of the coil coincides with the axis of the tube. The electrical conductivity and magnetic permeability of one or several conducting layers are power functions of the radial coordinate. It is shown in **Chapter 3** that different analytical solutions can be constructed depending on the choice of the constants characterizing the power functions. Analytical solutions are derived in terms of Bessel functions and confluent hypergeometric functions. Results of numerical calculations are presented (calculations are done with „Mathematica”).

Chapter 4 is devoted to the solution of an eddy current problem for a planar multilayer medium with exponentially varying electrical conductivity and magnetic permeability for the case where an excitation coil is modeled by means of a double conductor line (the coil is formed

by two infinitely long parallel wires). The double conductor line is a sufficiently accurate model for the case where a rectangular frame with current is located above a multilayer medium under the assumption that the ratio of the sides of the frame is 1:4 or smaller. In addition, one case where an excitation current in the coil is not alternating current but is in the form of a pulse is considered in detail. The corresponding boundary value problem is solved by the method of the Laplace and Fourier transforms. The inverse Laplace transform is found in closed form for the case where a double conductor line is located above a conducting half-space with constant electrical conductivity and magnetic permeability.

In **Chapter 5** the results obtained in **Chapter 2** are generalized for the case where an upper conducting layer is moving with constant velocity V in a horizontal direction. The problem is solved by means of the double Fourier transform in the two horizontal directions. The corresponding system of ordinary differential equations is solved analytically in terms of Bessel functions of complex argument. The change in impedance is found in closed form in terms of double integrals. Results of numerical calculations with „Mathematica” are presented. Solutions obtained in **Chapters 2 – 5** correspond to the case where a conducting medium is assumed to be infinite in one or two spatial directions. Integral transforms such as Hankel or Fourier integral transforms are used to solve the corresponding problems. Such an approach can be used in practice if a coil is sufficiently smaller than a conducting medium. However, the finite size of a conducting medium has to be taken into account if the size of the coil is comparable to the size of the conducting medium.

Chapter 6 is devoted to the analysis of axisymmetric problems where a coil with alternating current is located above a conducting medium of finite size. Three cases which are important for applications are considered in detail: (a) a coil above a conducting cylinder of finite size located in free space (this model is relevant for the analysis of eddy current problem in coin validators), (b) a coil above an infinite conducting plate with a bottom cylindrical hole coaxial with the axis of the coil (this problem can be used to estimate and model the effect of corrosion) and (c) a coil above a half-space with a coaxial conducting cylinder of finite size (this model is a sufficiently accurate model of spot welding where the cylinder represents a cast core which is formed in the conducting medium during welding process). Mathematical solution is based on the TREE methodology (TREE stands for „TRuncated Eigenfunktion Expansions”, see [60]). The main assumption in the TREE method is that the vector potential is exactly zero at a sufficiently large distance from the coils’ center (usual boundary conditions at infinity imply that the vector potential approaches zero as the geometrical coordinate tends to infinity). Using the TREE method quasi-analytical solution for all three problems mentioned above is constructed. We use the term „quasi-analytical solution” since the solution is found by the method of separation of variables, but there are two stages of the solution process where numerical methods should be used: (a) calculation of complex eigenvalues and (b) solution of the system of linear equations in order to determine integration constants. A computer program is developed on the basis of the algorithm described in [43] in order to compute complex eigenvalues without good initial guesses for the roots. The formula for the change in impedance in all three cases is derived. Results of numerical computations of the change in impedance using „Mathematica” are presented. Computational values of the change in impedance are compared with experimental

data for the case where a coil with alternating current is located above a conducting cylinder of finite size. Good agreement between theoretical and experimental values of the change in impedance is found.

Importance of the subject

Eddy current method is widely used in practice in order to control properties of electrically conducting materials. Applications include determination of electrical conductivity or magnetic permeability of conducting materials, estimation of thickness of metal coatings and analysis of the properties of coatings, detection of defects (such as voids or cracks) in a conducting medium. The solution of an inverse problem is usually required in order to solve eddy current testing problem. At this stage the difference (in some norm) between theoretical model and experimental data is minimized in order to estimate unknown parameters of the model (such as electrical conductivity of a conducting plate or parameters of the defect). Thus, in order to solve the inverse problem one has to have a convenient and reliable mathematical model for the solution of the direct problem.

Analytical solutions of eddy current testing problems for the case where the properties of the conducting medium (electrical conductivity and magnetic permeability) are constant are well-known in the literature. In addition, the conducting medium is assumed to be infinite in one or two spatial directions. In the thesis we generalize the approach developed for media with constant properties to the case where the properties of the medium vary with respect to one spatial coordinate (vertical coordinate for the case of a planar multilayer medium and radial coordinate for the case of a cylindrical multilayer tube). Analytical solutions are constructed in the thesis for the cases where the electrical conductivity and magnetic permeability of the medium are exponential functions of the vertical coordinate and power functions of the radial coordinate for planar and cylindrical cases, respectively. The solutions are found for different types of coils (circular coils and coils in the form of a double conductor line).

We also analyze axisymmetric cases where a circular coil is coaxial with cylindrical region of finite size. The solutions are found by the method of separation of variables where complex eigenvalues are determined by an efficient algorithm without prior knowledge of the initial guesses for the roots. Thus, the constructed solutions are „quasi-analytical”. The obtained solutions can be used in practice to develop algorithms for coin validators, to estimate the effect of corrosion in metal plates and to assess quality of spot welding.

The objectives of the thesis

All the suggested methods for eddy current testing problems analyzed in the thesis can be divided into two parts: (a) analytical solutions for planar and cylindrical multilayer medium under the assumption that the magnetic permeability and electrical conductivity of the medium are functions of one spatial coordinate and the medium is infinite in one or two spatial directions, and (b) the magnetic permeability and electrical conductivity of the conducting medium are constant but the medium is of finite size (axisymmetrical problems where the axis of the coil coincides with the axis of a cylindrical region of finite size).

The objectives of the work are as follows:

- (1) Obtain analytical solutions for eddy current problems where the excitation coil is a circular coil with alternating current and the properties of planar multilayer conducting medium (electrical conductivity and magnetic permeability) are exponential functions of the vertical coordinate;
- (2) Construct analytical solutions for the case of a planar multilayer medium with exponentially varying properties under the assumption that the coil is modeled by a double conductor line consisting of two parallel infinite wires;
- (3) Obtain analytical solutions for the cases where a circular coil with alternating current is located inside (or outside) a multilayer conducting tube. The coil is coaxial with the tube. In addition, the properties of conducting layers (the electrical conductivity and magnetic permeability) are power functions of the radial coordinate;
- (4) Develop analytical solution for the case where a double conductor line is located above a conducting half-space with varying electrical conductivity and magnetic permeability. The excitation current in the double conductor line is in the form of a pulse. Pulsed eddy current represent a convenient alternative to traditional eddy current method based on alternating excitation current;
- (5) Obtain analytical solutions for eddy current problem where a coil with current is located above a moving medium. Two cases of the moving medium are considered: (a) a moving half-space and (b) a moving two-layer medium where the upper layer is moving but the lower layer is fixed. The electrical conductivity and magnetic permeability of the moving medium are exponential functions of the vertical coordinate;
- (6) Construct quasi-analytical solutions for axisymmetric problems where a circular coil is coaxial with cylindrical region of finite size. Three axisymmetric problems are solved in the thesis: (a) a coil with alternating current above a conducting cylinder of finite size, (b) a coil with alternating current above a conducting plate with bottom hole in the form of a cylinder coaxial with the coil and (c) a coil with alternating current above a conducting half-space with a flaw in the form of a cylinder coaxial with the coil.

Research methodology

Methods of integral transforms such as Hankel, Fourier and Laplace transforms are used in **Chapters 2 -5** of the thesis in order to construct analytical solutions of eddy current testing problems in planar and cylindrical cases where the medium is assumed to be infinite in one or two spatial dimensions. The electrical conductivity and magnetic permeability of the medium are functions of one spatial coordinate (exponential functions of the vertical coordinate in the case of a planar medium and power functions of the radial coordinate in the case of a cylindrical medium). The solution of the corresponding ordinary differential equations in the transformed space is found in closed form by means of the Bessel functions and confluent hypergeometric functions. The results are presented in the form of the change in impedance of the coil (for the case of a single-turn coil and coil of finite dimensions). In addition, analytical solutions are also constructed for the case of a moving planar medium under the assumption that the magnetic

permeability and electrical conductivity are exponential functions of the vertical coordinate. Calculations are performed with “Mathematica”.

Method of separation of variables is used in **Chapter 6** in order to solve axisymmetric problems where a coil is located above a conducting medium which contains a cylinder of finite size coaxial with the coil. A fast and reliable algorithm for the determination of complex eigenvalues is implemented in the form of the “Mathematica” code. Note that no initial guess for a complex eigenvalue is required. The change in impedance is calculated in the form of a truncated series containing Bessel functions.

Scientific novelty and main results

The following new results are obtained in the thesis.

- (1) Analytical solution is obtained for eddy current problem where a single-turn circular coil or a coil in the form of a double conductor line is located above a conducting multilayer medium with varying electrical conductivity and magnetic permeability. The solution is found by the method of Hankel integral transform. The change in impedance of the coil is expressed in terms of improper integral containing Bessel functions. The particular cases of a conducting medium in the form of a half-space and two-layer medium are considered in detail. The change in impedance is also obtained for the case where a circular coil of finite dimensions is located above a multilayer medium. Results of numerical calculations with “Mathematica” are presented.
- (2) Method of the Fourier integral transform is used in the thesis in order to solve the eddy current problem where a coil with alternating current is located inside (or outside) a conducting multilayer tube. Analytical solution is found in the form of an improper integral containing Bessel or confluent hypergeometric functions. The change in impedance is computed with “Mathematica”.
- (3) Analytical solution is also found for the case where the excitation current in a double conductor line located above a multilayer conducting medium with constant electrical conductivity and magnetic permeability is assumed to be in the form of a pulse. A double conductor line is a sufficiently accurate model of a rectangular frame with current provided the ratio of the sides of the frame is 1:4 or smaller. Solution is found by the method of the Laplace transform.
- (4) Method of the double Fourier integral transform is used in the thesis in order to construct an analytical solution of the eddy current problem where a coil with alternating current is located above a conducting half-space or two-layer medium. The half-space (or the upper layer of the two-layer medium) is assumed to be moving in the horizontal direction with constant velocity V . The electrical conductivity and magnetic permeability of the moving medium are exponential functions of the vertical coordinate. The solution is found in closed form in terms of a double integral. Calculations of the change in impedance are performed with “Mathematica”.
- (5) Method of separation of variables is used in the thesis in order to construct quasi-analytical solution for eddy current problem where a circular coil is located above a conducting medium with constant properties under the assumption that the medium contains a cylindrical region

of finite size (a finite cylinder in free space, or a cylindrical hole in a conducting plate, or a cylinder of finite size in a conducting half-space). The axis of the coil coincides with the axis of the cylindrical region. Calculations show good agreement between experimental data and theoretical model.

Applications

Analytical and quasi-analytical methods for the solution of direct problems in eddy current testing developed in the thesis can be used in practice in order to control properties of materials and devices with the help of eddy current method.

Analytical solutions obtained in **Chapters 2, 3, 4 and 5** where the electrical conductivity and magnetic permeability are functions of one spatial coordinate can be used to analyze ferromagnetic metal processing methods (such as surface hardening). Experiments show that in this case a thin layer of reduced magnetic permeability is formed in the upper layer where the magnetic permeability can be accurately approximated by an exponential function of the vertical coordinate. Another application where the electrical conductivity varies with respect to the vertical coordinate is related to diffusion of aluminium in blades of gas turbines exposed to high temperatures.

Mathematical models developed in the thesis can be used to compute the change in impedance of the coil where the properties of the conducting medium (electrical conductivity and magnetic permeability) are exponential functions of the vertical coordinate.

Quasi-analytical solutions constructed in **Chapter 6** can be used in order to test quality of products and materials. The model developed in **Section 6.1** (a coil above a finite cylinder coaxial with the axis of the coil) can be implemented in coin validators. Eddy current method is one of the methods that can be used to validate coins by analyzing the electrical conductivity of a metal sample inserted into the validator. Mathematical model analyzed in **Section 6.2** (a coil above a conducting plate with a bottom hole in the form of a circular cylinder coaxial with the coil) can be used to estimate the effect of corrosion in metal plates. Finally, the model developed in **Section 6.3** (a coil above a conducting half-space with a flaw in the form of a circular cylinder of finite height coaxial with the coil) can be used to analyze the quality of spot welding. During welding process a cast core is formed whose properties (electrical conductivity, for example) are very close to the properties of the surrounding medium. Such a model can be quite useful in order to determine the properties of the cast core.

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1. BASIC EQUATIONS

1.1 Medium with constant properties

Maxwell's equations for a homogeneous isotropic medium have the form (see [40]):

$$\operatorname{curl} \tilde{E} = -\frac{\partial \tilde{B}}{\partial t}, \quad (1.1.1)$$

$$\operatorname{curl} \tilde{H} = \tilde{I} + \tilde{I}^e + \frac{\partial \tilde{D}}{\partial t}, \quad (1.1.2)$$

$$\operatorname{div} \tilde{B} = 0, \quad (1.1.3)$$

$$\operatorname{div} \tilde{D} = \tilde{\rho}, \quad (1.1.4)$$

$$\tilde{I} = \sigma \tilde{E}, \quad (1.1.5)$$

$$\tilde{B} = \mu_0 \mu \tilde{H}, \quad (1.1.6)$$

$$\tilde{D} = \varepsilon_0 \varepsilon \tilde{E}, \quad (1.1.7)$$

where \tilde{E} and \tilde{D} are the electric field and electric induction vectors, respectively; \tilde{H} and \tilde{B} are the magnetic field and magnetic induction vectors, respectively; \tilde{I} and \tilde{I}^e are the current and external current vectors, respectively; $\tilde{\rho}$ is the charge density; σ is the electrical conductivity; μ_0 and ε_0 are the magnetic and electric constants, respectively; μ and ε are the constant relative magnetic permeability and relative permittivity, respectively.

System (1.1.1) – (1.1.7) can be rewritten in more convenient form by introducing vector and scalar potentials of the electromagnetic field. The magnetic vector potential \tilde{A} is defined by the relation (see [40]):

$$\operatorname{curl} \tilde{A} = \tilde{B}. \quad (1.1.8)$$

Equations (1.1.1) and (1.1.8) give

$$\operatorname{curl} \left(\tilde{E} + \frac{\partial \tilde{A}}{\partial t} \right) = 0. \quad (1.1.9)$$

It follows from (1.1.9) that there exists a scalar potential of the electromagnetic field, $\tilde{\psi}$, which is defined as follows

$$\tilde{E} + \frac{\partial \tilde{A}}{\partial t} = -\operatorname{grad} \tilde{\psi}. \quad (1.1.10)$$

Using (1.1.6) and (1.1.8) we rewrite the left-hand side of equation (1.1.2) in the form

$$\operatorname{curl} \tilde{H} = \operatorname{curl} \left(\frac{1}{\mu_0 \mu} \tilde{B} \right) = \frac{1}{\mu_0 \mu} \operatorname{curl} \tilde{A} = \frac{1}{\mu_0 \mu} \left(\operatorname{grad} \operatorname{div} \tilde{A} - \Delta_3 \tilde{A} \right), \quad (1.1.11)$$

where $\Delta_3 \tilde{A}$ is the Laplacian in three dimensions.

It follows from (1.1.10) and (1.1.5) that

$$\tilde{I} = -\sigma \left(\text{grad } \tilde{\psi} + \frac{\partial \tilde{A}}{\partial t} \right). \quad (1.1.12)$$

Differentiating (1.1.7) with respect to t and using (1.1.10) we obtain

$$\frac{\partial \tilde{D}}{\partial t} = -\varepsilon_0 \varepsilon \left(\text{grad } \frac{\partial \tilde{\psi}}{\partial t} + \frac{\partial^2 \tilde{A}}{\partial t^2} \right). \quad (1.1.13)$$

Using (1.1.11) – (1.1.13) equation (1.1.2) can be rewritten in the form

$$\text{grad div } \tilde{A} - \Delta_3 \tilde{A} = -\mu_0 \mu \sigma \left(\text{grad } \tilde{\psi} + \frac{\partial \tilde{A}}{\partial t} \right) - \mu_0 \varepsilon_0 \mu \varepsilon \left(\text{grad } \frac{\partial \tilde{\psi}}{\partial t} + \frac{\partial^2 \tilde{A}}{\partial t^2} \right) + \mu_0 \mu \tilde{I}^e. \quad (1.1.14)$$

Equation (1.1.14) is derived under the assumption that the properties of a conducting medium (the electrical conductivity σ and magnetic permeability μ) are constant. It follows from (1.1.8) and (1.1.10) that \tilde{A} and $\tilde{\psi}$ are not uniquely defined (unless additional conditions on the vector potential \tilde{A} and scalar potential $\tilde{\psi}$ are imposed). These additional conditions are called *gauge conditions* (see the discussion on different gauge conditions in [40]).

In particular, the *Lorentz gauge* has the form

$$\text{div } \tilde{A} + \mu_0 \mu \sigma \tilde{\psi} + \mu_0 \varepsilon_0 \mu \varepsilon \frac{\partial \tilde{\psi}}{\partial t} = 0. \quad (1.1.15)$$

Using (1.1.14) and (1.1.15) we obtain

$$\Delta_3 \tilde{A} = \mu_0 \varepsilon_0 \mu \varepsilon \frac{\partial^2 \tilde{A}}{\partial t^2} + \mu_0 \mu \sigma \frac{\partial \tilde{A}}{\partial t} - \mu_0 \mu \tilde{I}^e. \quad (1.1.16)$$

Since the constants μ_0 and ε_0 are very small ($\mu_0 = 4\pi \cdot 10^{-7}$ H/m and $\varepsilon_0 = 8.85 \cdot 10^{-12}$ F/m, respectively), the first term on the right-hand side of (1.1.16) (which is usually referred to as the displacement current in the literature) is much smaller than the other terms and can be neglected for frequencies up to 100 MHz. Such high frequencies are not used in eddy current testing (the depth of penetration decreases with frequency). Thus, for most applications in eddy current testing for media with constant properties (1.1.16) can be simplified to the following form

$$\Delta_3 \tilde{A} = \mu_0 \mu \sigma \frac{\partial \tilde{A}}{\partial t} - \mu_0 \mu \tilde{I}^e. \quad (1.1.17)$$

If the excitation current is periodic with respect to time then all the functions in (1.1.1) – (1.1.10) are also periodic with respect to t :

$$\begin{aligned} \tilde{A} &= \vec{A} e^{j\omega t}, \quad \tilde{\psi} = \psi e^{j\omega t}, \quad \tilde{E} = \vec{E} e^{j\omega t}, \quad \tilde{D} = \vec{D} e^{j\omega t}, \\ \tilde{B} &= \vec{B} e^{j\omega t}, \quad \tilde{H} = \vec{H} e^{j\omega t}, \quad \tilde{I} = \vec{I} e^{j\omega t}, \quad \tilde{I}^e = \vec{I}^e e^{j\omega t}, \end{aligned} \quad (1.1.18)$$

where ω is the frequency.

In this case (1.1.17) reduces to a non-homogeneous *Helmoltz* equation

$$\Delta_3 \vec{A} + k^2 \vec{A} = -\mu_0 \mu \vec{I}^e, \quad (1.1.19)$$

where $k^2 = -j\omega\mu_0\mu\sigma$.

1.2 Medium with variable properties

1.2.1. Multilayer planar medium with varying electrical conductivity and magnetic permeability

In this section we consider an air core coil located above a multilayer conducting medium (see Fig. 1.1)

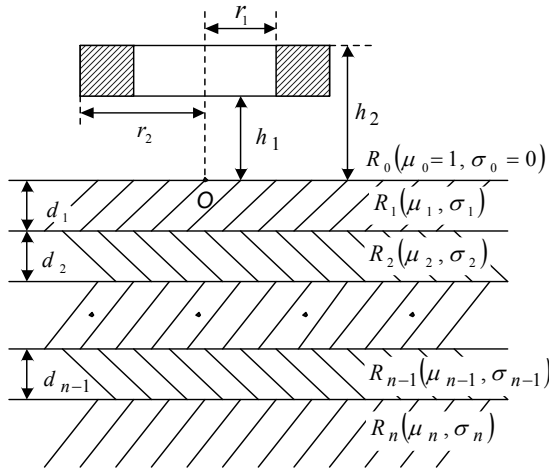


Fig. 1.1 A coil above a multilayer conducting medium.

The following parameters characterize the coil: r_1 and r_2 are the inner and outer radii of the coil, respectively, h_1 is the lift-off and $h_2 - h_1$ is the height of the coil. Each conducting layer R_i , $i=1,2,\dots,n$ is characterized by the electrical conductivity σ_i and relative magnetic permeability μ_i . We use a system of cylindrical polar coordinates (r, φ, z) centered at O, the z -axis is directed upwards. In this case due to axial symmetry the vector potential \vec{A}_i in region R_i has only one non-zero component of the form

$$\vec{A}_i(r, \varphi, z) = A_i(r, z) \vec{e}_\varphi, \quad i = 0, 1, 2, \dots, n \quad (1.2.1)$$

where \vec{e}_φ is the unit vector in the azimuthal direction. We assume that the electrical conductivity σ_i and relative magnetic permeability μ_i in each layer R_i , $i=1,2,\dots,n$ are functions of the vertical coordinate:

$$\sigma_i = \sigma(z), \quad \mu_i = \mu(z). \quad (1.2.2)$$

The subscript i in \vec{A}_i is omitted in sequel.

Using (1.1.10) and (1.1.18) we obtain

$$\vec{E} + j\omega\vec{A} = -\text{grad } \psi. \quad (1.2.3)$$

Using (1.1.18), (1.2.3) and (1.2.2) equations (1.1.5), (1.1.7) can be rewritten as follows

$$\vec{I} = -\sigma(z) (\text{grad } \psi + j\omega\vec{A}), \quad (1.2.4)$$

$$\vec{D} = -\varepsilon_0 \varepsilon (\text{grad } \psi + j\omega\vec{A}). \quad (1.2.5)$$

Equations (1.1.2) and (1.1.18) give

$$\text{curl } \vec{H} = \vec{I} + \vec{I}^e + j\omega\vec{D}. \quad (1.2.6)$$

The left-hand side of equation (1.2.6) can be rewritten using (1.1.6), (1.1.8), (1.1.18) in the form

$$\begin{aligned} \text{curl } \vec{H} &= \text{curl} \left(\frac{1}{\mu_0 \mu(z)} \text{curl } \vec{A} \right) = \frac{1}{\mu_0 \mu(z)} \text{curl curl } \vec{A} + \left[\text{grad} \frac{1}{\mu_0 \mu(z)}, \text{curl } \vec{A} \right] = \\ &= \frac{1}{\mu_0 \mu(z)} \left(\text{grad div } \vec{A} - \Delta \vec{A} \right) + \frac{1}{\mu_0 \mu^2(z)} \frac{d\mu}{dz} \frac{\partial A_\varphi}{\partial z} \vec{e}_\varphi. \end{aligned} \quad (1.2.7)$$

Using (1.2.4), (1.2.5) and (1.2.7) we transform (1.2.6) to the form

$$\begin{aligned} \text{grad div } \vec{A} - \Delta \vec{A} + \frac{1}{\mu(z)} \frac{d\mu}{dz} \frac{\partial A_\varphi}{\partial z} \vec{e}_\varphi &= -\mu_0 \mu(z) \sigma(z) (\text{grad } \psi + j\omega \vec{A}) + \mu_0 \mu(z) \vec{I}^e - \\ &- j\omega \mu_0 \varepsilon_0 \varepsilon \mu(z) (\text{grad } \psi + j\omega \vec{A}). \end{aligned} \quad (1.2.8)$$

Taking the projection of (1.2.8) on the φ -axis, imposing the condition

$$\text{grad div } \vec{A} + \mu_0 \mu(z) [\sigma(z) + j\omega \varepsilon_0 \varepsilon] \text{grad } \psi = 0, \quad (1.2.9)$$

using (1.2.1) and the fact that the constants μ_0 and ε_0 are very small ($\mu_0 = 4\pi \cdot 10^{-7}$ H/m and $\varepsilon_0 = 8.85 \cdot 10^{-12}$ F/m, respectively), that is, neglecting the displacement current if the frequency is not so high, we obtain

$$\Delta A - j\omega \sigma(z) \mu_0 \mu(z) A - \frac{1}{\mu(z)} \frac{d\mu}{dz} \frac{\partial A}{\partial z} = -\mu_0 \mu(z) I^e, \quad (1.2.10)$$

where $I^e(r, z)$ is the amplitude of the external current: $\vec{I}^e = I^e \vec{e}_\varphi$.

The operator ΔA in (1.2.10) has the form

$$\begin{aligned} \Delta A &= \Delta A_\varphi + \frac{2}{r^2} \frac{\partial A_r}{\partial \varphi} - \frac{A_\varphi}{r^2} = \left\| \Delta u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} \right\| = \\ &= \frac{1}{r} \frac{\partial A_\varphi}{\partial r} + \frac{\partial^2 A_\varphi}{\partial r^2} + \frac{\partial^2 A_\varphi}{\partial z^2} - \frac{A_\varphi}{r^2} \end{aligned} \quad (1.2.11)$$

Finally, using (1.2.10) and (1.2.11) we obtain the following equation for the amplitude of the vector potential in a conducting region with varying electric conductivity $\sigma(z)$ and relative magnetic permeability $\mu(z)$

$$\frac{\partial^2 A}{\partial r^2} + \frac{1}{r} \frac{\partial A}{\partial r} - \left(\frac{1}{r^2} + j\omega \sigma(z) \mu_0 \mu(z) \right) A - \frac{1}{\mu(z)} \frac{d\mu}{dz} \frac{\partial A}{\partial z} + \frac{\partial^2 A}{\partial z^2} = -\mu_0 \mu(z) I^e. \quad (1.2.12)$$

1.2.2. Multilayer cylindrical problems with varying electrical conductivity and magnetic permeability.

Consider a coil situated inside an infinitely long multilayer tube (see Fig. 1.2).

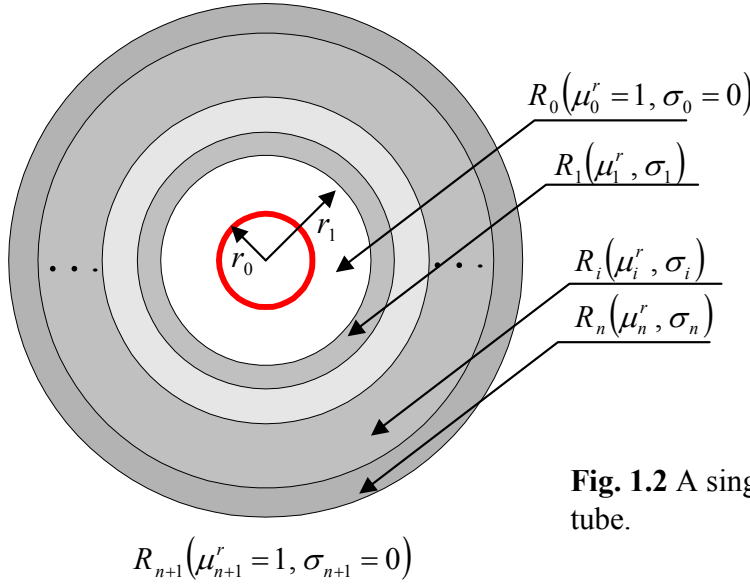


Fig. 1.2 A single-turn coil inside a multilayer tube.

The coil is located inside a multilayer tube where each coaxial layer (region R_i) is described by the inequalities: $R_i = \{r_i \leq r \leq r_{i+1}, 0 \leq \varphi \leq 2\pi, -\infty < z < +\infty\}, i=1,2,\dots,n$. Here r_i and r_{i+1} are the inner and outer radii of the cylindrical layer, respectively. Regions R_0 and R_{n+1} represent free space. In particular, $R_0 = \{0 \leq r \leq r_1, 0 \leq \varphi \leq 2\pi, -\infty < z < +\infty\}$ and

$R_{n+1} = \{r \geq r_{n+1}, 0 \leq \varphi \leq 2\pi, -\infty < z < +\infty\}$. Due to axial symmetry the vector potential has only one nonzero component in each region $R_i, i=0,1,2,\dots,n+1$ which is the function of r and z only:

$$\vec{A}_i(r, \varphi, z) = A_i(r, z)\vec{e}_\varphi, \quad i = 0,1,2,\dots,n+1. \quad (1.2.13)$$

Suppose that the electrical conductivity σ_i and magnetic permeability μ_i in each region $R_i, i=1,2,\dots,n$ are functions of the radial coordinate r :

$$\sigma_i = \sigma(r), \quad \mu_i = \mu(r). \quad (1.2.14)$$

The subscript i in \vec{A}_i is omitted in sequel.

It follows from (1.1.10) and (1.1.18) that

$$\vec{E} + j\omega\vec{A} = -\text{grad } \psi. \quad (1.2.15)$$

Using (1.1.18), (1.2.14) and (1.2.15) equations (1.1.5), (1.1.7) can be rewritten as follows

$$\vec{I} = -\sigma(r)(\text{grad } \psi + j\omega\vec{A}), \quad (1.2.16)$$

$$\vec{D} = -\varepsilon_0\varepsilon(\text{grad } \psi + j\omega\vec{A}). \quad (1.2.17)$$

Equations (1.1.2) and (1.1.18) give

$$\text{curl } \vec{H} = \vec{I} + \vec{I}^e + j\omega\vec{D}. \quad (1.2.18)$$

The left-hand side of equation (1.2.18) can be rewritten using (1.1.6), (1.1.8), (1.1.18) in the form

$$\text{curl } \vec{H} = \text{curl} \left(\frac{1}{\mu_0 \mu(r)} \text{curl } \vec{A} \right) = \frac{1}{\mu_0 \mu(r)} \text{curl curl } \vec{A} + \left[\text{grad} \frac{1}{\mu_0 \mu(r)}, \text{curl } \vec{A} \right]$$

$$\left\| \begin{aligned} \cdot \text{grad} \frac{1}{\mu_0 \mu(r)} &= -\frac{1}{\mu_0 \mu^2(r)} \frac{d\mu}{dr} \vec{e}_r - \frac{1}{r} \frac{1}{\mu_0 \mu^2(r)} \frac{d\mu}{dr} \vec{e}_\varphi \\ \cdot \text{curl } \vec{A} &= -\frac{\partial A_\varphi}{\partial z} \vec{e}_r + \frac{1}{r} \frac{\partial(rA_\varphi)}{\partial r} \vec{e}_z \\ \cdot \left[\text{grad} \frac{1}{\mu_0 \mu(r)}, \text{curl } \vec{A} \right] &= \frac{1}{r} \frac{1}{\mu_0 \mu^2(r)} \frac{d\mu}{dr} \frac{\partial(rA_\varphi)}{\partial r} \vec{e}_\varphi - \frac{1}{r} \frac{1}{\mu_0 \mu^2(r)} \frac{d\mu}{dr} \frac{\partial(A_\varphi)}{\partial z} \vec{e}_z - \\ &\quad - \frac{1}{r^2} \frac{1}{\mu_0 \mu^2(r)} \frac{d\mu}{dr} \frac{\partial(rA_\varphi)}{\partial r} \vec{e}_r \end{aligned} \right\|$$

$$\text{curl } \vec{H} = \frac{1}{\mu_0 \mu(r)} (\text{grad div } \vec{A} - \Delta \vec{A}) + \frac{1}{r} \frac{1}{\mu_0 \mu^2(r)} \frac{d\mu}{dr} \left(A_\varphi + r \frac{\partial A_\varphi}{\partial r} \right) \vec{e}_\varphi. \quad (1.2.19)$$

Using (1.2.16), (1.2.17) and (1.2.19) equation (1.2.18) can be rewritten as follows

$$\text{grad div } \vec{A} - \Delta \vec{A} + \frac{1}{\mu(r)} \frac{d\mu}{dr} \frac{\partial A_\varphi}{\partial r} \vec{e}_\varphi + \frac{1}{r} \frac{1}{\mu(r)} \frac{d\mu}{dr} A_\varphi \vec{e}_\varphi =$$

$$- \mu_0 \mu(r) \sigma(r) (\text{grad } \psi + j\omega \vec{A}) + \mu_0 \mu(r) \vec{I}^e - j\omega \mu_0 \varepsilon_0 \varepsilon \mu(r) (\text{grad } \psi + j\omega \vec{A}). \quad (1.2.20)$$

Imposing the condition

$$\text{grad div } \vec{A} + \mu_0 \mu(r) [\sigma(r) + j\omega \varepsilon_0 \varepsilon] \text{grad } \psi = 0, \quad (1.2.21)$$

using (1.2.13), neglecting the displacement current and taking the projection of (1.2.20) on the φ -axis we obtain

$$\Delta A - \frac{1}{\mu(r)} \frac{d\mu}{dr} \frac{\partial A}{\partial r} - \left(\frac{1}{r} \frac{1}{\mu(r)} \frac{d\mu}{dr} + j\omega \mu_0 \mu(r) \sigma(r) \right) A = -\mu_0 \mu(r) \vec{I}^e \quad (1.2.22)$$

where $I^e(r, z)$ is the amplitude of the external current: $\vec{I}^e = I^e \vec{e}_\varphi$.

Finally, using (1.2.22) and (1.2.11) we obtain the following equation for the amplitude of the vector potential in a conducting region with varying electric conductivity $\sigma(r)$ and relative magnetic permeability $\mu(r)$

$$\frac{\partial^2 A}{\partial r^2} + \left(\frac{1}{r} - \frac{1}{\mu(r)} \frac{d\mu}{dr} \right) \frac{\partial A}{\partial r} - \left(\frac{1}{r^2} + \frac{1}{r\mu(r)} \frac{d\mu}{dr} + j\omega \sigma(r) \mu_0 \mu(r) \right) A + \frac{\partial^2 A}{\partial z^2} = -\mu_0 \mu(r) I^e. \quad (1.2.23)$$

1.2.3. Multilayer medium with varying electrical conductivity and magnetic permeability in Cartesian coordinates

A double conductor line in the form of two infinitely long parallel wires is considered in the thesis as a simple model of a rectangular frame with current located parallel to the surface (we

assume that the ratio of the sides of the frame is 1:4 or smaller). The vector potential in this case has the form

$$\vec{A}(x, y, z) = A_x(x, y, z)\vec{i} + A_y(x, y, z)\vec{j}. \quad (1.2.24)$$

Using (1.1.6), (1.1.8), (1.1.18) we transform the left-hand side of equation (1.2.6) to the form

$$\begin{aligned} \text{curl } \vec{H} &= \text{curl} \left(\frac{1}{\mu_0 \mu(z)} \text{curl } \vec{A} \right) = \frac{1}{\mu_0 \mu(z)} \text{curl curl } \vec{A} + \left[\text{grad } \frac{1}{\mu_0 \mu(z)}, \text{curl } \vec{A} \right] = \\ & \left\| \begin{aligned} & \cdot \text{grad } \frac{1}{\mu_0 \mu(z)} = -\frac{1}{\mu_0 \mu^2(z)} \frac{d\mu}{dz} \vec{k} \\ & \cdot \text{curl } \vec{A} = -\frac{\partial A_y}{\partial z} \vec{i} + \frac{\partial A_x}{\partial z} \vec{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \vec{k} \end{aligned} \right\| \\ & = \frac{1}{\mu_0 \mu(z)} (\text{grad div } \vec{A} - \Delta \vec{A}) + \frac{1}{\mu_0 \mu^2(z)} \frac{d\mu}{dz} \left(\frac{\partial A_y}{\partial z} \vec{j} + \frac{\partial A_x}{\partial z} \vec{i} \right). \end{aligned} \quad (1.2.25)$$

Using (1.2.4), (1.2.5) and (1.2.25) equation (1.2.6) can be rewritten as follows

$$\begin{aligned} \text{grad div } \vec{A} - \Delta \vec{A} + \frac{1}{\mu(z)} \frac{d\mu}{dz} \left(\frac{\partial A_x}{\partial z} \vec{i} + \frac{\partial A_y}{\partial z} \vec{j} \right) &= -\mu_0 \mu(z) \sigma(z) (\text{grad } \psi + j\omega \vec{A}) + \\ &+ \mu_0 \mu(z) (I_x \vec{i} + I_y \vec{j}) - j\omega \mu_0 \varepsilon_0 \varepsilon \mu(z) (\text{grad } \psi + j\omega \vec{A}). \end{aligned} \quad (1.2.26)$$

Neglecting the displacement current and using (1.2.9) we obtain

$$\Delta \vec{A} - \frac{1}{\mu(z)} \frac{d\mu}{dz} \left(\frac{\partial A_x}{\partial z} \vec{i} + \frac{\partial A_y}{\partial z} \vec{j} \right) - j\omega \mu_0 \mu(z) \sigma(z) \vec{A} = -\mu_0 \mu(z) (I_x \vec{i} + I_y \vec{j}). \quad (1.2.27)$$

1.3 Literature survey

In this section we present a brief literature survey of the basic methods and models used in eddy current testing. Detailed references to each particular topic analyzed in the thesis are also given in the beginning of each chapter.

Methods of nondestructive testing are widely used in practice in order to estimate quality of products and materials. Eddy current method is one of the popular methods for nondestructive testing of electrically conducting materials. The method is based on the principle of electromagnetic induction discovered by M. Faraday in 1831. The idea of the method is as follows. Suppose that a coil (or any other source of alternating current) is located near an electrically conducting medium. The electromagnetic field generated by the currents in the coil is called the *primary field*. In accordance with the principle of electromagnetic induction eddy currents (also known as Foucault currents by the name of the French scientist Leon Foucault who discovered this phenomenon) are induced in the conducting medium. These currents represent the *secondary field*. Eddy currents induced in the conducting medium interact with the currents in the coil changing the primary field. As a result of the interaction the impedance of the coil also changes. The change in impedance of the coil can be measured experimentally. The following parameters characterizing the coil and the conducting medium affect the change in impedance: (a) geometrical size of the coil (the inner and outer radii of the coil, the height of the coil, the

number of turns and the distance from the bottom of the coil to the conducting medium being tested, known as the lift-off in the literature), (b) parameters of the medium (electrical conductivity and magnetic permeability) and (c) the frequency of the excitation current. In addition, if a conducting medium contains a flaw then the output signal of the coil would also depend on the properties of the flaw. Physical principles of eddy current testing are described, for example, in [7].

Mathematical models can also be used in order to compute the change in impedance of the coil. These models can be divided into two groups: analytical and numerical models. The term “analytical model” refers to the case where the system of Maxwell’s equations describing the interaction of the currents in the coil with a conducting medium is solved analytically (that is, the solution is expressed in closed form in terms of known special or elementary functions). However, the change in impedance of the coil is usually expressed in terms of improper integral so that at the last stage of the analysis the integral has to be evaluated numerically. Analytical models can be developed for the cases where the conducting medium is infinite in one or two spatial dimensions.

Numerical models are based on direct numerical modeling of the corresponding boundary-value problems. Finite element methods are widely used to model Maxwell’s equations in a conducting medium (see, for example, [22]).

Analytical or numerical methods are used in practice for the solution of a direct problem in eddy current testing. The term “direct problem” is referred to as the solution of Maxwell’s equations under the assumption that all parameters of the conducting medium (in particular, electrical conductivity and magnetic permeability) as well as geometrical parameters characterizing the medium (for example, the thickness of a conducting coating for the case of a two-layer medium) are known. In practice, however, these parameters may not be known and need to be determined from the solution of an inverse problem. Examples of inverse problems in eddy current testing include determination of electrical conductivity of metal plates or the thickness of metal coatings. Usually unknown parameters are obtained by minimizing the norm of the difference between measured and computed change in impedance. Thus, efficient and reliable methods for the solution of direct problems are necessary in order to solve inverse problems.

Mathematical models for eddy current testing of conducting media with constant electrical and magnetic properties are well-developed in the literature for the case where the medium is assumed to be infinite with respect to one or two special dimensions. The main results are summarized in monographs [3], [59], [60] and in several books published in Russian [17]-[19]. The solution process usually starts with the case where a single-turn circular coil is located above a conducting medium. Assuming that the Maxwell’s equations are solved for the case of a single-turn coil, the change in impedance of a coil of finite dimensions can be calculated using the superposition principle.

The system of Maxwell’s equations for the case where a single-turn coil with alternating current is located above a conducting half-space with constant electrical conductivity σ is solved in [55]. Later this solution was generalized for the case of a multilayer medium [15]. The problems in [55] and [15] are solved by the method of the Hankel integral transform. Several

papers are devoted to particular cases of the problem: a single-turn coil above a two-layer medium or a plate (the analysis of the corresponding solutions can be found, for example, in [60]). Different types of eddy current coils which can be used to locate cracks or other defects are described in [13], [14].

The solutions of direct problems for a multilayer medium are used in practice to solve a series of inverse problems. Thickness and conductivity of metallic layers are determined in [46]. Similar problem for determination of thickness and conductivity of thin nonmagnetic coatings on ferromagnetic conductive substrates is solved in [51]. Other applications include hardness testing of steel [45], [54], estimation of electrical conductivity of alloys [8], [53], thickness determination of metal plates using multi-frequency eddy current coils [69], [70].

Another geometrical configuration where analytical solutions can be constructed corresponds to the case where a coil with alternating current is located inside or outside a multilayer tube. Detailed solution of the problem is given in [16] and [19]. The problem has also important practical applications. For example, such coils are used to test quality of heat exchanger tubes in nuclear reactors [10]. Analytical solutions of the corresponding direct problems (under the additional assumption that the axis of a coil coincides with the axis of the tube) are obtained by the method of Fourier integral transform with respect to the longitudinal coordinate. The change in impedance of the coil in this case is obtained in terms of improper integrals containing Bessel functions.

There are also examples of analytical solutions for the case where a coil is located above a conducting multilayer sphere [17]. However, these problems have quite limited range of practical applications and will not be considered in sequel.

Some industrial applications require the analysis of a moving conducting medium [1], [48]. Examples include steel processing at a metallurgical plant or movement of a coin inside a coin validator.

In many cases analytical solutions described above are based on the assumption of an infinite conducting medium. From a practical point of view the medium can be considered infinite in one or two spatial dimensions if the size of the coil is much smaller than the size of an object of inspection. However, there are many applications where the size of the coil is comparable with the size of the object being tested. Solution procedure has to be modified for such cases in order to take into account finite size of a conducting medium. Recently one quasi-analytical method (TREE method) is proposed in [60]. The idea of the method is based on a physical assumption that electromagnetic field can be negligibly small far from the source of alternating current. It is assumed in a "classical" theory of eddy current testing that the vector potential and its derivatives approach zero at infinity. In the TREE method the vector potential is assumed to be exactly zero at a sufficiently large distance from the coil. The corresponding boundary value problem can be solved by the method of separation of variables. However, the solution procedure includes two steps which require numerical computations: (a) calculations of complex eigenvalues and (b) solution of a linear algebraic system. The solution is then represented in the form of a truncated eigenfunction expansion. In contrast with classical problems of mathematical physics the terms of the series cannot be represented in closed form, but can be computed provided that steps (a) and (b) described above are completed.

2. PLANAR MULTILAYER MEDIA WITH VARYING PROPERTIES

2.1 Introduction

Mathematical models of eddy current testing problems for planar multilayer medium with constant electrical and magnetic properties are well-developed in the literature [15], [17]-[19], [3], [59]-[60]. Analytical solutions of direct problems for multilayer medium are usually constructed by means of integral transforms (such as Hankel or Fourier integral transforms). The resulting system of ordinary differential equations in the transformed space is then solved analytically if the properties of each layer of the conducting medium are constant.

Alternative approach is developed in [63] where the magnetic field is assumed to be zero at a sufficiently large distance from the axis of a coil. Such a method is known as the TREE (TRuncated Eigenfunction Expansion) method in the literature [60]. In this case analytical and quasi-analytical solutions to eddy current problems can be constructed in the form of truncated series expansions.

Since the range of applications of eddy current method is quite wide it is not surprising to know that in some cases the electrical conductivity and magnetic permeability of a conducting medium can vary with respect to spatial coordinates. It is shown in [57] and [58] that special type of treatment of ferromagnetic metals (such as surface hardening) can lead to the presence of a surface layer with reduced magnetic permeability which varies exponentially with respect to the vertical coordinate. In order to optimize the performance of gas turbines it is necessary to increase firing temperatures. As a result, blades are usually protected from the exposure to high temperatures by layers containing aluminium and chrome in special proportions [41]. The depletion of aluminium in this case leads to the variation of electrical conductivity with respect to the vertical coordinate.

Hence, in order to adequately describe eddy current problems in the above mentioned cases it is necessary to develop mathematical models which take into account variability of the electrical conductivity and/or magnetic permeability with respect to one geometrical coordinate (the vertical coordinate in the case of a planar multilayer medium). Two methods are usually used in such cases. One method is based on the assumption that the variation of electrical and/or magnetic properties of the medium can be represented by piecewise constant functions. In other words, a conducting layer with varying properties is divided into a large number of relatively thin sub-layers where the electrical conductivity and/or magnetic permeability of each sub-layer are constant. This approach is used in [41], [66] for a rectangular coordinate system, in [65] for a cylindrical coordinate system and in [62] for a spherical coordinate system. Note that up to 50 layers are used in [64] and up to 20 layers in [41]. In addition, it is estimated in [41] that the use of many layers affects computational efficiency by increasing computational time.

Alternative approach is based on the assumption that simple model profiles (for example, in the form of exponential or power functions) can be used in order to represent variability of the properties of the conducting medium in one spatial direction (see, for example, [3], [23], [61], [24], [30], [39]).

In this chapter we construct analytical solutions for the case where a coil with alternating current is located above a multilayer conducting medium. It is assumed that the electrical conductivity and magnetic permeability of each layer can vary with respect to the vertical coordinate. In particular, the electrical conductivity and magnetic permeability in each layer are assumed to be exponential functions of the vertical coordinate. Some particular cases of the suggested solutions are considered in detail. Multilayer medium with many conducting layers is

rather a rare case in practice. In many applications the medium consists of one or two layers. Thus, the following particular cases of a general model described in **Section 2.3** are considered: (a) a coil above a half-space and (b) a coil above a conducting two-layer medium. Results of numerical calculations are presented.

2.2 A coil above a multilayer medium with varying properties

Consider an air core coil located above a multilayer medium (see **Fig. 1.1**). The outer and inner radii of the coil are r_2 and r_1 , respectively. The height of the coil is $h_2 - h_1$, where h_1 is the lift-off. The coil is located in free space (region R_0). The thickness of each conducting layer R_i is denoted by d_i , $i = 1, 2, \dots, n-1$. The bottom layer R_n is assumed to be infinite in the vertical direction.

Each conducting layer $R_i, i = 1, 2, \dots, n$ is characterized by the two parameters: the electrical conductivity σ_i and relative magnetic permeability μ_i^r .

We consider a system of cylindrical polar coordinates (r, φ, z) centered at O, the z -axis is directed upwards. The coil is carrying alternating current of the form

$$\vec{I}^e = I_0(r, h)e^{j\omega t}\vec{e}_\varphi, \quad (2.2.1)$$

where $r_1 \leq r \leq r_2, h_1 \leq h \leq h_2$, ω is the frequency, \vec{e}_φ is the unit vector in the φ -direction and the current density $I_0(r, h)$ is constant over the cross-section of the coil:

$$I_0(r, h) = \begin{cases} \frac{NI}{(r_2 - r_1)(h_2 - h_1)}, & r_1 \leq r \leq r_2, h_1 \leq h \leq h_2 \\ 0, & \text{otherwise} \end{cases}$$

where I is constant and N is the number of turns in the coil.

In this case it is natural to assume that the vector potential \vec{A}_i in each region $R_i, i = 0, 1, \dots, n$ has only one non-zero component of the form

$$\vec{A}_i = A_i(r, z)e^{j\omega t}\vec{e}_\varphi. \quad (2.2.2)$$

We assume that the electrical conductivity σ_i and magnetic permeability μ_i^r in each region R_i depends on the vertical coordinate. More precisely, σ_i and μ_i^r are modeled by the following relations:

$$\mu_i^r(z) = \mu_i^m \exp(\beta_i z), \quad \sigma_i(z) = \sigma_i^m \exp(\alpha_i z), \quad (2.2.3)$$

$$i = 1, 2, \dots, n,$$

where $\alpha_i, \beta_i, \mu_i^m, \sigma_i^m$ are constants.

2.3 A single-turn coil above a multilayer medium with varying properties

The first step in solving the problem is to find the solution for the case where a single-turn coil of radius r_c is located at distance h above a conducting multilayer medium [24]

(see **Fig. 2.1**).

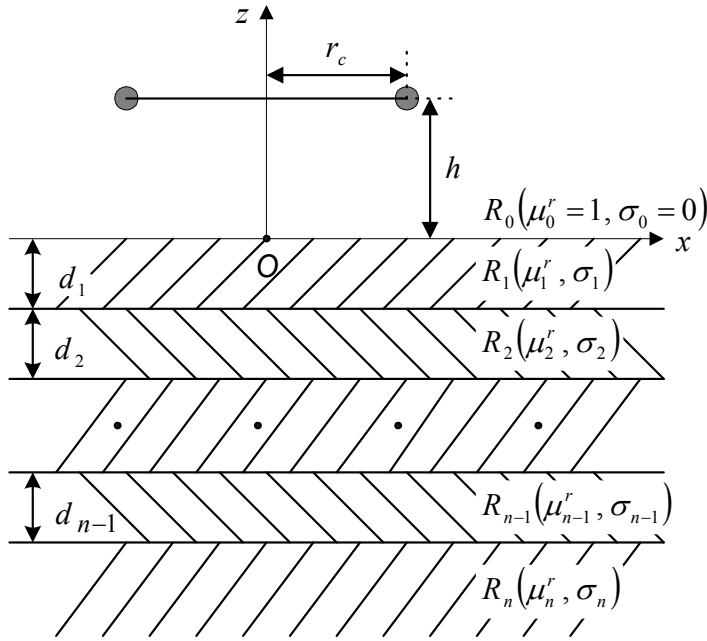


Fig. 2.1 A single-turn coil above multilayer medium.

Using (1.2.12) with $\mu_0^r = 1$, $\sigma_0 = 0$ and $I^e = I\delta(r - r_c)\delta(z - h)$ we obtain the equation for the amplitude of the vector potential in region R_0 :

$$\frac{\partial^2 A_0}{\partial r^2} + \frac{1}{r} \frac{\partial A_0}{\partial r} - \frac{A_0}{r^2} + \frac{\partial^2 A_0}{\partial z^2} = -\mu_0 I \delta(r - r_c) \delta(z - h), \quad (2.3.1)$$

where $\delta(x)$ is the Dirac delta-function.

Using (1.2.12) with $I = 0$, $\mu(z) = \mu_i^r(z)$ and $\sigma(z) = \sigma_i(z)$ we obtain the following system of equations for the amplitudes $A_i(r, z)$ of the vector potential in each region $R_i, i = 1, 2, \dots, n$:

$$\frac{\partial^2 A_i}{\partial r^2} + \frac{1}{r} \frac{\partial A_i}{\partial r} - \frac{A_i}{r^2} + \frac{\partial^2 A_i}{\partial z^2} - \frac{1}{\mu_i^r(z)} \frac{d\mu_i^r(z)}{dz} \frac{\partial A_i}{\partial z} - j\omega\sigma_i(z)\mu_0\mu_i^r(z)A_i = 0, \quad (2.3.2)$$

$i = 1, 2, \dots, n$.

Using (2.2.3) we rewrite equations (2.3.2) in the form

$$\frac{\partial^2 A_i}{\partial r^2} + \frac{1}{r} \frac{\partial A_i}{\partial r} - \frac{A_i}{r^2} + \frac{\partial^2 A_i}{\partial z^2} = \beta_i \frac{\partial A_i}{\partial z} + j\omega\sigma_i^m \mu_0 \mu_i^m e^{(\alpha_i + \beta_i)z} A_i, \quad (2.3.3)$$

$i = 1, 2, \dots, n$,

The boundary conditions are

$$A_0|_{z=0} = A_1|_{z=0}, \quad \frac{\partial A_0}{\partial z}|_{z=0} = \frac{1}{\mu_1^r(0)} \frac{\partial A_1}{\partial z}|_{z=0}, \quad (2.3.4)$$

$$A_i|_{z=-\hat{d}_i} = A_{i+1}|_{z=-\hat{d}_i},$$

$$\frac{1}{\mu_i^r(-\hat{d}_i)} \frac{\partial A_i}{\partial z}|_{z=-\hat{d}_i} = \frac{1}{\mu_{i+1}^r(-\hat{d}_i)} \frac{\partial A_{i+1}}{\partial z}|_{z=-\hat{d}_i}, \quad i = 1, 2, \dots, n-1, \quad (2.3.5)$$

where $\hat{d}_i = \sum_{k=1}^i d_k$.

The following conditions hold at infinity:

$$A_i, \frac{\partial A_i}{\partial r} \rightarrow 0 \text{ as } r \rightarrow \infty, \quad i = 0, 1, \dots, n, \quad (2.3.6)$$

$$A_0 \rightarrow 0 \text{ as } z \rightarrow +\infty, \quad A_n \rightarrow 0 \text{ as } z \rightarrow -\infty. \quad (2.3.7)$$

In order to solve (2.3.1) – (2.3.8) we use the Hankel integral transform of the form

$$\tilde{A}_i(\lambda, z) = \int_0^\infty A_i(r, z) r J_1(\lambda r) dr, \quad i = 0, 1, \dots, n. \quad (2.3.8)$$

Using (2.3.6), (2.3.8) we rewrite the left-hand side of equation (2.3.3) in the form (assuming uniform convergence of the corresponding integrals)

$$\int_0^\infty \left(\frac{\partial^2 A_i}{\partial r^2} + \frac{1}{r} \frac{\partial A_i}{\partial r} - \frac{A_i}{r^2} + \frac{\partial^2 A_i}{\partial z^2} \right) r J_1(\lambda r) dr = \int_0^\infty \left(\frac{\partial^2 A_i}{\partial r^2} + \frac{1}{r} \frac{\partial A_i}{\partial r} - \frac{A_i}{r^2} \right) r J_1(\lambda r) dr + \int_0^\infty \frac{\partial^2 A_i}{\partial z^2} r J_1(\lambda r) dr$$

$$\left\| \begin{aligned} & \cdot \int_0^\infty \frac{\partial^2 A_i}{\partial z^2} r J_1(\lambda r) dr = \frac{d^2 \tilde{A}_i}{dz^2} \\ & \cdot \int_0^\infty \left(\frac{\partial^2 A_i}{\partial r^2} + \frac{1}{r} \frac{\partial A_i}{\partial r} - \frac{A_i}{r^2} \right) r J_1(\lambda r) dr = \left| \begin{aligned} & u = r J_1(\lambda r) \rightarrow du = (J_1(\lambda r) + \lambda r J_1'(\lambda r)) dr \\ & dv = \frac{\partial^2 A_i}{\partial r^2} dr \rightarrow v = \frac{\partial A_i}{\partial r} \end{aligned} \right| = \\ & = r J_1(\lambda r) \frac{\partial A_i}{\partial r} \Big|_0^\infty - \int_0^\infty J_1(\lambda r) \frac{\partial A_i}{\partial r} dr - \lambda \int_0^\infty r J_1'(\lambda r) \frac{\partial A_i}{\partial r} dr + \int_0^\infty J_1(\lambda r) \frac{\partial A_i}{\partial r} dr - \int_0^\infty \frac{1}{r} J_1(\lambda r) A_i dr = \\ & = \left| \begin{aligned} & u = r J_1'(\lambda r) \rightarrow du = (J_1'(\lambda r) + \lambda r J_1''(\lambda r)) dr \\ & dv = \frac{\partial A_i}{\partial r} dr \rightarrow v = A_i \end{aligned} \right| = \\ & = -\lambda r J_1'(\lambda r) A_i \Big|_0^\infty + \lambda^2 \int_0^\infty \left(J_1''(\lambda r) + \frac{1}{\lambda r} J_1'(\lambda r) - \frac{1}{\lambda^2 r^2} J_1(\lambda r) \right) A_i r dr = \\ & = \left| \begin{aligned} & \cdot x = \lambda r \rightarrow J_1''(x) + \frac{1}{x} J_1'(x) - \frac{1}{x^2} J_1(x) \\ & y''(x) + \frac{1}{x} y'(x) + \left(1 - \frac{\nu^2}{x^2} \right) y(x) = 0 \rightarrow y = J_\nu(x) \\ & \cdot \nu = 1 \rightarrow y = J_1(x) \\ & J_1''(x) + \frac{1}{x} J_1'(x) - \frac{1}{x^2} J_1(x) = -J_1(x) \end{aligned} \right| = -\lambda^2 \tilde{A}_i \end{aligned} \right\|$$

$$\int_0^\infty \left(\frac{\partial^2 A_i}{\partial r^2} + \frac{1}{r} \frac{\partial A_i}{\partial r} - \frac{A_i}{r^2} + \frac{\partial^2 A_i}{\partial z^2} \right) r J_1(\lambda r) dr = \frac{d^2 \tilde{A}_i}{dz^2} - \lambda^2 \tilde{A}_i, \quad i = 0, 1, \dots, n, \quad (2.3.9)$$

The right-hand side of equation (2.3.1) can be rewritten using (2.3.8) in the form

$$-\mu_0 I \delta(z-h) \int_0^{\infty} \delta(r-r_c) r J_1(\lambda r) dr = -\mu_0 I r_c J_1(\lambda r_c) \delta(z-h) \quad (2.3.10)$$

Applying the Hankel transform (2.3.8) to the right-hand side of equation (2.3.3) we obtain

$$\int_0^{\infty} \left(\beta_i \frac{\partial A_i}{\partial z} + j\omega \sigma_i^m \mu_0 \mu_i^m e^{(\alpha_i + \beta_i)z} A_i \right) r J_1(\lambda r) dr = \beta_i \frac{d\tilde{A}_i}{dz} + j\omega \sigma_i^m \mu_0 \mu_i^m e^{(\alpha_i + \beta_i)z} \tilde{A}_i \quad (2.3.11)$$

$$i = 1, 2, \dots, n,$$

Using (2.3.9), (2.3.10) and (2.3.11) we obtain

$$\frac{d^2 \tilde{A}_0}{dz^2} - \lambda^2 \tilde{A}_0 = -\mu_0 I r_c J_1(\lambda r_c) \delta(z-h), \quad (2.3.12)$$

$$\frac{d^2 \tilde{A}_i}{dz^2} - \beta_i \frac{d\tilde{A}_i}{dz} - \left(\lambda^2 + j\omega \sigma_i^m \mu_0 \mu_i^m e^{(\alpha_i + \beta_i)z} \right) \tilde{A}_i = 0, \quad i = 1, 2, \dots, n. \quad (2.3.13)$$

Applying the Hankel integral transform (2.3.8) to the boundary conditions we obtain

$$\tilde{A}_0|_{z=0} = \tilde{A}_1|_{z=0}, \quad \frac{d\tilde{A}_0}{dz}|_{z=0} = \frac{1}{\mu_1^r(0)} \frac{d\tilde{A}_1}{dz}|_{z=0}, \quad (2.3.14)$$

$$\tilde{A}_i|_{z=-\hat{d}_i} = \tilde{A}_{i+1}|_{z=-\hat{d}_i}, \quad (2.3.15)$$

$$\frac{1}{\mu_i^r(-\hat{d}_i)} \frac{d\tilde{A}_i}{dz}|_{z=-\hat{d}_i} = \frac{1}{\mu_{i+1}^r(-\hat{d}_i)} \frac{d\tilde{A}_{i+1}}{dz}|_{z=-\hat{d}_i}, \quad i = 1, 2, \dots, n-1,$$

$$\tilde{A}_0 \rightarrow 0 \text{ as } z \rightarrow +\infty, \quad \tilde{A}_n \rightarrow 0 \text{ as } z \rightarrow -\infty. \quad (2.3.16)$$

In order to solve equation (2.3.12) we consider the following two sub-regions of R_0 : $0 < z < h$ and $z > h$. The solutions in these regions are denoted by \tilde{A}_{00} and \tilde{A}_{01} , respectively. Hence,

$$\frac{d^2 \tilde{A}_{00}}{dz^2} - \lambda^2 \tilde{A}_{00} = 0, \quad 0 < z < h, \quad (2.3.17)$$

$$\frac{d^2 \tilde{A}_{01}}{dz^2} - \lambda^2 \tilde{A}_{01} = 0, \quad z > h. \quad (2.3.18)$$

The general solution to (2.3.17) can be written in the form

$$\tilde{A}_{00} = C_1 e^{\lambda z} + C_2 e^{-\lambda z}. \quad (2.3.19)$$

The bounded solution to (2.3.18) is

$$\tilde{A}_{01} = C_3 e^{-\lambda z}. \quad (2.3.20)$$

The functions $\tilde{A}_{00}(\lambda, z)$ and $\tilde{A}_{01}(\lambda, z)$ satisfy the following condition at $z = h$:

$$\tilde{A}_{00}|_{z=h} = \tilde{A}_{01}|_{z=h}. \quad (2.3.21)$$

The condition (2.3.21) reflects the fact that the function $\tilde{A}_0(\lambda, z)$ is continuous at $z = h$.

The second condition is obtained by integrating (2.3.12) with respect to z from $z = h - \varepsilon$ to $z = h + \varepsilon$ and considering the limit in the resulting expression as $\varepsilon \rightarrow +0$:

$$\left\| \begin{aligned} & \cdot \int_{h-\varepsilon}^{h+\varepsilon} \frac{d^2 \tilde{A}_0}{dz^2} dz - \lambda^2 \int_{h-\varepsilon}^{h+\varepsilon} \tilde{A}_0 dz = -\mu_0 I r_c J_1(\lambda r_c) \int_{h-\varepsilon}^{h+\varepsilon} \delta(z-h) dz \\ & \frac{d\tilde{A}_0}{dz} \Big|_{h-\varepsilon}^{h+\varepsilon} - \lambda^2 \int_{h-\varepsilon}^{h+\varepsilon} \tilde{A}_0 dz = -\mu_0 I r_c J_1(\lambda r_c) \\ & \cdot -\lambda^2 \lim_{\varepsilon \rightarrow 0} \int_{h-\varepsilon}^{h+\varepsilon} \tilde{A}_0 dz = -\lambda^2 \lim_{\varepsilon \rightarrow 0} 2\varepsilon \tilde{A}_0(\lambda, z^*) = 0 \\ & \cdot \lim_{\varepsilon \rightarrow 0} \frac{d\tilde{A}_0}{dz} \Big|_{h-\varepsilon}^{h+\varepsilon} = \frac{d\tilde{A}_{01}}{dz} \Big|_{z=h} - \frac{d\tilde{A}_{00}}{dz} \Big|_{z=h} \end{aligned} \right\|$$

$$\frac{d\tilde{A}_{01}}{dz} \Big|_{z=h} - \frac{d\tilde{A}_{00}}{dz} \Big|_{z=h} = -\mu_0 I r_c J_1(\lambda r_c). \quad (2.3.22)$$

Using (2.3.19) – (2.3.22) we obtain

$$\begin{aligned} & \begin{cases} C_1 e^{\lambda h} + C_2 e^{-\lambda h} = C_3 e^{-\lambda h} \\ -\lambda C_3 e^{-\lambda h} - \lambda C_1 e^{\lambda h} + \lambda C_2 e^{-\lambda h} = -\mu_0 I r_c J_1(\lambda r_c) \end{cases} \Rightarrow \\ & \Rightarrow \begin{cases} C_1 e^{2\lambda h} + C_2 - C_3 = 0 \\ -C_1 e^{2\lambda h} + C_2 - C_3 = -\frac{\mu_0}{\lambda} I r_c J_1(\lambda r_c) e^{\lambda h} \end{cases} \Rightarrow \end{aligned}$$

$$\begin{cases} C_1 = \frac{\mu_0}{2\lambda} I r_c J_1(\lambda r_c) e^{-\lambda h} \\ C_3 = C_2 + \frac{\mu_0}{2\lambda} I r_c J_1(\lambda r_c) e^{\lambda h} \end{cases} \quad (2.3.23)$$

Using (2.3.23) we rewrite (2.3.19) and (2.3.20) in the following form

$$\begin{cases} \tilde{A}_{00} = C_2 e^{-\lambda z} + \frac{\mu_0}{2\lambda} I r_c J_1(\lambda r_c) e^{-\lambda(h-z)} \\ \tilde{A}_{01} = C_2 e^{-\lambda z} + \frac{\mu_0}{2\lambda} I r_c J_1(\lambda r_c) e^{-\lambda(z-h)} \end{cases} \Rightarrow \tilde{A}_0(\lambda, z) = C_2 e^{-\lambda z} + \frac{\mu_0}{2\lambda} I r_c J_1(\lambda r_c) e^{-\lambda|h-z|} \quad (2.3.24)$$

In order to understand the meaning of each of the two terms on the right-hand side of (2.3.24) we consider the case where a coil of radius r_c centered at the point $(0, h)$ is located in an unbounded free space. The corresponding component of the vector potential will be denoted by $A_0^{free}(r, z)$.

The equation for A_0^{free} is

$$\frac{\partial^2 A_0^{free}}{\partial r^2} + \frac{1}{r} \frac{\partial A_0^{free}}{\partial r} - \frac{A_0^{free}}{r^2} + \frac{\partial^2 A_0^{free}}{\partial z^2} = -\mu_0 I \delta(r - r_c) \delta(z - h), \quad (2.3.25)$$

where $\delta(x)$ is the Dirac delta-function.

In addition, A_0^{free} is bounded at $r = 0$ and A_0^{free} is bounded as $r \rightarrow \infty$.

Applying the Hankel integral transform (2.3.8), using (2.3.12) and (2.3.16) we get

$$\frac{d^2 \tilde{A}_0^{free}}{dz^2} - \lambda^2 \tilde{A}_0^{free} = -\mu_0 I_r J_1(\lambda r_c) \delta(z-h). \quad (2.3.26)$$

The bounded solutions in regions R_{00} and R_{01} are :

$$R_{00} : \frac{d^2 \tilde{A}_{00}^{free}}{dz^2} - \lambda^2 \tilde{A}_{00}^{free} = 0, \quad z < h, \\ \rightarrow \tilde{A}_{00}^{free}(r, \lambda) = C_1 e^{\lambda z}. \quad (2.3.27)$$

$$R_{01} : \frac{d^2 \tilde{A}_{01}^{free}}{dz^2} - \lambda^2 \tilde{A}_{01}^{free} = 0, \quad z > h. \\ \rightarrow \tilde{A}_{01}^{free}(r, \lambda) = C_3 e^{-\lambda z}. \quad (2.3.28)$$

Using (2.3.21) and (2.3.22) solutions (2.3.27) and (2.3.28) can be rewritten as follows

$$\tilde{A}_{00}^{free}(r, \lambda) = \frac{\mu_0}{2\lambda} I_r e^{-\lambda(h-z)} \quad (2.3.29)$$

$$\tilde{A}_{01}^{free}(r, \lambda) = \frac{\mu_0}{2\lambda} I_r e^{-\lambda(z-h)} \quad (2.3.30)$$

Comparing (2.3.24) and (2.3.29), (2.3.30) we see that the second term on the right-hand side of (2.3.24) represents the vector potential in an unbounded free space. Thus, the first term of the right-hand side of (2.3.24) represents the induced vector potential due to the presence of the conducting medium.

The solution to (2.3.13) can be expressed in terms of Bessel functions (see [50], formula 2.1.3.10, page 247):

$$\left\| \begin{aligned} y''_{xx} + ay'_x + (be^{\lambda x} + c)y = 0 &\Rightarrow y = e^{\frac{ax}{2}} \left[C_1 J_\nu \left(\frac{2\sqrt{b}}{\lambda} e^{\frac{\lambda x}{2}} \right) + C_2 Y_\nu \left(\frac{2\sqrt{b}}{\lambda} e^{\frac{\lambda x}{2}} \right) \right] \\ \nu = \frac{\sqrt{a^2 - 4c}}{\lambda} \end{aligned} \right\| \\ \tilde{A}_i(\lambda, z) = e^{\frac{\beta_i z}{2}} \left[C_{4_i} J_{\nu_i} \left(\frac{2\sqrt{b_i}}{\alpha_i + \beta_i} e^{\frac{(\alpha_i + \beta_i)z}{2}} \right) + C_{5_i} Y_{\nu_i} \left(\frac{2\sqrt{b_i}}{\alpha_i + \beta_i} e^{\frac{(\alpha_i + \beta_i)z}{2}} \right) \right], \quad i = 1, 2, \dots, n-1, \quad (2.3.31)$$

where $\nu_i = \frac{\sqrt{\beta_i^2 + 4\lambda^2}}{\alpha_i + \beta_i}$, $b_i = -j\omega\mu_0\mu_i^m\sigma_i^m$,

and J_{ν_i}, Y_{ν_i} are the Bessel functions of the first and second kind, respectively.

The bounded solution to (2.3.13) in region R_n is

$$\tilde{A}_n(\lambda, z) = C_{4_n} e^{\frac{\beta_n z}{2}} J_{\nu_n} \left(\frac{2\sqrt{b_n}}{\alpha_n + \beta_n} e^{\frac{(\alpha_n + \beta_n)z}{2}} \right). \quad (2.3.32)$$

Using (2.3.24), (2.3.31), (2.3.32) and the boundary conditions (2.3.14), (2.3.15) we obtain

$$C_2 + \frac{\mu_0}{2\lambda} Ir_c J_1(\lambda r_c) e^{-\lambda h} = C_{4_1} J_{\nu_1}(z_{0_1}) + C_{5_1} Y_{\nu_1}(z_{0_1}), \quad (2.3.33)$$

$$\begin{aligned} -\lambda C_2 + \frac{\mu_0}{2\lambda} Ir_c J_1(\lambda r_c) e^{-\lambda h} &= \\ &= \frac{C_{4_1}}{\mu_1^m} \left(\frac{\beta_1}{2} J_{\nu_1}(z_{0_1}) + \sqrt{b_1} J'_{\nu_1}(z_{0_1}) \right) + \frac{C_{5_1}}{\mu_1^m} \left(\frac{\beta_1}{2} Y_{\nu_1}(z_{0_1}) + \sqrt{b_1} Y'_{\nu_1}(z_{0_1}) \right), \end{aligned} \quad (2.3.34)$$

$$\begin{aligned} e^{-\frac{\beta_i \hat{d}_i}{2}} \left(C_{4_i} J_{\nu_i}(z_{1_i}(\hat{d}_i)) + C_{5_i} Y_{\nu_i}(z_{1_i}(\hat{d}_i)) \right) &= \\ &= e^{-\frac{\beta_{i+1} \hat{d}_{i+1}}{2}} \left(C_{4_{i+1}} J_{\nu_{i+1}}(z_{1_i}(\hat{d}_i)) + C_{5_{i+1}} Y_{\nu_{i+1}}(z_{1_i}(\hat{d}_i)) \right), \end{aligned} \quad (2.3.35)$$

$$\begin{aligned} \frac{e^{-\frac{\beta_i \hat{d}_i}{2}}}{\mu_i^m} \left(C_{4_i} \left(\frac{\beta_i}{2} J_{\nu_i}(z_{1_i}(\hat{d}_i)) + \sqrt{b_i} e^{-\frac{(\alpha_i + \beta_i) \hat{d}_i}{2}} J'_{\nu_i}(z_{1_i}(\hat{d}_i)) \right) + \right. \\ \left. + C_{5_i} \left(\frac{\beta_i}{2} Y_{\nu_i}(z_{1_i}(\hat{d}_i)) + \sqrt{b_i} e^{-\frac{(\alpha_i + \beta_i) \hat{d}_i}{2}} Y'_{\nu_i}(z_{1_i}(\hat{d}_i)) \right) \right) &= \end{aligned} \quad (2.3.36)$$

$$= \frac{e^{-\frac{\beta_{i+1} \hat{d}_{i+1}}{2}}}{\mu_{i+1}^m} \left(C_{4_{i+1}} \left(\frac{\beta_{i+1}}{2} J_{\nu_{i+1}}(z_{1_{i+1}}(\hat{d}_i)) + \sqrt{b_{i+1}} e^{-\frac{(\alpha_{i+1} + \beta_{i+1}) \hat{d}_{i+1}}{2}} J'_{\nu_{i+1}}(z_{1_{i+1}}(\hat{d}_i)) \right) + \right. \\ \left. + C_{5_{i+1}} \left(\frac{\beta_{i+1}}{2} Y_{\nu_{i+1}}(z_{1_{i+1}}(\hat{d}_i)) + \sqrt{b_{i+1}} e^{-\frac{(\alpha_{i+1} + \beta_{i+1}) \hat{d}_{i+1}}{2}} Y'_{\nu_{i+1}}(z_{1_{i+1}}(\hat{d}_i)) \right) \right)$$

$$\begin{aligned} e^{-\frac{\beta_{n-1} \hat{d}_{n-1}}{2}} \left(C_{4_{n-1}} J_{\nu_{n-1}}(z_{1_{n-1}}(-\hat{d}_{n-1})) + C_{5_{n-1}} Y_{\nu_{n-1}}(z_{1_{n-1}}(-\hat{d}_{n-1})) \right) &= \\ &= e^{-\frac{\beta_n \hat{d}_n}{2}} C_{4_n} J_{\nu_n}(z_{1_n}(-\hat{d}_{n-1})) \end{aligned} \quad (2.3.37)$$

$$\begin{aligned} \frac{e^{-\frac{\beta_{n-1} \hat{d}_{n-1}}{2}}}{\mu_{n-1}^m} \left(C_{4_{n-1}} \left(\frac{\beta_{n-1}}{2} J_{\nu_{n-1}}(z_{1_{n-1}}(\hat{d}_{n-1})) + \sqrt{b_{n-1}} e^{-\frac{(\alpha_{n-1} + \beta_{n-1}) \hat{d}_{n-1}}{2}} J'_{\nu_{n-1}}(z_{1_{n-1}}(\hat{d}_{n-1})) \right) + \right. \\ \left. + C_{5_{n-1}} \left(\frac{\beta_{n-1}}{2} Y_{\nu_{n-1}}(z_{1_{n-1}}(\hat{d}_{n-1})) + \sqrt{b_{n-1}} e^{-\frac{(\alpha_{n-1} + \beta_{n-1}) \hat{d}_{n-1}}{2}} Y'_{\nu_{n-1}}(z_{1_{n-1}}(\hat{d}_{n-1})) \right) \right) &= \end{aligned} \quad (2.3.38)$$

$$= \frac{e^{-\frac{\beta_n \hat{d}_n}{2}}}{\mu_n^m} C_{4_n} \left(\frac{\beta_n}{2} J_{\nu_n}(z_{1_n}(\hat{d}_{n-1})) + \sqrt{b_n} e^{-\frac{(\alpha_n + \beta_n) \hat{d}_{n-1}}{2}} J'_{\nu_n}(z_{1_n}(\hat{d}_{n-1})) \right),$$

where $z_{0_i} = \frac{2\sqrt{b_i}}{\alpha_i + \beta_i}$ $z_{1_i}(\hat{d}_i) = z_{0_i} e^{-\frac{(\alpha_i + \beta_i) \hat{d}_i}{2}}$ $\hat{d}_i = \sum_{k=1}^i d_{.k}$.

Solving the system (2.3.33) – (2.3.38), substituting the values of the constants into (2.3.24),

(2.3.31) and (2.3.32) we obtain the solution in each region R_i in the transformed space. The solution in each region $R_i (i = 0, 1, \dots, n)$ is then found by means of the inverse Hankel transform of the form

$$A_i(r, z) = \int_0^{\infty} \tilde{A}_i(\lambda, z) \lambda J_1(\lambda r) d\lambda, \quad i = 0, 1, \dots, n. \quad (2.3.39)$$

The method described above can be used for any finite number of conducting layers. In the next section we consider one particular case – a single-turn coil above a conducting half-space with varying electrical and magnetic properties.

2.4 A single-turn coil above a conducting half-space with varying properties

Consider a single-turn coil of radius r_c located at a distance h above a conducting half-space (see **Fig. 2.1** in the limit as $d_1 \rightarrow \infty$). The geometry of the problem is shown in **Fig. 2.2**, where $\mu_1^r(z) = \mu_m e^{\beta z}$, $\sigma_1(z) = \sigma_m e^{\alpha z}$ (see [39]).

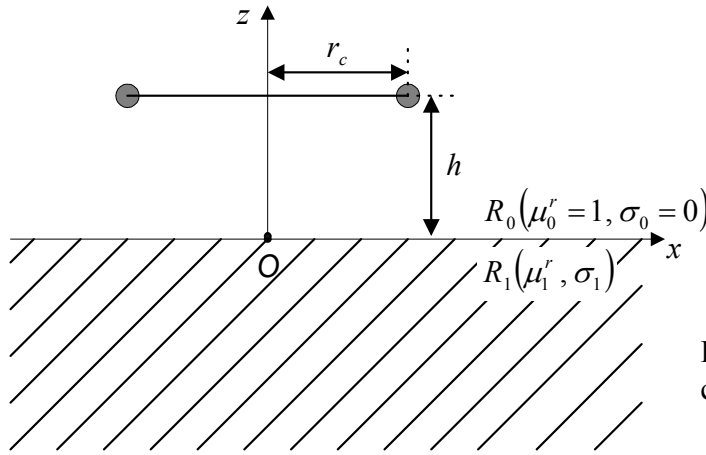


Fig. 2.2 A single-turn coil above a conducting half-space.

Using the results from **Section 2.3** we obtain the solutions in regions R_0 and R_1 (see (2.3.24) and (2.3.32)):

$$\begin{aligned} R_0: \quad \frac{d^2 \tilde{A}_0}{dz^2} - \lambda^2 \tilde{A}_0 &= -\mu_0 I r_c J_1(\lambda r_c) \delta(z-h) \rightarrow \\ &\rightarrow \tilde{A}_0(\lambda, z) = C_2 e^{-\lambda z} + \frac{\mu_0}{2\lambda} I r_c J_1(\lambda r_c) e^{-\lambda|h-z|}, \end{aligned} \quad (2.4.1)$$

$$\begin{aligned} R_1: \quad \frac{d^2 \tilde{A}_1}{dz^2} - \beta \frac{d\tilde{A}_1}{dz} - (\lambda^2 + j\omega\sigma_m\mu_0\mu_m e^{(\alpha+\beta)z}) \tilde{A}_1 &= 0 \rightarrow \\ &\rightarrow \tilde{A}_1(\lambda, z) = C_4 e^{\frac{\beta z}{2}} J_\nu \left(\frac{2\sqrt{\beta}}{\alpha+\beta} e^{\frac{(\alpha+\beta)z}{2}} \right). \end{aligned} \quad (2.4.2)$$

The boundary conditions are :

$$\tilde{A}_0|_{z=0} = \tilde{A}_1|_{z=0}, \quad \frac{d\tilde{A}_0}{dz}|_{z=0} = \frac{1}{\mu_m} \frac{d\tilde{A}_1}{dz}|_{z=0}, \quad (2.4.3)$$

$$\tilde{A}_0 \rightarrow 0 \text{ as } z \rightarrow +\infty, \quad \tilde{A}_1 \rightarrow 0 \text{ as } z \rightarrow -\infty. \quad (2.4.4)$$

Using (2.4.1) – (2.4.4) we obtain

$$\begin{cases} C_2 + \frac{\mu_0}{2\lambda} Ir_c J_1(\lambda r_c) e^{-\lambda h} = C_4 J_\nu(z_0), \\ -\lambda C_2 + \frac{\mu_0}{2} Ir_c J_1(\lambda r_c) e^{-\lambda h} = \frac{C_4}{\mu_m} \left(\frac{\beta}{2} J_\nu(z_0) + \sqrt{b} J'_\nu(z_0) \right), \end{cases} \quad (2.4.5)$$

$$\text{where } z_0 = \frac{2\sqrt{b}}{\alpha + \beta}, \quad b = -j\omega\mu_0\mu_m\sigma_m, \quad \nu = \frac{\sqrt{\beta^2 + 4\lambda^2}}{\alpha + \beta}.$$

It follows from (2.4.5) that

$$\begin{aligned} & \left\| \begin{aligned} & \cdot C_2 = C_4 J_\nu(z_0) - \frac{\mu_0}{2\lambda} Ir_c J_1(\lambda r_c) e^{-\lambda h} \\ & \cdot \left(-\lambda C_4 J_\nu(z_0) + \mu_0 Ir_c J_1(\lambda r_c) e^{-\lambda h} \right) \mu_m = C_4 \left(\frac{\beta}{2} J_\nu(z_0) + \sqrt{b} J'_\nu(z_0) \right) \end{aligned} \right\| \\ C_4 &= \frac{\mu_0 \mu_m Ir_c J_1(\lambda r_c) e^{-\lambda h}}{\left(\lambda \mu_m + \frac{\beta}{2} \right) J_\nu(z_0) + \sqrt{b} J'_\nu(z_0)}. \end{aligned} \quad (2.4.6)$$

In particular, the value of C_2 is

$$C_2 = \frac{\mu_0 Ir_c J_1(\lambda r_c) e^{-\lambda h} \left(\left(\lambda \mu_m - \frac{\beta}{2} \right) J_\nu(z_0) - \sqrt{b} J'_\nu(z_0) \right)}{2\lambda \left(\left(\lambda \mu_m + \frac{\beta}{2} \right) J_\nu(z_0) + \sqrt{b} J'_\nu(z_0) \right)}. \quad (2.4.7)$$

It has been shown in **Section 2.3** that the second term in (2.4.1) represents the vector potential for the case where a single-turn coil is located in an unbounded free space. Thus, the first term in (2.4.1) is the reaction of the conducting half-space on the current in the coil. This term is usually referred to as “the induced vector potential”.

The induced vector potential in region R_0 is

$$\tilde{A}_0^{ind}(\lambda, z) = C_2 e^{-\lambda z}, \quad (2.4.8)$$

where C_2 is given by (2.4.7).

Applying the inverse Hankel transform of the form (2.3.39) to (2.4.8) we obtain

$$A_0^{ind}(r, z) = \frac{\mu_0 Ir_c}{2} \int_0^\infty F(\lambda) J_1(\lambda r_c) J_1(\lambda r) e^{-\lambda(z+h)} d\lambda, \quad (2.4.9)$$

where

$$F(\lambda) = \frac{\left(\lambda \mu_m - \frac{\beta}{2} \right) J_\nu(z_0) - \sqrt{b} J'_\nu(z_0)}{\left(\lambda \mu_m + \frac{\beta}{2} \right) J_\nu(z_0) + \sqrt{b} J'_\nu(z_0)}.$$

The induced change in impedance of the coil is given by the formula (see [60])

$$Z^{ind} = \frac{j\omega}{I} \oint_L A_0^{ind}(r, z) dl, \quad (2.4.10)$$

where L is the contour of the coil.

Substituting (2.4.9) into (2.4.10) we obtain

$$Z^{ind}(r, z) = \omega \pi \mu_0 r_c^2 j \int_0^{\infty} F(\lambda) J_1^2(\lambda r_c) e^{-2\lambda h} d\lambda. \quad (2.4.11)$$

Using the following dimensionless parameters (2.4.11) can be rewritten as follows

$$\left\| \begin{array}{l} \cdot \lambda r_c = u \rightarrow \lambda = \frac{u}{r_c}, \\ \hat{\alpha} = \alpha r_c, \hat{\beta} = \beta r_c, \end{array} \right\} \Rightarrow \hat{v} = \frac{\sqrt{\hat{\beta}^2 + 4u^2}}{\hat{\alpha} + \hat{\beta}},$$

$$\left\| \begin{array}{l} \cdot \hat{b} = \omega \mu_0 \mu_m \sigma_m r_c^2, \hat{z}_0 = \frac{2\sqrt{-j\hat{b}}}{\hat{\alpha} + \hat{\beta}}, \hat{h} = \frac{h}{r_c}. \end{array} \right\|$$

$$Z^{ind}(r, z) = \omega \pi \mu_0 r_c Z, \quad (2.4.12)$$

where

$$Z = j \int_0^{\infty} \frac{\left(u \mu_m - \frac{\hat{\beta}}{2} \right) J_{\hat{v}}(\hat{z}_0) - \sqrt{-j\hat{b}} J'_{\hat{v}}(\hat{z}_0)}{\left(u \mu_m + \frac{\hat{\beta}}{2} \right) J_{\hat{v}}(\hat{z}_0) + \sqrt{-j\hat{b}} J'_{\hat{v}}(\hat{z}_0)} J_1^2(u) e^{-2u\hat{h}} du. \quad (2.4.13)$$

Fig. 2.3 plots the change in impedance Z for three different values of $\hat{\beta} = 1, 2, 3$. The other parameters of the problem are fixed at $\hat{\alpha} = 0, \hat{h} = 0.05$ and $\mu_m = 5$. The calculated points in **Fig. 2.3** correspond to different values of $\hat{b} = 1, 2, \dots, 40$ (from left to right). Computations are done with “Mathematica” (see **Appendix Fig.Ap.1**).

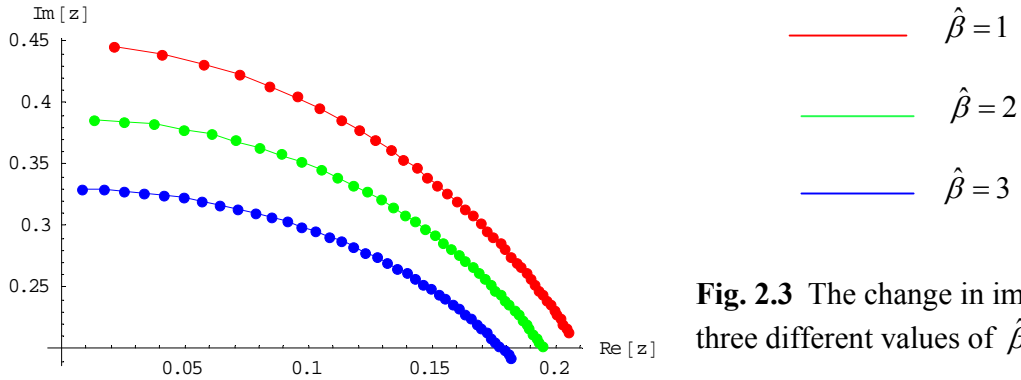


Fig. 2.3 The change in impedance Z for three different values of $\hat{\beta}$.

It is seen from the graph that the modulus of Z increases as the parameter $\hat{\beta}$ increases.

All calculations in the thesis are done with software package “Mathematica”. In particular, integral (2.4.13) is computed using “Mathematica” command NIntegrate. Users can specify infinity as the upper limit of integration in NIntegrate. However, it is known that computation of some improper integrals in “Mathematica” can be time-consuming. Similar problem is found in the thesis for calculation of some improper integrals. As a result, the

improper integrals in the form $\int_0^{\infty} f(x) dx$ are replaced by $\int_0^{\gamma} f(x) dx$, where the upper limit γ is

chosen in order to satisfy the desired accuracy of the calculations. Example of such computation is shown in **Table 2.1**. It is seen from **Table 2.1** that for $1 \leq \hat{b} \leq 11$ and $\hat{\beta} = 2$ in order to construct the graph it is more than enough to choose $\gamma = 60$. “Mathematica” code for the computation of integral in (2.4.13) is shown in **Appendix Fig.Ap.2**.

$\gamma \backslash \hat{b}$	1	3	5	7	9	11
20	0.01283 0.37593i	0.03767 0.37150i	0.06034 0.36348i	0.08022 0.35306i	0.09725 0.34128i	0.11170 0.32894i
40	0.01284 0.38579i	0.03768 0.38136i	0.06036 0.37335i	0.08025 0.36292i	0.09728 0.35114i	0.11173 0.33880i
60	0.01284 0.38652i	0.03768 0.38209i	0.06036 0.37408i	0.08025 0.36365i	0.09728 0.35187i	0.11173 0.33953i
80	0.01284 0.38658i	0.03768 0.38216i	0.06036 0.37414i	0.08025 0.36371i	0.09728 0.35193i	0.11173 0.33960i
100	0.01284 0.38659i	0.03768 0.38216i	0.06036 0.37414i	0.08025 0.36371i	0.09728 0.35194i	0.11173 0.33960i
120	0.01284 0.38659i	0.03768 0.38217i	0.06036 0.37415i	0.08025 0.36372i	0.09728 0.35195i	0.11173 0.33961i
140	0.01284 0.38659i	0.03768 0.38217i	0.06036 0.37415i	0.08025 0.36372i	0.09728 0.35194i	0.11173 0.33961i

Table 2.1

Calculated values of the change in impedance given by (2.4.13) for different values of γ and \hat{b} .

2.5 Coil of finite dimensions

In this section we compute the induced vector potential of a coil of finite dimensions using solution (2.4.9) for a filamentary coil. The geometry of the problem is shown in **Fig.2.4**.

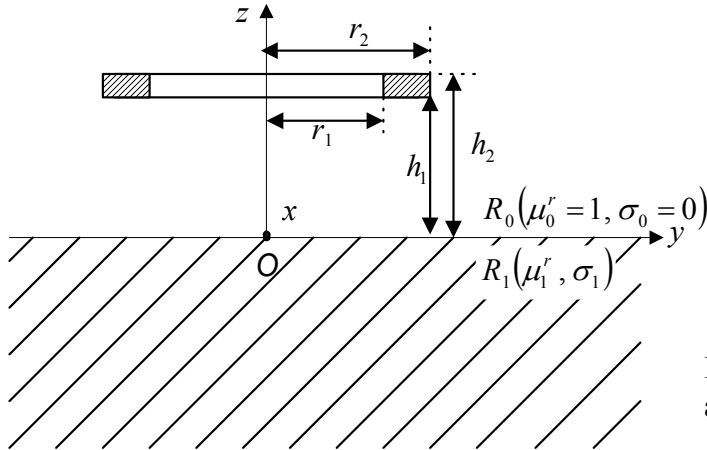


Fig. 2.4 A coil of finite dimensions above a conducting half-space.

Assume that N is the number of turns in the coil. The inner and outer radii of the coil are r_1 and r_2 , respectively. The distance from the coil to the top surface of the cylinder is denoted by h_1 . The height of the coil is $h_2 - h_1$. The induced vector potential in air due to currents in the whole coil is obtained as follows

$$A_{0coil}^{ind}(r, z) = \int_{r_1}^{r_2} \int_{h_1}^{h_2} A_0^{ind}(r, z, r_c, h) dr_c dh. \quad (2.5.1)$$

Substituting (2.4.9) into (2.5.1) and using the formulas

$$\left\| \begin{aligned} \cdot \int_{h_1}^{h_2} e^{-\lambda h} dh &= -\frac{1}{\lambda} (e^{-\lambda h_2} - e^{-\lambda h_1}) \\ \cdot \int_{r_1}^{r_2} r_c J_1(\lambda r_c) dr_c &= \left| y = \lambda r_c \right| = \frac{1}{\lambda^2} \int_{\lambda r_1}^{\lambda r_2} y J_1(y) dy \end{aligned} \right\| \quad (2.5.2)$$

we obtain the induced vector potential in air due to the presence of the conducting half-space (the current amplitude I in this case is replaced by the current density $\frac{NI}{(r_2 - r_1)(h_2 - h_1)}$) :

$$A_{0coil}^{ind}(r, z) = \frac{\mu_0 NI}{2(r_2 - r_1)(h_2 - h_1)} \int_0^\infty \left(\frac{F(\lambda)}{\lambda^3} (e^{-\lambda h_1} - e^{-\lambda h_2}) J_1(\lambda r) e^{-\lambda z} \int_{\lambda r_1}^{\lambda r_2} y J_1(y) dy \right) d\lambda, \quad (2.5.3)$$

where

$$F(\lambda) = \frac{\left(\lambda \mu_m - \frac{\beta}{2} \right) J_\nu(z_0) - \sqrt{b} J'_\nu(z_0)}{\left(\lambda \mu_m + \frac{\beta}{2} \right) J_\nu(z_0) + \sqrt{b} J'_\nu(z_0)}.$$

The integral with respect to y in (2.5.2) can be computed in terms of the Bessel and Struve functions [2] as follows

$$\int_{\lambda r_1}^{\lambda r_2} y J_1(y) dy = \left\{ \frac{\pi}{2} y [J_0(y) H_1(y) - J_1(y) H_0(y)] \right\} \Big|_{y=\lambda r_1}^{y=\lambda r_2} \quad (2.5.4)$$

The induced change in impedance of a coil of finite dimensions (see **Fig. 2.4**) is calculated by means of the following formula [15]:

$$\left\| Z^{ind} = \frac{j\omega}{I^2} \iiint_V \vec{A} \vec{I} dv = \frac{j\omega}{I^2} \int_0^{2\pi} d\varphi \int_{r_1}^{r_2} r dr \int_{h_1}^{h_2} A_{0coil}^{ind} dz \right\|$$

$$Z_{0coil}^{ind}(r, z) = \frac{2\pi j \omega}{I} \frac{N}{(r_2 - r_1)(h_2 - h_1)} \int_{r_1}^{r_2} \int_{h_1}^{h_2} r A_{0coil}^{ind}(r, z) dr dz. \quad (2.5.5)$$

Using (2.5.3) and (2.5.5) we obtain the induced change in impedance of the coil in the form

$$Z_{0coil}^{ind}(r, z) = \frac{j\omega \pi \mu_0 N^2}{(r_2 - r_1)^2 (h_2 - h_1)^2} \int_0^\infty \left(\frac{F(\lambda)}{\lambda^6} (e^{-\lambda h_1} - e^{-\lambda h_2})^2 \left(\int_{\lambda r_1}^{\lambda r_2} y J_1(y) dy \right)^2 \right) d\lambda. \quad (2.5.6)$$

Using the following dimensionless parameters (2.5.6) can be rewritten as follows

$$\left\| \begin{aligned} \cdot \lambda r_1 = u \rightarrow \lambda = \frac{u}{r_1}, \\ \hat{\alpha} = \alpha r_1, \hat{\beta} = \beta r_1, \end{aligned} \right\} \Rightarrow \hat{\nu} = \frac{\sqrt{\hat{\beta}^2 + 4u^2}}{\hat{\alpha} + \hat{\beta}},$$

$$\left\| \begin{aligned} \cdot \hat{b} = \omega \mu_0 \mu_m \sigma_m r_1^2, \hat{z}_0 = \frac{2\sqrt{-j\hat{b}}}{\hat{\alpha} + \hat{\beta}}, \hat{h}_1 = \frac{h_1}{r_1}, \hat{h}_2 = \frac{h_2}{r_1}, \hat{r}_2 = \frac{r_2}{r_1} \end{aligned} \right\|$$

$$Z_{0coil}^{ind}(r, z) = \omega \pi \mu_0 r_1 Z, \quad (2.5.7)$$

where

$$Z = \frac{N^2}{(\hat{h}_2 - \hat{h}_1)^2 (\hat{r}_2 - 1)^2} j \int_0^\infty \left(\frac{\hat{F}(u)}{u^6} (e^{-u\hat{h}_1} - e^{-u\hat{h}_2})^2 \left(\int_u^{\hat{r}_2} y J_1(y) dy \right)^2 \right) du, \quad (2.5.8)$$

and

$$\hat{F}(u) = \frac{\left(u\mu_m - \frac{\hat{\beta}}{2} \right) J_{\hat{\nu}}(\hat{z}_0) - \sqrt{-j\hat{b}} J'_{\hat{\nu}}(\hat{z}_0)}{\left(u\mu_m + \frac{\hat{\beta}}{2} \right) J_{\hat{\nu}}(\hat{z}_0) + \sqrt{-j\hat{b}} J'_{\hat{\nu}}(\hat{z}_0)}.$$

Fig. 2.5 plots the change in impedance Z for three different values of $\hat{\beta} = 1, 2, 3$. The other parameters of the problem are fixed at $\hat{\alpha} = 0, \hat{h}_2 = 1.6, \hat{h}_1 = 0.4, \hat{r}_2 = 2, N = 50$ and $\mu_m = 5$. The calculated points in **Fig. 2.5** correspond to different values of $\hat{b} = 1, 2, \dots, 20$ (from left to right). Computations are done with “Mathematica” (see **Appendix Fig.Ap.3**).

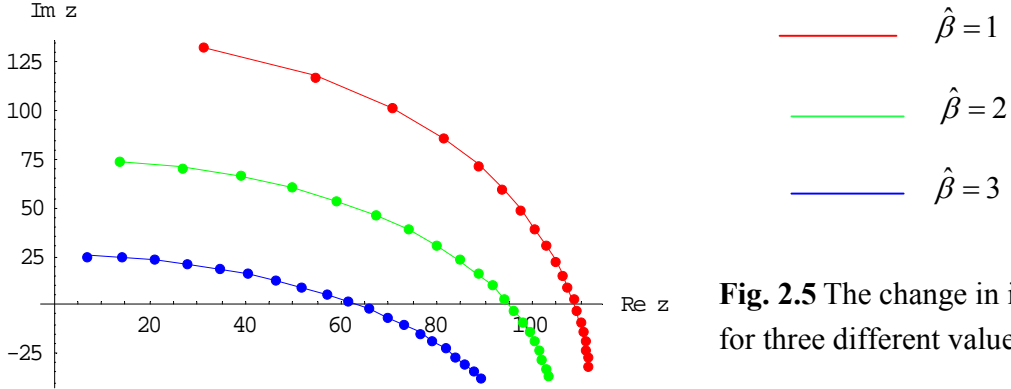


Fig. 2.5 The change in impedance Z for three different values of $\hat{\beta}$.

2.6 A single-turn coil above a two-layer medium with varying properties

In this section we consider another example of the general theory [30]. Consider a single-turn coil of radius r_c located at a distance h from a two-layer medium (see **Fig. 2.6**).

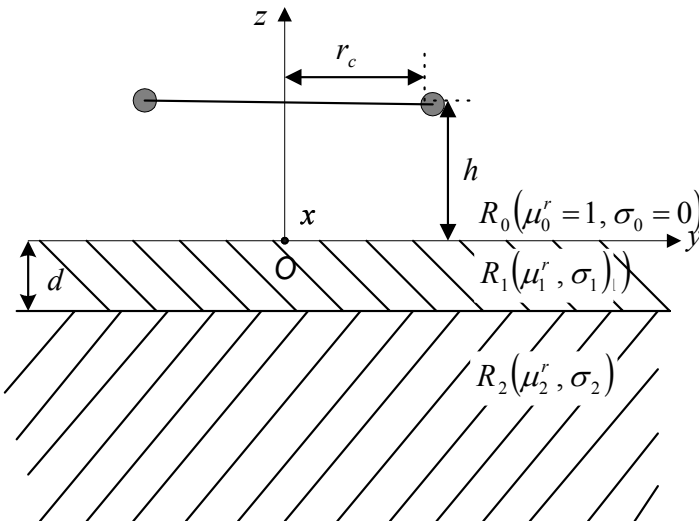


Fig. 2.6 A single-turn coil above a two-layer medium.

where $\mu_1^r(z) = \mu_1^m e^{\beta z}$, $\sigma_1(z) = \sigma_1^m e^{\alpha z}$,
 $\mu_2^r(z) = \mu_2^m$, $\sigma_2(z) = \sigma_2^m$.

Formulation of the problem follows from (2.3.1) – (2.3.7) and has the form

$$R_0: \frac{\partial^2 A_0}{\partial r^2} + \frac{1}{r} \frac{\partial A_0}{\partial r} - \frac{A_0}{r^2} + \frac{\partial^2 A_0}{\partial z^2} = -\mu_0 I \delta(r - r_c) \delta(z - h), \quad (2.6.1)$$

$$R_1: \frac{\partial^2 A_1}{\partial r^2} + \frac{1}{r} \frac{\partial A_1}{\partial r} - \frac{A_1}{r^2} + \frac{\partial^2 A_1}{\partial z^2} - \beta \frac{\partial A_1}{\partial z} - j\omega \sigma_1^m \mu_0 \mu_1^m e^{(\alpha+\beta)z} A_1 = 0, \quad (2.6.2)$$

$$R_2: \frac{\partial^2 A_2}{\partial r^2} + \frac{1}{r} \frac{\partial A_2}{\partial r} - \frac{A_2}{r^2} + \frac{\partial^2 A_2}{\partial z^2} = j\omega \sigma_2^m \mu_0 \mu_2^m A_2, \quad (2.6.3)$$

where σ_2^m and μ_2^m are, respectively, the constant electrical conductivity and relative magnetic permeability of region R_2 .

The boundary conditions are

$$A_0|_{z=0} = A_1|_{z=0}, \quad \frac{\partial A_0}{\partial z}|_{z=0} = \frac{1}{\mu_1^r(0)} \frac{\partial A_1}{\partial z}|_{z=0}, \quad (2.6.4)$$

$$A_1|_{z=-d} = A_2|_{z=-d}, \quad \frac{1}{\mu_1^r(-d)} \frac{\partial A_1}{\partial z}|_{z=-d} = \frac{1}{\mu_2^m} \frac{\partial A_2}{\partial z}|_{z=-d}. \quad (2.6.5)$$

The following conditions hold at infinity:

$$A_i, \frac{\partial A_i}{\partial r} \rightarrow 0 \text{ as } r \rightarrow \infty, \quad i = 0, 1, 2, \quad (2.6.6)$$

$$A_0 \rightarrow 0 \text{ as } z \rightarrow +\infty, \quad A_2 \rightarrow 0 \text{ as } z \rightarrow -\infty. \quad (2.6.7)$$

Applying the Hankel transform (2.3.8) to problem (2.6.1) – (2.6.7) we obtain the solution in regions $R_i, i = 0, 1$:

$$R_0: \frac{d^2 \tilde{A}_0}{dz^2} - \lambda^2 \tilde{A}_0 = -\mu_0 I r_c J_1(\lambda r_c) \delta(z - h) \rightarrow$$

$$\rightarrow \tilde{A}_0(\lambda, z) = C_2 e^{-\lambda z} + \frac{\mu_0}{2\lambda} I r_c J_1(\lambda r_c) e^{-\lambda|h-z|}, \quad (2.6.8)$$

$$R_1: \frac{d^2 \tilde{A}_1}{dz^2} - \beta_1 \frac{d\tilde{A}_1}{dz} - (\lambda^2 + j\omega \sigma_1^m \mu_0 \mu_1^m e^{(\alpha+\beta)z}) \tilde{A}_1 = 0 \rightarrow$$

$$\rightarrow \tilde{A}_1(\lambda, z) = e^{\frac{\beta z}{2}} \left[C_4 J_\nu \left(\frac{2\sqrt{b}}{\alpha+\beta} e^{\frac{(\alpha+\beta)z}{2}} \right) + C_5 Y_\nu \left(\frac{2\sqrt{b}}{\alpha+\beta} e^{\frac{(\alpha+\beta)z}{2}} \right) \right], \quad (2.6.9)$$

where $b = -j\omega \mu_0 \mu_1^m \sigma_1^m$, $\nu = \frac{\sqrt{\beta^2 + 4\lambda^2}}{\alpha + \beta}$.

The solution to (2.6.3) in the transformed space which remains bounded as $z \rightarrow -\infty$ is

$$R_2: \frac{d^2 \tilde{A}_2}{dz^2} - q_2^2 \tilde{A}_2 = 0 \quad \rightarrow \quad \tilde{A}_2(\lambda, z) = C_6 e^{q_2 z}, \quad (2.6.10)$$

where $q_2 = \sqrt{\lambda^2 + j\omega\sigma_2^m\mu_0\mu_2^m}$.

The boundary conditions are :

$$\tilde{A}_0|_{z=0} = \tilde{A}_1|_{z=0}, \quad \frac{d\tilde{A}_0}{dz}|_{z=0} = \frac{1}{\mu_1^m} \frac{d\tilde{A}_1}{dz}|_{z=0}, \quad (2.6.11)$$

$$\tilde{A}_1|_{z=-d} = \tilde{A}_2|_{z=-d}, \quad \frac{1}{\mu_1^m(-d)} \frac{d\tilde{A}_1}{dz}|_{z=-d} = \frac{1}{\mu_2^m} \frac{d\tilde{A}_2}{dz}|_{z=-d}. \quad (2.6.12)$$

$$\tilde{A}_0 \rightarrow 0 \text{ as } z \rightarrow +\infty, \quad \tilde{A}_2 \rightarrow 0 \text{ as } z \rightarrow -\infty. \quad (2.6.13)$$

Using (2.6.8) – (2.6.13) we obtain

$$C_2 + \frac{\mu_0}{2\lambda} Ir_c J_1(\lambda r_c) e^{-\lambda h} = C_4 J_\nu(z_0) + C_5 Y_\nu(z_0), \quad (2.6.14)$$

$$-\lambda C_2 + \frac{\mu_0}{2} Ir_c J_1(\lambda r_c) e^{-\lambda h} = \frac{C_4}{\mu_1^m} \left(\frac{\beta}{2} J_\nu(z_0) + \sqrt{b} J'_\nu(z_0) \right) + \frac{C_5}{\mu_1^m} \left(\frac{\beta}{2} Y_\nu(z_0) + \sqrt{b} Y'_\nu(z_0) \right), \quad (2.6.15)$$

$$C_6 e^{-q_2 d} = e^{-\frac{\beta d}{2}} (C_4 J_\nu(z_1) + C_5 Y_\nu(z_1)), \quad (2.6.16)$$

$$\begin{aligned} \frac{q_2}{\mu_2^m} C_6 e^{-q_2 d} = \\ = \frac{e^{-\frac{\beta d}{2}}}{\mu_1^m} \left(C_4 \left(\frac{\beta}{2} J_\nu(z_1) + \sqrt{b} e^{-\frac{(\alpha+\beta)d}{2}} J'_\nu(z_1) \right) + C_5 \left(\frac{\beta}{2} Y_\nu(z_1) + \sqrt{b} e^{-\frac{(\alpha+\beta)d}{2}} Y'_\nu(z_1) \right) \right), \end{aligned} \quad (2.6.17)$$

where $z_1 = z_0 e^{-\frac{(\alpha+\beta)d}{2}}$, $z_0 = \frac{2\sqrt{b}}{\alpha+\beta}$, $b = -j\omega\mu_0\mu_1^m\sigma_1^m$.

It follows from (2.6.14) – (2.6.15) that

$$\begin{aligned} \left\| \begin{aligned} \cdot C_2 = C_4 J_\nu(z_0) + C_5 Y_\nu(z_0) - \frac{\mu_0}{2\lambda} Ir_c J_1(\lambda r_c) e^{-\lambda h} \\ \cdot -\lambda C_4 J_\nu(z_0) - \lambda C_5 Y_\nu(z_0) + \frac{\mu_0}{2} Ir_c J_1(\lambda r_c) e^{-\lambda h} + \frac{\mu_0}{2} Ir_c J_1(\lambda r_c) e^{-\lambda h} = \\ = \frac{C_4}{\mu_1^m} \left(\frac{\beta}{2} J_\nu(z_0) + \sqrt{b} J'_\nu(z_0) \right) + \frac{C_5}{\mu_1^m} \left(\frac{\beta}{2} Y_\nu(z_0) + \sqrt{b} Y'_\nu(z_0) \right) \end{aligned} \right\| \\ C_4 = - \frac{C_5 \left(\left(\lambda \mu_1^m + \frac{\beta}{2} \right) Y_\nu(z_0) + \sqrt{b} Y'_\nu(z_0) \right) - \mu_0 \mu_1^m Ir_c J_1(\lambda r_c) e^{-\lambda h}}{\left(\lambda \mu_1^m + \frac{\beta}{2} \right) J_\nu(z_0) + \sqrt{b} J'_\nu(z_0)}. \end{aligned} \quad (2.6.18)$$

Similarly, conditions (2.6.16) – (2.6.17) can be rewritten as follows

$$\left\| \cdot C_6 = e^{\frac{(2q_2-\beta)d}{2}} (C_4 J_\nu(z_1) + C_5 Y_\nu(z_1)) \right\|$$

$$\begin{aligned}
& \left\| \cdot C_4 \left(\left(\frac{\beta}{2} e^{\beta d} \mu_2^m - \mu_1^m q_2 \right) J_\nu(z_1) + \sqrt{b} \mu_2^m e^{\frac{(\beta-\alpha)d}{2}} J'_\nu(z_1) \right) = \right. \\
& \quad \left. = -C_5 \left(\left(\frac{\beta}{2} e^{\beta d} \mu_2^m - \mu_1^m q_2 \right) Y_\nu(z_1) + \sqrt{b} \mu_2^m e^{\frac{(\beta-\alpha)d}{2}} Y'_\nu(z_1) \right) \right\| \\
C_4 = & - \frac{C_5 \left(\left(\frac{\beta}{2} e^{\beta d} \mu_2^m - \mu_1^m q_2 \right) Y_\nu(z_1) + \sqrt{b} \mu_2^m e^{\frac{(\beta-\alpha)d}{2}} Y'_\nu(z_1) \right)}{\left(\frac{\beta}{2} e^{\beta d} \mu_2^m - \mu_1^m q_2 \right) J_\nu(z_1) + \sqrt{b} \mu_2^m e^{\frac{(\beta-\alpha)d}{2}} J'_\nu(z_1)} \quad (2.6.19)
\end{aligned}$$

Constant C_5 is found from (2.6.18) – (2.6.19) and has the form

$$C_5 = - \frac{\mu_0 \mu_1^m Ir_c J_1(\lambda r_c) e^{-\lambda h} (D_1 J_\nu(z_1) + D_3 J'_\nu(z_1))}{(D_1 Y_\nu(z_1) + D_3 Y'_\nu(z_1)) (D_2 J_\nu(z_0) + \sqrt{b} J'_\nu(z_0)) - (D_1 J_\nu(z_1) + D_3 J'_\nu(z_1)) (D_2 Y_\nu(z_0) + \sqrt{b} Y'_\nu(z_0))}, \quad (2.6.20)$$

where

$$D_1 = \frac{\beta}{2} e^{\beta d} \mu_2^m - \mu_1^m q_2,$$

$$D_2 = \frac{\beta}{2} + \lambda \mu_1^m,$$

$$D_3 = \sqrt{b} \mu_2^m e^{\frac{(\beta-\alpha)d}{2}}.$$

Using (2.6.19) – (2.6.20) we obtain

$$C_4 = \frac{\mu_0 \mu_1^m Ir_c J_1(\lambda r_c) e^{-\lambda h} (D_1 Y_\nu(z_1) + D_3 Y'_\nu(z_1))}{(D_1 Y_\nu(z_1) + D_3 Y'_\nu(z_1)) (D_2 J_\nu(z_0) + \sqrt{b} J'_\nu(z_0)) - (D_1 J_\nu(z_1) + D_3 J'_\nu(z_1)) (D_2 Y_\nu(z_0) + \sqrt{b} Y'_\nu(z_0))}. \quad (2.6.21)$$

In particular, the value of C_2 is

$$\begin{aligned}
& \left\| \cdot C_2 = C_4 J_\nu(z_0) + C_5 Y_\nu(z_0) - \frac{\mu_0}{2\lambda} Ir_c J_1(\lambda r_c) e^{-\lambda h} \right\| \\
C_2 = & \frac{\mu_0 Ir_c J_1(\lambda r_c) e^{-\lambda h}}{2\lambda} \cdot \frac{B}{D}, \quad (2.6.22)
\end{aligned}$$

where

$$B = E \left((2\lambda \mu_1^m - D_2) J_\nu(z_0) - \sqrt{b} J'_\nu(z_0) \right) - F \left((2\lambda \mu_1^m - D_2) Y_\nu(z_0) - \sqrt{b} Y'_\nu(z_0) \right), \quad (2.6.23)$$

$$D = E \left(D_2 J_\nu(z_0) + \sqrt{b} J'_\nu(z_0) \right) - F \left(D_2 Y_\nu(z_0) + \sqrt{b} Y'_\nu(z_0) \right), \quad (2.6.24)$$

$$E = D_1 Y_\nu(z_1) + D_3 Y'_\nu(z_1), \quad (2.6.25)$$

$$F = D_1 J_\nu(z_1) + D_3 J'_\nu(z_1) \quad (2.6.26)$$

and

$$q_2 = \sqrt{\lambda^2 + j\omega \sigma_2^m \mu_0 \mu_2^m}, \quad b = -j\omega \mu_0 \mu_1^m \sigma_1^m$$

$$z_1 = z_0 e^{-\frac{(\alpha+\beta)d}{2}}, \quad z_0 = \frac{2\sqrt{b}}{\alpha + \beta}, \quad v = \frac{\sqrt{\beta^2 + 4\lambda^2}}{\alpha + \beta}.$$

The induced vector potential in region R_0 is

$$\tilde{A}_0^{ind}(\lambda, z) = C_2 e^{-\lambda z}, \quad (2.6.27)$$

where C_2 is given by (2.6.22).

Applying the inverse Hankel transform (2.3.39) to (2.6.27) we obtain the induced vector potential in the form

$$A_0^{ind}(r, z) = \frac{\mu_0 I r_c}{2} \int_0^\infty \frac{B}{D} J_1(\lambda r_c) J_1(\lambda r) e^{-\lambda(z+h)} d\lambda, \quad (2.6.28)$$

where B and D are given by (2.6.23) and (2.6.24), respectively.

The change in impedance of the coil is computed using (2.4.10) and (2.6.28):

$$Z^{ind}(r, z) = \omega \pi \mu_0 r_c^2 j \int_0^\infty \frac{B}{D} J_1^2(\lambda r_c) e^{-2\lambda h} d\lambda. \quad (2.6.29)$$

Using the following dimensionless parameters (2.6.29) can be rewritten as follows

$$\left\{ \begin{array}{l} \cdot \lambda r_c = u \rightarrow \lambda = \frac{u}{r_c}, \\ \hat{\alpha} = \alpha r_c, \quad \hat{\beta} = \beta r_c, \end{array} \right\} \Rightarrow \hat{v} = \frac{\sqrt{\hat{\beta}^2 + 4u^2}}{\hat{\alpha} + \hat{\beta}},$$

$$\cdot \hat{b} = \omega \mu_0 \mu_1^m \sigma_1^m r_c^2,$$

$$\cdot \hat{D}_1 = \frac{\hat{\beta}}{2} e^{\hat{\beta} d} \mu_2^m - \mu_1^m \hat{q}_2, \quad \text{kur } \hat{q}_2 = \sqrt{u^2 + jq}, \quad q = \omega \sigma_2^m \mu_0 \mu_2^m r_c^2,$$

$$\cdot \hat{D}_2 = \frac{\hat{\beta}}{2} + u \mu_1^m, \quad \cdot \hat{D}_3 = \sqrt{-j \hat{b}} \mu_2^m e^{\frac{(\hat{\beta}-\hat{\alpha})d}{2}},$$

$$\cdot \hat{z}_0 = \frac{2\sqrt{-j \hat{b}}}{\hat{\alpha} + \hat{\beta}}, \quad \hat{z}_1 = \hat{z}_0 e^{-\frac{(\hat{\alpha}+\hat{\beta})d}{2}}, \quad \hat{h} = \frac{h}{r_c},$$

$$\cdot \hat{E} = \hat{D}_1 Y_{\hat{v}}(\hat{z}_1) + \hat{D}_3 Y_{\hat{v}}'(\hat{z}_1), \quad \cdot \hat{F} = \hat{D}_1 J_{\hat{v}}(\hat{z}_1) + \hat{D}_3 J_{\hat{v}}'(\hat{z}_1),$$

$$\cdot \hat{B} = \hat{E} \left((2u \mu_1^m - \hat{D}_2) J_{\hat{v}}(\hat{z}_0) - \sqrt{-j \hat{b}} J_{\hat{v}}'(\hat{z}_0) \right) - \hat{F} \left((2u \mu_1^m - \hat{D}_2) Y_{\hat{v}}(\hat{z}_0) - \sqrt{-j \hat{b}} Y_{\hat{v}}'(\hat{z}_0) \right),$$

$$\cdot D = \hat{E} \left(\hat{D}_2 J_{\hat{v}}(\hat{z}_0) + \sqrt{-j \hat{b}} J_{\hat{v}}'(\hat{z}_0) \right) - \hat{F} \left(\hat{D}_2 Y_{\hat{v}}(\hat{z}_0) + \sqrt{-j \hat{b}} Y_{\hat{v}}'(\hat{z}_0) \right)$$

$$Z^{ind}(r, z) = \omega \pi \mu_0 r_c Z, \quad (2.6.30)$$

where

$$Z = j \int_0^\infty \frac{\hat{B}}{\hat{D}} J_1^2(u) e^{-2u \hat{h}} du.$$

Fig. 2.7 plots the change in impedance Z for three different values of $\hat{\beta} = 1, 2, 3$. The other

parameters of the problem are fixed at $\hat{d} = 0.2$, $\mu_1^m = 1$, $\mu_2^m = 100$, $\hat{h} = 0.05$, $\hat{\alpha} = 0$.

The calculated points in **Fig. 2.7** correspond to different values of $\hat{b} = 1, 2, \dots, 10$ (from left to right). Computations are done with “Mathematica” (see **Appendix Fig.Ap.4**).

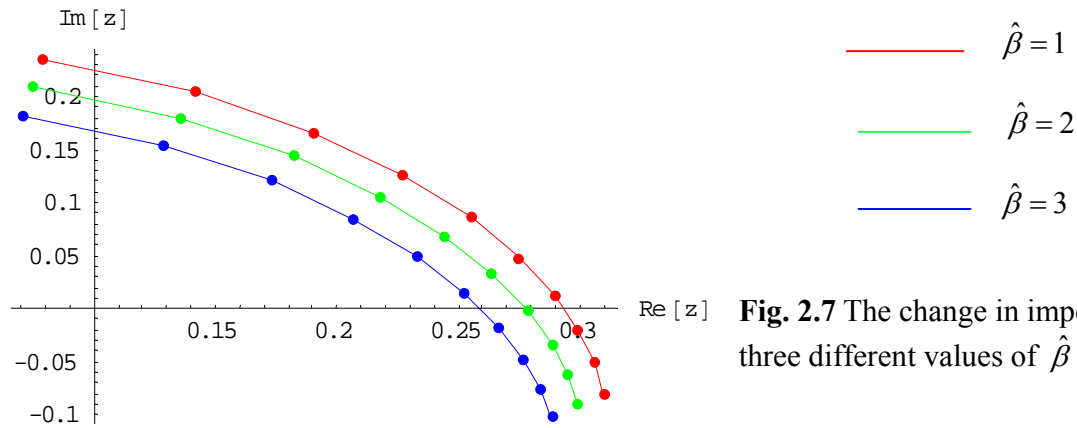


Fig. 2.7 The change in impedance Z for three different values of $\hat{\beta}$.

The solution for a single-turn coil can be easily generalized for the case of a coil of finite dimensions as it is done in **Section 2.5**. One can apply the general theory developed in **Section 2.3** to other examples with any finite number of layers. The transformed induced vector potential in free space in any case is given by the first term in formula (2.3.24) where the constant C_2 is obtained from the boundary conditions (2.3.14) – (2.3.15). The induced vector potential in free space is given by (2.3.39). The change in impedance of the single-turn coil is computed by (2.4.10). Finally, the change in impedance of a coil with finite dimensions is computed using (2.5.1) and (2.5.5).

3. CYLINDRICAL PROBLEMS FOR MULTILAYER MEDIA WITH VARYING PROPERTIES

3.1 Introduction

Eddy current methods are widely used in practice in order to control properties of conducting materials. All problems where eddy current coils are used for inspection of materials can be divided into the following two categories: (a) estimation of properties and/or other parameters of conducting media (for example, electric conductivity of a conducting layer or thickness of metal coatings) and (b) detection of defects (or flaws) in a conducting medium (for example, estimation of the effect of corrosion or presence of voids or other non-metallic inclusions in the medium). In both cases theoretical models (with some unknown parameters) are usually compared with experimental data. Optimization methods (for example, the least squares method) are then used to estimate unknown parameters of the medium (see, for example, [47], [49]).

Thus, a necessary step for the solution of an inverse problem (determination of unknown parameters of a medium) is the existence of a mathematical model describing the interaction of an alternating current in a coil with the conducting medium (direct problem).

Mathematical models for eddy current testing problems of conducting media with constant properties are well-known in the literature [60], [3]. Analytical solutions are presented in [15] for the case where a coil with alternating current is located above a multilayer medium. Similar problems for coils encircling multiple coaxial conductors or coils inside multiple coaxial conductors are analyzed in [16]. The properties of all media in [16] are assumed to be constant.

Some industrial applications (for example, surface hardening or decarbonization) modify the properties of a conducting medium (electric conductivity and/or magnetic permeability) which depend on geometrical coordinates. It is shown in [57], [58] that a thin layer of reduced magnetic permeability can exist in a medium which undergoes special treatment. In the case of a planar medium the magnetic permeability of the layer depends on the vertical coordinate.

Thus, in order to analyze such cases in practice one needs to develop mathematical models where electric conductivity and magnetic permeability of conducting layers depend on geometrical coordinates. There are at least two methods that can be used in order to take into account variability of the properties of the medium with respect to one geometrical coordinate. Regions where the properties of the medium are not constant can be divided into sufficiently large number of sub-layers where the electric conductivity and magnetic permeability are assumed to be constant. Therefore, in the whole region of interest the properties of the medium are piecewise constant functions of, say, depth. For example, up to 50 layers are used in [64] to model the change in electric conductivity with respect to a vertical coordinate.

Another approach is based on an attempt to use relatively simple electric conductivity and/or magnetic permeability profiles in order to model the variation of the properties of the medium with respect to one spatial coordinate. Some analytical solutions for the case where the properties of the medium depend on the vertical or radial coordinates are presented in [3].

In this chapter we follow the second approach. Analytical solutions are constructed for the case where a coil is located inside or outside a multilayer tube with varying properties. The electric conductivity and magnetic permeability are assumed to be power functions of the radial coordinate. The solution procedure is described for an arbitrary number of conducting layers with varying electric conductivity and magnetic permeability. Three cases are analyzed in detail: (a) the case of a coil inside one infinite outer layer, (b) the case of a coil inside a two-layer tube, and

(c) the case of a coil outside a two-layer tube

3.2 A coil located inside a multilayer medium with varying properties

Consider a single-turn coil of radius r_0 with alternating current located inside a multilayer tube where each coaxial layer (region R_i) is described by the inequalities: $R_i = \{r_i \leq r \leq r_{i+1}, 0 \leq \varphi \leq 2\pi, -\infty < z < +\infty\}, i = 1, 2, \dots, n$ [31]. Here r_i and r_{i+1} are the inner and outer radii of the cylindrical layer, respectively. Regions R_0 and R_{n+1} represent free space.

In particular,

$R_0 = \{0 \leq r \leq r_1, 0 \leq \varphi \leq 2\pi, -\infty < z < +\infty\}$ and $R_{n+1} = \{r \geq r_{n+1}, 0 \leq \varphi \leq 2\pi, -\infty < z < +\infty\}$. The coil is located in the plane $z = z_0$ perpendicular to the axis of the tubes (see **Fig. 3.1**).

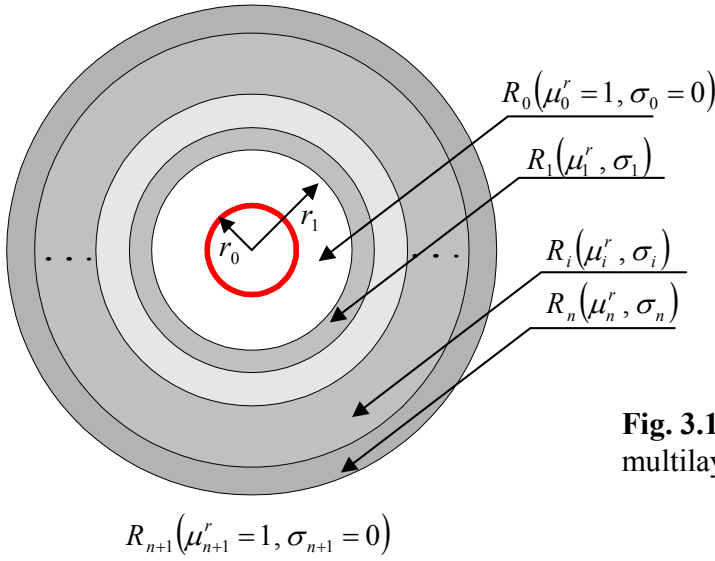


Fig. 3.1 A single-turn coil inside a multilayer tube.

Due to axial symmetry the vector potential has only one nonzero component in each region $R_i, i = 0, 1, \dots, n+1$ which is the function of r and z only. We assume that the electric conductivity $\sigma_i(r)$ and magnetic permeability $\mu_i(r)$ in region $R_i (i = 1, 2, \dots, n)$ are modeled by the following relations

$$\mu_i(r) = \mu_0 \mu_i^r(r) = \mu_0 \mu_i^m r^{\beta_i}, \quad \sigma_i(r) = \sigma_i^m r^{\alpha_i}, \quad i = 1, 2, \dots, n, \quad (3.2.1)$$

where $\alpha_i, \beta_i, \mu_i^m, \sigma_i^m$ are constants, and $\mu_i^r(r)$ is the relative magnetic permeability of region R_i .

Using (1.2.23) where $\mu_0^r = 1, \sigma_0 = 0$ and $I^e = I\delta(r - r_0)\delta(z - z_0)$ we obtain the equation for the vector potential in region R_0 :

$$\frac{\partial^2 A_0}{\partial r^2} + \frac{1}{r} \frac{\partial A_0}{\partial r} - \frac{A_0}{r^2} + \frac{\partial^2 A_0}{\partial z^2} = -\mu_0 I \delta(r - r_0) \delta(z - z_0), \quad (3.2.2)$$

where $\delta(x)$ is the Dirac delta-function.

Using (1.2.23) and $I = 0$ we obtain the following system of equations for the amplitudes $A_i(r, z)$ of the vector potential in each region $R_i, i = 1, 2, \dots, n$:

$$\frac{\partial^2 A_i}{\partial r^2} + \left(\frac{1}{r} - \frac{1}{\mu_i^r(r)} \frac{d\mu_i^r}{dr} \right) \frac{\partial A_i}{\partial r} - \left(\frac{1}{r^2} + \frac{1}{r\mu_i^r(r)} \frac{d\mu_i^r}{dr} + j\omega\sigma_i(r)\mu_0\mu_i^r(r) \right) A_i + \frac{\partial^2 A_i}{\partial z^2} = 0, \quad (3.2.3)$$

$i = 1, 2, \dots, n.$

It follows from (3.2.1) and (3.2.3) that

$$\frac{\partial^2 A_i}{\partial r^2} + \frac{1}{r} \frac{\partial A_i}{\partial r} - \frac{A_i}{r^2} + \frac{\partial^2 A_i}{\partial z^2} = \frac{\beta_i}{r} \frac{\partial A_i}{\partial r} + \left(\frac{\beta_i}{r^2} + j\omega\sigma_i^m \mu_0 \mu_i^m r^{\alpha_i + \beta_i} \right) A_i, \quad (3.2.4)$$

$i = 1, 2, \dots, n.$

Using (1.2.23) with $\mu_{n+1}^r = 1$, $\sigma_{n+1} = 0$ and $I = 0$ we obtain the equation for the vector potential in region R_{n+1} :

$$\frac{\partial^2 A_{n+1}}{\partial r^2} + \frac{1}{r} \frac{\partial A_{n+1}}{\partial r} - \frac{A_{n+1}}{r^2} + \frac{\partial^2 A_{n+1}}{\partial z^2} = 0. \quad (3.2.5)$$

The boundary conditions are

$$A_0|_{r=r_1} = A_1|_{r=r_1}, \quad \frac{\partial A_0}{\partial r}|_{r=r_1} = \frac{1}{\mu_1^r(r_1)} \frac{\partial A_1}{\partial r}|_{r=r_1}, \quad (3.2.6)$$

$$A_i|_{r=r_{i+1}} = A_{i+1}|_{r=r_{i+1}},$$

$$\frac{1}{\mu_i^r(r_{i+1})} \frac{\partial A_i}{\partial r}|_{r=r_{i+1}} = \frac{1}{\mu_{i+1}^r(r_{i+1})} \frac{\partial A_{i+1}}{\partial r}|_{r=r_{i+1}}, \quad i = 1, 2, \dots, n. \quad (3.2.7)$$

In addition, A_0 is bounded at $r = 0$ and A_{n+1} is bounded as $r \rightarrow \infty$.

The following conditions hold at infinity:

$$A_i, \frac{\partial A_i}{\partial z} \rightarrow 0 \text{ as } z \rightarrow \pm\infty, \quad i = 0, 1, \dots, n. \quad (3.2.8)$$

Applying the Fourier transform of the form

$$\tilde{A}_i(r, \lambda) = \int_{-\infty}^{\infty} A_i(r, z) e^{-j\lambda z} dz, \quad i = 0, 1, \dots, n+1. \quad (3.2.9)$$

to the left-hand side of equation (3.2.4) we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} \left(\frac{\partial^2 A_i}{\partial r^2} + \frac{1}{r} \frac{\partial A_i}{\partial r} - \frac{A_i}{r^2} + \frac{\partial^2 A_i}{\partial z^2} \right) e^{-j\lambda z} dz &= \int_{-\infty}^{\infty} \left(\frac{\partial^2 A_i}{\partial r^2} + \frac{1}{r} \frac{\partial A_i}{\partial r} - \frac{A_i}{r^2} \right) e^{-j\lambda z} dz + \int_{-\infty}^{\infty} \frac{\partial^2 A_i}{\partial z^2} e^{-j\lambda z} dz \\ &= \left\| \begin{aligned} \int_{-\infty}^{\infty} \left(\frac{\partial^2 A_i}{\partial r^2} + \frac{1}{r} \frac{\partial A_i}{\partial r} - \frac{A_i}{r^2} \right) e^{-j\lambda z} dz &= \frac{d^2 \tilde{A}_i}{dr^2} + \frac{1}{r} \frac{d\tilde{A}_i}{dr} - \frac{\tilde{A}_i}{r^2} \\ \int_{-\infty}^{\infty} \frac{\partial^2 A_i}{\partial z^2} e^{-j\lambda z} dz &= \left| \begin{array}{l} u = e^{-j\lambda z} \rightarrow du = -j\lambda e^{-j\lambda z} dz \\ dv = \frac{\partial^2 A_i}{\partial z^2} dz \rightarrow v = \frac{\partial A_i}{\partial z} \end{array} \right| = e^{-j\lambda z} \frac{\partial A_i}{\partial z} \Big|_{-\infty}^{\infty} + j\lambda \int_{-\infty}^{\infty} \frac{\partial A_i}{\partial z} e^{-j\lambda z} dz = \\ &= j\lambda e^{-j\lambda z} A_i \Big|_{-\infty}^{\infty} + j^2 \lambda^2 \int_{-\infty}^{\infty} A_i e^{-j\lambda z} dz = -\lambda^2 \tilde{A}_i \end{aligned} \right\| \\ \int_{-\infty}^{\infty} \left(\frac{\partial^2 A_i}{\partial r^2} + \frac{1}{r} \frac{\partial A_i}{\partial r} - \frac{A_i}{r^2} + \frac{\partial^2 A_i}{\partial z^2} \right) e^{-j\lambda z} dz &= \frac{d^2 \tilde{A}_i}{dr^2} + \frac{1}{r} \frac{d\tilde{A}_i}{dr} - \frac{\tilde{A}_i}{r^2} - \lambda^2 \tilde{A}_i \end{aligned} \quad (3.2.10)$$

Applying (3.2.9) to the right-hand side of equation (3.2.2) we obtain

$$-\mu_0 I \delta(r-r_0) \int_0^{\infty} \delta(z-z_0) e^{-j\lambda z} dz = -\mu_0 I e^{-j\lambda z_0} \delta(r-r_0) \quad (3.2.11)$$

Similarly, the right-hand side of equation (3.2.4) is transformed to the form

$$\begin{aligned} \int_{-\infty}^{\infty} \left(\frac{\beta_i}{r} \frac{\partial A_i}{\partial r} + \left(\frac{\beta_i}{r^2} + j\omega\sigma_i^m \mu_0 \mu_i^m r^{\alpha_i+\beta_i} \right) A_i \right) e^{-j\lambda z} dz = \\ = \frac{\beta_i}{r} \frac{d\tilde{A}_i}{dr} + \left(\frac{\beta_i}{r^2} + j\omega\sigma_i^m \mu_0 \mu_i^m r^{\alpha_i+\beta_i} \right) \tilde{A}_i, \quad i=1,2,\dots,n. \end{aligned} \quad (3.2.12)$$

It follows from (3.2.10), (3.2.11) and (3.2.12) that

$$\frac{d^2 \tilde{A}_0}{dr^2} + \frac{1}{r} \frac{d\tilde{A}_0}{dr} - \left(\frac{1}{r^2} + \lambda^2 \right) \tilde{A}_0 = -\mu_0 I e^{-j\lambda z_0} \delta(r-r_0), \quad (3.2.13)$$

$$\frac{d^2 \tilde{A}_i}{dr^2} + \frac{(1-\beta_i)}{r} \frac{d\tilde{A}_i}{dr} - \left(\frac{1+\beta_i}{r^2} + p_i^2 r^{\alpha_i+\beta_i} + \lambda^2 \right) \tilde{A}_i = 0, \quad i=1,2,\dots,n, \quad (3.2.14)$$

where $p_i = \sqrt{j\omega\mu_0\mu_i^m\sigma_i^m}$, j is the imaginary unit and ω is the frequency of the current in the coil.

The transformed equation in region R_n is

$$\frac{d^2 \tilde{A}_{n+1}}{dr^2} + \frac{1}{r} \frac{d\tilde{A}_{n+1}}{dr} - \left(\frac{1}{r^2} + \lambda^2 \right) \tilde{A}_{n+1} = 0. \quad (3.2.15)$$

Applying the Fourier transform to the boundary conditions we obtain

$$\tilde{A}_0|_{r=r_0} = \tilde{A}_1|_{r=r_1}, \quad \frac{d\tilde{A}_0}{dr}|_{r=r_0} = \frac{1}{\mu_1^r(r_1)} \frac{d\tilde{A}_1}{dr}|_{r=r_1}, \quad (3.2.16)$$

$$\tilde{A}_i|_{r=r_{i+1}} = \tilde{A}_{i+1}|_{r=r_{i+1}}, \quad \frac{1}{\mu_i^r(r_{i+1})} \frac{d\tilde{A}_i}{dr}|_{r=r_{i+1}} = \frac{1}{\mu_{i+1}^r(r_{i+1})} \frac{d\tilde{A}_{i+1}}{dr}|_{r=r_{i+1}}, \quad i=1,2,\dots,n. \quad (3.2.17)$$

$$\text{In addition, } \tilde{A}_0 \text{ is bounded at } r=0 \text{ and } \tilde{A}_{n+1} \text{ is bounded as } r \rightarrow \infty. \quad (3.2.18)$$

In order to find the solution to (3.2.13) we consider two sub-regions, R_{00} and R_{01} , of

region R_0 , namely, $R_{00} = \{0 \leq r < r_0, 0 \leq \varphi \leq 2\pi, -\infty < z < +\infty\}$ and

$R_{01} = \{r_0 < r \leq r_1, 0 \leq \varphi \leq 2\pi, -\infty < z < +\infty\}$. The solutions in regions R_{00} and R_{01} are denoted by

\tilde{A}_{00} and \tilde{A}_{01} , respectively. Hence,

$$\frac{d^2 \tilde{A}_{00}}{dr^2} + \frac{1}{r} \frac{d\tilde{A}_{00}}{dr} - \left(\frac{1}{r^2} + \lambda^2 \right) \tilde{A}_{00} = 0, \quad 0 \leq r < r_0 \quad (3.2.19)$$

$$\frac{d^2 \tilde{A}_{01}}{dr^2} + \frac{1}{r} \frac{d\tilde{A}_{01}}{dr} - \left(\frac{1}{r^2} + \lambda^2 \right) \tilde{A}_{01} = 0, \quad r > r_0 \quad (3.2.20)$$

The solution to (3.2.19) can be expressed in terms of modified Bessel functions:

$$\left\| \begin{aligned} & y''_{xx} + \frac{1}{x} y'_x - \left(1 + \frac{v^2}{x^2}\right) y = 0 \Rightarrow y = I_v(x) \\ & \cdot y = y(\lambda r) \rightarrow \begin{cases} y'_r = \lambda y'_r(\lambda r) \\ y''_r = \lambda^2 y''_r(\lambda r) \end{cases} \\ & \cdot \lambda^2 y''_{rr}(\lambda r) + \frac{\lambda}{r} y'_r(\lambda r) - \left(\frac{1}{r^2} + \lambda^2\right) y(\lambda r) = 0 \\ & \rightarrow y''_{rr}(\lambda r) + \frac{1}{\lambda r} y'_r(\lambda r) - \left(1 + \frac{1}{\lambda^2 r^2}\right) y(\lambda r) = 0 \rightarrow y = I_1(\lambda r) \end{aligned} \right\|$$

The general solution to (3.2.19) can be written in the form

$$\tilde{A}_{00} = B_1 I_1(\lambda r) + B_2 K_1(\lambda r),$$

where $I_1(\lambda r)$, $K_1(\lambda r)$ are the modified Bessel functions of the first and second kind, respectively.

The bounded solution to (3.2.19) in region $0 \leq r < r_0$ is

$$\tilde{A}_{00}(r, \lambda) = B_1 I_1(\lambda r). \quad (3.2.21)$$

The general solution to (3.2.20) can be written in the form

$$\tilde{A}_{01}(r, \lambda) = B_2 I_1(\lambda r) + B_3 K_1(\lambda r). \quad (3.2.22)$$

The functions $\tilde{A}_{00}(r, \lambda)$ and $\tilde{A}_{01}(r, \lambda)$ satisfy the following conditions at $r = r_0$:

$$\tilde{A}_{00}|_{r=r_0} = \tilde{A}_{01}|_{r=r_0}, \quad (3.2.23)$$

The condition (3.2.23) reflects the fact that the function $\tilde{A}_0(r, \lambda)$ is continuous at $r = r_0$.

The second condition is obtained integrating (3.2.13) with respect to r from $r = r_0 - \varepsilon$ to $r = r_0 + \varepsilon$ and considering the limit in the resulting expression as $\varepsilon \rightarrow +0$:

$$\left\| \begin{aligned} & \int_{r_0-\varepsilon}^{r_0+\varepsilon} \left(\frac{d^2 \tilde{A}_0}{dr^2} + \frac{1}{r} \frac{d\tilde{A}_0}{dr} - \left(\frac{1}{r^2} + \lambda^2 \right) \tilde{A}_0 \right) dr = -\mu_0 I e^{-j\lambda z_0} \int_{r_0-\varepsilon}^{r_0+\varepsilon} \delta(r - r_0) dr \\ & \frac{d\tilde{A}_0}{dr} \Big|_{r_0-\varepsilon}^{r_0+\varepsilon} + \int_{r_0-\varepsilon}^{r_0+\varepsilon} \left(\frac{1}{r} \frac{d\tilde{A}_0}{dr} - \left(\frac{1}{r^2} + \lambda^2 \right) \tilde{A}_0 \right) dr = -\mu_0 I e^{-j\lambda z_0} \\ & \left| \begin{aligned} & u = \frac{1}{r} \quad \rightarrow du = -\frac{1}{r^2} dr \\ & dv = \frac{d\tilde{A}_0}{dr} \rightarrow v = \tilde{A}_0 \end{aligned} \right| \\ & \frac{d\tilde{A}_0}{dr} \Big|_{r_0-\varepsilon}^{r_0+\varepsilon} + \frac{1}{r} \tilde{A}_0 \Big|_{r_0-\varepsilon}^{r_0+\varepsilon} + \int_{r_0-\varepsilon}^{r_0+\varepsilon} \frac{\tilde{A}_0}{r^2} dr - \int_{r_0-\varepsilon}^{r_0+\varepsilon} \frac{\tilde{A}_0}{r^2} dr - \int_{r_0-\varepsilon}^{r_0+\varepsilon} \lambda^2 \tilde{A}_0 dr = -\mu_0 I e^{-j\lambda z_0} \end{aligned} \right\|$$

$$\left\| \begin{aligned} & \cdot \lim_{\varepsilon \rightarrow 0} \frac{d\tilde{A}_0}{dr} \Big|_{r_0-\varepsilon}^{r_0+\varepsilon} = \frac{d\tilde{A}_{01}}{dr} \Big|_{r=r_0} - \frac{d\tilde{A}_{00}}{dr} \Big|_{r=r_0} \\ & \cdot \lim_{\varepsilon \rightarrow 0} \frac{\tilde{A}_0}{r} \Big|_{r_0-\varepsilon}^{r_0+\varepsilon} = \frac{1}{r} \left(\tilde{A}_{01} \Big|_{r=r_0} - \tilde{A}_{00} \Big|_{r=r_0} \right) = 0 \\ & \cdot -\lambda^2 \lim_{\varepsilon \rightarrow 0} \int_{r_0-\varepsilon}^{r_0+\varepsilon} \tilde{A}_0 dz = -\lambda^2 \lim_{\varepsilon \rightarrow 0} 2\varepsilon \tilde{A}_0(r^*, \lambda) = 0 \end{aligned} \right\|$$

$$\frac{d\tilde{A}_{01}}{dr} \Big|_{r=r_0} - \frac{d\tilde{A}_{00}}{dr} \Big|_{r=r_0} = -\mu_0 I e^{-j\lambda z_0}. \quad (3.2.24)$$

Using (3.2.21) – (3.2.24) we obtain

$$\begin{cases} B_1 I_1(\lambda r_0) = B_2 I_1(\lambda r_0) + B_3 K_1(\lambda r_0) \\ \lambda B_2 I_1'(\lambda r_0) + \lambda B_3 K_1'(\lambda r_0) - \lambda B_1 I_1'(\lambda r_0) = -\mu_0 I e^{-j\lambda z_0} \end{cases} \Rightarrow$$

$$\begin{cases} B_1 = B_2 + B_3 \frac{K_1(\lambda r_0)}{I_1(\lambda r_0)} \\ B_2 I_1'(\lambda r_0) + B_3 K_1'(\lambda r_0) - B_2 I_1'(\lambda r_0) - B_3 \frac{I_1'(\lambda r_0) K_1(\lambda r_0)}{I_1(\lambda r_0)} = -\frac{\mu_0}{\lambda} I e^{-j\lambda z_0} \end{cases} \Rightarrow$$

$$\left\| \begin{aligned} & \cdot B_3 (I_1(\lambda r_0) K_1'(\lambda r_0) - I_1'(\lambda r_0) K_1(\lambda r_0)) = -\frac{\mu_0}{\lambda} I e^{-j\lambda z_0} I_1(\lambda r_0) \\ & W\{I_\nu(z), K_\nu(z)\} = -\frac{1}{z} \end{aligned} \right\|$$

$$\begin{cases} B_3 = \mu_0 r_0 I e^{-j\lambda z_0} I_1(\lambda r_0) \\ B_1 = B_2 + \mu_0 r_0 I e^{-j\lambda z_0} K_1(\lambda r_0), \end{cases} \quad (3.2.25)$$

where $W\{I_\nu(z), K_\nu(z)\}$ is the Wronskian.

It follows from (3.2.21), (3.2.22) and (3.2.25) that

$$\tilde{A}_{00}(r, \lambda) = B_2 I_1(\lambda r) + \mu_0 r_0 I e^{-j\lambda z_0} K_1(\lambda r_0) I_1(\lambda r), \quad (3.2.26)$$

$$\tilde{A}_{01}(r, \lambda) = B_2 I_1(\lambda r) + \mu_0 r_0 I e^{-j\lambda z_0} I_1(\lambda r_0) K_1(\lambda r). \quad (3.2.27)$$

In order to understand the meaning of each of the two terms on the right-hand sides of (3.2.26) and (3.2.27) we consider the case where a coil of radius r_0 centered at the point $(0, z_0)$ is located in an unbounded free space. The corresponding component of the vector potential will be denoted by $A_0^{free}(r, z)$. The equation for A_0^{free} is

$$\frac{\partial^2 A_0^{free}}{\partial r^2} + \frac{1}{r} \frac{\partial A_0^{free}}{\partial r} - \frac{A_0^{free}}{r^2} + \frac{\partial^2 A_0^{free}}{\partial z^2} = -\mu_0 I \delta(r - r_0) \delta(z - z_0), \quad (3.2.28)$$

where $\delta(x)$ is the Dirac delta-function.

In addition, A_0^{free} is bounded at $r = 0$ and A_0^{free} is bounded as $r \rightarrow \infty$.

Applying the Fourier transform (3.2.9) and using (3.2.8) and (3.2.13) we get

$$\frac{d^2 \tilde{A}_0^{free}}{dr^2} + \frac{1}{r} \frac{d\tilde{A}_0^{free}}{dr} - \left(\frac{1}{r^2} + \lambda^2 \right) \tilde{A}_0^{free} = -\mu_0 I e^{-j\lambda z_0} \delta(r - r_0) \quad (3.2.29)$$

The bounded solutions in regions R_{00} and R_{01} are :

$$\begin{aligned} R_{00} : \frac{d^2 \tilde{A}_{00}^{free}}{dr^2} + \frac{1}{r} \frac{d\tilde{A}_{00}^{free}}{dr} - \left(\frac{1}{r^2} + \lambda^2 \right) \tilde{A}_{00}^{free} &= 0, \quad 0 \leq r < r_0 \\ &\rightarrow \tilde{A}_{00}^{free}(r, \lambda) = B_4 I_1(\lambda r). \end{aligned} \quad (3.2.30)$$

$$\begin{aligned} R_{01} : \frac{d^2 \tilde{A}_{01}^{free}}{dr^2} + \frac{1}{r} \frac{d\tilde{A}_{01}^{free}}{dr} - \left(\frac{1}{r^2} + \lambda^2 \right) \tilde{A}_{01}^{free} &= 0, \quad r > r_0 \\ &\rightarrow \tilde{A}_{01}^{free}(r, \lambda) = B_5 K_1(\lambda r). \end{aligned} \quad (3.2.31)$$

Using (3.2.23) and (3.2.24) solutions (3.2.30) and (3.2.31) can be rewritten as follows

$$\tilde{A}_{00}^{free}(r, \lambda) = \mu_0 r_0 I e^{-j\lambda z_0} K_1(\lambda r_0) I_1(\lambda r), \quad (3.2.32)$$

$$\tilde{A}_{01}^{free}(r, \lambda) = \mu_0 r_0 I e^{-j\lambda z_0} I_1(\lambda r_0) K_1(\lambda r). \quad (3.2.33)$$

Comparing (3.2.26), (3.2.27) and (3.2.32), (3.2.33) we see that the second terms on the right-hand sides of (3.2.26) and (3.2.27) represent the vector potential in an unbounded free space. Thus, the first term of the right-hand sides of (3.2.26) and (3.2.27) represents the induced vector potential due to the presence of the conducting medium.

The solution to (3.2.14) can be expressed in terms of different special functions. For example, the solution to (3.2.14) for the case $\alpha_i = -1$, $\beta_i = -1$ (see [50]) is

$$\frac{d^2 \tilde{A}_i}{dr^2} + \frac{2}{r} \frac{d\tilde{A}_i}{dr} - \left(\lambda^2 + \frac{p_i^2}{r^2} \right) \tilde{A}_i = 0, \quad i = 1, 2, \dots, n$$

$$\left\| \begin{aligned} &\cdot y''_{zz} + \frac{1-2a}{z} y'_z + \left(b^2 + \frac{a^2 - v^2}{z^2} \right) y = 0 \rightarrow y(z) = z^a Z_v(bz) \\ &\cdot 1 - 2a = 2 \rightarrow a = -\frac{1}{2} \\ &b^2 = \lambda^2 \rightarrow b = j\lambda \\ &p_i^2 = -a^2 + v_i^2 = -\frac{1}{4} + v_i^2 \rightarrow v_i = \sqrt{\frac{1}{4} + p_i^2} \end{aligned} \right\|$$

where $p_i = \sqrt{j\omega\mu_0\mu_i^m\sigma_i^m}$.

$$\tilde{A}_i(r, \lambda) = \frac{1}{\sqrt{r}} \left(B_{4_i} I_{v_i}(\lambda r) + B_{5_i} K_{v_i}(\lambda r) \right) \quad (3.2.34)$$

The bounded solution to (3.2.5) is

$$\tilde{A}_{n+1}(r, \lambda) = B_6 K_1(\lambda r). \quad (3.2.35)$$

Unknown constants in (3.2.26), (3.2.27), (3.2.34) and (3.2.35) can be found from the boundary conditions (3.2.16) – (3.2.18). Then the solution in each region R_i , $i = 0, 1, \dots, n+1$ can be found

by means of the inverse Fourier transform of the form

$$A_i(r, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{A}_i(r, \lambda) e^{j\lambda z} d\lambda. \quad (3.2.36)$$

It can be shown that the induced vector potential in free space due to multilayer conducting tubes has the form (see [3])

$$A_0^{ind}(r, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} B_2 I_1(\lambda r) e^{j\lambda z} d\lambda. \quad (3.2.37)$$

3.3 A coil inside a cylindrical region

In this section we consider one particular case of the problem analyzed in **Section 3.2** where the tube consists of one unbounded conducting layer [32]. Suppose that a single-turn coil with alternating current is located inside a cylindrical region $R_1 = \{r \geq r_1, 0 \leq \varphi \leq 2\pi, -\infty < z < +\infty\}$. The geometry of the problem is shown in **Fig. 3.2**,

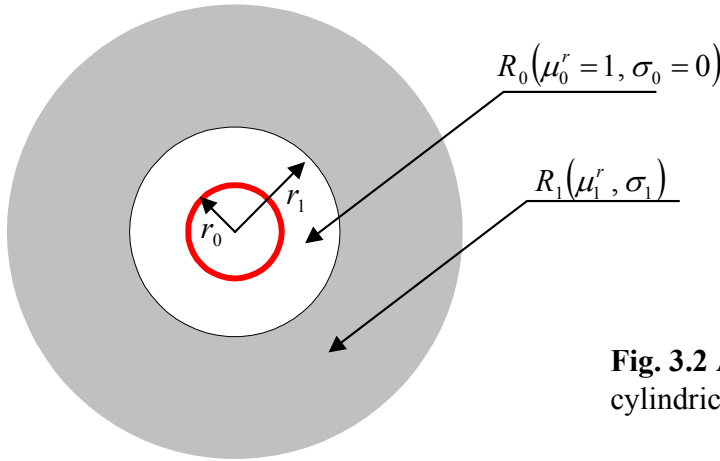


Fig. 3.2 A single-turn coil inside a cylindrical region.

where $\mu_0^r(r) = 1$, $\sigma_0(r) = 0$,

$$\mu_1^r(r) = \mu_1^m r^\beta, \quad \sigma_1(r) = \sigma_1^m r^\alpha,$$

Using the results from **Section 3.2** we obtain the following problem :

$$R_0: \quad \frac{\partial^2 A_0}{\partial r^2} + \frac{1}{r} \frac{\partial A_0}{\partial r} - \frac{A_0}{r^2} + \frac{\partial^2 A_0}{\partial z^2} = -\mu_0 I \delta(r - r_0) \delta(z - z_0), \quad (3.3.1)$$

$$R_1: \quad \frac{\partial^2 A_1}{\partial r^2} + \frac{1 - \beta}{r} \frac{\partial A_1}{\partial r} - \left(\frac{1 + \beta}{r^2} + j\omega \mu_0 \mu_1^m \sigma_1^m r^{\alpha + \beta} \right) A_1 + \frac{\partial^2 A_1}{\partial z^2} = 0. \quad (3.3.2)$$

The boundary conditions are

$$A_0|_{r=r_1} = A_1|_{r=r_1}, \quad \frac{\partial A_0}{\partial r}|_{r=r_1} = \frac{1}{\mu_1^r(r_1)} \frac{\partial A_1}{\partial r}|_{r=r_1}, \quad (3.3.3)$$

In addition, A_0 is bounded at $r = 0$ and A_1 is bounded as $r \rightarrow \infty$.

The following conditions hold at infinity:

$$A_i, \quad \frac{\partial A_i}{\partial z} \rightarrow 0 \quad \text{as } z \rightarrow \pm\infty, \quad i = 0, 1. \quad (3.3.4)$$

Applying the Fourier transform (3.2.9) to problem (3.3.1) – (3.3.4) and using (3.2.26) – (3.2.27), we obtain the solution in regions R_i , $i = 0, 1$:

$$R_0: \quad \frac{d^2 \tilde{A}_0}{dr^2} + \frac{1}{r} \frac{d\tilde{A}_0}{dr} - \left(\frac{1}{r^2} + \lambda^2 \right) \tilde{A}_0 = -\mu_0 I e^{-j\lambda z_0} \delta(r - r_0),$$

$$\rightarrow \begin{cases} \tilde{A}_{00}(r, \lambda) = B_2 I_1(\lambda r) + \mu_0 r_0 I e^{-j\lambda z_0} K_1(\lambda r_0) I_1(\lambda r), \\ \tilde{A}_{01}(r, \lambda) = B_2 I_1(\lambda r) + \mu_0 r_0 I e^{-j\lambda z_0} I_1(\lambda r_0) K_1(\lambda r), \end{cases} \quad (3.3.5)$$

$$R_1: \quad \frac{d^2 \tilde{A}_1}{dr^2} + \frac{(1-\beta)}{r} \frac{d\tilde{A}_1}{dr} - \left(\frac{1+\beta}{r^2} + p_1^2 r^{\alpha+\beta} + \lambda^2 \right) \tilde{A}_1 = 0.$$

This equation has an analytical solution, for example, for the case $\alpha = -1, \beta = -1$. The bounded general solution of the corresponding equation has the form

$$\tilde{A}_1(r, \lambda) = \frac{1}{\sqrt{r}} B_5 K_\nu(\lambda r), \quad (3.3.6)$$

where $p_1 = \sqrt{j\omega\mu_0\mu_1^m\sigma_1^m}$, $\nu = \sqrt{\frac{1}{4} + p_1^2}$.

The boundary conditions are

$$\tilde{A}_0|_{r=r_1} = \tilde{A}_1|_{r=r_1}, \quad \frac{d\tilde{A}_0}{dr}|_{r=r_1} = \frac{1}{\mu_1^m(r_1)} \frac{d\tilde{A}_1}{dr}|_{r=r_1}. \quad (3.3.7)$$

Using (3.3.5) – (3.3.7) we obtain

$$\begin{cases} B_2 I_1(\lambda r_1) + \mu_0 r_0 I e^{-j\lambda z_0} I_1(\lambda r_0) K_1(\lambda r_1) = B_5 \frac{1}{\sqrt{r_1}} K_\nu(\lambda r_1), \\ \lambda B_2 I_1'(\lambda r_1) + \lambda \mu_0 r_0 I e^{-j\lambda z_0} I_1(\lambda r_0) K_1'(\lambda r_1) = \frac{r_1}{\mu_1^m} \left(-\frac{1}{2r_1 \sqrt{r_1}} B_5 K_\nu(\lambda r_1) + \frac{\lambda}{\sqrt{r_1}} B_5 K_\nu'(\lambda r_1) \right). \end{cases} \quad (3.3.8)$$

$$\begin{aligned} & \left\| \begin{aligned} \cdot B_2 &= B_5 \frac{1}{\sqrt{r_1}} \frac{K_\nu(\lambda r_1)}{I_1(\lambda r_1)} - \mu_0 r_0 I e^{-j\lambda z_0} I_1(\lambda r_0) \frac{K_1(\lambda r_1)}{I_1(\lambda r_1)} \\ \cdot B_5 \frac{\lambda}{\sqrt{r_1}} I_1'(\lambda r_1) \frac{K_\nu(\lambda r_1)}{I_1(\lambda r_1)} - \lambda \mu_0 r_0 I e^{-j\lambda z_0} I_1(\lambda r_0) I_1'(\lambda r_1) \frac{K_1(\lambda r_1)}{I_1(\lambda r_1)} + \lambda \mu_0 r_0 I e^{-j\lambda z_0} I_1(\lambda r_0) K_1'(\lambda r_1) &= \\ &= -\frac{1}{2\mu_1^m \sqrt{r_1}} B_5 (K_\nu(\lambda r_1) - 2r_1 \lambda K_\nu'(\lambda r_1)) \end{aligned} \right\| \\ & \left\| \begin{aligned} \frac{1}{2\mu_1^m \sqrt{r_1}} B_5 (K_\nu(\lambda r_1) (2\mu_1^m \lambda I_1'(\lambda r_1) + I_1(\lambda r_1)) - 2r_1 \lambda K_\nu'(\lambda r_1) I_1(\lambda r_1)) &= \\ = \lambda \mu_0 r_0 I e^{-j\lambda z_0} I_1(\lambda r_0) (K_1(\lambda r_1) I_1'(\lambda r_1) - I_1(\lambda r_1) K_1'(\lambda r_1)) \end{aligned} \right\| \\ B_5 &= \frac{2\mu_1^m}{\sqrt{r_1}} \frac{\mu_0 r_0 I e^{-j\lambda z_0} I_1(\lambda r_0)}{(2\mu_1^m \lambda I_1'(\lambda r_1) + I_1(\lambda r_1)) K_\nu(\lambda r_1) - 2r_1 \lambda K_\nu'(\lambda r_1) I_1(\lambda r_1)}, \quad (3.3.9) \end{aligned}$$

where the formula for the Wronskian $W\{K_\nu(\lambda r_1), I_\nu(\lambda r_1)\} = \frac{1}{\lambda r_1}$ can be used.

In particular, the value of B_2 is

$$B_2 = \mu_0 r_0 I e^{-j\lambda z_0} \frac{I_1(\lambda r_0)}{I_1(\lambda r_1)} \left(\frac{2\mu_1^m}{r_1} \frac{K_\nu(\lambda r_1)}{D_1 K_\nu(\lambda r_1) - 2r_1 \lambda K'_\nu(\lambda r_1) I_1(\lambda r_1)} - K_1(\lambda r_1) \right), \quad (3.3.10)$$

where $D_1 = 2\mu_1^m \lambda I'_1(\lambda r_1) + I_1(\lambda r_1)$, $\nu = \sqrt{\frac{1}{4} + p_1^2}$, $p_1 = \sqrt{j\omega\mu_0\mu_1^m\sigma_1^m}$.

The induced vector potential in region R_0 is

$$\tilde{A}_0^{ind}(r, \lambda) = B_2 I_1(\lambda r), \quad (3.3.11)$$

where B_2 is given by (3.3.10).

Applying the inverse Fourier transform (3.2.36) to (3.3.11) we obtain the induced vector potential in free space due to the presence of a cylindrical region in the form

$$A_0^{ind}(r, z) = \frac{\mu_0 r_0 I}{2\pi} \int_{-\infty}^{\infty} \tilde{B}_2 I_1(\lambda r) \frac{I_1(\lambda r_0)}{I_1(\lambda r_1)} e^{j\lambda(z-z_0)} d\lambda, \quad (3.3.12)$$

where $\tilde{B}_2 = \frac{2\mu_1^m}{r_1} \frac{K_\nu(\lambda r_1)}{D_1 K_\nu(\lambda r_1) - 2r_1 \lambda K'_\nu(\lambda r_1) I_1(\lambda r_1)} - K_1(\lambda r_1)$

The induced change in impedance of the coil is given by the formula

$$Z^{ind} = \frac{j\omega}{I} \oint_L A_0^{ind}(r, z) dl, \quad (3.3.13)$$

where L is the contour of the coil. Substituting (3.3.12) into (3.3.13) we obtain

$$Z^{ind} = j\omega\mu_0 r_0^2 \int_{-\infty}^{\infty} \tilde{B}_2 \frac{I_1^2(\lambda r_0)}{I_1(\lambda r_1)} d\lambda. \quad (3.3.14)$$

Using the following dimensionless parameters (3.3.14) can be rewritten as follows

$$\left\{ \begin{array}{l} \cdot \lambda r_1 = u \rightarrow \lambda = \frac{u}{r_1}, \\ \hat{r}_0 = \frac{r_0}{r_1}, \mu_1^m = \hat{\mu}_1^m r_1, \sigma_1^m = \hat{\sigma}_1^m r_1, \\ \cdot \hat{D}_1 = 2\hat{\mu}_1^m u I'_1(u) + I_1(u) \end{array} \right\} \Rightarrow \hat{\nu} = \sqrt{\frac{1}{4} + j\hat{p}_1^2}, \cdot \hat{p}_1 = \sqrt{\omega\mu_0\hat{\mu}_1^m\hat{\sigma}_1^m r_1^2} \quad (3.3.15)$$

$$Z^{ind}(r, z) = \omega\mu_0 \frac{r_0^2}{r_1} Z, \quad (3.3.16)$$

and

$$\hat{B}_2 = \frac{2\hat{\mu}_1^m K_{\hat{\nu}}(u)}{\hat{D}_1 K_{\hat{\nu}}(u) - 2u K'_{\hat{\nu}}(u) I_1(u)} - K_1(u).$$

Formula (3.3.16) is used to compute the change in impedance of the coil. Calculations are performed with “Mathematica” since it has built-in functions to evaluate modified Bessel functions of complex order. “Mathematica” program which is used to compute the change in impedance (3.3.16) is shown in **Appendix Fig.Ap.5**. The results of computations are shown in

Fig. 3.3 . The calculated points (from top to bottom) for each curve correspond to the following values of $\hat{p}_1 = 1, 2, \dots, 15$. The parameter \hat{p}_1 is defined by $\hat{p}_1 = \sqrt{\omega \mu_0 \hat{\mu}_1^m \hat{\sigma}_1^m r_1^2}$. The three curves in **Fig. 3.3** (from right to left) correspond to the cases $\hat{r}_0 = 0.9, 0.8$, and 0.7 , respectively.

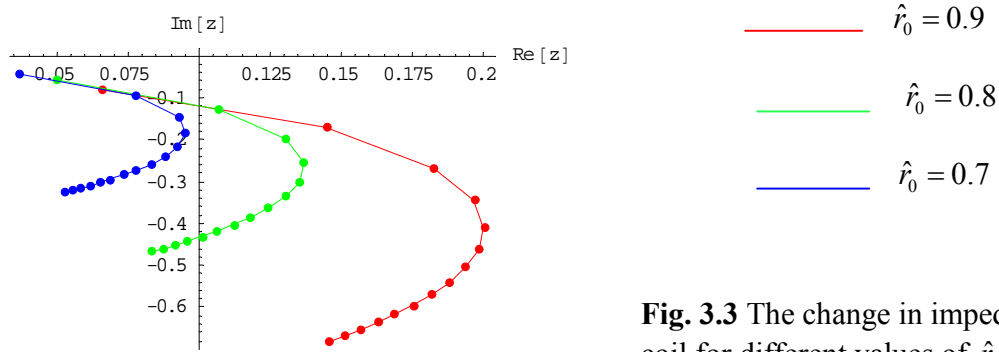


Fig. 3.3 The change in impedance of a coil for different values of \hat{r}_0 .

It is seen that the modulus of the change in impedance increases as the parameter \hat{p}_1 increases (that is, if the frequency increases). In addition, the change in impedance is larger when the coil is closer to the cylindrical region.

3.4 A coil inside a two-layer tube

Another example of important problem in applications is considered in this section. Consider a single-turn coil of radius r_0 located inside a two-layer tube [33].

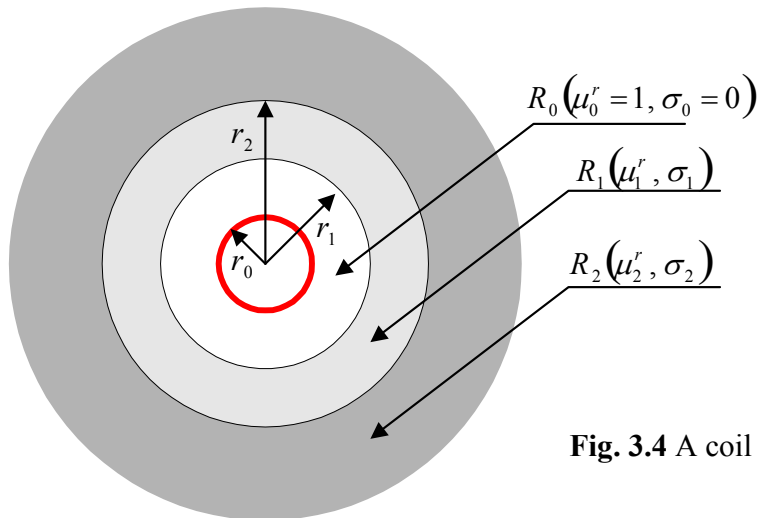


Fig. 3.4 A coil inside a two-layer tube.

Suppose that $\mu_0^r(r) = 1$, $\sigma_0(r) = 0$,

$$\mu_1^r(r) = \mu_1^m r^\beta, \quad \sigma_1(r) = \sigma_1^m r^\alpha,$$

$$\mu_2^r(r) = \mu_2^m, \quad \sigma_2(r) = \sigma_2^m.$$

The electric conductivity and magnetic permeability of the inner layer are given by formula (3.2.1) while the electric conductivity σ_2^m and magnetic permeability μ_2^m of the second layer (R_2) are constants. The outer layer is unbounded in the radial direction (see **Fig. 3.4**).

Using the results from **Section 3.2** we obtain the following equations:

$$R_0: \frac{\partial^2 A_0}{\partial r^2} + \frac{1}{r} \frac{\partial A_0}{\partial r} - \frac{A_0}{r^2} + \frac{\partial^2 A_0}{\partial z^2} = -\mu_0 I \delta(r-r_0) \delta(z-z_0), \quad (3.4.1)$$

$$R_1: \frac{\partial^2 A_1}{\partial r^2} + \frac{1-\beta}{r} \frac{\partial A_1}{\partial r} - \left(\frac{1+\beta}{r^2} + p_1^2 r^{\alpha+\beta} \right) A_1 + \frac{\partial^2 A_1}{\partial z^2} = 0, \quad (3.4.2)$$

where $p_1 = \sqrt{j\omega\mu_0\mu_1^m\sigma_1^m}$.

The equation in region R_2 is

$$R_2: \frac{\partial^2 A_2}{\partial r^2} + \frac{1}{r} \frac{\partial A_2}{\partial r} - \left(\frac{1}{r^2} + p_2^2 \right) A_2 + \frac{\partial^2 A_2}{\partial z^2} = 0, \quad (3.4.3)$$

where $p_2 = \sqrt{j\omega\mu_0\mu_2^m\sigma_2^m}$.

The boundary conditions are

$$A_0|_{r=r_1} = A_1|_{r=r_1}, \quad \frac{\partial A_0}{\partial r}|_{r=r_1} = \frac{1}{\mu_1^r(r_1)} \frac{\partial A_1}{\partial r}|_{r=r_1}, \quad (3.4.4)$$

$$A_1|_{r=r_2} = A_2|_{r=r_2}, \quad \frac{1}{\mu_1^r(r_2)} \frac{\partial A_1}{\partial r}|_{r=r_2} = \frac{1}{\mu_2^m} \frac{\partial A_2}{\partial r}|_{r=r_2}. \quad (3.4.5)$$

In addition, A_0 is bounded at $r=0$ and A_2 is bounded as $r \rightarrow \infty$.

The following conditions hold at infinity:

$$A_i, \frac{\partial A_i}{\partial z} \rightarrow 0 \text{ as } z \rightarrow \pm\infty, \quad i=0,1,2. \quad (3.4.6)$$

Applying the Fourier transform (3.2.9) to problem (3.4.1) - (3.4.6) we obtain the solution in region R_0 :

$$R_0: \frac{d^2 \tilde{A}_0}{dr^2} + \frac{1}{r} \frac{d\tilde{A}_0}{dr} - \left(\frac{1}{r^2} + \lambda^2 \right) \tilde{A}_0 = -\mu_0 I e^{-j\lambda z_0} \delta(r-r_0),$$

$$\rightarrow \begin{cases} \tilde{A}_{00}(r, \lambda) = B_2 I_1(\lambda r) + \mu_0 r_0 I e^{-j\lambda z_0} K_1(\lambda r_0) I_1(\lambda r), \\ \tilde{A}_{01}(r, \lambda) = B_2 I_1(\lambda r) + \mu_0 r_0 I e^{-j\lambda z_0} I_1(\lambda r_0) K_1(\lambda r). \end{cases} \quad (3.4.7)$$

The transformed equation in region R_1 is

$$R_1: \frac{d^2 \tilde{A}_1}{dr^2} + \frac{(1-\beta)}{r} \frac{d\tilde{A}_1}{dr} - \left(\frac{1+\beta}{r^2} + p_1^2 r^{\alpha+\beta} + \lambda^2 \right) \tilde{A}_1 = 0.$$

The general solution for the case $\alpha = -1, \beta = -1$ has the form

$$\rightarrow \tilde{A}_1(r, \lambda) = \frac{1}{\sqrt{r}} (B_4 I_\nu(\lambda r) + B_5 K_\nu(\lambda r)), \quad (3.4.8)$$

where $p_1 = \sqrt{j\omega\mu_0\mu_1^m\sigma_1^m}$, $\nu = \sqrt{\frac{1}{4} + p_1^2}$.

The transformed equation in region R_2 is

$$R_2: \frac{d^2 \tilde{A}_2}{dr^2} + \frac{1}{r} \frac{d\tilde{A}_2}{dr} - \left(\frac{1}{r^2} + q^2 \right) \tilde{A}_2 = 0,$$

so that the bounded solution of the above equation as $r \rightarrow \infty$ is given by

$$\rightarrow \tilde{A}_2(r, \lambda) = B_6 K_1(qr), \quad (3.4.9)$$

where $q = \sqrt{p_2^2 + \lambda^2}$, $p_2 = \sqrt{j\omega\mu_0\mu_2^m\sigma_2^m}$.

The boundary conditions are

$$\tilde{A}_0|_{r=r_1} = \tilde{A}_1|_{r=r_1}, \quad \frac{d\tilde{A}_0}{dr}|_{r=r_1} = \frac{1}{\mu_1^r(r_1)} \frac{d\tilde{A}_1}{dr}|_{r=r_1}, \quad (3.4.10)$$

$$\tilde{A}_1|_{r=r_2} = \tilde{A}_2|_{r=r_2}, \quad \frac{1}{\mu_1^r(r_2)} \frac{d\tilde{A}_1}{dr}|_{r=r_2} = \frac{1}{\mu_2^m} \frac{d\tilde{A}_2}{dr}|_{r=r_2}. \quad (3.4.11)$$

Using (3.4.7) – (3.4.11) we obtain

$$B_2 I_1(\lambda r_1) + \mu_0 r_0 I e^{-j\lambda z_0} I_1(\lambda r_0) K_1(\lambda r_1) = \frac{B_4}{\sqrt{r_1}} I_\nu(\lambda r_1) + \frac{B_5}{\sqrt{r_1}} K_\nu(\lambda r_1), \quad (3.4.12)$$

$$\lambda B_2 I_1'(\lambda r_1) + \lambda \mu_0 r_0 I e^{-j\lambda z_0} I_1(\lambda r_0) K_1'(\lambda r_1) = \frac{r_1}{\mu_1^m} \left(-\frac{1}{2r_1 \sqrt{r_1}} \right) \cdot (B_4 (I_\nu(\lambda r_1) - 2r_1 \lambda I_\nu'(\lambda r_1)) + B_5 (K_\nu(\lambda r_1) - 2r_1 \lambda K_\nu'(\lambda r_1))), \quad (3.4.13)$$

$$B_6 K_1(qr_2) = \frac{B_4}{\sqrt{r_2}} I_\nu(\lambda r_2) + \frac{B_5}{\sqrt{r_2}} K_\nu(\lambda r_2), \quad (3.4.14)$$

$$\frac{B_6}{\mu_2^m} q K_1'(qr_2) = \frac{r_2}{\mu_1^m} \left(-\frac{1}{2r_2 \sqrt{r_2}} \right) \cdot (B_4 (I_\nu(\lambda r_2) - 2r_2 \lambda I_\nu'(\lambda r_2)) + B_5 (K_\nu(\lambda r_2) - 2r_2 \lambda K_\nu'(\lambda r_2))). \quad (3.4.15)$$

It follows from (3.4.12) and (3.4.13) that

$$\left\| \begin{aligned} \cdot B_2 &= \frac{B_4}{\sqrt{r_1}} \frac{I_\nu(\lambda r_1)}{I_1(\lambda r_1)} + \frac{B_5}{\sqrt{r_1}} \frac{K_\nu(\lambda r_1)}{I_1(\lambda r_1)} - \mu_0 r_0 I e^{-j\lambda z_0} I_1(\lambda r_0) \frac{K_1(\lambda r_1)}{I_1(\lambda r_1)} \\ \cdot \frac{B_4 \lambda}{\sqrt{r_1}} \frac{I_\nu(\lambda r_1)}{I_1(\lambda r_1)} I_1'(\lambda r_1) + \frac{B_5 \lambda}{\sqrt{r_1}} \frac{K_\nu(\lambda r_1)}{I_1(\lambda r_1)} I_1'(\lambda r_1) - \lambda \mu_0 r_0 I e^{-j\lambda z_0} I_1(\lambda r_0) \frac{K_1(\lambda r_1)}{I_1(\lambda r_1)} I_1'(\lambda r_1) + \\ &+ \lambda \mu_0 r_0 I e^{-j\lambda z_0} I_1(\lambda r_0) K_1'(\lambda r_1) = \frac{r_1}{\mu_1^m} \left(-\frac{1}{2r_1 \sqrt{r_1}} \right) (B_4 (I_\nu(\lambda r_1) - 2r_1 \lambda I_\nu'(\lambda r_1)) \\ &+ B_5 (K_\nu(\lambda r_1) - 2r_1 \lambda K_\nu'(\lambda r_1))) \\ \cdot \frac{B_4}{2\mu_1^m \sqrt{r_1}} (I_\nu(\lambda r_1) (2\mu_1^m \lambda I_1'(\lambda r_1) + I_1(\lambda r_1)) - 2r_1 \lambda I_\nu'(\lambda r_1) I_1(\lambda r_1)) = \\ &= -\frac{B_5}{2\mu_1^m \sqrt{r_1}} (K_\nu(\lambda r_1) (2\mu_1^m \lambda I_1'(\lambda r_1) + I_1(\lambda r_1)) - 2r_1 \lambda K_\nu'(\lambda r_1) I_1(\lambda r_1)) + \\ &+ \lambda \mu_0 r_0 I e^{-j\lambda z_0} I_1(\lambda r_0) (K_1(\lambda r_1) I_1'(\lambda r_1) - I_1(\lambda r_1) K_1'(\lambda r_1)) \end{aligned} \right\|$$

$$B_4 = -B_5 \frac{K_v(\lambda r_1)(2\mu_1^m \lambda I_1'(\lambda r_1) + I_1(\lambda r_1)) - 2r_1 \lambda K_v'(\lambda r_1) I_1(\lambda r_1)}{I_v(\lambda r_1)(2\mu_1^m \lambda I_1'(\lambda r_1) + I_1(\lambda r_1)) - 2r_1 \lambda I_v'(\lambda r_1) I_1(\lambda r_1)} + \frac{2\mu_1^m}{\sqrt{r_1}} \frac{\mu_0 r_0 I e^{-j\lambda z_0} I_1(\lambda r_0)}{I_v(\lambda r_1)(2\mu_1^m \lambda I_1'(\lambda r_1) + I_1(\lambda r_1)) - 2r_1 \lambda I_v'(\lambda r_1) I_1(\lambda r_1)}, \quad (3.4.16)$$

where $W\{K_1(\lambda r_1), I_1(\lambda r_1)\} = \frac{1}{\lambda r_1}$.

Using (3.4.14) and (3.4.15) we obtain

$$\left\| \begin{aligned} \cdot B_6 &= \frac{B_4}{\sqrt{r_2}} \frac{I_v(\lambda r_2)}{K_1(qr_2)} + \frac{B_5}{\sqrt{r_2}} \frac{K_v(\lambda r_2)}{K_1(qr_2)} \\ \cdot \frac{B_4 q}{\mu_2^m \sqrt{r_2}} \frac{I_v(\lambda r_2)}{K_1(qr_2)} K_1'(qr_2) + \frac{B_5 q}{\mu_2^m \sqrt{r_2}} \frac{K_v(\lambda r_2)}{K_1(qr_2)} K_1'(qr_2) &= -\frac{B_4}{2\mu_1^m \sqrt{r_2}} (I_v(\lambda r_2) - 2r_2 \lambda I_v'(\lambda r_2)) - \\ &\quad - \frac{B_5}{2\mu_1^m \sqrt{r_2}} (K_v(\lambda r_2) - 2r_2 \lambda K_v'(\lambda r_2)) \\ \cdot B_4 \frac{1}{2\mu_1^m \mu_2^m} \left(\mu_2^m I_v(\lambda r_2) - 2r_2 \lambda \mu_2^m I_v'(\lambda r_2) + 2\mu_1^m q I_v(\lambda r_2) \frac{K_1'(qr_2)}{K_1(qr_2)} \right) &= \\ &= -B_5 \frac{1}{2\mu_1^m \mu_2^m} \left(\mu_2^m K_v(\lambda r_2) - 2r_2 \lambda \mu_2^m K_v'(\lambda r_2) + 2\mu_1^m q K_v(\lambda r_2) \frac{K_1'(qr_2)}{K_1(qr_2)} \right) \end{aligned} \right\|$$

$$B_4 = -B_5 \frac{\mu_2^m K_1(qr_2)(K_v(\lambda r_2) - 2r_2 \lambda K_v'(\lambda r_2)) + 2\mu_1^m q K_v(\lambda r_2) K_1'(qr_2)}{\mu_2^m K_1(qr_2)(I_v(\lambda r_2) - 2r_2 \lambda I_v'(\lambda r_2)) + 2\mu_1^m q I_v(\lambda r_2) K_1'(qr_2)}. \quad (3.4.17)$$

Solving equations (3.4.16) and (3.4.17) we obtain

$$B_5 = \frac{2\mu_1^m}{\sqrt{r_1}} \mu_0 r_0 I e^{-j\lambda z_0} I_1(\lambda r_0) \frac{C_1}{D}, \quad (3.4.18)$$

where

$$C_1 = \mu_2^m K_1(qr_2)(I_v(\lambda r_2) - 2r_2 \lambda I_v'(\lambda r_2)) + 2\mu_1^m q I_v(\lambda r_2) K_1'(qr_2),$$

$$D = \mu_2^m K_1(qr_2) D_5 (D_1 - 2r_2 \lambda D_2) + 2\mu_1^m q K_1'(qr_2) D_1 D_5 -$$

$$- 2r_1 \lambda \mu_2^m K_1(qr_2) I_1(\lambda r_1) (D_3 - 2r_2 \lambda D_4) - 4r_1 \mu_1^m \lambda q K_1'(qr_2) I_1(\lambda r_1) D_3.$$

$$\left\| \begin{aligned} D_1 &= I_v(\lambda r_2) K_v(\lambda r_1) - I_v(\lambda r_1) K_v(\lambda r_2) \\ D_2 &= I_v'(\lambda r_2) K_v(\lambda r_1) - I_v(\lambda r_1) K_v'(\lambda r_2) \\ D_3 &= I_v(\lambda r_2) K_v'(\lambda r_1) - I_v'(\lambda r_1) K_v(\lambda r_2) \\ D_4 &= I_v'(\lambda r_2) K_v'(\lambda r_1) - I_v'(\lambda r_1) K_v'(\lambda r_2) \\ D_5 &= 2\mu_1^m \lambda I_1'(\lambda r_1) + I_1(\lambda r_1) \end{aligned} \right\|$$

The constant B_4 is found from (3.4.17) and (3.4.18):

$$B_4 = -\frac{2\mu_1^m}{\sqrt{r_1}} \mu_0 r_0 I e^{-j\lambda z_0} I_1(\lambda r_0) \frac{C_2}{D}, \quad (3.4.19)$$

where

$$C_2 = \mu_2^m K_1(qr_2) (K_v(\lambda r_2) - 2r_2 \lambda K_v'(\lambda r_2)) + 2\mu_1^m q K_v(\lambda r_2) K_1'(qr_2).$$

The value of B_2 is

$$\left\| \cdot B_2 = \frac{B_4}{\sqrt{r_1}} \frac{I_v(\lambda r_1)}{I_1(\lambda r_1)} + \frac{B_5}{\sqrt{r_1}} \frac{K_v(\lambda r_1)}{I_1(\lambda r_1)} - \mu_0 r_0 I e^{-j\lambda z_0} I_1(\lambda r_0) \frac{K_1(\lambda r_1)}{I_1(\lambda r_1)} \right\|$$

$$B_2 = \mu_0 r_0 I e^{-j\lambda z_0} \frac{I_1(\lambda r_0)}{I_1(\lambda r_1)} \left(\frac{2\mu_1^m}{r_1} \frac{\mu_2^m K_1(qr_2) (D_1 - 2r_2 \lambda D_2) + 2\mu_1^m q K_1'(qr_2) D_1}{D} - K_1(\lambda r_1) \right). \quad (3.4.20)$$

The induced vector potential in region R_0 is

$$\tilde{A}_0^{ind}(r, \lambda) = B_2 I_1(\lambda r), \quad (3.4.21)$$

where B_2 is given by (3.4.20).

Applying the inverse Fourier transform (3.2.36) to (3.4.21) we obtain the induced vector potential in free space due to the presence of a cylindrical region in the form

$$A_0^{ind}(r, z) = \frac{\mu_0 r_0 I}{2\pi} \int_{-\infty}^{\infty} \tilde{B}_2 I_1(\lambda r) \frac{I_1(\lambda r_0)}{I_1(\lambda r_1)} e^{j\lambda(z-z_0)} d\lambda, \quad (3.4.22)$$

$$\text{where } \tilde{B}_2 = \frac{2\mu_1^m}{r_1} \frac{\mu_2^m K_1(qr_2) (D_1 - 2r_2 \lambda D_2) + 2\mu_1^m q K_1'(qr_2) D_1}{D} - K_1(\lambda r_1).$$

The induced change in impedance of the coil is given by the formula

$$Z^{ind} = \frac{j\omega}{I} \oint_L A_0^{ind}(r, z) dl, \quad (3.4.23)$$

where L is the contour of the coil. Substituting (3.4.22) into (3.4.23) we obtain

$$Z^{ind} = j\omega \mu_0 r_0^2 \int_{-\infty}^{\infty} \tilde{B}_2 \frac{I_1^2(\lambda r_0)}{I_1(\lambda r_1)} d\lambda. \quad (3.4.24)$$

Formula (3.4.24) can be rewritten as follows

$$\left\| \begin{aligned} \cdot \lambda r_1 = u \rightarrow \lambda = \frac{u}{r_1}, \\ \hat{r}_0 = \frac{r_0}{r_1}, \mu_1^m = \hat{\mu}_1^m r_1, \sigma_1^m = \hat{\sigma}_1^m r_1, \end{aligned} \right\} \Rightarrow \hat{v} = \sqrt{\frac{1}{4} + j\hat{p}_1^2}, \cdot \hat{p}_1 = \sqrt{\omega \mu_0 \hat{\mu}_1^m \hat{\sigma}_1^m r_1^2}$$

$$\cdot \hat{r}_2 = \frac{r_2}{r_1}, \hat{q} = \sqrt{u^2 + j\hat{p}_2^2}, \hat{p}_2 = \sqrt{\omega \mu_0 \mu_2^m \sigma_2^m r_1^2} = \sqrt{\omega \mu_0 r_1^2} \sqrt{\mu_2^m \sigma_2^m} = \hat{p}_1 \sqrt{\frac{\mu_2^m \sigma_2^m}{\hat{\mu}_1^m \hat{\sigma}_1^m}}$$

$$\cdot \hat{D}_1 = I_{\hat{v}}(u\hat{r}_2) K_{\hat{v}}(u) - I_{\hat{v}}(u) K_{\hat{v}}(u\hat{r}_2), \hat{D}_2 = I_{\hat{v}}'(u\hat{r}_2) K_{\hat{v}}(u) - I_{\hat{v}}(u) K_{\hat{v}}'(u\hat{r}_2)$$

$$\cdot \hat{D}_3 = I_{\hat{v}}(u\hat{r}_2) K_{\hat{v}}'(u) - I_{\hat{v}}'(u) K_{\hat{v}}(u\hat{r}_2), \hat{D}_4 = I_{\hat{v}}'(u\hat{r}_2) K_{\hat{v}}'(u) - I_{\hat{v}}'(u) K_{\hat{v}}'(u\hat{r}_2)$$

$$\cdot \hat{D}_5 = 2\hat{\mu}_1^m u I_1'(u) + I_1(u), \hat{D}_6 = \mu_2^m K_1(\hat{q}\hat{r}_2) (\hat{D}_1 - 2u\hat{r}_2 \hat{D}_2)$$

$$\cdot \hat{D}_7 = 2\hat{\mu}_1^m \hat{q} K_1'(\hat{q}\hat{r}_2) \hat{D}_1$$

$$Z^{ind}(r, z) = \omega \mu_0 \frac{r_0^2}{r_1} Z \quad (3.4.25)$$

$$\text{where } Z = j \int_{-\infty}^{\infty} \hat{B}_2 \frac{I_1^2(u \hat{r}_0)}{I_1(u)} du \quad (3.4.26)$$

and

$$\hat{B}_2 = \frac{2\hat{\mu}_1^m (\hat{D}_6 + \hat{D}_7)}{\hat{D}_5 (\hat{D}_6 + \hat{D}_7) - 2u\mu_2^m K_1(\hat{q}\hat{r}_2) I_1(u) (\hat{D}_3 - 2u\hat{r}_2 \hat{D}_4) - 4\hat{\mu}_1^m u \hat{q} K_1'(\hat{q}\hat{r}_2) I_1(u) \hat{D}_3} - K_1(u).$$

Formula (3.4.26) is used to compute the change in impedance of the coil for different values of the parameters of the problem. **Fig. 3.5** plots the real and imaginary parts of the change in impedance for three different values of \hat{r}_2 : 1.1, 1.2 and 1.3.

The curves correspond to the following values of $\hat{p}_1 = 3, 4, \dots, 10$ (from top to bottom).

The other parameters are $\mu_1 = 1, \hat{\mu}_2 = 6, \hat{r}_0 = 0.9, \sqrt{\frac{\mu_2^m \sigma_2^m}{\hat{\mu}_1^m \hat{\sigma}_1^m}} = 1.5$. Computations are done with

Mathematica (see **Appendix Fig.Ap.6**).

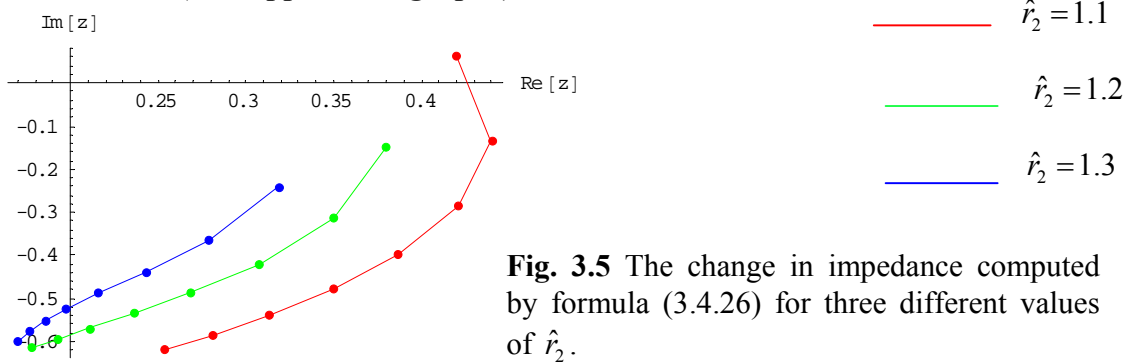


Fig. 3.5 The change in impedance computed by formula (3.4.26) for three different values of \hat{r}_2 .

It is seen from **Fig. 3.5** that for higher frequencies (larger values of \hat{p}_1) the modulus of the change in impedance decreases. The decrease is related to smaller values of the real part of the change in impedance. The values of Z for three different values of $\hat{\mu}_2$, namely, $\hat{\mu}_2 = 2, 4, 6$ are plotted in **Fig. 3.6**. The points on each curve correspond to the following values of $\hat{p}_1 = 3, 4, \dots, 10$ (from top to bottom).

The other parameters are set at $\mu_1 = 1, \hat{r}_2 = 1.1, \hat{r}_0 = 0.9, \sqrt{\frac{\mu_2^m \sigma_2^m}{\hat{\mu}_1^m \hat{\sigma}_1^m}} = 1.5$.

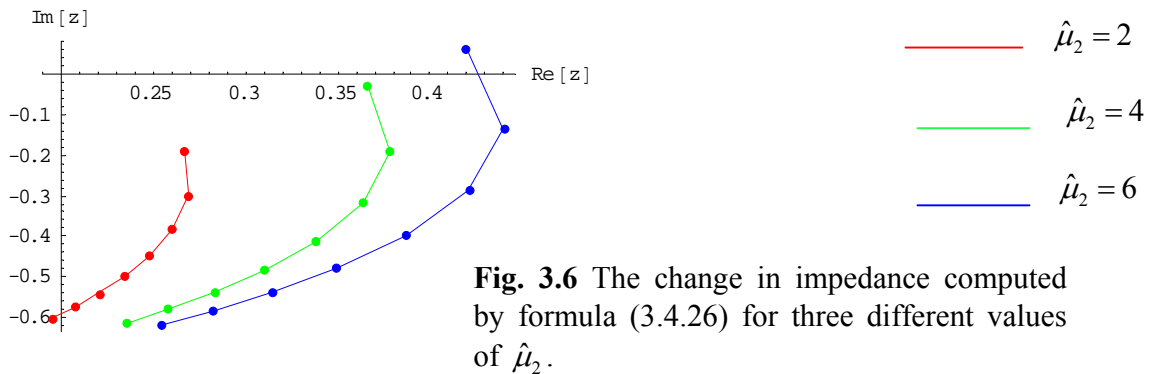


Fig. 3.6 The change in impedance computed by formula (3.4.26) for three different values of $\hat{\mu}_2$.

It follows from **Fig. 3.6** that for high frequencies (large values of \hat{p}_1) the calculated points are very close to one another as the parameter $\hat{\mu}_2$ increases.

3.5 A coil outside a multilayer tube

There are many different types of eddy current probes that are used to control the properties of objects with cylindrical symmetry. Encircling coils are widely used for inspecting cylinders, rods or tubes. Mathematical models described in the previous sections can also be applied for the case of encircling coils. Consider a multilayer tube described in **Section 3.2**. We assume that a single-turn coil of radius r_0 is located outside the tube ($r_0 > r_1$). The axis of the coil coincides with the axis of the tube [35] (see **Fig. 3.7**).

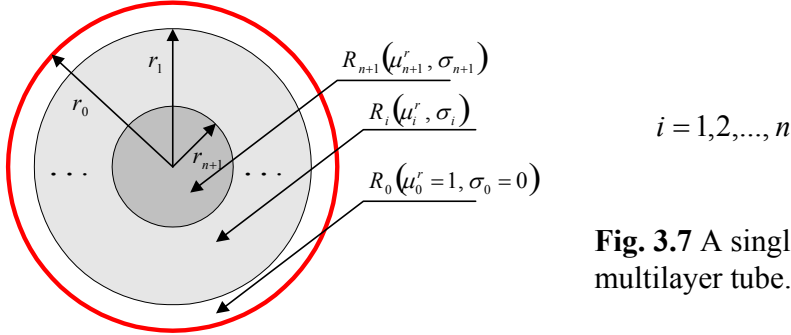


Fig. 3.7 A single-turn coil outside a multilayer tube.

The system of equations for the amplitudes of the vector potential in each region $R_i = \{r_i \leq r \leq r_{i+1}, 0 \leq \varphi \leq 2\pi, -\infty < z < +\infty\}, i = 1, 2, \dots, n$ is given by (3.2.2), (3.2.4) and (3.2.5) where r_0 (the radius of the coil) is larger than r_1 (the radius of the inner cylinder of the tube). The solution procedure is essentially the same as in **Section 3.2** with minor modifications. Applying the Fourier transform to (3.2.2), (3.2.4) and (3.2.5) we obtain the system of equations in the form (3.2.13) – (3.2.15). The boundary conditions (3.2.16), (3.2.17) are the same. In addition, \tilde{A}_0 is bounded at $r \rightarrow \infty$ and \tilde{A}_{n+1} is bounded as $r = 0$. As a result, the solution in regions R_0 and R_{n+1} has to be modified. We consider two sub-regions of region R_0 , namely, R_{00} and R_{01} , which are defined as follows: $R_{00} = \{r_1 \leq r < r_0, 0 \leq \varphi \leq 2\pi, -\infty < z < +\infty\}$, $R_{01} = \{r > r_0, 0 \leq \varphi \leq 2\pi, -\infty < z < +\infty\}$. The solutions in R_{00} and R_{01} are denoted, as before, by \tilde{A}_{00} and \tilde{A}_{01} , respectively. Solving (3.2.13) in regions R_{00} and R_{01} we obtain

$$R_0: \frac{d^2 \tilde{A}_0}{dr^2} + \frac{1}{r} \frac{d\tilde{A}_0}{dr} - \left(\frac{1}{r^2} + \lambda^2 \right) \tilde{A}_0 = -\mu_0 I e^{-j\lambda z_0} \delta(r - r_0), \quad (3.5.1)$$

$$\rightarrow \begin{cases} \tilde{A}_{00}(r, \lambda) = B_1 I_1(\lambda r) + B_2 K_1(\lambda r), \\ \tilde{A}_{01}(r, \lambda) = B_3 K_1(\lambda r). \end{cases} \quad (3.5.2)$$

The functions $\tilde{A}_{00}(r, \lambda)$ and $\tilde{A}_{01}(r, \lambda)$ satisfy the following conditions at $r = r_0$:

$$\tilde{A}_{00} |_{r=r_0} = \tilde{A}_{01} |_{r=r_0}. \quad (3.5.3)$$

The condition (3.5.3) reflects the fact that the function $\tilde{A}_0(r, \lambda)$ is continuous at $r = r_0$.

The second condition is obtained by integrating (3.5.1) with respect to r from $r = r_0 - \varepsilon$ to $r = r_0 + \varepsilon$ and considering the limit in the resulting expression as $\varepsilon \rightarrow +0$:

$$\left. \frac{d\tilde{A}_{01}}{dr} \right|_{r=r_0} - \left. \frac{d\tilde{A}_{00}}{dr} \right|_{r=r_0} = -\mu_0 I e^{-j\lambda z_0}. \quad (3.5.4)$$

Using (3.5.2) – (3.5.4) we obtain

$$\begin{cases} B_1 I_1(\lambda r_0) + B_2 K_1(\lambda r_0) = B_3 K_1(\lambda r_0), \\ \lambda B_3 K_1'(\lambda r_0) - \lambda B_1 I_1'(\lambda r_0) - \lambda B_2 K_1'(\lambda r_0) = -\mu_0 I e^{-j\lambda z_0}, \end{cases} \Rightarrow$$

$$\begin{cases} B_3 = B_2 + B_1 \frac{I_1(\lambda r_0)}{K_1(\lambda r_0)}, \\ B_2 K_1'(\lambda r_0) + B_1 \frac{I_1(\lambda r_0) K_1'(\lambda r_0)}{K_1(\lambda r_0)} - B_1 I_1'(\lambda r_0) - B_2 K_1'(\lambda r_0) = -\frac{\mu_0}{\lambda} I e^{-j\lambda z_0}, \end{cases} \Rightarrow$$

$$\left\| \begin{aligned} & \cdot B_1 (I_1(\lambda r_0) K_1'(\lambda r_0) - I_1'(\lambda r_0) K_1(\lambda r_0)) = -\frac{\mu_0}{\lambda} I e^{-j\lambda z_0} K_1(\lambda r_0) \\ & W\{I_v(\lambda r_0), K_v(\lambda r_0)\} = -\frac{1}{\lambda r_0} \end{aligned} \right\|$$

$$\begin{cases} B_1 = \mu_0 r_0 I e^{-j\lambda z_0} K_1(\lambda r_0), \\ B_3 = B_2 + \mu_0 r_0 I e^{-j\lambda z_0} I_1(\lambda r_0). \end{cases} \quad (3.5.5)$$

It follows from (3.5.2) and (3.5.5) that

$$\tilde{A}_{00}(r, \lambda) = B_2 K_1(\lambda r) + \mu_0 r_0 I e^{-j\lambda z_0} K_1(\lambda r_0) I_1(\lambda r), \quad (3.5.6)$$

$$\tilde{A}_{01}(r, \lambda) = B_2 K_1(\lambda r) + \mu_0 r_0 I e^{-j\lambda z_0} I_1(\lambda r_0) K_1(\lambda r). \quad (3.5.7)$$

The solution in other conducting layers can be constructed as in **Section 3.2**.

For example, if $\alpha = -1$, $\beta = -1$ the solution is

$$R_i: \quad \frac{d^2 \tilde{A}_i}{dr^2} + \frac{(1 - \beta_i)}{r} \frac{d\tilde{A}_i}{dr} - \left(\frac{1 + \beta_i}{r^2} + p_i^2 r^{\alpha_i + \beta_i} + \lambda^2 \right) \tilde{A}_i = 0,$$

$$\rightarrow \tilde{A}_i(r, \lambda) = \frac{1}{\sqrt{r}} (B_{4i} I_{\nu_i}(\lambda r) + B_{5i} K_{\nu_i}(\lambda r)) \quad (3.5.8)$$

where $\nu = \sqrt{\frac{1}{4} + p_i^2}$, $p_i = \sqrt{j\omega\mu_0\mu_i^m\sigma_i^m}$, $i = 1, 2, \dots, n$

In region R_{n+1} we have

$$R_{n+1}: \quad \frac{d^2 \tilde{A}_{n+1}}{dr^2} + \frac{1}{r} \frac{d\tilde{A}_{n+1}}{dr} - \left(\frac{1}{r^2} + q^2 \right) \tilde{A}_{n+1} = 0,$$

$$\rightarrow \tilde{A}_{n+1}(r, \lambda) = B_6 I_1(qr), \quad (3.5.9)$$

where $q = \sqrt{p_{n+1}^2 + \lambda^2}$, $p_2 = \sqrt{j\omega\mu_0\mu_{n+1}^m\sigma_{n+1}^m}$.

Then the solution in each region R_i , $i = 0, 1, \dots, n+1$ can be found by means of the inverse Fourier transform of the form

$$A_i(r, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{A}_i(r, \lambda) e^{j\lambda z} d\lambda. \quad (3.5.10)$$

It can be shown that the induced vector potential in free space due to multilayer conducting tubes has the form

$$A_0^{ind}(r, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} B_2 K_1(\lambda r) e^{j\lambda z} d\lambda. \quad (3.5.11)$$

As an example we consider the case where a single-turn coil of radius r_0 is located outside a two-layer coaxial conducting tube. The conducting layer (region R_1) is defined by the inequalities: $R_1 = \{r_1 \leq r \leq r_2, 0 \leq \varphi \leq 2\pi, -\infty < z < +\infty\}$. The electric conductivity and magnetic permeability of region R_1 are given by (3.2.1). The properties of region $R_2 = \{0 \leq r \leq r_2, 0 \leq \varphi \leq 2\pi, -\infty < z < +\infty\}$, σ_2 and μ_2 , are assumed to be constant. (see **Fig. 3.8**)

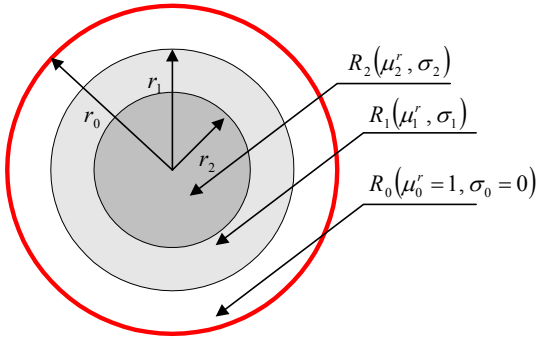


Fig. 3.8 A single-turn coil outside a two-layer tube.

The solutions in regions R_0, R_1 and R_2 can be written as follows

$$R_0: \rightarrow \begin{cases} \tilde{A}_{00}(r, \lambda) = B_2 K_1(\lambda r) + \mu_0 r_0 I e^{-j\lambda z_0} K_1(\lambda r_0) I_1(\lambda r), \\ \tilde{A}_{01}(r, \lambda) = B_2 K_1(\lambda r) + \mu_0 r_0 I e^{-j\lambda z_0} I_1(\lambda r_0) K_1(\lambda r), \end{cases} \quad (3.5.12)$$

$$R_1: \rightarrow \tilde{A}_1(r, \lambda) = \frac{1}{\sqrt{r}} (B_4 I_v(\lambda r) + B_5 K_v(\lambda r)), \quad (3.5.13)$$

$$R_2: \rightarrow \tilde{A}_2(r, \lambda) = B_6 I_1(qr). \quad (3.5.14)$$

Using (3.5.12) - (3.5.14) and the boundary conditions we obtain

$$B_2 K_1(\lambda r_1) + \mu_0 r_0 I e^{-j\lambda z_0} K_1(\lambda r_0) I_1(\lambda r_1) = \frac{B_4}{\sqrt{r_1}} I_v(\lambda r_1) + \frac{B_5}{\sqrt{r_1}} K_v(\lambda r_1), \quad (3.5.15)$$

$$\lambda B_2 K_1'(\lambda r_1) + \lambda \mu_0 r_0 I e^{-j\lambda z_0} K_1(\lambda r_0) I_1'(\lambda r_1) = \frac{r_1}{\mu_1^m} \left(-\frac{1}{2r_1 \sqrt{r_1}} \right) \cdot (B_4 (I_v(\lambda r_1) - 2r_1 \lambda I_v'(\lambda r_1)) + B_5 (K_v(\lambda r_1) - 2r_1 \lambda K_v'(\lambda r_1))), \quad (3.5.16)$$

$$B_6 I_1(qr_2) = \frac{B_4}{\sqrt{r_2}} I_v(\lambda r_2) + \frac{B_5}{\sqrt{r_2}} K_v(\lambda r_2), \quad (3.5.17)$$

$$\frac{B_6}{\mu_2^m} q I_1'(qr_2) = \frac{r_2}{\mu_1^m} \left(-\frac{1}{2r_2 \sqrt{r_2}} \right) \cdot (B_4 (I_v(\lambda r_2) - 2r_2 \lambda I_v'(\lambda r_2)) + B_5 (K_v(\lambda r_2) - 2r_2 \lambda K_v'(\lambda r_2))) \quad (3.5.18)$$

Eliminating B_2 from (3.5.15) and (3.5.16) we get

$$B_4 = -B_5 \frac{K_v(\lambda r_1) (2\mu_1^m \lambda K_1'(\lambda r_1) + K_1(\lambda r_1)) - 2r_1 \lambda K_v'(\lambda r_1) K_1(\lambda r_1)}{I_v(\lambda r_1) (2\mu_1^m \lambda K_1'(\lambda r_1) + K_1(\lambda r_1)) - 2r_1 \lambda I_v'(\lambda r_1) K_1(\lambda r_1)} - \frac{2\mu_1^m}{\sqrt{r_1}} \frac{\mu_0 r_0 I e^{-j\lambda z_0} K_1(\lambda r_0)}{I_v(\lambda r_1) (2\mu_1^m \lambda K_1'(\lambda r_1) + K_1(\lambda r_1)) - 2r_1 \lambda I_v'(\lambda r_1) K_1(\lambda r_1)}, \quad (3.5.19)$$

where $W\{I_1(\lambda r_1), K_1(\lambda r_1)\} = -\frac{1}{\lambda r_1}$.

Eliminating B_6 from (3.5.17) and (3.5.18) we obtain

$$\left\{ \begin{aligned} \cdot B_6 &= \frac{B_4}{\sqrt{r_2}} \frac{I_v(\lambda r_2)}{I_1(qr_2)} + \frac{B_5}{\sqrt{r_2}} \frac{K_v(\lambda r_2)}{I_1(qr_2)}, \\ \cdot \frac{B_4 q}{\mu_2^m \sqrt{r_2}} \frac{I_v(\lambda r_2)}{I_1(qr_2)} I_1'(qr_2) + \frac{B_5 q}{\mu_2^m \sqrt{r_2}} \frac{K_v(\lambda r_2)}{I_1(qr_2)} I_1'(qr_2) &= -\frac{B_4}{2\mu_1^m \sqrt{r_2}} (I_v(\lambda r_2) - 2r_2 \lambda I_v'(\lambda r_2)) - \\ &\quad - \frac{B_5}{2\mu_1^m \sqrt{r_2}} (K_v(\lambda r_2) - 2r_2 \lambda K_v'(\lambda r_2)), \\ \cdot B_4 \frac{1}{2\mu_1^m \mu_2^m} \left(\mu_2^m I_v(\lambda r_2) - 2r_2 \lambda \mu_2^m I_v'(\lambda r_2) + 2\mu_1^m q \frac{I_v(\lambda r_2)}{I_1(qr_2)} I_1'(qr_2) \right) &= \\ &= -B_5 \frac{1}{2\mu_1^m \mu_2^m} \left(\mu_2^m K_v(\lambda r_2) - 2r_2 \lambda \mu_2^m K_v'(\lambda r_2) + 2\mu_1^m q \frac{K_v(\lambda r_2)}{I_1(qr_2)} I_1'(qr_2) \right), \end{aligned} \right. \quad (3.5.20)$$

$$B_4 = -B_5 \frac{\mu_2^m I_1(qr_2) (K_v(\lambda r_2) - 2r_2 \lambda K_v'(\lambda r_2)) + 2\mu_1^m q K_v(\lambda r_2) I_1'(qr_2)}{\mu_2^m I_1(qr_2) (I_v(\lambda r_2) - 2r_2 \lambda I_v'(\lambda r_2)) + 2\mu_1^m q I_v(\lambda r_2) I_1'(qr_2)}. \quad (3.5.20)$$

Solving (3.5.19) and (3.5.20) we obtain

$$B_5 = -\frac{2r_0}{\sqrt{r_1}} \mu_0 \mu_1^m I e^{-j\lambda z_0} K_1(\lambda r_0) \frac{C_1}{D}, \quad (3.5.21)$$

where

$$\begin{aligned} C_1 &= \mu_2^m I_1(qr_2) (I_v(\lambda r_2) - 2r_2 \lambda I_v'(\lambda r_2)) + 2\mu_1^m q I_v(\lambda r_2) I_1'(qr_2), \\ D &= \mu_2^m I_1(qr_2) D_5 (D_1 - 2r_2 \lambda D_2) + 2\mu_1^m q I_1'(qr_2) D_1 D_5 - \\ &\quad - 2r_1 \lambda \mu_2^m I_1(qr_2) K_1(\lambda r_1) (D_3 - 2r_2 \lambda D_4) - 4r_1 \mu_1^m \lambda q I_1'(qr_2) K_1(\lambda r_1) D_3. \end{aligned}$$

$$\left\{ \begin{aligned} \cdot D_1 &= I_v(\lambda r_2) K_v(\lambda r_1) - I_v(\lambda r_1) K_v(\lambda r_2) & \cdot D_2 &= I_v'(\lambda r_2) K_v(\lambda r_1) - I_v(\lambda r_1) K_v'(\lambda r_2) \\ \cdot D_3 &= I_v(\lambda r_2) K_v'(\lambda r_1) - I_v'(\lambda r_1) K_v(\lambda r_2) & \cdot D_4 &= I_v'(\lambda r_2) K_v'(\lambda r_1) - I_v'(\lambda r_1) K_v'(\lambda r_2) \\ \cdot D_5 &= 2\mu_1^m \lambda K_1'(\lambda r_1) + K_1(\lambda r_1) \end{aligned} \right.$$

The constant B_2 is given by the formula

$$B_4 = \frac{2r_0}{\sqrt{r_1}} \mu_0 \mu_1^m I e^{-j\lambda z_0} K_1(\lambda r_0) \frac{C_2}{D}, \quad (3.5.22)$$

where

$$C_2 = \mu_2^m I_1(qr_2) (K_v(\lambda r_2) - 2r_2 \lambda K_v'(\lambda r_2)) + 2\mu_1^m q K_v(\lambda r_2) I_1'(qr_2).$$

Finally, the value of B_2 is

$$\left\| \cdot B_2 = \frac{B_4}{\sqrt{r_1}} \frac{I_v(\lambda r_1)}{K_1(\lambda r_1)} + \frac{B_5}{\sqrt{r_1}} \frac{K_v(\lambda r_1)}{K_1(\lambda r_1)} - \mu_0 r_0 I e^{-j\lambda z_0} K_1(\lambda r_0) \frac{I_1(\lambda r_1)}{K_1(\lambda r_1)} \right\|$$

$$B_2 = -\mu_0 r_0 I e^{-j\lambda z_0} \frac{K_1(\lambda r_0)}{K_1(\lambda r_1)} \left(\frac{2\mu_1^m \mu_2^m I_1(qr_2) (D_1 - 2r_2 \lambda D_2)}{r_1 D} + 2\mu_1^m q I_1'(qr_2) D_1 + I_1(\lambda r_1) \right). \quad (3.5.23)$$

The induced vector potential in region R_0 is given by

$$\tilde{A}_0^{ind}(r, \lambda) = B_2 K_1(\lambda r), \quad (3.5.24)$$

where B_2 is obtained from (3.5.23).

Applying the inverse Fourier transform (3.5.10) to (3.5.24) we obtain the induced vector potential in free space due to the presence of a cylindrical region in the form

$$A_0^{ind}(r, z) = \frac{\mu_0 r_0 I}{2\pi} \int_{-\infty}^{\infty} \tilde{B}_2 K_1(\lambda r) \frac{K_1(\lambda r_0)}{K_1(\lambda r_1)} e^{j\lambda(z-z_0)} d\lambda, \quad (3.5.25)$$

$$\text{where } \tilde{B}_2 = -\frac{2\mu_1^m \mu_2^m I_1(qr_2) (D_1 - 2r_2 \lambda D_2) + 2\mu_1^m q I_1'(qr_2) D_1 - I_1(\lambda r_1)}{r_1 D}$$

The induced change in impedance of the coil is given by the formula

$$Z^{ind} = \frac{j\omega}{I} \oint_L A_0^{ind}(r, z) dl, \quad (3.5.26)$$

where L is the contour of the coil. Substituting (3.5.25) into (3.5.26) we obtain

$$Z^{ind} = j\omega \mu_0 r_0^2 \int_{-\infty}^{\infty} \tilde{B}_2 \frac{K_1^2(\lambda r_0)}{K_1(\lambda r_1)} d\lambda. \quad (3.5.27)$$

Introducing dimensionless parameters and simplifying the resulting expression we rewrite (3.5.27) as follows

$$\left\| \cdot \lambda r_1 = u \rightarrow \lambda = \frac{u}{r_1}, \right. \\ \left. \hat{r}_0 = \frac{r_0}{r_1}, \mu_1^m = \hat{\mu}_1^m r_1, \sigma_1^m = \hat{\sigma}_1^m r_1, \right\} \Rightarrow \hat{v} = \sqrt{\frac{1}{4} + j\hat{p}_1^2}, \cdot \hat{p}_1 = \sqrt{\omega \mu_0 \hat{\mu}_1^m \hat{\sigma}_1^m r_1^2}$$

$$\cdot \hat{r}_2 = \frac{r_2}{r_1}, \hat{q} = \sqrt{u^2 + j\hat{p}_2^2}, \hat{p}_2 = \sqrt{\omega \mu_0 \mu_2^m \sigma_2^m r_1^2} = \sqrt{\omega \mu_0 r_1^2} \sqrt{\mu_2^m \sigma_2^m} = \hat{p}_1 \sqrt{\frac{\mu_2^m \sigma_2^m}{\hat{\mu}_1^m \hat{\sigma}_1^m}}$$

$$\left\| \begin{aligned} \cdot \hat{D}_1 &= I_{\hat{v}}(u\hat{r}_2)K_{\hat{v}}(u) - I_{\hat{v}}(u)K_{\hat{v}}(u\hat{r}_2), & \hat{D}_2 &= I_{\hat{v}}'(u\hat{r}_2)K_{\hat{v}}(u) - I_{\hat{v}}'(u)K_{\hat{v}}(u\hat{r}_2) \\ \cdot \hat{D}_3 &= I_{\hat{v}}(u\hat{r}_2)K_{\hat{v}}'(u) - I_{\hat{v}}'(u)K_{\hat{v}}(u\hat{r}_2), & \hat{D}_4 &= I_{\hat{v}}'(u\hat{r}_2)K_{\hat{v}}'(u) - I_{\hat{v}}'(u)K_{\hat{v}}'(u\hat{r}_2) \\ \cdot \hat{D}_5 &= 2\hat{\mu}_1^m u K_1'(u) + K_1(u), & \hat{D}_6 &= \mu_2^m I_1(\hat{q}\hat{r}_2) (\hat{D}_1 - 2u\hat{r}_2\hat{D}_2) \\ \cdot \hat{D}_7 &= 2\hat{\mu}_1^m \hat{q} I_1'(\hat{q}\hat{r}_2) \hat{D}_1 \end{aligned} \right\|$$

$$Z^{ind}(r, z) = \omega\mu_0 \frac{r_0^2}{r_1} Z, \quad (3.5.28)$$

$$\text{where} \quad Z = j \int_{-\infty}^{\infty} \hat{B}_2 \frac{K_1^2(u\hat{r}_0)}{K_1(u)} du \quad (3.5.29)$$

$$\text{and} \quad \hat{B}_2 = -\frac{2\hat{\mu}_1^m (\hat{D}_6 + \hat{D}_7)}{\hat{D}_5 (\hat{D}_6 + \hat{D}_7) - 2u\mu_2^m I_1(\hat{q}\hat{r}_2) K_1(u) (\hat{D}_3 - 2u\hat{r}_2\hat{D}_4) - 4\hat{\mu}_1^m u \hat{q} I_1'(\hat{q}\hat{r}_2) K_1(u) \hat{D}_3} - I_1(u).$$

Results of numerical calculations using formula (3.5.29) are shown in **Fig. 3.9**. The following values of the parameters are used for calculations: $\mu_1 = 1, \hat{\mu}_2 = 5, \hat{r}_2 = 0.9, \sqrt{\frac{\mu_2^m \sigma_2^m}{\hat{\mu}_1^m \hat{\sigma}_1^m}} = 1.5$. The points on each curve correspond to the values of $\hat{p}_1 = 1, 2, \dots, 10$ (from top to bottom). The graphs in **Fig. 3.9** are shown for the following three values of \hat{r}_0 : 1.1, 1.3 and 1.5. It is seen from **Fig. 3.9** that for larger values of \hat{r}_0 the induced change in impedance is weaker: the modulus of Z decreases as the distance from the coil to the tube increases. Computations are done with “Mathematica” (see **Appendix Fig.Ap.7**).

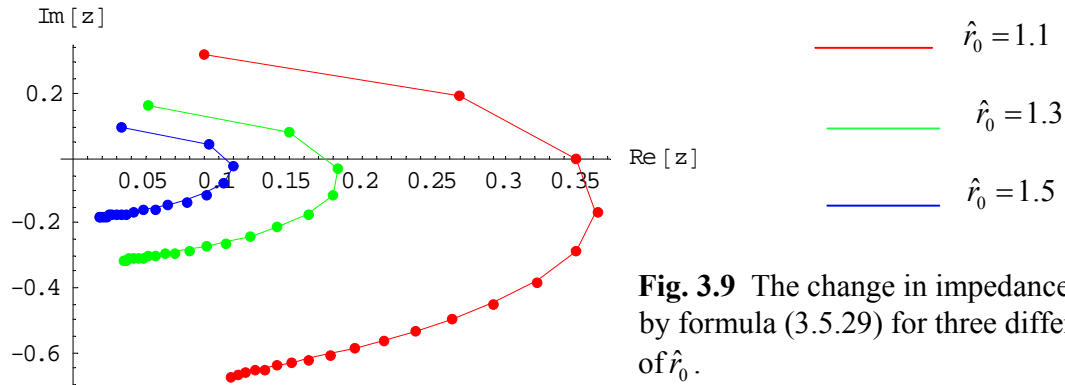


Fig. 3.9 The change in impedance computed by formula (3.5.29) for three different values of \hat{r}_0 .

3.6 Other analytical solutions

Equation (3.2.14) is the second order ordinary differential equation with variable coefficients that depend on two parameters α_i and β_i . As it is shown in the previous sections the solution to (3.2.14) for the case $\alpha_i = -1, \beta_i = -1$ is expressed in terms of the modified Bessel functions (see (3.2.34)). It is possible to construct closed-form solutions to (3.2.14) for other combinations of the parameters α_i and β_i .

If $\alpha_i = 0, \beta_i = -2$ equation (3.2.14) becomes

$$\frac{d^2 \tilde{A}_i}{dr^2} + \frac{3}{r} \frac{d\tilde{A}_i}{dr} + \left(\frac{1-p_i^2}{r^2} - \lambda^2 \right) \tilde{A}_i = 0, \quad i=1,2,\dots,n \quad (3.6.1)$$

Using formula 132 in [50], page 228, the general solution to (3.6.1) can be written in the form

$$\tilde{A}_i(r, \lambda) = \frac{1}{r} \left(D_{4_i} I_{p_i}(\lambda r) + D_{5_i} K_{p_i}(\lambda r) \right) \quad (3.6.2)$$

$$\left\| \begin{array}{l} y'' + \frac{a}{x} y' + \left(bx^{n-2} + \frac{c}{x^2} \right) y = 0 \Rightarrow \\ \cdot y = x^{\frac{1-a}{2}} \left(C_1 J_\nu \left(\frac{2}{n} \sqrt{bx^{\frac{n}{2}}} \right) + C_2 Y_\nu \left(\frac{2}{n} \sqrt{bx^{\frac{n}{2}}} \right) \right) \\ \cdot \nu = \frac{1}{n} \sqrt{(1-a)^2 - 4c} \end{array} \right\|$$

where $n=2$, $b=-\lambda^2$, $c=-p_i^2+1$, $a=3$

$$\nu = p_i, \quad \frac{2}{n} \sqrt{bx^{\frac{n}{2}}} = -j\lambda r.$$

If $\alpha_i + \beta_i = -2$ and $\beta_i \neq 1$ equation (3.2.14) is written as follows

$$\frac{d^2 \tilde{A}_i}{dr^2} + \frac{1-\beta_i}{r} \frac{d\tilde{A}_i}{dr} - \left(\frac{1+\beta_i+p_i^2}{r^2} + \lambda^2 \right) \tilde{A}_i = 0, \quad i=1,2,\dots,n \quad (3.6.3)$$

The general solution to (3.6.3) is (see [50], page 228, formula 132):

$$\tilde{A}_i(r, \lambda) = r^{\frac{\beta_i}{2}} \left(C_{4_i} I_{\nu_i}(\xi_i r) + C_{5_i} K_{\nu_i}(\xi_i r) \right) \quad (3.6.4)$$

where $\nu_i = \frac{1}{2} \sqrt{\beta_i^2 + 4\lambda^2}$, $\xi_i = \sqrt{1 + \beta_i + p_i^2}$.

In addition, if $\beta_i \neq 1$, $\alpha_i + \beta_i = -1$ equation (3.2.14) reduces to

$$\frac{d^2 \tilde{A}_i}{dr^2} + \frac{1-\beta_i}{r} \frac{d\tilde{A}_i}{dr} - \left(\frac{1+\beta_i}{r^2} + \frac{p_i^2}{r} + \lambda^2 \right) \tilde{A}_i = 0, \quad i=1,2,\dots,n \quad (3.6.5)$$

which is a particular form of the confluent hypergeometric equation

$$\frac{d^2 y}{dx^2} + \left(a + \frac{b}{x} \right) \frac{dy}{dx} + \left(c + \frac{d}{x} + \frac{e}{x^2} \right) y = 0$$

with $a=0$, $b=1-\beta_i$, $c=-\lambda^2$, $d=-p_i^2$, $e=-1-\beta_i$ (see [4], page 239).

In this case the solution to (3.2.14) can be expressed in terms of Whittaker functions.

This case is considered in detail in **Section 3.7**.

Probably, other analytical solutions of equation (3.2.14) can be constructed for other combinations of the parameters α_i and β_i .

3.7 Solution in terms of confluent hypergeometric function

One-parameter family of analytical solutions of the problem described in **Section 3.4** (see **Fig. 3.4**) for the case where $\alpha = -1 - \beta$ is obtained in the present section. Solution to (3.4.1) - (3.4.6) is sought by the method of the Fourier transform in the form (3.2.9) so that the

transformed equations are :

$$R_0: \frac{d^2 \tilde{A}_0}{dr^2} + \frac{1}{r} \frac{d\tilde{A}_0}{dr} - \left(\frac{1}{r^2} + \lambda^2 \right) \tilde{A}_0 = -\mu_0 I e^{-j\lambda z_0} \delta(r - r_0), \quad (3.7.1)$$

$$\rightarrow \begin{cases} \tilde{A}_{00}(r, \lambda) = B_2 I_1(\lambda r) + \mu_0 r_0 I e^{-j\lambda z_0} K_1(\lambda r_0) I_1(\lambda r), \\ \tilde{A}_{01}(r, \lambda) = B_2 I_1(\lambda r) + \mu_0 r_0 I e^{-j\lambda z_0} I_1(\lambda r_0) K_1(\lambda r), \end{cases} \quad (3.7.2)$$

$$R_1: \frac{d^2 \tilde{A}_1}{dr^2} + \frac{(1-\beta)}{r} \frac{d\tilde{A}_1}{dr} - \left(\frac{1+\beta}{r^2} + \frac{p_1^2}{r} + \lambda^2 \right) \tilde{A}_1 = 0, \quad (3.7.3)$$

where $p_1 = \sqrt{j\omega\mu_0\mu_1^m\sigma_1^m}$,

$$R_2: \frac{d^2 \tilde{A}_2}{dr^2} + \frac{1}{r} \frac{d\tilde{A}_2}{dr} - \left(\frac{1}{r^2} + q^2 \right) \tilde{A}_2 = 0, \quad (3.7.4)$$

$$\rightarrow \tilde{A}_2(r, \lambda) = B_6 K_1(qr), \quad (3.7.5)$$

and $q = \sqrt{p_2^2 + \lambda^2}$, $p_2 = \sqrt{j\omega\mu_0\mu_2^m\sigma_2^m}$.

The boundary conditions are

$$\tilde{A}_0|_{r=r_1} = \tilde{A}_1|_{r=r_1}, \quad \frac{d\tilde{A}_0}{dr}|_{r=r_1} = \frac{1}{\mu_1^r(r_1)} \frac{d\tilde{A}_1}{dr}|_{r=r_1}, \quad (3.7.6)$$

$$\tilde{A}_1|_{r=r_2} = \tilde{A}_2|_{r=r_2}, \quad \frac{1}{\mu_1^r(r_2)} \frac{d\tilde{A}_1}{dr}|_{r=r_2} = \frac{1}{\mu_2^m} \frac{d\tilde{A}_2}{dr}|_{r=r_2}. \quad (3.7.7)$$

Using the substitution

$$\tilde{A}_1 = r^{\frac{\beta-1}{2}} \tilde{B} \quad (3.7.8)$$

(the details are shown below)

$$\left\| \begin{aligned} & \cdot \tilde{A}_1 = r^{\frac{\beta-1}{2}} \tilde{B} \\ & \cdot \tilde{A}_{1r} = \frac{\beta-1}{2} r^{\frac{\beta-3}{2}} \tilde{B} + r^{\frac{\beta-1}{2}} \tilde{B}_r \quad \cdot \tilde{A}_{1rr} = r^{\frac{\beta-1}{2}} \tilde{B}_{rr} + (\beta-1) r^{\frac{\beta-3}{2}} \tilde{B}_r + \frac{\beta-1}{2} \frac{\beta-3}{2} r^{\frac{\beta-5}{2}} \tilde{B} \\ & \cdot r^{\frac{\beta-1}{2}} \tilde{B}_{rr} + (\beta-1) r^{\frac{\beta-3}{2}} \tilde{B}_r + \frac{\beta-1}{2} \frac{\beta-3}{2} r^{\frac{\beta-5}{2}} \tilde{B} - \frac{(\beta-1)^2}{2} r^{\frac{\beta-5}{2}} \tilde{B} - (\beta-1) r^{\frac{\beta-3}{2}} \tilde{B}_r - \\ & \quad - \left(\frac{1+\beta}{r^2} + \frac{p_1^2}{r} + \lambda^2 \right) r^{\frac{\beta-1}{2}} \tilde{B} = 0 \rightarrow \\ & \cdot \tilde{B}_{rr} + \frac{\beta-1}{2} \frac{\beta-3}{2} \frac{\tilde{B}}{r^2} - \frac{(\beta-1)^2}{2} \frac{\tilde{B}}{r^2} - \left(\frac{1+\beta}{r^2} + \frac{p_1^2}{r} + \lambda^2 \right) \tilde{B} = 0 \rightarrow \\ & \cdot \tilde{B}_{rr} + \lambda^2 \left(-1 - \frac{p_1^2}{r\lambda^2} + \frac{1}{4} - \left(\frac{\beta}{2} + 1 \right)^2 \right) \tilde{B} = 0 \end{aligned} \right\|$$

we transform (3.7.3) to the Whittaker's equation (see [4], page 237):

$$\tilde{B}_{xx} + \left(-\frac{1}{4} + \frac{k}{x} + \frac{\frac{1}{4} - \nu^2}{x^2} \right) \tilde{B} = 0, \quad (3.7.9)$$

where $x = 2\lambda r$, $k = -\frac{p_1^2}{2\lambda}$, $\nu = \frac{\beta}{2} + 1$.

Using the substitution

$$\tilde{B} = e^{-\frac{x}{2}} x^{\frac{c}{2}} y \quad (3.7.10)$$

$$\left\| \begin{aligned} \cdot \tilde{B} &= e^{-\frac{x}{2}} x^{\frac{c}{2}} y \\ \cdot \tilde{B}_x &= e^{-\frac{x}{2}} x^{\frac{c}{2}} \left(y' + \left(\frac{c}{2x} - \frac{1}{2} \right) y \right) \quad \cdot \tilde{B}_{xx} = e^{-\frac{x}{2}} x^{\frac{c}{2}} \left(y'' + \left(\frac{c}{x} - 1 \right) y' + \left(\frac{1}{4} + \frac{1}{x^2} \left(\frac{c^2}{4} - \frac{c}{2} \right) - \frac{c}{2x} \right) y \right) \end{aligned} \right\|$$

equation (3.7.9) can be written in the form

$$y'' + \left(\frac{c}{x} - 1 \right) y' - \frac{a}{x} y = 0, \quad (3.7.11)$$

where $a = \nu - k + \frac{1}{2}$ and $c = 2\nu + 1$.

General solution to (3.7.11) can be expressed in the form:

$$y = B_4 \psi(a, c; x) + B_5 e^x \psi(c - a, c; -x), \quad (3.7.12)$$

where $\psi(a, c; x)$ is the confluent hypergeometric function (see [4], page 241) and B_4, B_5 are arbitrary constants. Thus, a general solution to (3.7.3) can be written in the form

$$\tilde{A}_1 = B_4 r^{\beta+1} e^{-\lambda r} \psi(a, c; 2\lambda r) + B_5 r^{\beta+1} e^{\lambda r} \psi(c - a, c; -2\lambda r). \quad (3.7.13)$$

In order to simplify the notations we define the following functions

$$\varphi_1(r) = r^{\beta+1} e^{-\lambda r} \psi(a, c; 2\lambda r) \quad (3.7.14)$$

and

$$\varphi_2(r) = r^{\beta+1} e^{\lambda r} \psi(c - a, c; -2\lambda r). \quad (3.7.15)$$

Hence,

$$\tilde{A}_1 = B_4 \varphi_1(r) + B_5 \varphi_2(r). \quad (3.7.16)$$

Using (3.7.2), (3.7.5), (3.7.16) and the boundary conditions (3.7.6), (3.7.7) we obtain

$$B_2 I_1(\lambda r_1) + \mu_0 r_0 I e^{-j\lambda z_0} I_1(\lambda r_0) K_1(\lambda r_1) = B_4 \varphi_1(r_1) + B_5 \varphi_2(r_1), \quad (3.7.17)$$

$$\lambda B_2 I_1'(\lambda r_1) + \lambda \mu_0 r_0 I e^{-j\lambda z_0} I_1(\lambda r_0) K_1'(\lambda r_1) = \frac{1}{\mu_1^r(r_1)} \cdot (B_4 \varphi_1'(r_1) + B_5 \varphi_2'(r_1)), \quad (3.7.18)$$

$$B_6 K_1(qr_2) = B_4 \varphi_1(r_2) + B_5 \varphi_2(r_2), \quad (3.7.19)$$

$$\frac{B_6}{\mu_2^m} q K_1'(qr_2) = \frac{1}{\mu_1^r(r_2)} \cdot (B_4 \varphi_1'(r_2) + B_5 \varphi_2'(r_2)). \quad (3.7.20)$$

Using (3.7.17) and (3.7.18) we obtain

$$\left\| \begin{aligned} & \cdot B_2 = \frac{B_4}{I_1(\lambda r_1)} \varphi_1(r_1) + \frac{B_5}{I_1(\lambda r_1)} \varphi_2(r_1) - \mu_0 r_0 I e^{-j\lambda z_0} I_1(\lambda r_0) \frac{K_1(\lambda r_1)}{I_1(\lambda r_1)} \\ & \cdot B_4 \left(\frac{I_1'(\lambda r_1)}{I_1(\lambda r_1)} \varphi_1(r_1) - \frac{\varphi_1'(r_1)}{\lambda \mu_1^r(r_1)} \right) = -B_5 \left(\frac{I_1'(\lambda r_1)}{I_1(\lambda r_1)} \varphi_2(r_1) - \frac{\varphi_2'(r_1)}{\lambda \mu_1^r(r_1)} \right) + \\ & \quad + \mu_0 r_0 I e^{-j\lambda z_0} I_1(\lambda r_0) \frac{I_1'(\lambda r_1) K_1(\lambda r_1) - I_1(\lambda r_0) K_1'(\lambda r_1)}{I_1(\lambda r_1)} \\ & \cdot \mu_1^r(r) = \mu_1^m r^\beta \rightarrow \mu_1^r(r_1) = \mu_1^m r_1^\beta, \text{ where } \mu_1^m = \hat{\mu}_1^m r_1^{-\beta} \Rightarrow \mu_1^r(r_1) = \hat{\mu}_1^m \\ & \cdot \sigma_1^m = \tilde{\sigma}_1^m r_1^{\beta+1} \\ & \cdot p_1^2 = j\omega \mu_o \hat{\mu}_1^m \tilde{\sigma}_1^m r_1 \end{aligned} \right\|$$

$$B_4 = -B_5 \frac{\lambda \hat{\mu}_1^m I_1'(\lambda r_1) \varphi_2(r_1) - I_1(\lambda r_1) \varphi_2'(r_1)}{\lambda \hat{\mu}_1^m I_1(\lambda r_1) \varphi_1(r_1) - I_1(\lambda r_1) \varphi_1'(r_1)} + \frac{\hat{\mu}_1^m \mu_0 \frac{r_0}{r_1} I e^{-j\lambda z_0} I_1(\lambda r_0)}{\lambda \hat{\mu}_1^m I_1(\lambda r_1) \varphi_1(r_1) - I_1(\lambda r_1) \varphi_1'(r_1)}, \quad (3.7.21)$$

where $W\{K_1(\lambda r_1), I_1(\lambda r_1)\} = \frac{1}{\lambda r_1}$.

Eliminating B_6 from (3.7.19) and (3.7.20) we get

$$\left\| \begin{aligned} & \cdot B_6 = B_4 \frac{\varphi_1(r_2)}{K_1(qr_2)} + B_5 \frac{\varphi_2(r_2)}{K_1(qr_2)} \\ & \cdot B_4 \left(\frac{\varphi_1'(r_2)}{\mu_1^r(r_2)} - \frac{q}{\mu_2^m} \frac{\varphi_1(r_2)}{K_1(qr_2)} K_1'(qr_2) \right) = -B_5 \left(\frac{\varphi_2'(r_2)}{\mu_1^r(r_2)} - \frac{q}{\mu_2^m} \frac{\varphi_2(r_2)}{K_1(qr_2)} K_1'(qr_2) \right) \\ & \cdot \mu_1^r(r) = \mu_1^m r^\beta \rightarrow \mu_1^r(r_2) = \mu_1^m r_2^\beta, \text{ where } \mu_1^m = \hat{\mu}_1^m r_1^{-\beta} \Rightarrow \mu_1^r(r_2) = \hat{\mu}_1^m \hat{r}_2^\beta \rightarrow \\ & \cdot B_4 \left(\mu_2^m \varphi_1'(r_2) - \hat{\mu}_1^m \hat{r}_2^\beta q \frac{\varphi_1(r_2)}{K_1(qr_2)} K_1'(qr_2) \right) = -B_5 \left(\mu_2^m \varphi_2'(r_2) - \hat{\mu}_1^m \hat{r}_2^\beta q \frac{\varphi_2(r_2)}{K_1(qr_2)} K_1'(qr_2) \right) \end{aligned} \right\|$$

$$B_4 = -B_5 \frac{\mu_2^m \varphi_2'(r_2) K_1(qr_2) - q \hat{\mu}_1^m \hat{r}_2^\beta \varphi_2(r_2) K_1'(qr_2)}{\mu_2^m \varphi_1'(r_2) K_1(qr_2) - q \hat{\mu}_1^m \hat{r}_2^\beta \varphi_1(r_2) K_1'(qr_2)}, \quad (3.7.22)$$

and

$$B_5 = \hat{\mu}_1^m \mu_0 \frac{r_0}{r_1} I e^{-j\lambda z_0} I_1(\lambda r_0) \frac{C_1}{D}, \quad (3.7.23)$$

where

$$\begin{aligned} C_1 &= \mu_2^m \varphi_1'(r_2) K_1(qr_2) - q \hat{\mu}_1^m \hat{r}_2^\beta \varphi_1(r_2) K_1'(qr_2), \\ D &= \lambda \hat{\mu}_1^m \mu_2^m K_1(qr_2) I_1'(\lambda r_1) D_1 + \lambda q (\hat{\mu}_1^m)^2 \hat{r}_2^\beta K_1'(qr_2) I_1'(\lambda r_1) D_2 + \\ & \quad + \mu_2^m K_1(qr_2) I_1(\lambda r_1) D_3 + q \hat{\mu}_1^m \hat{r}_2^\beta K_1'(qr_2) I_1(\lambda r_1) D_4. \end{aligned}$$

$$\left\| \begin{aligned} & \cdot D_1 = \varphi_1'(r_2) \varphi_2(r_1) - \varphi_1(r_1) \varphi_2'(r_2) \quad \cdot D_2 = \varphi_1(r_1) \varphi_2(r_2) - \varphi_1(r_2) \varphi_2(r_1) \\ & \cdot D_3 = \varphi_1'(r_1) \varphi_2'(r_2) - \varphi_1'(r_2) \varphi_2'(r_1) \quad \cdot D_4 = \varphi_1(r_2) \varphi_2'(r_1) - \varphi_1'(r_1) \varphi_2(r_2) \end{aligned} \right\|$$

The constant B_4 is

$$B_4 = -\hat{\mu}_1^m \mu_0 \frac{r_0}{r_1} I e^{-j\lambda z_0} I_1(\lambda r_0) \frac{C_2}{D}, \quad (3.7.24)$$

where $C_2 = \mu_2^m \phi_2'(r_2) K_1(qr_2) - q \hat{\mu}_1^m \hat{r}_2^\beta \phi_2(r_2) K_1'(qr_2)$.

In particular, the value of B_2 is

$$B_2 = \mu_0 r_0 I e^{-j\lambda z_0} \frac{I_1(\lambda r_0)}{I_1(\lambda r_1)} \left(\frac{\hat{\mu}_1^m \mu_2^m K_1(qr_2) D_1 + q \hat{\mu}_1^m \hat{r}_2^\beta K_1'(qr_2) D_2}{D} - K_1(\lambda r_1) \right). \quad (3.7.25)$$

The induced change in impedance of the coil due to the presence of a conducting tube is given by the formula

$$\tilde{A}_0^{ind}(r, \lambda) = B_2 I_1(\lambda r), \quad (3.7.26)$$

where B_2 is given by (3.7.25).

Applying the inverse Fourier transform (3.2.36) to (3.7.26) we obtain the induced vector potential in free space due to the presence of a cylindrical region in the form

$$A_0^{ind}(r, z) = \frac{\mu_0 r_0 I}{2\pi} \int_{-\infty}^{\infty} \tilde{B}_2 I_1(\lambda r) \frac{I_1(\lambda r_0)}{I_1(\lambda r_1)} e^{j\lambda(z-z_0)} d\lambda, \quad (3.7.27)$$

where $\tilde{B}_2 = \frac{\hat{\mu}_1^m \mu_2^m K_1(qr_2) D_1 + q \hat{\mu}_1^m \hat{r}_2^\beta K_1'(qr_2) D_2}{r_1 D} - K_1(\lambda r_1)$

The change in impedance of the coil is given by

$$Z^{ind}(r, z) = \frac{j\omega}{I} \oint_L A_0^{ind}(r, z) dl, \quad (3.7.28)$$

where L is the contour of the coil. Substituting (3.7.27) into (3.7.28) we obtain

$$Z^{ind}(r, z) = j\omega \mu_0 r_0^2 \int_{-\infty}^{\infty} \tilde{B}_2 \frac{I_1^2(\lambda r_0)}{I_1(\lambda r_1)} d\lambda. \quad (3.7.29)$$

Using following dimensionless parameters (3.7.29) can be rewritten as follows

$$\left\| \begin{array}{l} \cdot \lambda r_1 = u \rightarrow \lambda = \frac{u}{r_1}, \\ \cdot \hat{r}_0 = \frac{r_0}{r_1}, \hat{r}_2 = \frac{r_2}{r_1}, \hat{r}_1 = \frac{r_1}{r_1} = 1, \cdot \hat{p}_1 = \sqrt{\omega \mu_0 \hat{\mu}_1^m \tilde{\sigma}_1^m r_1^2}, \\ \cdot \hat{p}_2 = \sqrt{\omega \mu_0 \mu_2^m \sigma_2^m r_1^2} = \sqrt{\omega \mu_0 r_1^2} \sqrt{\mu_2^m \sigma_2^m} = \hat{p}_1 \sqrt{\frac{\mu_2^m \sigma_2^m}{\hat{\mu}_1^m \tilde{\sigma}_1^m}}, \\ \cdot \hat{q} = \sqrt{u^2 + j\hat{p}_2^2}, \cdot c = \beta + 3, a = \frac{c}{2} + \frac{\hat{p}_1^2}{2u} \end{array} \right\|$$

$$\left\| \begin{array}{ll} \cdot \hat{\phi}_1(\hat{r}_2) = \hat{r}_2^{\beta+1} e^{-u\hat{r}_2} \psi(a, c; 2u\hat{r}_2) & \cdot \hat{\phi}_1(1) = e^{-u} \psi(a, c; 2u) \\ \cdot \hat{\phi}_2(\hat{r}_2) = \hat{r}_2^{\beta+1} e^{u\hat{r}_2} \psi(c-a, c; -2u\hat{r}_2) & \cdot \hat{\phi}_2(1) = e^u \psi(c-a, c; -2u) \\ \cdot \hat{D}_1 = \hat{\phi}_1'(\hat{r}_2) \hat{\phi}_2(1) - \hat{\phi}_1(1) \hat{\phi}_2'(\hat{r}_2) & \cdot \hat{D}_2 = \hat{\phi}_1(1) \hat{\phi}_2(\hat{r}_2) - \hat{\phi}_1(\hat{r}_2) \hat{\phi}_2(1) \\ \cdot \hat{D}_3 = \hat{\phi}_1'(1) \hat{\phi}_2'(\hat{r}_2) - \hat{\phi}_1'(\hat{r}_2) \hat{\phi}_2'(1) & \cdot \hat{D}_4 = \hat{\phi}_1(\hat{r}_2) \hat{\phi}_2'(1) - \hat{\phi}_1'(1) \hat{\phi}_2(\hat{r}_2) \end{array} \right\|$$

$$Z^{ind}(r, z) = \omega \mu_0 \frac{r_0^2}{r_1} Z, \quad (3.7.30)$$

$$\text{where } Z = j \int_{-\infty}^{\infty} \hat{B}_2 \frac{I_1^2(u \hat{r}_0)}{I_1(u)} du \quad (3.7.31)$$

and

$$\hat{B}_2 = \frac{\hat{\mu}_1^m (\mu_2^m K_1(\hat{q} \hat{r}_2) \hat{D}_1 + \hat{q} \hat{\mu}_1^m \hat{r}_2^\beta K_1'(\hat{q} \hat{r}_2) \hat{D}_2)}{\hat{D}}$$

$$\begin{aligned} \hat{D} = & u \hat{\mu}_1^m \mu_2^m K_1(\hat{q} \hat{r}_2) I_1'(u) \hat{D}_1 + u \hat{q} (\hat{\mu}_1^m)^\beta \hat{r}_2^\beta K_1'(\hat{q} \hat{r}_2) I_1'(u) \hat{D}_2 + \mu_2^m K_1(\hat{q} \hat{r}_2) I_1(u) \hat{D}_3 + \\ & + \hat{q} \hat{\mu}_1^m \hat{r}_2^\beta K_1'(\hat{q} \hat{r}_2) I_1(u) \hat{D}_4. \end{aligned}$$

The results of numerical computations of the change in impedance Z using formula (3.7.31) are shown in **Fig.3.10**. The three curves in **Fig. 3.10** correspond to three different values of β , namely, $\beta = 0, -1$ and -2 (from bottom to top). The following values of the other parameters are used for calculations: $\hat{\mu}_1^m = 1$, $\hat{r}_0 = 0.8$, $\hat{r}_2 = 1.2$. The points on each curve in **Fig.3.10** correspond to different values of the parameter $\hat{p}_1 = r_1 \sqrt{\omega \mu_0 \tilde{\sigma}_1 \hat{\mu}_1^m}$ (\hat{p}_1 increases from 1 to 3 from left to right in **Fig.3.10**). Calculations are done with “Mathematica” since confluent hypergeometric functions and Bessel functions are the built-in functions in “Mathematica”. (see **Appendix Fig.Ap.8**).

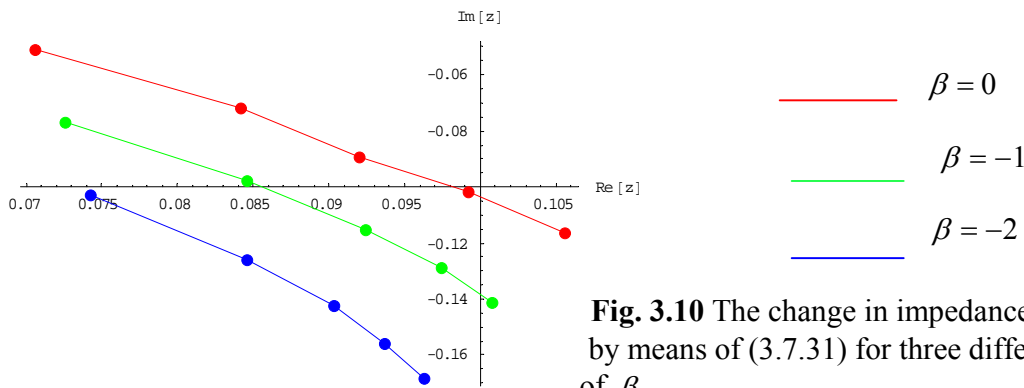


Fig. 3.10 The change in impedance computed by means of (3.7.31) for three different values of β .

In summary, the idea of using relatively simple model one-parameter electric conductivity and magnetic permeability profiles allows one to obtain different analytical solutions that can be used in eddy current testing of objects of cylindrical shapes with varying electric conductivity and magnetic permeability.

The change in impedance of a single-turn coil with alternating current located inside or outside a multilayer tube with arbitrary number of conducting layers is obtained in the present section. The electric conductivity and/or magnetic permeability of each conducting layer are assumed to be power functions of the radial coordinate. The closed-form solution is expressed in terms of improper integral containing Bessel functions. It is shown that for some combinations of the parameters the solution in a conducting layer with variable properties can be expressed in terms of different special functions (Bessel functions and Whittaker functions). Theoretical model is developed for an arbitrary number of concentric conducting layers. Three examples are considered in detail. The first two examples correspond to the case where a coil is located inside a multilayer tube: (a) the case of an infinite outer conducting layer with varying properties and

(b) the case of a two-layer tube where the electric conductivity and magnetic permeability depend on the radial coordinate. In addition, the case of a coil located outside a two-layer tube with varying properties considered as well.

Results of numerical calculations for all three examples are presented. Calculations are performed with “Mathematica”.

There are at least two cases where analytical solutions for eddy current testing problems can be helpful. First, analytical solutions suggested in the present chapter can be used to solve inverse problems in cylindrical geometry where the electric conductivity and magnetic permeability of each conducting layer depend on the radial coordinate. Second, analytical solutions are often used to test numerical algorithms developed for more complicated cases (examples include equations with variable coefficients where the coefficients depend on more than one variable or nonlinear equations). If necessary, one can obtain formulas for the change in impedance for a coil of finite dimensions, as it is done in **Chapter 2**.

4. DOUBLE CONDUCTOR LINE ABOVE A PLANAR MEDIUM WITH VARYING PROPERTIES

4.1 Double conductor line above a multilayer medium with varying electrical and magnetic properties

Different types of eddy current coils are used in applications: circular coils, rectangular coils, planar sensors of circular and rectangular shape. It is known [67], [72] that if the ratio of the sides of a rectangular frame is 1:4 or smaller, then a double conductor line can be considered as a sufficiently accurate model of the rectangular coil. In this chapter analytical solutions are found for the case where a double conductor line formed by two infinite parallel wires is located parallel to a multilayer conducting medium [27]. The electrical conductivity and magnetic permeability of each layer are exponential functions of the vertical coordinate. The solution is found by the method of Fourier integral transform. Corresponding system of ordinary differential equations in the transformed space is solved in closed form by means of the modified Bessel functions.

Suppose that two infinitely long parallel wires are located above a multilayer conducting medium shown in **Fig. 4.1**.

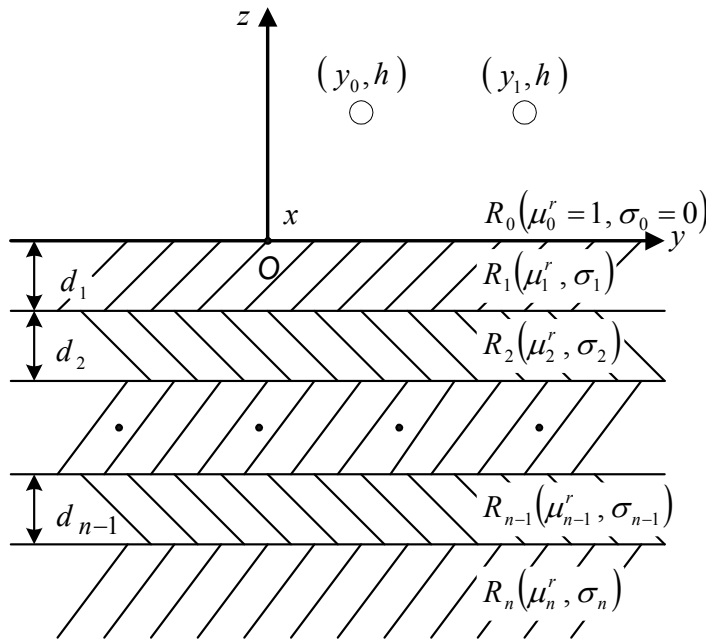


Fig. 4.1 A double conductor line above a multilayer medium.

The coordinates of the wires are (y_0, h) and (y_1, h) , respectively. The current in the wires is $\pm Ie^{j\omega t}$ where I is the amplitude of the current and ω is the frequency. Each conducting layer $R_i, i=1,2,\dots,n$ in **Fig. 4.1** is characterized by the two parameters: electrical conductivity σ_i and relative magnetic permeability μ_i^r . It is assumed that σ_i and μ_i^r are exponential functions of the vertical coordinate of the form

$$\mu_i^r(z) = \mu_i^m e^{\beta_i z}, \quad \sigma_i(z) = \sigma_i^m e^{\alpha_i z}, \quad i=1,2,\dots,n, \quad (4.1.1)$$

where $\alpha_i, \beta_i, \mu_i^m, \sigma_i^m$ are constants. The amplitude of the vector potential \vec{A}_i in each region $R_i, i = 0, 1, \dots, n$ has only one non-zero component in the x - direction which is a function of y and z only. In this case, \vec{A}_i is parallel to the x - axis so that

$$\vec{A}_i = A_i(y, z)\vec{e}_x. \quad (4.1.2)$$

Using (1.2.10) in Cartesian coordinates with $\mu_0^r = 1, \sigma_0 = 0$ and $I^e = I\delta(z-h)(\delta(y-y_1) - \delta(y-y_0))$ we obtain the equation for the amplitude of the vector potential in region R_0 :

$$\frac{\partial^2 A_0}{\partial y^2} + \frac{\partial^2 A_0}{\partial z^2} = \mu_0 I \delta(z-h)(\delta(y-y_1) - \delta(y-y_0)), \quad (4.1.3)$$

where $\delta(x)$ is the Dirac delta-function.

Using (1.2.10) in Cartesian coordinates with $I = 0$ we obtain the following system of equations for the amplitudes $A_i(r, z)$ of the vector potential in each region $R_i, i = 1, 2, \dots, n$:

$$\frac{\partial^2 A_i}{\partial y^2} + \frac{\partial^2 A_i}{\partial z^2} - \frac{1}{\mu_i^r(z)} \frac{d\mu_i^r(z)}{dz} \frac{\partial A_i}{\partial z} - j\omega\sigma_i(z)\mu_0\mu_i^r(z)A_i = 0, \quad (4.1.4)$$

$$i = 1, 2, \dots, n.$$

Substituting (4.1.1) into (4.1.4) we obtain

$$\frac{\partial^2 A_i}{\partial y^2} + \frac{\partial^2 A_i}{\partial z^2} = \beta_i \frac{\partial A_i}{\partial z} + j\omega\sigma_i^m \mu_0 \mu_i^m e^{(\alpha_i + \beta_i)z} A_i, \quad (4.1.5)$$

$$i = 1, 2, \dots, n.$$

The boundary conditions are

$$A_0|_{z=0} = A_1|_{z=0}, \quad \frac{\partial A_0}{\partial z}|_{z=0} = \frac{1}{\mu_1^r(0)} \frac{\partial A_1}{\partial z}|_{z=0}, \quad (4.1.6)$$

$$A_i|_{z=-\hat{d}_i} = A_{i+1}|_{z=-\hat{d}_i},$$

$$\frac{1}{\mu_i^r(-\hat{d}_i)} \frac{\partial A_i}{\partial z}|_{z=-\hat{d}_i} = \frac{1}{\mu_{i+1}^r(-\hat{d}_i)} \frac{\partial A_{i+1}}{\partial z}|_{z=-\hat{d}_i}, \quad i = 1, 2, \dots, n-1, \quad (4.1.7)$$

$$\text{where } \hat{d}_i = \sum_{k=1}^i d_k.$$

The following conditions are satisfied at infinity:

$$A_i, \quad \frac{\partial A_i}{\partial y} \rightarrow 0 \quad \text{as} \quad \sqrt{y^2 + z^2} \rightarrow \infty, \quad i = 0, 1, \dots, n \quad (4.1.8)$$

Problem (4.1.3), (4.1.5) - (4.1.7) is solved by the method of integral transforms. It is convenient to represent the solution A_i in each region R_i as the sum of even and odd components of the form $A_i(y, z) = A_i^{even}(y, z) + A_i^{odd}(y, z)$.

$$(4.1.9)$$

For this purpose, we represent the right-hand side of (4.1.3) in a similar form

$$\left\| f(y, z) = \frac{1}{2}[f(y, z) + f(-y, z)] + \frac{1}{2}[f(y, z) - f(-y, z)] = f^{even}(y, z) + f^{odd}(y, z) \right\|$$

$$\left\| \begin{aligned} f^{even}(y, z) &= \frac{\mu_0 I}{2} \delta(z-h) (\delta(y-y_1) - \delta(y-y_0) + \delta(y+y_1) - \delta(y+y_0)) \\ f^{odd}(y, z) &= \frac{\mu_0 I}{2} \delta(z-h) (\delta(y-y_1) - \delta(y-y_0) - \delta(y+y_1) + \delta(y+y_0)) \end{aligned} \right\|$$

$$f^{even}(y, z) = f^{odd}(y, z) = \frac{\mu_0 I}{2} \delta(z-h) (\delta(y-y_1) - \delta(y-y_0)) \quad (4.1.10)$$

The even component of the solution is found by applying the Fourier cosine transform of the form

$$\tilde{A}_i^{even}(\lambda, z) = \int_0^{\infty} A_i^{even}(y, z) \cos(\lambda y) dy, \quad i = 0, 1, \dots, n. \quad (4.1.11)$$

The left-hand side of equation (4.1.3) can be rewritten as follows

$$\int_0^{\infty} \left(\frac{\partial^2 A_i^{even}}{\partial y^2} + \frac{\partial^2 A_i^{even}}{\partial z^2} \right) \cos(\lambda y) dy = \int_0^{\infty} \frac{\partial^2 A_i^{even}}{\partial y^2} \cos(\lambda y) dy + \int_0^{\infty} \frac{\partial^2 A_i^{even}}{\partial z^2} \cos(\lambda y) dy$$

$$\left\| \begin{aligned} & \cdot \int_0^{\infty} \frac{\partial^2 A_i^{even}}{\partial z^2} \cos(\lambda y) dy = \frac{d^2 \tilde{A}_i^{even}}{dz^2} \\ & \cdot \int_0^{\infty} \frac{\partial^2 A_i^{even}}{\partial y^2} \cos(\lambda y) dy = \left| \begin{array}{l} u = \cos(\lambda y) \rightarrow du = -\lambda \sin(\lambda y) dy \\ dv = \frac{\partial^2 A_i^{even}}{\partial y^2} dy \rightarrow v = \frac{\partial A_i^{even}}{\partial y} \end{array} \right| = \\ & = \frac{\partial A_i^{even}}{\partial y} \cos(\lambda y) \Big|_0^{\infty} + \lambda \int_0^{\infty} \frac{\partial A_i^{even}}{\partial y} \sin(\lambda y) dy = \left| \begin{array}{l} u = \sin(\lambda y) \rightarrow du = \lambda \cos(\lambda y) dy \\ dv = \frac{\partial A_i^{even}}{\partial y} dy \rightarrow v = A_i^{even} \end{array} \right| = \\ & = \lambda A_i^{even} \sin(\lambda y) \Big|_0^{\infty} - \lambda^2 \int_0^{\infty} A_i^{even} \cos(\lambda y) dy = -\lambda^2 \tilde{A}_i^{even} \end{aligned} \right\| =$$

$$\int_0^{\infty} \left(\frac{\partial^2 A_i^{even}}{\partial y^2} + \frac{\partial^2 A_i^{even}}{\partial z^2} \right) \cos(\lambda y) dy = \frac{d^2 \tilde{A}_i^{even}}{dz^2} - \lambda^2 \tilde{A}_i^{even}, \quad i = 0, 1, \dots, n, \quad (4.1.12)$$

Applying the transform (4.1.11) to (4.1.3) we obtain

$$\frac{\mu_0 I}{2} \delta(z-h) \int_0^{\infty} (\delta(y-y_1) - \delta(y-y_0)) \cos(\lambda y) dy = \frac{\mu_0 I}{2} \delta(z-h) (\cos(\lambda y_1) - \cos(\lambda y_0)) \quad (4.1.13)$$

Similarly, the right-hand side of equation (4.1.5) is transformed to the form

$$\begin{aligned} \int_0^{\infty} \left(\beta_i \frac{\partial A_i^{even}}{\partial z} + j\omega \sigma_i^m \mu_0 \mu_i^m e^{(\alpha_i + \beta_i)z} A_i^{even} \right) \cos(\lambda y) dy = \\ = \beta_i \frac{d\tilde{A}_i^{even}}{dz} + j\omega \sigma_i^m \mu_0 \mu_i^m e^{(\alpha_i + \beta_i)z} \tilde{A}_i^{even}, \quad i = 1, 2, \dots, n. \end{aligned} \quad (4.1.14)$$

Using (4.1.12), (4.1.13) and (4.1.14) we obtain

$$\frac{d^2 \tilde{A}_0^{even}}{dz^2} - \lambda^2 \tilde{A}_0^{even} = \frac{\mu_0 I}{2} \delta(z-h) (\cos(\lambda y_1) - \cos(\lambda y_0)), \quad (4.1.15)$$

$$\frac{d^2 \tilde{A}_i^{even}}{dz^2} - \beta_i \frac{d\tilde{A}_i^{even}}{dz} - (\lambda^2 + j\omega\sigma_i^m \mu_0 \mu_i^m e^{(\alpha_i + \beta_i)z}) \tilde{A}_i^{even} = 0, \quad i = 1, 2, \dots, n. \quad (4.1.16)$$

Applying the Fourier cosine transform to the boundary conditions we get

$$\tilde{A}_0^{even} \Big|_{z=0} = \tilde{A}_1^{even} \Big|_{z=0}, \quad \frac{d\tilde{A}_0^{even}}{dz} \Big|_{z=0} = \frac{1}{\mu_1^r(0)} \frac{d\tilde{A}_1^{even}}{dz} \Big|_{z=0}, \quad (4.1.17)$$

$$\tilde{A}_i^{even} \Big|_{z=-\hat{d}_i} = \tilde{A}_{i+1}^{even} \Big|_{z=-\hat{d}_i}, \quad (4.1.18)$$

$$\frac{1}{\mu_i^r(-\hat{d}_i)} \frac{d\tilde{A}_i^{even}}{dz} \Big|_{z=-\hat{d}_i} = \frac{1}{\mu_{i+1}^r(-\hat{d}_i)} \frac{d\tilde{A}_{i+1}^{even}}{dz} \Big|_{z=-\hat{d}_i}, \quad i = 1, 2, \dots, n-1,$$

In order to solve equation (4.1.15) we consider the following two sub-regions of $R_0 : 0 < z < h$ and $z > h$. The solutions in these regions are denoted by \tilde{A}_{00}^{even} and \tilde{A}_{01}^{even} , respectively. Hence,

$$\frac{d^2 \tilde{A}_{00}^{even}}{dz^2} - \lambda^2 \tilde{A}_{00}^{even} = 0, \quad 0 < z < h, \quad (4.1.19)$$

$$\frac{d^2 \tilde{A}_{01}^{even}}{dz^2} - \lambda^2 \tilde{A}_{01}^{even} = 0, \quad z > h. \quad (4.1.20)$$

The general solution to (4.1.19) is

$$\tilde{A}_{00}^{even} = C_1 e^{\lambda z} + C_2 e^{-\lambda z}. \quad (4.1.21)$$

The bounded solution to (4.1.20) is given by

$$\tilde{A}_{01}^{even} = C_3 e^{-\lambda z}. \quad (4.1.22)$$

The functions $\tilde{A}_{00}^{even}(\lambda, z)$ and $\tilde{A}_{01}^{even}(\lambda, z)$ satisfy the following conditions at $z = h$:

$$\tilde{A}_{00}^{even} \Big|_{z=h} = \tilde{A}_{01}^{even} \Big|_{z=h}, \quad (4.1.23)$$

The condition (4.1.23) represents continuity of the function $\tilde{A}_0^{even}(\lambda, z)$ at $z = h$.

One more boundary condition is obtained by integrating (4.1.15) with respect to z from $z = h - \varepsilon$ to $z = h + \varepsilon$ and considering the limit in the resulting expression as $\varepsilon \rightarrow +0$.

$$\left\| \begin{aligned} & \cdot \int_{h-\varepsilon}^{h+\varepsilon} \frac{d^2 \tilde{A}_0^{even}}{dz^2} dz - \lambda^2 \int_{h-\varepsilon}^{h+\varepsilon} \tilde{A}_0^{even} dz = \frac{\mu_0 I}{2} (\cos(\lambda y_1) - \cos(\lambda y_0)) \int_{h-\varepsilon}^{h+\varepsilon} \delta(z-h) dz \\ & \frac{d\tilde{A}_0^{even}}{dz} \Big|_{h-\varepsilon}^{h+\varepsilon} - \lambda^2 \int_{h-\varepsilon}^{h+\varepsilon} \tilde{A}_0^{even} dz = \frac{\mu_0 I}{2} (\cos(\lambda y_1) - \cos(\lambda y_0)) \\ & \cdot - \lambda^2 \lim_{\varepsilon \rightarrow 0} \int_{h-\varepsilon}^{h+\varepsilon} \tilde{A}_0^{even} dz = - \lambda^2 \lim_{\varepsilon \rightarrow 0} 2\varepsilon \tilde{A}_0^{even}(\lambda, z^*) = 0 \\ & \cdot \lim_{\varepsilon \rightarrow 0} \frac{d\tilde{A}_0^{even}}{dz} \Big|_{h-\varepsilon}^{h+\varepsilon} = \frac{d\tilde{A}_{01}^{even}}{dz} \Big|_{z=h} - \frac{d\tilde{A}_{00}^{even}}{dz} \Big|_{z=h} \end{aligned} \right\|$$

$$\left. \frac{d\tilde{A}_{01}^{even}}{dz} \right|_{z=h} - \left. \frac{d\tilde{A}_{00}^{even}}{dz} \right|_{z=h} = \frac{\mu_0 I}{2} (\cos(\lambda y_1) - \cos(\lambda y_0)). \quad (4.1.24)$$

Using (4.1.21) – (4.1.24) we obtain

$$\begin{cases} C_1 e^{\lambda h} + C_2 e^{-\lambda h} = C_3 e^{-\lambda h} \\ -\lambda C_3 e^{-\lambda h} - \lambda C_1 e^{\lambda h} + \lambda C_2 e^{-\lambda h} = \frac{\mu_0 I}{2} (\cos(\lambda y_1) - \cos(\lambda y_0)) \end{cases} \Rightarrow$$

$$\left\| \begin{array}{l} C_1 e^{2\lambda h} + C_2 - C_3 = 0 \\ -C_1 e^{2\lambda h} + C_2 - C_3 = \frac{\mu_0 I}{2\lambda} (\cos(\lambda y_1) - \cos(\lambda y_0)) e^{2\lambda h} \end{array} \right\|$$

$$C_1 = \frac{\mu_0 I}{4\lambda} (\cos(\lambda y_0) - \cos(\lambda y_1)) e^{-\lambda h} \quad (4.1.25)$$

$$C_3 = C_2 + \frac{\mu_0 I}{4\lambda} (\cos(\lambda y_0) - \cos(\lambda y_1)) e^{\lambda h} \quad (4.1.26)$$

Using (4.1.21), (4.1.22), (4.1.25) and (4.1.26) we get

$$\begin{cases} \tilde{A}_{00}^{even} = C_2 e^{-\lambda z} + \frac{\mu_0 I}{4\lambda} (\cos(\lambda y_0) - \cos(\lambda y_1)) e^{-\lambda(h-z)} \\ \tilde{A}_{01}^{even} = C_2 e^{-\lambda z} + \frac{\mu_0 I}{4\lambda} (\cos(\lambda y_0) - \cos(\lambda y_1)) e^{-\lambda(z-h)} \end{cases} \Rightarrow$$

$$\tilde{A}_0^{even}(\lambda, z) = C_2 e^{-\lambda z} + \frac{\mu_0 I}{4\lambda} (\cos(\lambda y_0) - \cos(\lambda y_1)) e^{-\lambda|h-z|} \quad (4.1.27)$$

The solution to (4.1.16) can be expressed in terms of modified Bessel functions

(see [50], formula 2.1.3.10, page 247):

$$\left\| y''_{xx} + ay'_x + (be^{\lambda x} + c)y = 0 \Rightarrow y = e^{-\frac{ax}{2}} \left[C_1 I_\nu \left(\frac{2\sqrt{b}}{\lambda} e^{\frac{\lambda x}{2}} \right) + C_2 K_\nu \left(\frac{2\sqrt{b}}{\lambda} e^{\frac{\lambda x}{2}} \right) \right], \nu = \frac{\sqrt{a^2 - 4c}}{\lambda} \right\|$$

$$\tilde{A}_i^{even}(\lambda, z) = e^{\frac{\beta_i z}{2}} \left[C_{4_i} I_{\nu_i} \left(\frac{2\sqrt{b_i}}{\alpha_i + \beta_i} e^{\frac{(\alpha_i + \beta_i)}{2} z} \right) + C_{5_i} K_{\nu_i} \left(\frac{2\sqrt{b_i}}{\alpha_i + \beta_i} e^{\frac{(\alpha_i + \beta_i)}{2} z} \right) \right], \quad (4.1.28)$$

$$i = 1, 2, \dots, n-1, \text{ where } \nu_i = \frac{\sqrt{\beta_i^2 + 4\lambda^2}}{\alpha_i + \beta_i}, \quad b_i = -j\omega\mu_0\mu_i^m\sigma_i^m,$$

and I_{ν_i}, K_{ν_i} are the modified Bessel functions of the first and second kind, respectively.

The bounded solution to (4.1.16) in region R_n can be written as follows

$$\tilde{A}_n^{even}(\lambda, z) = C_{4_n} e^{\frac{\beta_n z}{2}} I_{\nu_n} \left(\frac{2\sqrt{b_n}}{\alpha_n + \beta_n} e^{\frac{(\alpha_n + \beta_n)}{2} z} \right) \quad (4.1.29)$$

The even component in each region $R_i, i = 0, 1, \dots, n$ can be determined by applying the inverse Fourier cosine transform of the form

$$A_i^{even}(y, z) = \frac{2}{\pi} \int_0^\infty \tilde{A}_i^{even}(\lambda, z) \cos(\lambda y) d\lambda. \quad (4.1.30)$$

The odd components of the solution, $A_i^{odd}(y, z)$ can be determined by the Fourier sine transform

$$\tilde{A}_i^{odd}(\lambda, z) = \int_0^{\infty} A_i^{odd}(y, z) \sin(\lambda y) dy \quad (4.1.31)$$

and the inverse Fourier sine transform

$$A_i^{odd}(y, z) = \frac{2}{\pi} \int_0^{\infty} \tilde{A}_i^{odd}(\lambda, z) \sin(\lambda y) d\lambda. \quad (4.1.32)$$

The solution to (4.1.3), (4.1.5) - (4.1.8) is then found by formula (4.1.9).

4.2 Double conductor line above a conducting half-space with varying electrical and magnetic properties

In this section we consider (as an example) the case, where $d_1 \rightarrow \infty$ (see **Fig. 4.1**). In other words, using the technique described above we solve the problem for the case of a conducting half-space with varying electrical and magnetic properties [37], where $\mu_1^r(z) = \mu_m e^{\beta z}$, $\sigma_1(z) = \sigma_m e^{\alpha z}$ (see **Fig. 4.2**).

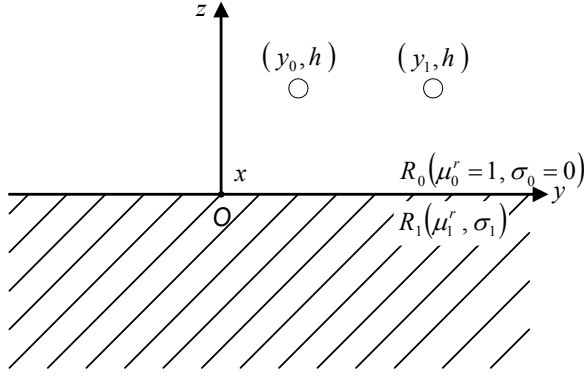


Fig. 4.2 A double conductor line above a conducting half-space.

Using the results from **Section 4.1** we obtain:

$$\begin{aligned} R_0: \quad \frac{d^2 \tilde{A}_0^{even}}{dz^2} - \lambda^2 \tilde{A}_0^{even} &= \frac{\mu_0 I}{2} \delta(z-h) (\cos(\lambda y_1) - \cos(\lambda y_0)) \rightarrow \\ &\rightarrow \tilde{A}_0^{even}(\lambda, z) = C_2 e^{-\lambda z} + \frac{\mu_0 I}{4\lambda} (\cos(\lambda y_0) - \cos(\lambda y_1)) e^{-\lambda|h-z|} \end{aligned} \quad (4.2.1)$$

$$\begin{aligned} R_1: \quad \frac{d^2 \tilde{A}_1^{even}}{dz^2} - \beta \frac{d\tilde{A}_1^{even}}{dz} - (\lambda^2 + j\omega\sigma_m\mu_0\mu_m e^{(\alpha+\beta)z}) \tilde{A}_1^{even} &= 0 \rightarrow \\ &\rightarrow \tilde{A}_1^{even}(\lambda, z) = C_4 e^{\frac{\beta z}{2}} I_\nu \left(\frac{2\sqrt{\beta}}{\alpha+\beta} e^{\frac{(\alpha+\beta)z}{2}} \right) \end{aligned} \quad (4.2.2)$$

The boundary conditions are

$$\tilde{A}_0^{even}|_{z=0} = \tilde{A}_1^{even}|_{z=0}, \quad \frac{d\tilde{A}_0^{even}}{dz}|_{z=0} = \frac{1}{\mu_1^r(0)} \frac{d\tilde{A}_1^{even}}{dz}|_{z=0}, \quad (4.2.3)$$

$$\begin{aligned}\tilde{A}_i^{even} \Big|_{z=-\hat{d}_i} &= \tilde{A}_{i+1}^{even} \Big|_{z=-\hat{d}_i}, \\ \frac{1}{\mu_i^r(-\hat{d}_i)} \frac{d\tilde{A}_i^{even}}{dz} \Big|_{z=-\hat{d}_i} &= \frac{1}{\mu_{i+1}^r(-\hat{d}_i)} \frac{d\tilde{A}_{i+1}^{even}}{dz} \Big|_{z=-\hat{d}_i}, \quad i=1,2,\dots,n-1,\end{aligned}\tag{4.2.4}$$

Using (4.2.1) – (4.2.4) we obtain

$$\begin{cases} C_2 + \frac{\mu_0 I}{4\lambda} (\cos(\lambda y_0) - \cos(\lambda y_1)) e^{-\lambda h} = C_4 I_\nu(z_0), \\ -\lambda C_2 + \frac{\mu_0 I}{4} (\cos(\lambda y_0) - \cos(\lambda y_1)) e^{-\lambda h} = \frac{C_4}{\mu_m} \left(\frac{\beta}{2} I_\nu(z_0) + \sqrt{b} I'_\nu(z_0) \right), \end{cases}\tag{4.2.5}$$

$$\text{where } z_0 = \frac{2\sqrt{b}}{\alpha + \beta}, \quad b = -j\omega\mu_0\mu_m\sigma_m, \quad \nu = \frac{\sqrt{\beta^2 + 4\lambda^2}}{\alpha + \beta}.$$

Solving (4.2.5) we get

$$C_4 = \frac{1}{2} \frac{\mu_0\mu_m I (\cos(\lambda y_0) - \cos(\lambda y_1)) e^{-\lambda h}}{\left(\lambda\mu_m + \frac{\beta}{2} \right) I_\nu(z_0) + \sqrt{b} I'_\nu(z_0)}\tag{4.2.6}$$

The constant C_2 is

$$C_2^{even} = \frac{\mu_0 I}{4\lambda} (\cos(\lambda y_0) - \cos(\lambda y_1)) e^{-\lambda h} \frac{\left(\lambda\mu_m - \frac{\beta}{2} \right) I_\nu(z_0) - \sqrt{b} I'_\nu(z_0)}{\left(\lambda\mu_m + \frac{\beta}{2} \right) I_\nu(z_0) + \sqrt{b} I'_\nu(z_0)}.\tag{4.2.7}$$

Applying the Fourier sine transform we obtain

$$C_2^{odd} = \frac{\mu_0 I}{4\lambda} (\sin(\lambda y_0) - \sin(\lambda y_1)) e^{-\lambda h} \frac{\left(\lambda\mu_m - \frac{\beta}{2} \right) I_\nu(z_0) - \sqrt{b} I'_\nu(z_0)}{\left(\lambda\mu_m + \frac{\beta}{2} \right) I_\nu(z_0) + \sqrt{b} I'_\nu(z_0)}.\tag{4.2.8}$$

The induced vector potential in region R_0 is

$$\tilde{A}_0^{ind}(\lambda, z) = (C_2^{even} + C_2^{odd}) e^{-\lambda z}.\tag{4.2.9}$$

Applying the inverse Fourier cosine transform (4.1.30) to the even component in (4.2.9) and inverse Fourier sine transform (4.1.32) to the odd component in (4.2.9) we obtain

$$A_0^{ind}(y, z) = \frac{\mu_0 I}{2\pi} \int_0^\infty F(\lambda) \frac{1}{\lambda} (\cos(\lambda(y - y_0)) - \cos(\lambda(y - y_1))) e^{-\lambda(z+h)} d\lambda,\tag{4.2.10}$$

where

$$F(\lambda) = \frac{\left(\lambda\mu_m - \frac{\beta}{2} \right) I_\nu(z_0) - \sqrt{b} I'_\nu(z_0)}{\left(\lambda\mu_m + \frac{\beta}{2} \right) I_\nu(z_0) + \sqrt{b} I'_\nu(z_0)}.\tag{4.2.11}$$

The induced change in impedance (per unit length) of the double conductor line is calculated by the formula

$$Z_{\text{per unit length}}^{\text{ind}}(y, z) = \frac{j\omega}{I} \oint_C A_0^{\text{ind}}(y, z) dx. \quad (4.2.12)$$

Substituting (4.2.10) into (4.2.12) we obtain the change in impedance in the form

$$Z_{\text{per unit length}}^{\text{ind}}(y, z) = \frac{\mu_0 \omega}{\pi} j \int_0^\infty \frac{F(\lambda)}{\lambda} (1 - \cos(\lambda(y_1 - y_0))) e^{-2\lambda h} d\lambda, \quad (4.2.13)$$

where $F(\lambda)$ is given by (4.2.11).

Formula (4.2.13) can be also rewritten as follows

$$Z_{\text{per unit length}}^{\text{ind}}(y, z) = \frac{\mu_0 \omega}{\pi} Z, \quad (4.2.14)$$

where

$$Z = j \int_0^\infty \frac{\left(u\mu_m - \frac{\hat{\beta}}{2} \right) I_{\hat{\nu}}(\hat{z}_0) - \sqrt{-j\hat{b}} I_{\hat{\nu}'}(\hat{z}_0)}{\left(u\mu_m + \frac{\hat{\beta}}{2} \right) I_{\hat{\nu}}(\hat{z}_0) + \sqrt{-j\hat{b}} I_{\hat{\nu}'}(\hat{z}_0)} (1 - \cos u) e^{-2u\hat{h}} \frac{du}{u} \quad (4.2.15)$$

and the dimensionless parameters are defined as follows

$$u = \lambda r_c, \quad \hat{b} = \omega \mu_0 \mu_m \sigma_m r_c^2, \quad \hat{z}_0 = \frac{2\sqrt{-j\hat{b}}}{\hat{\alpha} + \hat{\beta}}, \quad \hat{\nu} = \frac{\sqrt{\hat{\beta}^2 + 4\lambda^2}}{\hat{\alpha} + \hat{\beta}}, \quad \hat{\alpha} = \alpha r_c, \quad \hat{\beta} = \beta r_c, \quad \hat{h} = \frac{h}{r_c}.$$

Here $r_c = y_1 - y_0$ is the distance between the wires.

Fig. 4.3 plots the change in impedance Z for three different values of $\hat{\beta} = 1, 2, 3$. The other parameters of the problem are fixed at $\hat{\alpha} = 0, \hat{h} = 0.05$ and $\mu_m = 5$. The calculated points in **Fig. 4.3** correspond to different values of $\hat{b} = 1, 2, \dots, 80$ (from left to right). Computations are done with Mathematica (see **Appendix Fig.Ap.9**).

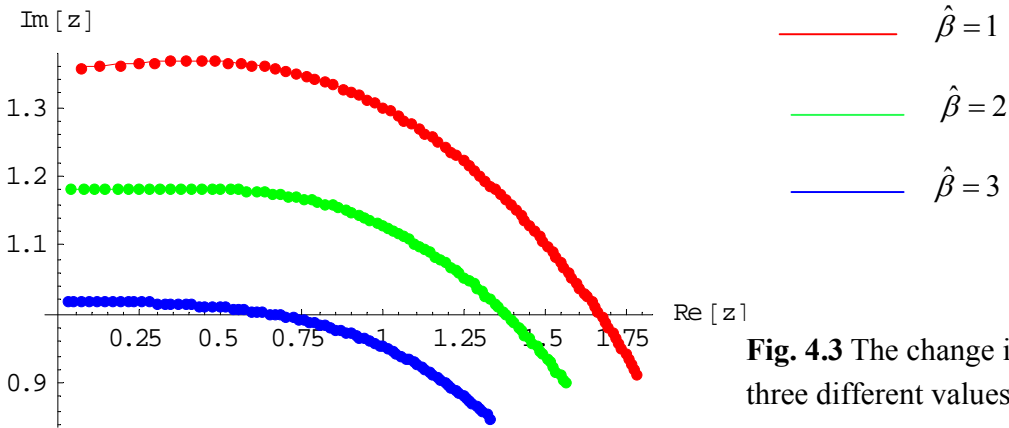


Fig. 4.3 The change in impedance Z for three different values of $\hat{\beta}$.

It is seen from the graph that the modulus of Z increases as the parameter $\hat{\beta}$ increases.

4.3 Double conductor line above a conducting two-layer medium with varying electrical and magnetic properties

Another example is discussed in this section (see [38]). Consider two parallel infinitely long wires with current located above a two-layer conducting medium (see **Fig. 4.4**).

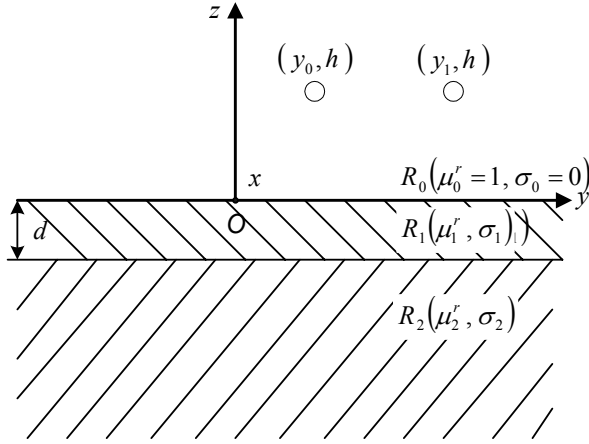


Fig. 4.4 A double conductor line above a two-layer medium.

We assume that the electrical conductivity and relative magnetic permeability of the upper layer are exponential functions of the vertical coordinate of the form

$$\mu_1^r(z) = \mu_1^m e^{\beta z}, \quad \sigma_1(z) = \sigma_1^m e^{\alpha z}. \quad (4.3.1)$$

In addition,

$$\mu_2^r(z) = \mu_2^m, \quad \sigma_2(z) = \sigma_2^m, \quad (4.3.2)$$

where $\sigma_1^m, \mu_1^m, \alpha, \beta$ are constants. The properties of region R_2 , that is, the electrical conductivity σ_2^m and the relative magnetic permeability μ_2^m are assumed to be constant.

The functions A_0, A_1 and A_2 satisfy the following system of equations:

$$R_0: \frac{\partial^2 A_0}{\partial y^2} + \frac{\partial^2 A_0}{\partial z^2} = \mu_0 I \delta(z-h) (\delta(y-y_1) - \delta(y-y_0)), \quad (4.3.3)$$

$$R_1: \frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2} - \beta \frac{\partial A_1}{\partial z} - j\omega \sigma_1^m \mu_0 \mu_1^m e^{(\alpha+\beta)z} A_1 = 0, \quad (4.3.4)$$

$$R_2: \frac{\partial^2 A_2}{\partial y^2} + \frac{\partial^2 A_2}{\partial z^2} - j\omega \sigma_2^m \mu_0 \mu_2^m A_2 = 0. \quad (4.3.5)$$

where $\delta(z)$ is the Dirac delta-function.

The boundary conditions are

$$A_0|_{z=0} = A_1|_{z=0}, \quad \frac{\partial A_0}{\partial z}|_{z=0} = \frac{1}{\mu_1^r(0)} \frac{\partial A_1}{\partial z}|_{z=0}, \quad (4.3.6)$$

$$A_1|_{z=-d} = A_2|_{z=-d}, \quad \frac{1}{\mu_1^r(-d)} \frac{\partial A_1}{\partial z}|_{z=-d} = \frac{1}{\mu_2^m} \frac{\partial A_2}{\partial z}|_{z=-d}. \quad (4.3.7)$$

The functions A_0, A_1, A_2 and their partial derivatives with respect to y tend to zero at infinity:

$$A_i, \quad \frac{\partial A_i}{\partial y} \rightarrow 0 \quad \text{as} \quad \sqrt{y^2 + z^2} \rightarrow \infty, \quad i = 0, 1, 2 \quad (4.3.8)$$

Problem (4.3.3) - (4.3.8) is solved by means of the Fourier cosine and sine integral transforms. Representing the solution in the form (4.1.9) and applying the Fourier cosine

transform (4.1.11) to the even part of the solution (using the results from **Section 4.1**) we obtain the system of ordinary differential equations and the solutions in regions R_0, R_1 and R_2 in the form

$$R_0: \frac{d^2 \tilde{A}_0^{even}}{dz^2} - \lambda^2 \tilde{A}_0^{even} = \frac{\mu_0 I}{2} \delta(z-h) (\cos(\lambda y_1) - \cos(\lambda y_0)) \rightarrow$$

$$\rightarrow \tilde{A}_0^{even}(\lambda, z) = C_2 e^{-\lambda z} + \frac{\mu_0 I}{4\lambda} (\cos(\lambda y_0) - \cos(\lambda y_1)) e^{-\lambda|h-z|}, \quad (4.3.9)$$

$$R_1: \frac{d^2 \tilde{A}_1^{even}}{dz^2} - \beta \frac{d\tilde{A}_1^{even}}{dz} - (\lambda^2 + j\omega\sigma_m \mu_0 \mu_m e^{(\alpha+\beta)z}) \tilde{A}_1^{even} = 0 \rightarrow$$

$$\rightarrow \tilde{A}_1^{even}(\lambda, z) = e^{\frac{\beta z}{2}} \left[C_4 I_\nu \left(\frac{2\sqrt{b}}{\alpha+\beta} e^{\frac{(\alpha+\beta)z}{2}} \right) + C_5 K_\nu \left(\frac{2\sqrt{b}}{\alpha+\beta} e^{\frac{(\alpha+\beta)z}{2}} \right) \right], \quad (4.3.10)$$

$$R_2: \frac{d^2 \tilde{A}_2^{even}}{dz^2} - q_2^2 \tilde{A}_2^{even} = 0 \rightarrow \tilde{A}_2^{even}(\lambda, z) = C_6 e^{q_2 z}, \quad (4.3.11)$$

where $\nu = \frac{\sqrt{\beta^2 + 4\lambda^2}}{\alpha + \beta}$, $b = -j\omega\mu_0\mu_1^m\sigma_1^m$, $q_2 = \sqrt{\lambda^2 + j\omega\sigma_2^m\mu_0\mu_2^m}$.

The boundary conditions are :

$$\tilde{A}_0^{even} |_{z=0} = \tilde{A}_1^{even} |_{z=0}, \quad \frac{d\tilde{A}_0^{even}}{dz} |_{z=0} = \frac{1}{\mu_1^r(0)} \frac{d\tilde{A}_1^{even}}{dz} |_{z=0},$$

$$\tilde{A}_1^{even} |_{z=-d} = \tilde{A}_2^{even} |_{z=-d}, \quad \frac{1}{\mu_1^r(-d)} \frac{d\tilde{A}_1^{even}}{dz} |_{z=-d} = \frac{1}{\mu_2^m} \frac{d\tilde{A}_2^{even}}{dz} |_{z=-d}. \quad (4.3.12)$$

Using (4.3.9) – (4.3.12) we obtain

$$C_2 + \frac{\mu_0 I}{4\lambda} (\cos(\lambda y_0) - \cos(\lambda y_1)) e^{-\lambda h} = C_4 I_\nu(z_0) + C_5 K_\nu(z_0), \quad (4.3.13)$$

$$-\lambda C_2 + \frac{\mu_0 I}{4} (\cos(\lambda y_0) - \cos(\lambda y_1)) e^{-\lambda h} =$$

$$= \frac{C_4}{\mu_1^m} \left(\frac{\beta}{2} I_\nu(z_0) + \sqrt{b} I_\nu'(z_0) \right) + \frac{C_5}{\mu_1^m} \left(\frac{\beta}{2} K_\nu(z_0) + \sqrt{b} K_\nu'(z_0) \right), \quad (4.3.14)$$

$$C_6 e^{-q_2 d} = e^{-\frac{\beta d}{2}} (C_4 I_\nu(z_1) + C_5 K_\nu(z_1)), \quad (4.3.15)$$

$$\frac{q_2}{\mu_2^m} C_6 e^{-q_2 d} =$$

$$= \frac{e^{\frac{\beta d}{2}}}{\mu_1^m} \left(C_4 \left(\frac{\beta}{2} I_\nu(z_1) + \sqrt{b} e^{-\frac{(\alpha+\beta)d}{2}} I_\nu'(z_1) \right) + C_5 \left(\frac{\beta}{2} K_\nu(z_1) + \sqrt{b} e^{-\frac{(\alpha+\beta)d}{2}} K_\nu'(z_1) \right) \right), \quad (4.3.16)$$

where $z_1 = z_0 e^{-\frac{(\alpha+\beta)d}{2}}$, $\mu_i b = -j\omega\mu_0\mu_1^m\sigma_1^m$.

Eliminating C_2 from (4.3.13) – (4.3.14) we obtain

$$C_4 = -\frac{C_5 \left((2\lambda\mu_1^m + \beta)K_\nu(z_0) + 2\sqrt{b}K'_\nu(z_0) \right) - \mu_0\mu_1^m I(\cos(\lambda y_0) - \cos(\lambda y_1))e^{-\lambda h}}{(2\lambda\mu_1^m + \beta)I_\nu(z_0) + 2\sqrt{b}I'_\nu(z_0)}. \quad (4.3.17)$$

Similarly, using (4.3.15) – (4.3.16) we get

$$C_4 = -\frac{C_5 \left(\left(\frac{\beta}{2} e^{\beta d} \mu_2^m - \mu_1^m q_2 \right) K_\nu(z_1) + \sqrt{b} \mu_2^m e^{\frac{(\beta-\alpha)d}{2}} K'_\nu(z_1) \right)}{\left(\frac{\beta}{2} e^{\beta d} \mu_2^m - \mu_1^m q_2 \right) I_\nu(z_1) + \sqrt{b} \mu_2^m e^{\frac{(\beta-\alpha)d}{2}} I'_\nu(z_1)}. \quad (4.3.18)$$

The solution to (4.3.17) – (4.3.18) is

$$C_5 = -\frac{\mu_0\mu_1^m I(\cos(\lambda y_0) - \cos(\lambda y_1))e^{-\lambda h} (D_1 I_\nu(z_1) + D_2 I'_\nu(z_1))}{D_1 D_3 D_4 + 2\sqrt{b} D_1 D_5 + D_2 D_3 D_6 + 2\sqrt{b} D_2 D_7}, \quad (4.3.19)$$

$$\text{where } D_1 = \frac{\beta}{2} e^{\beta d} \mu_2^m - \mu_1^m q_2, \quad D_2 = \sqrt{b} \mu_2^m e^{\frac{(\beta-\alpha)d}{2}}, \quad D_3 = 2\lambda\mu_1^m + \beta,$$

$$D_4 = I_\nu(z_0)K_\nu(z_1) - I_\nu(z_1)K_\nu(z_0), \quad D_6 = I_\nu(z_0)K'_\nu(z_1) - I'_\nu(z_1)K_\nu(z_0),$$

$$D_5 = I'_\nu(z_0)K_\nu(z_1) - I_\nu(z_1)K'_\nu(z_0), \quad D_7 = I'_\nu(z_0)K'_\nu(z_1) - I'_\nu(z_1)K'_\nu(z_0),$$

and

$$C_4 = \frac{\mu_0\mu_1^m I(\cos(\lambda y_0) - \cos(\lambda y_1))e^{-\lambda h} (D_1 K_\nu(z_1) + D_2 K'_\nu(z_1))}{D_1 D_3 D_4 + 2\sqrt{b} D_1 D_5 + D_2 D_3 D_6 + 2\sqrt{b} D_2 D_7}. \quad (4.3.20)$$

The value of C_2 is given by

$$\left\| C_2 = C_4 I_\nu(z_0) + C_5 K_\nu(z_0) - \frac{\mu_0 I(\cos(\lambda y_0) - \cos(\lambda y_1))e^{-\lambda h}}{4\lambda} \right\|$$

$$C_2^{\text{even}} = \mu_0 I(\cos(\lambda y_0) - \cos(\lambda y_1))e^{-\lambda h} \left(\frac{\mu_1^m (D_1 D_4 + D_2 D_6)}{D_1 D_3 D_4 + 2\sqrt{b} D_1 D_5 + D_2 D_3 D_6 + 2\sqrt{b} D_2 D_7} - \frac{1}{4\lambda} \right). \quad (4.3.21)$$

Similarly, we obtain

$$C_2^{\text{odd}} = \mu_0 I(\sin(\lambda y_0) - \sin(\lambda y_1))e^{-\lambda h} \left(\frac{\mu_1^m (D_1 D_4 + D_2 D_6)}{D_1 D_3 D_4 + 2\sqrt{b} D_1 D_5 + D_2 D_3 D_6 + 2\sqrt{b} D_2 D_7} - \frac{1}{4\lambda} \right). \quad (4.3.22)$$

The induced vector potential in region R_0 is

$$\tilde{A}_0^{\text{ind}}(\lambda, z) = (C_2^{\text{even}} + C_2^{\text{odd}})e^{-\lambda z}. \quad (4.3.23)$$

The solution in region R_0 can be written in the form

$$A_0(y, z) = A_0^{\text{free}}(y, z) + A_0^{\text{ind}}(y, z). \quad (4.3.24)$$

Applying the inverse Fourier cosine and sine transforms (4.1.30) and (4.1.32) to (4.3.23) we obtain

$$A_0^{\text{ind}}(y, z) = \frac{2\mu_0 I}{\pi} \int_0^\infty F(\lambda) (\cos(\lambda(y - y_0)) - \cos(\lambda(y - y_1))) e^{-\lambda(z+h)} d\lambda, \quad (4.3.25)$$

where

$$F(\lambda) = \frac{\mu_1^m (D_1 D_4 + D_2 D_6)}{D_1 D_3 D_4 + 2\sqrt{b} D_1 D_5 + D_2 D_3 D_6 + 2\sqrt{b} D_2 D_7} - \frac{1}{4\lambda}. \quad (4.3.26)$$

The induced change in impedance (per unit length) of the double conductor line is computed by means of the formula

$$Z_{\text{per unit length}}^{\text{ind}}(y, z) = \frac{j\omega}{I} \oint_C A_0^{\text{ind}}(y, z) dx. \quad (4.3.27)$$

Substituting (4.3.25) into (4.3.27) we obtain the change in impedance in the form

$$Z_{\text{per unit length}}^{\text{ind}}(y, z) = \frac{4\mu_0\omega}{\pi} j \int_0^\infty F(\lambda) (1 - \cos(\lambda(y_1 - y_0))) e^{-2\lambda h} d\lambda, \quad (4.3.28)$$

where $F(\lambda)$ is given by (4.3.26).

Formula (4.3.28) can be rewritten in the form

$$Z_{\text{per unit length}}^{\text{ind}}(y, z) = \frac{4\mu_0\omega}{\pi} Z, \quad (4.3.29)$$

where

$$Z = j \int_0^\infty \left(\frac{\mu_1^m (\hat{D}_1 \hat{D}_4 + \hat{D}_2 \hat{D}_6)}{\hat{D}_1 \hat{D}_3 \hat{D}_4 + 2\sqrt{-j\hat{b}} \hat{D}_1 \hat{D}_5 + \hat{D}_2 \hat{D}_3 \hat{D}_6 + 2\sqrt{-j\hat{b}} \hat{D}_2 \hat{D}_7} - \frac{1}{4u} \right) (1 - \cos u) e^{-2u\hat{h}} du, \quad (4.3.30)$$

$$\text{and } \hat{D}_1 = \frac{\hat{\beta}}{2} e^{\hat{\beta}\hat{d}} \mu_2^m - \mu_1^m \hat{q}_2, \quad \hat{D}_2 = \sqrt{-j\hat{b}} \mu_2^m e^{\frac{(\hat{\beta}-\hat{\alpha})\hat{d}}{2}}, \quad \hat{D}_3 = 2u\mu_1^m + \hat{\beta},$$

$$\begin{aligned} \hat{D}_4 &= I_\nu(\hat{z}_0) K_\nu(\hat{z}_1) - I_\nu(\hat{z}_1) K_\nu(\hat{z}_0), & \hat{D}_6 &= I'_\nu(\hat{z}_0) K'_\nu(\hat{z}_1) - I'_\nu(\hat{z}_1) K'_\nu(\hat{z}_0), \\ \hat{D}_5 &= I'_\nu(\hat{z}_0) K_\nu(\hat{z}_1) - I_\nu(\hat{z}_1) K'_\nu(\hat{z}_0), & \hat{D}_7 &= I'_\nu(\hat{z}_0) K'_\nu(\hat{z}_1) - I'_\nu(\hat{z}_1) K'_\nu(\hat{z}_0). \end{aligned}$$

The dimensionless parameters are defined as follows

$$\begin{aligned} u &= \lambda r_c, \quad \hat{b} = \omega\mu_0\mu_1^m \sigma_1^m r_c^2, \quad \hat{q}_2 = \sqrt{u^2 + j\omega\mu_0\mu_2^m \sigma_2^m r_c^2}, \quad \hat{\nu} = \frac{\sqrt{\hat{\beta}^2 + 4\lambda^2}}{\hat{\alpha} + \hat{\beta}}, \\ \hat{z}_1 &= \hat{z}_0 e^{-\frac{(\hat{\alpha}+\hat{\beta})\hat{d}}{2}}, \quad \hat{z}_0 = \frac{2\sqrt{-j\hat{b}}}{\hat{\alpha} + \hat{\beta}}, \\ \hat{\alpha} &= \alpha r_c, \quad \hat{\beta} = \beta r_c, \quad \hat{h} = \frac{h}{r_c}, \quad \hat{d} = \frac{d}{r_c}. \end{aligned}$$

Here $r_c = y_1 - y_0$ is the distance between the wires.

The change in impedance given by formula (4.3.30) is computed for different values of the parameters of the problem using package ‘‘Mathematica’’ since it allows one to calculate improper integrals and evaluate modified Bessel functions of variable order and complex argument. The results of calculations are presented in **Fig. 4.5**.

Fig. 4.5 plots the change in impedance Z for three different values of $\hat{\beta} = 1, 2, 3$. The other parameters of the problem are fixed at $\hat{\alpha} = 0, \hat{h} = 0.05, \hat{d} = 0.05$ and $\mu_1^m = 1$. The calculated points in **Fig. 4.5** correspond to different values of $\hat{b} = 1, 2, \dots, 10$ (from left to right). Computations are done with ‘‘Mathematica’’ (see **Appendix Fig.Ap.10**).

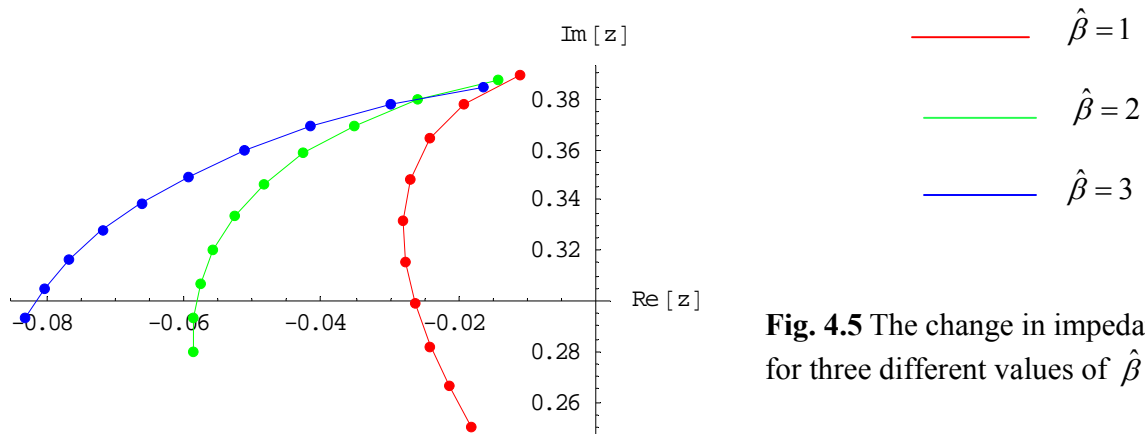


Fig. 4.5 The change in impedance Z for three different values of $\hat{\beta}$.

4.4 Pulsed eddy currents a conducting half-space

Theory of an eddy current method for the case where a coil is excited by an alternating current is well-developed in the literature [15], [60]. Current excitation in the form of a pulse represents an alternative to traditional eddy current methods. There are two basic methods that are usually used to analyze non-periodic time-dependent signals in eddy current testing: fast Fourier transform and Laplace transform. The difficulty in using the Laplace transform is that the inverse transform is not always available in closed form. However, several authors [68], [9], [11], [42] have obtained analytical solutions for problems where a single-turn coil or a coil with finite dimensions is located above a conducting half-space assuming that the excitation current is in the form of a step current or exponential source current. Analytical solutions are used in practice in order to estimate thickness and conductivity of metallic layers [56].

In this section we consider transient eddy current problem for the case where an excitation coil is assumed to be of the form of a double conductor line formed by two infinitely long wires located parallel to a conducting half-space [29]. The inverse Laplace transform is found in closed form for the case of an excitation current in the form of a unit step function.

Different types of eddy current coils are used in applications. One example is a coil in the form of a rectangular frame. If the ratio of the sides of the frame is 4:1 or larger then such a coil can be modeled by a double conductor line [67], [72] with a relatively small error. Suppose that two horizontal infinitely long parallel wires are situated above a conducting half-space with electrical conductivity σ and relative magnetic permeability μ (see **Fig. 4.6**)

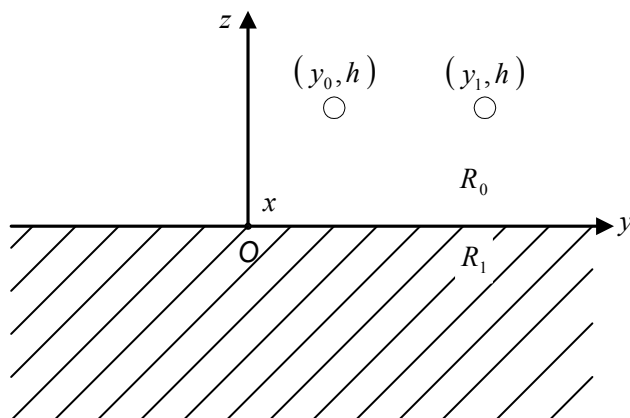


Fig. 4.6 A double conductor line above a conducting half-space.

Assume that the current in the wires is

$$I(t) = \pm I_0 \varphi(t) \quad (4.4.1)$$

at the points (y_0, h) and (y_1, h) , respectively, where I_0 is a constant, and $\varphi(t)$ is the Heaviside function of the form

$$\varphi(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \quad (4.4.2)$$

We assume that the electric field \vec{E}_0 and \vec{E}_1 in regions $R_0 = \{z > 0\}$ and $R_1 = \{z < 0\}$, respectively, has only one non-zero component in the x -direction:

$$\vec{E}_i = E_i(y, z, t) \vec{e}_x, \quad i = 0, 1, \quad (4.4.3)$$

where \vec{e}_x is the unit vector in the positive x -direction. The system of Maxwell's equations in this case can be transformed to one equation for the electric field which has the following form in regions R_0 and R_1 :

$$R_0: \frac{\partial^2 E_0}{\partial y^2} + \frac{\partial^2 E_0}{\partial z^2} = \mu_0 \frac{\partial I}{\partial t} \delta(z-h) (\delta(y-y_1) - \delta(y-y_0)), \quad (4.4.4)$$

$$R_1: \frac{\partial^2 E_1}{\partial y^2} + \frac{\partial^2 E_1}{\partial z^2} - \mu_0 \mu_1^m \sigma_1^m \frac{\partial E_1}{\partial t} = 0, \quad (4.4.5)$$

where $\delta(\xi)$ is the Dirac delta-function.

The boundary conditions are

$$E_0|_{z=0} = E_1|_{z=0}, \quad \frac{\partial E_0}{\partial z}|_{z=0} = \frac{1}{\mu_1^m} \frac{\partial E_1}{\partial z}|_{z=0}. \quad (4.4.6)$$

In addition, the following conditions hold at infinity:

$$E_0, E_1, \frac{\partial E_0}{\partial y}, \frac{\partial E_1}{\partial y} \rightarrow 0 \text{ as } \sqrt{y^2 + z^2} \rightarrow \infty. \quad (4.4.7)$$

The initial conditions are

$$E_0|_{t=0} = 0, \quad E_1|_{t=0} = 0. \quad (4.4.8)$$

Applying the Laplace transform to (4.4.4) - (4.4.8) we obtain

$$R_0: \frac{\partial^2 \bar{E}_0}{\partial y^2} + \frac{\partial^2 \bar{E}_0}{\partial z^2} = \mu_0 s \bar{I}(s) \delta(z-h) (\delta(y-y_1) - \delta(y-y_0)), \quad (4.4.9)$$

$$R_1: \frac{\partial^2 \bar{E}_1}{\partial y^2} + \frac{\partial^2 \bar{E}_1}{\partial z^2} - \mu_0 \mu_1^m \sigma_1^m s \bar{E}_1 = 0, \quad (4.4.10)$$

$$\bar{E}_0|_{z=0} = \bar{E}_1|_{z=0}, \quad \frac{\partial \bar{E}_0}{\partial z}|_{z=0} = \frac{1}{\mu_1^m} \frac{\partial \bar{E}_1}{\partial z}|_{z=0}, \quad (4.4.11)$$

$$\bar{E}_0, \bar{E}_1, \frac{\partial \bar{E}_0}{\partial y}, \frac{\partial \bar{E}_1}{\partial y} \rightarrow 0 \text{ as } \sqrt{y^2 + z^2} \rightarrow \infty, \quad (4.4.12)$$

where \bar{E}_0, \bar{E}_1 and \bar{I} are the Laplace transforms of the functions E_0, E_1 and I , respectively, s is the parameter of the Laplace transform.

Problem (4.4.9) - (4.4.12) is solved by the method of integral transforms. It is convenient to represent the solution \bar{E}_i in each region R_i as the sum of even and odd components of the form

$$\bar{E}_i(y, z, s) = \bar{E}_i^{even}(y, z, s) + \bar{E}_i^{odd}(y, z, s), \quad i = 0, 1. \quad (4.4.13)$$

For this purpose, we represent the right-hand side of (4.4.9) in a similar form

$$\left\| \begin{aligned} \bar{f}(y, z, s) &= \frac{1}{2} [\bar{f}(y, z, s) + \bar{f}(-y, z, s)] + \frac{1}{2} [\bar{f}(y, z, s) - \bar{f}(-y, z, s)] = \\ &= \bar{f}^{even}(y, z, s) + \bar{f}^{odd}(y, z, s) \end{aligned} \right\|$$

$$\bar{f}^{even}(y, z, s) = \frac{\mu_0 s \bar{I}(s)}{2} \delta(z-h) (\delta(y-y_1) - \delta(y-y_0) + \delta(y+y_1) - \delta(y+y_0)) \quad (4.4.14)$$

$$\bar{f}^{odd}(y, z, s) = \frac{\mu_0 s \bar{I}(s)}{2} \delta(z-h) (\delta(y-y_1) - \delta(y-y_0) - \delta(y+y_1) + \delta(y+y_0))$$

The following properties of the delta-function are used in order to derive (4.4.14) (see[40])

$$\left\| \begin{aligned} &\cdot \delta(-y) = \delta(y) \\ &\cdot \begin{cases} \delta(y+y_0) = 0 \\ \delta(y+y_1) = 0 \end{cases} \text{ if } y \geq 0, y_0 \geq 0, y_1 \geq 0 \end{aligned} \right\|$$

$$\bar{f}^{even}(y, z, s) = \bar{f}^{odd}(y, z, s) = \frac{\mu_0 s \bar{I}(s)}{2} \delta(z-h) (\delta(y-y_1) - \delta(y-y_0)) \quad (4.4.15)$$

The even component of the solution is found by applying the Fourier cosine transform of the form

$$\tilde{\bar{E}}_i^{even}(\lambda, z, s) = \int_0^{\infty} \bar{E}_i^{even}(y, z, s) \cos(\lambda y) dy, \quad i = 0, 1. \quad (4.4.16)$$

The odd component of the solution is found by applying the Fourier sine transform of the form

$$\tilde{\bar{E}}_i^{odd}(\lambda, z, s) = \int_0^{\infty} \bar{E}_i^{odd}(y, z, s) \sin(\lambda y) dy, \quad i = 0, 1. \quad (4.4.17)$$

Applying the Fourier cosine transform (4.4.16) to (4.4.9) - (4.4.12) we obtain

$$R_0: \frac{d^2 \tilde{\bar{E}}_0^{even}}{dz^2} - \lambda^2 \tilde{\bar{E}}_0^{even} = \frac{\mu_0 s \bar{I}(s)}{2} \delta(z-h) (\cos \lambda y_1 - \cos \lambda y_0), \quad (4.4.18)$$

$$R_1: \frac{d^2 \tilde{\bar{E}}_1^{even}}{dz^2} - q^2 \tilde{\bar{E}}_1^{even} = 0, \quad \text{where } q = \sqrt{\lambda^2 + \mu_0 \mu_1^m \sigma_1^m s}. \quad (4.4.19)$$

$$\tilde{\bar{E}}_0^{even} |_{z=0} = \tilde{\bar{E}}_1^{even} |_{z=0}, \quad \frac{d\tilde{\bar{E}}_0^{even}}{dz} |_{z=0} = \frac{1}{\mu_1^m} \frac{d\tilde{\bar{E}}_1^{even}}{dz} |_{z=0}, \quad (4.4.20)$$

In order to solve (4.4.18) we consider two sub-regions, $R_{00} = \{0 < z < h\}$ and

$R_{01} = \{z > h\}$, of region R_0 . The solutions to (4.4.18) in R_{00} and R_{01} are denoted by

$\tilde{\bar{E}}_{00}^{even}$ and $\tilde{\bar{E}}_{01}^{even}$, respectively.

The general solution to (4.4.18) in R_{00} is

$$\tilde{\bar{E}}_{00}^{even}(\lambda, z, s) = C_1 e^{\lambda z} + C_2 e^{-\lambda z}. \quad (4.4.21)$$

The general solution to (4.4.18) in R_{01} which is bounded as $z \rightarrow +\infty$ has the form

$$\tilde{\bar{E}}_{01}^{even}(\lambda, z, s) = C_3 e^{-\lambda z}. \quad (4.4.22)$$

Finally, the general solution to (4.4.19) which remains bounded as $z \rightarrow -\infty$ can be written as follows

$$\widetilde{E}_1^{even}(\lambda, z, s) = C_4 e^{qz}. \quad (4.4.23)$$

Since there are four unknown constants in (4.4.21) - (4.4.23) and only two boundary conditions (4.4.20), we need two additional conditions at $z = h$. One additional condition represents continuity of the electric field at $z = h$ and has the form

$$\widetilde{E}_{00}^{even} \Big|_{z=h} = \widetilde{E}_{01}^{even} \Big|_{z=h}. \quad (4.4.24)$$

Integrating (4.4.18) with respect to z from $h - \varepsilon$ to $h + \varepsilon$ and taking the limit as $\varepsilon \rightarrow +0$ we obtain the second additional condition in the form

$$\frac{d\widetilde{E}_{01}^{even}}{dz} \Big|_{z=h} - \frac{d\widetilde{E}_{00}^{even}}{dz} \Big|_{z=h} = \frac{\mu_0 s \bar{I}(s)}{2} (\cos(\lambda y_1) - \cos(\lambda y_0)). \quad (4.4.25)$$

Using (4.4.21), (4.4.22), (4.4.24), and (4.4.25) we obtain

$$\begin{cases} C_1 e^{\lambda h} + C_2 e^{-\lambda h} = C_3 e^{-\lambda h}, \\ -\lambda C_3 e^{-\lambda h} - \lambda C_1 e^{\lambda h} + \lambda C_2 e^{-\lambda h} = \frac{\mu_0 s \bar{I}(s)}{2} (\cos(\lambda y_1) - \cos(\lambda y_0)), \end{cases} \Rightarrow$$

$$\left\| \begin{array}{l} C_1 e^{2\lambda h} + C_2 - C_3 = 0 \\ -C_1 e^{2\lambda h} + C_2 - C_3 = \frac{\mu_0 s \bar{I}(s)}{2\lambda} (\cos(\lambda y_1) - \cos(\lambda y_0)) e^{\lambda h} \end{array} \right\|$$

$$C_1 = \frac{\mu_0 s \bar{I}(s)}{4\lambda} (\cos(\lambda y_0) - \cos(\lambda y_1)) e^{-\lambda h}, \quad (4.4.26)$$

$$C_3 = C_2 + \frac{\mu_0 s \bar{I}(s)}{4\lambda} (\cos(\lambda y_0) - \cos(\lambda y_1)) e^{\lambda h}. \quad (4.4.27)$$

Thus, solutions (4.4.21) and (4.4.22) can be rewritten as follows

$$\begin{cases} \widetilde{E}_{00}^{even} = C_2 e^{-\lambda z} + \frac{\mu_0 s \bar{I}(s)}{4\lambda} (\cos(\lambda y_0) - \cos(\lambda y_1)) e^{-\lambda(h-z)} \\ \widetilde{E}_{01}^{even} = C_2 e^{-\lambda z} + \frac{\mu_0 s \bar{I}(s)}{4\lambda} (\cos(\lambda y_0) - \cos(\lambda y_1)) e^{-\lambda(z-h)} \end{cases} \Rightarrow$$

$$\widetilde{E}_0^{even} = C_2 e^{-\lambda z} + \frac{\mu_0 s \bar{I}(s)}{4\lambda} (\cos(\lambda y_0) - \cos(\lambda y_1)) e^{-\lambda|h-z|}. \quad (4.4.28)$$

Using (4.4.20), (4.4.21) and (4.4.23) we obtain

$$\begin{cases} C_1 + C_2 = C_4 \\ C_1 - C_2 = \frac{q}{\lambda \mu_1^m} C_4 \end{cases} \Rightarrow \begin{cases} C_4 = \frac{2\lambda \mu_1^m}{\lambda \mu_1^m + q} C_1, \\ C_2 = \frac{\lambda \mu_1^m - q}{2\lambda \mu_1^m} C_4, \end{cases}$$

$$C_4 = \frac{\mu_0 \mu_1^m s \bar{I}(s)}{2(\lambda \mu_1^m + q)} (\cos(\lambda y_0) - \cos(\lambda y_1)) e^{-\lambda h}, \quad (4.4.29)$$

$$C_2 = \frac{\lambda\mu_1^m - q}{\lambda\mu_1^m + q} \frac{\mu_0 s \bar{I}(s)}{4\lambda} (\cos(\lambda y_0) - \cos(\lambda y_1)) e^{-\lambda h}. \quad (4.4.30)$$

Using the inverse Fourier cosine transform of the form

$$\bar{E}_i^{even}(y, z, s) = \frac{2}{\pi} \int_0^\infty \tilde{\bar{E}}_i^{even}(\lambda, z, s) \cos(\lambda y) d\lambda, \quad i = 0, 1 \quad (4.4.31)$$

we obtain the solution $\bar{E}_i^{even}(y, z, s)$ in regions R_0, R_1 :

$$R_0 : \bar{E}_0^{even}(y, z, s) = \bar{E}_{0(free)}^{even}(y, z, s) + \bar{E}_{0(ind)}^{even}(y, z, s), \quad (4.4.32)$$

where

$$\bar{E}_{0(free)}^{even}(y, z, s) = \frac{\mu_0 s \bar{I}(s)}{2\pi} \int_0^\infty \frac{e^{-\lambda|z-h|}}{\lambda} (\cos(\lambda y_0) - \cos(\lambda y_1)) \cos(\lambda y) d\lambda, \quad (4.4.33)$$

$$\bar{E}_{0(ind)}^{even}(y, z, s) = \frac{\mu_0 s \bar{I}(s)}{2\pi} \int_0^\infty \frac{\lambda\mu_1^m - q}{\lambda\mu_1^m + q} \frac{e^{-\lambda(z+h)}}{\lambda} (\cos(\lambda y_0) - \cos(\lambda y_1)) \cos(\lambda y) d\lambda. \quad (4.4.34)$$

$$R_1 : \bar{E}_1^{even}(y, z, s) = \frac{\mu_0 \mu_1^m s \bar{I}(s)}{\pi} \int_0^\infty \frac{e^{qz-\lambda h}}{\lambda\mu_1^m + q} (\cos(\lambda y_0) - \cos(\lambda y_1)) \cos(\lambda y) d\lambda. \quad (4.4.35)$$

Here $\bar{E}_{0(free)}^{even}(y, z, s)$ is the even component of the electric field in free space while

$\bar{E}_{0(ind)}^{even}(y, z, s)$ is the even component of the induced electric field due to the presence of the conducting half-space.

The Fourier sine transform of the form (4.4.17) and the inverse Fourier sine transform

$$\bar{E}_i^{odd}(y, z, s) = \frac{2}{\pi} \int_0^\infty \tilde{\bar{E}}_i^{odd}(\lambda, z, s) \sin(\lambda y) d\lambda, \quad i = 0, 1 \quad (4.4.36)$$

can be used in order to find the odd components of the solution. It can easily be shown that the odd components of the solution, $\bar{E}_0^{odd}(y, z, s)$ and $\bar{E}_1^{odd}(y, z, s)$ are given by (4.4.32) – (4.4.35) where cosine is replaced by sine.

Finally, the solution to (4.4.9) - (4.4.12) is obtained as the sum of the even and odd components and can be written as follows

$$\bar{E}_0(y, z, s) = \bar{E}_{0(free)}(y, z, s) + \bar{E}_{0(ind)}(y, z, s), \quad (4.4.37)$$

$$\bar{E}_1(y, z, s) = \frac{\mu_0 \mu_1^m s \bar{I}(s)}{\pi} \int_0^\infty \frac{e^{qz-\lambda h}}{\lambda\mu_1^m + q} (\cos(\lambda(y-y_0)) - \cos(\lambda(y-y_1))) d\lambda. \quad (4.4.38)$$

where

$$\bar{E}_{0(free)}(y, z, s) = \frac{\mu_0 s \bar{I}(s)}{2\pi} \int_0^\infty \frac{e^{-\lambda|z-h|}}{\lambda} (\cos(\lambda(y-y_0)) - \cos(\lambda(y-y_1))) d\lambda, \quad (4.4.39)$$

is the electric field of the double conductor line in free space and

$$\bar{E}_{0(ind)}(y, z, s) = \frac{\mu_0 s \bar{I}(s)}{2\pi} \int_0^\infty \frac{\lambda\mu_1^m - q}{\lambda\mu_1^m + q} \frac{e^{-\lambda(z+h)}}{\lambda} (\cos(\lambda(y-y_0)) - \cos(\lambda(y-y_1))) d\lambda \quad (4.4.40)$$

is the induced electric field in free space due to the presence of the conducting half-space.

The Laplace transform of the induced electromotive force (EMF) in a double conductor line per unit length due to the presence of the conducting half-space is given by (see [9]):

$$v(s) = \frac{1}{\bar{I}(s)} \int_C \bar{E}_0 (ind)(y, z, s) dl, \quad (4.4.41)$$

where C is the contour of the double conductor line of length one unit in the x – direction.

In order to determine the inverse Laplace transform of (4.4.40) we rewrite the expression

$$\frac{\lambda\mu_1^m - q}{\lambda\mu_1^m + q} \text{ in the form } \frac{\lambda\mu_1^m - q}{\lambda\mu_1^m + q} = \frac{2\mu_1^m}{\mu_1^m + \sqrt{1+s\tau}} - 1, \text{ where } \tau = \frac{\mu_0\mu_1^m\sigma_1^m}{\lambda^2}.$$

$$\left\| \cdot \frac{\lambda\mu_1^m - q}{\lambda\mu_1^m + q} = -\frac{q + \lambda\mu_1^m - 2\lambda\mu_1^m}{\lambda\mu_1^m + q} = -1 + \frac{2\lambda\mu_1^m}{\lambda\mu_1^m + \sqrt{\lambda^2 + \mu_0\mu_1^m\sigma_1^m s}} = \frac{2\mu_1^m}{\mu_1^m + \sqrt{1+s\tau}} - 1 \right\|$$

Consider the case of a step current in the form (4.4.1).

The Laplace transform of (4.4.1) is $\bar{I}(s) = \frac{I_0}{s}$ so that

$$\Phi(\lambda, s) = s\bar{I}(s) \frac{\lambda\mu_1^m - q}{\lambda\mu_1^m + q} = I_0 \left(\frac{2\mu_1^m}{\mu_1^m + \sqrt{1+s\tau}} - 1 \right). \quad (4.4.42)$$

The inverse Laplace transform of (4.4.42) can be obtained either by means of “Mathematica” command `InverseLaplaceTransform` or using tables of the Laplace transform (see [2], page 810):

$$\Phi(\lambda, t) = \frac{2\mu_1^m I_0}{\tau} e^{-\frac{t}{\tau}} \left(\sqrt{\frac{\tau}{\pi t}} - \mu_1^m e^{\frac{(\mu_1^m)^2 t}{\tau}} \operatorname{erfc} \left(\mu_1^m \sqrt{\frac{t}{\tau}} \right) \right) - I_0 \delta(t). \quad (4.4.43)$$

Using (4.4.40) - (4.4.42) we obtain the inverse Laplace transform of the induced electric field in region R_0 in the form

$$E_0 (ind)(y, z, t) = \frac{\mu_0 I_0}{2\pi} \int_0^\infty e^{-\lambda(z+h)} (\cos(\lambda(y-y_0)) - \cos(\lambda(y-y_1))) \cdot \left(\frac{2\mu_1^m}{\tau} e^{-\frac{t}{\tau}} \left(\sqrt{\frac{\tau}{\pi t}} - \mu_1^m e^{\frac{(\mu_1^m)^2 t}{\tau}} \operatorname{erfc} \left(\mu_1^m \sqrt{\frac{t}{\tau}} \right) \right) - \delta(t) \right) \frac{d\lambda}{\lambda} \quad (4.4.44)$$

The induced IMF is obtained from (4.4.41) and (4.4.44) in the form

$$v(t) = \frac{\mu_0}{\pi} \int_0^\infty e^{-2\lambda h} (1 - \cos(\lambda(y_1 - y_0))) \left(\frac{2\mu_1^m}{\tau} e^{-\frac{t}{\tau}} \left(\sqrt{\frac{\tau}{\pi t}} - \mu_1^m e^{\frac{(\mu_1^m)^2 t}{\tau}} \operatorname{erfc} \left(\mu_1^m \sqrt{\frac{t}{\tau}} \right) \right) - \delta(t) \right) \frac{d\lambda}{\lambda}. \quad (4.4.45)$$

Using (4.4.45) we compute the integral of the EMF with respect to time:

$$V(T) = \int_0^T v(t) dt. \quad (4.4.46)$$

The integral with respect to time can be computed, for example, by Mathematica command `Integrate`. The result is

$$V(T) = \frac{\mu_0}{\pi} \int_0^\infty e^{-2\lambda h} [\cos \lambda(y_1 - y_0) - 1] \psi(T, \tau, \mu) \frac{d\lambda}{\lambda}, \quad (4.4.47)$$

where

$$\psi(T, \tau, \mu) = -\frac{1}{2} + 2\mu \operatorname{erf}\left(\sqrt{\frac{T}{\tau}}\right) - \frac{2\mu^2 \left\{ -1 + \mu \operatorname{erf}\left(\sqrt{\frac{T}{\tau}}\right) + \exp\left[\frac{T(\mu^2 - 1)}{\tau}\right] \operatorname{erfc}\left[\mu\sqrt{\frac{T}{\tau}}\right] \right\}}{\mu^2 - 1}.$$

Integral (4.4.47) is computed by using “Mathematica” (see **Appendix Fig.Ap.11**) for the following values of the parameters of the problem: $y_1 - y_0 = 1$, $\mu = 2$, $h = 0.1$ and three values of σ , namely, $\sigma = 10^7 S/m$, $\sigma = 5 \cdot 10^7 S/m$ and $\sigma = 10^8 S/m$. The results are shown in **Fig. 4.7**.

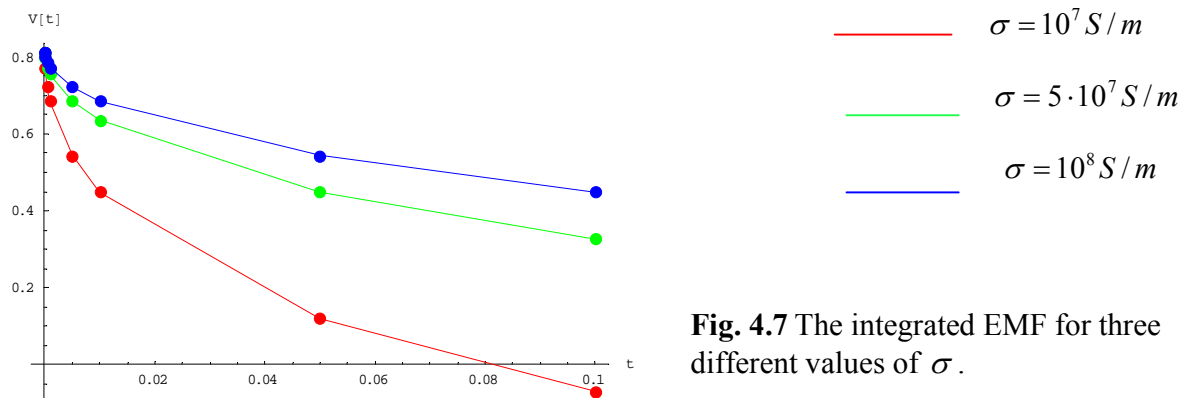


Fig. 4.7 The integrated EMF for three different values of σ .

5. MOVING PLANAR MEDIUM WITH VARYING PROPERTIES

5.1 Moving halfspace with varying electrical conductivity and magnetic permeability

In some cases (for example, during steel processing at metallurgical plants) a conducting medium is moving with constant velocity in a horizontal direction. Analytical solutions for moving media with constant properties are presented in [1], [18]. Such solutions can serve as a basis for the solution of the inverse problems. It is shown, for example, in [48] how to determine resistivity, permeability and thickness of a moving metal sheet simultaneously using a double coil method.

In the present section we calculate the change in impedance of a coil located above a conducting half-space [28]. The half-space is moving with constant velocity in a horizontal direction. The properties of a conducting half-space (the electrical conductivity and magnetic permeability) are assumed to vary with depth. In particular, analytical solution is constructed for the case where both electrical conductivity and relative magnetic permeability are exponential functions of the vertical coordinate. The problem is solved by the method of a double Fourier integral transform with respect to the x and y coordinates. The change in impedance of a single-turn coil is expressed in terms of a double integral containing the parameters of the problem. Results of numerical calculations are presented.

Consider a single-turn coil of radius r_c located above a conducting half-space moving with constant velocity V in the x direction (see **Fig. 5.1**).

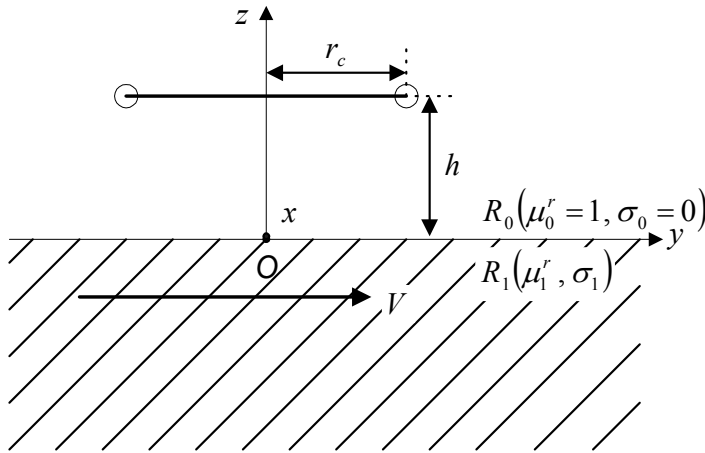


Fig. 5.1 A single-turn coil located above a conducting half-space.

In this case the vector potential has only two nonzero components of the form

$$\vec{A}(x, y, z) = A_x(x, y, z)\vec{e}_x + A_y(x, y, z)\vec{e}_y, \quad (5.1.1)$$

where \vec{e}_x and \vec{e}_y are the unit vectors in the x and y directions, respectively. The external current in the coil has the form

$$\vec{I}^e = I_\varphi \vec{e}_\varphi. \quad (5.1.2)$$

The components of the external current \vec{I}^e can be written in the form

$$\vec{I}^e = I_x \vec{e}_x + I_y \vec{e}_y = \left\| \begin{array}{l} \vec{e}_x = \vec{e}_r \cos \varphi_r - \vec{e}_\varphi \sin \varphi_r \\ \vec{e}_y = \vec{e}_r \sin \varphi_r + \vec{e}_\varphi \cos \varphi_r \\ I_\varphi = I \delta(z-h) \delta(r-r_c) \end{array} \right\| = -I_\varphi \sin \varphi_r \vec{e}_\varphi + I_\varphi \cos \varphi_r \vec{e}_\varphi. \quad (5.1.3)$$

It can be shown that the system of the Maxwell's equations in this case can be transformed to the form

$$\Delta \vec{A} - \frac{1}{\mu^r(z)} \frac{d\mu^r}{dz} \left(\frac{\partial A_x}{\partial z} \vec{e}_x + \frac{\partial A_y}{\partial z} \vec{e}_y \right) - j\omega\mu_0\mu^r(z)\sigma(z)\vec{A} - \mu_0\mu^r(z)\sigma(z)V \frac{\partial \vec{A}}{\partial x} = -\mu_0\mu^r(z)(I_x \vec{e}_x + I_y \vec{e}_y), \quad (5.1.4)$$

where the term containing V represents the effect of a moving halfspace.

Projecting (5.1.4) on the x and y axes we obtain

$$\left\{ \begin{array}{l} \Delta A_x - \frac{1}{\mu^r(z)} \frac{d\mu^r}{dz} \frac{\partial A_x}{\partial z} - j\omega\mu_0\mu^r(z)\sigma(z)A_x - \mu_0\mu^r(z)\sigma(z)V \frac{\partial A_x}{\partial x} = -\mu_0\mu^r(z)I_x, \\ \Delta A_y - \frac{1}{\mu^r(z)} \frac{d\mu^r}{dz} \frac{\partial A_y}{\partial z} - j\omega\mu_0\mu^r(z)\sigma(z)A_y - \mu_0\mu^r(z)\sigma(z)V \frac{\partial A_y}{\partial x} = -\mu_0\mu^r(z)I_y. \end{array} \right. \quad (5.1.5)$$

The x component of the vector potential in region R_0 (see **Fig. 5.1**) satisfies the following equation:

$$\begin{aligned} R_0 : \mu^r(z) &= 1, \quad \sigma(z) = 0. \\ \Delta A_{x_0} &= \mu_0 I \delta(z-h) \delta(r-r_c) \sin \varphi_r, \end{aligned} \quad (5.1.6)$$

where $\delta(x)$ is the Dirac delta-function.

The electrical conductivity $\sigma(z)$ and magnetic permeability $\mu(z)$ in region R_1 are assumed to be of the form

$$R_1 : \mu_1^r(z) = \mu_m e^{\beta z}, \quad \sigma(z) = \sigma_1(z) = \sigma_m e^{\alpha z}. \quad (5.1.7)$$

The equation for the x component of the vector potential in region R_1 is

$$\Delta A_{x_1} = \frac{1}{\mu_1^r(z)} \frac{d\mu_1^r}{dz} \frac{\partial A_{x_1}}{\partial z} - k_1^2 A_{x_1} + \tilde{V} \frac{\partial A_{x_1}}{\partial x}, \quad (5.1.8)$$

where $k_1^2 = -j\omega\mu_0\mu_1^r(z)\sigma_1(z)$, $\tilde{V} = \mu_0\mu_1^r(z)\sigma_1(z)V$.

The boundary conditions are

$$A_{x_0} |_{z=0} = A_{x_1} |_{z=0}, \quad \frac{\partial A_{x_0}}{\partial z} |_{z=0} = \frac{1}{\mu_1^r(0)} \frac{\partial A_{x_1}}{\partial z} |_{z=0}. \quad (5.1.9)$$

The following conditions hold at infinity:

$$A_{x_i}, \frac{\partial A_{x_i}}{\partial x}, \frac{\partial A_{x_i}}{\partial y} \rightarrow 0 \text{ as } x \rightarrow \pm\infty, \quad i = 0, 1. \quad (5.1.10)$$

Problem (5.1.6) - (5.1.10) is solved by the method of double Fourier transform with respect to the variables x and y of the form

$$\tilde{A}_{x_i}(\lambda_x, y, z) = \int_{-\infty}^{\infty} A_{x_i}(x, y, z) \exp(-j\lambda_x x) dx, \quad i = 0, 1, \quad (5.1.11)$$

$$\tilde{\tilde{A}}_{x_i}(\lambda_x, \lambda_y, z) = \int_{-\infty}^{\infty} \tilde{A}_{x_i}(\lambda_x, y, z) \exp(-j\lambda_y y) dy, \quad i = 0, 1. \quad (5.1.12)$$

Applying the transform (5.1.11) to ΔA_{x_i} we obtain

$$\begin{aligned} \Delta \tilde{A}_{x_i}(\lambda_x, y, z) &= \int_{-\infty}^{\infty} \left(\frac{\partial^2 A_{x_i}}{\partial x^2} + \frac{\partial^2 A_{x_i}}{\partial y^2} + \frac{\partial^2 A_{x_i}}{\partial z^2} \right) e^{-j\lambda_x x} dx = \\ &= \left\| \begin{array}{l} u = e^{-j\lambda_x x} \rightarrow du = -j\lambda_x e^{-j\lambda_x x} dx \\ dv = \frac{\partial^2 A_{x_i}}{\partial x^2} dx \rightarrow v = \frac{\partial A_{x_i}}{\partial x} \end{array} \right\| = j\lambda_x \int_{-\infty}^{\infty} \frac{\partial A_{x_i}}{\partial x} e^{-j\lambda_x x} dx + \frac{\partial^2 \tilde{A}_{x_i}}{\partial y^2} + \frac{\partial^2 \tilde{A}_{x_i}}{\partial z^2} = \\ &= \left\| \begin{array}{l} u = e^{-j\lambda_x x} \rightarrow du = -j\lambda_x e^{-j\lambda_x x} dx \\ dv = \frac{\partial A_{x_i}}{\partial x} dx \rightarrow v = A_{x_i} \end{array} \right\| = \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \lambda_x^2 \right) \tilde{A}_{x_i} \end{aligned} \quad (5.1.13)$$

Using (5.1.10), (5.1.12) and (5.1.13) we obtain

$$\Delta \tilde{\tilde{A}}_{x_i}(\lambda_x, \lambda_y, z) = \int_{-\infty}^{\infty} \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \lambda_x^2 \right) \tilde{A}_{x_i} e^{-j\lambda_y y} dy = \left(\frac{d^2}{dz^2} - \lambda^2 \right) \tilde{\tilde{A}}_{x_i} \quad (5.1.14)$$

where $\lambda^2 = \lambda_x^2 + \lambda_y^2$.

Applying the Fourier transforms (5.1.11), (5.1.12) to the right-hand side of equation (5.1.6) we obtain

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu_0 I \delta(z-h) \delta(r-r_c) \sin \varphi_r e^{-j(\lambda_x x + \lambda_y y)} dx dy = \\ &= \left\| \lambda_x x + \lambda_y y = \lambda r \cos \varphi_\lambda \cos \varphi_r + \lambda r \sin \varphi_\lambda \sin \varphi_r = \lambda r \cos(\varphi_\lambda - \varphi_r) \right\| = \\ &= \mu_0 I \delta(z-h) \int_0^{\infty} \delta(r-r_c) r dr \int_0^{2\pi} \sin \varphi_r e^{-j\lambda r \cos(\varphi_\lambda - \varphi_r)} d\varphi_r = \\ &= \left\| e^{jz \cos \varphi} = J_0(z) + 2 \sum_{k=1}^{\infty} j^k J_k(z) \cos k\varphi \right\| = \\ &= \left\| \int_0^{2\pi} \sin \varphi_r J_0(-\lambda r) d\varphi_r + 2 \sum_{k=1}^{\infty} j^k J_k(-\lambda r) \left(\cos k\varphi_\lambda \int_0^{2\pi} \cos k\varphi_r \sin \varphi_r d\varphi_r + \right. \right. \\ &\quad \left. \left. + \sin k\varphi_\lambda \int_0^{2\pi} \sin k\varphi_r \sin \varphi_r d\varphi_r \right) \right\| = \\ &= -2j\pi J_1(\lambda r) \sin \varphi_\lambda \\ &= -2j\mu_0 \pi \sin \varphi_\lambda I \delta(z-h) \int_0^{\infty} J_1(\lambda r) \delta(r-r_c) r dr = -2j\mu_0 \pi \sin \varphi_\lambda I J_1(\lambda r_c) r_c \delta(z-h) \Rightarrow \end{aligned}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu_0 I \delta(z-h) \delta(r-r_c) \sin \varphi_r e^{-j(\lambda_x x + \lambda_y y)} dx dy = -2j\mu_0 \pi \sin \varphi_\lambda I J_1(\lambda r_c) r_c \delta(z-h) \quad (5.1.15)$$

Using (5.1.14) and (5.1.15) we rewrite (5.1.6) in the form

$$\frac{d^2 \tilde{A}_{x_0}}{dz^2} - \lambda^2 \tilde{A}_{x_0} = \gamma \delta(z-h), \quad (5.1.16)$$

where $\gamma = -2j\mu_0 \pi \sin \varphi_\lambda I J_1(\lambda r_c) r_c$.

The right-hand side of equation (5.1.8) can be rewritten in the form

$$\left\| \int_{-\infty}^{\infty} \left(\beta \frac{\partial A_{x_1}}{\partial z} - k_1^2 A_{x_1} + \tilde{V} \frac{\partial A_{x_1}}{\partial x} \right) e^{-j\lambda_x x} dx = \beta \frac{\partial}{\partial z} \int_{-\infty}^{\infty} A_{x_1} e^{-j\lambda_x x} dx - k_1^2 \int_{-\infty}^{\infty} A_{x_1} e^{-j\lambda_x x} dx + \right. \\ \left. + \tilde{V} \int_{-\infty}^{\infty} \frac{\partial A_{x_1}}{\partial x} e^{-j\lambda_x x} dx = \beta \frac{\partial \tilde{A}_{x_1}}{\partial z} - k_1^2 \tilde{A}_{x_1} + j\lambda_x \tilde{V} \tilde{A}_{x_1} \right\| \\ \int_{-\infty}^{\infty} \left(\beta \frac{\partial \tilde{A}_{x_1}}{\partial z} - k_1^2 \tilde{A}_{x_1} + j\lambda_x \tilde{V} \tilde{A}_{x_1} \right) e^{-j\lambda_y y} dy = \beta \frac{d\tilde{A}_{x_1}}{dz} - (k_1^2 - j\lambda \cos \varphi_\lambda \tilde{V}) \tilde{A}_{x_1} \quad (5.1.17)$$

Using (5.1.11) and (5.1.14) we transform (5.1.5) to the form

$$\frac{d^2 \tilde{A}_{x_1}}{dz^2} - \beta \frac{d\tilde{A}_{x_1}}{dz} - q_1^2 \tilde{A}_{x_1} = 0, \quad (5.1.18)$$

where $q_1^2 = \lambda^2 - k_1^2 + j\lambda \cos \varphi_\lambda \tilde{V}$, $k_1^2 = -j\omega\mu_0\mu_1(z)\sigma_1(z)$, $\tilde{V} = \mu_0\mu_1(z)\sigma_1(z)V$.

Applying the Fourier transform with respect to x and y to the boundary conditions we obtain

$$\tilde{A}_{x_0} |_{z=0} = \tilde{A}_{x_1} |_{z=0}, \quad \frac{d\tilde{A}_{x_0}}{dz} |_{z=0} = \frac{1}{\mu_m} \frac{d\tilde{A}_{x_1}}{dz} |_{z=0}. \quad (5.1.19)$$

The following conditions hold at infinity:

$$\tilde{A}_{x_i} \rightarrow 0 \text{ as } x \rightarrow \pm\infty, \quad i = 0, 1. \quad (5.1.20)$$

In order to solve equation (5.1.16) we consider the following two sub-regions of R_0 : $0 < z < h$

and $z > h$. The solutions in these regions are denoted by $\tilde{A}_{x_{00}}$ and $\tilde{A}_{x_{01}}$, respectively.

Hence,

$$\frac{d^2 \tilde{A}_{x_{00}}}{dz^2} - \lambda^2 \tilde{A}_{x_{00}} = 0, \quad 0 < z < h, \quad (5.1.21)$$

$$\frac{d^2 \tilde{A}_{x_{01}}}{dz^2} - \lambda^2 \tilde{A}_{x_{01}} = 0, \quad z > h, \quad (5.1.22)$$

The general solution to (5.1.21) can be written in the form

$$\tilde{A}_{x_{00}} = C_1 e^{\lambda z} + C_2 e^{-\lambda z}. \quad (5.1.23)$$

The bounded solution to (5.1.22) is

$$\tilde{A}_{x_{01}} = C_3 e^{-\lambda z}. \quad (5.1.24)$$

The functions $\tilde{A}_{x_{00}}(\lambda_x, \lambda_y, z)$ and $\tilde{A}_{x_{01}}(\lambda_x, \lambda_y, z)$ satisfy the following conditions at $z = h$:

$$\tilde{A}_{x_{00}}|_{z=h} = \tilde{A}_{x_{01}}|_{z=h}, \quad \frac{d\tilde{A}_{x_{01}}}{dz}|_{z=h} - \frac{d\tilde{A}_{x_{00}}}{dz}|_{z=h} = \gamma. \quad (5.1.25)$$

The first condition in (5.1.25) reflects the fact that the function $\tilde{A}_{x_0}(\lambda_x, \lambda_y, z)$ is continuous at $z = h$ while the second condition in (5.1.25) is obtained integrating (5.1.16) with respect to z from $z = h - \varepsilon$ to $z = h + \varepsilon$ and considering the limit in the resulting expression as $\varepsilon \rightarrow +0$.

Using (5.1.23), (5.1.24) and (5.1.25) we obtain

$$\begin{cases} C_1 e^{\lambda h} + C_2 e^{-\lambda h} = C_3 e^{-\lambda h} \\ -\lambda C_3 e^{-\lambda h} - \lambda C_1 e^{\lambda h} + \lambda C_2 e^{-\lambda h} = \gamma \end{cases} \Rightarrow \left\| \begin{array}{l} C_1 e^{2\lambda h} + C_2 - C_3 = 0 \\ -C_1 e^{2\lambda h} + C_2 - C_3 = \frac{\gamma}{\lambda} e^{\lambda h} \end{array} \right\| \Rightarrow \begin{cases} C_1 = -\frac{\gamma}{2\lambda} e^{-\lambda h} \\ C_3 = C_2 - \frac{\gamma}{2\lambda} e^{\lambda h} \end{cases} \quad (5.1.26)$$

Hence, solutions (5.1.23) and (5.1.24) can be rewritten as follows

$$\begin{cases} \tilde{A}_{x_{00}} = C_2 e^{-\lambda z} - \frac{\gamma}{2\lambda} e^{-\lambda(h-z)} \\ \tilde{A}_{x_{01}} = C_2 e^{-\lambda z} - \frac{\gamma}{2\lambda} e^{-\lambda(z-h)} \end{cases} \Rightarrow \tilde{A}_{x_0} = C_2 e^{-\lambda z} - \frac{\gamma}{2\lambda} e^{-\lambda|h-z|}. \quad (5.1.27)$$

The solution to (5.1.18) can be expressed in terms of modified Bessel functions (see [50], formula 2.1.3.10, page 247):

$$\left\| \begin{array}{l} y''_{xx} + ay'_x + (be^{\lambda x} + c)y = 0 \Rightarrow y = e^{\frac{ax}{2}} \left[C_1 J_\nu \left(\frac{2\sqrt{b}}{\lambda} e^{\frac{\lambda x}{2}} \right) + C_2 Y_\nu \left(\frac{2\sqrt{b}}{\lambda} e^{\frac{\lambda x}{2}} \right) \right] \\ \nu = \frac{\sqrt{a^2 - 4c}}{\lambda} \end{array} \right\|$$

$$\tilde{A}_{x_1} = e^{\frac{\beta z}{2}} \left[C_4 J_\nu \left(\frac{2\sqrt{b}}{\alpha + \beta} e^{\frac{(\alpha + \beta)z}{2}} \right) + C_5 Y_\nu \left(\frac{2\sqrt{b}}{\alpha + \beta} e^{\frac{(\alpha + \beta)z}{2}} \right) \right].$$

The bounded solution to (5.1.18) is

$$\tilde{A}_{x_1} = e^{\frac{\beta z}{2}} C_4 J_\nu \left(\frac{2\sqrt{b}}{\alpha + \beta} e^{\frac{(\alpha + \beta)z}{2}} \right), \quad (5.1.28)$$

where $\nu = \frac{\sqrt{\beta^2 + 4\lambda^2}}{\alpha + \beta}$, $b = -j\mu_0\mu_m\sigma_m(\omega + \lambda \cos \varphi_\lambda \tilde{V})$, $\tilde{V} = \mu_0\mu_1(z)\sigma_1(z)V$.

Using (5.1.19) , (5.1.27) and (5.1.28) we obtain

$$\begin{aligned} C_2 - \frac{\gamma}{2\lambda} e^{-\lambda h} &= C_4 J_\nu(z_0), \\ -\lambda C_2 - \frac{\gamma}{2} e^{-\lambda h} &= \frac{C_4}{\mu_m} \left(\frac{\beta}{2} J_\nu(z_0) + \sqrt{b} J'_\nu(z_0) \right), \end{aligned} \quad (5.1.29)$$

$$\text{where } z_0 = \frac{2\sqrt{b}}{\alpha + \beta}$$

$$\left\| \begin{aligned} \cdot C_2 &= C_4 J_\nu(z_0) + \frac{\gamma}{2\lambda} e^{-\lambda h} \\ \cdot (-\lambda C_4 J_\nu(z_0) - \gamma e^{-\lambda h}) \mu_m &= C_4 \left(\frac{\beta}{2} J_\nu(z_0) + \sqrt{b} J'_\nu(z_0) \right) \end{aligned} \right\|$$

and

$$C_4 = - \frac{\gamma \mu_m e^{-\lambda h}}{\left(\lambda \mu_m + \frac{\beta}{2} \right) J_\nu(z_0) + \sqrt{b} J'_\nu(z_0)}, \quad (5.1.30)$$

$$C_2 = \frac{\gamma}{2\lambda} e^{-\lambda h} \frac{\left(\frac{\beta}{2} - \lambda \mu_m \right) J_\nu(z_0) + \sqrt{b} J'_\nu(z_0)}{\left(\frac{\beta}{2} + \lambda \mu_m \right) J_\nu(z_0) + \sqrt{b} J'_\nu(z_0)}, \quad (5.1.31)$$

$$\text{where } \gamma = -2j\mu_0\pi \sin \varphi_\lambda I J_1(\lambda r_c) r_c.$$

The induced vector potential in region R_0 is

$$\tilde{A}_{x_0}^{ind}(\lambda_x, \lambda_y, z) = C_2 e^{-\lambda z}, \quad (5.1.32)$$

where C_2 is given by (5.1.31).

The inverse Fourier transform is defined as follows

$$A_{x_i}(x, y, z) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{A}_{x_i}(\lambda_x, \lambda_y, z) \exp(j(\lambda_x x + \lambda_y y)) d\lambda_x d\lambda_y. \quad (5.1.33)$$

Applying (5.1.33) to $\tilde{A}_{x_0}^{ind}$ we obtain

$$\begin{aligned} A_{x_0}^{ind}(x, y, z) &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{A}_{x_0}^{ind}(\lambda_x, \lambda_y, z) e^{j(\lambda_x x + \lambda_y y)} d\lambda_x d\lambda_y = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C_2 e^{-\lambda z} e^{j(\lambda_x x + \lambda_y y)} d\lambda_x d\lambda_y = \\ &= \left\| \begin{aligned} \lambda_x x + \lambda_y y &= \lambda r \cos \varphi_\lambda \cos \varphi_r + \lambda r \sin \varphi_\lambda \sin \varphi_r = \lambda r \cos(\varphi_\lambda - \varphi_r) \\ \gamma &= -2j\mu_0\pi \sin \varphi_\lambda I J_1(\lambda r_c) r_c \\ D &= \frac{\left(\frac{\beta}{2} - \lambda \mu_m \right) J_\nu(z_0) + \sqrt{b} J'_\nu(z_0)}{\left(\frac{\beta}{2} + \lambda \mu_m \right) J_\nu(z_0) + \sqrt{b} J'_\nu(z_0)} \end{aligned} \right\| = \end{aligned}$$

$$= -\frac{1}{4\pi} jI r_c \mu_0 \int_0^\infty J_1(\lambda r_c) e^{-\lambda(z+h)} d\lambda \int_0^{2\pi} D \sin \varphi_\lambda e^{j\lambda r \cos(\varphi_\lambda - \varphi_r)} d\varphi_\lambda$$

$$\left\| e^{jz \cos \varphi} = J_0(z) + 2 \sum_{k=1}^\infty j^k J_k(z) \cos k\varphi \right\|$$

Consequently,

$$A_{x_0}^{ind}(x, y, z) =$$

$$= -\frac{1}{4\pi} jI r_c \mu_0 \int_0^\infty J_1(\lambda r_c) e^{-\lambda(z+h)} d\lambda \int_0^{2\pi} D \sin \varphi_\lambda \left(J_0(\lambda r) + 2 \sum_{k=1}^\infty j^k J_k(\lambda r) \cos k(\varphi_\lambda - \varphi_r) \right) d\varphi_\lambda. \quad (5.1.34)$$

Similarly, using (5.1.3) and (5.1.5) we obtain the equations for the y – components of the vector potential in regions R_0 and R_1 . Equation in region R_1 has the same form as (5.1.8), where A_{x_1} is replaced by A_{y_1} . Equation for the y – component of the vector potential in region R_0 has the form (5.1.6) where $\sin \varphi_r$ is replaced by $-\cos \varphi_r$. The solution of the corresponding problem for the y – component of the vector potential is obtained by means of the double Fourier transforms (5.1.11) and (5.1.12). Without repeating the calculations we present only the final result here:

$$A_{y_0}^{ind}(x, y, z) =$$

$$= \frac{1}{4\pi} jI r_c \mu_0 \int_0^\infty J_1(\lambda r_c) e^{-\lambda(z+h)} d\lambda \int_0^{2\pi} D \cos \varphi_\lambda \left(J_0(\lambda r) + 2 \sum_{k=1}^\infty j^k J_k(\lambda r) \cos k(\varphi_\lambda - \varphi_r) \right) d\varphi_\lambda. \quad (5.1.35)$$

The induced change in impedance of the coil is given by the formula

$$Z^{ind} = \frac{j\omega}{I} \oint_L \vec{A}_0^{ind}(x, y, z) d\vec{l}, \quad (5.1.36)$$

where L is the contour of the coil and

$$\vec{A}_0^{ind}(x, y, z) = A_{x_0}^{ind}(x, y, z) \vec{e}_x + A_{y_0}^{ind}(x, y, z) \vec{e}_y. \quad (5.1.37)$$

Substituting (5.1.34) and (5.1.35) into (5.1.36) we obtain

$$Z^{ind} = \frac{j\omega r_c}{I} \int_0^{2\pi} \left(A_{y_0}^{ind} \cos \varphi_r - A_{x_0}^{ind} \sin \varphi_r \right) d\varphi_r \Bigg|_{\substack{r=r_c \\ z=h}} =$$

$$= \frac{j\omega r_c^2}{4\pi I} jI \mu_0 \int_0^\infty J_1(\lambda r_c) e^{-\lambda(z+h)} d\lambda \int_0^{2\pi} D \left[\cos \varphi_\lambda \int_0^{2\pi} \cos \varphi_r \left(J_0(\lambda r) + 2 \sum_{k=1}^\infty j^k J_k(\lambda r) \cos k(\varphi_\lambda - \varphi_r) \right) d\varphi_r + \right. \\ \left. + \sin \varphi_\lambda \int_0^{2\pi} \sin \varphi_r \left(J_0(\lambda r) + 2 \sum_{k=1}^\infty j^k J_k(\lambda r) \cos k(\varphi_\lambda - \varphi_r) \right) d\varphi_r \right] d\varphi_\lambda =$$

$$= -\frac{\omega r_c^2}{4\pi} \mu_0 \int_0^\infty J_1(\lambda r_c) e^{-\lambda(z+h)} d\lambda \int_0^{2\pi} 2D j \pi J_1(\lambda r) d\varphi_\lambda \Bigg|_{\substack{r=r_c \\ z=h}}$$

where

$$D = \frac{\left(\frac{\beta}{2} - \lambda\mu_m\right)J_\nu(z_0) + \sqrt{b}J'_\nu(z_0)}{\left(\frac{\beta}{2} + \lambda\mu_m\right)J_\nu(z_0) + \sqrt{b}J'_\nu(z_0)}.$$

$$\left\| \begin{aligned} & \cdot J_0(\lambda r) \cos \varphi_\lambda \int_0^{2\pi} \cos \varphi_r d\varphi_r + \\ & + 2 \cos \varphi_\lambda \sum_{k=1}^{\infty} j^k J_k(\lambda r) \left[\cos k\varphi_\lambda \int_0^{2\pi} \cos k\varphi_r \cos \varphi_r d\varphi_r + \sin k\varphi_\lambda \int_0^{2\pi} \sin k\varphi_r \cos \varphi_r d\varphi_r \right] = \\ & = 2j\pi \cos^2 \varphi_\lambda J_1(\lambda r) \end{aligned} \right\|$$

$$\left\| \begin{aligned} & \cdot J_0(\lambda r) \sin \varphi_\lambda \int_0^{2\pi} \sin \varphi_r d\varphi_r + \\ & + 2 \sin \varphi_\lambda \sum_{k=1}^{\infty} j^k J_k(\lambda r) \left[\cos k\varphi_\lambda \int_0^{2\pi} \cos k\varphi_r \sin \varphi_r d\varphi_r + \sin k\varphi_\lambda \int_0^{2\pi} \sin k\varphi_r \sin \varphi_r d\varphi_r \right] = \\ & = 2j\pi \sin^2 \varphi_\lambda J_1(\lambda r) \end{aligned} \right\|$$

Hence,

$$Z^{ind} = -\frac{j}{2} \omega r_c^2 \mu_0 \int_0^{\infty} J_1^2(\lambda r_c) e^{-2\lambda h} d\lambda \int_0^{2\pi} \frac{\left(\frac{\beta}{2} - \lambda\mu_m\right)J_\nu(z_0) + \sqrt{b}J'_\nu(z_0)}{\left(\frac{\beta}{2} + \lambda\mu_m\right)J_\nu(z_0) + \sqrt{b}J'_\nu(z_0)} d\varphi_\lambda. \quad (5.1.38)$$

Using following dimensionless parameters (5.1.38) can be rewritten as follows

$$\left\| \begin{aligned} & \cdot \lambda r_c = u \rightarrow \lambda = \frac{u}{r_c}, \\ & \cdot \alpha = \frac{\hat{\alpha}}{r_c}, \beta = \frac{\hat{\beta}}{r_c}, \end{aligned} \right\} \Rightarrow \hat{\nu} = \frac{\sqrt{\hat{\beta}^2 + 4u^2}}{\hat{\alpha} + \hat{\beta}}$$

$$\cdot \sqrt{b} = \frac{1}{r_c} \sqrt{-j(\hat{\delta}_1^2 + \hat{\rho}u \cos \varphi_\lambda)} = \frac{\sqrt{-j}}{r_c} \sqrt{\hat{b}}, \text{ where } \begin{cases} \hat{\delta}_1 = r_c \sqrt{\omega \mu_0 \mu_m \sigma_m} \\ \hat{\rho} = r_c \mu_0 \mu_m \sigma_m V \end{cases}$$

$$\left\| \begin{aligned} & \cdot \frac{\beta}{2} + \lambda\mu_m = \frac{1}{r_c} \left(\frac{\hat{\beta}}{2} + u\mu_m \right), \quad \cdot \frac{\beta}{2} - \lambda\mu_m = \frac{1}{r_c} \left(\frac{\hat{\beta}}{2} - u\mu_m \right) \\ & \cdot \hat{z}_0 = \frac{2\sqrt{\hat{b}}}{\hat{\alpha} + \hat{\beta}} \end{aligned} \right\|$$

Using the previous formula we obtain

$$Z^{ind} = \omega r_c \mu_0 Z, \quad (5.1.39)$$

where

$$Z = -\frac{j}{2} \int_0^\infty J_1^2(u) e^{-2u\hat{h}} du \int_0^{2\pi} \frac{\left(\frac{\hat{\beta}}{2} - u\mu_m\right) J_{\hat{\nu}}(\hat{z}_0) + \sqrt{-j\hat{b}} J_{\hat{\nu}}'(\hat{z}_0)}{\left(\frac{\hat{\beta}}{2} + u\mu_m\right) J_{\hat{\nu}}(\hat{z}_0) + \sqrt{-j\hat{b}} J_{\hat{\nu}}'(\hat{z}_0)} d\varphi_\lambda \quad (5.1.40)$$

and the dimensionless parameters are defined as follows

$$\hat{\nu} = \frac{\sqrt{\hat{\beta}^2 + 4u^2}}{\hat{\alpha} + \hat{\beta}}, \quad \hat{b} = \hat{\delta}_1^2 + \hat{\rho} u \cos \varphi_\lambda, \quad \hat{\delta}_1 = r_c \sqrt{\omega \mu_0 \mu_m \sigma_m}, \quad \hat{\rho} = r_c \mu_0 \mu_m \sigma_m V.$$

Formula (5.1.40) is used to compute the change in impedance of a single-turn coil for the following values of the parameters of the problem:

$$\hat{\alpha} = 0, \quad \hat{\beta} = 2, \quad \hat{h} = 0.05, \quad d = 0.1, \quad r_0 = 5, \quad \mu_m = 5.$$

The results of the calculations are shown in **Fig. 5.2**. The points on each graph correspond to different values of $\hat{\delta}_1 = 1, 2, \dots, 20$ (from left to right).

Computations are done with “Mathematica” (see **Appendix Fig.Ap.12**).

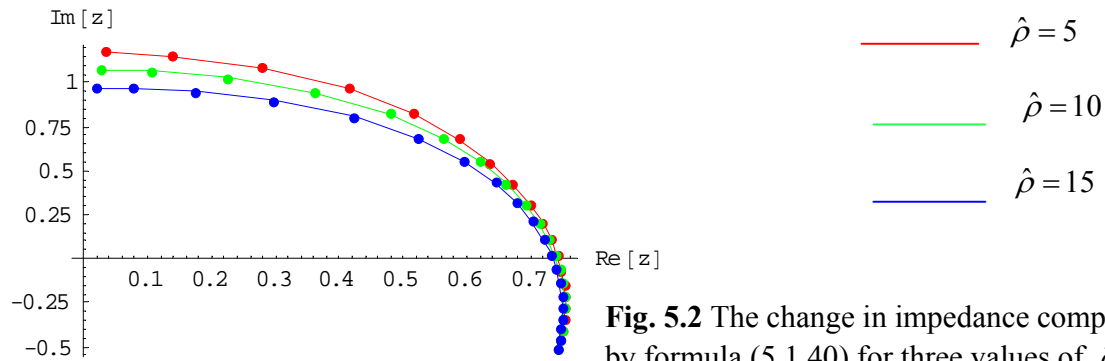


Fig. 5.2 The change in impedance computed by formula (5.1.40) for three values of $\hat{\rho}$

5.2 Moving two-layer medium with varying electrical conductivity and magnetic permeability

Consider the case where an upper layer of a two-layer medium is moving in the horizontal direction with constant velocity V while the lower half-space is fixed [26] (see **Fig. 5.3**).

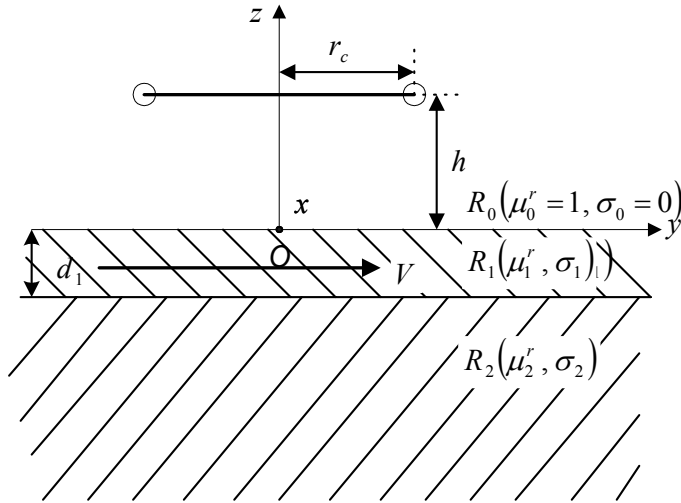


Fig. 5.3 A single-turn coil above a moving two-layer medium.

The electrical conductivity and magnetic permeability of the upper layer are exponential functions of the vertical coordinate:

$$\mu(z) = \mu_1(z) = \mu_m e^{\beta z}, \quad \sigma(z) = \sigma_1(z) = \sigma_m e^{\alpha z}, \quad (5.2.1)$$

while the conductivity σ_2 and relative magnetic permeability μ_2 of the lower half-space are assumed to be constant.

The components $A_{x_i}(x, y, z)$, $i = 0, 1, 2$ of the vector potential in regions R_0, R_1 and R_2 (see **Fig. 5.3**) satisfy the following system of partial differential equations:

$$R_0 : \Delta A_{x_0} = \mu_0 I \delta(z-h) \delta(r-r_c) \sin \varphi_r, \quad (5.2.2)$$

$$R_1 : \Delta A_{x_1} = \frac{1}{\mu_1(z)} \frac{d\mu_1}{dz} \frac{\partial A_{x_1}}{\partial z} - k_1^2 A_{x_1} + \tilde{V} \frac{\partial A_{x_1}}{\partial x}, \quad (5.2.3)$$

$$\text{where } k_1^2 = -j\omega\mu_0\mu_1(z)\sigma_1(z), \quad \tilde{V} = \mu_0\mu_1(z)\sigma_1(z)V,$$

$$R_2 : \Delta A_{x_2} = -k_2^2 A_{x_2}, \quad (5.2.4)$$

$$\text{where } k_2^2 = -j\omega\mu_0\mu_2\sigma_2.$$

The boundary conditions are

$$A_{x_0}|_{z=0} = A_{x_1}|_{z=0}, \quad \frac{\partial A_{x_0}}{\partial z}|_{z=0} = \frac{1}{\mu_1(0)} \frac{\partial A_{x_1}}{\partial z}|_{z=0}, \quad (5.2.5)$$

$$A_{x_1}|_{z=-d} = A_{x_2}|_{z=-d}, \quad \frac{1}{\mu_1(-d)} \frac{\partial A_{x_1}}{\partial z}|_{z=-d} = \frac{1}{\mu_2} \frac{\partial A_{x_2}}{\partial z}|_{z=-d}, \quad (5.2.6)$$

The following conditions hold at infinity:

$$A_{x_i}, \frac{\partial A_{x_i}}{\partial x}, \frac{\partial A_{x_i}}{\partial y} \rightarrow 0 \text{ as } x \rightarrow \pm\infty, \quad i = 0, 1, 2. \quad (5.2.7)$$

Applying the Fourier transforms (5.1.11) and (5.1.12) to (5.2.2)-(5.2.7) we obtain

$$\frac{d^2 \tilde{A}_{x_0}}{dz^2} - \lambda^2 \tilde{A}_{x_0} = \gamma \delta(z-h), \quad (5.2.8)$$

$$\frac{d^2 \tilde{A}_{x_1}}{dz^2} - \beta \frac{d \tilde{A}_{x_1}}{dz} - q_1^2 \tilde{A}_{x_1} = 0, \quad (5.2.9)$$

$$\frac{d^2 \tilde{A}_{x_2}}{dz^2} - q_2^2 \tilde{A}_{x_2} = 0, \quad (5.2.10)$$

where

$$\gamma = -2j\mu_0\pi \sin \varphi_\lambda I J_1(\lambda r_c) r_c, \quad q_1^2 = \lambda^2 - k_1^2 + j\lambda \cos \varphi_\lambda \tilde{V}, \quad k_1^2 = -j\omega\mu_0\mu_1(z)\sigma_1(z),$$

$$\tilde{V} = \mu_0\mu_1(z)\sigma_1(z)\mathcal{V}, \quad q_2^2 = \lambda^2 - k_2^2, \quad k_2^2 = -j\omega\mu_0\mu_2\sigma_2.$$

The boundary conditions in the transformed space can be written as follows:

$$\tilde{A}_{x_0} \Big|_{z=0} = \tilde{A}_{x_1} \Big|_{z=0}, \quad \frac{\partial \tilde{A}_{x_0}}{\partial z} \Big|_{z=0} = \frac{1}{\mu_m} \frac{\partial \tilde{A}_{x_1}}{\partial z} \Big|_{z=0}, \quad (5.2.11)$$

$$\tilde{A}_{x_1} \Big|_{z=-d} = \tilde{A}_{x_2} \Big|_{z=-d}, \quad \frac{e^{\beta d}}{\mu_m} \frac{\partial \tilde{A}_{x_1}}{\partial z} \Big|_{z=-d} = \frac{1}{\mu_2} \frac{\partial \tilde{A}_{x_2}}{\partial z} \Big|_{z=-d}. \quad (5.2.12)$$

The following conditions hold at infinity:

$$\tilde{A}_{x_i}, \frac{\partial \tilde{A}_{x_i}}{\partial x}, \frac{\partial \tilde{A}_{x_i}}{\partial y} \rightarrow 0 \text{ as } x \rightarrow \pm\infty, \quad i = 0,1,2$$

As in **Section 5.1**, the solution of (5.2.8) in free space is given by (5.1.27) where the term containing C_2 represents the induced vector potential in free space due to the presence of a two-layer conducting medium.

General solution to (5.2.9) can be expressed in terms of modified Bessel functions (see [50], formula 2.1.3.10, page 247):

$$\left\| \left\| \begin{aligned} y''_{xx} + ay'_x + (be^{\lambda x} + c)y = 0 &\Rightarrow y = e^{\frac{ax}{2}} \left[C_1 J_\nu \left(\frac{2\sqrt{b}}{\lambda} e^{\frac{\lambda x}{2}} \right) + C_2 Y_\nu \left(\frac{2\sqrt{b}}{\lambda} e^{\frac{\lambda x}{2}} \right) \right] \\ \nu = \frac{\sqrt{a^2 - 4c}}{\lambda} \end{aligned} \right\| \right\|$$

$$\tilde{A}_{x_1} = e^{\frac{\beta z}{2}} \left[C_4 J_\nu \left(\frac{2\sqrt{b}}{\alpha + \beta} e^{\frac{(\alpha + \beta)z}{2}} \right) + C_5 Y_\nu \left(\frac{2\sqrt{b}}{\alpha + \beta} e^{\frac{(\alpha + \beta)z}{2}} \right) \right], \quad (5.2.13)$$

$$\text{where } \nu = \frac{\sqrt{\beta^2 + 4\lambda^2}}{\alpha + \beta}, \quad b = -j\mu_0\mu_m\sigma_m(\omega + \lambda \cos \varphi_\lambda \tilde{V}), \quad \tilde{V} = \mu_0\mu_1(z)\sigma_1(z)\mathcal{V}.$$

The bounded solution to (5.2.10) is

$$\tilde{A}_{x_2} = C_6 e^{q_2 z}. \quad (5.2.14)$$

Using the boundary conditions (5.1.25) and (5.2.11)–(5.2.12) we obtain

$$C_2 - \frac{\gamma}{2\lambda} e^{-\lambda h} = C_4 J_\nu(z_0) + C_5 Y_\nu(z_0), \quad (5.2.15)$$

$$-\lambda C_2 - \frac{\gamma}{2} e^{-\lambda h} = \frac{C_4}{\mu_m} \left(\frac{\beta}{2} J_\nu(z_0) + \sqrt{b} J'_\nu(z_0) \right) + \frac{C_5}{\mu_m} \left(\frac{\beta}{2} Y_\nu(z_0) + \sqrt{b} Y'_\nu(z_0) \right), \quad (5.2.16)$$

$$C_6 e^{-q_2 d} = C_4 e^{-\frac{\beta d}{2}} J_\nu \left(z_0 e^{-\frac{(\alpha+\beta)d}{2}} \right) + C_5 e^{-\frac{\beta d}{2}} Y_\nu \left(z_0 e^{-\frac{(\alpha+\beta)d}{2}} \right), \quad (5.2.17)$$

where $z_0 = \frac{2\sqrt{b}}{\alpha + \beta}$.

It follows from (5.2.15) and (5.2.16) that

$$\left\| \begin{aligned} & \cdot C_2 = C_4 J_\nu(z_0) + C_5 Y_\nu(z_0) + \frac{\gamma}{2\lambda} e^{-\lambda h} \\ & \cdot \left(-\lambda C_4 J_\nu(z_0) - \lambda C_5 Y_\nu(z_0) - \gamma e^{-\lambda h} \right) \mu_m = C_4 \left(\frac{\beta}{2} J_\nu(z_0) + \sqrt{b} J'_\nu(z_0) \right) + C_5 \left(\frac{\beta}{2} Y_\nu(z_0) + \sqrt{b} Y'_\nu(z_0) \right) \end{aligned} \right\|$$

$$C_4 = - \frac{C_5 \left(\left(\lambda \mu_m + \frac{\beta}{2} \right) Y_\nu(z_0) + \sqrt{b} Y'_\nu(z_0) \right) + \gamma \mu_m e^{-\lambda h}}{\left(\lambda \mu_m + \frac{\beta}{2} \right) J_\nu(z_0) + \sqrt{b} J'_\nu(z_0)}. \quad (5.2.18)$$

Using (5.2.17) and (5.2.18) we obtain

$$\left\| \begin{aligned} & \cdot C_6 e^{-q_2 d} = C_4 e^{-\frac{\beta d}{2}} J_\nu(z_1) + C_5 e^{-\frac{\beta d}{2}} Y_\nu(z_1) \\ & \cdot C_6 e^{-q_2 d} = \frac{\mu_2 e^{\frac{\beta d}{2}}}{\mu_m q_2} \left(C_4 \left(\frac{\beta}{2} J_\nu(z_1) + \sqrt{b} e^{-\frac{(\alpha+\beta)d}{2}} J'_\nu(z_1) \right) + C_5 \left(\frac{\beta}{2} Y_\nu(z_1) + \sqrt{b} e^{-\frac{(\alpha+\beta)d}{2}} Y'_\nu(z_1) \right) \right) \\ & \cdot \mu_m q_2 \left(C_4 e^{-\frac{\beta d}{2}} J_\nu(z_1) + C_5 e^{-\frac{\beta d}{2}} Y_\nu(z_1) \right) = \\ & \quad = \mu_2 e^{\frac{\beta d}{2}} \left(C_4 \left(\frac{\beta}{2} J_\nu(z_1) + \sqrt{b} e^{-\frac{(\alpha+\beta)d}{2}} J'_\nu(z_1) \right) + C_5 \left(\frac{\beta}{2} Y_\nu(z_1) + \sqrt{b} e^{-\frac{(\alpha+\beta)d}{2}} Y'_\nu(z_1) \right) \right) \end{aligned} \right\|$$

$$C_4 = -C_5 \frac{\left(\frac{\beta}{2} \mu_2 e^{\beta d} - \mu_m q_2 \right) Y_\nu(z_1) + \mu_2 \sqrt{b} e^{\frac{(\beta-\alpha)d}{2}} Y'_\nu(z_1)}{\left(\frac{\beta}{2} \mu_2 e^{\beta d} - \mu_m q_2 \right) J_\nu(z_1) + \mu_2 \sqrt{b} e^{\frac{(\beta-\alpha)d}{2}} J'_\nu(z_1)}, \quad (5.2.19)$$

where $z_1 = z_0 e^{-\frac{(\alpha+\beta)d}{2}}$, $z_0 = \frac{2\sqrt{b}}{\alpha + \beta}$.

Eliminating C_4 from (5.2.18) and (5.2.19) we get

$$\begin{aligned}
& \cdot C_5 \left(\frac{\left(\frac{\beta}{2} \mu_2 e^{\beta d} - \mu_m q_2 \right) Y_\nu(z_1) + \mu_2 \sqrt{b} e^{\frac{(\beta-\alpha)d}{2}} Y'_\nu(z_1)}{\left(\frac{\beta}{2} \mu_2 e^{\beta d} - \mu_m q_2 \right) J_\nu(z_1) + \mu_2 \sqrt{b} e^{\frac{(\beta-\alpha)d}{2}} J'_\nu(z_1)} - \frac{\left(\lambda \mu_m + \frac{\beta}{2} \right) Y_\nu(z_0) + \sqrt{b} Y'_\nu(z_0)}{\left(\lambda \mu_m + \frac{\beta}{2} \right) J_\nu(z_0) + \sqrt{b} J'_\nu(z_0)} \right) = \\
& \qquad \qquad \qquad = \frac{\gamma \mu_m e^{-\lambda h}}{\left(\lambda \mu_m + \frac{\beta}{2} \right) J_\nu(z_0) + \sqrt{b} J'_\nu(z_0)} \\
& \cdot C_{skait} = \left(\left(\frac{\beta}{2} \mu_2 e^{\beta d} - \mu_m q_2 \right) Y_\nu(z_1) + \mu_2 \sqrt{b} e^{\frac{(\beta-\alpha)d}{2}} Y'_\nu(z_1) \right) \cdot \left(\left(\lambda \mu_m + \frac{\beta}{2} \right) J_\nu(z_0) + \sqrt{b} J'_\nu(z_0) \right) - \\
& \quad - \left(\left(\frac{\beta}{2} \mu_2 e^{\beta d} - \mu_m q_2 \right) J_\nu(z_1) + \mu_2 \sqrt{b} e^{\frac{(\beta-\alpha)d}{2}} J'_\nu(z_1) \right) \cdot \left(\left(\lambda \mu_m + \frac{\beta}{2} \right) Y_\nu(z_0) + \sqrt{b} Y'_\nu(z_0) \right) = \\
& \quad = \left(\frac{\beta}{2} \mu_2 e^{\beta d} - \mu_m q_2 \right) \left(\lambda \mu_m + \frac{\beta}{2} \right) (J_\nu(z_0) Y_\nu(z_1) - J_\nu(z_1) Y_\nu(z_0)) + \\
& \quad + \left(\frac{\beta}{2} \mu_2 e^{\beta d} - \mu_m q_2 \right) \sqrt{b} (J'_\nu(z_0) Y_\nu(z_1) - J'_\nu(z_1) Y_\nu(z_0)) + \\
& \quad + \left(\lambda \mu_m + \frac{\beta}{2} \right) \mu_2 \sqrt{b} e^{\frac{(\beta-\alpha)d}{2}} (J_\nu(z_0) Y'_\nu(z_1) - J'_\nu(z_1) Y_\nu(z_0)) + \mu_2 b e^{\frac{(\beta-\alpha)d}{2}} (J'_\nu(z_0) Y'_\nu(z_1) - J'_\nu(z_1) Y'_\nu(z_0)) \\
& \cdot C_{sauc} = \left(\left(\frac{\beta}{2} \mu_2 e^{\beta d} - \mu_m q_2 \right) J_\nu(z_1) + \mu_2 \sqrt{b} e^{\frac{(\beta-\alpha)d}{2}} J'_\nu(z_1) \right) \cdot \left(\left(\lambda \mu_m + \frac{\beta}{2} \right) J_\nu(z_0) + \sqrt{b} J'_\nu(z_0) \right)
\end{aligned}$$

$$C_5 = \gamma \mu_m e^{-\lambda h} \frac{S_1}{D}, \quad (5.2.20)$$

where

$$S_1 = \left(\frac{\beta}{2} \mu_2 e^{\beta d} - \mu_m q_2 \right) J_\nu(z_1) + \mu_2 \sqrt{b} e^{\frac{(\beta-\alpha)d}{2}} J'_\nu(z_1),$$

$$D = D_5 (D_6 D_1 + \sqrt{b} D_2) + \mu_2 \sqrt{b} e^{\frac{(\beta-\alpha)d}{2}} (D_6 D_3 + \sqrt{b} D_4),$$

$$\begin{aligned}
& \cdot D_1 = J_\nu(z_0) Y_\nu(z_1) - J_\nu(z_1) Y_\nu(z_0), & \cdot D_5 = \frac{\beta}{2} \mu_2 e^{\beta d} - \mu_m q_2, \\
& \cdot D_2 = J'_\nu(z_0) Y_\nu(z_1) - J'_\nu(z_1) Y_\nu(z_0), & \cdot D_6 = \lambda \mu_m + \frac{\beta}{2}. \\
& \cdot D_3 = J_\nu(z_0) Y'_\nu(z_1) - J'_\nu(z_1) Y_\nu(z_0), \\
& \cdot D_4 = J'_\nu(z_0) Y'_\nu(z_1) - J'_\nu(z_1) Y'_\nu(z_0),
\end{aligned}$$

The constant C_4 is given by

$$C_4 = -\gamma \mu_m e^{-\lambda h} \frac{S_2}{D}, \quad (5.2.21)$$

where

$$S_2 = \left(\frac{\beta}{2} \mu_2 e^{\beta d} - \mu_m q_2 \right) Y_\nu(z_1) + \mu_2 \sqrt{b} e^{\frac{(\beta-\alpha)d}{2}} Y_\nu'(z_1).$$

Finally, the value of C_2 is

$$\begin{aligned} & \left\| \cdot C_2 = C_4 J_\nu(z_0) + C_5 Y_\nu(z_0) + \frac{\gamma}{2\lambda} e^{-\lambda h} \right\| \\ C_2 &= \frac{\gamma}{2\lambda} e^{-\lambda h} \left[1 - \frac{2\lambda \mu_m \left(D_5 D_1 - \mu_2 \sqrt{b} e^{\frac{(\beta-\alpha)d}{2}} D_3 \right)}{D_5 (D_6 D_1 + \sqrt{b} D_2) + \mu_2 \sqrt{b} e^{\frac{(\beta-\alpha)d}{2}} (D_6 D_3 + \sqrt{b} D_4)} \right], \end{aligned} \quad (5.2.22)$$

where $\gamma = -2j\mu_0\pi \sin \varphi_\lambda I J_1(\lambda r_c) r_c$.

The induced vector potential in region R_0 is

$$\tilde{A}_{x_0}^{ind}(\lambda_x, \lambda_y, z) = C_2 e^{-\lambda z}, \quad (5.2.23)$$

where C_2 is given by (5.2.22).

Applying the inverse Fourier transform in the form

$$A_{x_i}(x, y, z) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{A}_x(\lambda_x, \lambda_y, z) \exp(j(\lambda_x x + \lambda_y y)) d\lambda_x d\lambda_y \quad (5.2.24)$$

to (5.2.23) we obtain

$$\begin{aligned} A_{x_0}^{ind}(x, y, z) &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{A}_{x_0}^{ind}(\lambda_x, \lambda_y, z) e^{j(\lambda_x x + \lambda_y y)} d\lambda_x d\lambda_y = \\ &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C_2 e^{-\lambda z} e^{j(\lambda_x x + \lambda_y y)} d\lambda_x d\lambda_y = \\ &= \left\| \begin{aligned} & \cdot C = 1 - \frac{2\lambda \mu_m \left(D_5 D_1 - \mu_2 \sqrt{b} e^{\frac{(\beta-\alpha)d}{2}} D_3 \right)}{D_5 (D_6 D_1 + \sqrt{b} D_2) + \mu_2 \sqrt{b} e^{\frac{(\beta-\alpha)d}{2}} (D_6 D_3 + \sqrt{b} D_4)} \\ & \cdot \lambda_x x + \lambda_y y = \lambda r \cos \varphi_\lambda \cos \varphi_r + \lambda r \sin \varphi_\lambda \sin \varphi_r = \lambda r \cos(\varphi_\lambda - \varphi_r) \\ & \cdot \gamma = -2j\mu_0\pi \sin \varphi_\lambda I J_1(\lambda r_c) r_c \end{aligned} \right\| = \\ &= -\frac{1}{4\pi} j I r_c \mu_0 \int_0^{\infty} J_1(\lambda r_c) e^{-\lambda(z+h)} d\lambda \int_0^{2\pi} C \sin \varphi_\lambda e^{j\lambda r \cos(\varphi_\lambda - \varphi_r)} d\varphi_\lambda = \\ &= \left\| e^{jz \cos \varphi} = J_0(z) + 2 \sum_{k=1}^{\infty} j^k J_k(z) \cos k\varphi \right\| = \end{aligned}$$

$$= -\frac{1}{4\pi} jI r_c \mu_0 \int_0^\infty J_1(\lambda r_c) e^{-\lambda(z+h)} d\lambda \int_0^{2\pi} C \sin \varphi_\lambda \left(J_0(\lambda r) + 2 \sum_{k=1}^\infty j^k J_k(\lambda r) \cos k(\varphi_\lambda - \varphi_r) \right) d\varphi_\lambda$$

Thus, the x -component of the induced vector potential is

$$A_{x_0}^{ind}(x, y, z) = -\frac{1}{4\pi} jI r_c \mu_0 \int_0^\infty J_1(\lambda r_c) e^{-\lambda(z+h)} d\lambda \int_0^{2\pi} C \sin \varphi_\lambda \left(J_0(\lambda r) + 2 \sum_{k=1}^\infty j^k J_k(\lambda r) \cos k(\varphi_\lambda - \varphi_r) \right) d\varphi_\lambda. \quad (5.2.25)$$

The y -component of the induced vector potential can be found similarly and has the form

$$A_{y_0}^{ind}(x, y, z) = \frac{1}{4\pi} jI r_c \mu_0 \int_0^\infty J_1(\lambda r_c) e^{-\lambda(z+h)} d\lambda \int_0^{2\pi} C \cos \varphi_\lambda \left(J_0(\lambda r) + 2 \sum_{k=1}^\infty j^k J_k(\lambda r) \cos k(\varphi_\lambda - \varphi_r) \right) d\varphi_\lambda. \quad (5.2.26)$$

The induced change in impedance of the coil is given by the formula

$$Z^{ind} = \frac{j\omega}{I} \oint_L \vec{A}_0^{ind}(x, y, z) d\vec{l}, \quad (5.2.27)$$

where L is the contour of the coil, and

$$\vec{A}_0^{ind}(x, y, z) = A_{x_0}^{ind}(x, y, z) \vec{e}_x + A_{y_0}^{ind}(x, y, z) \vec{e}_y. \quad (5.2.28)$$

Substituting (5.2.25) and (5.2.26) into (5.2.27) we obtain

$$\begin{aligned} Z^{ind} &= \frac{j\omega r_c}{I} \int_0^{2\pi} \left(A_{y_0}^{ind} \cos \varphi_r - A_{x_0}^{ind} \sin \varphi_r \right) d\varphi_r \Bigg|_{\substack{r=r_c \\ z=h}} = \\ &= \frac{j\omega r_c^2}{4\pi I} jI \mu_0 \int_0^\infty J_1(\lambda r_c) e^{-\lambda(z+h)} d\lambda \int_0^{2\pi} C \left[\begin{aligned} &\cos \varphi_\lambda \int_0^{2\pi} \cos \varphi_r \left(J_0(\lambda r) + 2 \sum_{k=1}^\infty j^k J_k(\lambda r) \cos k(\varphi_\lambda - \varphi_r) \right) d\varphi_r + \\ &+ \sin \varphi_\lambda \int_0^{2\pi} \sin \varphi_r \left(J_0(\lambda r) + 2 \sum_{k=1}^\infty j^k J_k(\lambda r) \cos k(\varphi_\lambda - \varphi_r) \right) d\varphi_r \end{aligned} \right] d\varphi_\lambda = \\ &\left[\begin{aligned} &\cdot J_0(\lambda r) \cos \varphi_\lambda \int_0^{2\pi} \cos \varphi_r d\varphi_r + \\ &+ 2 \cos \varphi_\lambda \sum_{k=1}^\infty j^k J_k(\lambda r) \left[\cos k\varphi_\lambda \int_0^{2\pi} \cos k\varphi_r \cos \varphi_r d\varphi_r + \sin k\varphi_\lambda \int_0^{2\pi} \sin k\varphi_r \cos \varphi_r d\varphi_r \right] = \\ &= 2j\pi \cos^2 \varphi_\lambda J_1(\lambda r) \\ &\cdot J_0(\lambda r) \sin \varphi_\lambda \int_0^{2\pi} \sin \varphi_r d\varphi_r + \\ &+ 2 \sin \varphi_\lambda \sum_{k=1}^\infty j^k J_k(\lambda r) \left[\cos k\varphi_\lambda \int_0^{2\pi} \cos k\varphi_r \sin \varphi_r d\varphi_r + \sin k\varphi_\lambda \int_0^{2\pi} \sin k\varphi_r \sin \varphi_r d\varphi_r \right] = \\ &= 2j\pi \sin^2 \varphi_\lambda J_1(\lambda r) \end{aligned} \right] \end{aligned}$$

$$= -\frac{\omega r_c^2}{4\pi} \mu_0 \int_0^\infty J_1(\lambda r_c) e^{-\lambda(z+h)} d\lambda \int_0^{2\pi} C 2j\pi J_1(\lambda r) d\varphi_\lambda \Big|_{\substack{r=r_c \\ z=h}}$$

Hence, the induced change in impedance of the coil is given by

$$Z^{ind} = -\frac{j}{2} \omega r_c^2 \mu_0 \int_0^\infty J_1^2(\lambda r_c) e^{-2\lambda h} d\lambda \int_0^{2\pi} C d\varphi_\lambda, \quad (5.2.29)$$

where

$$C = 1 - \frac{2\lambda\mu_m \left(D_5 D_1 - \mu_2 \sqrt{b} e^{\frac{(\beta-\alpha)d}{2}} D_3 \right)}{D_5 \left(D_6 D_1 + \sqrt{b} D_2 \right) + \mu_2 \sqrt{b} e^{\frac{(\beta-\alpha)d}{2}} \left(D_6 D_3 + \sqrt{b} D_4 \right)},$$

$$\left\{ \begin{array}{l} \cdot D_1 = J_\nu(z_0)Y_\nu(z_1) - J_\nu(z_1)Y_\nu(z_0), \quad \cdot D_5 = \frac{\beta}{2} \mu_2 e^{\beta d} - \mu_m q_2, \\ \cdot D_2 = J'_\nu(z_0)Y_\nu(z_1) - J'_\nu(z_1)Y'_\nu(z_0), \quad \cdot D_6 = \lambda \mu_m + \frac{\beta}{2}. \\ \cdot D_3 = J_\nu(z_0)Y'_\nu(z_1) - J'_\nu(z_1)Y_\nu(z_0), \\ \cdot D_4 = J'_\nu(z_0)Y'_\nu(z_1) - J'_\nu(z_1)Y'_\nu(z_1), \end{array} \right.$$

Using the following dimensionless parameters (5.2.29) can be rewritten as follows

$$\left\{ \begin{array}{l} \cdot \lambda r_c = u \rightarrow \lambda = \frac{u}{r_c}, \\ \cdot \alpha = \frac{\hat{\alpha}}{r_c}, \beta = \frac{\hat{\beta}}{r_c}, \end{array} \right\} \Rightarrow \hat{\nu} = \frac{\sqrt{\hat{\beta}^2 + 4u^2}}{\hat{\alpha} + \hat{\beta}}$$

$$\cdot \sqrt{b} = \frac{1}{r_c} \sqrt{-j(\hat{\delta}_1^2 + \hat{\rho} u \cos \varphi_\lambda)} = \frac{\sqrt{-j}}{r_c} \sqrt{\hat{b}}, \text{ where } \left\{ \begin{array}{l} \hat{\delta}_1 = r_c \sqrt{\omega \mu_0 \mu_m \sigma_m} \\ \hat{\rho} = r_c \mu_0 \mu_m \sigma_m V \end{array} \right.$$

$$\cdot q_2 = \frac{1}{r_c} \sqrt{u^2 + j\hat{\delta}_2^2} = \frac{1}{r_c} \hat{q}_2, \hat{z}_1 = \hat{z}_0 e^{\frac{(\hat{\alpha} + \hat{\beta})\hat{d}}{2}}, \hat{z}_0 = \frac{2\sqrt{-j\hat{b}}}{\hat{\alpha} + \hat{\beta}}$$

where $\hat{\delta}_2 = r_c \sqrt{\omega \mu_0 \mu_2 \sigma_2} = r_c \sqrt{\omega \mu_0} \sqrt{\mu_2 \sigma_2} = \hat{\delta}_1 \sqrt{\frac{\mu_2 \sigma_2}{\mu_m \sigma_m}}$,

$$\left\{ \begin{array}{l} \cdot \hat{D}_1 = J_{\hat{\nu}}(\hat{z}_0)Y_{\hat{\nu}}(\hat{z}_1) - J_{\hat{\nu}}(\hat{z}_1)Y_{\hat{\nu}}(\hat{z}_0), \hat{D}_2 = J'_{\hat{\nu}}(\hat{z}_0)Y_{\hat{\nu}}(\hat{z}_1) - J_{\hat{\nu}}(\hat{z}_1)Y'_{\hat{\nu}}(\hat{z}_0), \hat{D}_3 = J_{\hat{\nu}}(\hat{z}_0)Y'_{\hat{\nu}}(\hat{z}_1) - J'_{\hat{\nu}}(\hat{z}_1)Y_{\hat{\nu}}(\hat{z}_0), \\ \cdot \hat{D}_4 = J'_{\hat{\nu}}(\hat{z}_0)Y'_{\hat{\nu}}(\hat{z}_1) - J'_{\hat{\nu}}(\hat{z}_1)Y'_{\hat{\nu}}(\hat{z}_0), D_5 = \frac{1}{r_c} \hat{D}_5 = \frac{1}{r_c} \left(\frac{\hat{\beta}}{2} \mu_2 e^{\beta d} - \mu_m \hat{q}_2 \right), D_6 = \frac{1}{r_c} \hat{D}_6 = \frac{1}{r_c} \left(u \mu_m + \frac{\hat{\beta}}{2} \right), \\ \cdot D_7 = \frac{1}{r_c} \hat{D}_7 = \frac{1}{r_c} \mu_2 \sqrt{-j\hat{b}} e^{\frac{(\hat{\beta}-\hat{\alpha})\hat{d}}{2}} \end{array} \right.$$

Finally, the induced change in impedance of the coil can be rewritten as follows

$$Z^{ind} = \omega r_c \mu_0 Z, \quad (5.2.30)$$

where

$$Z = -\frac{j}{2} \int_0^\infty J_1^2(u) e^{-2u\hat{h}} du \int_0^{2\pi} \hat{C} d\varphi_\lambda, \quad (5.2.31)$$

$$\hat{C} = 1 - \frac{2u\mu_m(\hat{D}_5\hat{D}_1 - \hat{D}_7\hat{D}_3)}{\hat{D}_5\left(\hat{D}_6\hat{D}_1 + \sqrt{-j\hat{b}D_2}\right) + \hat{D}_7\left(\hat{D}_6\hat{D}_3 + \sqrt{-j\hat{b}D_4}\right)}$$

Formula (5.2.31) is used to compute the change in impedance of a single-turn coil for the following values of the parameters of the problem:

$\hat{\alpha} = 0$, $\hat{\beta} = 2$, $\hat{h} = 0.05$, $d = 0.1$, $r_0 = 5$, $\mu_m = 5$. The results of the calculations are shown in **Fig.**

5.4. The points on each graph correspond to different values of $\hat{\delta}_1 = 1, 2, \dots, 10$ (from left to right). Computations are done with “Mathematica” (see **Appendix Fig.Ap.13**) .

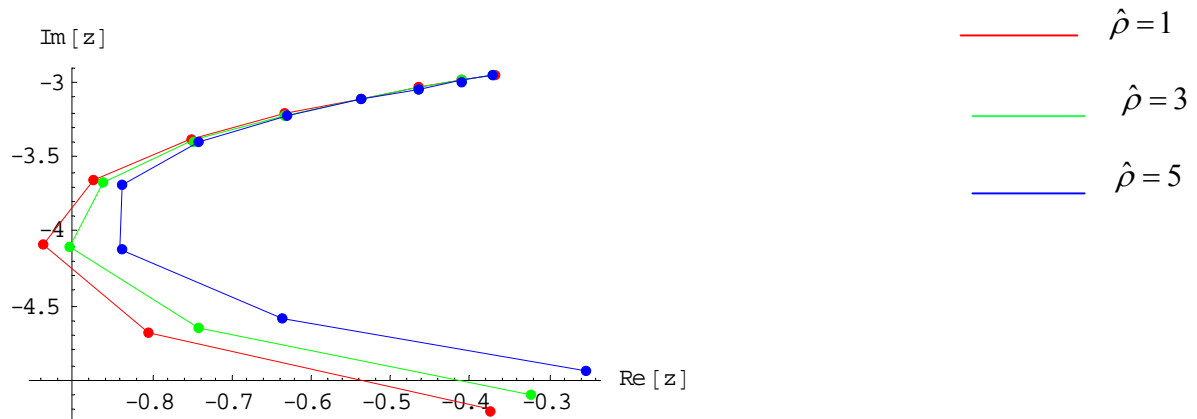


Fig. 5.4 The change in impedance computed by formula (5.2.31) for three values of $\hat{\rho}$

6. AXISYMMETRIC PROBLEMS FOR MEDIA OF FINITE SIZE

6.1 A single-turn coil above a conducting cylinder of finite size

Mathematical models of eddy current testing problems developed in the previous chapters are based on the assumption that a conducting medium is infinite in one or two spatial dimensions. Analytical solutions of the corresponding equations for the vector potential can be obtained in such cases by the method of integral transforms (for example, Fourier or Hankel integral transforms).

Recently a quasi-analytical approach for the solution of eddy current testing problems is suggested in [63]. The authors use the abbreviation TREE (TRuncated Eigenfunction Expansion) method. The main idea of the TREE method is that the vector potential is assumed to be exactly zero at a sufficiently large radial distance $r = b$ from eddy current coil (provided that there are no other sources of alternating current). Note that in the problem formulation described in the previous chapters (for the case of an unbounded medium) the vector potential approaches zero at infinity. From a physical point of view the assumption of the TREE method (the vector potential is equal to zero at a large distance from the coil) is quite reasonable. Recommendations on the selection of the value of b are given in [60].

Thus, a solution of an eddy current problem with the TREE method is expressed in terms of a series (rather than integrals). This is the reason the term “TRuncated Eigenfunction Expansion” is used in order to describe the method. The main advantage of the TREE method in comparison with other analytical methods used for infinite domains is that with the TREE method one can also construct quasi-analytical solutions for the cases where a conducting medium has a finite size. Such models are quite important for applications since one can also model the presence of inhomogeneities (flaws) of finite size in a conducting medium.

In the present section we construct a quasi-analytical solution for the problem shown in **Fig 6.1**. Suppose that a conducting cylinder of a finite radius c is located below a single-turn coil carrying alternating current. The axis of the coil coincides with the axis of the cylinder. The radius of the coil is r_0 and the distance between the coil and the cylinder is equal to h (see **Fig. 6.1**). Such a problem has an important practical application for coin validators. Many coin validators use eddy current method in order to compare the conductivity of an object inserted in the validator with the conductivity of a “typical” coin. The cylinder shown in **Fig. 6.1** can represent a coin.

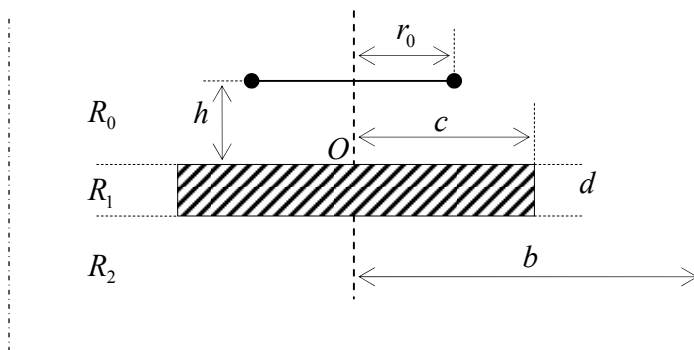


Fig. 6.1 A single-turn coil with alternating current above a conducting cylinder.

Consider the following three regions in a three-dimensional space (see **Fig. 6.1**):
 $R_0 = \{z > 0\}$, $R_1 = \{-d < z < 0\}$ and $R_2 = \{z < -d\}$, where d is the thickness of the cylinder. The vector potential (due to azimuthal symmetry) has only one nonzero component in the φ -direction in each of the regions R_0, R_1 and R_2 (the nonzero components of the vector potential in regions R_0, R_1 and R_2 are denoted by A_0, A_1 and A_2 , respectively). Since region R_1 is not homogeneous in the radial direction we use the notations A_1^{con} and A_1^{air} in order to determine the solutions in regions $0 \leq r \leq c$ and $c \leq r \leq b$, respectively. The system of equations for the functions A_0, A_1 and A_2 is obtained from (1.2.12) for the case where the conductivity σ is constant (note that $\sigma = 0$ in region R_1 where $c \leq r \leq b$) and the relative magnetic permeability μ is equal to one and has the form

$$R_0: \quad \mu(z) = \mu_0^r = 1, \quad \sigma(z) = 0, \quad I^e = I\delta(r - r_0)\delta(z - h),$$

$$\frac{\partial^2 A_0}{\partial r^2} + \frac{1}{r} \frac{\partial A_0}{\partial r} - \frac{A_0}{r^2} + \frac{\partial^2 A_0}{\partial z^2} = -\mu_0 I \delta(r - r_0) \delta(z - h), \quad (6.1.1)$$

$$R_1: \quad \begin{cases} \cdot 0 \leq r \leq c & \mu(z) = \mu_1^r = 1, \quad \sigma(z) = \sigma, \quad I = 0 \\ \cdot c \leq r \leq b & \mu(z) = \mu_1^r = 1, \quad \sigma(z) = 0, \quad I = 0 \end{cases}$$

$$\frac{\partial^2 A_1}{\partial r^2} + \frac{1}{r} \frac{\partial A_1}{\partial r} - \frac{A_1}{r^2} - j\omega\sigma\mu_0 A_1 + \frac{\partial^2 A_1}{\partial z^2} = 0, \quad (6.1.2)$$

$$R_2: \quad \mu(z) = \mu_2^r = 1, \quad \sigma(z) = 0, \quad I = 0$$

$$\frac{\partial^2 A_2}{\partial r^2} + \frac{1}{r} \frac{\partial A_2}{\partial r} - \frac{A_2}{r^2} + \frac{\partial^2 A_2}{\partial z^2} = 0, \quad (6.1.3)$$

where the current in the coil is assumed to be of the form

$$\vec{I}^e = Ie^{j\omega t} \vec{e}_\varphi. \quad (6.1.4)$$

Following the basic assumption in the TREE method we assume that the functions A_0, A_1 and A_2 are equal to zero at $r = b$:

$$A_i|_{r=b} = 0, \quad i = 0, 2, \quad A_1^{air}|_{r=b} = 0. \quad (6.1.5)$$

Continuity conditions are imposed at $r = c$:

$$A_1^{con}|_{r=c} = A_1^{air}|_{r=c}, \quad \frac{\partial A_1^{con}}{\partial r}|_{r=c} = \frac{\partial A_1^{air}}{\partial r}|_{r=c}. \quad (6.1.6)$$

The conditions at $z = 0$ and $z = -d$ are

$$A_0|_{z=0} = A_1|_{z=0}, \quad \frac{\partial A_0}{\partial z}|_{z=0} = \frac{\partial A_1}{\partial z}|_{z=0}, \quad (6.1.7)$$

$$A_1|_{z=-d} = A_2|_{z=-d}, \quad \frac{\partial A_1}{\partial z}|_{z=-d} = \frac{\partial A_2}{\partial z}|_{z=-d}. \quad (6.1.8)$$

Conditions (6.1.6)-(6.1.8) follow from the physical conditions of continuity of the tangent components for the intensity of electrical and magnetic field.

Let us consider the solution of the corresponding homogeneous equation (6.1.1) by the method of separation of variables. Assuming that

$$A_0(r, z) = R(r)Z(z) \quad (6.1.9)$$

we rewrite equation (6.1.1) with zero right-hand side in the form

$$R''Z + \frac{1}{r}R'Z - \frac{1}{r^2}RZ + RZ'' = 0. \quad (6.1.10)$$

Dividing (6.1.10) by RZ and separating the variables we obtain

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} - \frac{1}{r^2} + \lambda^2 = 0, \quad (6.1.11)$$

$$\frac{Z''}{Z} - \lambda^2 = 0, \quad (6.1.12)$$

where λ is the separation constant.

Equation (6.1.11) can be rewritten in the following form

$$R'' + \frac{1}{r}R' + \left(\lambda^2 - \frac{1}{r^2}\right)R = 0, \quad (6.1.13)$$

$$\left\| y'' + \frac{1}{x}y' + \left(\lambda^2 - \frac{v^2}{x^2}\right)y = 0 \rightarrow y = J_v(\lambda x) \right\|$$

General solution to (6.1.13) is

$$R(r) = C_1J_1(\lambda r) + C_2Y_1(\lambda r), \quad (6.1.14)$$

where $J_1(\lambda r)$ and $Y_1(\lambda r)$ are the Bessel functions of order one of the first and second kind, respectively.

Thus, bounded solution to (6.1.14) at $r = 0$ is

$$R(r) = C_1J_1(\lambda r). \quad (6.1.15)$$

General solution to (6.1.12) can be written in the form

$$Z(z) = D_1e^{-\lambda z} + D_2e^{\lambda z}. \quad (6.1.16)$$

Using (6.1.9), (6.1.15), (6.1.16) and the boundary condition $A_0|_{r=b} = 0$ we obtain

$$J_1(\lambda_i b) = 0, \quad i = 1, 2, \dots \quad \left\| \lambda_i = \frac{\alpha_i}{b} \right\| \quad (6.1.17)$$

where λ_i are eigenvalues and α_i , $i = 1, 2, \dots$ are the roots of the equation

$$J_1(\alpha_i) = 0. \quad (6.1.18)$$

In order to construct the solution to (6.1.1) we consider two sub-regions of region R_0 , namely,

$R_{00} = \{0 < z < h\}$ and $R_{01} = \{z > h\}$, respectively. The solutions in R_{00} and R_{01} are denoted by A_{00} and A_{01} , respectively. The right-hand side of (6.1.1) is equal to zero in R_{00} and R_{01} . Using the principle of superposition we represent the solutions of (6.1.1) in R_{00} and R_{01} in the form

$$R_{01} : A_{01}(r, z) = \sum_{i=1}^{\infty} D_{1i}e^{-\lambda_i z} J_1(\lambda_i r), \quad (6.1.19)$$

$$R_{00} : A_{00}(r, z) = \sum_{i=1}^{\infty} (D_{2i}e^{-\lambda_i z} + D_{3i}e^{\lambda_i z}) J_1(\lambda_i r), \quad (6.1.20)$$

where D_{1i} , D_{2i} and D_{3i} are arbitrary constants.

The vector potential is continuous at $z = h$:

$$A_{00} |_{z=h} = A_{01} |_{z=h} . \quad (6.1.21)$$

Integrating (6.1.1) with respect to z from $h - \varepsilon$ to $h + \varepsilon$ and considering the limit as $\varepsilon \rightarrow +0$ in the resulting equation we obtain

$$\frac{\partial A_{01}}{\partial z} |_{z=h} - \frac{\partial A_{00}}{\partial z} |_{z=h} = -\mu_0 I \delta(r - r_0). \quad (6.1.22)$$

It follows from (6.1.19)-(6.1.22) that

$$\sum_{i=1}^{\infty} D_{1i} e^{-\lambda_i h} J_1(\lambda_i r) = \sum_{i=1}^{\infty} (D_{2i} e^{-\lambda_i h} + D_{3i} e^{\lambda_i h}) J_1(\lambda_i r), \quad (6.1.23)$$

$$\sum_{i=1}^{\infty} \lambda_i D_{1i} e^{-\lambda_i h} J_1(\lambda_i r) + \sum_{i=1}^{\infty} (-\lambda_i D_{2i} e^{-\lambda_i h} + \lambda_i D_{3i} e^{\lambda_i h}) J_1(\lambda_i r) = \mu_0 I \delta(r - r_0). \quad (6.1.24)$$

Multiplying (6.1.23) by $r J_1(\lambda_j r)$, integrating the resulting equation with respect to r from 0 to b and using the orthogonality condition

$$\int_0^b r J_1(\lambda_j r) J_1(\lambda_i r) dr = \begin{cases} 0, & i \neq j \\ \frac{b^2}{2} J_0^2(\lambda_j b), & i = j \end{cases} \quad (6.1.25)$$

the following equation is obtained

$$D_{1j} e^{-\lambda_j h} = D_{2j} e^{-\lambda_j h} + D_{3j} e^{\lambda_j h}. \quad (6.1.26)$$

$$\| D_{1j} = D_{2j} + D_{3j} e^{2\lambda_j h} \|$$

Applying the same procedure to (6.1.24) we obtain

$$(D_{1j} e^{-\lambda_j h} - D_{2j} e^{-\lambda_j h} + D_{3j} e^{\lambda_j h}) \lambda_j \frac{b^2}{2} J_0^2(\lambda_j b) = \mu_0 I r_0 J_1(\lambda_j r_0). \quad (6.1.27)$$

Using (6.1.26) and (6.1.27) we get

$$D_{3j} = \frac{\mu_0 I r_0 J_1(\lambda_j r_0)}{\lambda_j b^2 J_0^2(\lambda_j b)} e^{-\lambda_j h}. \quad (6.1.28)$$

Substituting (6.1.26) and (6.1.28) into (6.1.19) and (6.1.20) we obtain

$$A_{00}(r, z) = \sum_{i=1}^{\infty} D_{2i} e^{-\lambda_i z} J_1(\lambda_i r) + \frac{\mu_0 I r_0}{b^2} \sum_{i=1}^{\infty} \frac{J_1(\lambda_i r_0)}{\lambda_i J_0^2(\lambda_i b)} e^{-\lambda_i (h-z)} J_1(\lambda_i r), \quad (6.1.29)$$

$$A_{01}(r, z) = \sum_{i=1}^{\infty} D_{2i} e^{-\lambda_i z} J_1(\lambda_i r) + \frac{\mu_0 I r_0}{b^2} \sum_{i=1}^{\infty} \frac{J_1(\lambda_i r_0)}{\lambda_i J_0^2(\lambda_i b)} e^{-\lambda_i (z-h)} J_1(\lambda_i r). \quad (6.1.30)$$

It can be shown (see [60] that the second term in (6.1.29), (6.1.30) represents the vector potential of a single-turn coil located in an unbounded space (no conducting medium is present). The first term in (6.1.29), (6.1.30) represents the induced component of the vector potential due to the presence of a conducting cylinder (see **Fig. 6.1**).

Let us now construct a solution to (6.1.2) in the conducting cylinder ($0 \leq r \leq c$) by the method of

separation of variables. Assuming the solution in the form

$$A_1^{con}(r, z) = R(r)Z(z)$$

we transform (6.1.2) as follows

$$\frac{R''}{R} + \frac{1}{r} \frac{R'}{R} - \frac{1}{r^2} + q^2 = 0, \quad (6.1.31)$$

$$\left\| R'' + \frac{1}{r} R' + \left(q^2 - \frac{1}{r^2} \right) R = 0 \right\|$$

$$\frac{Z''}{Z} - j\omega\sigma\mu_0 - q^2 = 0, \quad (6.1.32)$$

$$\left\| Z'' - (q^2 + j\omega\sigma\mu_0)Z = 0 \right\|$$

where q is the separation constant.

Solution to (6.1.31) which is bounded at $r = 0$ has the form

$$R(r) = C_3 J_1(qr). \quad (6.1.33)$$

General solution to (6.1.32) is

$$Z(z) = \tilde{D}_3 e^{pz} + \tilde{D}_4 e^{-pz}, \quad (6.1.34)$$

where

$$p = \sqrt{q^2 + j\omega\sigma\mu_0}.$$

Using (6.1.33), (6.1.34) and the superposition principle the solution to (6.1.2) in region $0 \leq r \leq c$ can be written in the form

$$A_1^{con}(r, z) = \sum_{i=1}^{\infty} (D_{4i} e^{p_i z} + D_{5i} e^{-p_i z}) J_1(q_i r), \quad (6.1.35)$$

where q_i are unknown eigenvalues and $p_i = \sqrt{q_i^2 + j\omega\sigma\mu_0}$.

In the non-conducting region ($c \leq r \leq b$) of R_1 equation (6.1.2) has the form

$$\frac{\partial^2 A_1^{air}}{\partial r^2} + \frac{1}{r} \frac{\partial A_1^{air}}{\partial r} - \frac{A_1^{air}}{r^2} + \frac{\partial^2 A_1^{air}}{\partial z^2} = 0. \quad (6.1.36)$$

Representing the solution to (6.1.36) in the form $A_1^{air}(r, z) = R(r)Z(z)$ and solving it by the method of separation of variables we obtain

$$\left\| \begin{aligned} \cdot R'' Z + \frac{1}{r} R' Z - \frac{1}{r^2} R Z + R Z'' = 0 \Big| : R Z \\ \cdot \frac{R''}{R} + \frac{1}{r} \frac{R'}{R} - \frac{1}{r^2} + \frac{Z''}{Z} = 0 \end{aligned} \right\|$$

$$\frac{R''}{R} + \frac{1}{r} \frac{R'}{R} - \frac{1}{r^2} + j\omega\sigma\mu_0 - j\omega\sigma\mu_0 + \frac{Z''}{Z} = 0. \quad (6.1.37)$$

Note that the parameter σ in (6.1.37) is equal to the conductivity of the cylinder (see **Fig. 6.1**).

$$\frac{R''}{R} + \frac{1}{r} \frac{R'}{R} - \frac{1}{r^2} + j\omega\sigma\mu_0 + q^2 = 0,$$

$$\left\| R'' + \frac{1}{r} R' + \left(p^2 - \frac{1}{r^2} \right) R = 0 \rightarrow R(r) = \tilde{C}_5 J_1(pr) + \tilde{C}_6 Y_1(pr) \right\|$$

$$\frac{Z''}{Z} - j\omega\sigma\mu_0 - q^2 = 0,$$

$$\left\| Z'' - p^2 Z = 0 \rightarrow Z(z) = \tilde{D}_5 e^{pz} + \tilde{D}_6 e^{-pz} \right\|$$

where q is the separation constant and $p^2 = q^2 + j\omega\sigma\mu_0$.

Using the principle of superposition we represent the solutions of (6.1.36) in R_1 ($c \leq r \leq b$) as follows

$$A_1^{air}(r, z) = \sum_{i=1}^{\infty} \left[(D_{6i} J_1(p_i r) + D_{7i} Y_1(p_i r)) e^{p_i z} + (D_{8i} J_1(p_i r) + D_{9i} Y_1(p_i r)) e^{-p_i z} \right] \quad (6.1.38)$$

The solution to (6.1.3) in region R_2 which is bounded as $z \rightarrow -\infty$ and satisfies the zero boundary condition at $r = b$ is

$$A_2(r, z) = \sum_{i=1}^{\infty} D_{10i} e^{\lambda_i z} J_1(\lambda_i r). \quad (6.1.39)$$

Using the zero boundary condition (6.1.5) for the function A_1^{air} and linear independence of the functions $e^{p_i z}$ and $e^{-p_i z}$ the following system is obtained from (6.1.38):

$$D_{6i} J_1(p_i b) + D_{7i} Y_1(p_i b) = 0, \quad (6.1.40)$$

$$D_{8i} J_1(p_i b) + D_{9i} Y_1(p_i b) = 0. \quad (6.1.41)$$

Eliminating D_{7i} and D_{9i} from (6.1.40) and (6.1.41) we obtain

$$D_{7i} = -D_{6i} \frac{J_1(p_i b)}{Y_1(p_i b)}, \quad (6.1.42)$$

$$D_{9i} = -D_{8i} \frac{J_1(p_i b)}{Y_1(p_i b)}. \quad (6.1.43)$$

Applying the first condition (6.1.6) to (6.1.35) and (6.1.38) gives

$$\begin{aligned} \sum_{i=1}^{\infty} [D_{4i} e^{p_i z} + D_{5i} e^{-p_i z}] J_1(q_i c) = \\ = \sum_{i=1}^{\infty} \left\{ [D_{6i} J_1(p_i c) + D_{7i} Y_1(p_i c)] e^{p_i z} + [D_{8i} J_1(p_i c) + D_{9i} Y_1(p_i c)] e^{-p_i z} \right\}. \end{aligned} \quad (6.1.44)$$

Now it becomes clear why do we need the transformation in (6.1.37): add and subtract $j\omega\sigma\mu_0$. In this case we obtain the same structure of the z -dependence of the solutions in regions $0 \leq r \leq c$ and $c \leq r \leq b$. Equating coefficients in front of $e^{p_i z}$ and $e^{-p_i z}$ in (6.1.44) the following two equations are obtained

$$D_{4i}J_1(q_i c) = D_{6i}J_1(p_i c) + D_{7i}Y_1(p_i c), \quad (6.1.45)$$

$$D_{5i}J_1(q_i c) = D_{7i}J_1(p_i c) + D_{9i}Y_1(p_i c). \quad (6.1.46)$$

Using (6.1.42) and (6.1.43) we rewrite (6.1.45) and (6.1.46) in the form

$$D_{4i} = \hat{D}_{6i}[J_1(p_i c)Y_1(p_i b) - J_1(p_i b)Y_1(p_i c)], \quad (6.1.47)$$

$$D_{5i} = \hat{D}_{8i}[J_1(p_i c)Y_1(p_i b) - J_1(p_i b)Y_1(p_i c)], \quad (6.1.48)$$

where

$$\hat{D}_{6i} = \frac{D_{6i}}{J_1(q_i c)Y_1(p_i b)}, \quad \hat{D}_{8i} = \frac{D_{8i}}{J_1(q_i c)Y_1(p_i b)}.$$

It follows from (6.1.47), (6.1.48) and (6.1.35) that

$$\begin{aligned} A_1^{con}(r, z) &= \sum_{i=1}^{\infty} \hat{D}_{6i}[J_1(p_i c)Y(p_i b) - J_1(p_i b)Y(p_i c)]e^{p_i z} J_1(q_i r) + \\ &+ \sum_{i=1}^{\infty} \hat{D}_{8i}[J_1(p_i c)Y(p_i b) - J_1(p_i b)Y(p_i c)]e^{-p_i z} J_1(q_i r). \end{aligned} \quad (6.1.49)$$

Similarly, using (6.1.42) and (6.1.43) we transform (6.1.38) to the form

$$\begin{aligned} A_1^{air}(r, z) &= \sum_{i=1}^{\infty} \hat{D}_{6i}J_1(q_i c)[J_1(p_i r)Y(p_i b) - J_1(p_i b)Y(p_i r)]e^{p_i z} + \\ &+ \sum_{i=1}^{\infty} \hat{D}_{8i}J_1(q_i c)[J_1(p_i r)Y(p_i b) - J_1(p_i b)Y(p_i r)]e^{-p_i z}. \end{aligned} \quad (6.1.50)$$

Introducing the notation

$$T_1(p_i c) = J_1(p_i c)Y_1(p_i b) - J_1(p_i b)Y_1(p_i c), \quad (6.1.51)$$

$$T_1(p_i r) = J_1(p_i r)Y_1(p_i b) - J_1(p_i b)Y_1(p_i r), \quad (6.1.52)$$

we rewrite (6.1.49) and (6.1.50) in a more compact form

$$A_1^{con}(r, z) = \sum_{i=1}^{\infty} J_1(q_i r)T_1(p_i c)(\hat{D}_{6i}e^{p_i z} + \hat{D}_{8i}e^{-p_i z}), \quad (6.1.53)$$

$$A_1^{air}(r, z) = \sum_{i=1}^{\infty} J_1(q_i c)T_1(p_i r)(\hat{D}_{6i}e^{p_i z} + \hat{D}_{8i}e^{-p_i z}). \quad (6.1.54)$$

Differentiating (6.1.53) and (6.1.54) with respect to r and evaluating the derivatives at $r = c$ we obtain

$$\frac{\partial A_1^{con}}{\partial r} \Big|_{r=c} = \sum_{i=1}^{\infty} q_i J_1'(q_i c)T_1(p_i c)(\hat{D}_{6i}e^{p_i z} + \hat{D}_{8i}e^{-p_i z}). \quad (6.1.55)$$

$$\frac{\partial A_1^{air}}{\partial r} \Big|_{r=c} = \sum_{i=1}^{\infty} p_i J_1(q_i c)T_1'(p_i c)(\hat{D}_{6i}e^{p_i z} + \hat{D}_{8i}e^{-p_i z}), \quad (6.1.56)$$

It follows from (6.1.55) and (6.1.56) and the second boundary condition in (6.1.6) that

$$p_i J_1(q_i c)T_1'(p_i c) = q_i J_1'(q_i c)T_1(p_i c). \quad (6.1.57)$$

Equation (6.1.57) is used to determine the eigenvalues p_i and related values q_i .

The relationship between p_i and q_i is $p_i = \sqrt{q_i^2 + j\omega\sigma\mu_0}$.

Thus, the solution in regions R_0, R_1 and R_2 is given by (6.1.29), (6.1.30), (6.1.53), (6.1.54) and (6.1.39). The four sets of constants in these formulas, namely, $D_{2i}, \hat{D}_{6i}, \hat{D}_{8i}$ and D_{10i} can be obtained from the boundary conditions (6.1.7) and (6.1.8).

Using the first condition in (6.1.7) we obtain

$$\sum_{i=1}^{\infty} D_{2i} J_1(\lambda_i r) + \frac{\mu_0 I r_0}{b^2} \sum_{i=1}^{\infty} \frac{J_1(\lambda_i r_0) e^{-\lambda_i h}}{\lambda_i J_0^2(\lambda_i b)} J_1(\lambda_i r) = \sum_{i=1}^{\infty} J_1(q_i r) T_1(p_i c) (\hat{D}_{6i} + \hat{D}_{8i}), \quad 0 \leq r \leq c, \quad (6.1.58)$$

$$\sum_{i=1}^{\infty} D_{2i} J_1(\lambda_i r) + \frac{\mu_0 I r_0}{b^2} \sum_{i=1}^{\infty} \frac{J_1(\lambda_i r_0) e^{-\lambda_i h}}{\lambda_i J_0^2(\lambda_i b)} J_1(\lambda_i r) = \sum_{i=1}^{\infty} J_1(q_i c) T_1(p_i r) (\hat{D}_{6i} + \hat{D}_{8i}), \quad c \leq r \leq b. \quad (6.1.59)$$

In order to determine the coefficients \hat{D}_{6i} and \hat{D}_{8i} the following procedure is used. First, equations (6.1.58) and (6.1.59) are combined into one equation where the right-hand side of the resulting equation is given by different expressions on the intervals $0 \leq r \leq c$ and $c \leq r \leq b$. These expressions are defined by the right-hand sides of (6.1.58) and (6.1.59), respectively. Second, the obtained equation is multiplied by $r J_1(\lambda_j r)$ and the resulting equation is integrated with respect to r from 0 to b . Third, we use the following orthogonality condition (6.1.25)

$$\left\| \int_0^b r J_1(\lambda_j r) J_1(\lambda_i r) dr = \begin{cases} 0, & i \neq j \\ \frac{b^2}{2} J_0^2(\lambda_j b), & i = j. \end{cases} \right\|$$

The result is

$$D_{2j} \frac{b^2}{2} J_0^2(\lambda_j b) + \frac{\mu_0 I r_0}{2\lambda_j} J_1(\lambda_j r_0) e^{-\lambda_j h} = \sum_{i=1}^{\infty} (\hat{D}_{6i} + \hat{D}_{8i}) a_{ji}, \quad (6.1.60)$$

where

$$a_{ji} = T_1(p_i c) \tilde{a}_{ji} + J_1(q_i c) \tilde{\tilde{a}}_{ji}. \quad (6.1.61)$$

The coefficients a_{ij} can be computed using the following formulas from [2]:

$$\int (k^2 - l^2) t J_1(kt) J_1(lt) dt = t [kJ_2(kt) J_1(lt) - lJ_1(kt) J_2(lt)], \quad (6.1.62)$$

$$\int (k^2 - l^2) t J_1(kt) Y_1(lt) dt = t [kJ_2(kt) Y_1(lt) - lJ_1(kt) Y_2(lt)]. \quad (6.1.63)$$

Thus, using (6.1.62), (6.1.63) and (6.1.52) we obtain

$$\left\| \begin{aligned} \cdot \tilde{a}_{ji} &= \int_0^c r J_1(\lambda_j r) J_1(q_i r) dr = \frac{c}{\lambda_j^2 - q_i^2} (\lambda_j J_2(\lambda_j c) J_1(q_i c) - q_i J_1(\lambda_j c) J_2(q_i c)) \\ \cdot \tilde{\tilde{a}}_{ji} &= \int_c^b r J_1(\lambda_j r) T_1(p_i r) dr = Y_1(p_i b) \int_c^b r J_1(\lambda_j r) J_1(p_i r) dr - J_1(p_i b) \int_c^b r J_1(\lambda_j r) Y_1(p_i r) dr = \\ &= \frac{1}{\lambda_j^2 - p_i^2} \left\{ \begin{aligned} &bp_i J_1(\lambda_j b) [J_1(p_i b) Y_2(p_i b) - J_2(p_i b) Y_1(p_i b)] + \\ &+ c \lambda_j J_2(\lambda_j c) [J_1(p_i b) Y_1(p_i c) - J_1(p_i c) Y_1(p_i b)] + \\ &+ cp_i J_1(\lambda_j c) [J_2(p_i c) Y_1(p_i b) - J_1(p_i b) Y_2(p_i c)] \end{aligned} \right\} \end{aligned} \right\|$$

Applying the same procedure and using the second condition in (6.1.7) we obtain

$$-\lambda_j D_{2j} \frac{b^2}{2} J_0^2(\lambda_j b) + \frac{\mu_0 I r_0}{2} J_1(\lambda_j r_0) e^{-\lambda_j h} = \sum_{i=1}^{\infty} (\hat{D}_{6i} - \hat{D}_{8i}) p_i a_{ji}. \quad (6.1.64)$$

Multiplying (6.1.60) by λ_j and adding with (6.1.64) gives

$$\sum_{i=1}^{\infty} [(\lambda_j + p_i) \hat{D}_{6i} + (\lambda_j - p_i) \hat{D}_{8i}] a_{ji} = \mu_0 I r_0 J_1(\lambda_j r_0) e^{-\lambda_j h}. \quad (6.1.65)$$

Two additional equations are obtained if the same procedure is applied to (6.1.53), (6.1.54) and (6.1.39) using boundary conditions (6.1.8). The result is shown below

$$D_{10j} e^{-\lambda_j d} \frac{b^2}{2} J_0^2(\lambda_j b) = \sum_{i=1}^{\infty} (\hat{D}_{6i} e^{-p_i d} + \hat{D}_{8i} e^{p_i d}) a_{ji}, \quad (6.1.66)$$

$$\lambda_j D_{10j} e^{-\lambda_j d} \frac{b^2}{2} J_0^2(\lambda_j b) = \sum_{i=1}^{\infty} (\hat{D}_{6i} e^{-p_i d} - \hat{D}_{8i} e^{p_i d}) p_i a_{ji}. \quad (6.1.67)$$

Multiplying (6.1.66) by $(-\lambda_j)$ and adding with (6.1.67) we obtain

$$\sum_{i=1}^{\infty} [(\lambda_j - p_i) e^{-p_i d} \hat{D}_{6i} + (\lambda_j + p_i) e^{p_i d} \hat{D}_{8i}] a_{ji} = 0. \quad (6.1.68)$$

The four sets of equations for the unknown coefficients D_{2j} , \hat{D}_{6i} , \hat{D}_{8i} and D_{10j} are given by (6.1.60), (6.1.65), (6.1.68) and (6.1.66). The solution of the system is obtained as follows. First, we consider only a finite number of terms n in (6.1.60), (6.1.65), (6.1.66) and (6.1.68).

Recommendations on the selection of the value of n are given in **Section 6.3**. Second, the coefficients \hat{D}_{6i} , \hat{D}_{8i} can be computed solving the system

$$\sum_{i=1}^n [(\lambda_j + p_i) a_{ji} \hat{D}_{6i} + (\lambda_j - p_i) a_{ji} \hat{D}_{8i}] = \mu_0 I r_0 J_1(\lambda_j r_0) e^{-\lambda_j h}, \quad (6.1.69)$$

$$\sum_{i=1}^n [(\lambda_j - p_i) a_{ji} e^{-p_i d} \hat{D}_{6i} + (\lambda_j + p_i) a_{ji} e^{p_i d} \hat{D}_{8i}] = 0. \quad (6.1.70)$$

where n is the number of eigenvalues p_i . The system (6.1.69)-(6.1.70) can be written in the matrix form

$$A \bar{X} = \bar{B} \quad (6.1.71)$$

where the coefficient matrix A is

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad (6.1.72)$$

and the block matrices A_{11} , A_{12} , A_{21} and A_{22} are

$$A_{11} = \begin{pmatrix} (\lambda_1 + p_1) a_{11} & (\lambda_1 + p_2) a_{12} & \dots & (\lambda_1 + p_n) a_{1n} \\ (\lambda_2 + p_1) a_{21} & (\lambda_2 + p_2) a_{22} & \dots & (\lambda_2 + p_n) a_{2n} \\ \dots & \dots & \dots & \dots \\ (\lambda_n + p_1) a_{n1} & (\lambda_n + p_2) a_{n2} & \dots & (\lambda_n + p_n) a_{nn} \end{pmatrix} \quad A_{12} = \begin{pmatrix} (\lambda_1 - p_1) a_{11} & (\lambda_1 - p_2) a_{12} & \dots & (\lambda_1 - p_n) a_{1n} \\ (\lambda_2 - p_1) a_{21} & (\lambda_2 - p_2) a_{22} & \dots & (\lambda_2 - p_n) a_{2n} \\ \dots & \dots & \dots & \dots \\ (\lambda_n - p_1) a_{n1} & (\lambda_n - p_2) a_{n2} & \dots & (\lambda_n - p_n) a_{nn} \end{pmatrix}$$

$$A_{21} = \begin{pmatrix} (\lambda_1 - p_1)a_{11}e^{-p_1d} & (\lambda_1 - p_2)a_{12}e^{-p_2d} & \dots & (\lambda_1 - p_n)a_{1n}e^{-p_nd} \\ (\lambda_2 - p_1)a_{21}e^{-p_1d} & (\lambda_2 - p_2)a_{22}e^{-p_2d} & \dots & (\lambda_2 - p_n)a_{2n}e^{-p_nd} \\ \dots & \dots & \dots & \dots \\ (\lambda_n - p_1)a_{n1}e^{-p_1d} & (\lambda_n - p_2)a_{n2}e^{-p_2d} & \dots & (\lambda_n - p_n)a_{nn}e^{-p_nd} \end{pmatrix}$$

$$A_{22} = \begin{pmatrix} (\lambda_1 + p_1)a_{11}e^{p_1d} & (\lambda_1 + p_2)a_{12}e^{p_2d} & \dots & (\lambda_1 + p_n)a_{1n}e^{p_nd} \\ (\lambda_2 + p_1)a_{21}e^{p_1d} & (\lambda_2 + p_2)a_{22}e^{p_2d} & \dots & (\lambda_2 + p_n)a_{2n}e^{p_nd} \\ \dots & \dots & \dots & \dots \\ (\lambda_n + p_1)a_{n1}e^{p_1d} & (\lambda_n + p_2)a_{n2}e^{p_2d} & \dots & (\lambda_n + p_n)a_{nn}e^{p_nd} \end{pmatrix}$$

The matrices \vec{X} and \vec{B} in (6.1.71) are

$$\vec{X} = \begin{pmatrix} \vec{X}_1 \\ \vec{X}_2 \end{pmatrix}, \quad \vec{B} = \begin{pmatrix} \vec{b} \\ 0 \end{pmatrix}, \quad (6.1.73)$$

where

$$\vec{X}_1 = \begin{pmatrix} \hat{D}_{61} \\ \hat{D}_{62} \\ \dots \\ \hat{D}_{6n} \end{pmatrix}, \quad \vec{X}_2 = \begin{pmatrix} \hat{D}_{81} \\ \hat{D}_{82} \\ \dots \\ \hat{D}_{8n} \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} \mu_0 I r_0 J_1(\lambda_1 r_0) e^{-\lambda_1 h} \\ \mu_0 I r_0 J_1(\lambda_2 r_0) e^{-\lambda_2 h} \\ \dots \\ \mu_0 I r_0 J_1(\lambda_n r_0) e^{-\lambda_n h} \end{pmatrix}. \quad (6.1.74)$$

Hence, (6.1.71) can be written in the form

$$\begin{cases} A_{11}\vec{X}_1 + A_{12}\vec{X}_2 = \vec{b} \\ A_{21}\vec{X}_1 + A_{22}\vec{X}_2 = 0 \end{cases} \quad (6.1.75)$$

Solving (6.1.75) we obtain all the coefficients \hat{D}_{6i} and \hat{D}_{8i} ($i = 1, 2, \dots, n$). Then D_{2j} ($j = 1, 2, \dots, n$) is calculated using

$$D_{2j} = \frac{2}{b^2 J_0^2(\lambda_j b)} \sum_{i=1}^n a_{ji} (\hat{D}_{6i} + \hat{D}_{8i}) - \frac{\mu_0 I r_0 J_1(\lambda_j r_0) e^{-\lambda_j h}}{\lambda_j b^2 J_0^2(\lambda_j b)}. \quad (6.1.76)$$

The induced change in impedance of the coil is given by the formula

$$Z^{ind} = \frac{j\omega}{I} 2\pi r_0 A_0^{ind}(r_0, h), \quad (6.1.77)$$

where

$$A_0^{ind}(r_0, h) = \sum_{j=1}^n D_{2j} e^{-\lambda_j h} J_1(\lambda_j r_0). \quad (6.1.78)$$

Formula (6.1.77) is used to compute the change in impedance of the coil. Calculations are performed with ‘‘Mathematica’’. The program which is used to compute the change in impedance (6.1.77) is shown in **Appendix Fig. Ap.14**. The following parameters of the problem are selected: $\mu_0 = 4 \cdot 10^{-7} \pi$, $\sigma = 9.6 \text{ Ms/m}$, $c = 19.75/2 \text{ mm}$, $r_0 = 4.5 \text{ mm}$, $h = 0.2 \text{ mm}$, $d = 1.93 \text{ mm}$,

$b = 60$ mm. The change in impedance is computed for the following five frequencies:
 $f = 1125, 1598, 2270, 3224$ and 4579 Hz.

The results of calculations are shown in **Fig. 6.2**. The calculated points (from top to bottom) correspond to the five frequencies (from smallest to largest). The upper limit of the summation index in (6.1.78) is fixed at $n = 68$. Comparison of the computational results obtained for other values of n showed that the chosen value of 68 is quite satisfactory in terms of calculation accuracy. More detailed analysis of convergence is presented in **Section 6.3**.

Several computational steps are necessary in order to calculate the induced change in impedance. First, the set of eigenvalues λ_i has to be calculated. This can easily be done in “Mathematica” using a built-in routine BesselZeros. Second, a set of complex roots of (6.1.57) should be computed. Calculations are based on the method described in **Section 6.3**. Third, several systems of linear equations have to be solved in order to determine expansion coefficients. Finally, the change in impedance is computed using (6.1.71) - (6.1.78).

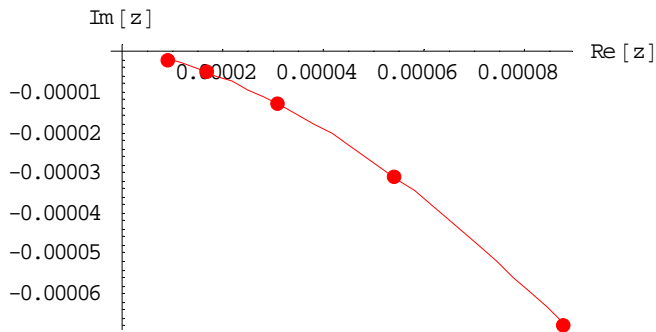


Fig. 6.2 The change in impedance of the coil for five frequencies.

6.2 A coil of finite dimensions above a conducting cylinder of finite size

The solution for a single-turn coil found in the previous section can be used to construct the vector potential due to a coil of finite dimensions. Consider a coil located above a conducting cylinder (see **Fig. 6.3**). Let N be the number of turns in the coil. The inner and outer radii of the coil are r_1 and r_2 , respectively. The distance from the coil to the top surface of the cylinder is denoted by z_1 . The height of the coil is $z_2 - z_1$.

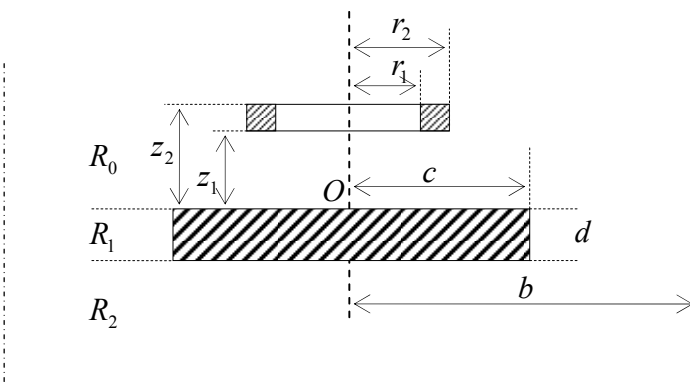


Fig. 6.3 A coil of finite dimensions above a conducting cylinder.

It can easily be shown that the second term in (6.1.29) and (6.1.30) represents the vector potential of a single-turn coil located in an unbounded free space. In addition, the first term in (6.1.29) and (6.1.30) is the induced vector potential in air due to the presence of a conducting cylinder:

$$A_0^{ind}(r, z, r_0, h) = \sum_{j=1}^n D_{2j} e^{-\lambda_j z} J_1(\lambda_j r), \quad (6.2.1)$$

where

$$D_{2j} = \frac{2}{b^2 J_0^2(\lambda_j b)} \sum_{i=1}^n a_{ji} (\hat{D}_{6i} + \hat{D}_{8i}) - \frac{\mu_0 I r_0 J_1(\lambda_j r_0) e^{-\lambda_j h}}{\lambda_j b^2 J_0^2(\lambda_j b)}$$

and the series are truncated at $i = n$.

The induced vector potential in air due to currents in the whole coil is obtained as follows

$$A_{0coil}^{ind}(r, z) = \int_{r_1}^{r_2} \int_{z_1}^{z_2} A_0^{ind}(r, z, r_0, h) dr_0 dh. \quad (6.2.2)$$

There is an important difference between the calculations for a single-turn coil and coil of finite dimensions. For the case of a single-turn coil (**Section 6.1**) the geometrical parameters of the coil (r_0 and h) are constants and the numerical values of r_0 and h can be used to solve the system (6.1.75). If a coil has finite dimensions then one needs to integrate the solution with respect to r_0 and h . Thus, numerical values to the parameters r_0 and h in (6.1.75) cannot be assigned. In this case, in order to obtain the induced vector potential of a single-turn coil we need to solve (6.1.75) in matrix form. Solving the second equation in (6.1.75) we obtain

$$\vec{X}_2 = -A_{22}^{-1} A_{21} \vec{X}_1. \quad (6.2.3)$$

Substituting (6.2.3) into the first equation in (6.1.75) gives

$$(A_{11} - A_{12} A_{22}^{-1} A_{21}) \vec{X}_1 = \vec{b}. \quad (6.2.4)$$

It follows from (6.2.3) and (6.2.4) that

$$\begin{cases} \vec{X}_1 = (A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1} \vec{b} \\ \vec{X}_2 = -A_{22}^{-1} A_{21} (A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1} \vec{b} \end{cases} \quad (6.2.5)$$

Equation (6.1.76) should be written in matrix form:

$$\vec{D}_2 = Y \vec{b}, \quad (6.2.6)$$

where

$$\vec{D}_2 = \begin{pmatrix} D_{21} \\ D_{22} \\ \dots \\ D_{2n} \end{pmatrix}, \quad Y = B_{11} (E - A_{22}^{-1} A_{21}) (A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1} - C_{diag}, \quad (6.2.7)$$

$$B_{11} = \begin{pmatrix} \frac{2a_{11}}{b^2 J_0^2(\lambda_1 b)} & \frac{2a_{12}}{b^2 J_0^2(\lambda_1 b)} & \dots & \frac{2a_{1n}}{b^2 J_0^2(\lambda_1 b)} \\ \frac{2a_{21}}{b^2 J_0^2(\lambda_2 b)} & \frac{2a_{22}}{b^2 J_0^2(\lambda_2 b)} & \dots & \frac{2a_{2n}}{b^2 J_0^2(\lambda_2 b)} \\ \dots & \dots & \dots & \dots \\ \frac{2a_{n1}}{b^2 J_0^2(\lambda_n b)} & \frac{2a_{n2}}{b^2 J_0^2(\lambda_n b)} & \dots & \frac{2a_{nn}}{b^2 J_0^2(\lambda_n b)} \end{pmatrix}, \quad C_{diag} = \begin{pmatrix} \frac{1}{\lambda_1 b^2 J_0^2(\lambda_1 b)} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{\lambda_2 b^2 J_0^2(\lambda_2 b)} & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \frac{1}{\lambda_n b^2 J_0^2(\lambda_n b)} \end{pmatrix}, \quad (6.2.8)$$

and \vec{b} is given by (6.1.74).

Formula (6.2.1) for the induced vector potential of a single-turn coil can be rewritten in the form

$$A_0^{ind}(r, z, r_0, h) = \vec{D}_2^T \vec{f}, \quad (6.2.9)$$

where

$$\vec{f} = \begin{pmatrix} J_1(\lambda_1 r) e^{-\lambda_1 z} \\ J_1(\lambda_2 r) e^{-\lambda_2 z} \\ \dots \\ J_1(\lambda_n r) e^{-\lambda_n z} \end{pmatrix}.$$

Substituting (6.2.9) into (6.2.2) and using the formulas

$$\left\| \begin{aligned} \int_{z_1}^{z_2} e^{-\lambda_j h} dh &= -\frac{1}{\lambda_j} (e^{-\lambda_j z_2} - e^{-\lambda_j z_1}) \\ \int_{r_1}^{r_2} r_0 J_1(\lambda_j r_0) dr_0 &= \left| \xi = \lambda_j r_0 \right| = \frac{1}{\lambda_j^2} \int_{\lambda_j r_1}^{\lambda_j r_2} \xi J_1(\xi) d\xi \end{aligned} \right\| \quad (6.2.10)$$

we obtain the induced vector potential in air due to the presence of the conducting cylinder (the current amplitude I in this case is replaced by the current density $\frac{NI}{(r_2 - r_1)(z_2 - z_1)}$) :

$$A_{0coil}^{ind}(r, z) = \frac{\mu_0 NI}{(r_2 - r_1)(z_2 - z_1)} \sum_{j=1}^n f_j \sum_{i=1}^n Y_{ji} \frac{(e^{-\lambda_i z_1} - e^{-\lambda_i z_2})}{\lambda_i^3} \int_{\lambda_i r_1}^{\lambda_i r_2} \xi J_1(\xi) d\xi. \quad (6.2.11)$$

The integral with respect to ξ in (6.2.10) can be computed in terms of the Bessel and Struve functions [2] as follows

$$\int_{\lambda_i r_1}^{\lambda_i r_2} \xi J_1(\xi) d\xi = \left\{ \frac{\pi}{2} \xi [J_0(\xi) H_1(\xi) - J_1(\xi) H_0(\xi)] \right\}_{\xi=\lambda_i r_1}^{\xi=\lambda_i r_2} \quad (6.2.12)$$

The induced change in impedance of a coil of finite dimensions (see **Fig. 6.3**) is calculated by means of the following formula [15]:

$$\left\| \begin{aligned} Z^{ind} &= \frac{j\omega}{I^2} \iiint_V \vec{A} \vec{I} dv = \frac{j\omega}{I^2} \int_0^{2\pi} d\varphi \int_{r_1}^{r_2} r dr \int_{z_1}^{z_2} A_{0coil}^{ind} dz \\ Z^{ind} &= \frac{2\pi j\omega}{I} \frac{N}{(r_2 - r_1)(z_2 - z_1)} \int_{r_1}^{r_2} \int_{z_1}^{z_2} r A_{0coil}^{ind}(r, z) dr dz. \end{aligned} \right\| \quad (6.2.13)$$

Using (6.2.11) and (6.2.13) we obtain the induced change in impedance of the coil in the form

$$Z^{ind} = \frac{2j\omega\pi\mu_0 N^2}{(r_2 - r_1)^2 (z_2 - z_1)^2} \sum_{j=1}^n \frac{(e^{-\lambda_j z_1} - e^{-\lambda_j z_2})}{\lambda_j^3} \int_{\lambda_j r_1}^{\lambda_j r_2} \xi J_1(\xi) d\xi \sum_{i=1}^n Y_{ji} \frac{(e^{-\lambda_i z_1} - e^{-\lambda_i z_2})}{\lambda_i^3} \int_{\lambda_i r_1}^{\lambda_i r_2} \xi J_1(\xi) d\xi \quad (6.2.14)$$

Formula (6.2.14) is used to compute the change in impedance of the coil. Calculations are performed with ‘‘Mathematica’’. Program which is used to compute the change in impedance (6.2.14) is shown in **Appendix Fig. Ap.15**. The following parameters of the problem are selected: $\mu_0 = 4 \cdot 10^{-7} \pi$, $\sigma = 9.6 \text{ Ms/m}$, $c = 19.75/2 \text{ mm}$, $b = 60 \text{ mm}$, $d = 1.93 \text{ mm}$, $r_2 = 6 \text{ mm}$, $r_1 = 3 \text{ mm}$, $z_2 = 0.39 \text{ mm}$, $z_1 = 0.06 \text{ mm}$, $N = 100$.

The results of calculations are shown in **Fig. 6.4**. The calculated points (from top to bottom) correspond to the following values of $f = 1125, 1598, 2270, 3224$ and 4579 Hz.

The upper limit of the summation index in (6.2.14) is fixed at $n = 68$. Comparison of the computational results obtained for other values of n showed that the chosen value of 68 is quite satisfactory in terms of calculation accuracy.

Several computational steps are necessary in order to calculate the induced change in impedance. First, the set of eigenvalues λ_i has to be calculated. This can easily be done in “Mathematica” using a built-in routine `BesselZeros`. Second, a set of complex roots of (6.1.57) should be computed. Calculations are based on the method described in **Section 6.3**. Third, several systems of linear equations have to be solved in order to determine expansion coefficients. Finally, the change in impedance is computed using (6.2.14). Details of the numerical aspects of the procedure are given in **Section 6.3**.

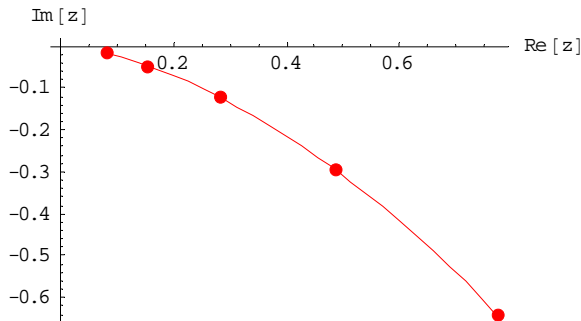


Fig. 6.4 The change in impedance of a coil (calculations are done using (6.2.14)).

Calculated change in impedance of the coil is compared with experimental data. Experiments are performed at Tallinn University of Technology as a part of the work for the project “Increasing EU citizen security by utilizing innovative intelligent signal processing systems for euro-coin validation and metal quality testing” in the framework of FP7 program for the period from 2010 till 2012. Measurement results are also published in [20].

The results of comparison are shown in **Fig. 6.5** for the same five frequencies as in **Fig. 6.4**. The parameters of the coil and conducting cylinder (coin) are the same as specified for **Fig. 6.4**. The larger points on the graph are experimental points while the smaller points represent theoretical calculations. As can be seen from the graph, good agreement is found between experimental data and theory.

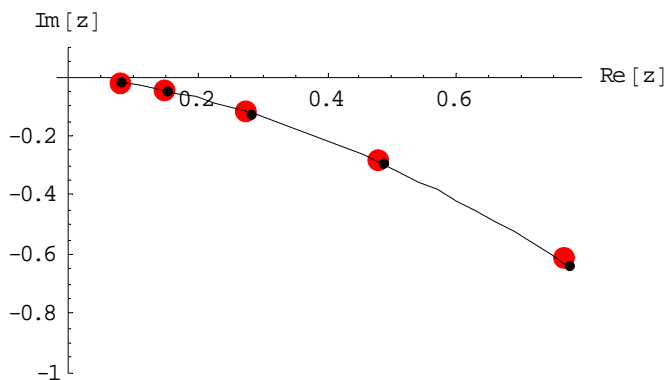


Fig. 6.5 Comparison of theoretical calculations with experimental data.

6.3 Computational aspects

The procedure for calculating the change in impedance of the coil by means of the formula (6.2.14) includes several computational steps. In this section we discuss each of the steps in detail.

The first decision has to be made with respect to the choice of the parameter b where the vector potential is assumed to be exactly zero (recall that b is the radial distance from the center of the coil). Recommendations on the selection of the parameter b are presented in [60] where it is recommended to choose b in the form $b = sr_2$, where r_2 is the outer radius of the coil and the factor s should be at least five. Larger values of s can be considered as well (in general, the choice of s is directly related to the desired accuracy of the calculations).

Next, we need to truncate the series in (6.2.14). In any realistic case the summation index i varies from 1 to some pre-specified value $n = N_s$. For example, N_s from 50 to 200 is used in [60] in order to compute the induced change in impedance for the case where a coil of finite dimensions is located above a conducting half-space. We have to pay the attention of the reader to the fact that the choice of the value of N_s also depends on the computational accuracy.

The third step includes computation of eigenvalues $\lambda_i, i = 1, 2, \dots, N_s$. The eigenvalues are computed by means of (6.1.17), where $\alpha_i, i = 1, 2, \dots, N_s$ are the roots of the equation (6.1.18). The roots α_i can be easily computed with “Mathematica”. The corresponding “Mathematica” script is shown in **Fig. 6.6** where the first ten zeros of (6.1.18) are shown

```
<< NumericalMath`BesselZeros`
Ns = 10; alfa = BesselJZeros[1, Ns]
{3.83171, 7.01559, 10.1735, 13.3237, 16.4706,
19.6159, 22.7601, 25.9037, 29.0468, 32.1897}
```

Fig. 6.6 “Mathematica” code for the computation of the zeros of (6.1.18).

The most complicated step of the procedure includes computation of complex eigenvalues $p_i, i = 1, 2, \dots, N_s$ and the related values of $q_i, i = 1, 2, \dots, N_s$ from equation (6.1.57). It is well-known that the most difficult step in calculating complex roots of an analytic function is to find a good initial approximation to the root. If a good initial approximation is known the value of the root with high accuracy can be easily computed by means of the Newton’s method. In order to locate the roots we follow the procedure described in [60] for other application of the TREE method. The method for finding complex roots of an analytic function is based on the two papers published in 1967: [12] and [43] and uses the Cauchy’s theorem. First, it is known from the theory of complex variables that the number of zeros, s_0 , of an analytic function $\phi(z)$ inside a closed contour C is equal to the following contour integral

$$s_0 = \frac{1}{2\pi i} \oint_C \frac{\phi'(z)}{\phi(z)} dz. \quad (6.3.1)$$

Another interesting result from the theory of complex variables is the following [12]. If C is a closed curve in the complex plane which does not pass through a zero of $\phi(z)$ and G is the interior of C , then

$$s_n = \frac{1}{2\pi i} \oint_C z^n \frac{\phi'(z)}{\phi(z)} dz = \sum_{i=1}^k \zeta_i^n, \quad (6.3.2)$$

where $\zeta_i, i = 1, 2, \dots, k$ are all zeros of $\phi(z)$ which lie in G and $n = 1, 2, \dots, k$. A multiple zero is counted according to its multiplicity in (6.3.2). Using (6.3.2) we can construct a polynomial of degree k which has the same zeros $\zeta_i, i = 1, 2, \dots, k$ as the function $\phi(z)$. Note that in this case the polynomial is not an approximation to $\phi(z)$, it simply has the same roots as the function $\phi(z)$. At this stage several possible strategies are available in order to find ζ_i . If the domain G is small enough and there is only one zero of $\phi(z)$ inside C , then $k = 1$ and the value s_1 from (6.3.2) gives us the only zero, ζ_1 , of the function $\phi(z)$ inside C . However, if there is more than one root of $\phi(z)$ inside C then a polynomial is constructed using (6.3.2) and the roots of this polynomial can be calculated (for example, by means of “Mathematica” command **Roots**). As mentioned before, the roots of the corresponding polynomial are exactly the same as the zeros of $\phi(z)$ inside C . It is known that the roots of higher order polynomials can be quite sensitive to the variation of the coefficients of the polynomial. Thus, the number of zeros of $\phi(z)$ inside C should not be too large (k should not exceed 5). As a result, if $k > 5$ then the domain G should be divided into smaller sub-domains so that the number of zeros of $\phi(z)$ in each sub-domain be smaller than 5.

The next practical issue is the shape of the contour C . Two basic shapes are recommended for calculations in [12]: circles and rectangles. The choice of the shape of the contour C is dictated by calculation of the contour integrals in (6.3.1) and (6.3.2). It is known that contour integrals can be conveniently calculated if C is a circle or rectangle. We choose rectangle as the basic shape of C since any rectangle can be easily sub-divided into smaller rectangles. Consider a rectangular contour $ABCD$ (see **Fig. 6.7**).

If $\phi(z)$ is an analytic function inside C (where C represents the contour $ABCD$ shown in **Fig. 6.7**) then

$$\oint_C \phi(z) dz = \int_{x_1}^{x_2} [\phi(x + jy_1) - \phi(x + jy_2)] dx + j \int_{y_1}^{y_2} [\phi(x_2 + jy) - \phi(x_1 + jy)] dy. \quad (6.3.3)$$

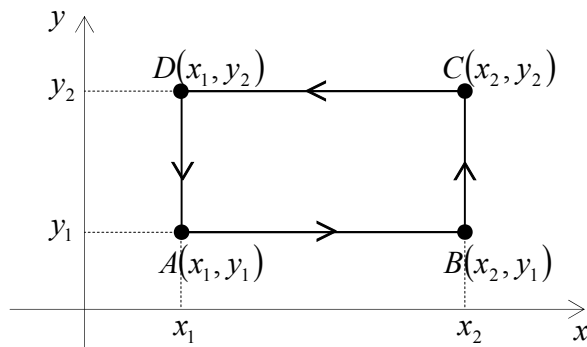


Fig. 6.7 Contour of integration.

In order to find eigenvalues p_i and the corresponding values q_i from equation (6.1.58) we define the function $F(p)$ by the formula (see (6.1.57)):

$$F(p) = pJ_1(qc)T_1'(pc) - qJ_1'(qc)T_1(pc). \quad (6.3.4)$$

Here $p = x + jy$ is an eigenvalue and $q = \sqrt{p^2 - j\omega\sigma\mu_0}$. Calculating the derivative of (6.3.4) with respect to p we obtain

$$\begin{aligned}
 F'(p) = & [J_1'(pc)Y_1(pb) - J_1(pb)Y_1'(pc)][J_1(qc) + p^2cq^{-1}J_1'(qc)] \\
 & + pJ_1'(qc)[cJ_1''(pc)Y_1(pb) + bJ_1'(pc)Y_1'(pb) - bJ_1'(pb)Y_1'(pc) - cY_1''(pc)J_1(pb)] \\
 & - [J_1'(pc)Y_1(pb) - J_1(pb)Y_1'(pc)][pq^{-1}J_1'(qc) + pcJ_1''(qc)] \\
 & - qJ_1'(qc)[cJ_1'(pc)Y_1(pb) + bJ_1'(pc)Y_1'(pb) - bJ_1'(pb)Y_1'(pc) - cY_1'(pc)J_1(pb)]
 \end{aligned} \tag{6.3.5}$$

Let $\phi(p) = \frac{F'(p)}{F(p)}$. Then the number of zeros of $\phi(p)$ inside C is given by (6.3.1) where the

contour integral is computed by means of (6.3.3). If there is only one zero, ζ_1 , of $F(p)$ inside C then it is calculated as follows (see (6.3.2))

$$s_1 = \frac{1}{2\pi i} \oint_C z \frac{\phi'(z)}{\phi(z)} dz = \zeta_1. \tag{6.3.6}$$

An example of numerical calculation of complex eigenvalues p_i is shown in **Fig. 6.8**.

Computations are done with “Mathematica” (see **Appendix Fig.Ap.16**). The parameters of the cylinder correspond to 10c euro coin (the outer radius of the coil is 10 mm). The algorithm is implemented for the case where there are at most two eigenvalues inside a rectangle. The program then subdivides the rectangle (if necessary) so that only one eigenvalue is inside a smaller rectangle. The precise value of the root is then found by means of the formula (6.3.6).

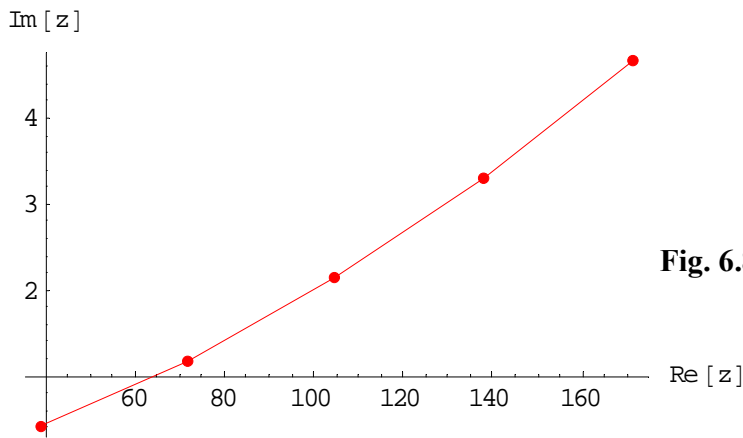


Fig. 6.8 The first five eigenvalues p_i .

The first five zeros of $F(p)$ are shown in **Fig. 6.8**.

As soon as all eigenvalues $p_i, i = 1, 2, \dots, N_s$ are calculated we can proceed to the next step of the procedure where the system of linear equations (6.1.72) - (6.1.75) is solved (note that the upper summation index in (6.1.72) - (6.1.75) is equal N_s).

Finally, the change in impedance of the coil is computed by means of the formula (6.2.14). Several calculations are done in order to estimate the effect of the parameters b and $n = N_s$ on the calculation accuracy. The parameters of the problem are as follows: $\sigma = 4\text{Ms/m}$, $c = 8\text{mm}$, $r_1 = 3\text{mm}$, $r_2 = 6\text{mm}$, $z_1 = 0.1\text{mm}$, $z_2 = 3.1\text{mm}$, $d = 2\text{mm}$. The first table shows

the change in impedance of the coil calculated by means of the formula (6.2.13) for the frequency $f = 1\text{kHz}$ for the case where the number of eigenvalues is $n = 68$ and four different values of b , namely, $b = 5r_2$, $10r_2$, $15r_2$, and $20r_2$ are chosen. The results show that $b = 10r_2$ is sufficient in order to calculate the induced change in impedance with the accuracy of three decimal places.

This value of b is used in all other calculations in the thesis.

$\begin{matrix} b \\ n \end{matrix}$	$5r_2$	$10r_2$	$15r_2$	$20r_2$
68	0.174015 -0.0124184i	0.176349 -0.0126594i	0.176578 -0.0126837i	0.17664 -0.0126913

Table 6.1

The second table shows the calculated induced change in impedance for different values of n and five different frequencies. The value of b is fixed at $b = 10r_2$. As can be seen from the table, $n = 68$ gives quite accurate value of the change in impedance. Convergence of the method is clearly seen from the table.

$\begin{matrix} f \\ n \end{matrix}$	1000	2000	3000	4000	5000
11	0.171115 -0.0128815i	0.673776 -0.101349i	1.47762 -0.332801i	2.53722 -0.760574i	3.79826 -1.41964i
19	0.17675 -0.012734i	0.695895 -0.100212i	1.52589 -0.329288i	2.61954 -0.752716i	3.92057 -1.40577i
27	0.176433 -0.0126717i	0.694685 -0.0997248i	1.52338 -0.32771i	2.61556 -0.74913i	3.91522 -1.39916i
34	0.176345 -0.0126615i	0.694348 -0.0996455i	1.52266 -0.327444i	2.61439 -0.748559i	3.91354 -1.39813i
47	0.17634 -0.0126593i	0.694329 -0.0996289i	1.5263 -0.327392i	2.61435 -0.748446i	3.91353 -1.39793i
68	0.176349 -0.0126594i	0.694366 -0.0996297i	1.52275 -0.327392i	2.6145 -0.748455i	3.91377 -1.39795i
102	0.176348 -0.0126592i	0.694362 -0.0996281i	1.52274 -0.327385i	2.61449 -0.748443i	3.91376 -1.39793i
137	0.176348 -0.0126592i	0.694363 -0.0996281i	1.52275 -0.327385i	2.6145 -0.748444i	3.91377 -1.39793i
171	0.176348 -0.0126592i	0.694363 -0.099628i	1.52274 -0.327385i	2.6145 -0.748443i	3.91377 -1.39793i
240	0.176348 -0.0126592i	0.694363 -0.099628i	1.52274 -0.327385i	2.6145 -0.748443i	3.91377 -1.39793i
309	0.176348 -0.0126592i	0.694363 -0.099628i	1.52274 -0.327385i	2.6145 -0.748443i	3.91377 -1.39793i
377	0.176348 -0.0126592i	0.694363 -0.099628i	1.52274 -0.327385i	2.6145 -0.748443i	3.91377 -1.39793i

Table 6.2

6.4 A coil above a conducting plate with a flaw in the form of a circular cylinder

In this section we consider a mathematical model of eddy current problem which can be used to model corrosion. Consider a coil of radius r_0 located at a distance h above a conducting plate (see **Fig. 6.9**). The plate has a cylindrical hole of height d_2 and radius c . The thickness of the plate is $d_1 + d_2$.

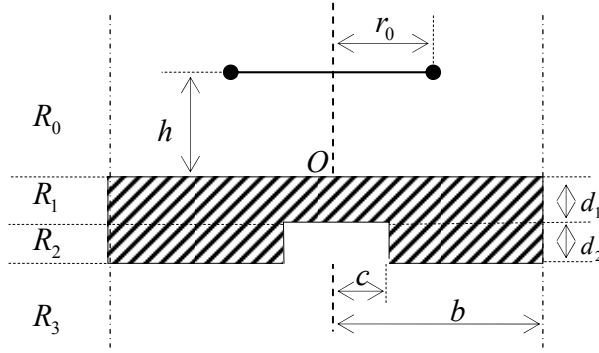


Fig. 6.9 A single-turn coil above a conducting plate with a flaw in the form of a circular cylinder.

We use the TREE method to solve the problem (the solution based on the layer approximation is given in [52]). The system of equations for the components of the vector potential in regions $R_i, i = 0,1,2,3$ has the form

$$R_0: \quad \mu(z) = \mu_0^r = 1, \quad \sigma(z) = 0, \quad I^e = I\delta(r - r_0)\delta(z - h)$$

$$\frac{\partial^2 A_0}{\partial r^2} + \frac{1}{r} \frac{\partial A_0}{\partial r} - \frac{A_0}{r^2} + \frac{\partial^2 A_0}{\partial z^2} = -\mu_0 I \delta(r - r_0) \delta(z - h), \quad (6.4.1)$$

$$R_1: \quad \mu(z) = \mu_1^r = 1, \quad \sigma(z) = \sigma, \quad I = 0$$

$$\frac{\partial^2 A_1}{\partial r^2} + \frac{1}{r} \frac{\partial A_1}{\partial r} - \frac{A_1}{r^2} - j\omega\sigma\mu_0 A_1 + \frac{\partial^2 A_1}{\partial z^2} = 0, \quad (6.4.2)$$

$$R_2: \quad \begin{cases} 0 \leq r \leq c & \mu(z) = \mu_2^r = 1, \quad \sigma(z) = 0, \quad I = 0 \\ c \leq r \leq b & \mu(z) = \mu_2^r = 1, \quad \sigma(z) = \sigma, \quad I = 0 \end{cases}$$

$$\frac{\partial^2 A_2}{\partial r^2} + \frac{1}{r} \frac{\partial A_2}{\partial r} - \frac{A_2}{r^2} - j\omega\sigma\mu_0 A_2 + \frac{\partial^2 A_2}{\partial z^2} = 0, \quad (6.4.3)$$

$$R_3: \quad \mu(z) = \mu_3^r = 1, \quad \sigma(z) = 0, \quad I = 0$$

$$\frac{\partial^2 A_3}{\partial r^2} + \frac{1}{r} \frac{\partial A_3}{\partial r} - \frac{A_3}{r^2} + \frac{\partial^2 A_3}{\partial z^2} = 0, \quad (6.4.4)$$

where A_i is the solution in region $R_i, i = 0,1,2,3$ (note that $\sigma = 0$ in region R_2 where $0 \leq r \leq c$).

As in **Section 6.1** we use the notations A_2^{air} and A_2^{con} in regions $0 \leq r \leq c$ and $c \leq r \leq b$, respectively. The boundary conditions are

$$A_i|_{r=b} = 0, \quad i = 0,1,3, \quad A_2^{con}|_{r=b} = 0, \quad (6.4.5)$$

$$A_0|_{z=0} = A_1|_{z=0}, \quad \frac{\partial A_0}{\partial z}|_{z=0} = \frac{\partial A_1}{\partial z}|_{z=0}, \quad 0 \leq r \leq b, \quad (6.4.6)$$

$$A_1|_{z=-d_1} = A_2^{air}|_{z=-d_1}, \quad \frac{\partial A_1}{\partial z}|_{z=-d_1} = \frac{\partial A_2^{air}}{\partial z}|_{z=-d_1}, \quad 0 \leq r \leq c, \quad (6.4.7)$$

$$A_1|_{z=-d_1} = A_2^{con}|_{z=-d_1}, \quad \frac{\partial A_1}{\partial z}|_{z=-d_1} = \frac{\partial A_2^{con}}{\partial z}|_{z=-d_1}, \quad c \leq r \leq b, \quad (6.4.8)$$

$$A_2^{air}|_{z=-d_1-d_2} = A_3|_{z=-d_1-d_2}, \quad \frac{\partial A_2^{air}}{\partial z}|_{z=-d_1-d_2} = \frac{\partial A_3}{\partial z}|_{z=-d_1-d_2}, \quad 0 \leq r \leq c, \quad (6.4.9)$$

$$A_2^{con}|_{z=-d_1-d_2} = A_3|_{z=-d_1-d_2}, \quad \frac{\partial A_2^{con}}{\partial z}|_{z=-d_1-d_2} = \frac{\partial A_3}{\partial z}|_{z=-d_1-d_2}, \quad c \leq r \leq b, \quad (6.4.10)$$

$$A_2^{con} \Big|_{r=c} = A_2^{air} \Big|_{r=c}, \quad \frac{\partial A_2^{con}}{\partial r} \Big|_{r=c} = \frac{\partial A_2^{air}}{\partial r} \Big|_{r=c}. \quad (6.4.11)$$

Solution to (6.4.1) is obtained as in **Section 6.1** and is given by (6.1.29) and (6.1.30) in regions $R_{00} = \{0 < z < h\}$ and $R_{01} = \{z > h\}$, respectively.

Using the method of separation of variables we obtain the general solution to (6.4.2) in region R_1 in the form

$$A_1(r, z) = \sum_{i=1}^{\infty} (D_{4i} e^{p_i z} + D_{5i} e^{-p_i z}) J_1(\lambda_i r), \quad (6.4.12)$$

where $p_i = \sqrt{\lambda_i^2 + j\omega\sigma\mu_0}$.

General solution to (6.4.3) can be written in the form (see the details in **Section 6.1**):

$$A_2^{air}(r, z) = \sum_{i=1}^{\infty} [D_{6i} e^{p_i z} + D_{7i} e^{-p_i z}] J_1(p_i r), \quad (6.4.13)$$

$$A_2^{con}(r, z) = \sum_{i=1}^{\infty} \left\{ [D_{8i} J_1(q_i r) + D_{9i} Y_1(q_i r)] e^{p_i z} + [D_{10i} J_1(q_i r) + D_{11i} Y_1(q_i r)] e^{-p_i z} \right\}, \quad (6.4.14)$$

where $p_i = \sqrt{q_i^2 + j\omega\sigma\mu_0}$.

General solution to (6.4.4) which is bounded as $z \rightarrow -\infty$ is written as follows

$$A_3(r, z) = \sum_{i=1}^{\infty} D_{12i} e^{\lambda_i z} J_1(\lambda_i r). \quad (6.4.15)$$

Using (6.4.14) and the last boundary condition in (6.4.5) we obtain the following two equations

$$D_{8i} J_1(q_i b) + D_{9i} Y_1(q_i b) = 0, \quad (6.4.16)$$

$$D_{10i} J_1(q_i b) + D_{11i} Y_1(q_i b) = 0. \quad (6.4.17)$$

Eliminating D_{9i} and D_{11i} from (6.4.16) and (6.4.17) we obtain

$$D_{9i} = -D_{8i} \frac{J_1(q_i b)}{Y_1(q_i b)}, \quad (6.4.18)$$

$$D_{11i} = -D_{10i} \frac{J_1(q_i b)}{Y_1(q_i b)}. \quad (6.4.19)$$

Using continuity of the functions A_2^{air} and A_2^{con} at $r = c$ (the first condition in (6.4.11)) we get

$$\begin{aligned} \sum_{i=1}^{\infty} [D_{6i} e^{p_i z} + D_{7i} e^{-p_i z}] J_1(p_i c) &= \\ &= \sum_{i=1}^{\infty} \left\{ [D_{8i} J_1(q_i c) + D_{9i} Y_1(q_i c)] e^{p_i z} + [D_{10i} J_1(q_i c) + D_{11i} Y_1(q_i c)] e^{-p_i z} \right\}. \end{aligned} \quad (6.4.20)$$

The following two relationships are obtained from (6.4.20):

$$D_{6i} J_1(p_i c) = D_{8i} J_1(q_i c) + D_{9i} Y_1(q_i c), \quad (6.4.21)$$

$$D_{7i} J_1(p_i c) = D_{10i} J_1(q_i c) + D_{11i} Y_1(q_i c). \quad (6.4.22)$$

Combining equations (6.4.18) and (6.4.21) we obtain

$$D_{6i} = \hat{D}_{8i}[J_1(q_i c)Y_1(q_i b) - J_1(q_i b)Y_1(q_i c)], \quad (6.4.23)$$

where

$$\hat{D}_{8i} = \frac{D_{8i}}{J_1(p_i c)Y_1(q_i b)}. \quad (6.4.24)$$

It follows from (6.4.19) and (6.4.22) that

$$D_{7i} = \hat{D}_{10i}[J_1(q_i c)Y_1(q_i b) - J_1(q_i b)Y_1(q_i c)], \quad (6.4.25)$$

where

$$\hat{D}_{10i} = \frac{D_{10i}}{J_1(p_i c)Y_1(q_i b)}. \quad (6.4.26)$$

Solutions (6.4.13) and (6.4.14) can be rewritten in the following form (here we use the relationships (6.4.18), (6.4.19), (6.4.23) and (6.4.25)):

$$A_2^{air}(r, z) = \sum_{i=1}^{\infty} T_1(q_i c)[\hat{D}_{8i}e^{p_i z} + \hat{D}_{10i}e^{-p_i z}]J_1(p_i r), \quad (6.4.27)$$

$$A_2^{con}(r, z) = \sum_{i=1}^{\infty} J_1(p_i c)[\hat{D}_{8i}e^{p_i z} + \hat{D}_{10i}e^{-p_i z}]T_1(q_i r), \quad (6.4.28)$$

where

$$T_1(q_i r) = J_1(q_i r)Y_1(q_i b) - J_1(q_i b)Y_1(q_i r), \quad (6.4.29)$$

$$T_1(q_i c) = J_1(q_i c)Y_1(q_i b) - J_1(q_i b)Y_1(q_i c). \quad (6.4.30)$$

Differentiating (6.4.27) and (6.4.28) with respect to r and evaluating the derivatives at $r = c$ we obtain

$$\frac{\partial A_2^{air}}{\partial r} \Big|_{r=c} = \sum_{i=1}^{\infty} p_i J_1'(p_i c) T_1(q_i c) (\hat{D}_{6i} e^{p_i z} + \hat{D}_{8i} e^{-p_i z}), \quad (6.4.31)$$

$$\frac{\partial A_2^{con}}{\partial r} \Big|_{r=c} = \sum_{i=1}^{\infty} q_i T_1'(q_i c) J_1(p_i c) (\hat{D}_{6i} e^{p_i z} + \hat{D}_{8i} e^{-p_i z}). \quad (6.4.32)$$

It follows from (6.4.31) and (6.4.32) and the second boundary condition in (6.4.11) that

$$p_i J_1'(p_i c) T_1(q_i c) = q_i T_1'(q_i c) J_1(p_i c). \quad (6.4.33)$$

Equation (6.4.33) is used to determine the eigenvalues p_i and related values q_i .

The relationship between p_i and q_i is $p_i = \sqrt{q_i^2 + j\omega\sigma\mu_0}$.

Thus, the solution in regions R_0, R_1, R_2 and R_3 is given by (6.1.29), (6.1.30), (6.4.12), (6.4.27), (6.4.28) and (6.4.15). The six sets of constants in these formulas, namely,

$D_{2i}, D_{4i}, D_{5i}, \hat{D}_{8i}, \hat{D}_{10i}$ and D_{12i} can be obtained from the boundary conditions (6.4.6) – (6.4.10).

Using the condition (6.4.6) we obtain

$$\sum_{i=1}^{\infty} D_{2i} J_1(\lambda_i r) + \frac{\mu_0 I r_0}{b^2} \sum_{i=1}^{\infty} \frac{J_1(\lambda_i r_0) e^{-\lambda_i h}}{\lambda_i J_0^2(\lambda_i b)} J_1(\lambda_i r) = \sum_{i=1}^{\infty} (D_{4i} + D_{5i}) J_1(\lambda_i r), \quad (6.4.34)$$

$$- \sum_{i=1}^{\infty} \lambda_i D_{2i} J_1(\lambda_i r) + \frac{\mu_0 I r_0}{b^2} \sum_{i=1}^{\infty} \frac{J_1(\lambda_i r_0) e^{-\lambda_i h}}{J_0^2(\lambda_i b)} J_1(\lambda_i r) = \sum_{i=1}^{\infty} p_{1i} (D_{4i} - D_{5i}) J_1(\lambda_i r). \quad (6.4.35)$$

The obtained equations are multiplied by $r J_1(\lambda_j r)$ and the resulting equation is integrated with respect to r from 0 to b . Using the orthogonality condition (6.1.25) we obtain

$$D_{2j} \frac{b^2}{2} J_0^2(\lambda_j b) + \frac{\mu_0 I r_0}{2 \lambda_j} J_1(\lambda_j r_0) e^{-\lambda_j h} = (D_{4j} + D_{5j}) \frac{b^2}{2} J_0^2(\lambda_j b), \quad (6.4.36)$$

$$- \lambda_j D_{2j} \frac{b^2}{2} J_0^2(\lambda_j b) + \frac{\mu_0 I r_0}{2} J_1(\lambda_j r_0) e^{-\lambda_j h} = p_{1j} (D_{4j} - D_{5j}) \frac{b^2}{2} J_0^2(\lambda_j b). \quad (6.4.37)$$

From (6.4.36) we find

$$D_{2j} = D_{4j} + D_{5j} - \mu_0 I r_0 \frac{J_1(\lambda_j r_0) e^{-\lambda_j h}}{\lambda_j b^2 J_0^2(\lambda_j b)}. \quad (6.4.38)$$

We multiply (6.4.36) by λ_j and add the resulting equation to (6.4.37):

$$(\lambda_j + p_{1j}) D_{4j} + (\lambda_j - p_{1j}) D_{5j} = \frac{2 \mu_0 I r_0 J_1(\lambda_j r_0) e^{-\lambda_j h}}{b^2 J_0^2(\lambda_j b)}. \quad (6.4.39)$$

Using the first conditions in (6.4.7) and in (6.4.8) we obtain

$$\begin{aligned} \sum_{i=1}^{\infty} (D_{4i} e^{-p_{1i} d_1} + D_{5i} e^{p_{1i} d_1}) J_1(\lambda_i r) = \\ = \sum_{i=1}^{\infty} T_1(q_i c) J_1(p_i r) (\hat{D}_{8i} e^{-p_i d_1} + \hat{D}_{10i} e^{p_i d_1}), \quad 0 \leq r \leq c, \end{aligned} \quad (6.4.40)$$

$$\begin{aligned} \sum_{i=1}^{\infty} (D_{4i} e^{-p_{1i} d_1} + D_{5i} e^{p_{1i} d_1}) J_1(\lambda_i r) = \\ = \sum_{i=1}^{\infty} T_1(q_i r) J_1(p_i c) (\hat{D}_{8i} e^{-p_i d_1} + \hat{D}_{10i} e^{p_i d_1}), \quad c \leq r \leq b. \end{aligned} \quad (6.4.41)$$

In order to determine the coefficients \hat{D}_{8i} and \hat{D}_{10i} the following procedure is used. First, equations (6.4.40) and (6.4.41) are combined into one equation where the right-hand side of the resulting equation is given by different expressions on the intervals $0 \leq r \leq c$ and $c \leq r \leq b$. These expressions are defined by the right-hand sides of (6.4.40) and (6.4.41), respectively. Second, the obtained equation is multiplied by $r J_1(\lambda_j r)$ and the resulting equation is integrated with respect to r from 0 to b . Third, we use the orthogonality condition (6.1.25) and formulas (6.4.29), (6.4.30), (6.1.62) and (6.1.63).

The result is

$$D_{4j} e^{-p_{1j} d_1} + D_{5j} e^{p_{1j} d_1} = \frac{2}{b^2 J_0^2(\lambda_j b)} \sum_{i=1}^{\infty} (\hat{D}_{8i} e^{-p_i d_1} + \hat{D}_{10i} e^{p_i d_1}) a_{ji}, \quad (6.4.42)$$

$$\text{where } a_{ji} = T_1(q_i c) \tilde{a}_{ji} + J_1(p_i c) \tilde{\tilde{a}}_{ji}. \quad (6.4.43)$$

Thus, using (6.1.62), (6.1.63) and (6.4.29) we obtain

$$\left\| \begin{aligned} \tilde{a}_{ji} &= \int_0^c r J_1(\lambda_j r) J_1(p_i r) dr = \frac{c}{\lambda_j^2 - p_i^2} (\lambda_j J_2(\lambda_j c) J_1(p_i c) - p_i J_1(\lambda_j c) J_2(p_i c)) \\ \tilde{\tilde{a}}_{ji} &= \int_c^b r J_1(\lambda_j r) I_1(q_i r) dr = Y_1(q_i b) \int_c^b r J_1(\lambda_j r) J_1(q_i r) dr - J_1(q_i b) \int_c^b r J_1(\lambda_j r) Y_1(q_i r) dr = \\ &= \frac{1}{\lambda_j^2 - q_i^2} \left\{ \begin{aligned} &b q_i J_1(\lambda_j b) [J_1(q_i b) Y_2(q_i b) - J_2(q_i b) Y_1(q_i b)] + \\ &+ c \lambda_j J_2(\lambda_j c) [J_1(q_i b) Y_1(q_i c) - J_1(q_i c) Y_1(q_i b)] + \\ &+ c q_i J_1(\lambda_j c) [J_2(q_i c) Y_1(q_i b) - J_1(q_i b) Y_2(q_i c)] \end{aligned} \right\} \end{aligned} \right\|$$

Using the same procedure and applying the second condition in (6.4.7) and in (6.4.8) we obtain

$$p_{1j} (D_{4j} e^{-p_{1j} d_1} - D_{5j} e^{p_{1j} d_1}) = \frac{2}{b^2 J_0^2(\lambda_j b)} \sum_{i=1}^{\infty} p_i (\hat{D}_{8i} e^{-p_i d_1} - \hat{D}_{10i} e^{p_i d_1}) a_{ji}. \quad (6.4.44)$$

We multiply (6.4.42) by p_{1j} and add the resulting equation to (6.4.44)

$$D_{4j} = \frac{e^{p_{1j} d_1}}{p_{1j} b^2 J_0^2(\lambda_j b)} \sum_{i=1}^{\infty} \left\{ \left((p_{1j} + p_i) e^{-p_i d_1} \hat{D}_{8i} + (p_{1j} - p_i) e^{p_i d_1} \hat{D}_{10i} \right) a_{ji} \right\} \quad (6.4.45)$$

We multiply (6.4.42) by $(-p_{1j})$ and add the resulting equation to (6.4.44)

$$D_{5j} = \frac{e^{-p_{1j} d_1}}{p_{1j} b^2 J_0^2(\lambda_j b)} \sum_{i=1}^{\infty} \left\{ \left((p_{1j} - p_i) e^{-p_i d_1} \hat{D}_{8i} + (p_{1j} + p_i) e^{p_i d_1} \hat{D}_{10i} \right) a_{ji} \right\} \quad (6.4.46)$$

Two additional equations are obtained if the same procedure is applied to (6.4.27), (6.4.28) and (6.4.15) using boundary conditions (6.4.9) and (6.4.10). The result is shown below

$$D_{12j} e^{-\lambda_j d_3} \frac{b^2}{2} J_0^2(\lambda_j b) = \sum_{i=1}^{\infty} (\hat{D}_{8i} e^{-p_i d_3} + \hat{D}_{10i} e^{p_i d_3}) a_{ji}, \quad (6.4.47)$$

$$\lambda_j D_{12j} e^{-\lambda_j d_3} \frac{b^2}{2} J_0^2(\lambda_j b) = \sum_{i=1}^{\infty} p_i (\hat{D}_{8i} e^{-p_i d_3} - \hat{D}_{10i} e^{p_i d_3}) a_{ji}, \quad (6.4.48)$$

where $d_3 = d_1 + d_2$

Similarly, multiplying (6.4.47) by $(-\lambda_j)$ and adding the resulting equation to (6.4.48) we obtain

$$\sum_{i=1}^{\infty} [(\lambda_j - p_i) e^{-p_i d_3} \hat{D}_{8i} + (\lambda_j + p_i) e^{p_i d_3} \hat{D}_{10i}] a_{ji} = 0. \quad (6.4.49)$$

Thus, using (6.4.45), (6.4.46) and (6.4.39) we obtain

$$\begin{aligned} \sum_{i=1}^{\infty} \left(\begin{aligned} &((\lambda_j + p_{1j})(p_{1j} + p_i) e^{(p_{1j} - p_i) d_1} + (\lambda_j - p_{1j})(p_{1j} - p_i) e^{-(p_{1j} + p_i) d_1}) a_{ji} \hat{D}_{8i} + \\ &+ ((\lambda_j + p_{1j})(p_{1j} - p_i) e^{(p_{1j} + p_i) d_1} + (\lambda_j - p_{1j})(p_{1j} + p_i) e^{-(p_{1j} - p_i) d_1}) a_{ji} \hat{D}_{10i} \end{aligned} \right) = \\ = 2 p_j \mu_0 I r_0 J_1(\lambda_j r_0) e^{-\lambda_j h} \end{aligned} \quad (6.4.50)$$

The six sets of equations for the unknown coefficients $D_{2j}, D_{4j}, D_{5j}, \hat{D}_{8i}, \hat{D}_{10i}$ and D_{12j} are given by (6.4.38), (6.4.45), (6.4.46), (6.4.49), (6.4.50) and (6.4.47). The solution of the system is obtained as follows. First, we consider only a finite number of terms n in (6.4.38), (6.4.45), (6.4.46), (6.4.47), (6.4.49) and (6.4.50). Recommendations on the selection of the value of n are given in **Section 6.3**. Second, the coefficients $\hat{D}_{8i}, \hat{D}_{10i}$ can be computed solving the system

$$\sum_{i=1}^n \left(\begin{aligned} & \left((\lambda_j + p_{1j})(p_{1j} + p_i) e^{(p_{1j}-p_i)d_1} + (\lambda_j - p_{1j})(p_{1j} - p_i) e^{-(p_{1j}+p_i)d_1} \right) a_{ji} \hat{D}_{8i} + \\ & \left((\lambda_j + p_{1j})(p_{1j} - p_i) e^{(p_{1j}+p_i)d_1} + (\lambda_j - p_{1j})(p_{1j} + p_i) e^{-(p_{1j}-p_i)d_1} \right) a_{ji} \hat{D}_{10i} \end{aligned} \right) = 2p_{1j}\mu_0 I r_0 J_1(\lambda_j r_0) e^{-\lambda_j h} \quad (6.4.51)$$

$$\sum_{i=1}^n [(\lambda_j - p_i) e^{-p_i d_3} \hat{D}_{8i} + (\lambda_j + p_i) e^{p_i d_3} \hat{D}_{10i}] a_{ji} = 0. \quad (6.4.52)$$

where n is the number of eigenvalues p_i . The system (6.4.51)-(6.4.52) can be written in the matrix form

$$A\bar{X} = \bar{B} \quad (6.4.53)$$

where the coefficient matrix A is

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad (6.4.54)$$

and the block matrices A_{11}, A_{12}, A_{21} and A_{22} are

$$A_{11} = \begin{pmatrix} \left((\lambda_1 + p_{11})(p_{11} + p_1) e^{(p_{11}-p_1)d_1} + (\lambda_1 - p_{11})(p_{11} - p_1) e^{-(p_{11}+p_1)d_1} \right) a_{11} & \dots & \dots \\ \dots \left((\lambda_2 + p_{12})(p_{12} + p_2) e^{(p_{12}-p_2)d_1} + (\lambda_2 - p_{12})(p_{12} - p_2) e^{-(p_{12}+p_2)d_1} \right) a_{22} & \dots & \dots \\ \dots & \dots & \dots \\ \dots \left((\lambda_n + p_{1n})(p_{1n} + p_n) e^{(p_{1n}-p_n)d_1} + (\lambda_n - p_{1n})(p_{1n} - p_n) e^{-(p_{1n}+p_n)d_1} \right) a_{nn} \end{pmatrix}$$

$$A_{12} = \begin{pmatrix} \left((\lambda_1 + p_{11})(p_{11} - p_1) e^{(p_{11}+p_1)d_1} + (\lambda_1 - p_{11})(p_{11} + p_1) e^{-(p_{11}-p_1)d_1} \right) a_{11} & \dots & \dots \\ \dots \left((\lambda_2 + p_{12})(p_{12} - p_2) e^{(p_{12}+p_2)d_1} + (\lambda_2 - p_{12})(p_{12} + p_2) e^{-(p_{12}-p_2)d_1} \right) a_{22} & \dots & \dots \\ \dots & \dots & \dots \\ \dots \left((\lambda_n + p_{1n})(p_{1n} - p_n) e^{(p_{1n}+p_n)d_1} + (\lambda_n - p_{1n})(p_{1n} + p_n) e^{-(p_{1n}-p_n)d_1} \right) a_{nn} \end{pmatrix}$$

$$A_{21} = \begin{pmatrix} (\lambda_1 - p_1) e^{-p_1 d_3} a_{11} & (\lambda_1 - p_2) e^{-p_2 d_3} a_{12} & \dots & (\lambda_1 - p_n) e^{-p_n d_3} a_{1n} \\ (\lambda_2 - p_1) e^{-p_1 d_3} a_{21} & (\lambda_2 - p_2) e^{-p_2 d_3} a_{22} & \dots & (\lambda_2 - p_n) e^{-p_n d_3} a_{2n} \\ \dots & \dots & \dots & \dots \\ (\lambda_n - p_1) e^{-p_1 d_3} a_{n1} & (\lambda_n - p_2) e^{-p_2 d_3} a_{n2} & \dots & (\lambda_n - p_n) e^{-p_n d_3} a_{nn} \end{pmatrix}$$

$$A_{22} = \begin{pmatrix} (\lambda_1 + p_1)e^{p_1 d_3} a_{11} & (\lambda_1 + p_2)e^{p_2 d_3} a_{12} & \dots & (\lambda_1 + p_n)e^{p_n d_3} a_{1n} \\ (\lambda_2 + p_1)e^{p_1 d_3} a_{21} & (\lambda_2 + p_2)e^{p_2 d_3} a_{22} & \dots & (\lambda_2 + p_n)e^{p_n d_3} a_{2n} \\ \dots & \dots & \dots & \dots \\ (\lambda_n + p_1)e^{p_1 d_3} a_{n1} & (\lambda_n + p_2)e^{p_2 d_3} a_{n2} & \dots & (\lambda_n + p_n)e^{p_n d_3} a_{nn} \end{pmatrix}$$

The matrices \bar{X} and \bar{B} in (6.4.53) are

$$\bar{X} = \begin{pmatrix} \bar{X}_1 \\ \bar{X}_2 \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} \bar{b} \\ 0 \end{pmatrix}, \quad (6.4.55)$$

where

$$\bar{X}_1 = \begin{pmatrix} \hat{D}_{8_1} \\ \hat{D}_{8_2} \\ \dots \\ \hat{D}_{8_n} \end{pmatrix}, \quad \bar{X}_2 = \begin{pmatrix} \hat{D}_{10_1} \\ \hat{D}_{10_2} \\ \dots \\ \hat{D}_{10_n} \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} 2p_1 \mu_0 I r_0 J_1(\lambda_1 r_0) e^{-\lambda_1 h} \\ 2p_2 \mu_0 I r_0 J_1(\lambda_2 r_0) e^{-\lambda_2 h} \\ \dots \\ 2p_n \mu_0 I r_0 J_1(\lambda_n r_0) e^{-\lambda_n h} \end{pmatrix}. \quad (6.4.56)$$

Hence, (6.4.53) can be written in the form

$$\begin{cases} A_{11} \bar{X}_1 + A_{12} \bar{X}_2 = \bar{b} \\ A_{21} \bar{X}_1 + A_{22} \bar{X}_2 = 0 \end{cases} \quad (6.4.57)$$

Solving (6.4.57) we obtain all the coefficients \hat{D}_{8_i} and \hat{D}_{10_i} ($i = 1, 2, \dots, n$). Then D_{4_j} and D_{5_j} ($j = 1, 2, \dots, n$) are calculated using

$$D_{4_j} = \sum_{i=1}^n \left\{ \left(\frac{(p_{1j} + p_i) e^{(p_{1j} - p_i) d_1} a_{ji}}{p_{1j} b^2 J_0^2(\lambda_j b)} \hat{D}_{8_i} + \frac{(p_{1j} - p_i) e^{(p_{1j} + p_i) d_1} a_{ji}}{p_{1j} b^2 J_0^2(\lambda_j b)} \hat{D}_{10_i} \right) \right\}, \quad (6.4.58)$$

$$D_{5_j} = \sum_{i=1}^n \left\{ \left(\frac{(p_{1j} - p_i) e^{-(p_{1j} + p_i) d_1} a_{ji}}{p_{1j} b^2 J_0^2(\lambda_j b)} \hat{D}_{8_i} + \frac{(p_{1j} + p_i) e^{-(p_{1j} - p_i) d_1} a_{ji}}{p_{1j} b^2 J_0^2(\lambda_j b)} \hat{D}_{10_i} \right) \right\}. \quad (6.4.59)$$

The system (6.4.58)-(6.4.59) can be written in the matrix form

$$\begin{cases} \bar{D}_4 = B_{11} \bar{X}_1 + B_{12} \bar{X}_2 \\ \bar{D}_5 = B_{21} \bar{X}_1 + B_{22} \bar{X}_2 \end{cases} \quad (6.4.60)$$

The matrices B_{11} , B_{12} , B_{21} and B_{22} are

$$B_{11} = \begin{pmatrix} \frac{(p_{1_1} + p_1) e^{(p_{1_1} - p_1) d_1} a_{11}}{p_{1_1} b^2 J_0^2(\lambda_1 b)} & \frac{(p_{1_1} + p_2) e^{(p_{1_1} - p_2) d_1} a_{12}}{p_{1_1} b^2 J_0^2(\lambda_1 b)} & \dots & \frac{(p_{1_1} + p_n) e^{(p_{1_1} - p_n) d_1} a_{1n}}{p_{1_1} b^2 J_0^2(\lambda_1 b)} \\ \frac{(p_{1_2} + p_1) e^{(p_{1_2} - p_1) d_1} a_{21}}{p_{1_2} b^2 J_0^2(\lambda_2 b)} & \frac{(p_{1_2} + p_2) e^{(p_{1_2} - p_2) d_1} a_{22}}{p_{1_2} b^2 J_0^2(\lambda_2 b)} & \dots & \frac{(p_{1_2} + p_n) e^{(p_{1_2} - p_n) d_1} a_{2n}}{p_{1_2} b^2 J_0^2(\lambda_2 b)} \\ \dots & \dots & \dots & \dots \\ \frac{(p_{1_n} + p_1) e^{(p_{1_n} - p_1) d_1} a_{n1}}{p_{1_n} b^2 J_0^2(\lambda_n b)} & \frac{(p_{1_n} + p_2) e^{(p_{1_n} - p_2) d_1} a_{n2}}{p_{1_n} b^2 J_0^2(\lambda_n b)} & \dots & \frac{(p_{1_n} + p_n) e^{(p_{1_n} - p_n) d_1} a_{nn}}{p_{1_n} b^2 J_0^2(\lambda_n b)} \end{pmatrix} \quad (6.4.61)$$

$$B_{12} = \begin{pmatrix} \frac{(p_{1_1} - p_1)e^{(p_{1_1} + p_1)d_1} a_{11}}{p_{1_1} b^2 J_0^2(\lambda_1 b)} & \frac{(p_{1_1} - p_2)e^{(p_{1_1} + p_2)d_1} a_{12}}{p_{1_1} b^2 J_0^2(\lambda_1 b)} & \cdots & \frac{(p_{1_1} - p_n)e^{(p_{1_1} + p_n)d_1} a_{1n}}{p_{1_1} b^2 J_0^2(\lambda_1 b)} \\ \frac{(p_{1_2} - p_1)e^{(p_{1_2} + p_1)d_1} a_{21}}{p_{1_2} b^2 J_0^2(\lambda_2 b)} & \frac{(p_{1_2} - p_2)e^{(p_{1_2} + p_2)d_1} a_{22}}{p_{1_2} b^2 J_0^2(\lambda_2 b)} & \cdots & \frac{(p_{1_2} - p_n)e^{(p_{1_2} + p_n)d_1} a_{2n}}{p_{1_2} b^2 J_0^2(\lambda_2 b)} \\ \dots & \dots & \dots & \dots \\ \frac{(p_{1_n} - p_1)e^{(p_{1_n} + p_1)d_1} a_{n1}}{p_{1_n} b^2 J_0^2(\lambda_n b)} & \frac{(p_{1_n} - p_2)e^{(p_{1_n} + p_2)d_1} a_{n2}}{p_{1_n} b^2 J_0^2(\lambda_n b)} & \cdots & \frac{(p_{1_n} - p_n)e^{(p_{1_n} + p_n)d_1} a_{nn}}{p_{1_n} b^2 J_0^2(\lambda_n b)} \end{pmatrix} \quad (6.4.62)$$

$$B_{21} = \begin{pmatrix} \frac{(p_{1_1} - p_1)e^{-(p_{1_1} + p_1)d_1} a_{11}}{p_{1_1} b^2 J_0^2(\lambda_1 b)} & \frac{(p_{1_1} - p_2)e^{-(p_{1_1} + p_2)d_1} a_{12}}{p_{1_1} b^2 J_0^2(\lambda_1 b)} & \cdots & \frac{(p_{1_1} - p_n)e^{-(p_{1_1} + p_n)d_1} a_{1n}}{p_{1_1} b^2 J_0^2(\lambda_1 b)} \\ \frac{(p_{1_2} - p_1)e^{-(p_{1_2} + p_1)d_1} a_{21}}{p_{1_2} b^2 J_0^2(\lambda_2 b)} & \frac{(p_{1_2} - p_2)e^{-(p_{1_2} + p_2)d_1} a_{22}}{p_{1_2} b^2 J_0^2(\lambda_2 b)} & \cdots & \frac{(p_{1_2} - p_n)e^{-(p_{1_2} + p_n)d_1} a_{2n}}{p_{1_2} b^2 J_0^2(\lambda_2 b)} \\ \dots & \dots & \dots & \dots \\ \frac{(p_{1_n} - p_1)e^{-(p_{1_n} + p_1)d_1} a_{n1}}{p_{1_n} b^2 J_0^2(\lambda_n b)} & \frac{(p_{1_n} - p_2)e^{-(p_{1_n} + p_2)d_1} a_{n2}}{p_{1_n} b^2 J_0^2(\lambda_n b)} & \cdots & \frac{(p_{1_n} - p_n)e^{-(p_{1_n} + p_n)d_1} a_{nn}}{p_{1_n} b^2 J_0^2(\lambda_n b)} \end{pmatrix} \quad (6.4.63)$$

$$B_{22} = \begin{pmatrix} \frac{(p_{1_1} + p_1)e^{-(p_{1_1} - p_1)d_1} a_{11}}{p_{1_1} b^2 J_0^2(\lambda_1 b)} & \frac{(p_{1_1} + p_2)e^{-(p_{1_1} - p_2)d_1} a_{12}}{p_{1_1} b^2 J_0^2(\lambda_1 b)} & \cdots & \frac{(p_{1_1} + p_n)e^{-(p_{1_1} - p_n)d_1} a_{1n}}{p_{1_1} b^2 J_0^2(\lambda_1 b)} \\ \frac{(p_{1_2} + p_1)e^{-(p_{1_2} - p_1)d_1} a_{21}}{p_{1_2} b^2 J_0^2(\lambda_2 b)} & \frac{(p_{1_2} + p_2)e^{-(p_{1_2} - p_2)d_1} a_{22}}{p_{1_2} b^2 J_0^2(\lambda_2 b)} & \cdots & \frac{(p_{1_2} + p_n)e^{-(p_{1_2} - p_n)d_1} a_{2n}}{p_{1_2} b^2 J_0^2(\lambda_2 b)} \\ \dots & \dots & \dots & \dots \\ \frac{(p_{1_n} + p_1)e^{-(p_{1_n} - p_1)d_1} a_{n1}}{p_{1_n} b^2 J_0^2(\lambda_n b)} & \frac{(p_{1_n} + p_2)e^{-(p_{1_n} - p_2)d_1} a_{n2}}{p_{1_n} b^2 J_0^2(\lambda_n b)} & \cdots & \frac{(p_{1_n} + p_n)e^{-(p_{1_n} - p_n)d_1} a_{nn}}{p_{1_n} b^2 J_0^2(\lambda_n b)} \end{pmatrix} \quad (6.4.64)$$

Solving (6.4.60) we obtain all the coefficients \hat{D}_{4_j} and \hat{D}_{5_j} ($j = 1, 2, \dots, n$).

Then D_{2_j} is calculated using

$$D_{2_j} = D_{4_j} + D_{5_j} - \mu_0 I r_0 \frac{J_1(\lambda_j r_0) e^{-\lambda_j h}}{\lambda_j b^2 J_0^2(\lambda_j b)}. \quad (6.4.65)$$

The induced change in impedance of the coil is given by the formula

$$Z^{ind} = \frac{j\omega}{I} 2\pi r_0 A_0^{ind}(r_0, h) \quad (6.4.66)$$

$$\text{where } A_0^{ind}(r_0, h) = \sum_{j=1}^n D_{2_j} e^{-\lambda_j h} J_1(\lambda_j r_0) \quad (6.4.67)$$

Formula (6.4.66) is used to compute the change in impedance of the coil. Calculations are performed with ‘‘Mathematica’’. The program which is used to compute the change in impedance

(6.4.66) is shown in **Appendix Fig. Ap.17**. The following parameters of the problem are selected: $\mu_0 = 4 \cdot 10^{-7} \pi$, $\sigma = 3 \text{ Ms/m}$, $c = 2.2 \text{ mm}$, $r_0 = 4.5 \text{ mm}$, $h = 1.4 \text{ mm}$, $d_1 = 0.7 \text{ mm}$, $d_2 = 0.3 \text{ mm}$, $b = 55 \text{ mm}$. The change in impedance is computed for the following seven frequencies: $f = 1000, 2000, 3000, 4000, 5000, 6000,$ and 7000 Hz . The results of calculations are shown in **Fig. 6.10**. The calculated points (from top to bottom) correspond to the seven frequencies (from smallest to largest). The upper limit of the summation index in (6.4.67) is fixed at $n = 62$. Comparison of the computational results obtained for other values of n showed that the chosen value of 62 is quite satisfactory in terms of calculation accuracy. More detailed analysis of convergence is presented in **Section 6.3**. Several computational steps are necessary in order to calculate the induced change in impedance. First, the set of eigenvalues λ_i has to be calculated. This can easily be done in “Mathematica” using a built-in routine BesselZeros. Second, a set of complex roots of (6.4.33) should be computed. Calculations are based on the method described in **Section 6.3**. Third, several systems of linear equations have to be solved in order to determine expansion coefficients. Finally, the change in impedance is computed using (6.4.53) -(6.4.67).

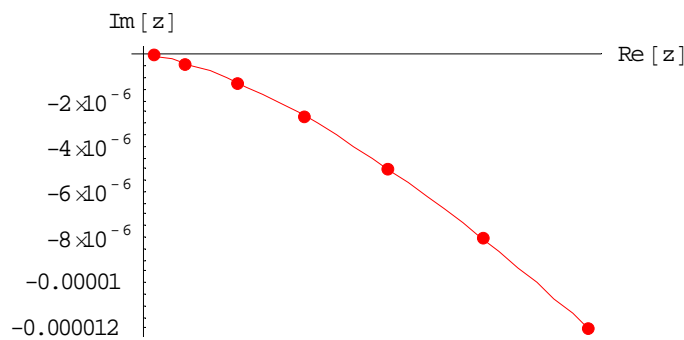


Fig. 6.10 The change in impedance of the coil for seven frequencies.

6.5 A coil of finite dimensions above a conducting plate with a flaw in the form of a circular cylinder

Consider the model shown in **Fig. 6.11**. The inner and outer radii of the coil are r_1 and r_2 , respectively. The bottom of the coil is located at the distance z_1 from the conducting plate. The height of the coil is $z_2 - z_1$ and the number of turns is N .

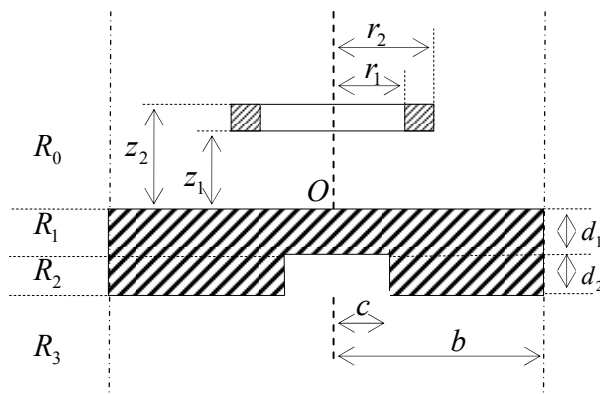


Fig. 6.11 A coil of finite dimensions above a conducting plate with a cylindrical hole.

It can easily be shown that the second term in (6.1.29) and (6.1.30) represents the vector potential of a single-turn coil located in an unbounded free space. In addition, the first term in (6.1.29) and (6.1.30) is the induced vector potential in air due to the presence of a conducting cylinder:

$$A_0^{ind}(r, z, r_0, h) = \sum_{j=1}^n D_{2j} e^{-\lambda_j z} J_1(\lambda_j r), \quad (6.5.1)$$

where

$$D_{2j} = D_{4j} + D_{5j} - \mu_0 I r_0 \frac{J_1(\lambda_j r_0) e^{-\lambda_j h}}{\lambda_j b^2 J_0^2(\lambda_j b)}.$$

The induced vector potential in air due to currents in the whole coil is obtained as follows

$$A_{0coil}^{ind}(r, z) = \int_{r_1}^{r_2} \int_{z_1}^{z_2} A_0^{ind}(r, z, r_0, h) dr_0 dh. \quad (6.5.2)$$

There is an important difference between the calculations for a single-turn coil and coil of finite dimensions. For the case of a single-turn coil (**Section 6.4**) the geometrical parameters of the coil (r_0 and h) are constants and the numerical values of r_0 and h can be used to solve the system (6.4.57). If a coil has finite dimensions then one needs to integrate the solution with respect to r_0 and h . Thus, numerical values to the parameters r_0 and h in (6.4.57) cannot be assigned. In this case, in order to obtain the induced vector potential of a single-turn coil we need to solve (6.4.57) in matrix form. Solving the second equation in (6.4.57) we obtain

$$\vec{X}_2 = -A_{22}^{-1} A_{21} \vec{X}_1. \quad (6.5.3)$$

Substituting (6.5.3) into the first equation in (6.4.57) gives

$$(A_{11} - A_{12} A_{22}^{-1} A_{21}) \vec{X}_1 = \vec{b}. \quad (6.5.4)$$

It follows from (6.5.3) and (6.5.4) that

$$\begin{cases} \vec{X}_1 = (A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1} \vec{b} \\ \vec{X}_2 = -A_{22}^{-1} A_{21} (A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1} \vec{b} \end{cases} \quad (6.5.5)$$

Using (6.4.60) - (6.4.64) equation (6.4.65) should be written in matrix form:

$$\vec{D}_2 = Y \vec{b}, \quad (6.5.6)$$

$$\text{where } Y = (B_{11} + B_{21} - (B_{12} + B_{22}) A_{22}^{-1} A_{21}) (A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1} - C_{diag}, \quad (6.5.7)$$

$$\vec{D}_2 = \begin{pmatrix} D_{2_1} \\ D_{2_2} \\ \dots \\ D_{2_n} \end{pmatrix}, \quad C_{diag} = \begin{pmatrix} \frac{1}{2p_{1_1} \lambda_1 b^2 J_0^2(\lambda_1 b)} & 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{2p_{1_2} \lambda_2 b^2 J_0^2(\lambda_2 b)} & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \frac{1}{2p_{1_n} \lambda_n b^2 J_0^2(\lambda_n b)} \end{pmatrix} \quad (6.5.8)$$

and \vec{b} is given by (6.4.56).

Formula (6.5.1) for the induced vector potential of a single-turn coil can be rewritten in the form

$$A_0^{ind}(r, z, r_0, h) = \vec{D}_2^T \vec{f}, \quad (6.5.9)$$

where

$$\vec{f} = \begin{pmatrix} J_1(\lambda_1 r) e^{-\lambda_1 z} \\ J_1(\lambda_2 r) e^{-\lambda_2 z} \\ \dots \\ J_1(\lambda_n r) e^{-\lambda_n z} \end{pmatrix}.$$

Substituting (6.5.9) into (6.5.2) and using the formulas

$$\left\| \begin{aligned} \int_{z_1}^{z_2} e^{-\lambda_j h} dh &= -\frac{1}{\lambda_j} (e^{-\lambda_j z_2} - e^{-\lambda_j z_1}) \\ \int_{r_1}^{r_2} r_0 J_1(\lambda_j r_0) dr_0 &= \left| \xi = \lambda_j r_0 \right| = \frac{1}{\lambda_j^2} \int_{\lambda_j r_1}^{\lambda_j r_2} \xi J_1(\xi) d\xi \end{aligned} \right\| \quad (6.5.10)$$

we obtain the induced vector potential in air due to the presence of the conducting cylinder (the current amplitude I in this case is replaced by the current

density $\frac{NI}{(r_2 - r_1)(z_2 - z_1)}$):

$$A_{0coil}^{ind}(r, z) = \frac{\mu_0 NI}{(r_2 - r_1)(z_2 - z_1)} \sum_{j=1}^n f_j \sum_{i=1}^n Y_{ji} \frac{(e^{-\lambda_i z_1} - e^{-\lambda_i z_2})^{\lambda_i r_2}}{\lambda_i^3} \int_{\lambda_i r_1}^{\lambda_i r_2} \xi J_1(\xi) d\xi. \quad (6.5.11)$$

The integral with respect to ξ in (6.5.10) can be computed in terms of the Bessel and Struve functions [2] as follows

$$\int_{\lambda_i r_1}^{\lambda_i r_2} \xi J_1(\xi) d\xi = \left\{ \frac{\pi}{2} \xi [J_0(\xi) H_1(\xi) - J_1(\xi) H_0(\xi)] \right\}_{\xi=\lambda_i r_1}^{\xi=\lambda_i r_2} \quad (6.5.12)$$

The induced change in impedance of a coil of finite dimensions (see **Fig. 6.11**) is calculated by means of the following formula [15]:

$$\left\| \begin{aligned} Z^{ind} &= \frac{j\omega}{I^2} \iiint_V \vec{A} \vec{I} dv = \frac{j\omega}{I^2} \int_0^{2\pi} d\varphi \int_{r_1}^{r_2} r dr \int_{z_1}^{z_2} A_{0coil}^{ind} dz \\ Z^{ind} &= \frac{2\pi j \omega}{I} \frac{N}{(r_2 - r_1)(z_2 - z_1)} \int_{r_1}^{r_2} \int_{z_1}^{z_2} r A_{0coil}^{ind}(r, z) dr dz. \end{aligned} \right\| \quad (6.5.13)$$

Using (6.5.11) and (6.5.13) we obtain the induced change in impedance of the coil in the form

$$Z^{ind} = \frac{2j\omega\pi\mu_0 N^2}{(r_2 - r_1)^2 (z_2 - z_1)^2} \sum_{j=1}^n \frac{(e^{-\lambda_j z_1} - e^{-\lambda_j z_2})^{\lambda_j r_2}}{\lambda_j^3} \int_{\lambda_j r_1}^{\lambda_j r_2} \xi J_1(\xi) d\xi \sum_{i=1}^n Y_{ji} \frac{(e^{-\lambda_i z_1} - e^{-\lambda_i z_2})^{\lambda_i r_2}}{\lambda_i^3} \int_{\lambda_i r_1}^{\lambda_i r_2} \xi J_1(\xi) d\xi \quad (6.5.14)$$

Formula (6.5.14) is used to compute the change in impedance of the coil. Calculations are performed with “Mathematica”. Program which is used to compute the change in impedance (6.5.14) is shown in **Appendix Fig. Ap.18**. The following parameters of the problem are selected: $\mu_0 = 4 \cdot 10^{-7} \pi$, $\sigma = 3 \text{ Ms/m}$, $c = 2.2 \text{ mm}$, $r_2 = 5.5 \text{ mm}$, $r_1 = 3.5 \text{ mm}$, $z_1 = 0.3 \text{ mm}$, $z_2 = 2.6 \text{ mm}$, $d_1 = 0.7 \text{ mm}$, $d_2 = 0.3 \text{ mm}$, $b = 55 \text{ mm}$, $N = 400$. The change in impedance is computed for the following seven frequencies: $f = 1000, 2000, 3000, 4000, 5000, 6000$, and 7000 Hz . The results of calculations are shown in **Fig. 6.12**. The calculated points (from top to bottom) correspond to the seven frequencies (from smallest to largest). The upper limit of the summation index in (6.5.14) is fixed at $n = 62$. Comparison of the computational results obtained for other values of n showed that the chosen value of 62 is quite satisfactory in terms of calculation accuracy. Several computational steps are necessary in order to calculate the induced change in impedance. First, the set of eigenvalues λ_i has to be calculated. This can easily be done in “Mathematica” using a built-in routine `BesselZeros`. Second, a set of complex roots of (6.4.33) should be computed. Calculations are based on the method described in **Section 6.3**. Third, several systems of linear equations have to be solved in order to determine expansion coefficients. Finally, the change in impedance is computed using (6.5.14). Details of the numerical aspects of the procedure are given in **Section 6.3**.

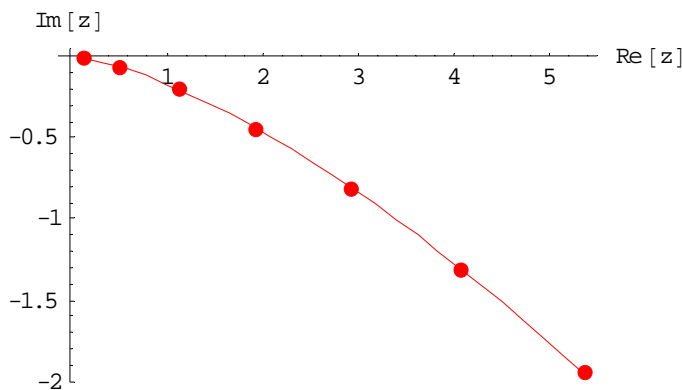


Fig. 6.12 The change in impedance of a coil (calculations are done using formula (6.5.14)).

6.6 A coil above a conducting half-space with a flaw in the form of a circular cylinder

In this section we consider a mathematical model of eddy current problem which can be used for quality control of spot welding [5], [6], [21], [71]. In this case metal parts are welded only at separate points (not along the whole surface of contact). Spot welding of two metal pieces produces cast core whose electrical conductivity is close to the conductivity of the surrounding medium.

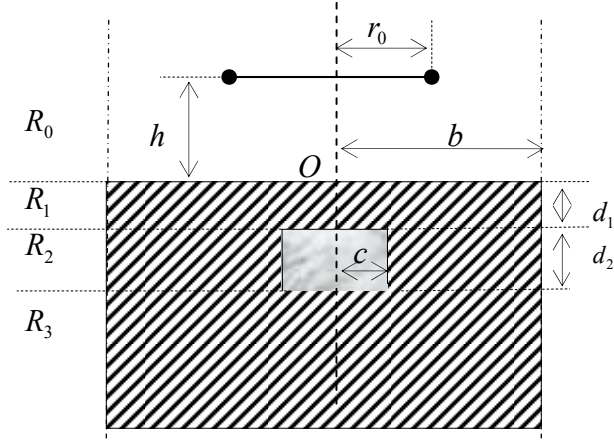


Fig. 6.13. A single-turn coil above a conducting plate.

Consider a coil of radius r_0 located at a distance h above a conducting plate (see Fig. 6.13). The plate has a cylindrical hole of height d_2 and radius c . The thickness of the plate is $d_1 + d_2$. We use the TREE method to solve the problem [25]. The system of equations for the components of the vector potential in regions $R_i, i = 0,1,2,3$ has the form

$$R_0: \quad \mu(z) = \mu_0' = 1, \quad \sigma(z) = 0, \quad I^e = I\delta(r - r_0)\delta(z - h)$$

$$\frac{\partial^2 A_0}{\partial r^2} + \frac{1}{r} \frac{\partial A_0}{\partial r} - \frac{A_0}{r^2} + \frac{\partial^2 A_0}{\partial z^2} = -\mu_0 I \delta(r - r_0) \delta(z - h), \quad (6.6.1)$$

$$R_1: \quad \mu(z) = \mu_1' = 1, \quad \sigma(z) = \sigma_1, \quad I = 0$$

$$\frac{\partial^2 A_1}{\partial r^2} + \frac{1}{r} \frac{\partial A_1}{\partial r} - \frac{A_1}{r^2} - j\omega\sigma_1\mu_0 A_1 + \frac{\partial^2 A_1}{\partial z^2} = 0, \quad (6.6.2)$$

$$R_2: \quad \begin{cases} 0 \leq r \leq c & \mu(z) = \mu_2' = 1, \quad \sigma(z) = \sigma_1 - \sigma_2, \quad I = 0 \\ c \leq r \leq b & \mu(z) = \mu_2' = 1, \quad \sigma(z) = \sigma_1, \quad I = 0 \end{cases}$$

$$\frac{\partial^2 A_2}{\partial r^2} + \frac{1}{r} \frac{\partial A_2}{\partial r} - \frac{A_2}{r^2} - j\omega\sigma\mu_0 A_2 + \frac{\partial^2 A_2}{\partial z^2} = 0, \quad (6.6.3)$$

$$R_3: \quad \mu(z) = \mu_3' = 1, \quad \sigma(z) = \sigma_1, \quad I = 0$$

$$\frac{\partial^2 A_3}{\partial r^2} + \frac{1}{r} \frac{\partial A_3}{\partial r} - \frac{A_3}{r^2} - j\omega\sigma_1\mu_0 A_3 + \frac{\partial^2 A_3}{\partial z^2} = 0, \quad (6.6.4)$$

where A_i is the solution in region $R_i, i = 0,1,2,3$ (note that $\sigma = \sigma_1 - \sigma_2$ and $\sigma = \sigma_1$ in region R_2 where $0 \leq r \leq c$ and $c \leq r \leq b$, respectively).

The boundary conditions are

$$A_i|_{r=b} = 0, \quad i = 0,1,3, \quad A_2^{con}|_{r=b} = 0, \quad (6.6.5)$$

$$A_0|_{z=0} = A_1|_{z=0}, \quad \frac{\partial A_0}{\partial z}|_{z=0} = \frac{\partial A_1}{\partial z}|_{z=0}, \quad 0 \leq r \leq b, \quad (6.6.6)$$

$$A_1|_{z=-d_1} = A_2^{cc}|_{z=-d_1}, \quad \frac{\partial A_1}{\partial z}|_{z=-d_1} = \frac{\partial A_2^{cc}}{\partial z}|_{z=-d_1}, \quad 0 \leq r \leq c, \quad (6.6.7)$$

$$A_1|_{z=-d_1} = A_2^{con}|_{z=-d_1}, \quad \frac{\partial A_1}{\partial z}|_{z=-d_1} = \frac{\partial A_2^{con}}{\partial z}|_{z=-d_1}, \quad c \leq r \leq b, \quad (6.6.8)$$

$$A_2^{cc}|_{z=-d_1-d_2} = A_3|_{z=-d_1-d_2}, \quad \frac{\partial A_2^{cc}}{\partial z}|_{z=-d_1-d_2} = \frac{\partial A_3}{\partial z}|_{z=-d_1-d_2}, \quad 0 \leq r \leq c, \quad (6.6.9)$$

$$A_2^{con}|_{z=-d_1-d_2} = A_3|_{z=-d_1-d_2}, \quad \frac{\partial A_2^{con}}{\partial z}|_{z=-d_1-d_2} = \frac{\partial A_3}{\partial z}|_{z=-d_1-d_2}, \quad c \leq r \leq b, \quad (6.6.10)$$

$$A_2^{con}|_{r=c} = A_2^{cc}|_{r=c}, \quad \frac{\partial A_2^{con}}{\partial r}|_{r=c} = \frac{\partial A_2^{cc}}{\partial r}|_{r=c}. \quad (6.6.11)$$

Here we used the abbreviations “cc” and “con” in region R_2 with the reference to conducting cylinder (cast core) and homogeneous conducting region, respectively. Solution to (6.6.1) is obtained as in **Section 6.1** and is given by (6.1.29) and (6.1.30) in regions $R_{00} = \{0 < z < h\}$ and $R_{01} = \{z > h\}$, respectively.

Using the method of separation of variables we obtain the general solution to (6.6.2) in region R_1 in the form

$$A_1(r, z) = \sum_{i=1}^{\infty} (D_{4i} e^{p_i z} + D_{5i} e^{-p_i z}) J_1(\lambda_i r), \quad (6.6.12)$$

where $p_i = \sqrt{\lambda_i^2 + j\omega\sigma_1\mu_0}$.

General solution to (6.6.3) can be written in the form (see the details in **Section 6.1**):

$$A_2^{cc}(r, z) = \sum_{i=1}^{\infty} [D_{10i} e^{p_{1i} z} + D_{11i} e^{-p_{1i} z}] J_1(p_{2i} r), \quad (6.6.13)$$

$$A_2^{con}(r, z) = \sum_{i=1}^{\infty} \{ [D_{6i} J_1(q_i r) + D_{7i} Y_1(q_i r)] e^{p_{1i} z} + [D_{8i} J_1(q_i r) + D_{9i} Y_1(q_i r)] e^{-p_{1i} z} \}. \quad (6.6.14)$$

where $p_{1i} = \sqrt{q_i^2 + j\omega\sigma_1\mu_0}$, $p_{2i} = \sqrt{q_i^2 + j\omega\sigma_2\mu_0}$.

General solution to (6.6.4) which is bounded as $z \rightarrow -\infty$ is written as follows

$$A_3(r, z) = \sum_{i=1}^{\infty} D_{12i} e^{p_i z} J_1(\lambda_i r). \quad (6.6.15)$$

Using (6.6.14) and the last boundary condition in (6.6.5) we obtain the following two equations

$$D_{6i} J_1(q_i b) + D_{7i} Y_1(q_i b) = 0, \quad (6.6.16)$$

$$D_{8i} J_1(q_i b) + D_{9i} Y_1(q_i b) = 0. \quad (6.6.17)$$

Eliminating D_{7i} and D_{9i} from (6.6.16) and (6.6.17) we obtain

$$D_{7i} = -D_{6i} \frac{J_1(q_i b)}{Y_1(q_i b)}, \quad (6.6.18)$$

$$D_{9i} = -D_{8i} \frac{J_1(q_i b)}{Y_1(q_i b)}. \quad (6.6.19)$$

Using continuity of the functions A_2^{cc} and A_2^{con} at $r = c$ (the first condition in (6.6.11)) we get

$$\begin{aligned} \sum_{i=1}^{\infty} [D_{10i} e^{p_{1i}z} + D_{11i} e^{-p_{1i}z}] J_1(p_{2i}c) = \\ = \sum_{i=1}^{\infty} \{ [D_{6i} J_1(q_i c) + D_{7i} Y_1(q_i c)] e^{p_{1i}z} + [D_{8i} J_1(q_i c) + D_{9i} Y_1(q_i c)] e^{-p_{1i}z} \}. \end{aligned} \quad (6.6.20)$$

The following two relationships are obtained from (6.6.20):

$$D_{10i} J_1(p_{2i}c) = D_{6i} J_1(q_i c) + D_{7i} Y_1(q_i c), \quad (6.6.21)$$

$$D_{11i} J_1(p_{2i}c) = D_{8i} J_1(q_i c) + D_{9i} Y_1(q_i c). \quad (6.6.22)$$

Combining equations (6.6.18) and (6.6.21) we obtain

$$D_{10i} = \hat{D}_{6i} [J_1(q_i c) Y_1(q_i b) - J_1(q_i b) Y_1(q_i c)], \quad (6.6.23)$$

where

$$\hat{D}_{6i} = \frac{D_{6i}}{J_1(p_{2i}c) Y_1(q_i b)}. \quad (6.6.24)$$

It follows from (6.6.19) and (6.6.22) that

$$D_{11i} = \hat{D}_{8i} [J_1(q_i c) Y_1(q_i b) - J_1(q_i b) Y_1(q_i c)], \quad (6.6.25)$$

where

$$\hat{D}_{8i} = \frac{D_{8i}}{J_1(p_{2i}c) Y_1(q_i b)}. \quad (6.6.26)$$

Solutions (6.6.13) and (6.6.14) can be rewritten in the following form (here we use the relationships (6.6.18), (6.6.19), (6.6.23) and (6.6.25)):

$$A_2^{cc}(r, z) = \sum_{i=1}^{\infty} T_1(q_i r) J_1(p_{2i}r) [\hat{D}_{6i} e^{p_{1i}z} + \hat{D}_{8i} e^{-p_{1i}z}], \quad (6.6.27)$$

$$A_2^{con}(r, z) = \sum_{i=1}^{\infty} T_1(q_i r) J_1(p_{2i}c) [\hat{D}_{6i} e^{p_{1i}z} + \hat{D}_{8i} e^{-p_{1i}z}], \quad (6.6.28)$$

where

$$T_1(q_i r) = J_1(q_i r) Y_1(q_i b) - J_1(q_i b) Y_1(q_i r). \quad (6.6.29)$$

$$T_1(q_i c) = J_1(q_i c) Y_1(q_i b) - J_1(q_i b) Y_1(q_i c) \quad (6.6.30)$$

Differentiating (6.6.27) and (6.6.28) with respect to r and evaluating the derivatives at $r = c$ we obtain

$$\frac{\partial A_2^{cc}}{\partial r} \Big|_{r=c} = \sum_{i=1}^{\infty} p_{2i} J_1'(p_{2i}c) T_1(q_i c) (\hat{D}_{6i} e^{p_{1i}z} + \hat{D}_{8i} e^{-p_{1i}z}), \quad (6.6.31)$$

$$\frac{\partial A_2^{con}}{\partial r} \Big|_{r=c} = \sum_{i=1}^{\infty} q_i T_1'(q_i c) J_1(p_{2i}c) (\hat{D}_{6i} e^{p_{1i}z} + \hat{D}_{8i} e^{-p_{1i}z}). \quad (6.6.32)$$

It follows from (6.6.31) and (6.6.32) and the second boundary condition in (6.6.11) that

$$p_{2i} J_1'(p_{2i}c) T_1(q_i c) = q_i T_1'(q_i c) J_1(p_{2i}c). \quad (6.6.33)$$

Equation (6.6.33) is used to determine the eigenvalues p_{2i} and related values q_i .

The relationship between p_{2i} and q_i is $p_{2i} = \sqrt{q_i^2 + j\omega\sigma_2\mu_0}$.

Thus, the solution in regions R_0, R_1, R_2 and R_3 is given by (6.1.29), (6.1.30), (6.6.12), (6.6.27), (6.6.28) and (6.6.15). The six sets of constants in these formulas, namely, $D_{2i}, D_{4i}, D_{5i}, \hat{D}_{6i}, \hat{D}_{8i}$ and D_{12i} can be obtained from the boundary conditions (6.6.6) – (6.6.10).

Using the condition (6.6.6) (see the details in **Section 6.4**) we obtain

$$D_{2j} = D_{4j} + D_{5j} - \frac{\mu_0 I r_0}{b^2 \lambda_j} \frac{J_1(\lambda_j r_0)}{J_0^2(\lambda_j b)} e^{-\lambda_j h}, \quad (6.6.34)$$

$$(\lambda_j + p_j)D_{4j} + (\lambda_j - p_j)D_{5j} = \frac{2\mu_0 I r_0 J_1(\lambda_j r_0) e^{-\lambda_j h}}{b^2 J_0^2(\lambda_j b)}. \quad (6.6.35)$$

where

$$p_j = \sqrt{\lambda_j^2 + j\omega\sigma_1\mu_0}. \quad (6.6.36)$$

Using the first conditions in (6.6.7) and in (6.6.8) we obtain

$$\begin{aligned} \sum_{i=1}^{\infty} (D_{4i} e^{-p_i d_1} + D_{5i} e^{p_i d_1}) J_1(\lambda_i r) &= \\ &= \sum_{i=1}^{\infty} T_1(q_i c) J_1(p_{2i} r) (\hat{D}_{6i} e^{-p_{1i} d_1} + \hat{D}_{8i} e^{p_{1i} d_1}), \quad 0 \leq r \leq c, \end{aligned} \quad (6.6.37)$$

$$\begin{aligned} \sum_{i=1}^{\infty} (D_{4i} e^{-p_i d_1} + D_{5i} e^{p_i d_1}) J_1(\lambda_i r) &= \\ &= \sum_{i=1}^{\infty} T_1(q_i r) J_1(p_{2i} c) (\hat{D}_{6i} e^{-p_{1i} d_1} + \hat{D}_{8i} e^{p_{1i} d_1}), \quad c \leq r \leq b, \end{aligned} \quad (6.6.38)$$

In order to determine the coefficients \hat{D}_{6i} and \hat{D}_{8i} the following procedure is used. First, equations (6.6.37) and (6.6.38) are combined into one equation where the right-hand side of the resulting equation is given by different expressions on the intervals $0 \leq r \leq c$ and $c \leq r \leq b$. These expressions are defined by the right-hand sides of (6.6.37) and (6.6.38), respectively. Second, the obtained equation is multiplied by $rJ_1(\lambda_j r)$ and the resulting equation is integrated with respect to r from 0 to b . Third, we use the orthogonality condition (6.1.25) and formulas (6.6.29), (6.6.30), (6.1.62) and (6.1.63).

The result is

$$D_{4j} e^{-p_j d_1} + D_{5j} e^{p_j d_1} = \frac{2}{b^2 J_0^2(\lambda_j b)} \sum_{i=1}^{\infty} (\hat{D}_{6i} e^{-p_{1i} d_1} + \hat{D}_{8i} e^{p_{1i} d_1}) a_{ji}, \quad (6.6.39)$$

$$\text{where } a_{ji} = T_1(q_i c) \tilde{a}_{ji} + J_1(p_{2i} c) \tilde{\tilde{a}}_{ji}. \quad (6.6.40)$$

Thus, using (6.1.64), (6.1.65) and (6.6.29) we obtain

$$\left\| \begin{aligned} \cdot \tilde{a}_{ji} &= \int_0^c r J_1(\lambda_j r) J_1(p_{2i} r) dr = \frac{c}{\lambda_j^2 - p_{2i}^2} (\lambda_j J_2(\lambda_j c) J_1(p_{2i} c) - p_{2i} J_1(\lambda_j c) J_2(p_{2i} c)) \\ \cdot \tilde{\tilde{a}}_{ji} &= \int_c^b r J_1(\lambda_j r) Y_1(q_i r) dr = Y_1(q_i b) \int_c^b r J_1(\lambda_j r) J_1(q_i r) dr - J_1(q_i b) \int_c^b r J_1(\lambda_j r) Y_1(q_i r) dr = \\ &= \frac{1}{\lambda_j^2 - q_i^2} \left\{ \begin{aligned} &b q_i J_1(\lambda_j b) [J_1(q_i b) Y_2(q_i b) - J_2(q_i b) Y_1(q_i b)] + \\ &+ c \lambda_j J_2(\lambda_j c) [J_1(q_i b) Y_1(q_i c) - J_1(q_i c) Y_1(q_i b)] + \\ &+ c q_i J_1(\lambda_j c) [J_2(q_i c) Y_1(q_i b) - J_1(q_i b) Y_2(q_i c)] \end{aligned} \right\} \end{aligned} \right\|$$

Using the same procedure and applying the second condition in (6.6.7) and in (6.6.8) we obtain

$$(D_{4j} e^{-p_j d_1} - D_{5j} e^{p_j d_1}) p_j = \frac{2}{b^2 J_0^2(\lambda_j b)} \sum_{i=1}^{\infty} p_{1i} (\hat{D}_{6i} e^{-p_{1i} d_1} - \hat{D}_{8i} e^{p_{1i} d_1}) a_{ji}, \quad (6.6.41)$$

Two additional equations are obtained if the same procedure is applied to (6.6.27), (6.6.28) and (6.6.15) using boundary conditions (6.6.9) and (6.6.10). The result is shown below

$$D_{12j} e^{-p_j d_3} \frac{b^2 J_0^2(\lambda_j b)}{2} = \sum_{i=1}^{\infty} (\hat{D}_{6i} e^{-p_{1i} d_3} + \hat{D}_{8i} e^{p_{1i} d_3}) a_{ji}, \quad (6.6.42)$$

$$D_{12j} p_j e^{-p_j d_3} \frac{b^2 J_0^2(\lambda_j b)}{2} = \sum_{i=1}^{\infty} p_{1i} (\hat{D}_{6i} e^{-p_{1i} d_3} - \hat{D}_{8i} e^{p_{1i} d_3}) a_{ji}, \quad (6.6.43)$$

where $d_3 = d_1 + d_2$.

Multiplying (6.6.39) by p_j and adding with (6.6.41) we obtain

$$D_{4j} = \frac{e^{p_j d_1}}{p_j b^2 J_0^2(\lambda_j b)} \sum_{i=1}^{\infty} \left\{ \left((p_j + p_{1i}) e^{-p_{1i} d_1} \hat{D}_{6i} + (p_j - p_{1i}) e^{p_{1i} d_1} \hat{D}_{8i} \right) a_{ji} \right\} \quad (6.6.44)$$

Multiplying (6.6.39) by $(-p_j)$ and adding with (6.6.41) we obtain

$$D_{5j} = \frac{e^{-p_j d_1}}{p_j b^2 J_0^2(\lambda_j b)} \sum_{i=1}^{\infty} \left\{ \left((p_j - p_{1i}) e^{-p_{1i} d_1} \hat{D}_{6i} + (p_j + p_{1i}) e^{p_{1i} d_1} \hat{D}_{8i} \right) a_{ji} \right\} \quad (6.6.45)$$

Thus, using (6.6.44), (6.6.45) and (6.6.35) we obtain

$$\begin{aligned} \sum_{i=1}^{\infty} \left(\begin{aligned} &\left((\lambda_j + p_j)(p_j + p_{1i}) e^{(p_j - p_{1i}) d_1} + (\lambda_j - p_j)(p_j - p_{1i}) e^{-(p_j + p_{1i}) d_1} \right) a_{ji} \hat{D}_{6i} + \\ &+ \left((\lambda_j + p_j)(p_j - p_{1i}) e^{(p_j + p_{1i}) d_1} + (\lambda_j - p_j)(p_j + p_{1i}) e^{-(p_j - p_{1i}) d_1} \right) a_{ji} \hat{D}_{8i} \end{aligned} \right) = \\ = 2 p_j \mu_0 I r_0 J_1(\lambda_j r_0) e^{-\lambda_j h} \end{aligned} \quad (6.6.46)$$

Multiplying (6.6.42) by $(-p_j)$ and adding with (6.6.43) we obtain

$$\sum_{i=1}^{\infty} \left((p_j - p_{1i}) e^{-p_{1i} d_3} a_{ji} \hat{D}_{6i} + (p_j + p_{1i}) e^{p_{1i} d_3} a_{ji} \hat{D}_{8i} \right) = 0 \quad (6.6.47)$$

The six sets of equations for the unknown coefficients $D_{2j}, D_{4j}, D_{5j}, \hat{D}_{6i}, \hat{D}_{8i}$ and D_{12j} are given by (6.6.34), (6.6.44), (6.6.45), (6.6.46), (6.6.47) and (6.6.42). The solution of the system is obtained as follows. First, we consider only a finite number of terms n in (6.6.34), (6.6.44), (6.6.45), (6.6.46), (6.6.47) and (6.6.42). Recommendations on the selection of the value of n are given in **Section 6.3**. Second, the coefficients $\hat{D}_{6i}, \hat{D}_{8i}$ can be computed solving the system

$$\sum_{i=1}^n \left(\begin{aligned} & \left((\lambda_j + p_j)(p_j + p_{1i}) e^{(p_j - p_{1i}) d_1} + (\lambda_j - p_j)(p_j - p_{1i}) e^{-(p_j + p_{1i}) d_1} \right) a_{ji} \hat{D}_{6i} + \\ & \left((\lambda_j + p_j)(p_j - p_{1i}) e^{(p_j + p_{1i}) d_1} + (\lambda_j - p_j)(p_j + p_{1i}) e^{-(p_j - p_{1i}) d_1} \right) a_{ji} \hat{D}_{8i} \end{aligned} \right) = 2 p_j \mu_0 I r_0 J_1(\lambda_j r_0) e^{-\lambda_j h} \quad (6.6.48)$$

$$\sum_{i=1}^n [(p_j - p_{1i}) e^{-p_{1i} d_3} a_{ji} \hat{D}_{6i} + (p_j + p_{1i}) e^{p_{1i} d_3} a_{ji} \hat{D}_{8i}] = 0, \quad (6.6.49)$$

where n is the number of eigenvalues p_{2i} . The system (6.6.48)-(6.6.49) can be written in the matrix form

$$A \bar{X} = \bar{B} \quad (6.6.50)$$

where the coefficient matrix A is

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad (6.6.51)$$

and the block matrices A_{11}, A_{12}, A_{21} and A_{22} are

$$A_{11} = \begin{pmatrix} \left((\lambda_1 + p_1)(p_1 + p_{11}) e^{(p_1 - p_{11}) d_1} + (\lambda_1 - p_1)(p_1 - p_{11}) e^{-(p_1 + p_{11}) d_1} \right) a_{11} & \dots & \dots \\ \dots \left((\lambda_2 + p_2)(p_2 + p_{12}) e^{(p_2 - p_{12}) d_1} + (\lambda_2 - p_2)(p_2 - p_{12}) e^{-(p_2 + p_{12}) d_1} \right) a_{22} & \dots & \dots \\ \dots & \dots & \dots \\ \dots \left((\lambda_n + p_n)(p_n + p_{1n}) e^{(p_n - p_{1n}) d_1} + (\lambda_n - p_n)(p_n - p_{1n}) e^{-(p_n + p_{1n}) d_1} \right) a_{nn} & \dots & \dots \end{pmatrix}$$

$$A_{12} = \begin{pmatrix} \left((\lambda_1 + p_1)(p_1 - p_{11}) e^{(p_1 + p_{11}) d_1} + (\lambda_1 - p_1)(p_1 + p_{11}) e^{-(p_1 - p_{11}) d_1} \right) a_{11} & \dots & \dots \\ \dots \left((\lambda_2 + p_2)(p_2 - p_{12}) e^{(p_2 + p_{12}) d_1} + (\lambda_2 - p_2)(p_2 + p_{12}) e^{-(p_2 - p_{12}) d_1} \right) a_{22} & \dots & \dots \\ \dots & \dots & \dots \\ \dots \left((\lambda_n + p_n)(p_n - p_{1n}) e^{(p_n + p_{1n}) d_1} + (\lambda_n - p_n)(p_n + p_{1n}) e^{-(p_n - p_{1n}) d_1} \right) a_{nn} & \dots & \dots \end{pmatrix}$$

$$A_{21} = \begin{pmatrix} (p_1 - p_{11}) e^{-p_{11} d_3} a_{11} & (p_1 - p_{12}) e^{-p_{12} d_3} a_{12} & \dots & \dots & (p_1 - p_{1n}) e^{-p_{1n} d_3} a_{1n} \\ (p_2 - p_{11}) e^{-p_{11} d_3} a_{21} & (p_2 - p_{12}) e^{-p_{12} d_3} a_{22} & \dots & \dots & (p_2 - p_{1n}) e^{-p_{1n} d_3} a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ (p_n - p_{11}) e^{-p_{11} d_3} a_{n1} & (p_n - p_{12}) e^{-p_{12} d_3} a_{n2} & \dots & \dots & (p_n - p_{1n}) e^{-p_{1n} d_3} a_{nm} \end{pmatrix}$$

$$A_{22} = \begin{pmatrix} (p_1 + p_{1_1})e^{p_{1_1}d_3} a_{11} & (p_1 + p_{1_2})e^{p_{1_2}d_3} a_{12} & \dots & (p_1 + p_{1_n})e^{p_{1_n}d_3} a_{1n} \\ (p_2 + p_{1_1})e^{p_{1_1}d_3} a_{21} & (p_2 + p_{1_2})e^{p_{1_2}d_3} a_{22} & \dots & (p_2 + p_{1_n})e^{p_{1_n}d_3} a_{2n} \\ \dots & \dots & \dots & \dots \\ (p_n + p_{1_1})e^{p_{1_1}d_3} a_{n1} & (p_n + p_{1_2})e^{p_{1_2}d_3} a_{n2} & \dots & (p_n + p_{1_n})e^{p_{1_n}d_3} a_{nn} \end{pmatrix}$$

The matrices \bar{X} and \bar{B} in (6.6.50) are

$$\bar{X} = \begin{pmatrix} \bar{X}_1 \\ \bar{X}_2 \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} \bar{b} \\ 0 \end{pmatrix}, \quad (6.6.52)$$

where

$$\bar{X}_1 = \begin{pmatrix} \hat{D}_{6_1} \\ \hat{D}_{6_2} \\ \dots \\ \hat{D}_{6_n} \end{pmatrix}, \quad \bar{X}_2 = \begin{pmatrix} \hat{D}_{8_1} \\ \hat{D}_{8_2} \\ \dots \\ \hat{D}_{8_n} \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} 2p_1\mu_0 Ir_0 J_1(\lambda_1 r_0) e^{-\lambda_1 h} \\ 2p_2\mu_0 Ir_0 J_1(\lambda_2 r_0) e^{-\lambda_2 h} \\ \dots \\ 2p_n\mu_0 Ir_0 J_1(\lambda_n r_0) e^{-\lambda_n h} \end{pmatrix}. \quad (6.6.53)$$

Hence, (6.6.50) can be written in the form

$$\begin{cases} A_{11}\bar{X}_1 + A_{12}\bar{X}_2 = \bar{b} \\ A_{21}\bar{X}_1 + A_{22}\bar{X}_2 = 0 \end{cases} \quad (6.6.54)$$

Solving (6.6.54) we obtain all the coefficients \hat{D}_{6_i} and \hat{D}_{8_i} ($i=1,2,\dots,n$). Then

D_{4_j} and D_{5_j} ($j=1,2,\dots,n$) is calculated using

$$D_{4_j} = \sum_{i=1}^n \left\{ \left(\frac{(p_j + p_{1_i})e^{(p_j - p_{1_i})d_1} a_{ji}}{p_j b^2 J_0^2(\lambda_j b)} \hat{D}_{6_i} + \frac{(p_j - p_{1_i})e^{(p_j + p_{1_i})d_1} a_{ji}}{p_j b^2 J_0^2(\lambda_j b)} \hat{D}_{8_i} \right) \right\}, \quad (6.6.55)$$

$$D_{5_j} = \sum_{i=1}^n \left\{ \left(\frac{(p_j - p_{1_i})e^{-(p_j + p_{1_i})d_1} a_{ji}}{p_j b^2 J_0^2(\lambda_j b)} \hat{D}_{6_i} + \frac{(p_j + p_{1_i})e^{-(p_j - p_{1_i})d_1} a_{ji}}{p_j b^2 J_0^2(\lambda_j b)} \hat{D}_{8_i} \right) \right\}. \quad (6.6.56)$$

The system (6.6.55)-(6.6.57) can be written in the matrix form

$$\begin{cases} \bar{D}_4 = B_{11}\bar{X}_1 + B_{12}\bar{X}_2 \\ \bar{D}_5 = B_{21}\bar{X}_1 + B_{22}\bar{X}_2 \end{cases} \quad (6.6.57)$$

The matrices B_{11} , B_{12} , B_{21} and B_{22} are

$$B_{11} = \begin{pmatrix} \frac{(p_1 + p_{1_1})e^{(p_1 - p_{1_1})d_1} a_{11}}{p_1 b^2 J_0^2(\lambda_1 b)} & \frac{(p_1 + p_{1_2})e^{(p_1 - p_{1_2})d_1} a_{12}}{p_1 b^2 J_0^2(\lambda_1 b)} & \dots & \frac{(p_1 + p_{1_n})e^{(p_1 - p_{1_n})d_1} a_{1n}}{p_1 b^2 J_0^2(\lambda_1 b)} \\ \frac{(p_2 + p_{1_1})e^{(p_2 - p_{1_1})d_1} a_{21}}{p_2 b^2 J_0^2(\lambda_2 b)} & \frac{(p_2 + p_{1_2})e^{(p_2 - p_{1_2})d_1} a_{22}}{p_2 b^2 J_0^2(\lambda_2 b)} & \dots & \frac{(p_2 + p_{1_n})e^{(p_2 - p_{1_n})d_1} a_{2n}}{p_2 b^2 J_0^2(\lambda_2 b)} \\ \dots & \dots & \dots & \dots \\ \frac{(p_n + p_{1_1})e^{(p_n - p_{1_1})d_1} a_{n1}}{p_n b^2 J_0^2(\lambda_n b)} & \frac{(p_n + p_{1_2})e^{(p_n - p_{1_2})d_1} a_{n2}}{p_n b^2 J_0^2(\lambda_n b)} & \dots & \frac{(p_n + p_{1_n})e^{(p_n - p_{1_n})d_1} a_{nn}}{p_n b^2 J_0^2(\lambda_n b)} \end{pmatrix} \quad (6.6.58)$$

$$B_{12} = \begin{pmatrix} \frac{(p_1 - p_{1_1})e^{(p_1+p_{1_1})d_1} a_{11}}{p_1 b^2 J_0^2(\lambda_1 b)} & \frac{(p_1 - p_{1_2})e^{(p_1+p_{1_2})d_1} a_{12}}{p_1 b^2 J_0^2(\lambda_1 b)} & \dots & \frac{(p_1 - p_{1_n})e^{(p_1+p_{1_n})d_1} a_{1n}}{p_1 b^2 J_0^2(\lambda_1 b)} \\ \frac{(p_2 - p_{1_1})e^{(p_2+p_{1_1})d_1} a_{21}}{p_2 b^2 J_0^2(\lambda_2 b)} & \frac{(p_2 - p_{1_2})e^{(p_2+p_{1_2})d_1} a_{22}}{p_2 b^2 J_0^2(\lambda_2 b)} & \dots & \frac{(p_2 - p_{1_n})e^{(p_2+p_{1_n})d_1} a_{2n}}{p_2 b^2 J_0^2(\lambda_2 b)} \\ \dots & \dots & \dots & \dots \\ \frac{(p_n - p_{1_1})e^{(p_n+p_{1_1})d_1} a_{n1}}{p_n b^2 J_0^2(\lambda_n b)} & \frac{(p_n - p_{1_2})e^{(p_n+p_{1_2})d_1} a_{n2}}{p_n b^2 J_0^2(\lambda_n b)} & \dots & \frac{(p_n - p_{1_n})e^{(p_n+p_{1_n})d_1} a_{nn}}{p_n b^2 J_0^2(\lambda_n b)} \end{pmatrix} \quad (6.6.59)$$

$$B_{21} = \begin{pmatrix} \frac{(p_1 - p_{1_1})e^{-(p_1+p_{1_1})d_1} a_{11}}{p_1 b^2 J_0^2(\lambda_1 b)} & \frac{(p_1 - p_{1_2})e^{-(p_1+p_{1_2})d_1} a_{12}}{p_1 b^2 J_0^2(\lambda_1 b)} & \dots & \frac{(p_1 - p_{1_n})e^{-(p_1+p_{1_n})d_1} a_{1n}}{p_1 b^2 J_0^2(\lambda_1 b)} \\ \frac{(p_2 - p_{1_1})e^{-(p_2+p_{1_1})d_1} a_{21}}{p_2 b^2 J_0^2(\lambda_2 b)} & \frac{(p_2 - p_{1_2})e^{-(p_2+p_{1_2})d_1} a_{22}}{p_2 b^2 J_0^2(\lambda_2 b)} & \dots & \frac{(p_2 - p_{1_n})e^{-(p_2+p_{1_n})d_1} a_{2n}}{p_2 b^2 J_0^2(\lambda_2 b)} \\ \dots & \dots & \dots & \dots \\ \frac{(p_n - p_{1_1})e^{-(p_n+p_{1_1})d_1} a_{n1}}{p_n b^2 J_0^2(\lambda_n b)} & \frac{(p_n - p_{1_2})e^{-(p_n+p_{1_2})d_1} a_{n2}}{p_n b^2 J_0^2(\lambda_n b)} & \dots & \frac{(p_n - p_{1_n})e^{-(p_n+p_{1_n})d_1} a_{nn}}{p_n b^2 J_0^2(\lambda_n b)} \end{pmatrix} \quad (6.6.60)$$

$$B_{22} = \begin{pmatrix} \frac{(p_1 + p_{1_1})e^{-(p_1-p_{1_1})d_1} a_{11}}{p_1 b^2 J_0^2(\lambda_1 b)} & \frac{(p_1 + p_{1_2})e^{-(p_1-p_{1_2})d_1} a_{12}}{p_1 b^2 J_0^2(\lambda_1 b)} & \dots & \frac{(p_1 + p_{1_n})e^{-(p_1-p_{1_n})d_1} a_{1n}}{p_1 b^2 J_0^2(\lambda_1 b)} \\ \frac{(p_2 + p_{1_1})e^{-(p_2-p_{1_1})d_1} a_{21}}{p_2 b^2 J_0^2(\lambda_2 b)} & \frac{(p_2 + p_{1_2})e^{-(p_2-p_{1_2})d_1} a_{22}}{p_2 b^2 J_0^2(\lambda_2 b)} & \dots & \frac{(p_2 + p_{1_n})e^{-(p_2-p_{1_n})d_1} a_{2n}}{p_2 b^2 J_0^2(\lambda_2 b)} \\ \dots & \dots & \dots & \dots \\ \frac{(p_n + p_{1_1})e^{-(p_n-p_{1_1})d_1} a_{n1}}{p_n b^2 J_0^2(\lambda_n b)} & \frac{(p_n + p_{1_2})e^{-(p_n-p_{1_2})d_1} a_{n2}}{p_n b^2 J_0^2(\lambda_n b)} & \dots & \frac{(p_n + p_{1_n})e^{-(p_n-p_{1_n})d_1} a_{nn}}{p_n b^2 J_0^2(\lambda_n b)} \end{pmatrix} \quad (6.6.61)$$

Solving (6.6.57) we obtain all the coefficients \hat{D}_{4j} and \hat{D}_{5j} ($j = 1, 2, \dots, n$).

Then D_{2j} is calculated using

$$D_{2j} = D_{4j} + D_{5j} - \mu_0 I r_0 \frac{J_1(\lambda_j r_0) e^{-\lambda_j h}}{\lambda_j b^2 J_0^2(\lambda_j b)}. \quad (6.6.62)$$

The induced change in impedance of the coil is given by the formula

$$Z^{ind} = \frac{j\omega}{I} 2\pi r_0 A_0^{ind}(r_0, h) \quad (6.6.63)$$

$$\text{where } A_0^{ind}(r_0, h) = \sum_{j=1}^n D_{2j} e^{-\lambda_j h} J_1(\lambda_j r_0) \quad (6.6.64)$$

Formula (6.6.63) is used to compute the change in impedance of the coil. Calculations are performed with ‘‘Mathematica’’. The program which is used to compute the change in impedance (6.6.63) is shown in **Appendix Fig. Ap.19**. The following parameters of the problem are selected: $\mu_0 = 4 \cdot 10^{-7} \pi$, $\sigma_1 = 18.5 \text{ Ms/m}$, $\sigma_2 = 3 \text{ Ms/m}$, $c = 2.2 \text{ mm}$, $r_0 = 4.5 \text{ mm}$, $h = 1.4 \text{ mm}$,

$d_1 = 0.7$ mm, $d_2 = 0.3$ mm, $b = 55$ mm. The change in impedance is computed for the following seven frequencies: $f = 1000, 2000, 3000, 4000, 5000, 6000,$ and 7000 Hz. The results of calculations are shown in **Fig. 6.14**. The calculated points (from top to bottom) correspond to the seven frequencies (from smallest to largest). The upper limit of the summation index in (6.6.64) is fixed at $n = 62$. Comparison of the computational results obtained for other values of n showed that the chosen value of 62 is quite satisfactory in terms of calculation accuracy. More detailed analysis of convergence is presented in **Section 6.3**. Several computational steps are necessary in order to calculate the induced change in impedance. First, the set of eigenvalues λ_i has to be calculated. This can easily be done in Mathematica using a built-in routine `BesselZeros`. Second, a set of complex roots of (6.6.33) should be computed. Calculations are based on the method described in **Section 6.3**. Third, several systems of linear equations have to be solved in order to determine expansion coefficients. Finally, the change in impedance is computed using (6.6.50) - (6.6.64).

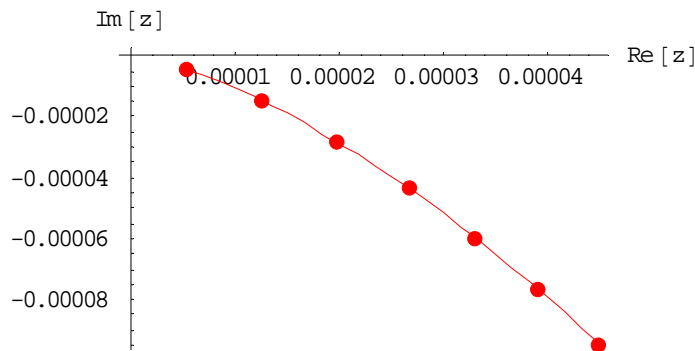


Fig. 6.14 The change in impedance of the coil for seven frequencies.

6.7 A coil of finite dimensions above a conducting half-space with a flaw in the form of a circular cylinder

Consider the model shown in **Fig. 6.15**. The inner and outer radii of the coil are r_1 and r_2 , respectively. The bottom of the coil is located at the distance z_1 from the conducting plate. The height of the coil is $z_2 - z_1$ and the number of turns is N .

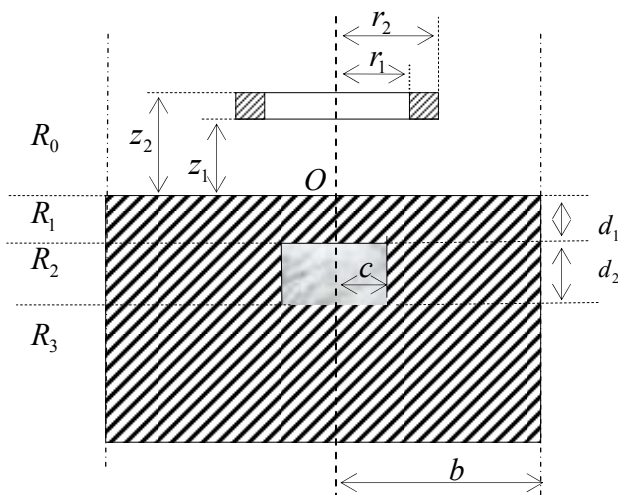


Fig. 6.15 A coil of finite dimensions above a conducting plate.

It can easily be shown that the second term in (6.1.29) and (6.1.30) represents the vector potential of a single-turn coil located in an unbounded free space. In addition, the first term in (6.1.29) and (6.1.30) is the induced vector potential in air due to the presence of a conducting cylinder:

$$A_0^{ind}(r, z, r_0, h) = \sum_{j=1}^n D_{2j} e^{-\lambda_j z} J_1(\lambda_j r), \quad (6.7.1)$$

where

$$D_{2j} = D_{4j} + D_{5j} - \mu_0 I r_0 \frac{J_1(\lambda_j r_0) e^{-\lambda_j h}}{\lambda_j b^2 J_0^2(\lambda_j b)}.$$

The induced vector potential in air due to currents in the whole coil is obtained as follows

$$A_{0coil}^{ind}(r, z) = \int_{r_1}^{r_2} \int_{z_1}^{z_2} A_0^{ind}(r, z, r_0, h) dr_0 dh. \quad (6.7.2)$$

There is an important difference between the calculations for a single-turn coil and coil of finite dimensions. For the case of a single-turn coil (**Section 6.6**) the geometrical parameters of the coil (r_0 and h) are constants and the numerical values of r_0 and h can be used to solve the system (6.6.57). If a coil has finite dimensions then one needs to integrate the solution with respect to r_0 and h . Thus, numerical values to the parameters r_0 and h in (6.6.57) cannot be assigned. In this case, in order to obtain the induced vector potential of a single-turn coil we need to solve (6.6.57) in matrix form. Solving the second equation in (6.6.57) we obtain

$$\vec{X}_2 = -A_{22}^{-1} A_{21} \vec{X}_1. \quad (6.7.3)$$

Substituting (6.7.3) into the first equation in (6.6.57) gives

$$(A_{11} - A_{12} A_{22}^{-1} A_{21}) \vec{X}_1 = \vec{b}. \quad (6.7.4)$$

It follows from (6.7.3) and (6.7.4) that

$$\begin{cases} \vec{X}_1 = (A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1} \vec{b} \\ \vec{X}_2 = -A_{22}^{-1} A_{21} (A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1} \vec{b} \end{cases} \quad (6.7.5)$$

Using (6.6.57) - (6.6.61) equation (6.6.62) should be written in matrix form:

$$\vec{D}_2 = Y \vec{b}, \quad (6.7.6)$$

$$\text{where } Y = (B_{11} + B_{21} - (B_{12} + B_{22}) A_{22}^{-1} A_{21}) (A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1} - C_{diag} \quad (6.7.7)$$

$$\vec{D}_2 = \begin{pmatrix} D_{2_1} \\ D_{2_2} \\ \dots \\ D_{2_n} \end{pmatrix}, \quad C_{diag} = \begin{pmatrix} \frac{1}{2 p_1 \lambda_1 b^2 J_0^2(\lambda_1 b)} & 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{2 p_2 \lambda_2 b^2 J_0^2(\lambda_2 b)} & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \frac{1}{2 p_n \lambda_n b^2 J_0^2(\lambda_n b)} \end{pmatrix} \quad (6.7.8)$$

and \vec{b} is given by (6.6.53).

Formula (6.7.1) for the induced vector potential of a single-turn coil can be rewritten in the form

$$A_0^{ind}(r, z, r_0, h) = \vec{D}_2^T \vec{f}, \quad (6.7.9)$$

where

$$\vec{f} = \begin{pmatrix} J_1(\lambda_1 r) e^{-\lambda_1 z} \\ J_1(\lambda_2 r) e^{-\lambda_2 z} \\ \dots \\ J_1(\lambda_n r) e^{-\lambda_n z} \end{pmatrix}.$$

Substituting (6.7.9) into (6.7.2) and using the formulas

$$\left\| \begin{aligned} \int_{z_1}^{z_2} e^{-\lambda_j h} dh &= -\frac{1}{\lambda_j} (e^{-\lambda_j z_2} - e^{-\lambda_j z_1}) \\ \int_{r_1}^{r_2} r_0 J_1(\lambda_j r_0) dr_0 &= \left| \xi = \lambda_j r_0 \right| = \frac{1}{\lambda_j^2} \int_{\lambda_j r_1}^{\lambda_j r_2} \xi J_1(\xi) d\xi \end{aligned} \right\| \quad (6.7.10)$$

we obtain the induced vector potential in air due to the presence of the conducting cylinder (the current amplitude I in this case is replaced by the current

density $\frac{NI}{(r_2 - r_1)(z_2 - z_1)}$):

$$A_{0coil}^{ind}(r, z) = \frac{\mu_0 NI}{(r_2 - r_1)(z_2 - z_1)} \sum_{j=1}^n f_j \sum_{i=1}^n Y_{ji} \frac{(e^{-\lambda_i z_1} - e^{-\lambda_i z_2})}{\lambda_i^3} \int_{\lambda_i r_1}^{\lambda_i r_2} \xi J_1(\xi) d\xi. \quad (6.7.11)$$

The integral with respect to ξ in (6.7.10) can be computed in terms of the Bessel and Struve functions [2] as follows

$$\int_{\lambda_i r_1}^{\lambda_i r_2} \xi J_1(\xi) d\xi = \left\{ \frac{\pi}{2} \xi [J_0(\xi) H_1(\xi) - J_1(\xi) H_0(\xi)] \right\}_{\xi=\lambda_i r_1}^{\xi=\lambda_i r_2} \quad (6.7.12)$$

The induced change in impedance of a coil of finite dimensions (see **Fig. 6.15**) is calculated by means of the following formula [15]:

$$\left\| \begin{aligned} Z^{ind} &= \frac{j\omega}{I^2} \iiint_V \vec{A} \vec{I} dv = \frac{j\omega}{I^2} \int_0^{2\pi} d\varphi \int_{r_1}^{r_2} r dr \int_{z_1}^{z_2} A_{0coil}^{ind} dz \\ Z^{ind} &= \frac{2\pi j\omega}{I} \frac{N}{(r_2 - r_1)(z_2 - z_1)} \int_{r_1}^{r_2} \int_{z_1}^{z_2} r A_{0coil}^{ind}(r, z) dr dz. \end{aligned} \right\| \quad (6.7.13)$$

Using (6.7.11) and (6.7.13) we obtain the induced change in impedance of the coil in the form

$$Z^{ind} = \frac{2j\omega\pi\mu_0 N^2}{(r_2 - r_1)^2 (z_2 - z_1)^2} \sum_{j=1}^n \frac{(e^{-\lambda_j z_1} - e^{-\lambda_j z_2})}{\lambda_j^3} \int_{\lambda_j r_1}^{\lambda_j r_2} \xi J_1(\xi) d\xi \sum_{i=1}^n Y_{ji} \frac{(e^{-\lambda_i z_1} - e^{-\lambda_i z_2})}{\lambda_i^3} \int_{\lambda_i r_1}^{\lambda_i r_2} \xi J_1(\xi) d\xi \quad (6.7.14)$$

Formula (6.7.14) is used to compute the change in impedance of the coil. Calculations are performed with “Mathematica”. Program which is used to compute the change in impedance (6.7.14) is shown in **Appendix Fig. Ap.20**. The following parameters of the problem are

selected: $\mu_0 = 4 \cdot 10^{-7} \pi$, $\sigma_1 = 18.5 \text{ Ms/m}$, $\sigma_2 = 3 \text{ Ms/m}$, $c = 2.2 \text{ mm}$, $r_2 = 5.5 \text{ mm}$, $r_1 = 3.5 \text{ mm}$, $z_1 = 0.3 \text{ mm}$, $z_2 = 2.6 \text{ mm}$, $d_1 = 0.7 \text{ mm}$, $d_2 = 0.3 \text{ mm}$, $b = 55 \text{ mm}$, $N = 200$. The change in impedance is computed for the following seven frequencies: $f = 1000, 2000, 3000, 4000, 5000, 6000, \text{ and } 7000 \text{ Hz}$. The results of calculations are shown in **Fig. 6.16**. The calculated points (from top to bottom) correspond to the seven frequencies (from smallest to largest). The upper limit of the summation index in (6.7.14) is fixed at $n = 62$. Comparison of the computational results obtained for other values of n showed that the chosen value of 62 is quite satisfactory in terms of calculation accuracy. Several computational steps are necessary in order to calculate the induced change in impedance. First, the set of eigenvalues λ_i has to be calculated. This can easily be done in “Mathematica” using a built-in routine BesselZeros. Second, a set of complex roots of (6.6.33) should be computed. Calculations are based on the method described in **Section 6.3**. Third, several systems of linear equations have to be solved in order to determine expansion coefficients. Finally, the change in impedance is computed using (6.7.14). Details of the numerical aspects of the procedure are given in **Section 6.3**.

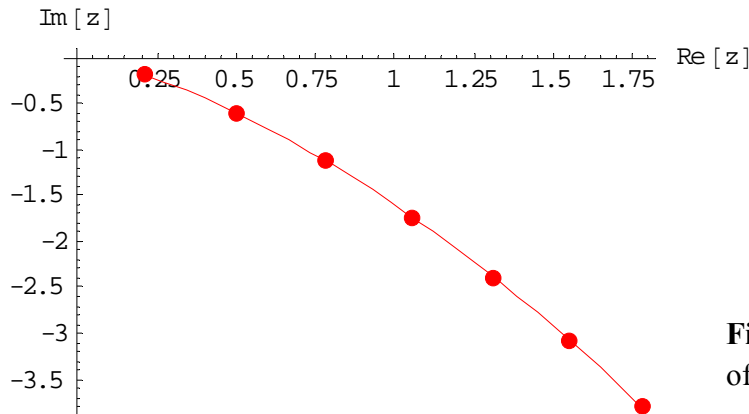


Fig. 6.16 The change in impedance of the coil for seven frequencies.

CONCLUSIONS

The thesis is devoted to the development and analysis of mathematical models of eddy current testing problems. Eddy current method is widely used in practice in order to control properties of conducting materials. The idea of the method is based on the principle of electromagnetic induction and can be explain as follows. Suppose that a coil with alternating current is located above an electrically conducting medium being tested. The currents in the coil induce eddy currents (or Foucault currents) in the conducting medium. The eddy currents, in turn, interact with the currents in the coil and change the impedance of the coil. By monitoring the change in impedance of the coil one can make conclusions with respect to the quality of the conducting medium. In particular, parameters of the medium (such as the electrical conductivity and magnetic permeability) can be estimated. In addition, if the medium contains defects (or flaws) then eddy current method can be used to estimate the parameters of the flaw (geometrical size or electrical conductivity).

The determination of unknown parameters of the conducting medium requires the solution of the inverse problem. Usually in such cases theoretical model is compared with experimental data and the difference between theoretical model and experimental data is minimized (in some norm). Thus, in order to solve the inverse problem one needs to have an accurate and reliable mathematical model for the solution of the direct problem. The thesis is devoted to the development and analysis of direct problems that occur in eddy current testing.

Two major groups of problems are analyzed in the thesis. The first group deals with the case where a coil with alternating current is located in the vicinity of a conducting medium. It is assumed that the properties of the conducting medium are not constant but vary with respect to one spatial coordinate. Such a situation often occurs in applications. One important example is metal processing (such as surface hardening) at metallurgical plants. It is known that surface hardening can produce a thin upper layer of reduced magnetic permeability. In this case models based on the assumption of constant properties of the conducting material (for example, electrical conductivity) do not work. Another example is related to the electric industry where the thermal efficiency of gas turbines can be considerably improved by increasing firing temperatures. In this case blades of gas turbines have to be protected from hot corrosion and high temperature oxidation. This is achieved using protective coatings containing aluminium and chrome. However, operation under extreme conditions may lead to loss of aluminium in the coating and may result in blade failure. Experiments show that in this case the electrical conductivity of the alloy varies with depth. Thus, the two above mentioned problems demonstrate the necessity of developing mathematical models where the electric conductivity and magnetic permeability are functions of one spatial coordinate. Two different geometrical configurations are considered in the thesis: (a) a coil with alternating current located above a conducting multilayer medium where the electrical conductivity and magnetic permeability of the upper layer are exponential functions of the vertical coordinate and (b) a coil inside or outside a conducting multilayer tube where the conducting layer closest to the coil has varying electrical conductivity and magnetic permeability (in particular, the electrical conductivity and magnetic permeability are power functions of the radial coordinate). Two different types of excitation coils are considered in part (a): a circular coil of finite cross-section and a double conductor line consisting of two infinitely long parallel wires. Analytical solutions of the above mentioned problems are constructed in the thesis. The methods of integral transforms (such as Hankel or Fourier integral transforms) are used to reduce the problem to the system of ordinary differential equations with variable coefficients. The solutions of the corresponding ordinary differential equations are found in

closed form in terms of different special functions such as Bessel functions or confluent hypergeometric functions. The change in impedance of the coil is computed in closed form in terms of improper integrals containing special functions. Thus, solution of direct eddy current testing problems for media with varying electrical conductivity and magnetic permeability is obtained in closed form. There is another advantage of using analytical solutions for media with varying properties. These solutions can be used as benchmark for more complicated problem where the solution can be found by numerical methods such as finite element method. Often analytical solutions are used to test numerical algorithms developed for more complicated cases where both electrical conductivity and magnetic permeability are functions of several spatial coordinates. In Chapter 5 eddy current problems analyzed in Chapter 2 (multilayer planar conducting medium) are generalized for the case where the conducting medium is moving in the horizontal direction with constant velocity. Analytical solutions are found for the case where the electrical conductivity and magnetic permeability of the upper layer are exponential functions of the vertical coordinate and the layer is moving in the horizontal direction with constant velocity. The change in impedance is found in closed form in terms of double integrals.

The second group of problems analyzed in the thesis is related to the solution of axisymmetric problems for the case of a circular coil located above a conducting medium with constant properties containing a cylinder of finite size whose axis coincides with the axis of the coil. Three problems are considered in the thesis: (i) a circular coil above a conducting cylinder of finite radius and height, (ii) a circular coil above an infinite conducting plate with a bottom cylindrical hole and (iii) a circular coil above a conducting half-space with a flaw in the form of a circular cylinder of finite height. All three problems have important practical applications. Problem in part (i) is relevant to coin validators. One of the principles which is used in coin validators is based on the comparison of the electrical conductivity of a conducting sample inserted into the validator (“fake coin”) with the electrical conductivity of real coin (such as Euro coin, for example). The model analyzed in part (i) can be used to compute the change in impedance of a coil in cases where the size of the coil is comparable to the size of a conducting sample (such as coin). Problem described in part (ii) can be used in order to estimate the effect of corrosion of metal plates (in this case the parameter of interest is the height of the cylindrical hole). Finally, problem analyzed in part (iii) can be used to test the quality of spot welding. In spot welding, metal pieces are welded only at separate points (not along the whole surface). A cast core is formed as the result of welding process. The properties of the cast core are of great interest to engineers since the quality of welding process can be directly assessed by analyzing the size of the core and its electrical conductivity. Solution developed in the thesis can be served as the solution for the direct problem which is used to model spot welding and assess its quality.

Mathematical basis for the second group of problems is so-called TREE method (“TRuncated Eigenfunction Expansions”) suggested by Prof. T. Theodoulidis. In any “classical” eddy current problem the vector potential approaches zero as the distance from the coil tends to infinity (assuming that there are no other sources of current except the current in the coil). “Classical” eddy current problems are usually solved by means of integral transform method (such as Fourier or Hankel integral transform). The idea of the TREE method is that one assumes that the vector potential is exactly zero at a sufficiently large distance from the coil. As a result, the solution is sought in finite domain and the corresponding eddy current problem can be solved by the method of separation of variables. Thus, the solution (for example, the change in impedance) is constructed in terms of series. There are several numerical aspects of the procedure which need to be taken care of in the TREE method. First, the distance b (usually the radial distance) from the

coil where the vector potential is zero should be specified. The choice of b depends on the accuracy desired and can be controlled by the user. Second, a set of eigenvalues (in many cases, complex eigenvalues) should be computed. As a result, one needs a fast and reliable procedure for calculation of eigenvalues. In addition, nothing is usually known about initial guesses for the roots. Thus, Newton's method for the calculation of eigenvalues cannot be used. An algorithm is implemented in the thesis which is based on theory of complex variable. Using well-known argument principle we can compute the number of zeros of an analytic function inside a contour of integration (usually rectangle with sides parallel to the coordinate axes). If the number of roots inside the contour is larger than one then a smaller contours of integration are considered. Finally, if there is only one root of an analytic function inside the contour, then explicit formula (in terms of contour integral) can be used to compute the root with any desired accuracy. A computer program in "Mathematica" is developed in the thesis which can be used to locate and compute the roots in case where the number of roots inside the contour is at most two. Third, constants of integration are obtained solving the corresponding linear system of equations. Finally, the change in impedance of the coil is computed in the form of a series where the number of terms in the series is equal to the number of complex eigenvalues (this number can be adjusted, that is, increased, if higher accuracy is desired). As one can see, the TREE method can be called quasi-analytical since the solution is obtained in the form of a series expansion but there are some elements of the procedure which require the use of numerical methods (computation of complex eigenvalues and solution of the system of linear equations). The author believes that the TREE method has high potential for solving eddy current problems since the class of problems which can be solved by the TREE method is much wider than the class of problems which can be solved by the method of integral transforms.

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APPENDIX.

Fig. Ap.1 (Section 2.4) “Mathematica” program for numerical calculation of the change in impedance Z for three different values of $\hat{\beta}$.

```

Remove["Global`*"]
 $\alpha = 0.0$ ;  $\beta = 1$ ;  $h = 0.05$ ;  $\mu m = 5$ ;
Do[
  {Do[
    {If[n == 1, {r1 == 1, r2 == 0, r3 == 0},
      If[n == 2, {r1 == 0, r2 == 1, r3 == 0}, {r1 == 0, r2 == 0, r3 == 1}]}];
    v[x_] :=  $\frac{\text{Sqrt}[\beta^2 + 4 * x^2]}{\alpha + \beta}$ ;
    sqrtb = Sqrt[-i * b];
    z0[x_] :=  $\frac{2 * \text{sqrtb}}{\alpha + \beta}$ ;
    D1[x_] :=  $\mu m * x + \frac{\beta}{2}$ ;
    D2[x_] :=  $\mu m * x - \frac{\beta}{2}$ ;
    Jz0[x_] := BesselJ[v[x], z0[x]];
    Jz01[x_] := BesselJ[v[x] + 1, z0[x]];
    Jz0pr[x_] := -Jz01[x] +  $\frac{v[x]}{z0[x]} * Jz0[x]$ ;
    skait[x_] := D2[x] * Jz0[x] - sqrtb * Jz0pr[x];
    sauc[x_] := D1[x] * Jz0[x] + sqrtb * Jz0pr[x];
    f[x_] := (BesselJ[1, x])^2 * Exp[-2 * x * h] *  $\frac{\text{skait}[x]}{\text{sauc}[x]}$ ;
    Z = i * NIntegrate[f[x], {x, 0, 100}, MaxRecursion -> 50, WorkingPrecision -> 8];
    data1[b] = Re[Z];
    data2[b] = Im[Z]}, {b, 1, 40}];
data = Table[If[m == 1, data1[k], data2[k]], {k, 1, 40}, {m, 1, 2}];
gr1[n] = ListPlot[data, PlotStyle -> {PointSize[0.02], RGBColor[r1, r2, r3]},
  AxesLabel -> {Re[z], Im[z]}, DisplayFunction -> Identity];
gr2[n] = ListPlot[data, AxesLabel -> {Re[z], Im[z]}, PlotJoined -> True,
  PlotStyle -> RGBColor[r1, r2, r3], DisplayFunction -> Identity];  $\beta = \beta + 1$ ,
{n, 1, 3}]; Show[gr1[1], gr2[1], gr1[2], gr2[2], gr1[3], gr2[3],
DisplayFunction -> $DisplayFunction]

```

Fig. Ap.2 (Section 2.4) “Mathematica” program for the selection of the upper limit of integration in (2.4.13).

```

Remove["Global`*"]
α = 0.0; β = 2; h = 0.05; μm = 5;
Do[
  {Do[
    {If[n == 1, {r1 = 1, r2 = 0, r3 = 0},
      If[n == 2, {r1 = 0, r2 = 1, r3 = 0}, {r1 = 0, r2 = 0, r3 = 1}]}];
    v[x_] :=  $\frac{\text{Sqrt}[\beta^2 + 4 * x^2]}{\alpha + \beta}$ ;
    sqrtb = Sqrt[-i * b];
    z0[x_] :=  $\frac{2 * \text{Sqrt}[-i * b]}{\alpha + \beta}$ ;
    D1[x_] := μm * x +  $\frac{\beta}{2}$ ;
    D2[x_] := μm * x -  $\frac{\beta}{2}$ ;
    Jz0[x_] := BesselJ[v[x], z0[x]];
    Jz01[x_] := BesselJ[v[x] + 1, z0[x]];
    Jz0pr[x_] := -Jz01[x] +  $\frac{v[x]}{z0[x]} * Jz0[x]$ ;
    skait[x_] := D2[x] * Jz0[x] - sqrtb * Jz0pr[x];
    sauc[x_] := D1[x] * Jz0[x] + sqrtb * Jz0pr[x];
    f[x_] := (BesselJ[1, x])^2 * Exp[-2 * x * h] *  $\frac{\text{skait}[x]}{\text{sauc}[x]}$ ;
    Z = i * NIntegrate[f[x], {x, 0, kk}, MaxRecursion → 50, WorkingPrecision → 12];
    data1[b] = Re[Z];
    data2[b] = Im[Z], {b, 1, 11, 2}];
  data = Table[If[m == 1, data1[k], data2[k]], {k, 1, 11, 2}, {m, 1, 2}];
  Print[data]; β = β}, {kk, 20, 140, 20}]

```

Fig. Ap.3 (Section 2.5) “Mathematica” program for the calculation of the change in impedance Z by means of (2.5.8) for three different values of $\hat{\beta}$.

```

Remove["Global`*"]
α = 0.0; β = 1; z2 = 1.6; z1 = 0.4; rr2 = 2;
μm = 5; nConst = 50;
Do[
  {Do[
    {If[n == 1, {r1 = 1, r2 = 0, r3 = 0},
      If[n == 2, {r1 = 0, r2 = 1, r3 = 0}, {r1 = 0, r2 = 0, r3 = 1}]}];
    v[x_] :=  $\frac{\text{Sqrt}[\beta^2 + 4 * x^2]}{\alpha + \beta}$ ;
    sqrtb = Sqrt[-i * b];
    z0[x_] :=  $\frac{2 * \text{sqrtb}}{\alpha + \beta}$ ;
    D1[x_] :=  $\mu m * x + \frac{\beta}{2}$ ;
    D2[x_] :=  $\mu m * x - \frac{\beta}{2}$ ;
    Jz0[x_] := BesselJ[v[x], z0[x]];
    Jz01[x_] := BesselJ[v[x] + 1, z0[x]];
    Jz0pr[x_] := -Jz01[x] +  $\frac{v[x]}{z0[x]} * Jz0[x]$ ;
    skait[x_] := D2[x] * Jz0[x] - sqrtb * Jz0pr[x];
    sauc[x_] := D1[x] * Jz0[x] + sqrtb * Jz0pr[x];
    gint[y_] =
      π / 2 * y * (BesselJ[0, y] * StruveH[1, y] - BesselJ[1, y] * StruveH[0, y]);
    gint1[x_] = (gint[x * rr2] - gint[x]) ^ 2;
    f[x_] :=  $\frac{\text{skait}[x] * (\text{Exp}[-x * z1] - \text{Exp}[-x * z2])^2 * \text{gint1}[x]}{\text{sauc}[x] * x^6}$ ;
    Z1 = i * NIntegrate[f[x], {x, 0, 100}, MaxRecursion → 50,
      WorkingPrecision → 8];
    Z =  $\frac{\text{nConst}^2}{(\text{rr2} - 1)^2 * (z2 - z1)^2} * Z1$ ;
    data1[b] = Re[Z];
    data2[b] = Im[Z], {b, 1, 20}];
    data = Table[If[m == 1, data1[k], data2[k]], {k, 1, 20}, {m, 1, 2}];
    gr1[n] = ListPlot[data, PlotStyle → {PointSize[0.02], RGBColor[r1, r2, r3]},
      AxesLabel → {Re(z), Im(z)}, DisplayFunction → Identity];
    gr2[n] = ListPlot[data, AxesLabel → {Re(z), Im(z)}, PlotJoined → True,
      PlotStyle → RGBColor[r1, r2, r3], DisplayFunction → Identity]; β = β + 1},
  {n, 1, 3}]; Show[gr1[1], gr2[1], gr1[2], gr2[2], gr1[3], gr2[3],
  DisplayFunction → $DisplayFunction]

```

Fig. Ap.4 (Section 2.6) “Mathematica” program for the calculation of the change in impedance Z for three different values of $\hat{\beta}$.

```

Remove["Global`*"]
 $\alpha = 0.0$ ;  $\beta = 1$ ;  $h = 0.05$ ;  $d = 0.2$ ;  $\mu_1 = 1$ ;  $\mu_2 = 100$ ;  $koef = 1.5$ ;
Do[{Do[{If[n == 1, {r1 = 1, r2 = 0, r3 = 0}, If[n == 2, {r1 = 0, r2 = 1, r3 = 0},
      {r1 = 0, r2 = 0, r3 = 1}]}];
  v[x_] :=  $\frac{\text{Sqrt}[\beta^2 + 4 * x^2]}{\alpha + \beta}$ ;
  sqrtb = Sqrt[-i * b];
  z0[x_] :=  $\frac{2 * \text{sqrtb}}{\alpha + \beta}$ ;
  z1[x_] := z0[x] * Exp[- $\frac{\alpha + \beta}{2} * d$ ];
  D1[x_] :=  $\frac{\beta}{2} * \text{Exp}[\beta * d] * \mu_2 - \mu_1 * \sqrt{x^2 + i * b * koef}$ ;
  D2[x_] :=  $\mu_1 * x + \frac{\beta}{2}$ ;
  D3[x_] :=  $\mu_2 * \text{sqrtb} * \text{Exp}[\frac{\beta - \alpha}{2} * d]$ ;
  Jz0[x_] := BesselJ[v[x], z0[x]];
  Jz01[x_] := BesselJ[v[x] + 1, z0[x]];
  Jz0pr[x_] := -Jz01[x] +  $\frac{v[x]}{z0[x]} * Jz0[x]$ ;
  Yz0[x_] := BesselY[v[x], z0[x]];
  Yz01[x_] := BesselY[v[x] + 1, z0[x]];
  Yz0pr[x_] := -Yz01[x] +  $\frac{v[x]}{z0[x]} * Yz0[x]$ ;
  Jz1[x_] := BesselJ[v[x], z1[x]];
  Jz11[x_] := BesselJ[v[x] + 1, z1[x]];
  Jz1pr[x_] := -Jz11[x] +  $\frac{v[x]}{z1[x]} * Jz1[x]$ ;
  Yz1[x_] := BesselY[v[x], z1[x]];
  Yz11[x_] := BesselY[v[x] + 1, z1[x]];
  Yz1pr[x_] := -Yz11[x] +  $\frac{v[x]}{z1[x]} * Yz1[x]$ ;
  R[x_] = D1[x] * Yz1[x] + D3[x] * Yz1pr[x];
  F[x_] = D1[x] * Jz1[x] + D3[x] * Jz1pr[x];
  skait[x_] := R[x] * ((2 * x *  $\mu_1$  - D2[x]) * Jz0[x] - sqrtb * Jz0pr[x]) -
    F[x] * ((2 * x *  $\mu_1$  - D2[x]) * Yz0[x] - sqrtb * Yz0pr[x]);
  sauc[x_] := R[x] * (D2[x] * Jz0[x] + sqrtb * Jz0pr[x]) - F[x] * (D2[x] * Yz0[x] + sqrtb * Yz0pr[x]);
  f[x_] := (BesselJ[1, x])^2 * Exp[-2 * x * h] *  $\frac{\text{skait}[x]}{\text{sauc}[x]}$ ;
  Z = i * NIntegrate[f[x], {x, 0, 100}, MaxRecursion -> 50, WorkingPrecision -> 8];
  data1[b] = Re[Z];
  data2[b] = Im[Z], {b, 1, 10}];
data = Table[If[m == 1, data1[k], data2[k]], {k, 10}, {m, 1, 2}];
gr1[n] = ListPlot[data, PlotStyle -> {PointSize[0.02], RGBColor[r1, r2, r3]},
  AxesLabel -> {Re[z], Im[z]}, DisplayFunction -> Identity];
gr2[n] = ListPlot[data, AxesLabel -> {Re[z], Im[z]}, PlotJoined -> True,
  PlotStyle -> RGBColor[r1, r2, r3], DisplayFunction -> Identity];  $\beta = \beta + 1$ }, {n, 1, 3}];
Show[gr1[1], gr2[1], gr1[2], gr2[2], gr1[3], gr2[3], DisplayFunction -> $DisplayFunction]

```

Fig. Ap.5 (Section 3.3) “Mathematica” program for numerical calculation of the change in impedance given by formula (3.3.16).

```

Remove["Global`*"]
r0 = 0.9;  $\mu$ 1 = 1;
Do[
  {Do[{If[n == 1, {r1 = 1, r2 = 0, r3 = 0},
    If[n == 2, {r1 = 0, r2 = 1, r3 = 0}, {r1 = 0, r2 = 0, r3 = 1}]}];
  v = Sqrt[1/4 + i * p1^2];
  Kv[x_] := BesselK[v, x];
  Kv1[x_] := BesselK[v + 1, x];
  Kvpr[x_] := -Kv1[x] +  $\frac{v}{x}$  * Kv[x];
  I1[x_] := BesselI[1, x];
  I2[x_] := BesselI[2, x];
  I1pr[x_] := I2[x] +  $\frac{1}{x}$  * I1[x];
  I1r0[x_] := BesselI[1, x * r0];
  K1[x_] := BesselK[1, x];
  D1[x_] := 2 *  $\mu$ 1 * x * I1pr[x] + I1[x];
  skait[x_] := 2 *  $\mu$ 1 * Kv[x];
  sauc[x_] := D1[x] * Kv[x] - 2 * x * Kvpr[x] * I1[x];
  f[x_] :=  $\frac{I1r0[x]^2}{I1[x]}$  *  $\left(\frac{skait[x]}{sauc[x]} - K1[x]\right)$ ;
  Z = i * NIntegrate[f[x], {x, 0.0001, 100}, MaxRecursion -> 50,
    WorkingPrecision -> 8];
  data1[p1] = Re[Z];
  data2[p1] = Im[Z], {p1, 1, 15}];
data = Table[If[m == 1, data1[k], data2[k]], {k, 15}, {m, 1, 2}];
gr1[n] = ListPlot[data, PlotStyle -> {PointSize[0.02], RGBColor[r1, r2, r3]},
  AxesLabel -> {Re[z], Im[z]}, DisplayFunction -> Identity];
gr2[n] = ListPlot[data, AxesLabel -> {Re[z], Im[z]}, PlotJoined -> True,
  PlotStyle -> RGBColor[r1, r2, r3], DisplayFunction -> Identity];
r0 = r0 - 0.1}, {n, 1, 3}];
Show[gr1[1], gr2[1], gr1[2], gr2[2], gr1[3], gr2[3],
  DisplayFunction -> $DisplayFunction]

```

Fig. Ap.6 (Section 3.4) “Mathematica” program for numerical calculation of the change in impedance given by formula (3.4.26).

```

Remove["Global`*"]
r0 = 0.9; r02 = 1.1; μ1 = 1; μ2 = 6; koef = 1.5;
Do[
  {Do[
    {If[n == 1, {r1 = 1, r2 = 0, r3 = 0}, If[n == 2, {r1 = 0, r2 = 1, r3 = 0},
      {r1 = 0, r2 = 0, r3 = 1}]}];
    p2 = p1 * koef;
    q[x_] = Sqrt[x^2 + i * p2^2];
    v = Sqrt[1/4 + i * p1^2];
    Kvu[x_] := BesselK[v, x];
    Kvu1[x_] := BesselK[v + 1, x];
    Kvupr[x_] := -Kvu1[x] +  $\frac{v}{x}$  * Kvu[x];
    Kvr2[x_] := BesselK[v, x * r02];
    Kvlr2[x_] := BesselK[v + 1, x * r02];
    Kvr2pr[x_] := -Kvlr2[x] +  $\frac{v}{x * r02}$  * Kvr2[x];
    Ivu[x_] := BesselI[v, x];
    Ivu1[x_] := BesselI[v + 1, x];
    Ivupr[x_] := Ivu1[x] +  $\frac{v}{x}$  * Ivu[x];
    Ivr2[x_] := BesselI[v, x * r02];
    Ivlr2[x_] := BesselI[v + 1, x * r02];
    Ivr2pr[x_] := Ivlr2[x] +  $\frac{v}{x * r02}$  * Ivr2[x];
    I1[x_] := BesselI[1, x];
    I2[x_] := BesselI[2, x];
    I1pr[x_] := I2[x] +  $\frac{1}{x}$  * I1[x];
    I1r0[x_] := BesselI[1, x * r0];
    K1[x_] := BesselK[1, x];

    K1r2[x_] := BesselK[1, q[x] * r02];
    K2r2[x_] := BesselK[2, q[x] * r02];
    K1r2pr[x_] := -K2r2[x] +  $\frac{1}{q[x] * r02}$  * K1r2[x];
    D1[x_] := Ivr2[x] * Kvu[x] - Ivu[x] * Kvr2[x];
    D2[x_] := Ivr2pr[x] * Kvu[x] - Ivu[x] * Kvr2pr[x];
    D3[x_] := Ivr2[x] * Kvupr[x] - Ivupr[x] * Kvr2[x];
    D4[x_] := Ivr2pr[x] * Kvupr[x] - Ivupr[x] * Kvr2pr[x];
    D5[x_] := 2 * μ1 * x * I1pr[x] + I1[x];
    D6[x_] := μ2 * K1r2[x] * (D1[x] - 2 * x * r02 * D2[x]);
    D7[x_] := 2 * μ1 * q[x] * K1r2pr[x] * D1[x];

    skait[x_] := 2 * μ1 * (D6[x] + D7[x]);
    sauc[x_] := D5[x] * (D6[x] + D7[x]) - 2 * x * μ2 * K1r2[x] * I1[x] * (D3[x] - 2 * x * r02 * D4[x]) -
      4 * μ1 * x * q[x] * K1r2pr[x] * I1[x] * D3[x];
    f[x_] :=  $\frac{I1r0[x]^2}{I1[x]}$  *  $\left( \frac{skait[x]}{sauc[x]} - K1[x] \right)$ ;
    Z = i * NIntegrate[f[x], {x, 0.0001, 100}, MaxRecursion → 50, WorkingPrecision → 8];
    data1[p1] = Re[Z];
    data2[p1] = Im[Z], {p1, 3, 10}];
    data = Table[If[m == 1, data1[k], data2[k]], {k, 3, 10}, {m, 1, 2}];
    gr1[n] = ListPlot[data, PlotStyle → {PointSize[0.02], RGBColor[r1, r2, r3]},
      AxesLabel → {Re[z], Im[z]}, DisplayFunction → Identity];
    gr2[n] = ListPlot[data, AxesLabel → {Re[z], Im[z]}, PlotJoined → True,
      PlotStyle → RGBColor[r1, r2, r3], DisplayFunction → Identity]; r02 = r02 + 0.1, {n, 1, 3}];
Show[gr1[1], gr2[1], gr1[2], gr2[2], gr1[3], gr2[3], DisplayFunction → $DisplayFunction]

```

Fig. Ap.7 (Section 3.5) “Mathematica” program for numerical calculation of the change in impedance given by formula (3.5.29).

```

Remove["Global`*"]
r0 = 1.1; r02 = 0.9;  $\mu$ 1 = 1;  $\mu$ 2 = 5; koef = 1.5;
Do[
  {Do[
    {If[n == 1, {r1 = 1, r2 = 0, r3 = 0},
      If[n == 2, {r1 = 0, r2 = 1, r3 = 0}, {r1 = 0, r2 = 0, r3 = 1}]}];
    p2 = p1 * koef;
    q[x_] = Sqrt[x^2 + i * p2^2];
    v = Sqrt[1/4 + i * p1^2];
    Kvu[x_] := BesselK[v, x]; Kvu1[x_] := BesselK[v + 1, x];
    Kvu1pr[x_] := -Kvu1[x] +  $\frac{v}{x}$  * Kvu[x];
    Kvr2[x_] := BesselK[v, x * r02]; Kvr1r2[x_] := BesselK[v + 1, x * r02];
    Kvr2pr[x_] := -Kvr1r2[x] +  $\frac{v}{x * r02}$  * Kvr2[x];
    Ivu[x_] := BesselI[v, x]; Ivu1[x_] := BesselI[v + 1, x];
    Ivu1pr[x_] := Ivu1[x] +  $\frac{v}{x}$  * Ivu[x];
    Ivr2[x_] := BesselI[v, x * r02]; Ivr1r2[x_] := BesselI[v + 1, x * r02];
    Ivr2pr[x_] := Ivr1r2[x] +  $\frac{v}{x * r02}$  * Ivr2[x];
    K1[x_] := BesselK[1, x]; K2[x_] := BesselK[2, x];
    K1pr[x_] := -K2[x] +  $\frac{1}{x}$  * K1[x];
    K1r0[x_] := BesselK[1, x * r0];
    K1[x_] := BesselK[1, x];
    I1[x_] := BesselI[1, x];
    I1r2[x_] := BesselI[1, q[x] * r02];
    I2r2[x_] := BesselI[2, q[x] * r02];
    I1r2pr[x_] := I2r2[x] +  $\frac{1}{q[x] * r02}$  * I1r2[x];
    D1[x_] := Ivr2[x] * Kvu[x] - Ivu[x] * Kvr2[x];
    D2[x_] := Ivr2pr[x] * Kvu[x] - Ivu[x] * Kvr2pr[x];
    D3[x_] := Ivr2[x] * Kvu1pr[x] - Ivu1pr[x] * Kvr2[x];
    D4[x_] := Ivr2pr[x] * Kvu1pr[x] - Ivu1pr[x] * Kvr2pr[x];
    D5[x_] := 2 *  $\mu$ 1 * x * K1pr[x] + K1[x];
    D6[x] :=  $\mu$ 2 * I1r2[x] * (D1[x] - 2 * x * r02 * D2[x]);
    D7[x] := 2 *  $\mu$ 1 * q[x] * I1r2pr[x] * D1[x];
    skait[x_] := 2 *  $\mu$ 1 * (D6[x] + D7[x]);
    sauc[x_] := D5[x] * (D6[x] + D7[x]) -
      2 * x *  $\mu$ 2 * I1r2[x] * K1[x] * (D3[x] - 2 * x * r02 * D4[x]) -
      4 *  $\mu$ 1 * x * q[x] * I1r2pr[x] * K1[x] * D3[x];
    f[x_] :=  $\frac{K1r0[x]^2}{K1[x]}$  *  $\left( -\frac{skait[x]}{sauc[x]} - I1[x] \right)$ ;
    Z = i * NIntegrate[f[x], {x, 0.0001, 100}, MaxRecursion -> 50,
      WorkingPrecision -> 8];
    data1[p1] = Re[Z];
    data2[p1] = Im[Z], {p1, 1, 20}];
    data = Table[If[m == 1, data1[k], data2[k]], {k, 1, 20}, {m, 1, 2}];
    gr1[n] = ListPlot[data, PlotStyle -> {PointSize[0.02], RGBColor[r1, r2, r3]},
      AxesLabel -> {Re[z], Im[z]}, DisplayFunction -> Identity];
    gr2[n] = ListPlot[data, AxesLabel -> {Re[z], Im[z]}, PlotJoined -> True,
      PlotStyle -> RGBColor[r1, r2, r3], DisplayFunction -> Identity]; r0 = r0 + 0.2},
  {n, 1, 3}]; Show[gr1[1], gr2[1], gr1[2], gr2[2], gr1[3], gr2[3],
  DisplayFunction -> $DisplayFunction]

```

Fig. Ap.8 (Section 3.7) “Mathematica” program for numerical calculation of the change in impedance given by formula (3.7.31).

```

Remove["Global`*"]
r0 = 0.8; r2 = 1.2;  $\mu_1 = 1$ ;  $\beta = 0$ ; koef = 1.5;
pk = 1;
Do[
  {Do[{If[n = 1, {rr1 = 1, rr2 = 0, rr3 = 0},
    If[n = 2, {rr1 = 0, rr2 = 1, rr3 = 0}, {rr1 = 0, rr2 = 0, rr3 = 1}]}];
     $\mu_2 = \mu_1 * r_2^\beta$ ;
    p2 = p1 * koef;
    q[x_] = Sqrt[x^2 + i * p2];
    c =  $\beta + 3$ ;
    a[x_] :=  $\frac{c}{2} + \frac{i * p1}{2 * x}$ ;
    f1[x_, y_] := HypergeometricU[a[x], c, 2 * x * y];
    f1pr[x_, y_] := -a[x] * HypergeometricU[a[x] + 1, c + 1, 2 * x * y];
    fi1[x_, y_] :=  $y^{\beta+1} * \text{Exp}[-x * y] * f1[x, y]$ ;
    fi1pr[x_, y_] :=  $y^\beta * \text{Exp}[-x * y] * ((\beta + 1 - x * y) * f1[x, y] + 2 * x * y * f1pr[x, y])$ ;
    f2[x_, y_] := HypergeometricU[c - a[x], c, -2 * x * y];
    f2pr[x_, y_] := (a[x] - c) * HypergeometricU[c - a[x] + 1, c + 1, -2 * x * y];
    (* (c - a[x]) *)
    fi2[x_, y_] :=  $y^{\beta+1} * \text{Exp}[x * y] * f2[x, y]$ ;
    fi2pr[x_, y_] :=  $y^\beta * \text{Exp}[x * y] * ((\beta + 1 + x * y) * f2[x, y] - 2 * x * y * f2pr[x, y])$ ;
    K1[x_] := BesselK[1, x];
    K2[x_] := BesselK[2, x];
    K1pr[x_] := -K2[x] +  $\frac{1}{x} * K1[x]$ ;
    I1[x_] := BesselI[1, x];
    I2[x_] := BesselI[2, x];
    I1pr[x_] := I2[x] +  $\frac{1}{x} * I1[x]$ ;
    I1r0[x_] := BesselI[1, x * r0];
    D1[x_] := fi1pr[x, r2] * fi2[x, 1] - fi1[x, 1] * fi2pr[x, r2];
    D2[x_] := fi1[x, 1] * fi2[x, r2] - fi1[x, r2] * fi2[x, 1];
    D3[x_] := fi1pr[x, 1] * fi2pr[x, r2] - fi2pr[x, 1] * fi1pr[x, r2];
    D4[x_] := fi1[x, r2] * fi2pr[x, 1] - fi1pr[x, 1] * fi2[x, r2];
    K1r2[x_] := BesselK[1, q[x] * r2];
    K2r2[x_] := BesselK[2, q[x] * r2];
    K1r2pr[x_] := -K2r2[x] +  $\frac{1}{q[x] * r2} * K1r2[x]$ ;
    skait[x_] :=  $\mu_2 * K1r2[x] * D1[x] + q[x] * \mu_1 * (r_2)^\beta * K1r2pr[x] * D2[x]$ ;
    sauc[x_] :=  $x * \mu_2 * \mu_1 * K1r2[x] * I1pr[x] * D1[x] +$ 
       $x * q[x] * (\mu_1)^2 * (r_2)^\beta * K1r2pr[x] * I1pr[x] * D2[x] + \mu_2 * K1r2[x] * I1[x] * D3[x] +$ 
       $q[x] * \mu_1 * (r_2)^\beta * K1r2pr[x] * I1[x] * D4[x]$ ;
    f[x_] :=  $\frac{I1r0[x]^\wedge 2}{I1[x]} * \left( \frac{\mu_1 * \text{skait}[x] - K1[x] * \text{sauc}[x]}{\text{sauc}[x]} \right)$ ;
    Z = i * NIntegrate[f[x], {x, 0.001, 100}, PrecisionGoal -> 4];
    data1[pk] = Re[Z];
    data2[pk] = Im[Z]; pk = pk + 1, {p1, 1, 3, 0.5}];
data = Table[If[m = 1, data1[kk], data2[kk]], {kk, 5}, {m, 1, 2}];
gr1[n] = ListPlot[data, PlotStyle -> {PointSize[0.02], RGBColor[rr1, rr2, rr3]},
  AxesLabel -> {Re[z], Im[z]}, DisplayFunction -> Identity];
gr2[n] = ListPlot[data, AxesLabel -> {Re[z], Im[z]}, PlotJoined -> True,
  PlotStyle -> RGBColor[rr1, rr2, rr3], DisplayFunction -> Identity];  $\beta = \beta - 1$ ; pk = 1},
{n, 1, 3}];
Show[{gr1[1], gr2[1], gr1[2], gr2[2], gr1[3], gr2[3]},
  DisplayFunction -> $DisplayFunction]

```


Fig. Ap.9 (Section 4.2) “Mathematica” program for numerical calculation of the change in impedance given by formula (4.2.15).

```

Remove["Global`*"]
α = 0.0; β = 1; h = 0.05; μm = 5;
Do[
  {Do[{If[n == 1, {r1 = 1, r2 = 0, r3 = 0},
    If[n == 2, {r1 = 0, r2 = 1, r3 = 0}, {r1 = 0, r2 = 0, r3 = 1}]}];
  v[x_] :=  $\frac{\text{Sqrt}[\beta^2 + 4 * x^2]}{\alpha + \beta}$ ;
  sqrtb = Sqrt[-i * b];
  z0[x_] :=  $\frac{2 * \text{sqrtb}}{\alpha + \beta}$ ;
  D1[x_] :=  $\mu m * x + \frac{\beta}{2}$ ;
  D2[x_] :=  $\mu m * x - \frac{\beta}{2}$ ;
  Iz0[x_] := BesselI[v[x], z0[x]];
  Iz01[x_] := BesselI[v[x] + 1, z0[x]];
  Iz0pr[x_] := -Iz01[x] +  $\frac{v[x]}{z0[x]} * Iz0[x]$ ;
  skait[x_] := D2[x] * Iz0[x] - sqrtb * Iz0pr[x];
  sauc[x_] := D1[x] * Iz0[x] + sqrtb * Iz0pr[x];
  f[x_] :=  $\frac{1 - \text{Cos}[x]}{x} * \text{Exp}[-2 * x * h] * \frac{\text{skait}[x]}{\text{sauc}[x]}$ ;
  Z = i * NIntegrate[f[x], {x, 0, 100}, MaxRecursion → 100,
    WorkingPrecision → 8];
  data1[b] = Re[Z];
  data2[b] = Im[Z]}, {b, 1, 60}];
data = Table[If[m == 1, data1[k], data2[k]], {k, 1, 60}, {m, 1, 2}];
gr1[n] = ListPlot[data,
  PlotStyle → {PointSize[0.02], RGBColor[r1, r2, r3]},
  AxesLabel → {Re[z], Im[z]}, DisplayFunction → Identity];
gr2[n] = ListPlot[data, AxesLabel → {Re[z], Im[z]}, PlotJoined → True,
  PlotStyle → RGBColor[r1, r2, r3], DisplayFunction → Identity];
β = β + 1}, {n, 1, 3}];
Show[gr1[1], gr2[1], gr1[2], gr2[2], gr1[3], gr2[3],
  DisplayFunction → $DisplayFunction]

```

Fig. Ap.10 (Section 4.3) “Mathematica” program for numerical calculation of the change in impedance given by formula (4.3.30).

```

Remove["Global`*"]
rc = 0.05;
α = 0.0; β = 1; h = 0.05; d = 0.05; μ1 = 1; μ2 = 100; koef = 1.5;
Do[
  {Do[
    {If[n == 1, {r1 = 1, r2 = 0, r3 = 0},
      If[n == 2, {r1 = 0, r2 = 1, r3 = 0}, {r1 = 0, r2 = 0, r3 = 1}]];
    v[x_] :=  $\frac{\text{Sqrt}[\beta^2 + 4 * x^2]}{\alpha + \beta}$ ;
    sqrtb = Sqrt[-i * b];
    z0[x_] :=  $\frac{2 * \text{sqrtb}}{\alpha + \beta}$ ;
    z1[x_] := z0[x] * Exp[- $\frac{\alpha + \beta}{2} * d$ ];
    D1[x_] :=  $\frac{\beta}{2} * \mu2 * \text{Exp}[\beta * d] - \mu1 * \sqrt{x^2 + i * b * \text{koef}}$ ;
    D2[x_] :=  $\mu2 * \text{sqrtb} * \text{Exp}[\frac{\beta - \alpha}{2} * d]$ ;
    D3[x_] := 2 * x * μ1 + β;
    Iz0[x_] := BesselI[v[x], z0[x]]; Iz01[x_] := BesselI[v[x] + 1, z0[x]];
    Iz0pr[x_] := -Iz01[x] +  $\frac{v[x]}{z0[x]} * \text{Iz0}[x]$ ;
    Kz0[x_] := BesselK[v[x], z0[x]]; Kz01[x_] := BesselK[v[x] + 1, z0[x]];
    Kz0pr[x_] := -Kz01[x] +  $\frac{v[x]}{z0[x]} * \text{Kz0}[x]$ ;
    Iz1[x_] := BesselI[v[x], z1[x]]; Iz11[x_] := BesselI[v[x] + 1, z1[x]];
    Iz1pr[x_] := -Iz11[x] +  $\frac{v[x]}{z1[x]} * \text{Iz1}[x]$ ;
    Kz1[x_] := BesselK[v[x], z1[x]]; Kz11[x_] := BesselK[v[x] + 1, z1[x]];
    Kz1pr[x_] := -Kz11[x] +  $\frac{v[x]}{z1[x]} * \text{Kz1}[x]$ ;
    D4[x_] = Iz0[x] * Kz1[x] - Iz1[x] * Kz0[x];
    D5[x_] = Iz0pr[x] * Kz1[x] - Iz1[x] * Kz0pr[x];
    D6[x_] = Iz0[x] * Kz1pr[x] - Iz1pr[x] * Kz0[x];
    D7[x_] = Iz0pr[x] * Kz1pr[x] - Iz1pr[x] * Kz0pr[x];
    skait[x_] := μ1 * (D1[x] * D4[x] + D2[x] * D6[x]);
    sauc[x_] := D1[x] * D3[x] * D4[x] + 2 * sqrtb * D1[x] * D5[x] + D2[x] * D3[x] * D6[x] +
      2 * sqrtb * D2[x] * D7[x];
    f[x_] :=  $\left(\frac{\text{skait}[x]}{\text{sauc}[x]} - \frac{1}{4 * x}\right) * \text{Exp}[-2 * x * h] * (1 - \text{Cos}[x])$ ;
    Z = i * NIntegrate[f[x], {x, 0, 100}, MaxRecursion → 50, WorkingPrecision → 8];
    data1[b] = Re[Z];
    data2[b] = Im[Z], {b, 1, 10}];
  data = Table[If[m == 1, data1[k], data2[k]], {k, 10}, {m, 1, 2}];
  gr1[n] = ListPlot[data, PlotStyle → {PointSize[0.02], RGBColor[r1, r2, r3]},
    AxesLabel → {Re[z], Im[z]}, DisplayFunction → Identity];
  gr2[n] = ListPlot[data, AxesLabel → {Re[z], Im[z]}, PlotJoined → True,
    PlotStyle → RGBColor[r1, r2, r3], DisplayFunction → Identity]; β = β + 1,
{n, 1, 3}]; Show[gr1[1], gr2[1], gr1[2], gr2[2], gr1[3], gr2[3],
DisplayFunction → $DisplayFunction]

```

Fig. Ap.11 (Section 4.4) “Mathematica” program for numerical calculation of the integrated EMF given by formula (4.4.47).

```

Remove["Global`*"]
masT = {10-6, 10-5, 10-4, 5 * 10-4, 10-3, 5 * 10-3, 10-2, 5 * 10-2, 10-1};
masσ = {1 * 107, 5 * 107, 10 * 107};
μ = 2; h = 0.1;
Do[
  {σ = masσ[[n]]};
  Do[{If[n == 1, {rr1 = 1, rr2 = 0, rr3 = 0},
    If[n == 2, {rr1 = 0, rr2 = 1, rr3 = 0}, {rr1 = 0, rr2 = 0, rr3 = 1}]}];
  T = masT[[k]] // N;
  τ[x_] = 4 * π * 10-7 * μ * σ / x2;
  f[t_, τ_, μ_] = InverseLaplaceTransform[2 * μ / (μ + Sqrt[1 + s * τ]) - 1, s, t];
  f1[τ_, μ_, T_] = Integrate[f[t, τ, μ], {t, 0, T}];
  aa = NIntegrate[Exp[-2 * x * h] * (Cos[x] - 1) / x * (f1[τ[x], μ, T]), {x, 0, 2000}];
  data1[k] = T;
  data2[k] = aa}, {k, 1, 9}];
data = Table[If[m == 1, data1[kk], data2[kk]], {kk, 9}, {m, 1, 2}];
gr1[n] = ListPlot[data, PlotStyle → {PointSize[0.02], RGBColor[rr1, rr2, rr3]},
  AxesLabel → {t, V[t]}, DisplayFunction → Identity];
gr2[n] = ListPlot[data, AxesLabel → {T, V[T]}, PlotJoined → True,
  PlotStyle → RGBColor[rr1, rr2, rr3], DisplayFunction → Identity], {n, 1, 3}];
Show[{gr1[1], gr2[1], gr1[2], gr2[2], gr1[3], gr2[3]},
  DisplayFunction → $DisplayFunction]

```

Fig. Ap.12 (Section 5.1) “Mathematica” program for numerical calculation of the change in impedance given by formula (5.1.40).

```

Remove["Global`*"]
 $\alpha = 0.0; \beta = 2; h = 0.05; d = 0.1; ro = 5; \mu m = 5;$ 
Do[
  {Do[{If[n == 1, {r1 = 1, r2 = 0, r3 = 0},
    If[n == 2, {r1 = 0, r2 = 1, r3 = 0}, {r1 = 0, r2 = 0, r3 = 1}]}];
    v[x_, y_] :=  $\frac{\text{Sqrt}[\beta^2 + 4 * x^2]}{\alpha + \beta};$ 
    bsqrt[x_, y_] :=  $\text{Sqrt}[-i * (\delta 1^2 + ro * x * \text{Cos}[y])];$ 
    D1[x_, y_] :=  $\frac{\beta}{2} + \mu m * x;$ 
    D2[x_, y_] :=  $\frac{\beta}{2} - \mu m * x;$ 
    z0[x_, y_] :=  $\frac{2 * \text{bsqrt}[x, y]}{\alpha + \beta};$ 

    Jz0[x_, y_] :=  $\text{BesselJ}[v[x, y], z0[x, y]];
    Jz01[x_, y_] :=  $\text{BesselJ}[v[x, y] + 1, z0[x, y]];
    Jz0pr[x_, y_] :=  $-\text{Jz01}[x, y] + \frac{v[x, y]}{z0[x, y]} * \text{Jz0}[x, y];$ 

    skait[x_, y_] :=  $\text{D2}[x, y] * \text{Jz0}[x, y] + \text{bsqrt}[x, y] * \text{Jz0pr}[x, y];
    sauc[x_, y_] :=  $\text{D1}[x, y] * \text{Jz0}[x, y] + \text{bsqrt}[x, y] * \text{Jz0pr}[x, y];$ 

    f[x_, y_] :=  $\text{BesselJ}[1, x]^2 * \text{Exp}[-2 * x * h] * \frac{\text{skait}[x, y]}{\text{sauc}[x, y]}$ ;
    Z =  $-\frac{i}{2} * \text{NIntegrate}[f[x, y], \{x, 0.000, 100\}, \{y, 0.000, 2 * \pi\},$ 
      PrecisionGoal -> 4];
    data1[ $\delta 1$ ] =  $\text{Re}[Z];$ 
    data2[ $\delta 1$ ] =  $\text{Im}[Z];$ }, { $\delta 1$ , 1, 20}];
    data =  $\text{Table}[\text{If}[m == 1, \text{data1}[k], \text{data2}[k]], \{k, 1, 20\}, \{m, 1, 2\}];$ 
    gr1[n] =  $\text{ListPlot}[\text{data},$ 
      PlotStyle -> {PointSize[0.02], RGBColor[r1, r2, r3]},
      AxesLabel -> {Re[z], Im[z]}, DisplayFunction -> Identity];
    gr2[n] =  $\text{ListPlot}[\text{data}, \text{AxesLabel} -> \{\text{Re}[z], \text{Im}[z]\}, \text{PlotJoined} -> \text{True},$ 
      PlotStyle -> RGBColor[r1, r2, r3], DisplayFunction -> Identity];
    ro = ro + 5}, {n, 1, 3}];
Show[gr1[1], gr1[2], gr2[2], gr1[3], gr2[3],
  DisplayFunction -> $DisplayFunction]$$$ 
```

Fig. Ap.13 (Section 5.2) “Mathematica” program for numerical calculation of the change in impedance given by formula (5.2.31).

```

Remove["Global`*"]
α = 0.0; β = 1; h = 0.05; d = 0.2; kδ = 1.5; ro = 1; μm = 1; μ2 = 5;
Do[
  {Do[
    {If[n == 1, {r1 = 1, r2 = 0, r3 = 0}, If[n == 2, {r1 = 0, r2 = 1, r3 = 0}, {r1 = 0, r2 = 0, r3 = 1}]}];

    δ2 = kδ * δ1;
    v[x_, y_] :=  $\frac{\text{Sqrt}[\beta^2 + 4 * x^2]}{\alpha + \beta}$ ;
    bsqrt[x_, y_] :=  $\text{Sqrt}[-i * (\delta1^2 + ro * x * \text{Cos}[y])]$ ;
    q2[x_, y_] :=  $\text{Sqrt}[x^2 + i * \delta2^2]$ ;
    z0[x_, y_] :=  $\frac{2 * \text{bsqrt}[x, y]}{\alpha + \beta}$ ;

    z1[x_, y_] :=  $z0[x, y] * \text{Exp}\left[\frac{-(\alpha + \beta) * d}{2}\right]$ ;

    Jz0[x_, y_] :=  $\text{BesselJ}[v[x, y], z0[x, y]]$ ;
    Jz01[x_, y_] :=  $\text{BesselJ}[v[x, y] + 1, z0[x, y]]$ ;
    Jz0pr[x_, y_] :=  $-\text{Jz01}[x, y] + \frac{v[x, y]}{z0[x, y]} * \text{Jz0}[x, y]$ ;
    Jz1[x_, y_] :=  $\text{BesselJ}[v[x, y], z1[x, y]]$ ;
    Jz11[x_, y_] :=  $\text{BesselJ}[v[x, y] + 1, z1[x, y]]$ ;
    Jz1pr[x_, y_] :=  $-\text{Jz11}[x, y] + \frac{v[x, y]}{z1[x, y]} * \text{Jz1}[x, y]$ ;
    Yz0[x_, y_] :=  $\text{BesselY}[v[x, y], z0[x, y]]$ ;
    Yz01[x_, y_] :=  $\text{BesselY}[v[x, y] + 1, z0[x, y]]$ ;
    Yz0pr[x_, y_] :=  $-\text{Yz01}[x, y] + \frac{v[x, y]}{z0[x, y]} * \text{Yz0}[x, y]$ ;
    Yz1[x_, y_] :=  $\text{BesselY}[v[x, y], z1[x, y]]$ ;
    Yz11[x_, y_] :=  $\text{BesselY}[v[x, y] + 1, z1[x, y]]$ ;
    Yz1pr[x_, y_] :=  $-\text{Yz11}[x, y] + \frac{v[x, y]}{z1[x, y]} * \text{Yz1}[x, y]$ ;
    D1[x_, y_] :=  $\text{Jz0}[x, y] * \text{Yz1}[x, y] - \text{Jz1}[x, y] * \text{Yz0}[x, y]$ ;
    D2[x_, y_] :=  $\text{Jz0pr}[x, y] * \text{Yz1}[x, y] - \text{Jz1}[x, y] * \text{Yz0pr}[x, y]$ ;
    D3[x_, y_] :=  $\text{Jz0}[x, y] * \text{Yz1pr}[x, y] - \text{Jz1pr}[x, y] * \text{Yz0}[x, y]$ ;
    D4[x_, y_] :=  $\text{Jz0pr}[x, y] * \text{Yz1pr}[x, y] - \text{Jz1pr}[x, y] * \text{Yz0pr}[x, y]$ ;
    D5[x_, y_] :=  $\frac{\mu2 * \beta}{2} * \text{Exp}[\beta * d] - \mu m * q2[x, y]$ ;
    D6[x_, y_] :=  $\mu m * x + \frac{\beta}{2}$ ;
    D7[x_, y_] :=  $\mu2 * \text{bsqrt}[x, y] * \text{Exp}\left[\frac{(\beta - \alpha) * d}{2}\right]$ ;
    skait[x_, y_] :=  $2 * x * \mu m * (D5[x, y] * D1[x, y] - D7[x, y] * D3[x, y])$ ;

    sauc[x_, y_] :=  $D5[x, y] * (D6[x, y] * D1[x, y] + \text{bsqrt}[x, y] * D2[x, y]) +$ 
       $D7[x, y] * (D6[x, y] * D3[x, y] + \text{bsqrt}[x, y] * D4[x, y])$ ;
    f[x_, y_] :=  $\text{BesselJ}[1, x]^2 * \text{Exp}[-2 * x * h] * \left(1 - \frac{\text{skait}[x, y]}{\text{sauc}[x, y]}\right)$ ;

    Z =  $-\frac{i}{2} * \text{NIntegrate}[f[x, y], \{x, 0.001, 50\}, \{y, 0.000, 2 * \pi\}]$ ;
    data1[δ1] =  $\text{Re}[Z]$ ;
    data2[δ1] =  $\text{Im}[Z]$ ;
    data =  $\text{Table}[\text{If}[m == 1, \text{data1}[k], \text{data2}[k]], \{k, 1, 10\}, \{m, 1, 2\}]$ ;
    gr1[n] =  $\text{ListPlot}[\text{data}, \text{PlotStyle} \rightarrow \{\text{PointSize}[0.02], \text{RGBColor}[r1, r2, r3]\},$ 
       $\text{AxesLabel} \rightarrow \{\text{Re}[z], \text{Im}[z]\}, \text{DisplayFunction} \rightarrow \text{Identity}]$ ;
    gr2[n] =  $\text{ListPlot}[\text{data}, \text{AxesLabel} \rightarrow \{\text{Re}[z], \text{Im}[z]\}, \text{PlotJoined} \rightarrow \text{True},$ 
       $\text{PlotStyle} \rightarrow \text{RGBColor}[r1, r2, r3], \text{DisplayFunction} \rightarrow \text{Identity}]$ ; ro = ro + 2}, {n, 1, 3}];
    Show[gr1[1], gr2[1], gr1[2], gr2[2], gr1[3], gr2[3],  $\text{DisplayFunction} \rightarrow \text{\$DisplayFunction}$ ]
  }
]

```

Fig. Ap.14 (Section 6.1) “Mathematica” program for numerical calculation of the change in impedance given by formula (6.1.77).

```

Remove["Global`*"]
mm = 10-3; MSm = 106; kHz = 103;
r0 = 4.5 * mm;
h = 0.2 * mm;
σ = 9.6 * MSm; μ0 = 4 * π * 10-7; ff = 2269.9059; nConst = 100;
ω = 2 * π * ff; c = 19.75 / 2 * mm; r2 = 6 * mm; r1 = 3 * mm; b = 10 * r2;
z2 = 0.39 * mm; z1 = 0.06 * mm;
d = 1.93 * mm;

pdata = {67.00875183470953` + 1.2984687655904985` ĩ, 125.7226613564663` + 3.313199300880373` ĩ,
185.26797461012538` + 5.552151080078192` ĩ, 245.4513397389452` + 7.886303003911224` ĩ,
306.1811733439874` + 10.160249304197377` ĩ, 367.2191824109314` + 12.080764736395478` ĩ,
427.87542043912464` + 13.649336114655455` ĩ, 487.52256884428806` + 15.77775453323662` ĩ,
546.0575269481018` + 20.309151366631184` ĩ, 599.3933177946013` + 33.04022627046605` ĩ,
371.161733985946` + 184.13053693428637` ĩ, 632.994932851322` + 31.1555629729415` ĩ,
686.8240879632289` + 19.44069396726628` ĩ, 743.0023811242852` + 15.560192051270032` ĩ,
798.357863313516` + 14.651451579730805` ĩ, 852.3397533788766` + 15.591457610303891` ĩ,
904.2069105583389` + 17.01615578586213` ĩ, 954.6318525239519` + 16.64506260361474` ĩ,
1006.1008882269491` + 14.454024526942483` ĩ, 1059.0183179232567` + 12.483338703200928` ĩ,
1112.3684283128523` + 11.602421080194741` ĩ, 1165.3787366618303` + 11.679764620140228` ĩ,
1217.6585972232187` + 12.05560530821854` ĩ, 1269.4123119845153` + 11.891456260181204` ĩ,
1321.351079404242` + 10.995238689769918` ĩ, 1373.8252323146492` + 9.967652182283146` ĩ,
1426.593992255994` + 9.36820680251447` ĩ, 1479.2767129803842` + 9.291283334691602` ĩ,
1531.6382027455247` + 9.428909210493893` ĩ, 1583.7345483415509` + 9.337124176993564` ĩ,
1635.8676959906277` + 8.851834990987964` ĩ, 1688.2474561075826` + 8.227269622881996` ĩ,
1740.8078145995062` + 7.806050502013001` ĩ, 1793.3514629420179` + 7.704072754100186` ĩ,
1845.7309961023407` + 7.762613161322348` ĩ, 1897.9559742082515` + 7.707171939284005` ĩ,
1950.179816990255` + 7.4064443932604` ĩ, 2002.5338491536913` + 6.987598578296566` ĩ,
2055.003597153803` + 6.676178499558322` ĩ, 2107.4781906770722` + 6.577008958534012` ĩ,
2159.860450418002` + 6.602805798281011` ĩ, 2212.144283023319` + 6.568150260488544` ĩ,
2264.41611977299` + 6.36675392146837` ĩ, 2316.762890578386` + 6.06720252354737` ĩ,
2369.187581341504` + 5.827102354639212` ĩ, 2421.6238646500615` + 5.736616095862516` ĩ,
2474.0047525045966` + 5.7466399884555655` ĩ, 2526.3190643639173` + 5.724660173565664` ĩ,
2578.618788764458` + 5.582926146972814` ĩ, 2630.964165140325` + 5.358959336062282` ĩ,
2683.36422037534` + 5.167741449353327` ĩ, 2735.7774522855866` + 5.086367189604264` ĩ,
2788.155982455466` + 5.087887730298888` ĩ, 2840.4874766161543` + 5.073932090767937` ĩ,
2892.804576916251` + 4.970759244690764` ĩ, 2945.1505478417525` + 4.797845025743435` ĩ,
2997.536164915454` + 4.641685969794301` ĩ, 3049.9346475338543` + 4.568518774767272` ĩ,
3102.310773531464` + 4.565037065899655` ĩ, 3154.652566568355` + 4.5562711908908575` ĩ,
3206.9810883827386` + 4.479355980835865` ĩ, 3259.3282918463915` + 4.3426200071251095` ĩ,
3311.704993147244` + 4.212560296494019` ĩ, 3364.093575748427` + 4.146485966997286` ĩ,
3416.4675262592045` + 4.139852368583515` ĩ, 3468.815781357907` + 4.134472852010781` ĩ,
3521.1521336231694` + 4.0761437759461785` ĩ, 3573.5007047974364` + 3.9660168956784103` ĩ};

kol = 10 + 24 + 34;
<< LinearAlgebra`MatrixManipulation`
<< NumericalMath`BesselZeros`
λdata =  $\frac{1}{b}$  * BesselJZeros[1, kol];

```

```

nm = 0;
Do[ { λ = λdata[ [nj] ];
  λc = λ * c; j1λc = BesselJ[1, λc]; (* J1 (λj c) *)
  j2λc = BesselJ[2, λc]; (* J2 (λj c) *)
  j0λb = BesselJ[0, λb]; (* J0 (λj b) *)

  j1λr0 = BesselJ[1, λ * r0]; (* J1 (λj r0) *)

  λb = λ * b; j1λb = BesselJ[1, λb]; (* J1 (λj b) *)
  j0λb = BesselJ[0, λb]; (* J0 (λj b) *)
  j2λb = BesselJ[2, λb]; (* J2 (λj b) *)

  p = pdata[ [ni] ]; (* p saknes no programmas *)
  qvert = Sqrt[p^2 - μ * σ * μ0];

  (* pc *)
  pc1 = p * c; j1pc = BesselJ[1, pc1]; (* J1 (pi c) *)
  j2pc = BesselJ[2, pc1]; (* J2 (pi c) *)
  y1pc = BesselY[1, pc1]; (* Y1 (pi c) *)
  y2pc = BesselY[2, pc1]; (* Y2 (pi c) *)
  (* pb *)
  pb1 = p * b; j1pb = BesselJ[1, pb1]; (* J1 (pi b) *)
  j2pb = BesselJ[2, pb1]; (* J2 (pi b) *)
  y1pb = BesselY[1, pb1]; (* Y1 (pi b) *)
  y2pb = BesselY[2, pb1]; (* Y2 (pi b) *)

  (* qc *)
  qc1 = qvert * c; j1qc = BesselJ[1, qc1]; (* J1 (qi c) *)
  j2qc = BesselJ[2, qc1]; (* J2 (qi c) *)
  y1qc = BesselY[1, qc1]; (* Y1 (qi c) *)
  y2qc = BesselY[2, qc1]; (* Y2 (qi c) *)
  (* qb *)
  qb1 = qvert * b; j1qb = BesselJ[1, qb1]; (* J1 (qi b) *)
  j2qb = BesselJ[2, qb1]; (* J2 (qi b) *)
  y1qb = BesselY[1, qb1]; (* Y1 (qi b) *)
  y2qb = BesselY[2, qb1]; (* Y2 (qi b) *)

  (* T1 (pi c) = J1 (pi c) * Y1 (pi b) - J1 (pi b) * Y1 (pi c) *)
  T1 = j1pc * y1pb - j1pb * y1pc;
  aa1 =  $\frac{c}{\lambda^2 - qvert^2} * (\lambda * j2λc * j1qc - qvert * j1λc * j2qc)$ ; (* a21ij *)
  aa2 =  $\frac{1}{\lambda^2 - p^2} * (\lambda c * j2λc * (j1pb * y1pc - j1pc * y1pb) + pc1 * j1λc * (j2pc * y1pb - j1pb * y2pc))$ ;
  (* a22ij *)
  a2 = T1 * aa1 + j1qc * aa2; (* a2ij *)
  nm = nm + 1;
  a2data[ [nm] ] = a2;
  A1[ [nm] ] = (λ + p) * a2; A2[ [nm] ] = (λ - p) * a2;
  A3[ [nm] ] = (λ - p) * Exp[-p * d] * a2; A4[ [nm] ] = (λ + p) * Exp[p * d] * a2;
  znam = b^2 * j0λb^2;
  B1[ [nm] ] =  $\frac{2 * a2}{znam}$ ;
  EJ[ [nj] ] = j1λr0 * Exp[-λ * h];

  Bvert[ [nj] ] = μ0 * r0 * EJ[ [nj] ];
  Dvert[ [nj] ] =  $\frac{Bvert[ [nj] ]}{\lambda * znam}$ ;
}, {nj, 1, kol}, {ni, 1, kol} ];

```

```

A11 = Table[A1[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
A12 = Table[A2[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
A21 = Table[A3[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
A22 = Table[A4[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
B11 = Table[B1[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
EEJ = Table[EJ[k], {k, 1, kol}];

AA = BlockMatrix[{{A11, A12}, {A21, A22}}];
BB = Table[If[i ≤ kol, Bvert[i], 0], {i, 1, 2 * kol}];
sol = LinearSolve[AA, BB];

Do[{D6[k] = sol[[k]]; D8[k] = sol[[kol + k]]}, {k, 1, kol}];
DD6 = Table[D6[k], {k, 1, kol}];
DD8 = Table[D8[k], {k, 1, kol}];
DDvert = Table[Dvert[k], {k, 1, kol}];
DD2 = B11.(DD6 + DD8) - DDvert;

A0ind = DD2.EEJ;
Zind = 2 * π * r0 * i * ω * A0ind;
Print["kol = ", kol, " ff = ", ff, " A0ind = ", A0ind, " Zind = ",
Zind]

kol = 68 ff = 2269.91 A0ind = -3.16713×10-8 - 7.67604×10-8 i Zind = 0.000030954 - 0.0000127716 i

```


Fig. Ap.15 (Section 6.2) “Mathematica” program for numerical calculation of the change in impedance given by formula (6.2.14).

```

Remove["Global`*"]
mm = 10-3; MSm = 106; kHz = 103;
σ = 9.6 * MSm; μ0 = 4 * π * 10-7; ff = 2269.9059; nConst = 100;
ω = 2 * π * ff; c = 19.75 / 2 * mm; r2 = 6 * mm; r1 = 3 * mm; b = 10 * r2;
z2 = 0.39 * mm; z1 = 0.06 * mm;
d = 1.93 * mm;

pdata = {67.00875183470953` + 1.2984687655904985` i, 125.7226613564663` + 3.313199300880373` i,
185.26797461012538` + 5.552151080078192` i, 245.4513397389452` + 7.886303003911224` i,
306.1811733439874` + 10.160249304197377` i, 367.2191824109314` + 12.080764736395478` i,
427.87542043912464` + 13.649336114655455` i, 487.52256884428806` + 15.77775453323662` i,
546.0575269481018` + 20.309151366631184` i, 599.3933177946013` + 33.04022627046605` i,
371.161733985946` + 184.13053693428637` i, 632.994932851322` + 31.1555629729415` i,
686.8240879632289` + 19.44069396726628` i, 743.0023811242852` + 15.560192051270032` i,
798.357863313516` + 14.651451579730805` i, 852.3397533788766` + 15.591457610303891` i,
904.2069105583389` + 17.01615578586213` i, 954.6318525239519` + 16.64506260361474` i,
1006.1008882269491` + 14.454024526942483` i, 1059.0183179232567` + 12.483338703200928` i,
1112.3684283128523` + 11.602421080194741` i, 1165.3787366618303` + 11.679764620140228` i,
1217.6585972232187` + 12.05560530821854` i, 1269.4123119845153` + 11.891456260181204` i,
1321.351079404242` + 10.995238689769918` i, 1373.8252323146492` + 9.967652182283146` i,
1426.593992255994` + 9.36820680251447` i, 1479.2767129803842` + 9.291283334691602` i,
1531.6382027455247` + 9.428909210493893` i, 1583.7345483415509` + 9.337124176993564` i,
1635.8676959906277` + 8.851834990987964` i, 1688.2474561075826` + 8.227269622881996` i,
1740.8078145995062` + 7.806050502013001` i, 1793.3514629420179` + 7.704072754100186` i,
1845.7309961023407` + 7.762613161322348` i, 1897.9559742082515` + 7.707171939284005` i,
1950.179816990255` + 7.4064443932604` i, 2002.5338491536913` + 6.987598578296566` i,
2055.003597153803` + 6.676178499558322` i, 2107.4781906770722` + 6.577008958534012` i,
2159.860450418002` + 6.602805798281011` i, 2212.144283023319` + 6.568150260488544` i,
2264.41611977299` + 6.36675392146837` i, 2316.762890578386` + 6.06720252354737` i,
2369.187581341504` + 5.827102354639212` i, 2421.6238646500615` + 5.736616095862516` i,
2474.0047525045966` + 5.7466399884555655` i, 2526.3190643639173` + 5.724660173565664` i,
2578.618788764458` + 5.582926146972814` i, 2630.964165140325` + 5.358959336062282` i,
2683.36422037534` + 5.167741449353327` i, 2735.7774522855866` + 5.086367189604264` i,
2788.155982455466` + 5.087887730298888` i, 2840.4874766161543` + 5.073932090767937` i,
2892.804576916251` + 4.970759244690764` i, 2945.1505478417525` + 4.797845025743435` i,
2997.536164915454` + 4.641685969794301` i, 3049.9346475338543` + 4.568518774767272` i,
3102.310773531464` + 4.565037065899655` i, 3154.652566568355` + 4.5562711908908575` i,
3206.9810883827386` + 4.479355980835865` i, 3259.3282918463915` + 4.3426200071251095` i,
3311.704993147244` + 4.212560296494019` i, 3364.093575748427` + 4.146485966997286` i,
3416.4675262592045` + 4.139852368583515` i, 3468.815781357907` + 4.134472852010781` i,
3521.1521336231694` + 4.0761437759461785` i, 3573.5007047974364` + 3.9660168956784103` i};

kol = 68;
<< LinearAlgebra`MatrixManipulation`
<< NumericalMath`BesselZeros`
λdata =  $\frac{1}{b}$  * BesselJZeros[1, kol];

```

```

nm = 0;
Do[{λ = λdata[[nj]];
  (*      λc      *)
  λc = λ*c; λb = λ*b;
  j1λc = BesselJ[1, λc];          (* J1(λj c) *)
  j2λc = BesselJ[2, λc];          (* J2(λj c) *)
  (*      λb      *)
  j1λb = BesselJ[1, λb];          (* J1(λj b) *)
  j0λb = BesselJ[0, λb];          (* J0(λj b) *)
  j2λb = BesselJ[2, λb];          (* J2(λj b) *)

  p = pdata[[ni]];                (* p saknes no programmas *)
  qvert = Sqrt[p^2 - ĩ*o*σ*μ0];

  (*      pc      *)
  pc1 = p*c; j1pc = BesselJ[1, pc1]; (* J1(pi c) *)
  j2pc = BesselJ[2, pc1];          (* J2(pi c) *)
  y1pc = BesselY[1, pc1];          (* Y1(pi c) *)
  y2pc = BesselY[2, pc1];          (* Y2(pi c) *)
  (*      pb      *)
  pb1 = p*b; j1pb = BesselJ[1, pb1]; (* J1(pi b) *)
  j2pb = BesselJ[2, pb1];          (* J2(pi b) *)
  y1pb = BesselY[1, pb1];          (* Y1(pi b) *)
  y2pb = BesselY[2, pb1];          (* Y2(pi b) *)
  (*      qc      *)
  qc1 = qvert*c; j1qc = BesselJ[1, qc1]; (* J1(qi c) *)
  j2qc = BesselJ[2, qc1];          (* J2(qi c) *)
  y1qc = BesselY[1, qc1];          (* Y1(qi c) *)
  y2qc = BesselY[2, qc1];          (* Y2(qi c) *)
  (*      qb      *)
  qb1 = qvert*b; j1qb = BesselJ[1, qb1]; (* J1(qi b) *)
  j2qb = BesselJ[2, qb1];          (* J2(qi b) *)
  y1qb = BesselY[1, qb1];          (* Y1(qi b) *)
  y2qb = BesselY[2, qb1];          (* Y2(qi b) *)

  (*      T1(pi c) = J1(pi c)*Y1(pi b) - J1(pi b)*Y1(pi c) *)
  T1 = j1pc*y1pb - j1pb*y1pc;
  aa1 =  $\frac{c}{\lambda^2 - qvert^2} * (\lambda * j2\lambda c * j1qc - qvert * j1\lambda c * j2qc);$  (* a21ij *)
  aa2 =  $\frac{1}{\lambda^2 - p^2} * (\lambda c * j2\lambda c * (j1pb * y1pc - j1pc * y1pb) + pc1 * j1\lambda c * (j2pc * y1pb - j1pb * y2pc));$  (* a22ij *)
  a2 = T1*aa1 + j1qc*aa2; (* a2ij *)

  nm = nm + 1;
  a2data[nm] = a2;
  A1[nm] = (λ + p) * a2;          A2[nm] = (λ - p) * a2;
  A3[nm] = (λ - p) * Exp[-p*d] * a2;  A4[nm] = (λ + p) * Exp[p*d] * a2;
  znam = b^2 * j0λb^2;
  B1[nm] =  $\frac{2 * a2}{znam};$           Cdiag[nj] =  $\frac{1}{\lambda * znam}$ 
}, {nj, 1, kol}, {ni, 1, kol}];

```

```

A11 = Table[A1[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
A12 = Table[A2[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
A21 = Table[A3[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
A22 = Table[A4[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
B11 = Table[B1[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
C0diag = Table[Cdiag[k], {k, 1, kol}];
C1diag = DiagonalMatrix[C0diag];
EE = IdentityMatrix[kol];

A22inv = Inverse[A22];
Y1 = A11 - A12.A22inv.A21;
Y1inv = Inverse[Y1];
Y = B11.(EE - A22inv.A21).Y1inv - C1diag;

Do[{λi = λdata[[i]]; mnoz1 =  $\frac{\text{Exp}[-\lambda_i * z1] - \text{Exp}[-\lambda_i * z2]}{\lambda_i^3}$  * (gint[λi * r2] - gint[λi * r1]);
    bb[i] =  $\frac{\mu0 * nConst * mnoz1}{(r2 - r1) * (z2 - z1)}$ ; fm[i] = mnoz1}, {i, 1, kol}];
bb1 = Table[bb[k], {k, 1, kol}];
fm1 = Table[fm[k], {k, 1, kol}];
a1nd = fm1.(Y.bb1);
z1nd =  $\frac{2 * \pi * i * \omega * nConst}{(r2 - r1) * (z2 - z1)}$  * a1nd ;
Print[" kol = ", kol, " ff = ", ff, " z1nd = ", z1nd]

kol = 68 ff = 2269.91 z1nd = 0.281031 - 0.122532 i

```

Fig. Ap.16 (Section 6.3) “Mathematica” program for numerical calculation of the first five eigenvalues p_i .

```

Remove["Global`*"]
prog[x1_, x2_, y1_, y2_] := Module[{xx1 = x1, xx2 = x2, yy1 = y1, yy2 = y2, mm = 10-3,
  MSm = 106, kHz = 103},
  σ = 9.6 * MSm; μ0 = 4 * π * 10-7; ff = 1 * kHz;
  ω = 2 * π * ff; c = 19.75 / 2 * mm; r2 = 10 * mm; b = 10 * r2;
  p[x_, y_] := x + i * y;
  pc[x_, y_] := p[x, y] * c;
  pb[x_, y_] := p[x, y] * b;
  qc[x_, y_] := √((x + i * y)2 - i * ω * σ * μ0);
  qc[x_, y_] := qc[x, y] * c;
  j0q[x_, y_] := BesselJ[0, qc[x, y]];
  j1q[x_, y_] := BesselJ[1, qc[x, y]];
  j1qpr1[x_, y_] := j0q[x, y] -  $\frac{1}{qc[x, y]}$  * j1q[x, y];
  j1qpr2[x_, y_] :=  $\left(\frac{1}{(qc[x, y])^2} - 1\right)$  * j1q[x, y] -  $\frac{1}{qc[x, y]}$  * j1qpr1[x, y];
  j0p[x_, y_] := BesselJ[0, pc[x, y]];
  j1p[x_, y_] := BesselJ[1, pc[x, y]];
  y0p[x_, y_] := BesselY[0, pc[x, y]];
  y1p[x_, y_] := BesselY[1, pc[x, y]];
  j1ppr1[x_, y_] := j0p[x, y] -  $\frac{1}{pc[x, y]}$  * j1p[x, y];
  y1ppr1[x_, y_] := y0p[x, y] -  $\frac{1}{pc[x, y]}$  * y1p[x, y];
  j0b[x_, y_] := BesselJ[0, pb[x, y]];
  j1b[x_, y_] := BesselJ[1, pb[x, y]];
  y0b[x_, y_] := BesselY[0, pb[x, y]];
  y1b[x_, y_] := BesselY[1, pb[x, y]];
  j1bpr1[x_, y_] := j0b[x, y] -  $\frac{1}{pb[x, y]}$  * j1b[x, y];
  y1bpr1[x_, y_] := y0b[x, y] -  $\frac{1}{pb[x, y]}$  * y1b[x, y];
  j1ppr2[x_, y_] :=  $\left(\frac{1}{(pc[x, y])^2} - 1\right)$  * j1p[x, y] -  $\frac{1}{pc[x, y]}$  * j1ppr1[x, y];
  y1ppr2[x_, y_] :=  $\left(\frac{1}{(pc[x, y])^2} - 1\right)$  * y1p[x, y] -  $\frac{1}{pc[x, y]}$  * y1ppr1[x, y];
  f1[x_, y_] := p[x, y] * j1q[x, y] * (j1ppr1[x, y] * y1b[x, y] - j1b[x, y] * y1ppr1[x, y])
    - q[x, y] * j1qpr1[x, y] * (j1p[x, y] * y1b[x, y] - j1b[x, y] * y1p[x, y]);
  f2[x_, y_] := j1q[x, y] * (j1ppr1[x, y] * y1b[x, y] - j1b[x, y] * y1ppr1[x, y]) +
    p[x, y] *  $\frac{P[x, y]}{q[x, y]}$  * c * j1qpr1[x, y] * (j1ppr1[x, y] * y1b[x, y]
    - j1b[x, y] * y1ppr1[x, y]) + p[x, y] * j1q[x, y] * (c * j1ppr2[x, y] * y1b[x, y]
    + b * j1ppr1[x, y] * y1bpr1[x, y]
    - b * j1bpr1[x, y] * y1ppr1[x, y] - c * j1b[x, y] * y1ppr2[x, y])
    -  $\frac{p[x, y]}{q[x, y]}$  * j1qpr1[x, y] * (j1p[x, y] * y1b[x, y] - j1b[x, y] * y1p[x, y])
    - p[x, y] * c * j1qpr2[x, y] * (j1p[x, y] * y1b[x, y] - j1b[x, y] * y1p[x, y]) -
    q[x, y] * j1qpr1[x, y] *
    (c * j1ppr1[x, y] * y1b[x, y] + b * j1p[x, y] * y1bpr1[x, y]
    - b * j1bpr1[x, y] * y1p[x, y] - c * j1b[x, y] * y1ppr1[x, y]);
  f[x_, y_] :=  $\frac{f2[x, y]}{f1[x, y]}$ ;
  a1 = NIntegrate[f[xx, yy1] - f[xx, yy2], {xx, xx1, xx2}, MaxRecursion -> 30];
  a2 = NIntegrate[f[xx2, yy] - f[xx1, yy], {yy, yy1, yy2}, MaxRecursion -> 30];
  int0 =  $\frac{1}{2 * \pi * i}$  * (a1 + i * a2); nrooll = Re[int0];
  a3 = NIntegrate[(xx + i * yy1) * f[xx, yy1] - (xx + i * yy2) * f[xx, yy2],
    {xx, xx1, xx2}, MaxRecursion -> 30];
  a4 = NIntegrate[(xx2 + i * yy) * f[xx2, yy] - (xx1 + i * yy) * f[xx1, yy],
    {yy, yy1, yy2}, MaxRecursion -> 30];
  z =  $\frac{1}{2 * \pi * i}$  * (a3 + i * a4)

```

```

av = 0.001; bv = 200; ch = 0.001; dh = 200; n1 = 10; n2 = 4;
kol = 0;

ysak1 = ch; ysak = ch;
ybeig = dh; nvert = n2;
hvert1 =  $\frac{ybeig - ysak}{nvert}$ ;

Do[{xsak1 = av; xsak = av; xbeig = bv; nhorz = n1; hhorz =  $\frac{xbeig - xsak}{nhorz}$ ; hvert = hvert1;
Do[
{x1 = xsak; x2 = x1 + hhorz; y1 = ysak; y2 = y1 + hvert;
prog[x1, x2, y1, y2];
If[nrcoll < 0.5, xsak = x2,
If[nrcoll < 1.5, {
kol = kol + 1;
data1[kol] = Re[Z];
data2[kol] = Im[Z]; xsak = x2,
{ny = 1; hvert =  $\frac{1}{2}$  * hvert;
While[ny == 1, {y1 = ysak; y2 = y1 + hvert;
prog[x1, x2, y1, y2];
If[nrcoll < 0.5, ysak = y2,
If[nrcoll < 1.5,
{ Do[{y1 = y1; y2 = y1 + hvert; prog[x1, x2, y1, y2];
kol = kol + 1;
data1[kol] = Re[Z];
data2[kol] = Im[Z]; y1 = y2,
{k, 1, 2}];
ysak = ysak1; hvert = ybeig; xsak = x2; ny = 2,
{(* Print[" "] *) hvert =  $\frac{1}{2}$  * hvert}}]}]}],
{nh, 1, nhorz}; ysak = ysak1 + nv * hvert1}, {nv, 1, nvert}]]
data = Table[If[m == 1, data1[k], data2[k]], {k, 1, kol}, {m, 1, 2}];
Print[data];
gr1[1] = ListPlot[data, PlotStyle -> {PointSize[0.02], RGBColor[1, 0, 0]}, AxesLabel -> {Re[z], Im[z]}, DisplayFunction -> Identity];
gr2[1] = ListPlot[data, AxesLabel -> {Re[z], Im[z]}, PlotJoined -> True, PlotStyle -> RGBColor[1, 0, 0], DisplayFunction -> Identity];
Show[gr1[1], gr2[1], DisplayFunction -> $DisplayFunction]

{{38.8055, 0.400362}, {71.6711, 1.16421}, {104.675, 2.15072}, {137.882, 3.30863}, {171.276, 4.661}}

```

Fig. Ap.17 (Section 6.4) “Mathematica” program for numerical calculation of the change in impedance given by formula (6.4.66).

```

Remove["Global`*"]
mm = 10-3; MSm = 106; kHz = 103;
ff = 7000.0; ω = 2 * π * ff;
σ = 3 * MSm;
μ0 = 4 * π * 10-7;
c = 2.2 * mm;
r0 = 4.5 * mm;
h = 1.4 * mm;
d1 = 0.7 * mm; d2 = 0.3 * mm;
d3 = d1 + d2;
b = 10 * 5.5 * mm;

pdata = {549.096978208783` + 149.52882915593767` ĩ, 499.18441867917574` + 164.8928681563969` ĩ,
  452.3539478271909` + 182.34844874431516` ĩ, 409.8019254811801` + 201.62650986187526` ĩ,
  372.80299587630515` + 221.92705542132163` ĩ, 318.9432124824753` + 259.8039780539326` ĩ,
  342.3702713381244` + 241.87990331798355` ĩ, 302.367867070953` + 274.1384122330393` ĩ,
  292.17279133053717` + 283.7441814738398` ĩ, 1329.2329935083806` + 59.51820672620454` ĩ,
  1386.176899830915` + 57.07846853412605` ĩ, 1443.1521694836172` + 54.853265444223474` ĩ,
  1500.153708349577` + 52.81465732097078` ĩ, 1557.1770768284669` + 50.93820547868496` ĩ,
  1614.2183861042176` + 49.20258902407101` ĩ, 1671.2742479303836` + 47.58928827419457` ĩ,
  1728.341748001099` + 46.08231630970696` ĩ, 1785.4184238706928` + 44.66798522501355` ĩ,
  932.2636597156051` + 85.99535053116698` ĩ, 988.6895165784267` + 80.82599871742124` ĩ,
  1045.2493344562758` + 76.23235039498023` ĩ, 1101.9120088964116` + 72.13507861242243` ĩ,
  1158.6555874522467` + 68.46764351428901` ĩ, 1215.4642778520304` + 65.1735684868936` ĩ,
  1272.3263995768627` + 62.2044006471691` ĩ, 709.1578128269097` + 114.68598773045271` ĩ,
  764.3464128585051` + 106.02330603799754` ĩ, 820.0124599371625` + 98.4645794170425` ĩ,
  876.0164075402721` + 91.83606613094479` ĩ, 601.1462479928093` + 136.1905499831746` ĩ,
  654.654257529524` + 124.6613429691535` ĩ, 2013.7816227969483` + 39.73243201692545` ĩ,
  2070.8805961764897` + 38.64248418085895` ĩ, 2127.9813206239915` + 37.60034869307359` ĩ,
  2185.0833358697837` + 36.60304578626898` ĩ, 2242.186295088276` + 35.64838590298651` ĩ,
  2299.289943418036` + 34.73478815649102` ĩ, 2356.394104688071` + 33.861106702647305` ĩ,
  2413.4986737077165` + 33.02646934712094` ĩ, 2470.603610900386` + 32.230132512338336` ĩ,
  2527.7089364020503` + 31.471356300511875` ĩ, 2584.814721508555` + 30.749302700264685` ĩ,
  2641.9210765357175` + 30.06295907630328` ĩ, 2699.0281353790883` + 29.41108802705787` ĩ,
  2756.1360381229124` + 28.79220339811314` ĩ, 2813.2449135969177` + 28.204571042901538` ĩ,
  2870.354864075254` + 27.64623162054064` ĩ, 2927.465953845413` + 27.115041793491308` ĩ,
  2984.578202890122` + 26.60872935297439` ĩ, 3041.691585891972` + 26.1249574890313` ĩ,
  3098.806036144899` + 25.661393348013625` ĩ, 3155.9214531653906` + 25.215776285384656` ĩ,
  3213.0377125028467` + 24.785981697292794` ĩ, 3270.154676197324` + 24.370076986812528` ĩ,
  3327.2722024755867` + 23.966366889370978` ĩ, 3384.390153780836` + 23.573426233681722` ĩ,
  3441.508402647483` + 23.190118931945594` ĩ, 3498.6268354166996` + 22.81560278488201` ĩ,
  3555.74535419817` + 22.449320508855575` ĩ, 1842.50223700856` + 43.33469666412491` ĩ,
  1899.591535133496` + 42.07274809049757` ĩ, 1956.6850055138937` + 40.87414762226893` ĩ};
kol = 62;
<< LinearAlgebra`MatrixManipulation`
<< NumericalMath`BesselZeros`
λdata =  $\frac{1}{b}$  * BesselJZeros[1, kol];
nn = 0;

```

```

Do[{λ = λdata[[nj]];
  λc = λ * c; j1λc = BesselJ[1, λc]; (* J1 (λj c) *)
  j2λc = BesselJ[2, λc]; (* J2 (λj c) *)
  j0λb = BesselJ[0, λb]; (* J0 (λj b) *)

  j1λr0 = BesselJ[1, λ * r0]; (* J1 (λj r0) *)

  λb = λ * b; j1λb = BesselJ[1, λb]; (* J1 (λj b) *)
  j0λb = BesselJ[0, λb]; (* J0 (λj b) *)
  j2λb = BesselJ[2, λb]; (* J2 (λj b) *)

  p1 = Sqrt[λ^2 + i * ω * σ * μ0]; (* p1 no R1 *)
  p = pdata[[ni]]; (* p saknes no programmas *)
  qvert = Sqrt[p^2 - i * ω * σ * μ0];

  (* pc *)
  pc1 = p * c; j1pc = BesselJ[1, pc1]; (* J1 (P2i c) *)
  j2pc = BesselJ[2, pc1]; (* J2 (P2i c) *)
  y1pc = BesselY[1, pc1]; (* Y1 (P2i c) *)
  y2pc = BesselY[2, pc1]; (* Y2 (P2i c) *)
  (* pb *)
  pb1 = p * b; j1pb = BesselJ[1, pb1]; (* J1 (P2i b) *)
  j2pb = BesselJ[2, pb1]; (* J2 (P2i b) *)
  y1pb = BesselY[1, pb1]; (* Y1 (P2i b) *)
  y2pb = BesselY[2, pb1]; (* Y1 (P2i b) *)
  (* qc *)
  qc1 = qvert * c; j1qc = BesselJ[1, qc1]; (* J1 (qi c) *)
  j2qc = BesselJ[2, qc1]; (* J2 (qi c) *)
  y1qc = BesselY[1, qc1]; (* Y1 (qi c) *)
  y2qc = BesselY[2, qc1]; (* Y2 (qi c) *)
  (* qb *)
  qb1 = qvert * b; j1qb = BesselJ[1, qb1]; (* J1 (qi b) *)
  j2qb = BesselJ[2, qb1]; (* J2 (qi b) *)
  y1qb = BesselY[1, qb1]; (* Y1 (qi b) *)
  y2qb = BesselY[2, qb1]; (* Y2 (qi b) *)
  (* T1 (qi c) = J1 (qi c) * Y1 (qi b) - J1 (qi b) * Y1 (qi c) *)
  T1 = j1qc * y1qb - j1qb * y1qc;
  aa1 =  $\frac{c}{\lambda^2 - p^2} * (\lambda * j2\lambda c * j1pc - p * j1\lambda c * j2pc);$  (* a21ij *)
  aa2 =  $\frac{1}{\lambda^2 - qvert^2} * (\lambda c * j2\lambda c * (j1qb * y1qc - j1qc * y1qb) + qc1 * j1\lambda c * (j2qc * y1qb - j1qb * y2qc));$  (* a22ij *)
  a2 = T1 * aa1 + j1pc * aa2; (* a2ij *)
  nn = nn + 1;
  a2data[nn] = a2;
  A1[nn] = ((λ + p1) * (p1 + p) * Exp[(p1 - p) * d1] + (λ - p1) * (p1 - p) * Exp[-(p1 + p) * d1]) * a2;
  A2[nn] = ((λ + p1) * (p1 - p) * Exp[(p1 + p) * d1] + (λ - p1) * (p1 + p) * Exp[-(p1 - p) * d1]) * a2;
  A3[nn] = (λ - p) * Exp[-p * d3] * a2; A4[nn] = (λ + p) * Exp[p * d3] * a2;
  znam = b^2 * j0λb^2 * p1;
  B1[nn] =  $\frac{(p1 + p) * Exp[(p1 - p) * d1] * a2}{znam};$  B2[nn] =  $\frac{(p1 - p) * Exp[(p1 + p) * d1] * a2}{znam};$ 
  B3[nn] =  $\frac{(p1 - p) * Exp[-(p1 + p) * d1] * a2}{znam};$  B4[nn] =  $\frac{(p1 + p) * Exp[-(p1 - p) * d1] * a2}{znam};$ 
  EJ[nj] = j1λr0 * Exp[-λ * h];
  Bvert[nj] = 2 * μ0 * r0 * p1 * EJ[nj];
  Dvert[nj] =  $\frac{\mu0 * r0 * EJ[nj]}{\lambda * b^2 * j0\lambda b^2}$ 
}, {nj, 1, kol}, {ni, 1, kol}];

```

```

A11 = Table[A1[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
A12 = Table[A2[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
A21 = Table[A3[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
A22 = Table[A4[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
B11 = Table[B1[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
B12 = Table[B2[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
B21 = Table[B3[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
B22 = Table[B4[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];

AA = BlockMatrix[{{A11, A12}, {A21, A22}}];
BB = Table[If[i ≤ kol, Bvert[i], 0], {i, 1, 2 * kol}];
sol = LinearSolve[AA, BB];

Do[{D8[k] = sol[[k]]; D10[k] = sol[[kol + k]]}, {k, 1, kol}];
DD8 = Table[D8[k], {k, 1, kol}];
DD10 = Table[D10[k], {k, 1, kol}];
DDvert = Table[Dvert[k], {k, 1, kol}];
EEJ = Table[EJ[k], {k, 1, kol}];
DD4 = B11.DD8 + B12.DD10; DD5 = B21.DD8 + B22.DD10;
DD2 = DD4 + DD5 - DDvert;

A0ind = Dot[DD2, EEJ];
Zind = 2 * π * r0 * i * ω * A0ind;
Print["kol = ", kol, "   A0ind = ", A0ind, "   Zind = ", Zind]

```

```

kol = 62   A0ind = -9.6869 × 10-9 - 2.67753 × 10-8 i   Zind = 0.0000332969 - 0.0000120463 i

```


Fig. Ap.18 (Section 6.5) “Mathematica” program for numerical calculation of the change in impedance given by formula (6.5.14).

```

Remove["Global`*"]
mm = 10-3; MSm = 106; kHz = 103;
ff = 7000.0; ω = 2 * π * ff;
σ = 3 * MSm;
μ0 = 4 * π * 10-7;
nConst = 400;
c = 2.2 * mm;
r2 = 5.5 * mm; r1 = 3.5 * mm;
z2 = 2.6 * mm; z1 = 0.3 * mm;
d1 = 0.7 * mm; d2 = 0.3 * mm;
d3 = d1 + d2; b = 10 * r2;

pdata = {549.096978208783` + 149.52882915593767` i, 499.18441867917574` + 164.8928681563969` i,
452.3539478271909` + 182.34844874431516` i, 409.8019254811801` + 201.62650986187526` i,
372.80299587630515` + 221.92705542132163` i, 318.9432124824753` + 259.8039780539326` i,
342.3702713381244` + 241.87990331798355` i, 302.367867070953` + 274.1384122330393` i,
292.17279133053717` + 283.7441814738398` i, 1329.2329935083806` + 59.51820672620454` i,
1386.176899830915` + 57.07846853412605` i, 1443.1521694836172` + 54.853265444223474` i,
1500.153708349577` + 52.81465732097078` i, 1557.1770768284669` + 50.93820547868496` i,
1614.2183861042176` + 49.20258902407101` i, 1671.2742479303836` + 47.58928827419457` i,
1728.341748001099` + 46.08231630970696` i, 1785.4184238706928` + 44.66798522501355` i,
932.2636597156051` + 85.99535053116698` i, 988.6895165784267` + 80.82599871742124` i,
1045.2493344562758` + 76.23235039498023` i, 1101.9120088964116` + 72.13507861242243` i,
1158.6555874522467` + 68.46764351428901` i, 1215.4642778520304` + 65.1735684868936` i,
1272.3263995768627` + 62.2044006471691` i, 709.1578128269097` + 114.68598773045271` i,
764.3464128585051` + 106.02330603799754` i, 820.0124599371625` + 98.4645794170425` i,
876.0164075402721` + 91.83606613094479` i, 601.1462479928093` + 136.1905499831746` i, 654.654257529524` + 124.6613429691535` i,
2013.7816227969483` + 39.73243201692545` i, 2070.8805961764897` + 38.64248418085895` i,
2127.9813206239915` + 37.60034869307359` i, 2185.0833358697837` + 36.60304578626898` i,
2242.186295088276` + 35.64838590298651` i, 2299.289943418036` + 34.73478815649102` i,
2356.394104688071` + 33.861106702647305` i, 2413.4986737077165` + 33.02646934712094` i,
2470.603610900386` + 32.230132512338336` i, 2527.7089364020503` + 31.471356300511875` i,
2584.814721508555` + 30.749302700264685` i, 2641.9210765357175` + 30.06295907630328` i,
2699.0281353790883` + 29.41108802705787` i, 2756.1360381229124` + 28.79220339811314` i,
2813.2449135969177` + 28.204571042901538` i, 2870.354864075254` + 27.64623162054064` i,
2927.465953845413` + 27.115041793491308` i, 2984.578202890122` + 26.60872935297439` i,
3041.691585891972` + 26.1249574890313` i, 3098.806036144899` + 25.661393348013625` i,
3155.9214531653906` + 25.215776285384656` i, 3213.0377125028467` + 24.785981697292794` i,
3270.154676197324` + 24.370076986812528` i, 3327.2722024755867` + 23.966366889370978` i,
3384.390153780836` + 23.573426233681722` i, 3441.508402647483` + 23.190118931945594` i,
3498.6268354166996` + 22.81560278488201` i, 3555.74535419817` + 22.449320508855575` i,
1842.50223700856` + 43.33469666412491` i, 1899.591535133496` + 42.07274809049757` i,
1956.6850055138937` + 40.87414762226893` i};

kol = 62;
<< LinearAlgebra`MatrixManipulation`
<< NumericalMath`BesselZeros`
λdata =  $\frac{1}{b}$  * BesselJZeros[1, kol];
nm = 0;

```

```

Do[{λ = λdata[[nj]];
  λc = λ*c; j1λc = BesselJ[1, λc]; (* J1(λjc) *)
  j2λc = BesselJ[2, λc]; (* J2(λjc) *)
  j0λb = BesselJ[0, λb]; (* J0(λjb) *)

  j1λr0 = BesselJ[1, λ*r0]; (* J1(λjr0) *)

  λb = λ*b; j1λb = BesselJ[1, λb]; (* J1(λjb) *)
  j0λb = BesselJ[0, λb]; (* J0(λjb) *)
  j2λb = BesselJ[2, λb]; (* J2(λjb) *)

  p1 = Sqrt[λ^2 + i*ω*σ*μ0]; (* p1 no R1 *)
  p = pdata[[ni]]; (* p saknes no programmas *)
  qvert = Sqrt[p^2 - i*ω*σ*μ0];
  (* pc *)
  pc1 = p*c; j1pc = BesselJ[1, pc1]; (* J1(p2ic) *)
  j2pc = BesselJ[2, pc1]; (* J2(p2ic) *)
  y1pc = BesselY[1, pc1]; (* Y1(p2ic) *)
  y2pc = BesselY[2, pc1]; (* Y2(p2ic) *)
  (* pb *)
  pb1 = p*b; j1pb = BesselJ[1, pb1]; (* J1(p2ib) *)
  j2pb = BesselJ[2, pb1]; (* J2(p2ib) *)
  y1pb = BesselY[1, pb1]; (* Y1(p2ib) *)
  y2pb = BesselY[2, pb1]; (* Y2(p2ib) *)
  (* qc *)
  qc1 = qvert*c; j1qc = BesselJ[1, qc1]; (* J1(qic) *)
  j2qc = BesselJ[2, qc1]; (* J2(qic) *)
  y1qc = BesselY[1, qc1]; (* Y1(qic) *)
  y2qc = BesselY[2, qc1]; (* Y2(qic) *)
  (* qb *)
  qb1 = qvert*b; j1qb = BesselJ[1, qb1]; (* J1(qib) *)
  j2qb = BesselJ[2, qb1]; (* J2(qib) *)
  y1qb = BesselY[1, qb1]; (* Y1(qib) *)
  y2qb = BesselY[2, qb1]; (* Y2(qib) *)
  (* T1(qic) = J1(qic)*Y1(qib) - J1(qib)*Y1(qic) *)
  T1 = j1qc*y1qb - j1qb*y1qc;
  aa1 =  $\frac{c}{\lambda^2 - p^2} * (\lambda * j2\lambda c * j1pc - p * j1\lambda c * j2pc)$ ; (* a21ij *)
  aa2 =  $\frac{1}{\lambda^2 - qvert^2} * (\lambda c * j2\lambda c * (j1qb*y1qc - j1qc*y1qb) + qc1 * j1\lambda c * (j2qc*y1qb - j1qb*y2qc))$ ; (* a22ij *)
  a2 = T1*aa1 + j1pc*aa2; (* a2ij *)
  nn = nn + 1;
  a2data[nn] = a2;
  A1[nn] = ((λ + p1) * (p1 + p) * Exp[(p1 - p) * d1] + (λ - p1) * (p1 - p) * Exp[-(p1 + p) * d1]) * a2;
  A2[nn] = ((λ + p1) * (p1 - p) * Exp[(p1 + p) * d1] + (λ - p1) * (p1 + p) * Exp[-(p1 - p) * d1]) * a2;
  A3[nn] = (λ - p) * Exp[-p*d3] * a2; A4[nn] = (λ + p) * Exp[p*d3] * a2;
  znam = b^2 * j0λb^2 * p1;
  B1[nn] =  $\frac{(p1 + p) * Exp[(p1 - p) * d1] * a2}{znam}$ ; B2[nn] =  $\frac{(p1 - p) * Exp[(p1 + p) * d1] * a2}{znam}$ ;
  B3[nn] =  $\frac{(p1 - p) * Exp[-(p1 + p) * d1] * a2}{znam}$ ; B4[nn] =  $\frac{(p1 + p) * Exp[-(p1 - p) * d1] * a2}{znam}$ ;
  Bdiag[nj] =  $\frac{1}{2 * \lambda * znam}$ 
}, {nj, 1, kol}, {ni, 1, kol}];

```

```

A11 = Table[A1[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
A12 = Table[A2[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
A21 = Table[A3[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
A22 = Table[A4[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
B11 = Table[B1[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
B12 = Table[B2[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
B21 = Table[B3[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
B22 = Table[B4[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
B0diag = Table[Bdiag[k], {k, 1, kol}];
B1diag = DiagonalMatrix[B0diag];

Y1 = B11 + B21;
A22inv = Inverse[A22];
Y2 = (B12 + B22).A22inv.A21;
Y3 = A11 - A12.A22inv.A21;
Y3inv = Inverse[Y3];
Y = (Y1 - Y2).Y3inv - B1diag;

gint[x_] =  $\pi/2 * x * (\text{BesselJ}[1, x] * \text{StruveH}[0, x] - \text{BesselJ}[0, x] * \text{StruveH}[1, x])$ ;
Do[{ $\lambda_i = \lambda\text{data}[[i]]$ ; p1 = Sqrt[ $\lambda_i^2 + i * \omega * \sigma * \mu_0$ ];
  mnoz1 =  $\frac{\text{Exp}[-\lambda_i * z1] - \text{Exp}[-\lambda_i * z2]}{\lambda_i^3} * (\text{gint}[\lambda_i * r2] - \text{gint}[\lambda_i * r1])$ ;
  bb[i] =  $\frac{2 * \mu_0 * p1 * n\text{Const} * \text{mnoz1}}{(r2 - r1) * (z2 - z1)}$ ; fm[i] = mnoz1}, {i, 1, kol}];
bb1 = Table[bb[k], {k, 1, kol}];
fm1 = Table[fm[k], {k, 1, kol}];
a1nd = fm1.(Y.bb1);
Z1nd =  $\frac{2 * \pi * i * \omega * n\text{Const}}{(r2 - r1) * (z2 - z1)} * \text{a1nd}$ ;
Print[" kol = ", kol, " ff = ", ff, " zind = ", Z1nd]

kol = 62 ff = 7000. zind = 5.35381 - 1.95226 i

```

Fig. Ap.19 (Section 6.6) “Mathematica” program for numerical calculation of the change in impedance given by formula (6.6.63).

```

Remove["Global`*"]
mm = 10-3; MSm = 106; kHz = 103;
ff = 1000.0; ω = 2 * π * ff;
c1 = 18.5 * MSm; c2 = 3 * MSm;
μ0 = 4 * π * 10-7;
nConst = 200;
c = 2.2 * mm;
r0 = 4.5 * mm;
h = 1.4 * mm;
d1 = 0.7 * mm; d2 = 0.3 * mm;
d3 = d1 + d2;
b = 10 * 5.5 * mm;

pdata = {585.6125666407848` + 19.96753914713313` i, 302.0138186894424` + 39.13318585213058` i,
358.1735464898753` + 32.953203961568846` i, 414.79663571326194` + 28.405271653801893` i,
471.63900542251565` + 24.92812718013798` i, 528.5945185780903` + 22.185617494793377` i,
246.9415593824931` + 47.90637239178142` i, 150.0069857013391` + 78.93953779439951` i,
194.70766373837137` + 60.795800878154935` i, 120.48253234046189` + 98.2979316591395` i,
642.6671139296317` + 18.136776865011754` i, 699.7439083636791` + 16.600475527500386` i,
756.8347425004006` + 15.293670753812087` i, 813.9346912119364` + 14.169598222564355` i,
871.0407000798174` + 13.193731915181857` i, 928.1508255334398` + 12.33998930479806` i,
985.2638064650472` + 11.588243421526323` i, 1042.378812402543` + 10.922650125853187` i,
1099.4952897599107` + 10.330499606789072` i, 1156.6128648345505` + 9.801414252695084` i,
1213.7312809750824` + 9.326780942590808` i, 1270.8503572355248` + 8.899345214259181` i,
1327.969961058612` + 8.512919133492588` i, 1385.0899904578364` + 8.162169966082372` i,
1442.210362743044` + 7.842466737127328` i, 1499.3310078060854` + 7.549768312423858` i,
1556.4518644391555` + 7.280541150936903` i, 1613.572878663931` + 7.031697970442959` i,
1670.6940031335075` + 6.800550868066859` i, 1727.8151970928461` + 6.584774023845818` i,
1784.936426406056` + 6.3823724232052035` i, 1842.057663438685` + 6.191653990520207` i,
1899.1788866769662` + 6.011203281806632` i, 1956.300080092903` + 5.839855541107693` i,
2013.4212323225302` + 5.676670433579974` i, 2070.5423357745317` + 5.520905130471578` i,
2127.6633857819543` + 5.371986838811965` i, 2184.784379871219` + 5.229485021844497` i,
2241.9053172177773` + 5.0930838157464855` i, 2299.0261982752354` + 4.962555204740818` i,
2356.147024575969` + 4.8377335904359215` i, 2413.2677986236667` + 4.718492390137807` i,
2470.3885238299167` + 4.604723227363309` i, 2527.509204437955` + 4.496318199478199` i,
2584.6298453857157` + 4.393155588568525` i, 2641.7504520923376` + 4.2950892212793805` i,
2698.8710301787273` + 4.201941548142413` i, 2755.9915851497167` + 4.113500335435322` i,
2813.1121220669866` + 4.02951877240534` i, 2870.2326452607126` + 3.9497185362435037` i,
2927.353158118439` + 3.87379541354091` i, 2984.473662979242` + 3.801426863363942` i, 3041.594161117447` + 3.732280911591989` i,
3098.7146528383437` + 3.666025718944301` i, 3155.8351376366986` + 3.6023392336681486` i,
3212.955614410991` + 3.540918299493317` i, 3270.076081674544` + 3.4814867269152945` i,
3327.1965377706665` + 3.423801896184088` i, 3384.316981038191` + 3.367659578676023` i,
3441.4374099554675` + 3.3128967757137113` i, 3498.5578231908785` + 3.2593925080324735` i,
3555.6782196812883` + 3.20706656693612` i};

kol = 62;
<< LinearAlgebra`MatrixManipulation`
<< NumericalMath`BesselZeros`
λdata =  $\frac{1}{b}$  * BesselJZeros[1, kol];
mn = 0;

```

```

Do[{λ = λdata[[nj]];
  λc = λ * c; j1λc = BesselJ[1, λc]; (* J1(λjc) *)
  j2λc = BesselJ[2, λc]; (* J2(λjc) *)
  j0λb = BesselJ[0, λb]; (* J0(λjb) *)
  j1λr0 = BesselJ[1, λ * r0]; (* J1(λjr0) *)
  λb = λ * b; j1λb = BesselJ[1, λb]; (* J1(λjb) *)
  j0λb = BesselJ[0, λb]; (* J0(λjb) *)
  j2λb = BesselJ[2, λb]; (* J2(λjb) *)
  pj = Sqrt[λ^2 + i * ω * c1 * μ0]; (* p no R1 *)
  p2 = pdata[[ni]]; (* p2 saknes no programmas *)
  qvert = Sqrt[p2^2 - i * ω * c2 * μ0];
  p1 = Sqrt[qvert^2 + i * ω * c1 * μ0]; (* p1 no R2 *)
  (* p2c *)
  pc1 = p2 * c; j1pc = BesselJ[1, pc1]; (* J1(p2i c) *)
  j2pc = BesselJ[2, pc1]; (* J2(p2i c) *)
  y1pc = BesselY[1, pc1]; (* Y1(p2i c) *)
  y2pc = BesselY[2, pc1]; (* Y2(p2i c) *)
  (* p2b *)
  pb1 = p2 * b; j1pb = BesselJ[1, pb1]; (* J1(p2i b) *)
  j2pb = BesselJ[2, pb1]; (* J2(p2i b) *)
  y1pb = BesselY[1, pb1]; (* Y1(p2i b) *)
  y2pb = BesselY[2, pb1]; (* Y1(p2i b) *)
  (* qc *)
  qc1 = qvert * c; j1qc = BesselJ[1, qc1]; (* J1(qic) *)
  j2qc = BesselJ[2, qc1]; (* J2(qic) *)
  y1qc = BesselY[1, qc1]; (* Y1(qic) *)
  y2qc = BesselY[2, qc1]; (* Y2(qic) *)
  (* qb *)
  qb1 = qvert * b; j1qb = BesselJ[1, qb1]; (* J1(qib) *)
  j2qb = BesselJ[2, qb1]; (* J2(qib) *)
  y1qb = BesselY[1, qb1]; (* Y1(qib) *)
  y2qb = BesselY[2, qb1]; (* Y2(qib) *)

  (* T1(qic) = J1(qic)*Y1(qib) - J1(qib)*Y1(qic) *)
  T1 = j1qc * y1qb - j1qb * y1qc;
  aa1 =  $\frac{c}{\lambda^2 - p2^2} * (\lambda * j2\lambda c * j1pc - p2 * j1\lambda c * j2pc)$ ; (* a21ij *)
  aa2 =  $\frac{1}{\lambda^2 - qvert^2} * (\lambda c * j2\lambda c * (j1qb * y1qc - j1qc * y1qb) + qc1 * j1\lambda c * (j2qc * y1qb - j1qb * y2qc))$ ; (* a22ij *)
  a2 = T1 * aa1 + j1pc * aa2; (* a2ij *)
  nn = nn + 1;
  a2data[nn] = a2;
  A1[nn] = ((λ + pj) * (pj + p1) * Exp[(pj - p1) * d1] + (λ - pj) * (pj - p1) * Exp[-(pj + p1) * d1]) * a2;
  A2[nn] = ((λ + pj) * (pj - p1) * Exp[(pj + p1) * d1] + (λ - pj) * (pj + p1) * Exp[-(pj - p1) * d1]) * a2;
  A3[nn] = (pj - p1) * Exp[-p1 * d3] * a2; A4[nn] = (pj + p1) * Exp[p1 * d3] * a2;
  znam = b^2 * j0λb^2 * pj;
  B1[nn] =  $\frac{(pj + p1) * Exp[(pj - p1) * d1] * a2}{znam}$ ; B2[nn] =  $\frac{(pj - p1) * Exp[(pj + p1) * d1] * a2}{znam}$ ;
  B3[nn] =  $\frac{(pj - p1) * Exp[-(pj + p1) * d1] * a2}{znam}$ ; B4[nn] =  $\frac{(pj + p1) * Exp[-(pj - p1) * d1] * a2}{znam}$ ;
  EJ[nj] = j1λr0 * Exp[-λ * h];
  Bvert[nj] = 2 * μ0 * r0 * pj * EJ[nj];
  Dvert[nj] =  $\frac{\mu0 * r0 * EJ[nj]}{\lambda * b^2 * j0\lambda b^2}$ 
}, {nj, 1, kol}, {ni, 1, kol}];

```

```

A11 = Table[A1[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
A12 = Table[A2[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
A21 = Table[A3[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
A22 = Table[A4[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
B11 = Table[B1[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
B12 = Table[B2[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
B21 = Table[B3[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
B22 = Table[B4[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
DDvert = Table[Dvert[k], {k, 1, kol}];
EEJ = Table[EJ[k], {k, 1, kol}];

AA = BlockMatrix[{{A11, A12}, {A21, A22}}];
BB = Table[If[i ≤ kol, Bvert[i], 0], {i, 1, 2 * kol}];
sol = LinearSolve[AA, BB];
Do[{D6[k] = sol[[k]]; D8[k] = sol[[kol + k]]}, {k, 1, kol}];
DD6 = Table[D6[k], {k, 1, kol}];
DD8 = Table[D8[k], {k, 1, kol}];
DD4 = B11.DD6 + B12.DD8; DD5 = B21.DD6 + B22.DD8;
DD2 = DD4 + DD5 - DDvert;
A0ind = DD2.EEJ;
Zind = 2 * π * r0 * i * ω * A0ind;
Print["kol = ", kol, "   A0ind = ", A0ind, "   Zind = ", Zind]

```

kol = 62 A0ind = $-2.53139 \times 10^{-8} - 2.91717 \times 10^{-8} i$ Zind = $5.18243 \times 10^{-6} - 4.49709 \times 10^{-6} i$

Fig. Ap.20 (Section 6.7) “Mathematica” program for numerical calculation of the change in impedance given by formula (6.7.14).

```

Remove["Global`*"]
mm = 10-3; MSm = 106; kHz = 103;
ff = 1000.0; ω = 2 * π * ff;
σ1 = 18.5 * MSm; σ2 = 3 * MSm;
μ0 = 4 * π * 10-7;
nConst = 200;
c = 2.2 * mm;
r2 = 5.5 * mm; r1 = 3.5 * mm;
z2 = 2.6 * mm; z1 = 0.3 * mm;
d1 = 0.7 * mm; d2 = 0.3 * mm;
d3 = d1 + d2; b = 10 * r2;

pdata = {585.6125666407848` + 19.96753914713313` i, 302.0138186894424` + 39.13318585213058` i,
358.1735464898753` + 32.953203961568846` i, 414.79663571326194` + 28.405271653801893` i,
471.63900542251565` + 24.92812718013798` i, 528.5945185780903` + 22.185617494793377` i,
246.9415593824931` + 47.90637239178142` i, 150.0069857013391` + 78.93953779439951` i,
194.70766373837137` + 60.795800878154935` i, 120.48253234046189` + 98.2979316591395` i,
642.6671139296317` + 18.136776865011754` i, 699.7439083636791` + 16.600475527500386` i,
756.8347425004006` + 15.293670753812087` i, 813.9346912119364` + 14.169598222564355` i,
871.0407000798174` + 13.193731915181857` i, 928.1508255334398` + 12.33998930479806` i,
985.2638064650472` + 11.588243421526323` i, 1042.378812402543` + 10.922650125853187` i,
1099.4952897599107` + 10.330499606789072` i, 1156.6128648345505` + 9.801414252695084` i,
1213.7312809750824` + 9.326780942590808` i, 1270.8503572355248` + 8.899345214259181` i,
1327.969961058612` + 8.512919133492588` i, 1385.0899904578364` + 8.162169966082372` i,
1442.210362743044` + 7.842466737127328` i, 1499.3310078060854` + 7.549768312423858` i,
1556.4518644391555` + 7.280541150936903` i, 1613.572878663931` + 7.031697970442959` i,
1670.6940031335075` + 6.800550868066859` i, 1727.8151970928461` + 6.584774023845818` i,
1784.936426406056` + 6.3823724232052035` i, 1842.057663438685` + 6.191653990520207` i,
1899.1788866769662` + 6.011203281806632` i, 1956.300080092903` + 5.839855541107693` i,
2013.4212323225302` + 5.676670433579974` i, 2070.5423357745317` + 5.520905130471578` i,
2127.6633857819543` + 5.371986838811965` i, 2184.784379871219` + 5.229485021844497` i,
2241.9053172177773` + 5.0930838157464855` i, 2299.0261982752354` + 4.962555204740818` i,
2356.147024575969` + 4.8377335904359215` i, 2413.2677986236667` + 4.718492390137807` i,
2470.3885238299167` + 4.604723227363309` i, 2527.509204437955` + 4.496318199478199` i,
2584.6298453857157` + 4.393155588568525` i, 2641.7504520923376` + 4.2950892212793805` i,
2698.8710301787273` + 4.201941548142413` i, 2755.9915851497167` + 4.113500335435322` i,
2813.1121220669866` + 4.02951877240534` i, 2870.2326452607126` + 3.9497185362435037` i,
2927.353158118439` + 3.87379541354091` i, 2984.473662979242` + 3.801426863363942` i,
3041.594161117447` + 3.732280911591989` i, 3098.7146528383437` + 3.666025718944301` i,
3155.8351376366986` + 3.6023392336681486` i, 3212.955614410991` + 3.540918299493317` i,
3270.076081674544` + 3.4814867269152945` i, 3327.1965377706665` + 3.423801896184088` i,
3384.316981038191` + 3.367659578676023` i, 3441.4374099554675` + 3.3128967757137113` i,
3498.5578231908785` + 3.2593925080324735` i, 3555.6782196812883` + 3.20706656693612` i};

kol = 62;
<< LinearAlgebra`MatrixManipulation`
<< NumericalMath`BesselZeros`
λdata =  $\frac{1}{b}$  * BesselJZeros[1, kol];
nn = 0;

```

```

Do[{λ = λdata[[nj]];
  λc = λ * c; λb = λ * b;
  j1λc = BesselJ[1, λc]; (* J1 (λj c) *)
  j2λc = BesselJ[2, λc]; (* J2 (λj c) *)
  j0λb = BesselJ[0, λb]; (* J0 (λj b) *)
  j1λb = BesselJ[1, λb]; (* J1 (λj b) *)
  j0λb = BesselJ[0, λb]; (* J0 (λj b) *)
  j2λb = BesselJ[2, λb]; (* J2 (λj b) *)
  pj = Sqrt[λ^2 + i * ω * c1 * μ0]; (* p no R1 *)
  p2 = pdata[[ni]]; (* p2 saknes no programmas *)
  qvert = Sqrt[p2^2 - i * ω * c2 * μ0];
  p1 = Sqrt[qvert^2 + i * ω * c1 * μ0]; (* p1 no R2 *)
  (* p2c *)
  pc1 = p2 * c; j1pc = BesselJ[1, pc1]; (* J1 (p2i c) *)
  j2pc = BesselJ[2, pc1]; (* J2 (p2i c) *)
  y1pc = BesselY[1, pc1]; (* Y1 (p2i c) *)
  y2pc = BesselY[2, pc1]; (* Y2 (p2i c) *)
  (* p2b *)
  pb1 = p2 * b; j1pb = BesselJ[1, pb1]; (* J1 (p2i b) *)
  j2pb = BesselJ[2, pb1]; (* J2 (p2i b) *)
  y1pb = BesselY[1, pb1]; (* Y1 (p2i b) *)
  y2pb = BesselY[2, pb1]; (* Y1 (p2i b) *)
  (* qc *)
  qc1 = qvert * c; j1qc = BesselJ[1, qc1]; (* J1 (qi c) *)
  j2qc = BesselJ[2, qc1]; (* J2 (qi c) *)
  y1qc = BesselY[1, qc1]; (* Y1 (qi c) *)
  y2qc = BesselY[2, qc1]; (* Y2 (qi c) *)
  (* qb *)
  qb1 = qvert * b; j1qb = BesselJ[1, qb1]; (* J1 (qi b) *)
  j2qb = BesselJ[2, qb1]; (* J2 (qi b) *)
  y1qb = BesselY[1, qb1]; (* Y1 (qi b) *)
  y2qb = BesselY[2, qb1]; (* Y2 (qi b) *)
  (* T1 (qi c) = J1 (qi c) * Y1 (qi b) - J1 (qi b) * Y1 (qi c) *)
  T1 = j1qc * y1qb - j1qb * y1qc;
  aa1 =  $\frac{c}{\lambda^2 - p2^2} * (\lambda * j2\lambda c * j1pc - p2 * j1\lambda c * j2pc)$ ; (* a21ij *)
  aa2 =  $\frac{1}{\lambda^2 - qvert^2} * (\lambda c * j2\lambda c * (j1qb * y1qc - j1qc * y1qb) + qc1 * j1\lambda c * (j2qc * y1qb - j1qb * y2qc))$ ; (* a22ij *)
  a2 = T1 * aa1 + j1pc * aa2; (* a2ij *)
  nn = nn + 1;
  a2data[nn] = a2;
  A1[nn] = ((λ + pj) * (pj + p1) * Exp[(pj - p1) * d1] + (λ - pj) * (pj - p1) * Exp[-(pj + p1) * d1]) * a2;
  A2[nn] = ((λ + pj) * (pj - p1) * Exp[(pj + p1) * d1] + (λ - pj) * (pj + p1) * Exp[-(pj - p1) * d1]) * a2;
  A3[nn] = (pj - p1) * Exp[-p1 * d3] * a2; A4[nn] = (pj + p1) * Exp[p1 * d3] * a2;
  znam = b^2 * j0λb^2 * pj;
  B1[nn] =  $\frac{(pj + p1) * Exp[(pj - p1) * d1] * a2}{znam}$ ; B2[nn] =  $\frac{(pj - p1) * Exp[(pj + p1) * d1] * a2}{znam}$ ;
  B3[nn] =  $\frac{(pj - p1) * Exp[-(pj + p1) * d1] * a2}{znam}$ ; B4[nn] =  $\frac{(pj + p1) * Exp[-(pj - p1) * d1] * a2}{znam}$ ;
  Bdiag[nj] =  $\frac{1}{2 * \lambda * znam}$ 
}], {nj, 1, kol}, {ni, 1, kol}];

```



```

A11 = Table[A1[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
A12 = Table[A2[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
A21 = Table[A3[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
A22 = Table[A4[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
B11 = Table[B1[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
B12 = Table[B2[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
B21 = Table[B3[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
B22 = Table[B4[(kk - 1) * kol + k], {kk, 1, kol}, {k, 1, kol}];
B0diag = Table[Bdiag[k], {k, 1, kol}];
B1diag = DiagonalMatrix[B0diag];

Y1 = B11 + B21;
A22inv = Inverse[A22];
Y2 = (B12 + B22).A22inv.A21;
Y3 = A11 - A12.A22inv.A21;
Y3inv = Inverse[Y3];
Y = (Y1 - Y2).Y3inv - B1diag;

gint[x_] =  $\pi/2 * x * (\text{BesselJ}[1, x] * \text{StruveH}[0, x] - \text{BesselJ}[0, x] * \text{StruveH}[1, x])$ ;
Do[{ $\lambda_i = \lambda\text{data}[[i]$ ];  $p_j = \text{Sqrt}[\lambda_i^2 + i * \omega * \sigma_1 * \mu_0]$ ;
  mnoz1 =  $\frac{\text{Exp}[-\lambda_i * z_1] - \text{Exp}[-\lambda_i * z_2]}{\lambda_i^3} * (\text{gint}[\lambda_i * r_2] - \text{gint}[\lambda_i * r_1])$ ;
  bb[i] =  $\frac{2 * \mu_0 * p_j * n\text{Const} * \text{mnoz1}}{(r_2 - r_1) * (z_2 - z_1)}$ ;  $\text{fm}[i] = \text{mnoz1}$ }, {i, 1, kol}];
bb1 = Table[bb[k], {k, 1, kol}];
fm1 = Table[fm[k], {k, 1, kol}];
aind = fm1.(Y.bb1);
Zind =  $\frac{2 * \pi * i * \omega * n\text{Const}}{(r_2 - r_1) * (z_2 - z_1)} * \text{aind}$ ;
Print[" kol = ", kol, " ff = ", ff, " zind = ", Zind]

```

```

kol = 62 ff = 1000. zind = 0.207724 - 0.181782 i

```