Using 2-colorings in the theory of uniquely Hamiltonian graphs

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Abstract

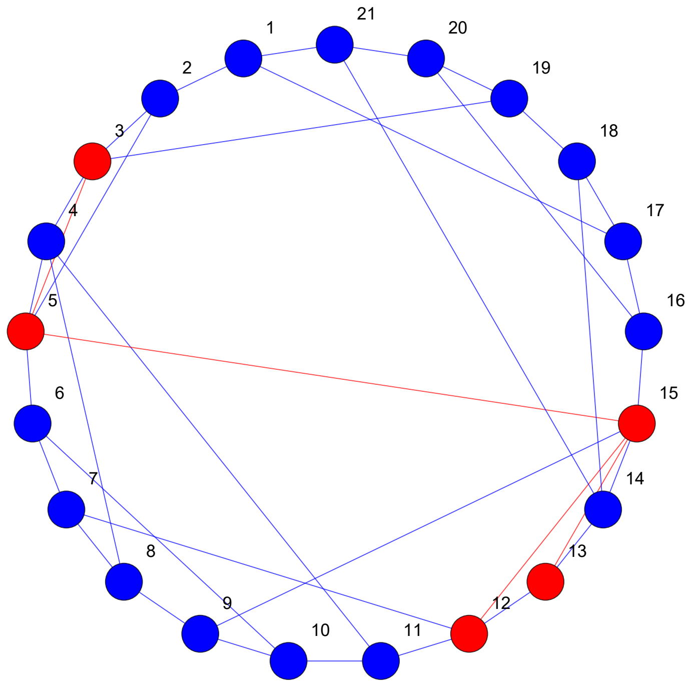
We use the concept of 2-coloring in analyzing UH3 graphs and building exact specifications of functions to find new UH3 graphs by Hamiltonian cycle edge extractions.

Introduction

We consider graphs that are uniquely Hamiltonian with minimal degree of vertices 3, calling them UH3 graphs. We go on on building theory around this subject following [1-10]. Here we introduce coloring of these graphs in 2 colors, coloring both vertices and edges. We find that this 2-coloring as we name it is very useful, and it gave us possibility to specify how to get ascendants (also UH3 graphs) of some UH3 graph, that we receive by new Hamiltonian cycle edge extraction from a vertex.

Two-colored, or 2-colored, UH3 graphs

We introduce coloring of vertices and edges of UH3 graph G in two colors as follows. Vertex u is colored blue if graph G\u is Hamiltonian, i.e, vertex u is deleted, otherwise u is colored red. Edge e in G is colored blue if G\e is Hamiltonian, i.e. edge e is contracted, otherwise edge is colored red. Below example of two colored UH3 graph with 21 vertices:

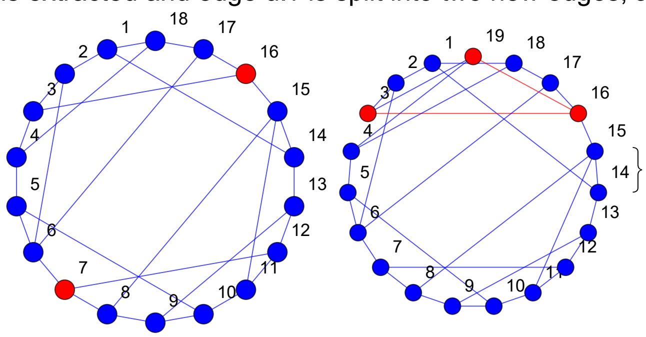


We choose these colors blue and red just following our previous choice in [10]. Of course, for more generality we could call these elements Hamiltonian or non-Hamiltonian, or like to this, as in [10]. Actually we are interested in coloring only chords of UH3 graph, because rim edges should be blue always, for obvious reason. In what follows we ignore, or drop, coloring of rim edges.
In [10] we called blue elements Hamiltonian vertices and edges, but previously, in [7,8,9], - Thomassian. The reason for the last is that the Thomassian conjecture requires that in the UH3 graph should be at least one blue chord, ie., such that its contraction leads to Hamiltonian graph.

We observe that red edges have both ends red. Let us state as a preposition.
Proposition Both ends of red edge in 2-coloring of UH3 graphs are colored red.
Proof It is easier to prove another equivalent statement. If chord has blue end then it is always blue. But it is easy to see this. If vertices u and v are incident to a chord, and u is blue, then there is n-1 vertices cycle without u, but deletion of u from this cycle means that edges incident with u can’t contribute to this cycle, that it is equivalent with the contraction the chord u.v, what says that chord u.v should be colored blue. The proof is **done**.
It is easy to see that blue chords may have either blue colored ends or differently colored ends. If we don’t have in the UH3 graph chords with differently colored ends then we call such 2-coloring **separable**. Truly, the proposition above says that chord graph induced from red edges cover all red vertices, which statement isn’t true for blue edges, but in case of separable 2-coloring blue edges behave like red edges, i.e., their induced subgraph cover all blue vertices. In [8] we discussed situation around separable 2-colorings and critical such UH3 graphs, ie., graphs with 34-n blue vertices, call special graph. This notion turns useful in considering Thomassen’s conjecture counterexample which should be special graph with 34 vertices and so with 34-34 blue vertices, and so no possibility to have blue chord in there.

The split of the blue edge with differently colored ends

Let us have in UH3 graph G blue edge u.v with differently colored ends, blue u and red v.
Let us consider the simplest case when degree of u is 3. But this simplest case is very common, i.e., rarely we find different situation. We now split vertex u on the Hamiltonian cycle so that new cycle edge is extracted and edge u.v is split into two new edges, as it can be seen in the example below.



We see that blue edge u.v split produced two red edges. We are to prove that this occurs always under specified conditions.

Theorem 1 Splitting vertex u with new rim edge u_1.u_2 extraction with (accompanied) edge u.v split into two new edges u_1.v and u_2.v leads to a new UH3 graph with these new elements, new edges and new vertices colored red.

Proof

Because vertex u is blue its deletion gives Hamiltonian graph, G\u. I.e., in all cases of eventual new (n-1)-vertices cycles, it takes vertex v, but excludes vertex u. Let us consider a new graph now according the premise of the theorem, that says, where new two vertices, u_1, u_2 and two new edges u_1.v and u_2.v arise. It is easy to see that the new graph is UH3 graph. It is Hamiltonian because rim edge is extracted, and the old cycle with the new rim edge makes Hamiltonian cycle. But it is the only Hamiltonian cycle. Let us suppose the opposite, and there exist some other Hamiltonian cycle. If this cycle takes rim edge u_1.u_2, then we come to contradiction because its contraction would give just such a cycle which we excluded with our previous argument, because vertex v was red.

If this other Hamiltonian cycle doesn’t take the new rim edge u_1.u_2, then it goes through new chords u_1.v and u_2.v, arisen from the split of the chord u.v, taking between vertices u_1 and u_2 vertex v, as a piece of assumable (n-1)-cycle, Hamiltonian cycle of G\u. It leads to assumption that G\{u,v} has (n-1) path from vertex u-1 to vertex u+2, where under vertex u-1 we understand vertex before u on the rim, and vertex u+1 as vertex after u on the rim. But this is equivalent to require Hamiltonian cycle for graph G\v, but, remember, v was read. Contradiction.

Now remains to prove that the new two chords are red, i.e., vertices u_1, u_2 are read.

Let us assume opposite and suppose that both new vertices are blue. Then, deletion of, say, u_1 would give Hamiltonian graph. But actually this assumption would be equivalent with the previous case, n-path in G\{u,v}+ u-1.u+1 from u-1 to u+2 only with one rim edge contracted, $u-1.u_1$, in this case. Contradiction from previous contradiction.

The proof is **done**.

If the degree of the vertex u in the theorem above is greater than 3, then our assumption to be proved would be as follows. There exist such a split of vertex u with edge u.v split that we get UH3 graph and new elements are colored red as before. See case 1 in appendix.

Perfect splits producing new UH3 graphs

We call vertex split and new rim edge extraction perfect if this operation leads to one or more new UH3 graphs.
Computer data using 2-colorings allows to specify perfect splits. Our analysis is based of cases with first family of UH3 graphs, with n=18.

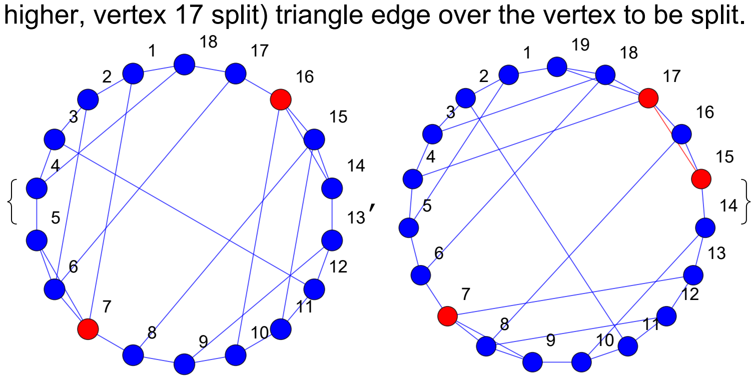
We distinguish three cases:
1) standard case, that what we considered higher, of split of blue vertex with adjoined red vertex with the split of this incident edge;
2) rim edge with different colored ends split, some special case of standard case;
3) split of red vertex of degree d>3.
All these cases lead to new UH3 graphs, and we discuss them below

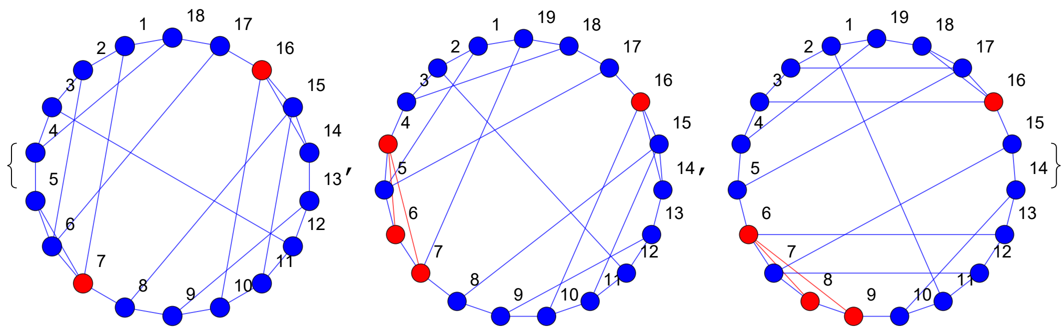
Standard case of perfect split of vertex

In n=18 family there is only one case when blue vertex, see Theorem 1, has degree d=4, and in this case edge to be split is that forming triangle with rim edges. As we see, case 1 in appendix, we get two new UH3 graphs, standard case, and other case, with one edge less.

Rim edge perfect splits

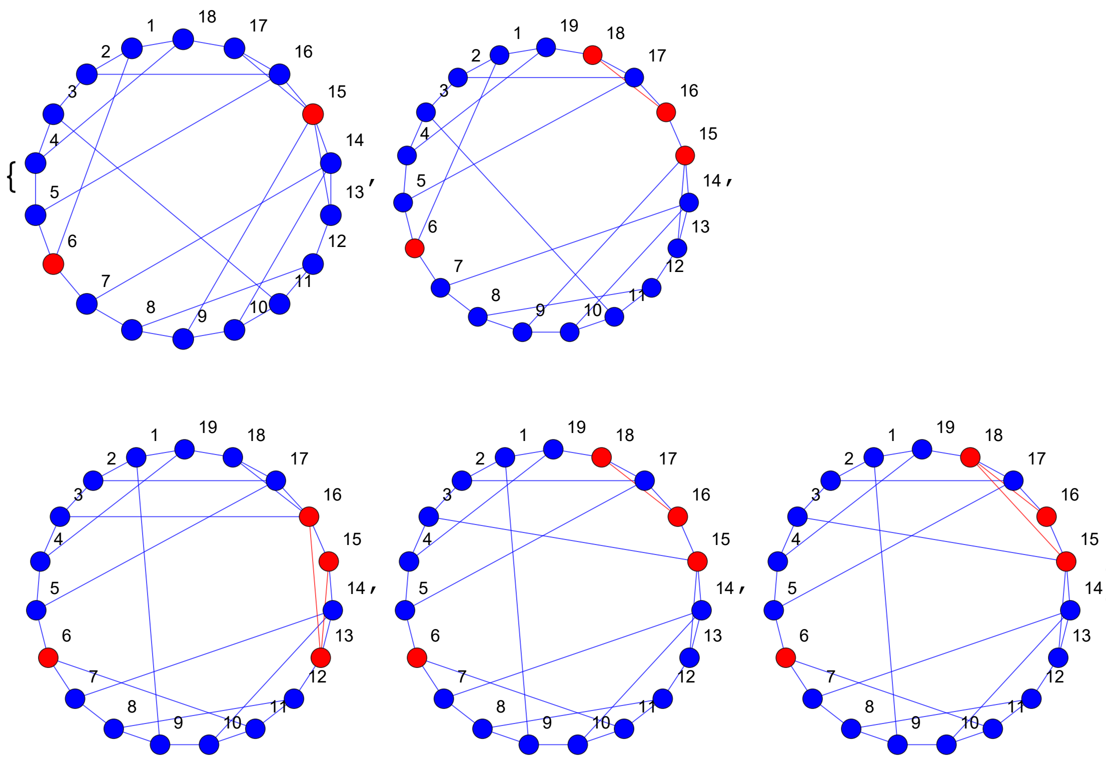
Rim edge splits are special case of standard case, but is important for characterization of other cases
We find two cases for perfect rim edge split. Cases differ with (see lower, vertex 8 split) or without (see higher, vertex 17 split) triangle edge over the vertex to be split.





Split of red vertex of degree d>3.

Splitting red vertex with d> 3 gives new UH3 graphs, for greater d we receive many graphs. Below see e.g. perfect split of vertex 15, d=5, , giving 4 new UH3 graphs



Plus quam perfect splits and rim edge extractions

We call vertex split with new rim edge extraction plus quam perfect, pq-split, if it gives rise of two new red vertices at least locally. If red vertices disappear for any unknown reason for us, or, as we say, disappear non-localy, i.e., in elements of graph not being engaged in the split process, we can’t some-how predict, or specify.

Actually for first our experiments we need to specify just pg-splits to get graphs with maximal number of red vertices on each step on advance in getting graphs with increased number of vertices.

We are ready to specify 3 types of pq-splits just for 3 cases in getting perfect splits, for standard case, i.e., blue edge split into 2 new red edges, for rim edge split, and for vertex with d>3 split.

The standard case gives already pq-split, and nothing to add.

What concerns rim edge split, if blue vertex is not covered with triangle edge, then we get rise with only one red new vertex.
(It is easy to characterize. If u’ is different neighbor of u on rim from v, and w is a new vertex to be added, then this split is achieved by replacing edge u.v with three edges w.u, w.u’ and w.v.)
But, if such cover is present, we get two pq-splits, performed correspondingly to left and to right. It is demonstrated by the code line:

```
splitCoveredRimEdge = If[EdgeQ[#, Mod[#2 - 1, n, 1] ⟷ Mod[#2 + 1, n, 1]],
      {splitRimEdge[#1, Mod[#2 - 1, n, 1], splitRimEdge[#1, Mod[#2 + 1, n, 1]}, {}] &;
```

In the third case, for vertex d>3 split, works the code, where similarly two little modified rim edge extractions are performed as in covered rim edge extraction case:

```
splitRedVertex[gr_, v_] :=
  Module[{n, g, u1, u2}, n = VertexCount[gr]; g = VertexAdd[gr, u2 = n + 1];
   If[EdgeQ[v, Mod[u3 = v - 2, n, 1] ⟷ v], u2 = Mod[v - 1, n, 1], u2 = Mod[v + 1, n, 1]];
   g = EdgeDelete[g, {u1 ⟷ v, u3 ⟷ v}];
   {g = EdgeAdd[g, {u2 ⟷ u3, u1 ⟷ u2, u2 ⟷ v}], EdgeAdd[g, u3 ⟷ v]}];
```

In place of splitRimEdge is to be used splitSimpleChord routine, at least it works as expected.

Illustrating pictures are to be added afterwards.

Procedure for generating ascendants for UH3 graph with pq-splits

To calculate new UH3 graphs ascending from already found families we use full local exhaustive search, as in [8]. Now we are able to replace LES with precise functions, as specified higher. Putting this all together, we may suggest new routine for ascending UH3 families, that do almost the same what did LES.

```
allAntisEx[gr_] := Module[{},
   scg1 =
    With[{el = #}, splitSimpleChord[gr, el[[1]], el[[2]]]] & /@ blueRedchords[gr];
   scg2 = Flatten[splitCoveredRimEdge[gr, #] & /@ blueRedRim[gr]];
   scg3 = Flatten[splitRedVertex[gr, #] & /@ redVertices[gr]];
   Join[scg1, scg2, scg3] // selIsom
  ];
```

Tests of built procedures

Running on computer newly obtained procedures give partly sufficient results. Firstly, all obtained new graphs are UH3 graphs both what concerns their Hamiltoniacity and absence of defective vertices, i.e., of degree two. The only fault seems that we receive little smaller amount of UH3 graphs that gave LES.

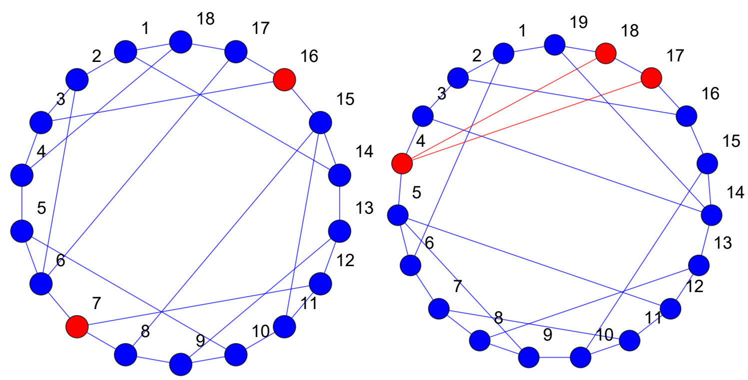
Obviously, we loose some portion of graphs by red vertex splits.
Looking on our procedures there is a temptation to replace our suggested procedures by one simle, i.e., to apply splitSimpleChord for all edges, not only for chords, and all. We tried this “temptation”. Firstly, it gave fewer UH3 graph, and, besides, it gave many graphs with defective vertices. At level n=24 we received more then thousend graphs, but only 4 UH3 graphs. How defective vertices appeared there? Obviously, rim edge split with cover gives this fault, if the eventual split ignores this cover aspect.
Repeating the process of [8], in place of 6 hours we did the same in less than 10 minutes.

Violation of locality

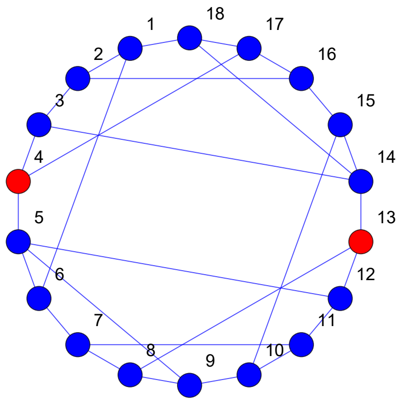
By violation of locality we understand cases when red vertices appear, or disappear in elements of graph that was not engaged in the operation we consider. Locality should be defined precisely, but we here give only some sense of this notion, we hope, very concise. We don’t try at this stage to give precise definition, because we don’t know how to limit the “locality”. In examples it seems clear, but we may have situations that would get out of our control.

Cases of violation of locality: examples

Here vertex 4 is split, left two new vertices 17, 18. Vertex 7 lost its red coloring. In this example vertex looses red coloring.



Here complicate case of violation of locality, see second graph. Vertex 14 were split, giving new vertices 14 and 15, together with rim edge 13.14 split. But this is not the case for the theorem 1, because vertex degree of 14 is 4. Two non-locality aspects: new red vertex 16 and disappearance of red vertex 4. But in case we would try to split vertex 17, we would receive the graph most right, with only non-locality of loss of red vertex 12.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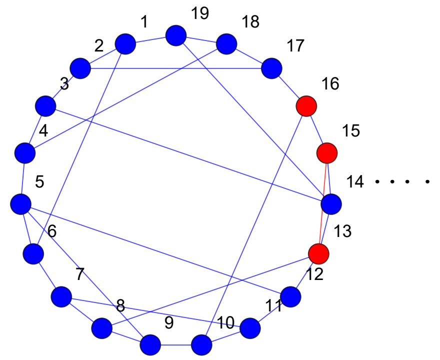
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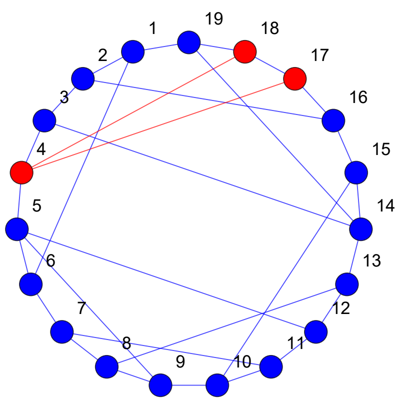
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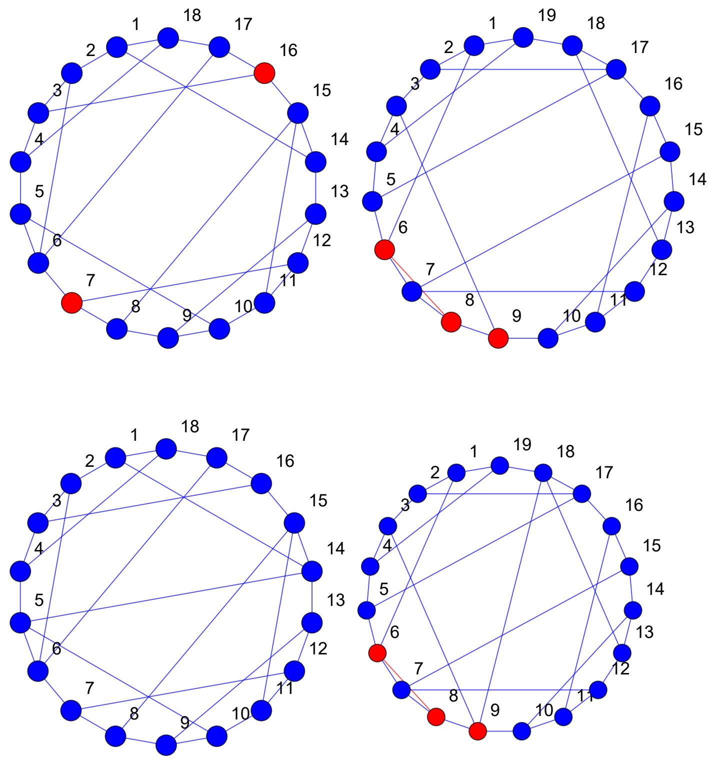
Locality violation explained?

A case explaining locality violation



Here we took the minimum UH3 graph, that stands here in the first position. Next to right UH3 graph, augmented by adding one edge, 5.14, to this graph. And below we show in both these cases vertex 6 split. Receiving the same result with the group of 3 vertices we may judge that the split done is the same. The only difference is the standing aside red vertex, 16 upper and 4 below, otherwise all the same.

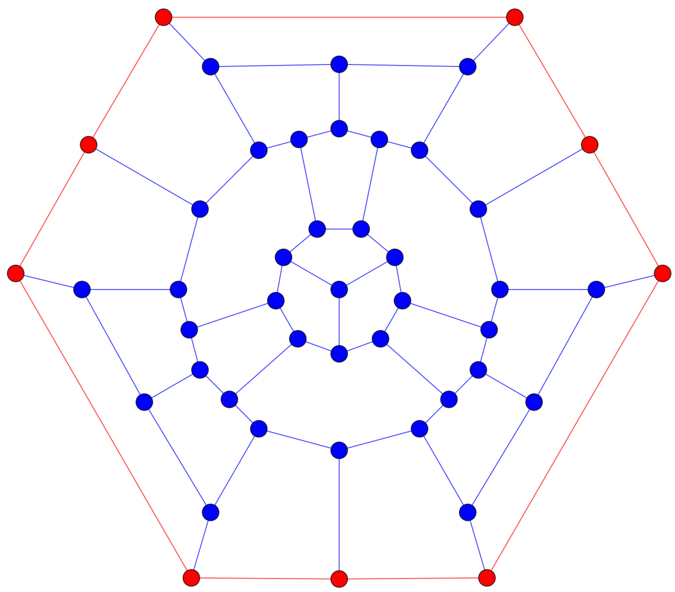
How to interpret this result? In the augmented graph the red vertices disappeared because of the added edge, but they remain in some “hidden” way. That is, when we perform the split of the same vertex, 6 in our case, the result obtained is such as if this vertex were red.

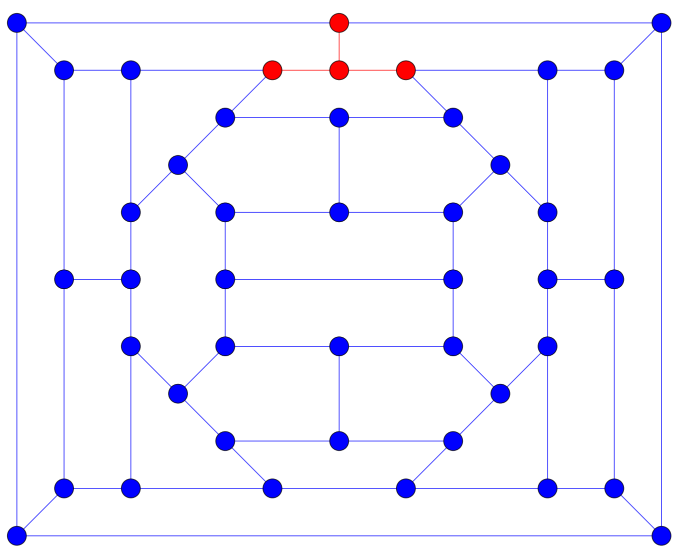
The use of 2-coloring not only for UH3 graphs

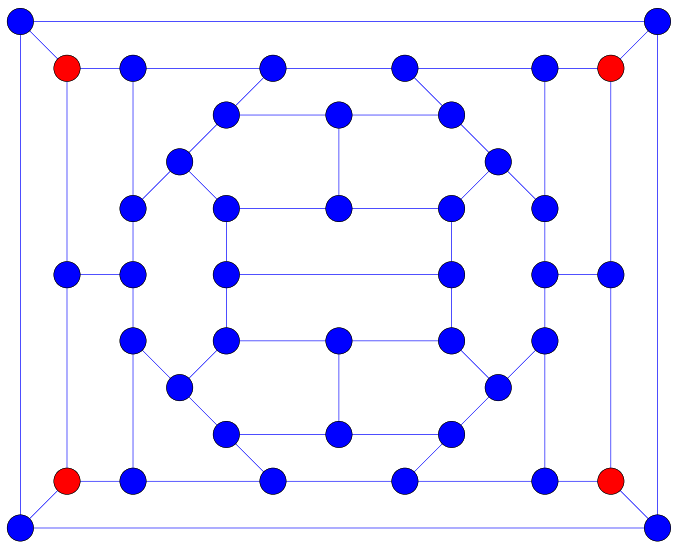
The 2-coloring may be used both for non-Hamiltonian graphs as like for Hamiltonian graphs with arbitrary number of Hamiltonian cycles. In the second case it is hardly expectable to find red vertices in them. We give exempli gratia some non-Hamiltonian graphs. Now we don’t have such category as chords there, and all edges are to be tried.

Grinbergs' graphs:

```
grg46 = GraphData["GrinbergGraph46"] // colGraph
grg44 = GraphData["GrinbergGraph44"] // colGraph
grg42 = GraphData["GrinbergGraph42"] // colGraph
```







These graphs may be tried to build planar preparations [2,3,4] for building of UH3 graphs. Grinbergs asked, [2], does there exist such planar preparations?

Aknowledgements

I give thanks to professor of UWA Gordon Royle who turned my attention to this problem after so many years since I worked on this subject with Emanuels Grinbergs, and generously gave me his data he computed in several few last years. Besides, G. Royle wellcomed my work and helped much in giving worthy advices and instructions at critical points.

References

[1]Jan Goedgebeura, Barbara Meersman, Carol T. Zamfirescu,Graphs with few Hamiltonian Cycles, 2018, https://arxiv.org/pdf/1812.05650.pdf

[2]Emanuels Grinbergs, On threeconnected graphs with unique Hamiltonian cycle, The archive of Emanuels Grinbergs manuscripts, University of Latvia, https://dspace.lu.lv/dspace/handle/7/2191

[3]E. Grinbergs, D.Zeps, Threeconnected graphs with only one Hamiltonian circuit, 1986, 2012, https://www.researchgate.net/publication/238736369, http://susurs.mii.lu.lv/Graphlab/Education/grafuTeorijaLatvija/Grinbergs/Grinbergs_ham_un.pdf

[4] Gordon Royle, UNIQUELY HAMILTONIAN GRAPHS, THE UNIVERSITY OF WESTERN AUSTRALIA, talk given in Singapour, 27. September, 2018.

[5] G. F. Royle. The smallest uniquely hamiltonian graph with minimum degree at least 3. https://mathoverflow.net/questions/255784/

what-is-the-smallest-uniquely-hamiltonian-graph-with-minimum-degree-at-least-3/, 2017.

[6] Gordon Royle, in personal communication professor of UWA Gordon Royle sent me three collections of UH3 graphs: 11 graphs with 18 vertices; 82 graphs with 19 vertices, and 1480 graphs with 20 vertices. First two families are most possibly complete, but the third must be about 1700 graphs large, thus it is not complete.

[7] Dainis Zeps, Uniquely Hamiltonian Graphs and Thomassen's conjecture, University of Latvia, 2019, https://www.researchgate.net/publication/330676066

[8] Dainis Zeps, How to compute critical UH3 graphs? 2019, https://www.researchgate.net/publication/330683627

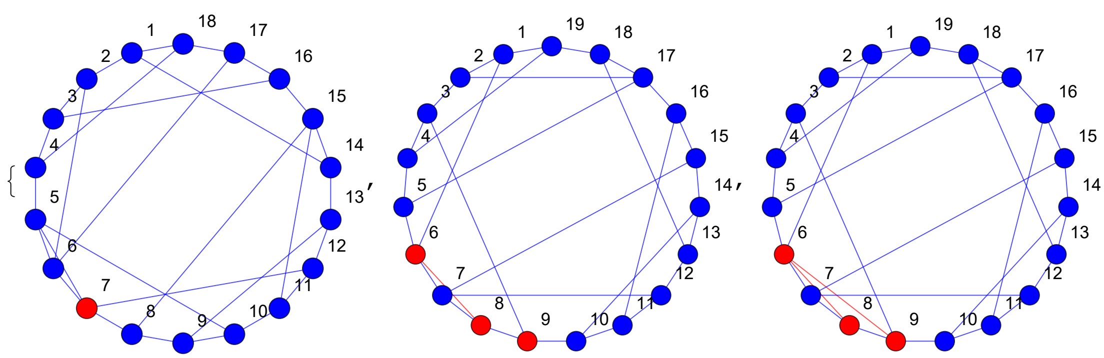
[9] Dainis Zeps, Two-separable spike subgraphs in UH3 graphs, 2019, https://www.researchgate.net/publication/330778877

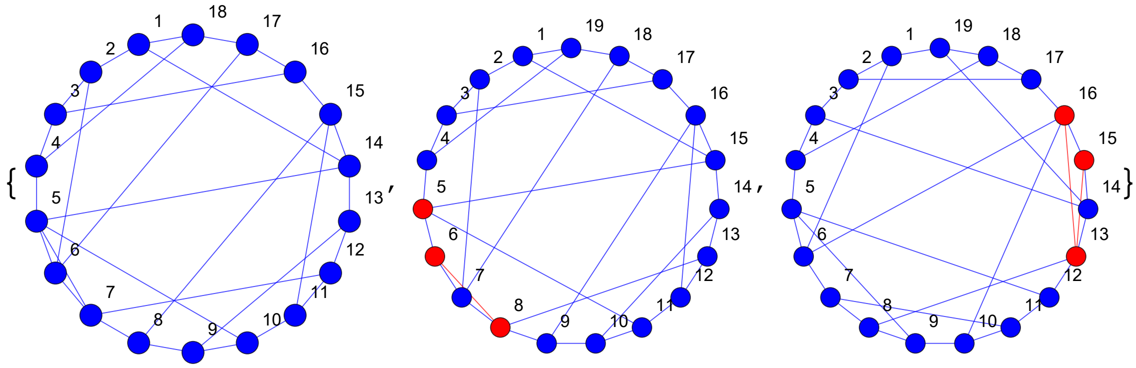
[10] Dainis Zeps, Hamiltonian vertices and Hamiltonian edges in uniquely Hamiltonian graphs, 2019, https://www.researchgate.net/publication/330846342

Appendix 1

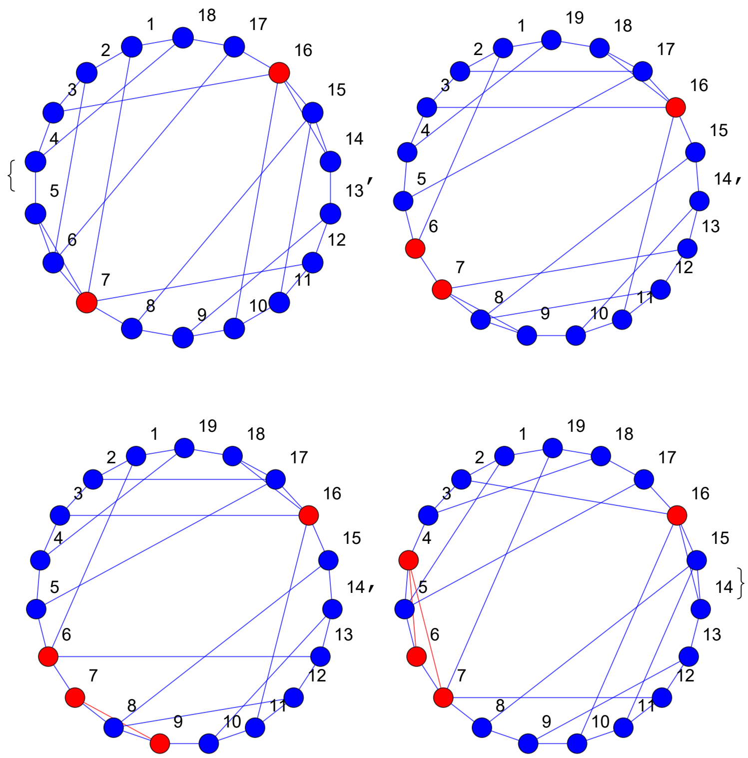
Case 1:

with u with degree > 3 split of 5

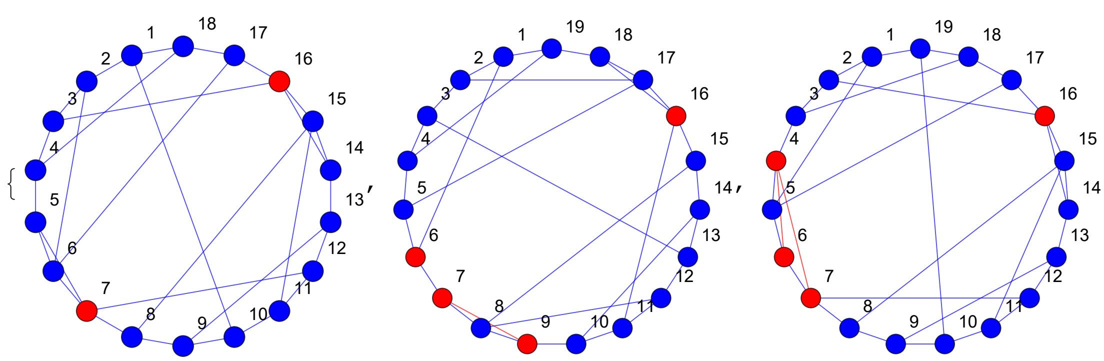




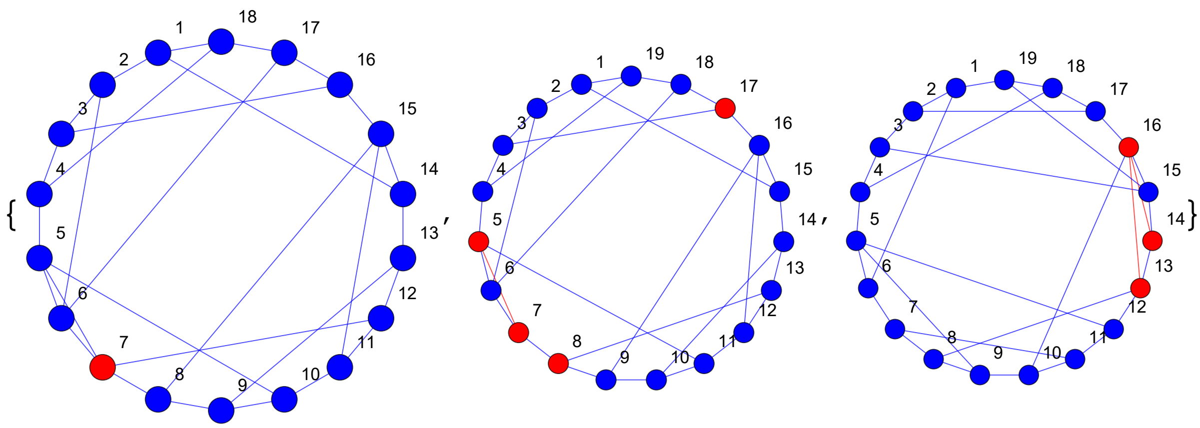
Graph 2, vertex split 7



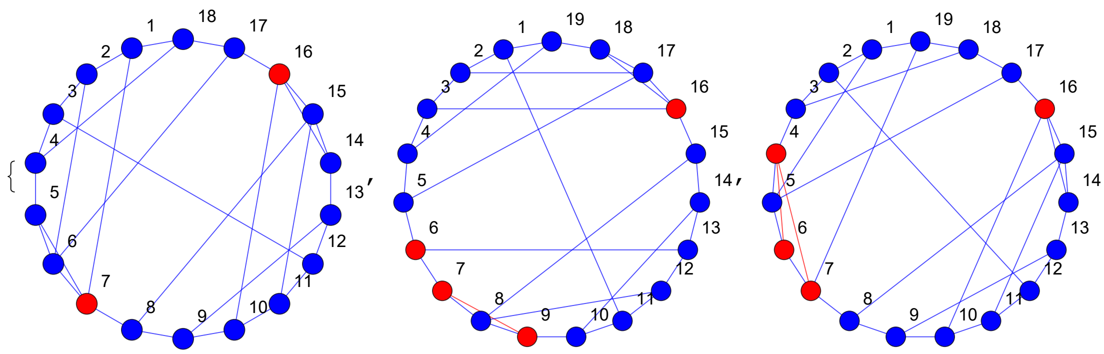
Graph 3, vertex split 7



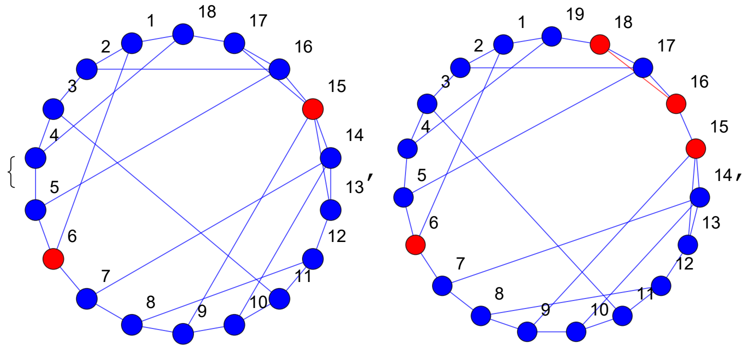
Graph 4, vertex split 7

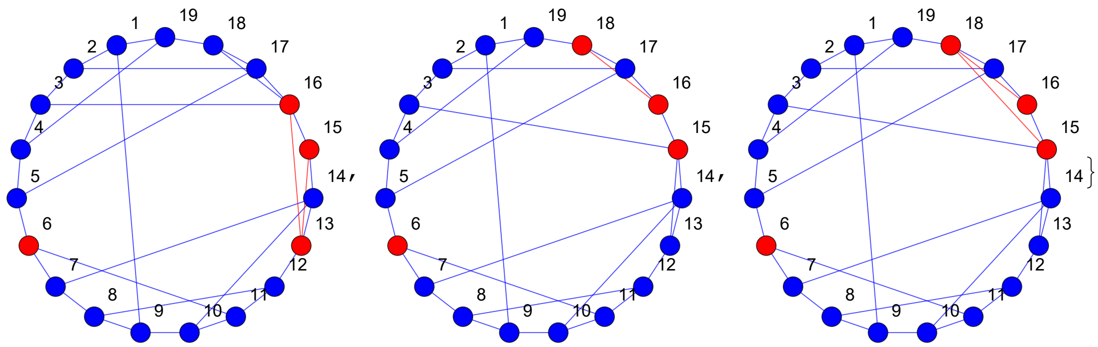


Graph 7, vertex split 7

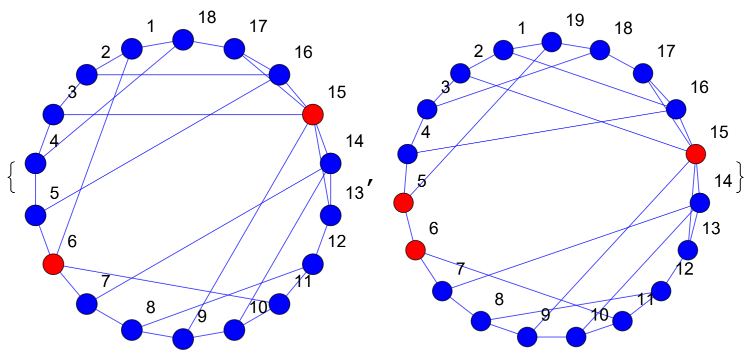


Graph 8, vertex split 15

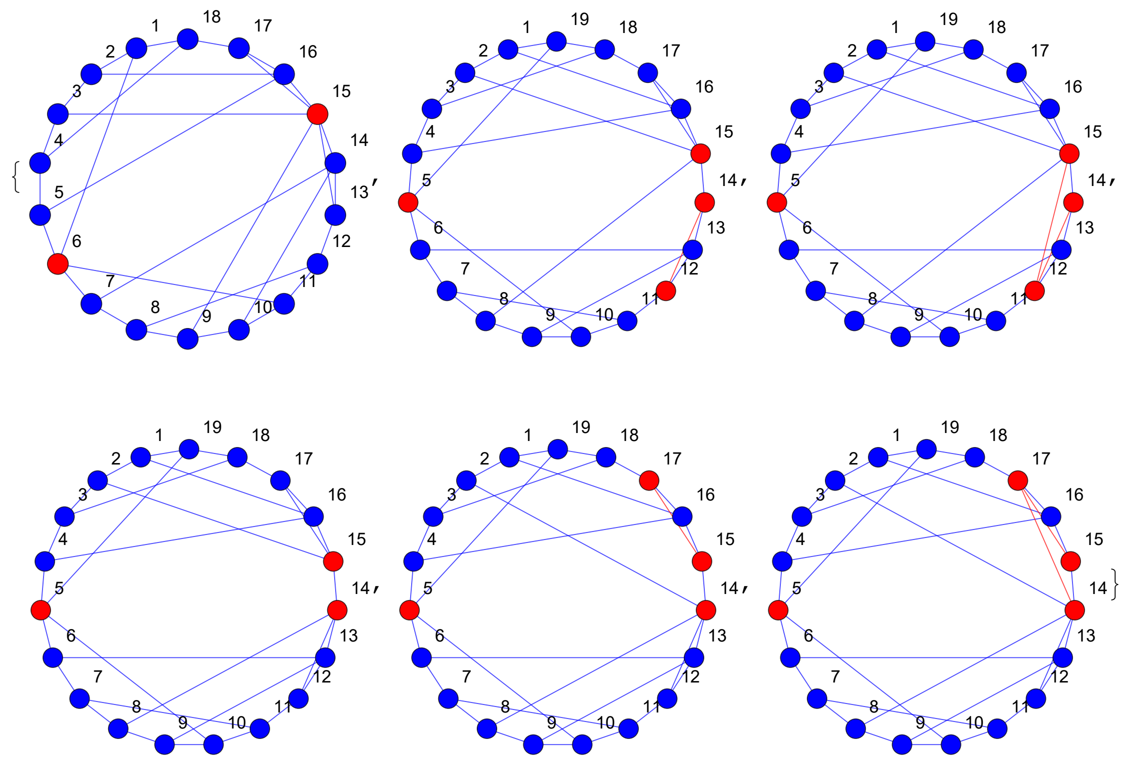




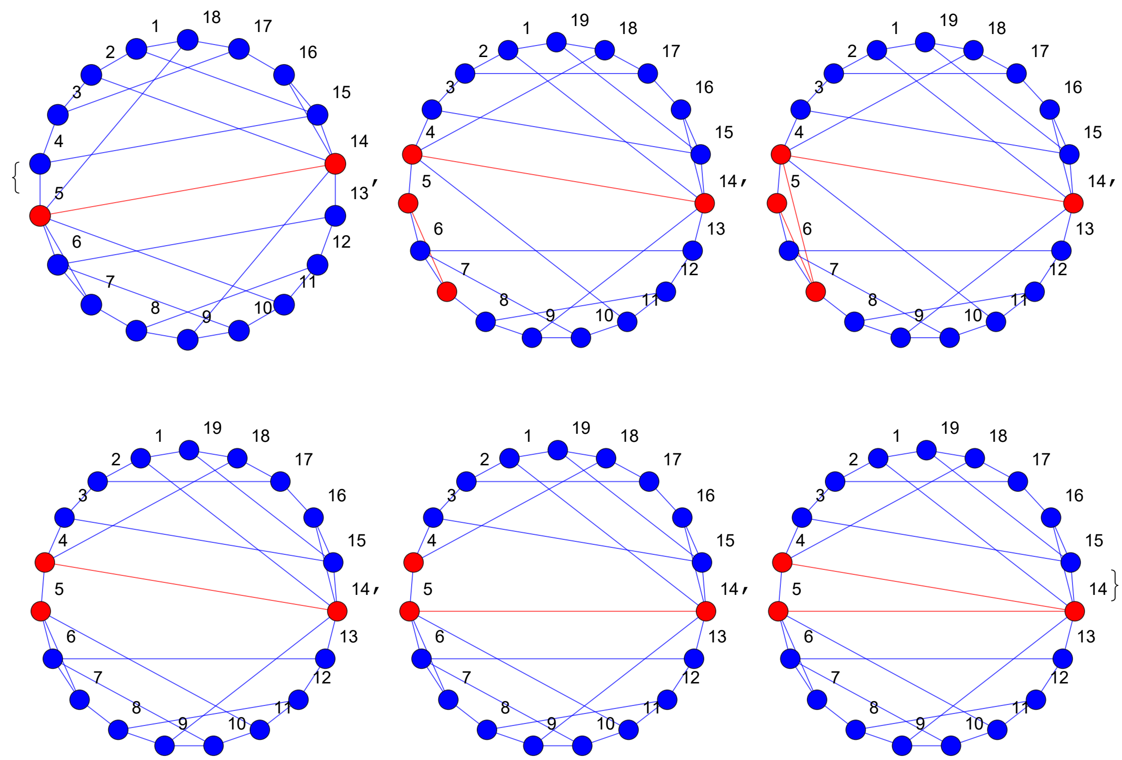
Graph 9, vertex split 6



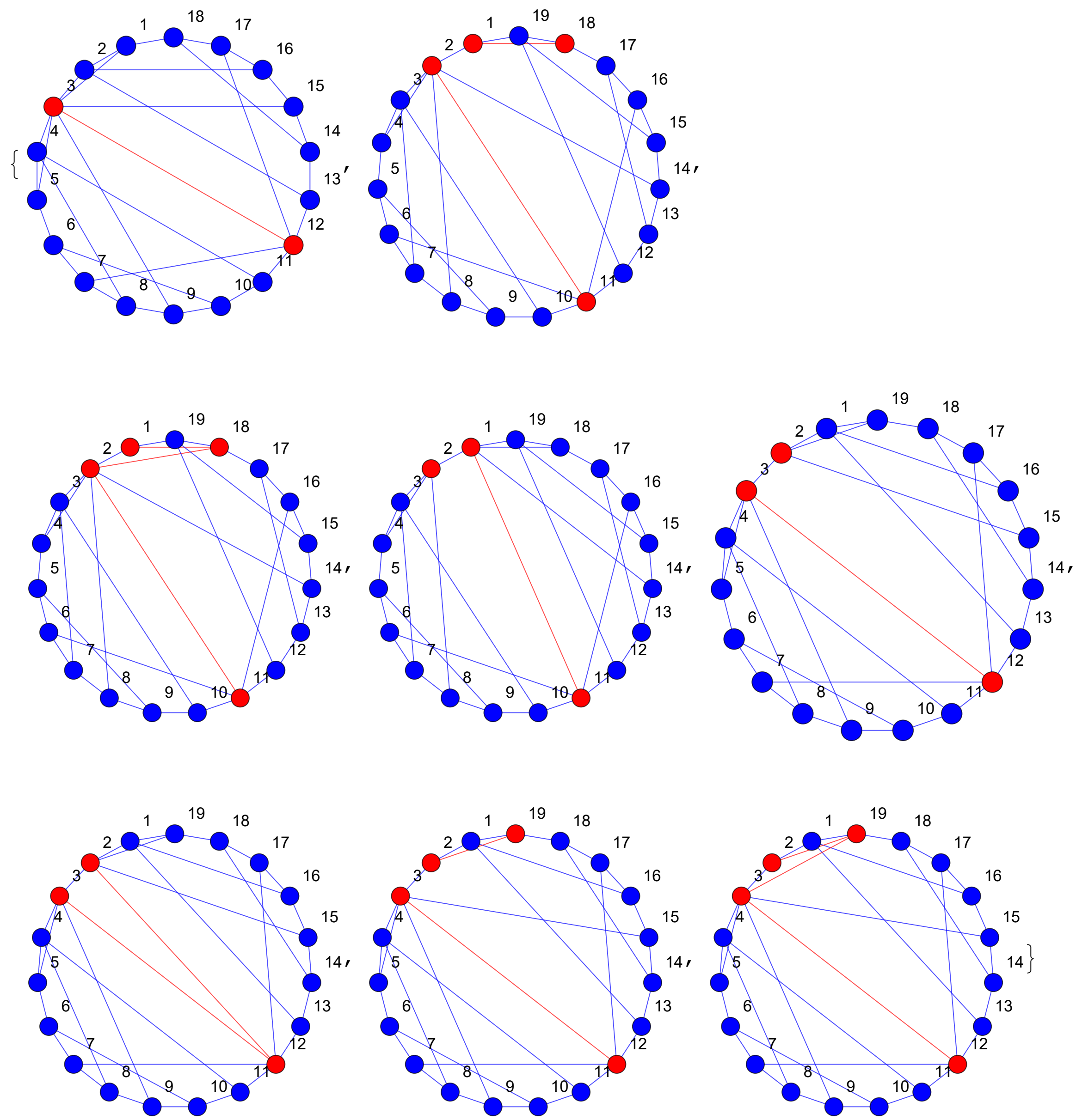
Graph 9, vertex split 15



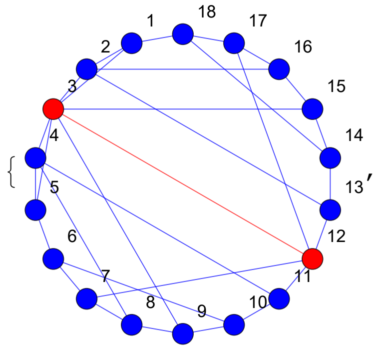
Graph 10, vertex split 5



Graph 11, vertex split 3



Graph 11, vertex split 12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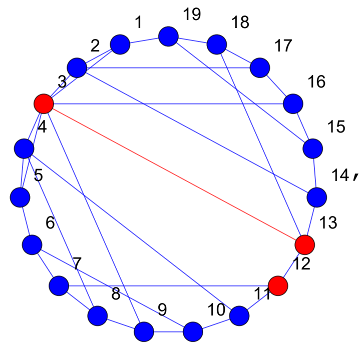
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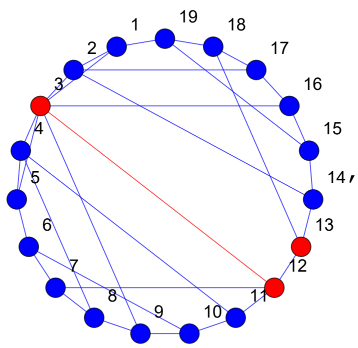
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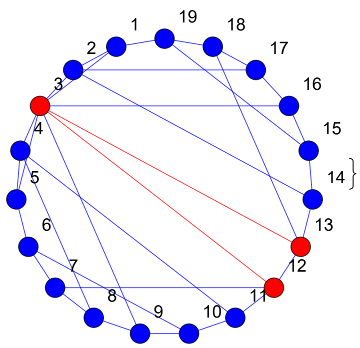
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