University of Latvia
Faculty of Physics and Mathematics
Department of Mathematics

# Dissertation

# Fuzzy matrices and generalized aggregation operators: theoretical foundations and possible applications

by Julija Lebedinska

Scientific supervisor: professor, Dr. Habil. Math.
Aleksandrs Šostaks

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## Annotation

According to the author's observations two mainstreams in the development of the theory of fuzzy sets can be isolated: the fuzzification of already known notions and the development of notions which either were originated in the frame of the theory of fuzzy sets or are tightly related to the theory. Many approaches and notions in topology, algebra, financial calculus and other fields were generalized by using fuzzy sets. Under the second mainstream we can mention such notions as the extension principle, a t-norm, a possibility distribution and others.
The goal of the thesis is to contribute to the both mainstreams. The following task is completed in the thesis: the theory of fuzzy matrices and the theory of generalized aggregation operators are developed and possible practical applications of the obtained results are outlined.
Years over years fuzzy sets community comes with a plenty of new and interesting results in the theory of fuzzy sets. Introduction of new and bright results is the complimentary but not easy task. This contribution has already interested at least one scientist from the community, i.e. the scientific supervisor of the thesis, thus the author considers that its development was not useless.

**MSC:** 15A09, 65G30, 94D05, 03E72, 91B99, 62P20, 62P99.

**Key words and phrases:** Interval matrix, interval inverse matrix, fuzzy matrix, fuzzy inverse matrix, system of interval linear equations, system of fuzzy linear equations, fuzzy input-output model, aggregation operator, generalized aggregation operator, pointwise extension, t-norm, $T$-extension.

## Anotācija

Pēc autores domām nestrikto kopu teorijas (NKT) attīstībā var iezīmēt divus konceptuāli dažādus virzienus:
1. jau zināmo jēdzienu un koncepciju fazifikācija un
2. jēdzienu, kas radušies NKT kontekstā vai arī ir saistīti ar NKT, attīstība.
Daudzi topoloģijas, algebras, finanšu matemātikas un citu nozaru jēdzieni ir vispārināti izmantojot nestriktas kopas. Pie otrā virziena mēs nosacīti varam pieskaitīt turpinājuma principu, t-normas, iespējamību sadalījumu un citus jēdzienus.
Disertācijas mērķis ir sniegt ieguldījumu abu virzienu attīstībā. Darbā ir realizēts sekojos uzdevums: attīstīti nestriktu matricu un vispārināto agregācijas operatoru teorijas un iezīmēti praktisko lietojumu sfēras.
Gadu pēc gada arvien jauni rezultāti paceļ NKT jaunā attīstības līmenī. Jauno un interesanto rezultātu izstrāde ir apsveicams uzdevums, bet tas nav viegls. Darbā iegūtie rezultāti ir ieinteresējuši vismaz vienu cilvēku, kas pieder pie NKT kopienas, t.i prof. A.Šostaku, kas ir darba zinātniskais vadītājs, un tāpēc autore uzskata, ka izvirzītais mērķis ir sasniegts.



**MSC:** 15A09, 65G30, 94D05, 03E72, 91B99, 62P20, 62P99.

**Atslēgas vārdi:** intervāla matrica, inversā intervāla matrica, nestrikta matrica, inversā nestrikta matrica, lineāro intervālo vienādojumu sistēma, lineāro nestrikto vienādojumu sistēma, nestriktais input-output modelis, agregācijas operators, vispārinātais agregācijas operators, punktveida turpinājums, t-norma, T-turpinājums.

# Contents

## 0.1 Introduction

Processing of inexact data is one of the challenging problems for modern engineers and practitioners in different areas. Since Kolmogorov has introduced the axiomatic of the modern probability theory in 1933 and even before it was the main tool used by scientists. But the apparatus of probability theory can not treat all kinds of vagueness, thus necessity of new approach was obvious.

At the beginning of 20th century philosophers actively discussed impossibility of putting real processes or objects into the strict frames based on the principles of bivalent logic (C. Peirce, B. Russell , I.M.Copilowish, C.G.Hempel. M. Black and others). On the other hand active work in this direction was performed by logicians (J. Lukasiewicz and his school), who have developed logics with intermediary truth values.

Polish mathematician and painter L. Chwistek did not believed in philosopher's attempts to reform traditional philosophy by means of mathematical logic (he believed that there are many realities therefore it is impossible to describe the reality by a homogeneous system). But his idea of using intervals instead of single numbers was developed and extended in different directions by many other scientists and as a one direction the interval mathematics (see e.g. Moore [32]) can be mentioned. The theory allowed to look at inexact data from the another perspective.

Provided intensive and fruitful work in 20th century in the circles of philosophers and mathematicians the prompt emergency of the theory of fuzzy sets was obvious. And it was crystallized in Zadeh's pioneering paper [53] in 1965 where foundations of the theory of fuzzy sets were presented. Two years later in 1967 Goguen ([14]) put the basis for the theory of L-fuzzy sets thus extending the theory proposed by Zadeh. On the one hand fuzzy set is a generalization of an interval, on the other it models a set with vague borders and it also can be treated as a function. The opulence of the structure of a fuzzy set makes the theory rich and flexible. Since these historical publications the era of the theory of fuzzy sets started ([7]). Also other close theories such as Dempster - Shafer theory of evidence, Pawlak rough sets theory and others appeared.

In the last forty five years many contributions have developed the theory of fuzzy sets to the impressive theory with the extensive list of notions, tools, theoretical results and the broad area of applications. Such a charming combination of the significant theoretical base and the admirable suitability of the theoretical results for the modeling of real world processes forms the unquestionable **motivation** for scientists-practitioners. The author includes herself to this community and thus would like **to contribute to the theoretical part of the fuzzy sets theory, which can be used for practical applications** and this is the **goal** of the thesis.

In order to reach the goal the following **task** was defined: **to develop the theory of fuzzy matrices and the theory of generalized aggregation operators and to outline possible applications of the obtained results.**

Results provided in the contribution can be divided into two mainstreams: generalization of classical mathematical notions by means of fuzzy sets and contri-

bution to the integral part of the fuzzy sets theory.
The **structure** of the thesis is the following: we recall the basic notions and results of the fuzzy sets theory necessary for our further study in the first chapter; the second chapter is the author's contribution to the generalization of the theory of matrices and tight connection between interval computations and fuzzy sets is highlighted; and the last chapter is devoted to the development of the theory of generalized aggregation operators. The second and the third chapters contain independent results, but they employ the same idea, i.e. generalization of the notion by choosing the next complexity level for the input objects, and namely - a fuzzy set.
For enjoyable reading the second and the third chapters start with detailed description of the structure with brief overview of existing results and the author's contribution.


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# 1 Preliminaries

We flash results of the theory of fuzzy sets in this chapter. We focus mainly on the theoretical results required for our further considerations. Sets, which are called $[0,1]$-sets in the modern terminology, are considered, although all presented results or their modifications hold for L-sets, where L is an arbitrary infinitely distributive lattice. Mainly the source [46] in the bibliography list is cited, but other sources, e.g. [7, 26] can be used.

## 1.1 Fuzzy sets theory: basic definitions

**Definition 1** ([46])**.** A mapping $M : X \to [0,1]$ is called a fuzzy subset of the set $X$ or simply a fuzzy set.

The set of all fuzzy subsets of $X$ will be denoted $F(X)$.
If $M, N : X \to [0,1]$ are fuzzy sets then operations of union, intersection and complement are defined in the following way ([46]):

$$(M \vee N)(x) = \max(M(x), N(x))$$

$$(M \wedge N)(x) = \min(M(x), N(x))$$

$$M^c(x) = 1 - M(x).$$

**Definition 2** ([46])**.** A fuzzy set $M$ is a subset of a fuzzy set $N$ if $M(x) \leq N(x)\ \forall x \in X$.

## 1.2 Decomposition of fuzzy sets

Let a fuzzy set $M$ be given, if we fix $\alpha \in [0,1]$ then:

**Definition 3** ([46])**.** $M^\alpha = \{x : M(x) \geq \alpha\}$ is called the $\alpha$-cut of the fuzzy set $M$.

**Definition 4** ([46])**.** $M_\alpha = \{x : M(x) > \alpha\}$ is called a strict $\alpha$-cut of the fuzzy set $M$.

$M_0$ is called the support of $M$.
If $\alpha, \beta \in [0,1]$ and $\alpha \leq \beta$ then the following inclusions hold:

$$M^\beta \subset M^\alpha, M_\beta \subset M_\alpha, M_\alpha \subset M^\alpha$$

Results formulated below hold for $\alpha$-cuts and strict $\alpha$-cuts:

**Theorem 1.1** ([46], 1.3.6.)**.** *Fuzzy sets $M, N : X \to [0,1]$ are given and $\alpha \in [0,1]$ then*

$$(M \vee N)^\alpha = M^\alpha \cup N^\alpha$$

*and*

$$(M \wedge N)^\alpha = M^\alpha \cap N^\alpha$$

**Theorem 1.2** ([46], 1.3.9.)**.** *Fuzzy sets* $M, N : X \to [0,1]$ *are given and* $\alpha \in [0,1]$ *then*

$$(M \vee N)_\alpha = M_\alpha \cup N_\alpha$$

*and*

$$(M \wedge N)_\alpha = M_\alpha \cap N_\alpha$$

## 1.3 The extension principle

Extension principle is one of the ways how to extend known results to more general cases, and in particular to fuzzy sets.

**Definition 5** ([46])**.** Let a mapping $\varphi : X \times Y \to Z$ be given, then the mapping $\tilde{\varphi} : F(X) \times F(Y) \to F(Z)$ defined by the formula

$$\tilde{\varphi}(M,N)(z) = \sup\{\min(M(x), N(y)) | x \in X, y \in Y, \varphi(x,y) = z\},$$

where $M \in F(X)$, $N \in F(Y)$,
is called the extension of the function $\varphi(x,y)$ to the sets $F(X), F(Y)$.

The extension of the $n$-argument function is done in the same manner:

**Definition 6** ([46])**.** Let a mapping $\varphi : X_1 \times ... \times X_n \to Z$ be given, then the mapping $\tilde{\varphi} : F(X_1) \times ... \times F(X_n) \to F(Z)$ defined by the formula

$$\tilde{\varphi}(M_1, ..., M_n)(z) = \sup\{\min(M_1(x_1), ..., M_n(x_n)) | x_1 \in X_1, ..., x_n \in X_n,$$
$$\varphi(x_1, ..., x_n) = z\},$$

where $M_1 \in F(X_1)$,..., $M_n \in F(X_n)$,
is called the extension of the function $\varphi(x_1, ..., x_n)$ to the sets $F(X_1), ..., F(X_n)$.

Any arithmetic operation can be extended to the operation on fuzzy sets of real numbers.
Although definitions 5 and 6 employ min t-norm other t-norms can be used also, and

$$\tilde{\varphi}(M_1, ..., M_n)(z) = \sup\{T(M_1(x_1), ..., M_n(x_n)) | x_1 \in X_1, ..., x_n \in X_n,$$
$$\varphi(x_1, ..., x_n) = z\} \tag{1}$$

is extension of the function $\varphi$ via an arbitrary t-norm.

## 1.4 Fuzzy quantities, fuzzy intervals and fuzzy numbers

Different properties of fuzzy sets play an important role in the theory of fuzzy sets:

**Definition 7** ([46])**.** A fuzzy set $M : \mathbb{R} \to [0,1]$ is convex provided that

$$\forall x, y, z : x \leq y \leq z \Rightarrow M(y) \geq \min(M(x), M(z))$$

The notion of upper semicontinuous fuzzy set is extensively used in our further study:

**Definition 8** ([46])**.** A mapping $f : X \to \mathbb{R}$ is upper semicontinuous, if for all $t \in \mathbb{R}$ the set $\{x | f(x) \geq t\}$ is closed.

It is a known fact that $M : \mathbb{R} \to [0, 1]$ is an upper semicontinuous fuzzy set if and only if for $\alpha > 0$ $M^\alpha$ are closed sets.
Theorems formulated below characterize upper semicontinuous fuzzy sets from the prospective of application of a continuous operation:

**Theorem 1.3** ([46], theorem 6.2.3.)**.** *If* $\circ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is a continuous operation and* $P, Q \in F(\mathbb{R})$ *are upper semicontinuous fuzzy sets with bounded* $\alpha$*-cuts* $\forall \alpha > 0$ *then for all* $z \in \mathbb{R}$*,* $z = x \circ y$ $\exists x_0, y_0 \in \mathbb{R}$ *such that* $z = x_0 \circ y_0$ *and* $(P \circ Q)(z) = \min(P(x_0), Q(y_0))$

**Theorem 1.4** ([46], theorem 6.2.5.)**.** *If* $\circ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is a continuous operation and* $P, Q \in F(\mathbb{R})$ *are upper semicontinuous fuzzy sets with bounded* $\alpha$*-cuts* $\forall \alpha > 0$ *then*

$$(P \circ Q)^\alpha = P^\alpha \circ Q^\alpha$$

*for all* $\alpha > 0$

Fuzzy quantities is a special class of fuzzy sets:

**Definition 9** ([46])**.** A convex, upper semicontinuous fuzzy set $M : \mathbb{R} \to [0, 1]$ with bounded $\alpha$-cuts for all $\alpha > 0$ is called a fuzzy quantity.

The class of all fuzzy quantities will be denoted $FQ(\mathbb{R})$ and it is characterized by the following results:

**Theorem 1.5** ([46], theorem 6.3.2.)**.** $P \in FQ(\mathbb{R}) \Leftrightarrow \forall \alpha > 0, P^\alpha = \{x : P(x) \geq \alpha\}$ *is a closed interval.*

**Theorem 1.6** ([46], theorem 6.3.3.)**.** *If* $\circ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is a continuous operation and* $P, Q \in FQ(\mathbb{R})$ *then* $P \circ Q \in FQ(\mathbb{R})$

Further in section 3.4.1 we generalize results of theorems 1.3, 1.4, 1.6 for an arbitrary continuous t-norm.
Fuzzy intervals ($FI(\mathbb{R})$) and fuzzy numbers ($FN(\mathbb{R})$) are subclasses of fuzzy quantities:

**Definition 10** ([46])**.** A fuzzy quantity $P$ is called a fuzzy interval if $\exists I = [a, b] \subseteq (-\infty, +\infty) : P(x) = 1 \Leftrightarrow x \in I$. Interval I is called the vertex of $P$.

**Definition 11** ([46])**.** Fuzzy quantity $P$ is called a fuzzy number if $\exists! x \in \mathbb{R} : P(x) = 1$. Point $x$ is called the vertex of $P$.

Triangular fuzzy numbers are fuzzy sets, which have a form of a triangle, and are defined in the following way:

$$M(x) = \begin{cases} 0, \text{ if } x < m_1 \\ \frac{x-m_1}{m_2-m_1}, x \in [m_1; m_2] \\ \frac{x-m_3}{m_2-m_3}, x \in [m_2; m_3] \\ 0, \text{ if } x > m_3 \end{cases}$$

for some $m_1, m_2, m_3 \in \mathbb{R}$, s.t. $m_1 \leq m_2 \leq m_3$ and:

$$M(m_1) = \frac{m_2 - m_1}{m_2 - m_1} = 1, \text{ when } m_1 = m_2$$

$$M(m_2) = \frac{m_3 - m_2}{m_3 - m_2} = 1, \text{ when } m_2 = m_3$$

and sometimes we use notation $M = (m_1, m_2, m_3)$ for a fuzzy triangular number.
Trapezoidal intervals are fuzzy sets, which have a form of a trapeze:

$$N(x) = \begin{cases} 0, \text{ if } x < n_1 \\ \frac{x-n_1}{n_2-n_1}, x \in [n_1; n_2] \\ 1, x \in [n_2, n_3] \\ \frac{x-n_4}{n_3-n_4}, x \in [n_3; n_4] \\ 0, \text{ if } x > n_4 \end{cases}$$

for some $n_1, n_2, n_3, n_4 \in \mathbb{R}$ s.t. $n_1 \leq n_2 \leq n_3 \leq n_4$ and:

$$N(n_1) = \frac{n_2 - n_1}{n_2 - n_1} = 1, \text{ when } n_1 = n_2$$

$$N(n_3) = \frac{n_3 - n_4}{n_3 - n_4} = 1, \text{ when } n_3 = n_4$$

and sometimes we use notation $N = (n_1, n_2, n_3, n_4)$ for a fuzzy trapezoidal interval.

## 1.5 t-norms

The notion of a t-norm is fundamental in different areas of fuzzy sets theory, and it plays an important role in our study. Detailed information on t-norms can be found e.g. in [22, 46]:

**Definition 12** ([46])**.** A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a t-norm provided that:
(1) $T(x, y) = T(y, x)$ - symmetry
(2) $T(T(x, y), z) = T(x, T(y, z))$ - associativity
(3) $x_1 \leq x_2 \Rightarrow T(x_1, y) \leq T(x_2, y))$ - monotonicity
(4) $T(x, 1) = x$

According to definition 12 an arbitrary t-norm fulfills:

$$(3^{'})\text{ if } x_1 \leq x_2, y_1 \leq y_2 \text{ then } T(x_1, y_1) \leq T(x_2, y_2)$$

$$(4^{'})\ T(1, y) = y$$

$$(5)\ T(x, 0) = T(0, y) = 0.$$

**Examples of t-norms:**
Min t-norm:

$$T_M(x, y) = \min(x, y)$$

Drastic t-norm:

$$T_W(x, y) = \begin{cases} \min(x, y), \text{ if } \max(x, y) = 1 \\ 0, \text{ otherwise} \end{cases}$$

Product t-norm:

$$T_P(x, y) = x \cdot y$$

Lukasiewicz t-norm:

$$T_L(x, y) = \max\{x + y - 1, 0\}.$$

For an arbitrary t-norm $T$ the following holds:

$$T_W \leq T \leq T_M$$

**Continuous t-norms:**

**Definition 13** ([46])**.** A t-norm $T : [0, 1] \times [0, 1] \to [0, 1]$ is continuous if it is continuous as the first argument function.

Symmetry of t-norm and definition 13 imply t-norm's continuity as the second argument function. If t-norm is continuous in the sense of definition 13 then it is continuous as the two arguments function.

**Definition 14** ([46])**.** A t-norm $T : [0, 1] \times [0, 1] \to [0, 1]$ is lower semicontinuous, if for an arbitrary sequence $(x_i)_{i \in I} \subset [0, 1]$ and for an arbitrary $y \in [0, 1]$ :

$$\sup_{i \in I} T(x_i, y) = T(\sup_{i \in I} x_i, y).$$

**Definition 15** ([46])**.** A t-norm $T : [0, 1] \times [0, 1] \to [0, 1]$ is upper semicontinuous, if for an arbitrary sequence $(x_i)_{i \in I} \subset [0, 1]$ and for an arbitrary $y \in [0, 1]$ :

$$\inf_{i \in I} T(x_i, y) = T(\inf_{i \in I} x_i, y).$$

A t-norm is continuous if and only if it is lower and upper semicontinuous simultaneously.

**Remark 1.** In [22] we can find the following result:

**Proposition 1.7** ([22], p.17)**.** A non-decreasing function $F : [0, 1]^2 \to [0, 1]$ is lower semiontinuous if and only if it is left-continuous in each component.

Since an arbitrary t-norm is non-decreasing and commutative, it is lower semicontinuous if and only if it is left-continuous in its first component.

# 2 The theory of fuzzy matrices: theoretical foundations and practical applications

The modern mathematics is considering more and more general problems. Different types of generalization are proposed. Classical notions of mathematics need to be generalized as well in order to provide tool for further investigation of generalized problems, otherwise successful development is impossible. The theory of fuzzy matrices, foundations of which are developed in this chapter, is author's small contribution to the field of generalization of classical notions of mathematics.

Although notion of fuzzy matrix rather often appears in literature, mainly in relation to systems of fuzzy linear equations, it is not studied by itself. We carry over the crisp notion to the fuzzy case and study the most important notions related to the theory of matrices. We call a matrix with entries in the form of fuzzy numbers a fuzzy matrix. Apologies should be addressed to W.B. Vasantha Kandasamy, Florentin Smarandache, K. Ilanthenral and others who have developed theory of fuzzy matrices ([49]). Although our object and approaches differ from the mentioned above we use the same term as it reflects the essence of the topic. The chapter is divided into five sections devoted to different aspects of the theory of fuzzy matrices. Mentioned sections include conclusion.

We introduce the notion of a fuzzy matrix in **Section 1**. We also define mathematical operations with fuzzy matrices and prove some related results. Throughout the chapter we assume that extension principle is performed via min t-norm.

**Section 2** is devoted to the notion of the inverse of a fuzzy matrix. The notion of the inverse matrix of an interval matrix ([41]) is the base of the fuzzy notion introduced further, and we recall necessary results at first. Afterwards we contribute to the fuzzy inverse. The notion of regularity of a fuzzy matrix is tightly related to the notion of regularity of an interval matrix ([37, 41]). Calculation of the fuzzy inverse in some special cases is provided. In the case when calculation of a fuzzy inverse is not reasonable due to time resources its estimation can be used, and we provide it here. Examples of the fuzzy inverse and some useful notes on calculation of the fuzzy inverse conclude the section.

**Section 3** deals with the fuzzy analogue of identity matrix, so called fuzzy identity matrix. We briefly review and prove some properties of fuzzy identity matrix and provide its estimation.

We specify practical applications of the theoretical results obtained in this chapter in **Section 4**. Fuzzy approximate solution (FAS) for the system of fuzzy linear equations is introduced. Applicability of the fuzzy inverse to the economic tasks is outlined.

And we conclude the chapter in the last section.

Throughout the work an interval matrix and a fuzzy matrix are typed as boldface uppercase letters with lower indices $I$ and $F$ correspondingly. We provide clarifications if we use different notations for these notions. Matrices with real values are called matrices or crisp matrices and usually are denoted with uppercase letters.

## 2.1 Fuzzy matrix: basic notions

### 2.1.1 Definition

We introduce the notion of a fuzzy matrix:

**Definition 16.** A matrix $\mathbf{A}_F = (A_{ij})_{m\times n}$, where $A_{ij}$ $\forall i,j$ is a fuzzy number is called a fuzzy matrix.

We remind that the set of fuzzy numbers is denoted $FN(\mathbb{R})$.
$(\mathbf{A}_F)^T$ denotes transposed fuzzy matrix. One column matrix $\mathbf{A}_F = (A_{ij})_{m\times 1}$ is called fuzzy vector. Evidently each crisp matrix is a fuzzy matrix.
We introduce definitions of lower and upper dominants of a fuzzy matrix. Let fuzzy matrices $\mathbf{A}_F = (A_{ij})_{n\times n}$, $\mathbf{A}_F^U = (A_{ij}^U)_{n\times n}$ and $\mathbf{A}_F^L = (A_{ij}^L)_{n\times n}$ be given, we say that:

**Definition 17.** $\mathbf{A}_F^U$ is an upper dominant of $\mathbf{A}_F$, if $A_{ij}(x) \leq A_{ij}^U(x)$, $\forall x \in \mathbb{R}, \forall i,j \in \overline{1,n}$.

**Definition 18.** $\mathbf{A}_F^L$ is a lower dominant of $\mathbf{A}_F$, if $A_{ij}(x) \geq A_{ij}^L(x)$, $\forall x \in \mathbb{R}, \forall i,j \in \overline{1,n}$.

The set of all $n\times m$ fuzzy matrices further will be denoted $\mathbb{M}_{n\times m}$. We are mainly interested in square fuzzy matrices, thus if index $n$ indicating the dimension of a fuzzy matrix is omitted we by default assume that fuzzy matrix is square and its dimension is $n$, the corresponding set of fuzzy matrices is denoted $\mathbb{M}$.

### 2.1.2 Operations with fuzzy matrices

This section is devoted to the definition of operations with fuzzy matrices. We remind that all operations with fuzzy numbers are extension of classical operations via min t-norm.
Similarly like in the crisp case fuzzy matrices with the same dimension can be summed:

**Definition 19.** The sum of fuzzy matrices $\mathbf{A}_F = (A_{ij})_{m\times n}$, $\mathbf{B}_F = (B_{ij})_{m\times n}$ is a fuzzy matrix

$$\mathbf{C}_F = \mathbf{A}_F + \mathbf{B}_F,$$

where $\mathbf{C}_F = (C_{ij})_{m\times n} = (A_{ij} + B_{ij})_{m\times n}$.

We define fuzzy matrix multiplication with a fuzzy number $C$ in the following way:

**Definition 20.** Multiplication of a fuzzy matrix $\mathbf{A}_F = (A_{ij})_{m\times n}$ with a fuzzy number $C$ is a fuzzy matrix

$$\mathbf{B}_F = C\mathbf{A}_F,$$

where $\mathbf{B}_F = (B_{ij})_{m\times n} = (CA_{ij})_{m\times n}$.

Evidently in the role of $C$ we can take a crisp number, then we obtain a fuzzy matrix multiplication with a constant from $\mathbb{R}$.
Multiplication of a fuzzy matrix $\mathbf{A}_F$ with $-1 \in \mathbb{R}$ and its subsequent addition to a fuzzy matrix $\mathbf{B}_F$ with the same dimension defines subtraction of two fuzzy matrices.
Product of two fuzzy matrices is defined in the following way:

**Definition 21.** Multiplication of two fuzzy matrices $\mathbf{A}_F = (A_{ij})_{m\times n}$, $\mathbf{B}_F = (B_{ij})_{n\times l}$ is a fuzzy matrix:

$$\mathbf{C}_F = \mathbf{A}_F \mathbf{B}_F,$$

where $\mathbf{C}_F = (C_{ij})_{m\times l} = (\sum\limits_{k=1}^{n} A_{ik}B_{kj})_{m\times l}$.

The following result shows that definitions 19, 20 and 21 define fuzzy matrices:

**Proposition 2.1.** The set $\mathbb{M}$ is closed w.r.t. fuzzy matrices addition, multiplication and multiplication with $C \in FN(\mathbb{R})$.

*Proof.* Fuzzy numbers are special class of fuzzy quantities and operations of addition and multiplication are continuous therefore according to theorem 1.6 $T_M$-extension of a corresponding operation to the set of fuzzy numbers is a fuzzy number. $\square$

## 2.2 Fuzzy inverse matrix

### 2.2.1 The inverse matrix of an interval matrix

We define the fuzzy inverse matrix in the next section, the notion of which is based on the inverse of an interval matrix, therefore we devote this section to the main results on the inverse of an interval matrix. Moore's pioneering work on interval arithmetic [32] can serve as reference on the basic of interval arithmetic. We use approach provided by J. Rohn on the inverse matrix of an interval matrix and all theoretical results presented in this section can be found in [25], [37], [38] and [41].
An interval matrix $\mathbf{A}_I$ is a matrix, whose elements are closed intervals. $\mathbf{A}_I$ can be written by means of the lower bounds matrix $\underline{A}$ and the upper bounds matrix $\overline{A}$:

$$\mathbf{A}_I = [\underline{A}, \overline{A}].$$

The centre matrix $A_c$ of an interval matrix $\mathbf{A}_I$ and its radius matrix $\Delta$ are defined in the following way:

$$A_c = \frac{1}{2}(\underline{A} + \overline{A})$$

$$\Delta = \frac{1}{2}(\overline{A} - \underline{A}).$$

An interval matrix is called regular ([41]) if all $A \in \mathbf{A}_I$ are non-singular (here A is matrix, whose elements belong to the corresponding intervals of the matrix

$\mathbf{A}_I$).
For each regular interval matrix we define the inverse matrix ([41]) as the narrowest interval matrix, which contains all inverse matrices: $(\mathbf{A}_I)^{-1} = \{A^{-1} : A \in A_I\}$.
J.Rohn has proved that we do not need to use all $2^{n^2}$ vertex matrices in order to find the inverse matrix of a regular interval matrix: it is sufficient to verify only $2^{2n-1}$ vertex matrices of a special type $A_{yz}$ (the result holds for other properties as well). Later V.Kreinovich ([25]) showed that further reduction is impossible: without checking all $2^{2n-1}$ matrices $A_{yz}$ one cannot guarantee that desired property holds for all $A \in \mathbf{A}_I$ for an arbitrary $n$ and $\mathbf{A}_I$. Thus, these special vertex matrices provide an optimal finite characterization of linear problems with inexact data.
In the sequel we will provide the main results by J.Rohn, which are necessary for determination of the inverse matrix of an interval matrix.
First we recall the definition of vertex matrices $A_{yz}$ ([38]):
given an $m \times n$ interval matrix $\mathbf{A}_I$, $y \in Y_m$, $z \in Y_n$, where $Y_m, Y_n$ are sets of all m-dimensional and n-dimensional vectors with components of 1 and -1 then:

$$(A_{yz})_{ij} = (A_c)_{ij} - y_i \Delta_{ij} z_j = \begin{cases} \overline{a}_{ij}, & \text{if } y_i z_j = -1, \\ \underline{a}_{ij}, & \text{if } y_i z_j = 1 \end{cases}$$

**Theorem 2.2.** *([38], p.17) An $n \times n$ elements interval matrix $\boldsymbol{A}_I = [\underline{A}; \overline{A}]$ is regular if and only if*

$$(\det A_{yz})(\det A_{y'z'}) > 0,$$

$\forall y, z, y', z' \in Y_n$.

**Example 1.** Let an interval matrix $\mathbf{A}_I = \begin{pmatrix} [1;2] & [3;4] \\ [-9;1] & [8;10] \end{pmatrix}$ be given.
Then $\underline{A} = \begin{pmatrix} 1 & 3 \\ -9 & 8 \end{pmatrix}$ and $\overline{A} = \begin{pmatrix} 2 & 4 \\ 1 & 10 \end{pmatrix}$, $A_c = \begin{pmatrix} 1,5 & 3,5 \\ -4 & 9 \end{pmatrix}$, $\Delta = \begin{pmatrix} 0,5 & 0,5 \\ 5 & 1 \end{pmatrix}$, $Y = ((1,1)^T, (-1,1)^T, (1,-1)^T, (-1,-1)^T)$
If we take $y = (1,1)^T$ and $z = (-1,-1)^T$ then corresponding $A_{yz}$ is:

$$A_{yz} = \begin{pmatrix} 2 & 4 \\ 1 & 10 \end{pmatrix}$$

Similarly we receive other matrices $A_{yz}$:

$$\begin{pmatrix} 1 & 3 \\ -9 & 8 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ -9 & 10 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 1 & 8 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 3 \\ -9 & 10 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ -9 & 8 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 1 & 10 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 1 & 8 \end{pmatrix}.$$

Determinants of matrices $A_{yz}$ are positive and therefore according to theorem 2.2 $\mathbf{A}_I$ is regular.

**Theorem 2.3.** *([38], p. 20 ) If $\boldsymbol{A}_I$ is an $n \times n$ regular interval matrix then its inverse matrix $(\boldsymbol{A}_I)^{-1} = \boldsymbol{B}_I = [\underline{B}; \overline{B}]$ can be determined in the following way:*

$$\underline{B} = \min_{y,z \in Y_n} A_{yz}^{-1}$$

*and*

$$\overline{B} = \max_{y,z \in Y_n} A_{yz}^{-1},$$

*where min and max are determined componentwise.*

According to theorem 2.3 the inverse matrix of the interval matrix given in example 1 is the following:

$$\mathbf{B}_I = \begin{pmatrix} [0.154; 2] & [-1; -0.0638] \\ [-0.25; 0.257] & [0.022; 0.25] \end{pmatrix}$$

Inversion of an interval matrix can be very time consuming, therefore sometimes estimation of a matrix which includes the inverse is more reasonable. Algorithms for bounding the inverse of an interval matrix can be found e.g. in [18].

### 2.2.2 The inverse matrix of a fuzzy matrix

Let's consider a square fuzzy matrix $\mathbf{A}_F = (A_{ij})_{n \times n}$. If one fixes an arbitrary $\alpha \in (0; 1]$ then $\mathbf{A}_F^\alpha = \left(\left[(\underline{a})_{ij}^\alpha, (\overline{a})_{ij}^\alpha\right]\right)_{n \times n}$, is an interval matrix, whose elements are the corresponding $\alpha$-cuts of the elements $A_{ij}$. When $\alpha = 0$ we take the closure of the strict $\alpha$-cut of $A_{ij}, \forall i, j$ in the role of elements of $\mathbf{A}_F^0$, thus obtaining an interval matrix.
An infinite sequence $0, \alpha_1, ..., \alpha_n, ...1$ induces a sequence of interval matrices

$$\mathbf{A}_F^0, \mathbf{A}_F^{\alpha_1}, ..., \mathbf{A}_F^{\alpha_n}, ...A_F^1$$

The corresponding sequence of interval inverse matrices is

$$\mathbf{B}_F^0, \mathbf{B}_F^{\alpha_1}, ..., \mathbf{B}_F^{\alpha_n}, ...\mathbf{B}_F^1,$$

where $\mathbf{B}_F^\alpha = (\mathbf{A}_F^\alpha)^{-1}$; $ij$-th element is denoted $[(\underline{b})_{ij}^\alpha; (\overline{b})_{ij}^\alpha]$.
Now we introduce the notion of regularity of a fuzzy matrix :

**Definition 22.** A fuzzy matrix $\mathbf{A}_F$ is regular if for each $\alpha \in [0; 1]$ the interval matrix $\mathbf{A}_F^\alpha$ is regular.

We will show that for verification of regularity of a matrix $\mathbf{A}_F$ it is enough to verify regularity of $\mathbf{A}_F^0$.

**Proposition 2.4.** A fuzzy matrix $\mathbf{A}_F$ is regular if and only if the interval matrix $\mathbf{A}_F^0$ is regular interval matrix.

*Proof.* Let's assume that $\mathbf{A}_F^0$ is regular. It is true that

$$(A_F^1)_{ij} \subseteq ...(A_F^\alpha)_{ij}.... \subseteq (A_F^0)_{ij}, \forall\alpha \in [0;1], \forall i,j \in \overline{1;n}.$$

Therefore we can write

$$\mathbf{A}_F^1 \subseteq ...\mathbf{A}_F^\alpha.... \subseteq \mathbf{A}_F^0, \forall\alpha \in [0;1]$$

understanding the inclusion of interval matrices as the inclusion of all corresponding elements.
If we take an arbitrary $A \in \mathbf{A}_F^\alpha$ and an arbitrary $\alpha \in [0,1]$ then $A$ is regular, since $A \in \mathbf{A}_F^\alpha \subseteq \mathbf{A}_F^0$ and each $A$ from $\mathbf{A}_F^0$ is non singular. As it is true for an arbitrary $A$ then we can state that $\mathbf{A}_F^\alpha$ is a regular interval matrix for an arbitrary $\alpha \in [0,1]$. And this means that $\mathbf{A}_F$ is regular.
The regularity of $\mathbf{A}_F^0$ follows from the definition of regularity of $\mathbf{A}_F$ .

□

Now we are ready to give definition of the fuzzy inverse. We will define it by means of inversion of corresponding interval matrices:

$$\mathbf{A}_F \longrightarrow \mathbf{A}_F^0, ...\mathbf{A}_F^\alpha, ...\mathbf{A}_F^1 \longrightarrow \mathbf{B}_F^0, ...\mathbf{B}_F^\alpha, ...\mathbf{B}_F^1 \longrightarrow \mathbf{B}_F$$

**Definition 23.** A matrix $\mathbf{B}_F = (B_{ij})_{n\times n}$ with its elements $B_{ij} : \mathbb{R} \to [0,1]$ defined in the following way:

$$B_{ij}(x) = \max\{\alpha : x \in [(\underline{b})_{ij}^\alpha; (\overline{b})_{ij}^\alpha]\} \ \forall x \in \mathbb{R}$$

is called the inverse matrix of a regular fuzzy matrix $\mathbf{A}_F = (A_{ij})_{n\times n}$.

The following assertion holds:

**Proposition 2.5.** $\mathbf{B}_F$ is a fuzzy matrix.

*Proof.* We will show that mappings $B_{ij}(x)\ \forall i,j = \overline{1,n}$ are fuzzy numbers.
According to the definition elements of $\mathbf{B}_F^\alpha, \alpha \in [0,1]$ are closed intervals, thus using definition 23 we obtain that $\alpha$-cuts of $B_{ij}(x)$ are closed intervals. So, $B_{ij}(x)$ are fuzzy quantities (see theorem 1.5).
Since $(\underline{b})_{ij}^1 = (\overline{b})_{ij}^1$, because $\mathbf{A}_F^1$ and correspondingly $\mathbf{B}_F^1$ are crisp matrices, we have that $B_{ij}((\underline{b})_{ij}^1) = B_{ij}((\overline{b})_{ij}^1) = 1$.
For all other $x \in \mathbb{R}$ $B_{ij}(x) < 1$. □

Let's $\mathbf{A}_F^{'}$ be an upper dominnat of $\mathbf{A}_F$, and $(\mathbf{A}_F^{'})^{-1} = \mathbf{B}_F^{'}$, $(\mathbf{A}_F)^{-1} = \mathbf{B}_F$ then:

**Proposition 2.6.** $\mathbf{B}_F^{'}$ is an upper dominant of $\mathbf{B}_F$.

*Proof.* $\mathbf{A}_F \leq \mathbf{A}_F^{'}$ componentwise and this implies that $\mathbf{A}_F^\alpha \subseteq \mathbf{A}_F^{'\alpha}$ for arbitrary $\alpha$.
Let's assume that there exists $\alpha$ such that $\mathbf{B}_F^\alpha \subseteq \mathbf{B}_F^{'\alpha}$ is not true. This means that there exists $B \in \mathbf{B}_F^\alpha$ and $B \notin \mathbf{B}_F^{'\alpha}$. We can find $A$ such that $B = A^{-1}$, $A \in \mathbf{A}_F^\alpha$ but $A \notin \mathbf{A}_F^{'\alpha}$, because otherwise $B$ would be in $\mathbf{B}_F^{'\alpha}$. We received contradictory statement, so, our assumption was not true. □

The same result holds for lower dominant of a fuzzy matrix and its inverse. Calculation of the fuzzy inverse is related to the calculation of the inverse matrices of interval matrices, and this is a time consuming task, when $n$ is large. We contributed to the question of simplification of calculation of the fuzzy inverse in the next section.

### 2.2.3 Calculation of the fuzzy inverse in the case of $2 \times 2$ fuzzy matrix with the same sign pattern elements

Numerous calculations and specificity of matrices $A_{yz}$ design allow us to raise hypotheses that for special type interval matrices the inverse matrix is not defined by all $2^{2n-1}$ $A_{yz}$ matrices. The fuzzy inverse calculation is based on inversion of interval matrices, therefore the main results of this section refer to interval matrices and the same result for fuzzy matrices follows as a corollary from the main result.
We consider a $2 \times 2$ interval matrix with the same sign pattern elements and we introduce the careful notation system, i.e. we fix matrices $A_{yz}$ design:
let's use $\mathbf{A} = ([\underline{a_{ij}}; \overline{a_{ij}}])_{n \times n}$ for interval matrix notation, the corresponding inverse matrix will be $\mathbf{B} = \mathbf{A}^{-1} = ([\underline{b_{ij}}; \overline{b_{ij}}])_{n \times n}$.
When $n = 2$ we need 8 matrices $\overline{A}_{yz}$, let's numerate them from 1 to 8 in the following way:

$$A_1 = \begin{pmatrix} \underline{a_{11}} & \underline{a_{12}} \\ \underline{a_{21}} & \underline{a_{22}} \end{pmatrix}, A_2 = \begin{pmatrix} \underline{a_{11}} & \overline{a_{12}} \\ \underline{a_{21}} & \overline{a_{22}} \end{pmatrix}, A_3 = \begin{pmatrix} \overline{a_{11}} & \underline{a_{12}} \\ \overline{a_{21}} & \underline{a_{22}} \end{pmatrix},$$

$$A_4 = \begin{pmatrix} \overline{a_{11}} & \overline{a_{12}} \\ \overline{a_{21}} & \overline{a_{22}} \end{pmatrix}, A_5 = \begin{pmatrix} \overline{a_{11}} & \underline{a_{12}} \\ \underline{a_{21}} & \overline{a_{22}} \end{pmatrix}, A_6 = \begin{pmatrix} \overline{a_{11}} & \overline{a_{12}} \\ \underline{a_{21}} & \underline{a_{22}} \end{pmatrix},$$

$$A_7 = \begin{pmatrix} \underline{a_{11}} & \underline{a_{12}} \\ \overline{a_{21}} & \overline{a_{22}} \end{pmatrix}, A_8 = \begin{pmatrix} \underline{a_{11}} & \overline{a_{12}} \\ \overline{a_{21}} & \underline{a_{22}} \end{pmatrix}.$$

Numeration does not apply any restrictions, and further results will not depend on it.
Corresponding inverse matrices will be denoted by $B_j, j = \overline{1,8}$ and $\Delta_i, i = \overline{1,8}$ will denote determinants of $A_i$.
It can be shown that in the case, when interval matrix elements are positive, the following statement is valid:

$$\Delta_8 \leq \Delta_i \leq \Delta_5, i = 1, 2, 3, 4, 6, 7.$$

And in the case of the negative sign, we have:

$$\Delta_8 \geq \Delta_i \geq \Delta_5, i = 1, 2, 3, 4, 6, 7.$$

Basically for this special type interval matrices in order to check regularity it is enough to verify $\Delta_8$ ($\Delta_5$) or both $\Delta_8$ and $\Delta_5$.

**Proposition 2.7.** If $\mathbf{A} = ([\underline{a_{ij}};\overline{a_{ij}}])_{2\times 2}$ is a regular interval matrix with the same sign pattern elements then

$$\mathbf{B} = \mathbf{A}^{-1} = ([\underline{b_{ij}};\overline{b_{ij}}])_{2\times 2} = \begin{pmatrix} [b_{11}^5; b_{11}^8] & [b_{12}^8; b_{12}^5] \\ [b_{21}^8; b_{21}^5] & [b_{22}^5; b_{22}^8] \end{pmatrix},$$

where $b_{ij}^5$ are elements of $B_5$ and $b_{ij}^8$ are elements of $B_8$ .

*Proof.* We will consider the case when all elements of $\mathbf{A}$ are positive intervals. When dimension is 2 then $B_j, j = \overline{1,8}$ can be written in the analytical form:

$$B_1 = \frac{1}{\Delta_1}\begin{pmatrix} \underline{a_{22}} & \underline{-a_{12}} \\ \underline{-a_{21}} & \underline{a_{11}} \end{pmatrix}, B_2 = \frac{1}{\Delta_2}\begin{pmatrix} \overline{a_{22}} & \overline{-a_{12}} \\ \underline{-a_{21}} & \underline{a_{11}} \end{pmatrix}, B_3 = \frac{1}{\Delta_3}\begin{pmatrix} \underline{a_{22}} & \underline{-a_{12}} \\ \overline{-a_{21}} & \overline{a_{11}} \end{pmatrix},$$

$$B_4 = \frac{1}{\Delta_4}\begin{pmatrix} \overline{a_{22}} & \overline{-a_{12}} \\ \overline{-a_{21}} & \overline{a_{11}} \end{pmatrix}, B_5 = \frac{1}{\Delta_5}\begin{pmatrix} \overline{a_{22}} & \underline{-a_{12}} \\ \underline{-a_{21}} & \overline{a_{11}} \end{pmatrix}, B_6 = \frac{1}{\Delta_6}\begin{pmatrix} \underline{a_{22}} & \overline{-a_{12}} \\ \underline{-a_{21}} & \overline{a_{11}} \end{pmatrix},$$

$$B_7 = \frac{1}{\Delta_7}\begin{pmatrix} \overline{a_{22}} & \underline{-a_{12}} \\ \overline{-a_{21}} & \underline{a_{11}} \end{pmatrix}, B_8 = \frac{1}{\Delta_8}\begin{pmatrix} \underline{a_{22}} & \overline{-a_{12}} \\ \overline{-a_{21}} & \underline{a_{11}} \end{pmatrix}.$$

Elements, which can qualify for $b_{ij}$ $i, j = 1, 2$, can be split into 2 sets: $X$ defined by $\overline{a_{ij}}$ and Y defined by $\underline{a_{ij}}$.
For $b_{11}$ corresponding sets are:

$$X = \left\{\frac{\overline{a_{22}}}{\Delta_2}, \frac{\overline{a_{22}}}{\Delta_4}, \frac{\overline{a_{22}}}{\Delta_5}, \frac{\overline{a_{22}}}{\Delta_7}\right\}, Y = \left\{\frac{\underline{a_{22}}}{\Delta_1}, \frac{\underline{a_{22}}}{\Delta_3}, \frac{\underline{a_{22}}}{\Delta_6}, \frac{\underline{a_{22}}}{\Delta_8}\right\} \tag{2}$$

$$b_{12} : X = \left\{\frac{\overline{-a_{12}}}{\Delta_2}, \frac{\overline{-a_{12}}}{\Delta_4}, \frac{\overline{-a_{12}}}{\Delta_6}, \frac{\overline{-a_{12}}}{\Delta_8}\right\}, Y = \left\{\frac{\underline{-a_{12}}}{\Delta_1}, \frac{\underline{-a_{12}}}{\Delta_3}, \frac{\underline{-a_{12}}}{\Delta_5}, \frac{\underline{-a_{12}}}{\Delta_7}\right\} \tag{3}$$

$$b_{21} : X = \left\{\frac{\overline{-a_{21}}}{\Delta_3}, \frac{\overline{-a_{21}}}{\Delta_4}, \frac{\overline{-a_{21}}}{\Delta_7}, \frac{\overline{-a_{21}}}{\Delta_8}\right\}, Y = \left\{\frac{\underline{-a_{21}}}{\Delta_1}, \frac{\underline{-a_{21}}}{\Delta_2}, \frac{\underline{-a_{21}}}{\Delta_5}, \frac{\underline{-a_{21}}}{\Delta_6}\right\} \tag{4}$$

$$b_{22} : X = \left\{\frac{\overline{a_{11}}}{\Delta_3}, \frac{\overline{a_{11}}}{\Delta_4}, \frac{\overline{a_{11}}}{\Delta_5}, \frac{\overline{a_{11}}}{\Delta_6}\right\}, Y = \left\{\frac{\underline{a_{11}}}{\Delta_1}, \frac{\underline{a_{11}}}{\Delta_2}, \frac{\underline{a_{11}}}{\Delta_7}, \frac{\underline{a_{11}}}{\Delta_8}\right\}. \tag{5}$$

According to (2) $\min X = \frac{\overline{a_{22}}}{\Delta_5}$ as $\Delta_5$ is the greatest determinant.
We will show that $\frac{\overline{a_{22}}}{\Delta_5} < \frac{\underline{a_{22}}}{\Delta_i}, i = 1, 3, 6, 8$.

$$\frac{\overline{a_{22}}}{\Delta_5} < \frac{\underline{a_{22}}}{\Delta_i} \Leftrightarrow \overline{a_{22}}\Delta_i < \underline{a_{22}}\Delta_5.$$

If one performs multiplication he can see that minuend of the left part is equal to $a_{11}\underline{a_{22}}\overline{a_{22}}$ , (here and after $a_{ij}$ without underline or overline means that $a_{ij}$ can take both values $\underline{a_{ij}}$ or $\overline{a_{ij}}$). Minuend of the right part is equal to $\overline{a_{11}}\underline{a_{22}}\overline{a_{22}}$. Always $a_{11}\underline{a_{22}}\overline{a_{22}} \leq \overline{a_{11}}\underline{a_{22}}\overline{a_{22}}$.
Subtrahend of the left part is equal to $a_{12}a_{21}\overline{a_{22}}$. Subtrahend of the right part is equal to $\underline{a_{12}a_{21}a_{22}}$. Always $a_{12}a_{21}\overline{a_{22}} > \underline{a_{12}a_{21}a_{22}}$.
This means that $\frac{\overline{a_{22}}}{\Delta_5} < \frac{\underline{a_{22}}}{\Delta_i}, i = 1, 3, 6, 8$ and $\underline{b_{11}} = \frac{\overline{a_{22}}}{\Delta_5}$.
According to (2) $\max Y = \frac{\underline{a_{22}}}{\Delta_8}$ as $\Delta_8$ is the smallest determinant.
We will show that $\frac{\underline{a_{22}}}{\Delta_8} > \frac{\overline{a_{22}}}{\Delta_i}, i = 2, 4, 5, 7$.

$$\frac{\underline{a_{22}}}{\Delta_8} > \frac{\overline{a_{22}}}{\Delta_i} \Leftrightarrow \underline{a_{22}}\Delta_i > \overline{a_{22}}\Delta_8.$$

Minuend of the left part is equal to $a_{11}\underline{a_{22}}\overline{a_{22}}$. Minuend of the right part is equal to $\underline{a_{11}}\underline{a_{22}}\overline{a_{22}}$. Always $a_{11}\underline{a_{22}}\overline{a_{22}} \geq \underline{a_{11}}\underline{a_{22}}\overline{a_{22}}$.
Subtrahend of the left part is equal to $a_{12}a_{21}\underline{a_{22}}$. Subtrahend of the right part is equal to $\overline{a_{12}a_{21}a_{22}}$. Always $a_{12}a_{21}\underline{a_{22}} < \overline{a_{12}a_{21}a_{22}}$.
This means that $\frac{\underline{a_{22}}}{\Delta_8} > \frac{\overline{a_{22}}}{\Delta_i}, i = 2, 4, 5, 7$ and $\overline{b_{11}} = \frac{\underline{a_{22}}}{\Delta_8}$.
Proof of the rest part (in the case of negative interval matrix) is analogous, and can be performed just carrying out careful calculations. □

Proposition 2.7 allows us to simplify the calculation of the interval inverse in the case of the same sign pattern elements. But the same is true for the fuzzy inverse as it's definition is based on the interval inverse. Only two inverse matrices $B_5$ and $B_8$ on each $\alpha$-cut need to be calculated.

### 2.2.4 Calculation of the fuzzy inverse in the case of an M - fuzzy matrix

We consider another special case in this section and namely the class of M-matrices. The class of M-matrices was originally proposed by Alexander Ostrowski in 1937 ([12]). A symmetric M-matrix is sometimes called a Stieltjes matrix. M-matrices have found broad application in economics and they are also known as Metzler matrices.
One of the important characteristics of a crisp M-matrix is positivity of its inverse, but in economics as well as other sciences, the inverse-positivity of real square matrices has been an important topic.
Many different definitions of a class of M-matrices can be found in the literature (see e.g. [1]) for the further characterization of a class of M-matrices notions of Z-matrix and P-matrix will be required:

in linear algebra the class of Z-matrices includes matrices whose off-diagonal entries are less than or equal to zero; that is, a Z-matrix $A$ satisfies

$$A = (a_{ij})_{n \times n},\ a_{ij} \leq 0,\ i \neq j.$$

A P-matrix is a complex square matrix such that every principal minor is positive (principal minor is determinant of the submatrix, where the same row and the same column are eliminated). A closely related class is the class of $P_0$-matrices, i.e. the class of P-matrices, such that every principal minor is $\geq 0$.

**Definition 24.** [1] An M-matrix is a Z-matrix with eigenvalues whose real parts are positive.

Another common characterizations of a crisp M-matrix are given below, however we have not find in the literature any prove of equivalence of the following statements:
1. An M -matrix is a non-singular, inverse-positive Z-matrix;
2. All matrices that are both Z-matrices and P-matrices are nonsingular M-matrices;
3. An M-matrix is a square matrix with non-positive off-diagonal entries, positive diagonal entries, non-negative row sums, and at least one positive row sum.
4. $A = (a_{ij})_{n \times n}$ is an M-matrix, if $a_{ij} \leq 0, \forall i \neq j$ and $Au > 0$ for some positive vector $u$.
The inverse positivity of an M-matrix is crucial for us.
Later the notion of an interval M-matrix appeared in the literature (e.g. [34] ):

**Definition 25.** [34] An interval matrix $\mathbf{A}$ is an interval M-matrix if $\forall A \in \mathbf{A}$ is an M-matrix.

We introduce the notion of a fuzzy M-matrix:

**Definition 26.** A fuzzy matrix $\mathbf{A}_F$ is a fuzzy M-matrix if $\forall \alpha \in [0, 1]$ the interval matrix $\mathbf{A}_F^\alpha$ is an interval M-matrix.

The following results for interval matrices can be found in [34]:

**Theorem 2.8.** *[34] An interval matrix $\boldsymbol{A}$ is an interval M-matrix if and only if $\underline{A}$ and $\overline{A}$ are M-matrices.*

As a corollary from this result we obtain the following theorem:

**Theorem 2.9.** *A fuzzy matrix $\boldsymbol{A}_F$ is a fuzzy M-matrix if and only if the interval matrix $\boldsymbol{A}_F^0$ is an interval M-matrix.*

**Theorem 2.10.** *[34] If $\boldsymbol{A}$ is an interval M-matrix then $\boldsymbol{A}^{-1} = [\overline{A}^{-1}, \underline{A}^{-1}]$.*

So, regardless the dimension of $\mathbf{A}_F$, only 2 inverse matrices need to be calculated on each $\alpha$ level.

### 2.2.5 Estimation of the fuzzy inverse

Calculation of the fuzzy inverse is time consuming in an arbitrary case, especially when $n$ is large and no simplification can be used. In such cases estimation of the fuzzy inverse is more reasonable than direct calculation. For further reasoning we need the following auxiliary lemma:

**Lemma 2.11.** For determinants of vertex matrices $A_{yz}$ (introduced in section 2.2.1) the following inequality holds:

$$\Delta_1 \leq \det A_{yz} \leq \Delta_2,$$

where

$$\Delta_1 = \begin{cases} \frac{n!}{2}(\underline{a}^n - \overline{a}^n), \text{ if } \underline{a}, \overline{a} \geq 0 \\ \frac{n!}{2}(\underline{a}^n - \overline{a}^n), \text{ if } \underline{a}, \overline{a} \leq 0 \text{ and } n = 2k+1, k \in \mathbb{N} \\ \frac{n!}{2}(\overline{a}^n - \underline{a}^n), \text{ if } \underline{a}, \overline{a} \leq 0 \text{ and } n = 2k, k \in \mathbb{N} \\ -n!a^n, \text{ if } \underline{a} \leq 0, \overline{a} \geq 0 \end{cases}$$

$$\Delta_2 = \begin{cases} \frac{n!}{2}(\overline{a}^n - \underline{a}^n), \text{ if } \underline{a}, \overline{a} \geq 0 \\ \frac{n!}{2}(\overline{a}^n - \underline{a}^n), \text{ if } \underline{a}, \overline{a} \leq 0 \text{ and } n = 2k+1, k \in \mathbb{N} \\ \frac{n!}{2}(\underline{a}^n - \overline{a}^n), \text{ if } \underline{a}, \overline{a} \leq 0 \text{ and } n = 2k, k \in \mathbb{N} \\ n!a^n, \text{ if } \underline{a} \leq 0, \overline{a} \geq 0, \end{cases}$$

where $\underline{a} = \min_{i,j} \underline{a}_{ij}$, $\overline{a} = \max_{i,j} \overline{a}_{ij}$, $a = \max\{|\underline{a}|, |\overline{a}|\}$.

*Proof.* For arbitrary crisp matrix ([19], p. 20)$A = (a_{ij})_{n \times n}$ we have:

$$\det A = \sum_{\sigma} sgn\sigma \prod_{i=1}^{n} a_{i\sigma(i)},$$

where $\sigma$ runs over the set of all $n!$ permutations of the set $\{1, 2, ..., n\}$;
$sgn\sigma$ denotes the sign of permutation $\sigma$: it is $+$ $(-)$ if number of transpositions (number of necessary rearrangements in the set $\sigma$ in order to receive set $\{1, 2, ..., n\}$) in the permutation $\sigma$ is even (odd);
$\sigma(i)$ is number on the i-th position in $\sigma$.
So, each product $a_{1\sigma(1)}a_{2\sigma(2)}...a_{n\sigma(n)}$ enters $\det A$ with $+$ or $-$ sign. Moreover, exactly $n!/2$ products have sign of permutation (not final sign though)$+$ and the same number have sign $-$.
Matrices $A_{yz}$ have elements $\underline{a}_{ij}$ and $\overline{a}_{ij}$. Evidently, $\underline{a}$ $(\overline{a})$ is the smallest (the greatest) element, which can appear in any matrix $A_{yz}$ and $\underline{a} \leq \overline{a}$. Further we consider different cases:
a) $\underline{a}$ ,$\overline{a} \geq 0$:
then each product $a_{1\sigma(1)}a_{2\sigma(2)}...a_{n\sigma(n)} \geq 0$ ($a_{ij}$ denotes $\underline{a}_{ij}$ or $\overline{a}_{ij}$) and it $n!/2$ times enters $\det A_{yz}$ with $+$ sign and it $n!/2$ times enters $\det A_{yz}$ with $-$ sign.
The inequality:

$$\underline{a}^n \leq a_{1\sigma(1)}a_{2\sigma(2)}...a_{n\sigma(n)} \leq \overline{a}^n$$

is true.
Using provided inequality we obtain the upper estimation:

$$\det A_{yz} \leq \frac{n!}{2}\overline{a}^n - \frac{n!}{2}\underline{a}^n = \frac{n!}{2}(\overline{a}^n - \underline{a}^n)$$

the same inequality gives us the following lower estimation:

$$\det A_{yz} \geq \frac{n!}{2}\underline{a}^n - \frac{n!}{2}\overline{a}^n = \frac{n!}{2}(\underline{a}^n - \overline{a}^n).$$

So,

$$\frac{n!}{2}(\underline{a}^n - \overline{a}^n) \leq \det A_{yz} \leq \frac{n!}{2}(\overline{a}^n - \underline{a}^n).$$

b) $\underline{a}$ ,$\overline{a} \leq 0$, $n = 2k, k \in \mathbb{N}$:
then $a_{1\sigma(1)}a_{2\sigma(2)}...a_{n\sigma(n)} \geq 0$ and for an arbitrary $a_{1\sigma(1)}a_{2\sigma(2)}...a_{n\sigma(n)}$ the following estimations hold ($|\underline{a}| \geq |\overline{a}|$, because $\underline{a} \leq \overline{a} \leq 0$):

$$\overline{a}^n \leq a_{1\sigma(1)}a_{2\sigma(2)}...a_{n\sigma(n)} \leq \underline{a}^n,$$

then $\det A_{yz}$ from the above is estimated in the following way:

$$\det A_{yz} \leq \frac{n!}{2}\underline{a}^n - \frac{n!}{2}\overline{a}^n = \frac{n!}{2}(\underline{a}^n - \overline{a}^n).$$

In the same manner we get estimation from the bellow:

$$\det A_{yz} \geq \frac{n!}{2}\overline{a}^n - \frac{n!}{2}\underline{a}^n = \frac{n!}{2}(\overline{a}^n - \underline{a}^n).$$

So,

$$\frac{n!}{2}(\overline{a}^n - \underline{a}^n) \leq \det A_{yz} \leq \frac{n!}{2}(\underline{a}^n - \overline{a}^n).$$

c) $\underline{a}$ ,$\overline{a} \leq 0$, $n = 2k+1, k \in \mathbb{N}$:
then $a_{1\sigma(1)}a_{2\sigma(2)}...a_{n\sigma(n)} \leq 0$ and for an arbitrary $a_{1\sigma(1)}a_{2\sigma(2)}...a_{n\sigma(n)}$ the following estimations hold:

$$\underline{a}^n \leq a_{1\sigma(1)}a_{2\sigma(2)}...a_{n\sigma(n)} \leq \overline{a}^n,$$

then $\det A_{yz}$ from the above is estimated in the following way:

$$\det A_{yz} \leq \frac{n!}{2}\overline{a}^n - \frac{n!}{2}\underline{a}^n = \frac{n!}{2}(\overline{a}^n - \underline{a}^n).$$

In the same manner we get estimation form the bellow:

$$\det A_{yz} \geq \frac{n!}{2}\underline{a}^n - \frac{n!}{2}\overline{a}^n = \frac{n!}{2}(\underline{a}^n - \overline{a}^n).$$

So,

$$\frac{n!}{2}(\underline{a}^n - \overline{a}^n) \leq \det A_{yz} \leq \frac{n!}{2}(\overline{a}^n - \underline{a}^n).$$

d)$\underline{a} \leq 0$, $\overline{a} \geq 0$:
we introduce notation: $a = \max\{|\underline{a}|, |\overline{a}|\}$; an arbitrary product $a_{1\sigma(1)}a_{2\sigma(2)}...a_{n\sigma(n)}$ is $\geq 0$ or $\leq 0$, but definitely inequality

$$-a^n \leq a_{1\sigma(1)}a_{2\sigma(2)}...a_{n\sigma(n)} \leq a^n$$

is true. So we receive:

$$-n!a^n \leq \det A_{yz} \leq n!a^n.$$

□

Now we start construction of an upper dominant of the fuzzy inverse. At first lemma estimating the interval inverse is given, after that we build upper dominant of the fuzzy inverse.

**Lemma 2.12.** Let $\mathbf{A}_I = ([\underline{a}_{ij}, \overline{a}_{ij}])_{n\times n}$ be an arbitrary regular interval matrix and $\mathbf{B}_I = ([\underline{b}_{ij}, \overline{b}_{ij}])_{n\times n}$ be its inverse matrix, then for $[\underline{b}_{ij}, \overline{b}_{ij}]$ the following estimations hold:

$$[\underline{b}_{ij}, \overline{b}_{ij}] \subseteq \left[\underline{a}_{ij}^{-1} - 2\frac{\Delta_2^{n-1}}{\Delta}, \underline{a}_{ij}^{-1} + 2\frac{\Delta_2^{n-1}}{\Delta}\right] \bigcap \left[\overline{a}_{ij}^{-1} - 2\frac{\Delta_2^{n-1}}{\Delta}, \overline{a}_{ij}^{-1} + 2\frac{\Delta_2^{n-1}}{\Delta}\right],$$

where
$\underline{A}^{-1} = (\underline{a}_{ij}^{-1})_{n\times n}$, $\overline{A}^{-1} = (\overline{a}_{ij}^{-1})_{n\times n}$,
$\Delta_2^{n-1}$ is estimation from lemma 2.11, which is based on $(n-1)$-dimensional matrix, $\Delta = \min_{k=1,...,2^{2n-1}}(|\det A^k|)$.

*Proof.* According to Rohn's results we have $2^{2n-1}$ vertex matrices, numerated in an arbitrary way:

$$A^1, A^2, ..., A^{2^{2n-1}}$$

and their inverses

$$B^1, B^2, ...B^k = (A^k)^{-1}, ...B^{2^{2n-1}}$$

should be calculated in order to calculate $\mathbf{B}_I$.
We use notation

$$B^k = \frac{1}{\det A^k}\begin{pmatrix} A_{11}^k & A_{21}^k... & A_{n1}^k \\ ... & ... & ... \\ A_{1n}^k & A_{2n}^k... & A_{nn}^k \end{pmatrix},$$

where $A_{ij}^k$ is cofactor of matrix

$$A^k = \begin{pmatrix} a_{11}^k & a_{12}^k... & a_{1n}^k \\ ... & ... & ... \\ a_{n1}^k & a_{n2}^k... & a_{nn}^k \end{pmatrix}$$

and $a_{ij}^k$ is equal to $\underline{a_{ij}}$ or $\overline{a_{ij}}$ according to the definition of vertex matrices.
According to the definition of the inverse matrix and theorem 2.3 $ij$-th element of $\mathbf{B}_I$ is equal to:

$$[\underline{b}_{ij}, \overline{b}_{ij}] = \left[\frac{A_{ji}^s}{\det A^s}; \frac{A_{ji}^t}{\det A^t}\right], \tag{6}$$

where

$$\frac{A^s_{ji}}{\det A^s} = \min_{k=1,\dots,2^{2n-1}} \frac{A^k_{ji}}{\det A^k}$$

and

$$\frac{A^t_{ji}}{\det A^t} = \max_{k=1,\dots,2^{2n-1}} \frac{A^k_{ji}}{\det A^k}.$$

First we determine the maximal possible length of intervals of the interval inverse matrix:

$$\overline{b}_{ij} - \underline{b}_{ij} \le |\overline{b}_{ij}| + |\underline{b}_{ij}| = \left|\frac{A^t_{ji}}{\det A^t}\right| + \left|\frac{A^s_{ji}}{\det A^s}\right| \tag{7}$$

we assume that $\left|\frac{A^t_{ji}}{\det A^t}\right| \ge \left|\frac{A^s_{ji}}{\det A^s}\right|$ then formula (7) can be continued in the following way:

$$\overline{b}_{ij} - \underline{b}_{ij} \le 2\left|\frac{A^t_{ji}}{\det A^t}\right| \tag{8}$$

$A^t_{ji}$ is cofactor of the vertex matrix $A^t$, thus estimations from lemma 2.11 hold, the only thing what need to be noticed that dimension is $n-1$. Using these estimations and applying $\Delta = \min_{k=1,\dots,2^{2n-1}}(|\det A^k|)$ for the estimation of the denominator we obtain:

$$\overline{b}_{ij} - \underline{b}_{ij} \le 2\frac{\Delta_2^{n-1}}{\Delta}. \tag{9}$$

Thus, formula (9) gives us an upper estimation of the length of an arbitrary interval of the inverse interval matrix.
$\underline{A}^{-1} = (\underline{a}^{-1}_{ij})_{n\times n}, \overline{A}^{-1} = (\overline{a}^{-1}_{ij})_{n\times n} \in \mathbf{B}_I$ thus the following estimations hold:

$$[\underline{b}_{ij}, \overline{b}_{ij}] \subseteq \left[\underline{a}^{-1}_{ij} - 2\frac{\Delta_2^{n-1}}{\Delta}, \underline{a}^{-1}_{ij} + 2\frac{\Delta_2^{n-1}}{\Delta}\right] \tag{10}$$

$$[\underline{b}_{ij}, \overline{b}_{ij}] \subseteq \left[\overline{a}^{-1}_{ij} - 2\frac{\Delta_2^{n-1}}{\Delta}, \overline{a}^{-1}_{ij} + 2\frac{\Delta_2^{n-1}}{\Delta}\right] \tag{11}$$

and as a result:

$$[\underline{b}_{ij}, \overline{b}_{ij}] \subseteq \left[\underline{a}^{-1}_{ij} - 2\frac{\Delta_2^{n-1}}{|\Delta|}, \underline{a}^{-1}_{ij} + 2\frac{\Delta_2^{n-1}}{|\Delta|}\right] \bigcap \left[\overline{a}^{-1}_{ij} - 2\frac{\Delta_2^{n-1}}{|\Delta|}, \overline{a}^{-1}_{ij} + 2\frac{\Delta_2^{n-1}}{|\Delta|}\right] \tag{12}$$

□

Now we provide construction, which builds an upper dominant of the fuzzy inverse:

**Construction 1.** We have an arbitrary fuzzy matrix $\mathbf{A}_F$ and corresponding set of interval matrices:

$$\mathbf{A}^0_F, \dots, \mathbf{A}^\alpha_F = [\underline{A}^\alpha, \overline{A}^\alpha] = ([\underline{a}^\alpha_{ij}, \overline{a}^\alpha_{ij}])_{n\times n}, \dots, \mathbf{A}^1_F.$$

According to lemma 2.12 for an arbitrary element of interval matrix $(\mathbf{A}_F^\alpha)^{-1} = \mathbf{B}_F^\alpha = ([\underline{b}_{ij}^\alpha, \overline{b}_{ij}^\alpha])_{n\times n}$ $\alpha \in [0,1)$ the following estimation holds:

$$[\underline{b}_{ij}^\alpha, \overline{b}_{ij}^\alpha] \subseteq \left[(\underline{a}_{ij}^\alpha)^{-1} - 2\frac{\Delta_2^{n-1}}{\Delta}, (\underline{a}_{ij}^\alpha)^{-1} + 2\frac{\Delta_2^{n-1}}{\Delta}\right] \bigcap$$

$$\bigcap \left[(\overline{a}_{ij}^\alpha)^{-1} - 2\frac{\Delta_2^{n-1}}{\Delta}, (\overline{a}_{ij}^\alpha)^{-1} + 2\frac{\Delta_2^{n-1}}{\Delta}\right], \tag{13}$$

where
$(\underline{A}^\alpha)^{-1} = ((\underline{a}_{ij}^\alpha)^{-1})_{n\times n}$, $(\overline{A}^\alpha)^{-1} = ((\overline{a}_{ij}^\alpha)^{-1})_{n\times n}$,
$\Delta_2^{n-1}$ is estimation from lemma 2.11, which is based on $n-1$-dimensional matrix,
$\Delta = \min_{k=1,\dots,2^{2n-1}}(|\det A^k|)$ and $A^k$ are vertex matrices of interval matrix $\mathbf{A}_F^\alpha$.
When $\alpha = 1$ elements of the crisp matrix $\mathbf{B}_F^1 = (b_{ij}^1)_{n\times n}$ are evaluated by means of corresponding elements of the crisp matrix:

$$(\mathbf{A}_F^1)^{-1} = ((a_{ij}^1)^{-1})_{n\times n} \tag{14}$$

We denote $I_\alpha$, $\alpha \in [0,1]$ interval, which includes $[\underline{b}_{ij}^\alpha, \overline{b}_{ij}^\alpha]$ according to formulas (13) and (14).
For all $x \in \mathbb{R}$ we assign the set of indices $N_x$ in the following way:

$$\alpha \in N_x \Leftrightarrow x \in I_\alpha \tag{15}$$

The upper dominant $\mathbf{B}_F^U = (B_{ij}^U)_{n\times n}$ of the fuzzy inverse matrix $\mathbf{B}_F = (B_{ij})_{n\times n}$ is defined in the following way:

$$B_{ij}^U(x) = \max_{\alpha\in N_x} \alpha,\ \forall i,j = 1,...,n. \tag{16}$$

Obviously $B_{ij}(x) \leq B_{ij}^U(x)$ $\forall x$ $\forall i,j = 1,...,n$ and we finish construction here.

Construction of $\mathbf{B}_F^L = (B_{ij}^L)_{n\times n}$ i.e. the lower dominant of $\mathbf{B}_F$ is accomplished in the same manner. At first we build intervals, which are included into elements of the interval inverse and then we apply construction 1 (namely formulas (15), (16)).
Let's consider an arbitrary interval matrix $\mathbf{A}_F^\alpha$, $\alpha \in [0,1)$ which coincides with fuzzy matrix $\mathbf{A}_F$.
$(\underline{A}^\alpha)^{-1}, (\overline{A}^\alpha)^{-1} \in \mathbf{B}_F^\alpha$, therefore the following inclusion holds:

$$[\underline{b}_{ij}^\alpha, \overline{b}_{ij}^\alpha] \supseteq [\min\{(\underline{a}_{ij}^\alpha)^{-1}, (\overline{a}_{ij}^\alpha)^{-1}\}, \max\{(\underline{a}_{ij}^\alpha)^{-1}, (\overline{a}_{ij}^\alpha)^{-1}\}] \tag{17}$$

Elements of the crisp matrix $\mathbf{B}_F^1 = (b_{ij}^1)_{n\times n}$ are calculated directly $\mathbf{B}_F^1 = (\mathbf{A}_F^1)^{-1} = ((a_{ij}^1)^{-1})_{n\times n}$.
We use estimation (17), elements of $\mathbf{B}_F^1$ and apply construction 1, as a result we get $\mathbf{B}_F^L$.

### 2.2.6 Examples of the fuzzy inverse

We give examples of the fuzzy inverse in this section. Values of the fuzzy inverse are given on $\alpha$-cuts. $\underline{b_{ij}}$ and $\overline{b_{ij}}$ denote correspondingly the maximal and the minimal elements of $\alpha$-cut.

**Example 2.** Let $\mathbf{A}_{F_1} = \begin{pmatrix} (0.5;\ 1.5;\ 2) & (3;\ 3.5;\ 4) \\ (9;\ 13;\ 20) & (8;\ 8.5;\ 10) \end{pmatrix}$ be a fuzzy matrix with entries in the form of positive triangular numbers.
Then its inverse is $\mathbf{B}_{F_1}$:

| $\alpha$ | $\underline{b_{11}}$ | $\overline{b_{11}}$ | $\underline{b_{12}}$ | $\overline{b_{12}}$ | $\underline{b_{21}}$ | $\overline{b_{21}}$ | $\underline{b_{22}}$ | $\overline{b_{22}}$ |
|---|---|---|---|---|---|---|---|---|
| 0.0 | -1.43 | -0.11 | 0.05 | 0.43 | 0.26 | 1.29 | -0.29 | -0.01 |
| 0.1 | -1.04 | -0.11 | 0.06 | 0.32 | 0.27 | 0.99 | -0.21 | -0.01 |
| 0.2 | -0.81 | -0.12 | 0.06 | 0.26 | 0.28 | 0.82 | -0.16 | -0.01 |
| 0.3 | -0.66 | -0.13 | 0.06 | 0.22 | 0.29 | 0.71 | -0.13 | -0.01 |
| 0.4 | -0.55 | -0.14 | 0.07 | 0.19 | 0.30 | 0.62 | -0.11 | -0.02 |
| 0.5 | -0.47 | -0.15 | 0.07 | 0.17 | 0.31 | 0.56 | -0.09 | -0.02 |
| 0.6 | -0.41 | -0.17 | 0.08 | 0.15 | 0.32 | 0.51 | -0.08 | -0.02 |
| 0.7 | -0.36 | -0.19 | 0.08 | 0.14 | 0.33 | 0.48 | -0.07 | -0.03 |
| 0.8 | -0.32 | -0.21 | 0.09 | 0.12 | 0.35 | 0.45 | -0.06 | -0.03 |
| 0.9 | -0.29 | -0.23 | 0.10 | 0.11 | 0.37 | 0.42 | -0.05 | -0.04 |
| 1.0 | -0.26 | -0.26 | 0.11 | 0.11 | 0.40 | 0.40 | -0.05 | -0.05 |

**Example 3.** Let $\mathbf{A}_{F_2} = \begin{pmatrix} (-2;\ -1.5;\ -0.5) & (-5.5;\ -3.5;\ -3) \\ (-18;\ -13;\ -9) & (-11;\ -8.5;\ -8) \end{pmatrix}$ be a fuzzy matrix with entries in the form of negative triangular numbers.
Then its inverse is $\mathbf{B}_{F_2}$:

| $\alpha$ | $\underline{b_{11}}$ | $\overline{b_{11}}$ | $\underline{b_{12}}$ | $\overline{b_{12}}$ | $\underline{b_{21}}$ | $\overline{b_{21}}$ | $\underline{b_{22}}$ | $\overline{b_{22}}$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0.08 | 2.20 | -0.60 | -0.06 | -1.80 | -0.19 | 0.01 | 0.40 |
| 0.1 | 0.09 | 1.39 | -0.40 | -0.06 | -1.22 | -0.20 | 0.01 | 0.25 |
| 0.2 | 0.10 | 1.01 | -0.30 | -0.06 | -0.94 | -0.21 | 0.01 | 0.18 |
| 0.3 | 0.11 | 0.78 | -0.24 | -0.07 | -0.77 | -0.22 | 0.01 | 0.14 |
| 0.4 | 0.12 | 0.63 | -0.20 | -0.07 | -0.67 | -0.24 | 0.01 | 0.11 |
| 0.5 | 0.13 | 0.52 | -0.17 | -0.07 | -0.59 | -0.25 | 0.02 | 0.09 |
| 0.6 | 0.15 | 0.44 | -0.15 | -0.08 | -0.53 | -0.27 | 0.02 | 0.08 |
| 0.7 | 0.17 | 0.38 | -0.14 | -0.08 | -0.49 | -0.29 | 0.02 | 0.07 |
| 0.8 | 0.19 | 0.33 | -0.13 | -0.09 | -0.45 | -0.32 | 0.03 | 0.06 |
| 0.9 | 0.22 | 0.29 | -0.12 | -0.10 | -0.42 | -0.35 | 0.04 | 0.05 |
| 1 | 0.26 | 0.26 | -0.11 | -0.11 | -0.40 | -0.40 | 0.05 | 0.05 |

**Example 4.** Let $\mathbf{A}_{F_3} = \begin{pmatrix} (-1;\ 1.5;\ 2) & (3;\ 3.5;\ 4) \\ (9;\ 13;\ 20) & (8;\ 8.5;\ 10) \end{pmatrix}$ be a fuzzy matrix with entries in the form of triangular numbers.
Then its inverse is $\mathbf{B}_{F_3}$:

| $\alpha$ | $\underline{b_{11}}$ | $\overline{b_{11}}$ | $\underline{b_{12}}$ | $\overline{b_{12}}$ | $\underline{b_{21}}$ | $\overline{b_{21}}$ | $\underline{b_{22}}$ | $\overline{b_{22}}$ |
|---|---|---|---|---|---|---|---|---|
| 0 | -1.43 | -0.09 | 0.04 | 0.43 | 0.20 | 1.29 | -0.29 | 0.03 |
| 0.1 | -1.04 | -0.10 | 0.05 | 0.32 | 0.21 | 0.99 | -0.21 | 0.02 |
| 0.2 | -0.81 | -0.11 | 0.05 | 0.26 | 0.23 | 0.82 | -0.16 | 0.01 |
| 0.3 | -0.66 | -0.11 | 0.05 | 0.22 | 0.24 | 0.71 | -0.13 | 0.01 |
| 0.4 | -0.55 | -0.13 | 0.06 | 0.19 | 0.26 | 0.62 | -0.11 | 0.00 |
| 0.5 | -0.47 | -0.14 | 0.06 | 0.17 | 0.28 | 0.56 | -0.09 | 0.00 |
| 0.6 | -0.41 | -0.15 | 0.07 | 0.15 | 0.29 | 0.51 | -0.08 | -0.01 |
| 0.7 | -0.36 | -0.17 | 0.07 | 0.14 | 0.31 | 0.48 | -0.07 | -0.02 |
| 0.8 | -0.32 | -0.19 | 0.08 | 0.12 | 0.33 | 0.45 | -0.06 | -0.02 |
| 0.9 | -0.29 | -0.22 | 0.09 | 0.11 | 0.36 | 0.42 | -0.05 | -0.03 |
| 1 | -0.26 | -0.26 | 0.11 | 0.11 | 0.40 | 0.40 | -0.05 | -0.05 |

#### 2.2.7 Concluding remarks on the fuzzy inverse

Practical calculations show that in the case of elements with the same sign pattern the interval inverse for an arbitrary $\mathbf{C}_I \subseteq \mathbf{A}_I$ ($\subseteq$ we understand here as componentwise intervals inclusion) is based on the same design vertex matrices adjusted to the matrix $\mathbf{C}_I$. We use an arbitrary numeration of vertex matrices. This can be explained on the following theoretical example:

**Example 5.** Interval matrices $\mathbf{A}_I, \mathbf{C}_I$ such that

$$\mathbf{C}_I \subseteq \mathbf{A}_I$$

and

$$\mathbf{A}_I \geq 0$$

are given.
Let's assume that

$$\mathbf{A}_I^{-1} = ([a_{ij}^l, a_{ij}^k])_{n\times n},$$

where indices $l$ and $k$ indicate vertex matrices, where corresponding element is coming from. Then the inverse of $\mathbf{C}_I$ is the following:

$$\mathbf{C}_I^{-1} = ([c_{ij}^l, c_{ij}^k])_{n\times n},$$

where indices $l$ and $k$ are the same as for $\mathbf{A}_I^{-1}$.

The hypothesis risen above would be very useful if its statement is proven. We can take an arbitrary interval matrix $\mathbf{A}_I$ with sufficiently big radius matrix $\Delta$, calculate its inverse and fix numbers of vertex matrices $A^k$, which are necessary for calculation of the inverse. Then for any other interval matrix $\mathbf{C}_I$, $\mathbf{C}_I \subseteq \mathbf{A}_I$ only inverse matrices of vertex matrices with fixed numbers need to be calculated. This evidently decreases time of calculation of $\mathbf{C}_I^{-1}$.
This result is directly applicable to the simplification of the calculation of the fuzzy inverse. For any arbitrary fuzzy matrix $\mathbf{A}_F$, $\mathbf{A}_F^1 \subseteq \mathbf{A}_F^\alpha \subseteq \mathbf{A}_F^0, \alpha \in (0,1)$, therefore only for $(\mathbf{A}_F^0)^{-1}$ all $2^{2n-1}$ inverses of vertex matrices need to be calculated. Further we use knowledge obtained before, and not more than $2n^2$

inverses of vertex matrices need to be calculated.
Although the statement risen above is not carefully proven, Sherman-Morrison and Woodbury formulas ([3]) for crisp matrices advocate for its correctness.
Suppose $A$ is an invertible square matrix and $u, v$ are vectors. Suppose furthermore that $1 + v^T A^{-1} u \neq 0$. Then the Sherman$-$Morrison formula states that

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^T A^{-1}}{1 + v^T A^{-1} u} \tag{18}$$

Generalized Sherman-Morrison formula is called Sherman-Morrison-Woodbury or just Woodbury formula:

$$(A + UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1}, \tag{19}$$

where $A = (a_{ij})_{n\times n}, U = (u_{ij})_{n\times k}, C = (c_{ij})_{k\times k}$ and $V = (a_{ij})_{k\times n}$ matrices.
In the special case where $C$ is constant Woodbury formula reduces to the Sherman-Morrison formula.
Evidently when change of initial matrix is small then the inverse of the new matrix tends to be very close to the inverse of initial matrix. In the case of an interval matrix it is applicable in the following way: let's assume that intervals are reduced a little in the initial interval matrix, and the inverse of the vertex matrix $A^k$ determines the lower bound of the $ij - th$ element of the interval inverse. Then the same design matrix $A^k$ (only adjusted to the new interval matrix) determines the lower bound of the $ij$-th element of the inverse matrix of the new interval matrix.
In any case formulas (18) and (19) provide a significant assistance in calculation of the fuzzy inverse. All vertex matrices based on $\mathbf{A}_F^0$ need to be inverted, further for $\alpha > 0$ matrix inversion can be substituted with less time consuming operations, i.e. matrix multiplication and addition. Thus using formulas (18) if appropriate or (19) we simplify calculation of the fuzzy inverse.

## 2.3 Fuzzy identity matrix

### 2.3.1 Definition and properties

Multiplication of a matrix with its inverse gives identity matrix in the crisp case. Multiplication of $\mathbf{A}_F$ and its inverse matrix $\mathbf{B}_F$ is identity matrix analogue, we call it fuzzy identity matrix, and this section is devoted to its definition, properties and estimation.

**Definition 27.** We say that $\mathbf{A}_F\mathbf{B}_F$ is right fuzzy identity matrix, and correspondingly $\mathbf{B}_F\mathbf{A}_F$ is left fuzzy identity matrix.

**Proposition 2.13.** Generally $\mathbf{A}_F\mathbf{B}_F \neq \mathbf{B}_F\mathbf{A}_F$.

*Proof.* The assertion of the proposition is implied by the fact that multiplication of crisp matrices is not symmetric. □

**Proposition 2.14.** $\mathbf{A}_F\mathbf{B}_F, \mathbf{B}_F\mathbf{A}_F$ are fuzzy matrices.

*Proof.* See proposition 2.1. □

Let $\mathbf{A}_F\mathbf{B}_F = (E^r_{ij})_{n\times n}$ and $\mathbf{B}_F\mathbf{A}_F = (E^l_{ij})_{n\times n}$ and according to the definition of $\mathbf{B}_F$ the following hold:

$$E^r_{ij}(1) = E^l_{ij}(1) = 1 \text{ if } i = j$$

$$E^r_{ij}(0) = E^l_{ij}(0) = 1 \text{ if } i \neq j$$

### 2.3.2 Estimation of a fuzzy identity matrix

Various calculations performed for two dimensional fuzzy matrices showed interesting feature of fuzzy identity matrices, namely if we take an arbitrary element of $\mathbf{A}_F\mathbf{B}_F$ and the same element of $\mathbf{B}_F\mathbf{A}_F$ then either the former is fuzzy subset of the latest or vice versa. If we take $\mathbf{A}_{F_1}$ from example 2 in the pior section and perform multiplication $\mathbf{A}_{F_1}\mathbf{B}_{F_1} = (AB_{ij})_{2\times 2}$ and $\mathbf{B}_{F_1}\mathbf{A}_{F_1} = (BA_{ij})_{2\times 2}$ we receive the following results:

$$AB_{11}(x) \leq BA_{11}(x)\ \forall x$$

$$AB_{12}(x) \leq BA_{12}(x)\ \forall x$$

$$AB_{21}(x) \geq BA_{21}(x)\ \forall x$$

$$AB_{22}(x) \geq BA_{22}(x)\ \forall x$$

Calculation of fuzzy identity matrix is the same complexity task as calculation of the fuzzy inverse, therefore evaluation of fuzzy identity matrix maybe useful. The following results hold:

**Proposition 2.15.** If $\circ$ is one of the following operations: addition, multiplication or multiplication with fuzzy number, $\mathbf{A}_F$ and $\mathbf{B}_F$ are arbitrary fuzzy matrices and $C$ is an arbitrary fuzzy number then the following assertions are true:
(1) $\mathbf{A}^U_F \circ \mathbf{B}^U_F$, $\mathbf{A}^U_F \circ \mathbf{B}_F$, $\mathbf{A}_F \circ \mathbf{B}^U_F$ are upper dominants of $\mathbf{A}_F \circ \mathbf{B}_F$
(2) $\mathbf{A}^L_F \circ \mathbf{B}^L_F$, $\mathbf{A}^L_F \circ \mathbf{B}_F$, $\mathbf{A}_F \circ \mathbf{B}^L_F$ are lower dominants of $\mathbf{A}_F \circ \mathbf{B}_F$
(3) $\mathbf{B}^U_F \circ \mathbf{A}^U_F$, $\mathbf{B}^U_F \circ \mathbf{A}_F$, $\mathbf{B}_F \circ \mathbf{A}^U_F$ are upper dominants of $\mathbf{B}_F \circ \mathbf{A}_F$
(4) $\mathbf{B}^L_F \circ \mathbf{A}^L_F$, $\mathbf{B}^L_F \circ \mathbf{A}_F$, $\mathbf{B}_F \circ \mathbf{A}^L_F$ are lower dominants of $\mathbf{B}_F \circ \mathbf{A}_F$
(5) $C^U \circ \mathbf{A}^U_F$, $C^U \circ \mathbf{A}_F$, $C \circ \mathbf{A}^U_F$ are upper dominants of $C \circ \mathbf{A}_F$
(6) $C^L \circ \mathbf{A}^L_F$, $C^L \circ \mathbf{A}_F$, $C \circ \mathbf{A}^L_F$ are lower dominants of $C \circ \mathbf{A}_F$,
where $\mathbf{A}^U_F, \mathbf{B}^U_F, C^U$ are upper dominants correspondingly of $\mathbf{A}_F, \mathbf{B}_F$ and $C$; $\mathbf{A}^L_F, \mathbf{B}^L_F, C^L$ are lower dominants correspondingly of $\mathbf{A}_F, \mathbf{B}_F$ and $C$.

Proof of proposition 2.15 is evident and is implied by the definition of extension principle.
Now given a fuzzy matrix $\mathbf{A}_F$ and using results of proposition 2.15, upper and lower dominants of the fuzzy inverse matrix obtained in section 2.2.5 we can build lower and upper dominants of left and right fuzzy identity matrices.

## 2.4 Application of the fuzzy inverse

The main results of the second chapter are related to the inverse of a fuzzy matrix. Therefore the question "are these results a mathematical abstraction only or not?" naturally arises. We provide possible practical applications of the theoretical results obtained in this chapter and thus give the negative answer on the question above.

### 2.4.1 Estimation of the approximate fuzzy solution of the system of fuzzy linear equations

Finding the solution of the system of the fuzzy linear equations (hereinafter SFLE) has been extensively studied after [11] appeared in 1998. Since then many conceptually different methods and algorithms has been proposed for the finding the solution of SFLE (e.g. [28], [29], [33], [36], [44], [51] and others.) Personal contribution and very good extensive summary of the results of other authors in the field of SFLE is provided in [45]. We contribute to this topic and define different types of solutions of SFLE and outline the possible calculation methods. Also we define the approximate fuzzy solution (AFS), provide its estimations, but calculation of AFS directly follows from theoretical results of this chapter.
Solution of the system of interval linear equations (SILE) is the base for many method and algorithms of finding SFLE. For the basic results on SILE interested reader can refer e.g. to [27], [38], [39], [40].
We recall that different types of solutions are defined in the case of SILE, e.g, solution set (or united solution set):

$$\varphi = \{x \in \mathbb{R}^n / \exists A \in \mathbf{A}, \exists b \in \mathbf{b} : Ax = b\},$$

tolerable solution set:

$$\pi = \{x \in \mathbb{R}^n / \forall A \in \mathbf{A}, \exists b \in \mathbf{b} : Ax = b\},$$

the hull solution:

$$\mathbf{A}^H \mathbf{b} \text{ i.e., the smallest box that encloses the solution set,}$$

the interval solution:

$$\mathbf{A}^{-1}\mathbf{b}$$

and others.
By the analogue with interval case different types of solutions are defined for SFLE (e.g. [45]).
Let's consider SFLE:

$$\mathbf{A}_F x = \mathbf{c}_F, \tag{20}$$

where $\mathbf{A}_F \in \mathbb{M}$ is an $n \times n$ matrix, $\mathbf{c}_F \in \mathbb{M}$ is one column matrix.
It can be decomposed into spectrum of SILE:

$$\mathbf{A}_F^{\alpha} x = \mathbf{c}_F^{\alpha}, \tag{21}$$

where $\alpha \in [0,1]$ and $\mathbf{A}_F^\alpha, \mathbf{c}_F^\alpha$ are corresponding interval matrices.
In particular cases for reasonable considerations we can choose final number of SILE from the spectrum, and this number is determined by the required accurateness.
Let $S_\pi^\alpha, S_{hull}^\alpha, S_{interval}^\alpha$ denote correspondingly the smallest box that encloses the tolerable solution set (further the tolerable solution set), the hull solution and the interval solution of SILE (21):

**Definition 28.** A fuzzy vector $S_{hull} = (S_1, ..., S_n)^T$, where for all $i \in \{1, ..., n\}$ $S_i : \mathbb{R} \to [0,1]$ is a fuzzy set and

$$S_i(x) = \begin{cases} 0, \text{ if } \neg\exists (x_1, ..., x_n) \in S_{hull}^0 : x = x_i \\ \sup\{\alpha | x = x_i \text{ for some } (x_1, ..., x_n) \in S_{hull}^\alpha\} \text{ otherwise} \end{cases}$$

is called fuzzy solution, which coincides with solution sets $S_{hull}^\alpha, \alpha \in [0,1]$.

Hereinafter in this section vectors, which elements are fuzzy sets (not only fuzzy numbers) will be called fuzzy vectors.
Fuzzy solutions of SFLE (20), which coincide with the tolerable solution set and the interval solution are defined similarly and are denoted $S_\pi, S_{interval}$. These fuzzy solutions are based on solutions of SILE therefore methods described in previously mentioned references can be used for calculation.
Now we define AFS:

**Definition 29.** A fuzzy vector

$$x = \mathbf{A}_F^{-1}\mathbf{c}_F = \mathbf{B}_F\mathbf{c}_F$$

is called AFS of SFLE.

We have shown previously that $\mathbf{B}_F \in \mathbb{M}$, thus AFS $\mathbf{B}_F\mathbf{c}_F \in \mathbb{M}$ because $\mathbb{M}$ is closed w.r.t. to continuous operations.
As it can be seen from the definition of AFS it is upper dominant of fuzzy solutions $S_\pi, S_{hull}, S_{interval}$.
If $\mathbf{B}_F$ is available we directly apply it to the calculation of AFS, otherwise upper and lower dominants of $\mathbf{B}_F$ can be built and according to proposition 2.15:

$$AFS^U = \mathbf{B}_F^U\mathbf{c}_F \tag{22}$$

$$AFS^L = \mathbf{B}_F^L\mathbf{c}_F \tag{23}$$

Practical calculations of $AFS^U, AFS^L$ show not very good results sometimes, but these evaluations can serve as the start and the end points for iterative algorithms dedicated to calculation of ASF.
When we speak about another types of fuzzy solutions only $AFS^U$ can be taken in the role of the start point for iterative algorithms, because $AFS^L$ is not a lower dominant (neither an upper) for an arbitrary type of solution of SFLE apart from AFS.

### 2.4.2 Fuzzy input-output analysis

This section is devoted to the fuzzy inverse in economical applications. Fuzzified approach for input-output analysis is considered. Different methods of processing of inexact data in input-output analysis can be found in the literature, the interest to this topic is explained by high practical value of such a view on economical problem. Rohn in [42] and Jerrell in [20], [21] offer an interval approach for input-output analysis, Buckley in [4] considers the next level of uncertainty and uses a fuzzy model. We extend the fuzzified approach and it differs from the one in [4].

First we give economical background of input-output model (more detailed explanation can be found e.g. in [13], [31]) and after that we explain the fuzzified model.

The interactions of linkages between sectors of economics are significant for planners, who simultaneously need to keep overall macroeconomic balances in view to ensure consistency. The tool designed to accomplish these tasks is the input-output or interindustry table. Its two inventors suggest its flexibility and usefulness. The Russian-born economist Wassily Leontief developed input-output tables at Harvard during the 1930s to help understand the working of a modern economy and later to help with postwar planning in the United States. About the same time, though working independently, the Russian economist Leonid Kantorovich developed the same tool to help planners in his country set quantity targets for Soviet production, allowing both for final demands and for the use of intermediate products within industry. The two economists eventually won Nobel prizes for their efforts.

The essence of an input-output table is to display the flow of output from one industry to another and to final users (consumers, investors, and exporters).

Let's assume that there are $n \geq 2$ sectors in a national economy and each sector produces a single kind of goods. A part (possibly zero) of the gross output of each sector is consumed by some other sectors as inputs for their own productions; the amount of the $i$-th goods consumed by the $j$-th sector is supposed to be proportional to the gross output $x_j$ of the $j$-th sector with the coefficient of proportionality $a^0_{ij}$. Thus the total amount of the $i$-th goods consumed for production purposes within the national economy is equal to $\sum\limits_{j=1}^{n} a^0_{ij} x_j$. Hence we have:

$$x_i = \sum_{j=1}^{n} a^0_{ij} x_j + y_i, \; i = 1, ..., n, \tag{24}$$

where, for each $i$, $x_i$ denotes the gross output and $y_i$ the net output of the $i$-th sector (measured ususally in monetary units). The $a^0_{ij}$ are called input coefficients and are assumed to be constant. Taking $A = (a^0_{ij})_{n \times n}$, $x = (x_1, ..., x_n)^T$, $y = (y_1, ..., y_n)^T$ the matrix form of (24) is the following:

$$(E - A_0)x = y \tag{25}$$

which is the basic equation in the input-output analysis. Matrix $E - A_0$, where $E$ is identity matrix, is usually called Leontief matrix.
Clearly, $A_0$ and $x$ are nonnegative and so is $y$ in case the workability of all sectors is supposed. The profitability of the $j$-th sector means that the value of its production is greater than that of components used, i.e.

$$x_j > \sum_{j=1}^{n} a_{ij}^0 x_j, \tag{26}$$

which leads to

$$1 > \sum_{j=1}^{n} a_{ij}^0. \tag{27}$$

Thus the profitability of all sectors can be written as

$$e^T(E - A_0) > 0^T, \tag{28}$$

where $e = (1, ..., 1)^T$.
For a given net output $y$ the number $\sum_{i=1}^{n} y_i = e^T y$ is called the national income.
The model (25) is used to solve the two main problems in planning:
($P_1$) to find a gross output $x$ which yields a given net output $y$
($P_2$) to find a net output $y$ corresponding to a given gross output $x$.
The national economy is completely characterized by the matrix $A_0$, but, in practice, it is difficult to find exact values of $a_{ij}^0$, because the data from which they are determined is both inexact and incomplete.
We assume that $a_{ij}^0$ are not known precisely, but each of them can be estimated by fuzzy number $A_{ij}$, such that $A(a_{ij}) > 0$ and thus the matrix of technical coefficients $A_0$ can be substituted with fuzzy matrix $\mathbf{A}_F = (A_{ij})_{n\times n}$.
$\mathbf{A}_F^0, ..., \mathbf{A}_F^\alpha = ([\underline{A}^\alpha, \overline{A}^\alpha]) = ([\underline{a}_{ij}^\alpha, \overline{a}_{ij}^\alpha])_{n\times n}, ..., \mathbf{A}_F^1$ are corresponding interval matrices.
We assume that additionally for $\mathbf{A}_F$ the following holds:

$$[\underline{A}^0, \overline{A}^0] \geq 0 \tag{29}$$

$$e^T(E - \mathbf{A}_F^0) > 0 \tag{30}$$

These conditions translated into language of economics mean profitability of all sectors in fuzzy input-output model and consistency of linkages.
Rohn has shown in [42], that (29) and (30) are equivalent to:

$$A_0 \geq 0 \tag{31}$$

$$e^T(E - A_0) > 0^T \tag{32}$$

Solving problems ($P_1$) or ($P_2$) we assume that some knowledge on vectors $x$ and $y$ is also available. Namely, that in the fuzzy model $x$ and $y$ can be evaluated by means of fuzzy vectors $\mathbf{X}_F = (X_1, ..., X_n)^T$ and $\mathbf{Y}_F = (Y_1, ..., Y_n)^T$ and the following conditions hold:

$$X_i(x_i) > 0, i = 1, ..., n \tag{33}$$

and

$$Y_i(y_i) > 0, i = 1, ..., n \tag{34}$$

$\mathbf{X}_F^0, ..., \mathbf{X}_F^\alpha = ([\underline{x}_1^\alpha, \overline{x}_1^\alpha], ..., [\underline{x}_n^\alpha, \overline{x}_n^\alpha]), ..., \mathbf{X}_F^0$ and
$\mathbf{Y}_F^0, ..., \mathbf{Y}_F^\alpha = ([\underline{y}_1^\alpha, \overline{y}_1^\alpha], ..., [\underline{y}_n^\alpha, \overline{y}_n^\alpha]), ..., \mathbf{Y}_F^0$ are interval vectors corresponding to fuzzy vectors $\mathbf{X}_F$ and $\mathbf{Y}_F$ (i.e. elements of interval vectors are $\alpha$-cuts of the corresponding fuzzy numbers).
Thus we have all elements of fuzzy input-output model:

$$(E - \mathbf{A}_F)\mathbf{X}_F = \mathbf{Y}_F, \tag{35}$$

where $E$ is the crisp identity matrix.
If $(P_2)$ need to be solved then we apply extension principle and calculate $\mathbf{Y}_F$. Interpretability of $\mathbf{Y}_F$ is tightly related to assumptions put on $\mathbf{A}_F$ and $\mathbf{X}_F$.
Problem $(P_1)$ is more complicated and we consider it in more details further. First we define fuzzy feasible solution (FFS) of fuzzy input-output equation (35):

**Definition 30.** A fuzzy vector $\mathbf{X}_F$ is FFS if it is a fuzzy solution, which coincides with tolerable solution set for all $\alpha \in [0, 1]$

Definition of FFS ensures that $\forall \alpha \in [0, 1]$

$$(E - \mathbf{A}_F^\alpha)\mathbf{X}_F^\alpha \subseteq \mathbf{Y}_F^\alpha \tag{36}$$

and as a result for $(E - \mathbf{A}_F)\mathbf{X}_F = (Z_1, ..., Z_n)^T$ and $\mathbf{Y}_F = (Y_1, ..., Y_n)^T$ the following holds:

$$Z_i(x) \leq Y_i(x) \forall x \in \mathbb{R}, \forall i = 1, ..., n \tag{37}$$

If for some $\alpha \in [0, 1]$

$$(E - \mathbf{A}_F^\alpha)\mathbf{X}_F^\alpha \supset \mathbf{Y}_F^\alpha \tag{38}$$

then there exists $x \in \mathbf{X}_F^\alpha$ which leads out of the apriory given $\mathbf{Y}_F^\alpha$ and thus $\mathbf{X}_F^\alpha$ is not a solution of the corresponding interval problem and as a result $\mathbf{X}_F$ is not a solution of (35).
Calculation and estimation of FFS can be performed in different ways. E.g., (35) can be decomposed into SILE:

$$(E - \mathbf{A}_F^\alpha)\mathbf{X}_F^\alpha = \mathbf{Y}_F^\alpha, \alpha \in [0, 1] \tag{39}$$

and tolerable solution for each $\alpha$ needs to be found or estimated. Different algorithms can be applied for these purposes, but we advise to follow reference [42], because the author considers this solution in the frame of input-output analysis and extensive explanations on interpretability of obtained results are also available in [42].

Another approach for the estimation of $\mathbf{X}_F$ is to apply the fuzzy inverse of $(E - \mathbf{A}_F)$ and use AFS of (35), which is defined in the previous section.

$$(E - \mathbf{A}_F)^{-1}\mathbf{Y}_F = (Z_1^{'}, ..., Z_n^{'})^T \tag{40}$$

According to the construction of AFS $Z_i^{'}(x) \geq X_i(x), \forall x \in \mathbb{R}, \forall i = 1, ..., n$. Calculation of the fuzzy inverse within this economic background is simplified, because for $\mathbf{A}_F^0$ the following result is provided in [42]:

**Proposition 2.16.** [42] Let $[\underline{A}^0, \overline{A}^0]$ be a nonnegative interval matrix and $e^T(E - \mathbf{A}_F^0) > 0^T$ then (1) and (2) hold:
(1) $E - \mathbf{A}_F^0$ is nonnegatively invertible
(2) $(E - \mathbf{A}_F^0)^{-1} \subseteq [(E - \underline{A}^0)^{-1}, (E - \overline{A}^0)^{-1}]$.

Accordingly the calculation of the fuzzy inverse is simplified.
Although AFS overestimates FFS it provides an idea about gross output required to satisfy the needs of the national economics, and it maybe useful on the pre-planning stage.

### 2.4.3 Fuzzy economic multipliers

We briefly outline another possible application of the fuzzy inverse in this subsection, namely estimation of economic multipliers.
In the most general sense, an economic multiplier is a quantitative measure of economic impact that explicitly recognizes that economies (local, state, regional, national, or global) are interconnected networks of interdependent activity. When a change takes place in one part of such a network, its effects propagate throughout the system. These effects typically result in a larger total impact than the original change would have caused in isolation. More detailed theory on economic multipliers can be found e.g. in [31], [35].
Ordinarily economic multipliers are calculated on the basis of mathematical model of the relationships in that economy, e.g. input-output modeling technique. Let's assume we have an input-output model in the matrix form:

$$(E - A_0)x = y \tag{41}$$

Much knowledge can be gained without knowing net output vector $y$, by considering how economic activity will change if there is a change in the vector $y$. This does not require computing the level of economic activity, only the change. One method of examining how a change in net output will affect an economy is to construct multipliers.
There are different types of multipliers, e.g. typeI, typeII and typeIII which differ in the way what kind of effects (direct, indirect, induced) and in what way are taken into calculation. Another classification is based on the area where multiplier can be applied, e.g. income estimation, employment estimation and others. Despite the economic interpretation the generalized mathematical form of the economic multiplier based on (41) is the following:

$$M = (L^{-1})^{'}(1, 1, ..., 1), \tag{42}$$

where $L$ is Leontief matrix $E - A_0$.
In particular as an example income and employment multipliers can be calculated:

$$I_i = \frac{[(L^{-1})^{'}h]_i}{h_i}, \tag{43}$$

where $h_i = \frac{i_i}{s_i}$ and $i_i$- household income generated by industry $i$, $s_i$ - total outlays in industry $i$.

$$E_i = \frac{[(L^{-1})^{'}e]_i}{e_i}, \tag{44}$$

where $e_i$- employment in industry $i$.
Employment multiplier application can be found in the following example usually given by economists. Let's assume that company A has hired 100 additional workers in order to increase production and meet increased demand. But suppliers of company A need to increase human resources as well in order to satisfy company's A demand (e.g. produce more materials for A). We can therefore define the employment multiplier as the number that is multiplied by the number of jobs directly involved in company A to yield the total number of jobs created, directly and indirectly, as a result of the increased demand of company A production. If the total number of jobs created were 200, the employment multiplier in this example would be 2.
There are two main sources of error in economic multipliers derived from input-output models:
1) incomplete and imperfect data used to estimate the input-output coefficients of the model
2) imperfections of the design and structure of the model itself,
therefore substitution of the Leontief matrix $L$ by its fuzzy estimation, i.e. $\mathbf{L}_F$ is reasonable. Thus fuzzy economic multiplier, which can be considered as estimation of crisp analogue, can be presented in the following form:

$$\mathbf{M} = (\mathbf{L}_F^{-1})^{'}(1, 1, ..., 1) \tag{45}$$

Fuzzy multiplier is a fuzzy vector and its economic interpretation is based on assumptions put on $\mathbf{L}_F$.

## 2.5 Concluding remarks on fuzzy matrices

The notion of the fuzzy inverse is central in this chapter because it has the largest practical value. As the first priority directions for further study we outline the following:

- development of simpler algorithms for calculation of the fuzzy inverse
- development of algorithms for more accurate estimation of the fuzzy inverse.

Also it is interesting to know what happens with the fuzzy inverse if operations with fuzzy matrices are defined by means of some other t-norm, not only $T_M$.

# 3 Generalized aggregation: theoretical foundations and practical applications

This chapter is dealing with questions related to the generalized aggregation. Term "generalized" refers to the generalized input and output of these objects, i.e. generalized aggregation operators (hereinafter gagops) aggregate fuzzy sets. We require that the input elements should follow the following:
(i) upper semicontinuous fuzzy sets
(ii) fuzzy sets with bounded $\alpha$-cuts, for all $\alpha > 0$.
Aggregation operators (hereinafter agops) according to the tradition can now be called crisp agops versus generalized ones mentioned above.
We study gagops from the two main aspects: representative properties and preservation of boundary conditions and monotonicity property.
We also introduce and study in the work a new class of agops, namely $\gamma$-agops. $\gamma$-agops also generalize the notion of agop in some sense. But this generalization differs from the one mentioned before.
The chapter is divided into nine sections including conclusion. Materials of Section 1 and partly of Section 3 and of Section 5 recall known results, all other sections are author's contribution to the topic.
**Section 1** provides the fundamental results on agops, which are necessary for our further study. We refer to the source [5] from the list of references and we flash agops from the two aspects, namely the main results and their properties. For more information on recent results in the field of agops the reader can refer e.g. to [5],[6].
**Section 2** is author's contribution to the theory of agops. We introduce the notion of a $\gamma$-agop, which generalizes the notion of an agop, if we consider it for an arbitrary $\gamma$ and we study it. Although $\gamma$ agops have some disadvantages they simplify the aggregation process. And this is due to the equivalence relation induced on the $[0, 1]^n$.
**Section 3** is the library of order relations which are used further in the study. We summarize already known order relations (which can be found in e.g. [46],[47]) and also we introduce new classes. The notion of order is tightly related to the notion of gagop, because the monotonicity property should be considered within the framework of some order.
**Section 4** contains generalization of some results from [46] for an arbitrary continuous t-norm. Also auxiliary construction is provided.
We start the topic of generalized aggregation with **Section 5**. First we give definition and construction methods of generalized agops ([47]). We conclude the section with definitions of properties of gagops, which are adopted to the fuzzy input.
We continue the topic of generalized aggregation in **Section 6**, where we contribute to generalized aggregation with study on pointwise extension of an agop. We investigate pointwise extension according to the following scheme:

- possible sets of inputs

- properties
- preservation of boundary conditions and the monotnicity property.

We show that some properties of pointwise extension mainly depend on the same properties of the extended agop. In the list of observed properties shift-invariance distinguished by a more complicated nature. And this is naturally, because additional operation with fuzzy sets (apart from extension of an agop) appears. And we conclude this section with results related to the boundary conditions and monotonicity preservation w.r.t. the order relations presented in Section 3.

**Section 7** is devoted to the study of $T$-extension (i.e. extension of an arbitrary agop by means of a continuous t-norm). $T$-extension is an another method of construction of a gagop. The scheme of investigation is the same like in the case of pointwise extension. Properties of $T$-extension depend on properties of the t-norm and the agop, which we extend. We show that some properties have a place only when we use a particular t-norm, e.g. idempotence of $T$-extension hold only for $T_M$. The section is concluded by results on preservation of the boundary conditions and monotnicity w.r.t. order relations introduced before.

The section before the last one contains a brief outline of possible practical applications of the apparatus investigated in this chapter.

And we conclude the chapter by **Section 9**, where we outline directions of the further research.

## 3.1 Fundamentals on aggregation operators

The aim of this section is to give a look at aggregation domain. We use the source [5] for reference and try to hold its style of notations and pitching and also we try to follow the same conventions if any is required. In the author's opinion the source [5] is well structured and it contains the recent results. Also many important information related to the aggregation operators can be found e.g. in [6]. Information in some specific information domain, particularly in the triangular norms, which play an important role in this work can be found e.g. in [22].

Generalized aggregation, which is the central matter of the third chapter, is broadly flashed from the aspect of its properties, and therefore we pay attention to properties of agops in this section. Other important parts of the aggregation domain such as classes and construction methods can be found in the already mentioned sources.

Problem of aggregation is very broad in general, and we use the following two restrictions in the work: the number of input values is finite and $I = [0, 1]$ is the set of inputs and outputs. If the second restriction is a matter of rescaling then the first divides the global aggregation into two parts, i.e aggregation of finite number of inputs and aggregation of infinite number of inputs. But even with a such restriction the problem of aggregation is still very general.

### 3.1.1 Definition

**Definition 31** ([5])**.** A mapping $A : \cup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ is an agop on the unit interval if for every $n \in \mathbb{N}$ the following conditions hold:
(A1) $A(0,...,0) = 0$
(A2) $A(1,...,1) = 1$
(A3) $(\forall i = \overline{1,n})\ (x_i \le y_i) \Rightarrow A(x_1,x_2,...,x_n) \le A(y_1,y_2,...,y_n)$

Conditions (A1) and (A2) are called boundary conditions, and they ensure that aggregation of completely bad (good) results will give the completely bad (good) output. Condition (A3) resembles the monotonicity property of $A$.
In general, the number of the input values to be aggregated is unknown, and therefore an agop can be presented as a family $A = (A_{(n)})_{n\in\mathbb{N}}$, where $A_{(n)} = A|_{[0,1]^n}$. Operators $A_{(n)}$ and $A_{(m)}$ for different $n$ and $m$ need not be related.
A specific case is the aggregation of a singleton, i.e., the unary operator $A_{(1)} : [0,1] \to [0,1]$. For many scientists the aggregation of a singleton is not an aggregation and they propose as a convention $A_{(1)}(x) = x, x \in [0,1]$. Throughout the work we will follow this convention.
This framework is enough general to include most of the relevant operators used for the fusion of the input data, however e.g. Godel implication $I_G$ is not an agop in the sense of definition 31:

$$I_G(x,y) = \begin{cases} 1, & \text{if } x \le y, \\ y, & \text{else} \end{cases}$$

**Definition 32** ([5])**.** Let $A, B : \cup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ be two agops. We say that $A$ is weaker than $B$, with notation $A \le B$, if

$$\forall n \in \mathbb{N}, \forall (x_1,...,x_n) \in [0,1]^n : A(x_1,...,x_n) \le B(x_1,...,x_n)$$

The weakest and the strongest agops are defined correspondingly:

$$\forall n \ge 2, (x_1,...,x_n) \ne (1,...,1) : A_w(x_1,...,x_n) = 0$$

and

$$\forall n \ge 2, (x_1,...,x_n) \ne (0,...,0) : A_s(x_1,...,x_n) = 1.$$

For any agop $A$ we have:

$$A_w \le A \le A_s.$$

For the agops

$$\Pi(x_1,...,x_n) = \Pi_{i=1}^n x_i,$$

$$M(x_1,...,x_n) = \frac{1}{n}\sum_{i=1}^n x_i,$$

$$max(x_1,...,x_n) = \max(x_1,...,x_n),$$

$$min(x_1,...,x_n) = \min(x_1,...,x_n)$$

we have:

$$A_w \leq \Pi \leq min \leq M \leq max \leq A_s.$$

Important examples of agops are projections $P_F$ (the projection of the first coordinate) and $P_L$ (the projection of the last coordinate):

$$P_F(x_1, ..., x_n) = x_1,\ P_L(x_1, ..., x_n) = x_n.$$

Often only a binary form $A_{(2)}$ of an agop is known. The ternary form $A_{(3)}$ of that operator needs not to have any relationship with $A_{(2)}$ in general. However if only $A_{(2)}$ is known, we have several ways for extending it to a complete agop. One possibility is the backward inductive extension of the binary operator, i.e.,

$$A^*(x_1, ..., x_n) = A_{(2)}(x_1, A_{(2)}(...A_{(2)}(x_{n-1}, x_n)...)), n > 2$$

and $A^*(x) = x$. An alternative approach is the forward inductive extension of the binary operator, i.e.,

$$A_*(x_1, ..., x_n) = A_{(2)}(...(A_{(2)}(A_{(2)}(x_1, x_2)x_3))..., x_n), n > 2$$

and $A_*(x) = x$. Observe that $A^* = A_*$ iff $A_{(2)}$ is associative.

### 3.1.2 The main properties

**Idempotence**

**Definition 33** ([5])**.** An element $x \in [0; 1]$ is called an $A$-idempotent element whenever $A_{(n)}(x, ..., x) = x, \forall n \in \mathbb{N}$. A is called an idempotent agop if each $x \in [0; 1]$ is an idempotent element of $A$.

Boundary conditions mean that 0 and 1 are $A$-idempotent elements for each agop $A$. Therefore 0 and 1 are called trivial idempotent elements. Idempotent agops are also called averaging operators. For agops in the sence of definition 31 the idempotency of an operator $A$ is equivalent to the so called compensation property:

$$\min \leq A \leq \max$$

Now it is evident that $\min, \max, M$ are idempotent agops while $A_w, \Pi, A_s$ are not. Observe that agop constructed in the spirit given by $A^*$ and $A_*$ based on an idempotent binary agop $A_{(2)}$ are idempotent as well. The compensation property also ensures another important feature of idempotent operators: for any interval $[c, d] \subset [0, 1]$, any idempotent agop $A$ and any n-tulpe $(x_1, ..., x_n) \in [c, d]^n$, also the value $A(x_1, ..., x_n) \in [c, d]$. Consequently, $A|_{\cup_{n \in \mathbb{N}}[c,d]^n}$ is an idempotent agop acting on $[c, d]$. Obviously for a general agop $A$ and for fixed $c, d$ the last claim (without idempotecy) is true iff $c$ and $d$ are idempotent elements of $A$.

**Continuity**

The continuity of an agop $A$ is simply the continuity of all $n$-ary operators $A_{(n)}$ in the standard sense of the continuity of real functions of $n$ variables.

**Definition 34** ([5]). An agop $A : \cup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ is called a continuous agop if for all $n \in \mathbb{N}$ the operators $A_{(n)} : [0,1]^n \to [0,1]$ are continuous, that is, if

$$\forall x_1,...,x_n \in [0,1], \forall (x_{1j})_{j\in\mathbb{N}},...,(x_{nj})_{j\in\mathbb{N}} \in [0,1]^{\mathbb{N}} : \lim_{j\to\infty} x_{ij} = x_i$$

for $i = 1,...,n$ then

$$\lim_{j\to\infty} A_{(n)}(x_{1j},...,x_{nj}) = A_{(n)}(x_1,...,x_n).$$

In engineering applications continuous agops are usually applied, reflecting the property that a 'small' error in inputs cannot cause a 'big' error in the output. From the mathematical point of view, because of the compactness of domains $[0,1]^n, n \in \mathbb{N}$, the continuity of an agop $A$ is equivalent to its uniform continuity:

$$\forall \epsilon > 0, \forall n \in \mathbb{N}, \exists \delta > 0 : |x_i - y_i| < \delta, i = 1,...,n \Rightarrow$$

$$\Rightarrow |A(x_1,...,x_n) - A(y_1,...,y_n)| < \epsilon.$$

Because of the monotonicity condition the continuity of an agop $A$ is also equivalent to the intermediate value property:

**Definition 35** ([5]). Let $(x_1,...,x_n),(y_1,...,y_n) \in [0,1]^n, n \in \mathbb{N}$ any n-tulpes such that $x_i \leq y_i, i = 1,...,n$. An agop $A$ has the intermediate value property if

$$\forall z \in [A(x_1,...,x_n), A(y_1,...,y_n)]$$

$$\exists z_i \in [x_i,y_i], i = 1,...,n : A(z_1,...,z_n) = z$$

The intermediate value property allows to introduce the equivalent of continuity in the case of agops acting on ordinal (discrete) scales.
An agop $A$ fulfils the Lipschitz property with a constant $c \in (0,\infty)$ if

$$\forall n \in \mathbb{N}, \forall (x_1,...,x_n),(y_1,...,y_n) \in [0,1]^n$$

$$|A(x_1,...,x_n) - A(y_1,...,y_n)| \leq c \sum_{i=1}^{n} |x_i - y_i|$$

Clearly, the Lipschitz property with an arbitrary $c$ ensures continuity but not vice-versa. Among the already introduced agops, the operators $A_w$ and $A_s$ are examples of non-continuous agops.
An agop $A$ is called lower (upper) semicontinuous if for all $n \in \mathbb{N}$, the operator $A_{(n)}$ is lower (upper) semicontinuous. Recall that:
$A_{(n)}$ is lower semicontinuous if:

$$\forall (x_{1j})_{j\in\mathbb{N}},...,(x_{nj})_{j\in\mathbb{N}} \in [0,1]^{\mathbb{N}} :$$

$$\sup_j A_{(n)}(x_{1j}, ..., x_{nj}) = A_{(n)}(\sup_j x_{1j}, ..., \sup_j x_{nj})$$

and upper semicontinuous if

$$\forall (x_{1j})_{j\in\mathbb{N}}, ..., (x_{nj})_{j\in\mathbb{N}} \in [0,1]^{\mathbb{N}} :$$

$$\inf_j A_{(n)}(x_{1j}, ..., x_{nj}) = A_{(n)}(\inf_j x_{1j}, ..., \inf_j x_{nj})$$

As usually, the continuity of an agop $A$ is equivalent to its simultaneous lower and upper semicontinuity.

### Symmetry

**Definition 36** ([5])**.** An agop $A : \cup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ is called a symmetric agop if

$$\forall n \in \mathbb{N}, \forall x_1, ..., x_n \in [0;1] : A(x_1, ..., x_n) = A(x_{\pi(1)}, ..., x_{\pi(n)})$$

for all permutations $\pi = (\pi(1), ..., \pi(n))$ of $(1, ..., n)$.

A weighted mean $W(x_1, ..., x_n) = \sum\limits_{i=1}^{n} w_i x_i$, where $\sum\limits_{i=1}^{n} w_i = 1$ is an example of a nonsymmetric agop.
Any non-symmetric agop can be symmetrised in the following way: $\forall n \in \mathbb{N}, \forall x_1, ..., x_n \in [0,1]$ let vector $(x_1^{'}, ..., x_n^{'})$ be a non-decreasing (non-increasing) permutation of the input vector $(x_1, ..., x_n)$, and define $A^{'}(x_1, ..., x_n) = A(x_1^{'}, ..., x_n^{'})$. Obviously, $A^{'}$ is a symmetric agop and $A^{'} = A$ iff $A$ is symmetric.

### Associativity

**Definition 37** ([5])**.** An agop $A : \cup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ is associative if

$$\forall n, m \in \mathbb{N}, \forall x_1, ..., x_n, y_1, ..., y_m \in [0;1] :$$

$$A(x_1, ..., x_n, y_1, ..., y_m) = A(A(x_1, ..., x_n), A(y_1, ..., y_m))$$

The associativity of an agop allows to aggregate first some subsystems of all inputs, and then the partial outputs. For practical purposes we can start with aggregation procedure before knowing all inputs to be aggregated. New (additional) input data are then simply aggregated with the actual aggregated output. From the structural point of view, an associative agop is uniquely determined by $A_{(2)}$. As examples of associative agops recall $A_w, A_s, \min, \max, \Pi$; non-associative agops are $M$ and geometric mean:

$$G(x_1, ..., x_n) = (\Pi_{i=1}^{n} x_i)^{1/n}.$$

Associativity is too strong and rather restrictive property, therefore sometimes some modifications of associativity preserving its advantages (from the computational aspect) are introduced.

### Bisymmetry

**Definition 38** ([5])**.** An agop $A : \cup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ is bisymmetric if

$$\forall n, m \in \mathbb{N}, \forall x_{11}, ..., x_{mn} \in [0;1] :$$

$$A_{(mn)}(x_{11}, ..., x_{mn}) = A_{(m)}(A_{(n)}(x_{11}, ..., x_{1n}), ..., A_{(n)}(x_{m1}, ..., x_{mn})) =$$
$$= A_{(n)}(A_{(m)}(x_{11}, ..., x_{m1}), ..., A_{(m)}(x_{1n}, ..., x_{mn}))$$

The bisymmetry allows to aggregate first rows and then partial outputs or first columns and then partial outputs if information is stored in the form of a matrix. Bisymmetry is implied by associativity and symmetry, but neither symmetry, nor associativity is implied by bisymmetry. Operators $A_w, A_s, \Pi, \min$ and $\max$ are examples of symmetric, associative and as a result bisymmetric agops. $M$ and $G$ are symmetric and bisymmetric but not associative.

### Neutral element

**Definition 39** ([5])**.** An element $e \in [0;1]$ is called a neutral element of A if $\forall n \in \mathbb{N}, \forall x_1, ..., x_n, \in [0;1]$ if $x_i = e$ for some $i \in \{1, ..., n\}$ then

$$A(x_1, ..., x_n) = A(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$$

So, the neutral element can be omitted from aggregation inputs without influencing the final output.
A typical examples are the product $\Pi$ with $e = 1$, $min$ with $e = 1$ and $max$ with $e = 0$. The existence of the neutral element is not related to the previous properties as continuity, symmetry, associativity or bisymmetry. $M, G, A_s, A_w$ are examples of agops without neutral element.

### Absorbing element

**Definition 40** ([5])**.** An element $a \in [0;1]$ is called an absorbing element of A if

$$\forall n \in \mathbb{N}, \forall x_1, ..., x_n, \in [0;1] : a \in \{x_1, ..., x_n\} \Rightarrow A(x_1, ..., x_n) = a$$

Operators $A_w, \Pi, G, min$ are examples of agops with $a = 0$ the operators $A_s, max$ have $a = 1$.
Note that $a$ is necessarily an A-idempotent element, that is $a$ is a trivial idempotent element. Note also that an agop $A$ with absorbing element $a \in (0,1)$ cannot have neutral element $e$. However, $A$ may have a neutral element $e$ if its absorbing element $a \in \{0,1\}$, and $e \neq a$.

**Some other properties**

**Definition 41** ([5])**.** An agop $A : \cup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ is said to be:
(1) shift-invariant if

$$\forall n \in \mathbb{N}, \forall b \in (0,1), \forall x_1, ..., x_n \in [0, 1-b] :$$

$$A(x_1 + b, ..., x_n + b) = A(x_1, ..., x_n) + b$$

(2) homogeneous if

$$\forall n \in \mathbb{N}, \forall b \in (0,1), \forall x_1, ..., x_n \in [0, 1] :$$

$$A(bx_1, ..., bx_n) = bA(x_1, ..., x_n)$$

(3) linear if it homogeneous and shift-invariant
(4) additive if

$$\forall n \in \mathbb{N}, \forall x_1, ..., x_n, y_1, ..., y_n \in [0,1] \text{ such that } x_1 + y_1, ..., x_n + y_n \in [0,1] :$$

$$A(x_1 + y_1, ..., x_n + y_n) = A(x_1, ..., x_n) + A(y_1, ..., y_n)$$

Note that additivity ensures linearity. Because of the boundary conditions any agop fulfilling at least one of the properties introduced in definition 41 is idempotent.

## 3.2 New class of aggregation operators: $\gamma$-agops

### 3.2.1 Definition of $\gamma$-agop

This section is devoted to $\gamma$-agops, which are a generalization of the class of agops in some sense.
We introduce the notion of a $\gamma$-agop by means of additional property ($\mathrm{A}_\gamma$). Let $\gamma \in [0;1]$ and $\varphi_\gamma : [0,1] \to \{0\} \cup [\gamma, 1]$ be defined in the following way:

$$\varphi_\gamma(x) = \begin{cases} 0, \text{ if } x < \gamma, \\ x, \text{ if } x \geq \gamma \end{cases}$$

**Definition 42.** $A : \cup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ is an $\gamma$-agop on the unit interval if the following conditions hold:
(A1) $A(0, ..., 0) = 0$
(A2) $A(1, ..., 1) = 1$
($\mathrm{A}_\gamma$) $(\forall i = \overline{1,n}, \gamma \in [0,1])$ $(\varphi_\gamma(x_i) \leq \varphi_\gamma(y_i)) \Rightarrow A(x_1, ...x_n) \leq A(y_1, ..., y_n)$

**Remark 2.** In case $\gamma = 0$ $\varphi_0(x) = x$ and condition ($\mathrm{A}_\gamma$) reduces to condition (A3) in definition 31.

**Proposition 3.1.** If A satisfies $(A_\gamma)$ and $\gamma > \gamma'$ then A satisfies $(A_{\gamma'})$.

*Proof.* Let's take arbitrary $(x_1, ..., x_n)$, $(y_1, ..., y_n)$ satisfying

$$\varphi_{\gamma'}(x_i) \leq \varphi_{\gamma'}(y_i),\ \forall i = \overline{1, n}.$$

Since $\varphi_{\gamma'}(x_i) \leq \varphi_{\gamma'}(y_i)$ and $\gamma > \gamma^{'}$, from the definition of $\varphi_\gamma$ it follows that

$$\varphi_\gamma(x_i) \leq \varphi_\gamma(y_i)$$

Therefore by condition $(A_\gamma)$:

$$A(x_1, ..., x_n) \leq A(y_1, ..., y_n)$$

and thus $(A_{\gamma'})$ is satisfied. $\square$

**Proposition 3.2.** Each $\gamma$- agop $A$ satisfies (A3), and hence is an agop.

*Proof.* Proof immediately follows from the definition of $\varphi_\gamma$ $\square$

It is intuitively clear that the formula of $\gamma$-agop should neutralize all arguments less than $\gamma$. Otherwise the left part of implication $(A_\gamma)$ will be true, but the right part will contradict the monotonicity property. The following example illustrates this fact:

**Example 6.** Let's consider vectors $x = (0.3, 0.2)$, $y = (0.3, 0.1)$, $\varphi_{0.3}(x)$ and an arbitrary binary 0.3-agop $A$. If we apply transformation $\varphi_{0.3}(x)$ to vectors $x$ and $y$ we obtain correspondingly vectors $x^{'} = (0.3, 0)$ and $y^{'} = (0.3, 0)$. Since $y^{'} \leq x^{'}$ according to $(A_\gamma)$ $A(y) \leq A(x)$, but this contradicts the monotonicity property.

The class of $\gamma$-agops in the case of some fixed $\gamma > 0$ is narrower than the class of agops. As an example we can mention usual arithmetic mean, which does not satisfy $(A_\gamma)$ for any $\gamma > 0$. But if we consider $\gamma$-agops for an arbitrary $\gamma \in [0, 1]$ we obtain a class which contains agops. Although $\gamma$-agops miss many important properties they can be useful as we show in the section related to the generalized aggregation.

### 3.2.2 Examples of $\gamma$- agops

We provide examples of $\gamma$-agops. We consider $n$-ary form for an arbitrary $n \in \mathbb{N}$:

**Example 7.**

$$A_1(x_1, ..., x_n) = \sum_{i=1}^{n} w_i x_i,$$

where
$w_i = \begin{cases} 0, \text{ if } x_i < \gamma, \\ 1/n, \text{ if } x_i \geq \gamma \end{cases}$

**Example 8.**

$$A_2(x_1, ..., x_n) = \min(w_1 x_1, ..., w_n x_n),$$

where
$w_i = \begin{cases} 0, \text{ if } x_i < \gamma, \\ 1, \text{ if } x_i \geq \gamma \end{cases}$

**Example 9.**

$$A_3(x_1, ..., x_n) = \max(w_1 x_1, ..., w_n x_n),$$

where
$w_i = \begin{cases} 0, \text{ if } x_i < \gamma, \\ 1, \text{ if } x_i \geq \gamma \end{cases}$

We see that above provided $\gamma$-agops are defined on the base of agops $(M, min, max)$.

### 3.2.3 Equivalence relation induced by $\varphi_\gamma$

Let's introduce relation $\equiv_{\varphi_\gamma}$ on $[0,1]^n$ in the following way:

$$(x_1, ..., x_n) \equiv_{\varphi_\gamma} (y_1, ..., y_n) \Leftrightarrow$$
$$\Leftrightarrow (\varphi_\gamma(x_1), ..., \varphi_\gamma(x_n)) = (\varphi_\gamma(y_1), ..., \varphi_\gamma(y_n)). \tag{46}$$

Further we show that $\equiv_{\varphi_\gamma}$ is an equivalence relation:
**reflexivity**:

$$(\varphi_\gamma(x_1), ..., \varphi_\gamma(x_n)) = (\varphi_\gamma(x_1), ..., \varphi_\gamma(x_n)) \Rightarrow (x_1, ..., x_n) \equiv_{\varphi_\gamma} (x_1, ..., x_n)$$

**symmetry**:

$$\begin{gathered}(x_1, ..., x_n) \equiv_{\varphi_\gamma} (y_1, ..., y_n) \Rightarrow \\ (\varphi_\gamma(x_1), ..., \varphi_\gamma(x_n)) = (\varphi_\gamma(y_1), ..., \varphi_\gamma(y_n)) \Rightarrow \\ (\varphi_\gamma(y_1), ..., \varphi_\gamma(y_n)) = (\varphi_\gamma(x_1), ..., \varphi_\gamma(x_n)) \Rightarrow \\ (y_1, ..., y_n) \equiv_{\varphi_\gamma} (x_1, ..., x_n)\end{gathered}$$

**transitivity**:

$$\begin{gathered}(x_1, ..., x_n) \equiv_{\varphi_\gamma} (y_1, ..., y_n) \text{ and } (y_1, ..., y_n) \equiv_{\varphi_\gamma} (z_1, ..., z_n) \Rightarrow \\ (\varphi_\gamma(x_1), ..., \varphi_\gamma(x_n)) = (\varphi_\gamma(y_1), ..., \varphi_\gamma(y_n)) \text{ and} \\ (\varphi_\gamma(y_1), ..., \varphi_\gamma(y_n)) = (\varphi_\gamma(z_1), ..., \varphi_\gamma(z_n)) \Rightarrow \\ (\varphi_\gamma(x_1), ..., \varphi_\gamma(x_n)) = (\varphi_\gamma(z_1), ..., \varphi_\gamma(z_n)) \Rightarrow \\ (x_1, ..., x_n) \equiv_{\varphi_\gamma} (z_1, ..., z_n).\end{gathered}$$

When $\gamma < 1$ the number of equivalence classes $s$ is infinite. In the particular case, when $\gamma = 1$, $s$ is finite and is determined by the formula:

$$s = 1 + C_n^{n-1} + C_n^{n-2} + ... + C_n^1 + 1. \tag{47}$$

When $n = 2$ then we have to deal only with 4 classes with the following representatives:

$$(0; 0), (0; 1), (1; 0), (1; 1).$$

We will denote equivalence classes $X_k$, $k = 1, 2, ...$

**Proposition 3.3.** If $(x_1, ..., x_n), (y_1, ..., y_n) \in X_k$, $A$ is $\gamma$-agop then $A(x_1, ..., x_n) = A(y_1, .., y_n)$

*Proof.* Let's take $(x_1, ..., x_n), (y_1, ..., y_n) \in X_k$, then according to the definition of $\equiv_\gamma$ we can write:

$$\varphi_\gamma(x_1, ..., x_n) = \varphi_\gamma(y_1, ..., y_n).$$

Let's assume that

$$A(x_1, ..., x_n) \neq A(y_1, ..., y_n).$$

Then $(A_\gamma)$ implies:

$$A(x_1, ..., x_n) < A(y_1, ..., y_n).$$

The same reasoning will lead us to the inequality:

$$A(y_1, ..., y_n) < A(x_1, ..., x_n).$$

The above derived inequalities cannot be true at the same time, and this means that our assumption on $A(x_1, ..., x_n) \neq A(y_1, ..., y_n)$ is not true. □

As corollaries from proposition 3.3 we obtain the following results:

**Corollary 1.** $\gamma$-agops $\forall \gamma > 0$ are not idempotent.

*Proof.* Proof immediately follows from the result of proposition 3.3 and the definition of $\varphi_\gamma$:
$\forall (x, ..., x): \ 0 < x < \gamma$,

$$A_{(n)}(x, ..., x) = A_{(n)}(0, ..., 0) = 0 \neq x$$

□

**Corollary 2.** $\gamma$-agops $\forall \gamma > 0$ are not shift-invariant, are not homogeneous and hence are not linear.

*Proof.* **Shift-invariance**
According to proposition 3.3:
$\forall (x, ..., x) \in [0, 1 - b]^n$, $b \in (0, 1)$: $x < \gamma$ and $x + b = 1$ we have

$$A_{(n)}(x + b, ..., x + b) = A_{(n)}(1, ..., 1) = 1$$

$$A_{(n)}(x, ..., x) + b = A_{(n)}(0, ..., 0) + b = b < 1$$

and thus

$$A_{(n)}(x + b, ..., x + b) \neq A_{(n)}(x, ..., x) + b$$

**Homogeneity**
According to proposition 3.3:
$\forall(x,...,x) \in [0,1]^n$, $b \in (0,1)$: $x = 1$ and $bx < \gamma$ we have

$$A_{(n)}(bx,...,bx) = A_{(n)}(0,...,0) = 0$$

$$bA_{(n)}(x,...,x) = bA_{(n)}(1...,1) = b > 0$$

and thus

$$A_{(n)}(bx,...,bx) \neq bA_{(n)}(x,...,x)$$

Absence of linearity follows from the preceding reasoning. □

**Corollary 3.** If $A_\gamma$, $\gamma \in (0,1]$ is a $\gamma$-agop and $a$ is its absorbing element then $a = 0$ or $a > \gamma$.

*Proof.* Let's assume that $a \in (0;\gamma]$, then $\forall(x_1,...,x_n)$: $x_i < \gamma$ and $a \in \{x_1,...,x_n\}$ we obtain
on the one hand according to the definition of absorbing element:

$$A(x_1,...,x_n) = a \neq 0$$

but on the other hand according to proposition 3.3:

$$A(x_1,...,x_n) = A_{(n)}(0,...,0) = 0$$

we have obtained a contradiction. □

**Remark 3.** $\gamma$-agops provided in the examples 8 and 9 have absorbing elements, correspondingly 0 and 1, and this coincides with the assertion of corollary 3.

## 3.3 Order relations

In this section we summarize order relations considered further in the work. Some of them are reflexive, antisymmetric and transitive. While others are transitive and asymmetric (strict order relation). For the sake of brevity all of them are called further order relations.
All observed order relations are divided into two groups: vertical orders and horizontal orders. If order relation is based on comparison of fuzzy sets values then we call it vertical order relation, because it acts on vertical axis $y$. If order relation compares $x$ values then we call it horizontal order relation, because horizontal axis is used. We refer to already known order relations and we also introduce new classes of order relations. Together with the definition of an order relation we specify the greatest and the least elements (w.r.t. the order relation) denoted correspondingly $\tilde{1}(x)$ and $\tilde{0}(x)$. Sometimes we define the whole classes. Hereinafter in this section $F(\mathbb{R}) = \{P|P : \mathbb{R} \to [0,1]\}$ is the set of all fuzzy subsets of $\mathbb{R}$. Different order relations may have particular requirements to the elements of $F(\mathbb{R})$ and also to the domain. We introduce clarifications if it is required.

### 3.3.1 Vertical order relations $\subseteq_{F_1}$ and $\subseteq^{\alpha}_{F_1}$

We recall definition of fuzzy sets order by inclusion. It is naturally based on the classical definition of fuzzy subsets (definition 2), see e.g. [47]:

**Definition 43.** [47] $P, Q \in F(\mathbb{R})$,

$$P \subseteq_{F_1} Q \Leftrightarrow (\forall x \in \mathbb{R})(P(x) \leq Q(x)).$$

Then

$$\tilde{1}(x) = 1, \forall x \in \mathbb{R},$$
$$\tilde{0}(x) = 0, \forall x \in \mathbb{R}$$

are correspondingly the greatest and the least elements w.r.t. $\subseteq_{F_1}$.
We introduce order $\subseteq^{\alpha}_{F1}$ on $F(\mathbb{R})$ in the following way:

**Definition 44.** Let $\alpha \in [0, 1]$, $P, Q \in F(\mathbb{R})$

$$P \subseteq^{\alpha}_{F1} Q \Leftrightarrow (\forall x \in \mathbb{R})(P(x) \geq \alpha \Rightarrow P(x) \leq Q(x)).$$

The greatest element w.r.t. $\subseteq^{\alpha}_{F1}$ is defined in the following way:

$$\tilde{1}(x) = 1, \forall x \in \mathbb{R}. \tag{48}$$

Let

$$\Theta = \{\tilde{0}(x) | \tilde{0}(x) \leq \alpha, \forall x \in \mathbb{R}\}. \tag{49}$$

Capital $\Theta$ denotes the class of elements, where $\tilde{0}(x) = 0, \forall x \in \mathbb{R}$ is the least.
Provided the essence of the parameter $\alpha$ (it "ignores" value if it is less than $\alpha$) we consider all elements of $\Theta$ to be equivalent. Further speaking about boundary condition of a generalized agop (w.r.t.$\subseteq^{\alpha}_{F_1}$) we require that $\forall n \in \mathbb{N}$ $n$-ary aggregation of arbitrary elements from this class should be equal to an element from this class, then we say that the boundary condition is satisfied (accurate requirement is provided in the sequel).
Further we call $\Theta$ the class of minimal elements.
As one can see $\tilde{1}$ for $\subseteq_{F1}$ and $\subseteq^{\alpha}_{F1}$ is defined in the same manner. And $\tilde{0}$ for $\subseteq_{F1}$ is one particular representative of the class of minimal elements for $\subseteq^{\alpha}_{F1}$.
It immediately follows from order definitions that $\subseteq_{F1}$ is a particular case of $\subseteq^{\alpha}_{F1}$ (when $\alpha = 0$).

**Remark 4.** Order relation $\subseteq_{F1}$ is reflexive, antisymmetric and transitive. While $\subseteq^{\alpha}_{F1}$ is transitive and asymmetric.

### 3.3.2 Horizontal order relations $\prec_I$ and $\subseteq^{\alpha}_{F_2}$

Order relations presented in this section act on fuzzy sets defined on the real line closed intervals, therefore the assumption $[a; b] \subseteq \mathbb{R}$ and correspondingly $F([a, b]) = \{P | P : [a, b] \to [0, 1]\}$ is required.
At the beginning we recall order relation introduced by Bodenhofer in [2] and modified by Takači in [47]. We adopt these definitions for $X = [a, b] \subseteq \mathbb{R}$:

**Definition 45.** [2] Let $P \in F([a,b])$, a fuzzy superset of P denoted by LTR(P) is defined as:

$$LTR_P(x) = \sup\{P(y)|y < x\}$$

Similarly, RTL(P) is defined as:

$$RTL_P(x) = \sup\{P(y)|x < y\}$$

LTR(P) is actually the smallest fuzzy superset of P with a non-decreasing membership function. Likewise, LTR(P) is the smallest fuzzy superset of P with a non-increasing membership function.

**Definition 46.** [2] If $P, Q \in F([a,b])$ then we define an ordering $\prec_I$ on F([a,b]) in the following way:

$$P \prec_I Q \Leftrightarrow LTR_Q \subseteq_{F_1} LTR_P \text{ and } RTL_P \subseteq_{F_1} RTL_Q$$

It can be easily seen that two fuzzy sets with different heights cannot be compared. Thus, a new ordering $\prec_I^{"}$ was proposed by Takači. First, a new set $\lceil P \rceil$ is defined:

$$\lceil P \rceil (x) = \begin{cases} 1, \text{ if } x : P(x) = height(P), \\ P(x), \text{ otherwise} \end{cases}$$

For an arbitrary $P, Q \in F([a,b])$, a new ordering which can be applied to larger number of fuzzy sets was proposed:

**Definition 47.** [47] If $P, Q \in F([a,b])$ then we define an ordering $\prec_I^{"}$ on F([a,b]) in the folllowing way:

$$P \prec_I^{"} Q \Leftrightarrow \lceil P \rceil \prec_I \lceil Q \rceil$$

$$\tilde{0}(x) = \begin{cases} 1, \text{ if } x = a \\ 0 \text{ otherwise.} \end{cases}$$

$$\tilde{1}(x) = \begin{cases} 1, \text{ if } x = b \\ 0 \text{ otherwise.} \end{cases}$$

Now we introduce another horizontal order relation $\subseteq_{F_2}^{\alpha}$ defined on $F([a,b])$ in the following way:

**Definition 48.** Let $\alpha \in (0,1]$, $P, Q \in F([a,b])$

$$P \subseteq_{F_2}^{\alpha} Q \Leftrightarrow \overline{P}^{\alpha} \leq \underline{Q}^{\alpha},$$

where

$$P^{\alpha} = \{x : P(x) \geq \alpha\}, \qquad \min P^{\alpha} = \underline{P}^{\alpha}, \qquad \max P^{\alpha} = \overline{P}^{\alpha}$$
$$Q^{\alpha} = \{x : Q(x) \geq \alpha\}, \qquad \min Q^{\alpha} = \underline{Q}^{\alpha}, \qquad \max Q^{\alpha} = \overline{Q}^{\alpha}.$$

The classes

$$\Theta = \{\tilde{0}(x)|\tilde{0}(x) = 1, \text{ if } x = a \text{ and } \tilde{0}(x) < \alpha \text{ if } x \in (a, b]\},$$

$$\Sigma = \{\tilde{1}(x)|\tilde{1}(x) = 1, \text{ if } x = b \text{ and } \tilde{1}(x) < \alpha \text{ if } x \in [a, b)\}$$

we will call correspondingly the class of minimal and maximal elements. The least element is defined in the following way:

$$\tilde{0}(x) = \begin{cases} 1, & \text{if } x = a \\ 0, & \text{otherwise} \end{cases}$$

but the greatest element does not exist.
The necessity of the whole class instead of just one the least (or the greatest) element is motivated by the essence of parameter $\alpha$. Further in the context of generalized aggregation (w.r.t.$\subseteq^{\alpha}_{F_2}$) we require that $\forall n \in \mathbb{N}$ n-ary aggregation of an arbitrary element from the class of minimal (maximal) elements is equal to an arbitrary element from the same class, if this holds we say that boundary condition is satisfied.
It is clear that $\forall\tilde{0} \in \Theta$ (correspondingly $\forall\tilde{1} \in \Sigma$), $\forall\alpha^* \in [\alpha, 1]$ $\alpha^*$-cut contains exactly one element - $a$ (correspondingly $b$):

$$\tilde{0}^{\alpha^*} = \{a\} \text{ and } a = \underline{\tilde{0}}^{\alpha^*} = \overline{\tilde{0}}^{\alpha^*},$$

$$\tilde{1}^{\alpha^*} = \{b\} \text{ and } b = \underline{\tilde{1}}^{\alpha^*} = \overline{\tilde{1}}^{\alpha^*}.$$

Property of $\alpha$-cuts implies the following result:

$$P \subseteq^{\alpha_1}_{F_2} Q \Rightarrow P \subseteq^{\alpha_2}_{F_2} Q, \ \forall\alpha_2 > \alpha_1.$$

**Remark 5.** Order relation $\prec^{"}_I$ is reflexive, antisymmetric and transitive. While $\subseteq^{\alpha}_{F_2}$ is trasitive and asymmetric.

## 3.4 Auxiliary results

### 3.4.1 Generalization of some results for continuous t-norms

Further we generalize results provided in section 1.4 and namely results of theorems 1.3, 1.4 and 1.6 are formulated for an arbitrary continuous t-norm. So we extend operations to the set $F(\mathbb{R})$ by means of continuous t-norm.

**Theorem 3.4.** *If $\circ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous operation, $T$ is a continuous t-norm and $P, Q \in F(\mathbb{R})$ are upper semicontinuous fuzzy sets with bounded $\alpha$-cuts $\forall\alpha > 0$ then for all $z \in \mathbb{R}$, $z = x \circ y$ $\exists x_0, y_0 \in \mathbb{R}$ such that $z = x_0 \circ y_0$ and $(P \circ Q)(z) = T(P(x_0), Q(y_0))$.*

*Proof.* According to the extension principle

$$(P \circ Q)(z) = \sup_{x \circ y = z} T(P(x), Q(y)).$$

The case when $(P \circ Q)(z) = 0$ is evident. Therefore we assume that $(P \circ Q)(z) = \alpha > 0$ and

$$T(P(x), Q(y)) < \alpha = \sup_{x,y \in \mathbb{R}: x \circ y = z} T(P(x), Q(y))$$

for all $x, y : x \circ y = z$.
According to the definition of supremum there exists a sequence $(\alpha_n)$: $\alpha_n \to \alpha$ from below and moreover we can construct sequences $(x_n), (y_n) : \forall n \; x_n \circ y_n = z$ and $T(P(x_n), Q(y_n)) \geq \alpha_n$.
$P, Q$ are upper semicontinuous fuzzy sets with bounded $\alpha$-cuts $P^\alpha, Q^\alpha$ $\forall \alpha > 0$, this implies that $\forall \alpha > 0$ $\alpha$-cuts are closed and bounded intervals and as a result sequences $(x_n), (y_n)$ are bounded. It is a known fact that a bounded sequence has a convergent subsequence, therefore $\exists (x_{n_k}) \subseteq (x_n)$ which converges to some point $x_0$. Obviously

$$x_0 \in \bigcap_n P^{\alpha_n}.$$

Further we consider $(y_{n_k})$ a subsequence of $(y_n)$ with corresponding to $(x_{n_k})$ numbers. Again $(y_{n_k})$ is a bounded sequence in compact sets $Q^{\alpha_n}$ and we can extract $(y_{n_{k_l}})$:

$$(y_{n_{k_l}}) \subseteq (y_{n_k}) \text{ and } y_{n_{k_l}} \to y_0 \text{ when } l \to \infty.$$

We go back to $(x_{n_k})$ and extract subsequence $(x_{n_{k_l}})$ with corresponding to $(y_{n_{k_l}})$ numbers. $x_{n_{k_l}} \to x_0$ (as a subsequence of the convergent sequence). The continuity of $\circ$ and constructions of $(x_n), (y_n)$ allow us to state that $x_0 \circ y_0 = z$.
Further we assume that

$$P(x_{n_{k_l}}) = \beta_{n_{k_l}}$$
$$Q(y_{n_{k_l}}) = \gamma_{n_{k_l}}.$$

As $(\beta_{n_{k_l}}), (\gamma_{n_{k_l}})$ are bounded sequences, then we can extract convergent subsequences (similar reasoning like above allows us to extract subsequences with the same index numbers):

$$(\beta_m) \subseteq (\beta_{n_{k_l}}) \text{ and } \beta_m \to \beta \text{ when } m \to \infty$$

$$(\gamma_m) \subseteq (\gamma_{n_{k_l}}) \text{ and } \gamma_m \to \gamma \text{ when } m \to \infty.$$

By construction of $(\beta_m), (\gamma_m)$ we have:

$$T(\beta_m, \gamma_m) \geq \alpha_m.$$

By construction of the sequence $(\alpha_m)$ $\alpha_m \to \alpha$ from below.
By continuity of $\circ$:

$$x_0 \circ y_0 = z,$$

by continuity of $T$:

$$T(\beta, \gamma) \geq \alpha.$$

Since $P(x_m) \geq \beta_m, \forall m$ this implies that $P(x_0) \geq \beta_m, \forall m$ and as a result $P(x_0) \geq \beta$, and similarly $Q(y_0) \geq \gamma$.
Using monotonicity of $T$ we can write:

$$T(P(x_0), Q(y_0)) \geq T(\beta, \gamma) \geq \alpha,$$

thus we have obtained that $T(P(x_0), Q(y_0)) \geq \alpha$, but $\alpha$ is the supremum, so only the equality is possible. $\square$

**Theorem 3.5.** *If* $\circ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is a continuous operation,* $T$ *is a continuous t-norm and* $P, Q \in F(\mathbb{R})$ *are upper semicontinuous fuzzy sets with bounded* $\alpha$*-cuts* $\forall \alpha > 0$ *then*

$$(P \circ Q)^{T(\alpha,\beta)} = P^\alpha \circ Q^\beta.$$

*Proof.* We take $z \in (P \circ Q)^{T(\alpha,\beta)}$ then according to theorem 3.4 $\exists x, y \in \mathbb{R}:$

$$z = x \circ y, x \in P^\alpha, y \in Q^\beta.$$

The extension principle implies that $z \in P^\alpha \circ Q^\beta$ and thus

$$(P \circ Q)^{T(\alpha,\beta)} \subset P^\alpha \circ Q^\beta.$$

Now we assume that $z \in P^\alpha \circ Q^\beta$ then $\exists x \in P^\alpha$ and $y \in Q^\beta : z = x \circ y$, but the extension principle implies that $(P \circ Q)(z) \geq T(\alpha, \beta)$ and thus $z \in (P \circ Q)^{T(\alpha,\beta)}$ $\square$

**Theorem 3.6.** *If* $\circ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is a continuous operation,* $T$ *is a continuous t-norm and* $P, Q \in FQ(\mathbb{R})$ *then* $P \circ Q \in FQ(\mathbb{R})$.

*Proof.* First we show that if $P = [a, b]$ and $Q = [c, d]$ then $P \circ Q$ is a closed interval.
Indeed, if $P, Q$ are closed intervals then $P \times Q$ is a closed rectangle in the plane $\mathbb{R}^2$. It is known from Topology that rectangle is a compact and connected set.
$\circ$ is a continuous two argument function then the image of $P \times Q$
$\circ(P \times Q) = P \circ Q$ is a compact $\mathbb{R}$ subset.
On the other hand the image of connected set $P \times Q$ is a connected $\mathbb{R}$ subset, and thus $P \circ Q$ is a compact and connected set.
But it is known from Topology that only closed bounded intervals are subsets of $\mathbb{R}$, which are compact and connected simultaneously.
Now we consider an arbitrary situation when $P, Q \in FQ(\mathbb{R})$, then according to theorem 3.5 $(P \circ Q)^{T(\alpha,\beta)} = P^\alpha \circ Q^\beta, \forall \alpha, \beta > 0$.
Since $P, Q \in FQ(\mathbb{R})$ therefore $\forall \alpha, \beta > 0$ the cuts $P^\alpha, Q^\beta$ are closed and bounded intervals.
Above proved allows us to state that $\forall T(\alpha, \beta) > 0$

$$(P \circ Q)^{T(\alpha,\beta)} = P^\alpha \circ Q^\beta$$

are closed and bounded intervals as thus $P \circ Q \in FQ(\mathbb{R})$. $\square$

**Remark 6.** Results of theorems 3.4, 3.5 and 3.6 hold for the case $n > 2$. We do not provide proofs for those cases here as they employ the same ideas and are much more complicated from the notations prospective.

### 3.4.2 Upper semicontinuous transformation of a function

We provide construction which transforms an arbitrary non continuous function into an upper semicontinuous function in the sense of definition 8.
The idea of construction was proposed by professor A.Šostaks.
Further the construction will be used for transformation of an output of gagop into an upper semicontinuous fuzzy set.
Let's assume that $f : \mathbb{R} \to [a, b]$ is an arbitrary function, we define the class of functions

$$Z = \{z : \mathbb{R} \to [a, b] | z(x) \geq f(x) \forall x \in \mathbb{R} \text{ and } z(x) \text{ is upper semicontinuous}\}. \tag{50}$$

Now we take pointwise infimum of the class $Z$ for all $x \in \mathbb{R}$:

$$\inf_{z \in Z} z(x_0) = \tilde{f}(x_0), \forall x_0 \in \mathbb{R} \tag{51}$$

According to R. Engelking ([10], p.87) $\tilde{f}(x)$ is an upper semicontinuous function and it is the smallest upper semicontinuous function which is greater or equal to $f(x)$. Thus $\tilde{f}(x)$ coincides with $f(x)$ in the points where the latest is continuous or upper semicontinuous and $\tilde{f}(x)$ "makes" $f(x)$ upper semicontinuous otherwise.

## 3.5 Generalized aggregation: introduction

The problem of aggregation can be generalized if we use fuzzy subsets as input information. Functions are aggregated in this case. We are further developing this approach, which is initiated by Takači in [47]. However other interesting, conceptually different approaches of generalization can be found in literature, e.g. in [30],[43],[52] and others.
General results related to the gagops, such as definition, definition of properties and construction methods, employ the set $F(\mathbb{R})$, i.e. the set of all fuzzy subsets defined on $\mathbb{R}$. Within the framework of detailed study on pointwise extension and $T$-extension (sections 3.6-3.9) we put additional requirements on elements of $F(\mathbb{R})$. Namely $F(\mathbb{R}) = \{P | P : \mathbb{R} \to [0, 1]\}$ in the sequel will denote the set of all upper semicontinuous fuzzy sets with bounded $\alpha$-cuts for every $\alpha > 0$.

### 3.5.1 Definition of a generalized agop

We give the definition of a gagop ([47]). This notion is the base of our further considerations.
Let $\prec$ be some order relation on $F(\mathbb{R})$ with the least element $\tilde{0} \in F(\mathbb{R})$ and the greatest element $\tilde{1} \in F(\mathbb{R})$.

**Definition 49.** [47] A mapping $\tilde{A} : \cup_{n \in \mathbb{N}} F(\mathbb{R})^n \to F(\mathbb{R})$ is called a generalized aggregation operator w.r.t. the order relation $\prec$, if for every $n \in \mathbb{N}$ the following conditions hold:
$(\tilde{A}1)$ $\tilde{A}(\tilde{0}, ..., \tilde{0}) = \tilde{0}$

$(\tilde{A}2)$ $\tilde{A}(\tilde{1}, ..., \tilde{1}) = \tilde{1}$
$(\tilde{A}3)$ $(\forall i = \overline{1,n})$ $(P_i \prec Q_i) \Rightarrow \tilde{A}(P_1, ..., P_n) \prec \tilde{A}(Q_1, ..., Q_n)$,
where $P_1, ..., P_n, Q_1, ..., Q_n \in F(\mathbb{R})$.

Gagop can be presented as a family $\tilde{A} = (\tilde{A}_{(n)})_{n\in\mathbb{N}}$, usually we consider an arbitrary $n$-ary restriction of this family.
We use convention $\tilde{A}_{(1)}(P(x)) = P(x)$ $\forall P(x) \in F(\mathbb{R})$.

**Remark 7.** We do not restrict definition 49 to the particular type of order relations. Relation $\prec$ can be a partial ordering with properties of reflexivity, antisymmetry and transitivity or it can be a strict ordering satisfying transitivity and asymmetry.

Further in the work we consider also the following sets of inputs: $FQ(\mathbb{R})$ - fuzzy quantities, $FI(\mathbb{R})$ - fuzzy intervals and $FN(\mathbb{R})$ - fuzzy numbers. Sometimes we consider special subclass of $FI(\mathbb{R})$, i.e., the set of trapezoidal intervals, denoted $FTI(\mathbb{R})$, and special subclass of $FN(\mathbb{R})$, i.e., the set of triangular numbers, denoted $FTN(\mathbb{R})$.
According to definition 49 the set of output values of a gagop should be the same like the class of input objects. Therefore additionally to properties $(\tilde{A}1) - (\tilde{A}3)$ we should verify the coherence of the set of inputs and definition of gagop.
For the mentioned sets the following inclusions hold:

$$FTN(\mathbb{R}) \subseteq FN(\mathbb{R}) \subseteq FI(\mathbb{R}) \subseteq FQ(\mathbb{R}) \subseteq F(\mathbb{R})$$

and

$$FTI(\mathbb{R}) \subseteq FI(\mathbb{R}).$$

Such a narrowing of set of inputs is necessary in the cases when a property does not hold for $F(\mathbb{R})$ but it holds for the smaller class.
The study given in the subsequent sections flashes gagops from the most important aspects and provides foundations of the theory of gagops.

### 3.5.2 Generalization methods

Takači in [47] summarizes methods of generalization of agops. We recall these results here and adopt them to the fuzzy sets defined on the real line. The author in [47] employs an arbitrary universe $X$ for the definition of the input information.
Let $P_1, ..., P_n \in F(\mathbb{R})$, $\tilde{A} : \cup_{n\in\mathbb{N}} F(\mathbb{R})^n \to F(\mathbb{R})$ and $A$ be an ordinary agop on the unit interval:

**Definition 50.** [47] $\tilde{A}$ is a pointwise extension of $A$ provided that:

$$\forall x \in \mathbb{R}\ \tilde{A}(P_1, ..., P_n)(x) = A(P_1(x), ..., P_n(x)).$$

$\tilde{A}(P_1, ..., P_n)$ is the fuzzy set obtained as a result of application of the operator $\tilde{A}$ to the fuzzy sets $P_1, ..., P_n$.

**Definition 51.** [47] Let $T$ be an arbitrary t-norm. $\tilde{A}$ is a $T$-extension of an agop $A$ provided that:

$$\tilde{A}(P_1, ..., P_n)(x) = \sup\{T(P_1(x_1), ..., P_n(x_n)) | (x_1, ..., x_n) \in \mathbb{R}^n : A(x_1, ..., x_n) = x\}.$$

**Definition 52.** [47] $\tilde{A}$ is defined as an A-extension of some increasing operator $\phi : \mathbb{R}^n \to \mathbb{R}$ if:

$$\tilde{A}(P_1, ..., P_n)(x) = \sup\{A(P_1(x_1), ..., P_n(x_n)) | x = \phi(x_1, ..., x_n), x_i \in P_i\}.$$

It is worth to mention that definitions 50, 51, 52 provide a possible construction method of a gagop. And some order relation $\prec$ should be specified in order to determine if they define a gagop.

### 3.5.3 Properties of a generalized agop

We define mathematical properties of a gagop $\tilde{A} : \cup_{n \in \mathbb{N}} F(\mathbb{R})^n \to F(\mathbb{R})$ w.r.t. some order relation $\prec$ with the least element $\tilde{0} \in F(\mathbb{R})$ and the greatest element $\tilde{1} \in F(\mathbb{R})$. Crisp case definitions from section 3.1.2 are careffully generalized and adopted. Also some related results are proven.
$\tilde{A}_{(n)}$ stands for an $n$-ary restriction of a gagop. Normally we will not use the index $n$, when we consider an arbitrary restriction, only in some cases when misunderstanding can arise. E.g. we use the following notations:

$$\tilde{A}(P_1, ..., P_n)$$

$$\tilde{A}_{(n)}(P, ..., P).$$

The number of arguments in the first case shows that it is a restriction to an $n$-ary aggregation. The second case needs clarifications therefore we use the index $n$.

**Definition 53** (IDEMPOTENCE)**.** An element $P \in F(\mathbb{R})$ is called an $\tilde{A}$-idempotent element whenever $\tilde{A}_{(n)}(P, ..., P) = P, \forall n \in \mathbb{N}$. $\tilde{A}$ is called an idempotent gagop if each $P \in F(\mathbb{R})$ is an idempotent element of $\tilde{A}$.

**Definition 54** (SYMMETRY)**.** A gagop $\tilde{A} : \cup_{n \in \mathbb{N}} F(\mathbb{R})^n \to F(\mathbb{R})$ is called a symmetric gagop if

$$\forall n \in \mathbb{N}, \forall P_1, ..., P_n \in F(\mathbb{R}) : \tilde{A}(P_1, ..., P_n) = \tilde{A}(P_{\pi(1)}, ..., P_{\pi(n)})$$

for all permutations $\pi = (\pi(1), ..., \pi(n))$ of $(1, ..., n)$.

**Definition 55** (ASSOCIATIVITY)**.** A gagop $\tilde{A} : \cup_{n \in \mathbb{N}} F(\mathbb{R})^n \to F(\mathbb{R})$ is associative if

$$\forall n, m \in \mathbb{N}, \forall P_1, ..., P_n, Q_1, ..., Q_m \in F(\mathbb{R}) :$$
$$\tilde{A}(P_1, ..., P_n, Q_1, ..., Q_m) = \tilde{A}(\tilde{A}(P_1, ..., P_n), \tilde{A}(Q_1, ..., Q_m)).$$

**Definition 56** (BISYMMETRY)**.** A gagop $\tilde{A} : \cup_{n\in\mathbb{N}} F(\mathbb{R})^n \to F(\mathbb{R})$ is bisymmetric if

$$\forall n, m \in \mathbb{N}, \forall P_{11}, ..., P_{mn} \in F(\mathbb{R}) :$$
$$\tilde{A}_{(mn)}(P_{11}, ..., P_{mn}) = \tilde{A}_{(m)}(\tilde{A}_{(n)}(P_{11}, ..., P_{1n}), ..., \tilde{A}_{(n)}(P_{m1}, ..., P_{mn})) =$$
$$= \tilde{A}_{(n)}(\tilde{A}_{(m)}(P_{11}, ..., P_{m1}), ..., \tilde{A}_{(m)}(P_{1n}, ..., P_{mn})).$$

It directly follows from definitions 54, 55 and 56 that bisymmetry of a gagop is implied by associativity and symmetry regardless the way of construction.

**Definition 57** (NEUTRAL ELEMENT)**.** An element $E \in F(\mathbb{R})$ is called a neutral element of $\tilde{A}$ if $\forall n \in \mathbb{N}, \forall P_1, ..., P_n, \in F(\mathbb{R})$ if $P_i = E$ for some $i \in \{1, ..., n\}$ then

$$\tilde{A}(P_1, ..., P_n) = \tilde{A}_{(n-1)}(P_1, ..., P_{i-1}, P_{i+1}, ..., P_n).$$

The following result holds for the neutral element of a gagop:

**Proposition 3.7.** If $\tilde{A} : \cup_{n\in\mathbb{N}} F(\mathbb{R})^n \to F(\mathbb{R})$ is a gagop w.r.t. $\prec$ and $E \in F(\mathbb{R})$ is a neutral element of $\tilde{A}$ then it is unique.

*Proof.* Let's assume that $E$ and $E^*$ are neutral elements of $\tilde{A}$ and $E \neq E^*$. We consider an arbitrary $n \in \mathbb{N}$ and vector $(P_1, ..., P_n)$ s.t.:

$$P_i = \begin{cases} E, \text{ if } i \in I_1 \\ E^*, \text{ if } i \in I_2, \end{cases}$$

where $I_1 \neq \emptyset$, $I_2 \neq \emptyset$ and $I_1 \cup I_2 = \{1, ..., n\}$.
Using neutrality of $\tilde{A}$ versus $E$ we obtain:

$$\tilde{A}(P_1, ..., P_n) = \tilde{A}_{(n-1)} = ... = \tilde{A}_{(k)}(E^*, ..., E^*) \tag{52}$$

now we apply neutrality of $E^*$, convention $\tilde{A}_{(1)}(P) = P$ and continue (52):

$$\tilde{A}_{(k)}(E^*, ..., E^*) = \tilde{A}_{(k-1)}(E^*, ..., E^*) = ... = \tilde{A}_{(1)}(E^*) = E^*. \tag{53}$$

In the same way first employing neutrality of $E^*$ and then neutrality of $E$ we obtain:

$$\tilde{A}(P_1, ..., P_n) = \tilde{A}_{(n-1)} = ... = \tilde{A}_{(s)}(E, ..., E)$$
$$\tilde{A}_{(s)}(E, ..., E) = \tilde{A}_{(s-1)}(E, ..., E) = ... = \tilde{A}_{(1)}(E) = E. \tag{54}$$

We have obtained a contradiction. □

**Definition 58** (ABSORBING ELEMENT)**.** An element $R \in F(\mathbb{R})$ is called an absorbing element of $\tilde{A}$ if

$$\forall n \in \mathbb{N}, \forall P_1, ..., P_n, \in F(\mathbb{R}) : R \in \{P_1, ..., P_n\} \Rightarrow \tilde{A}(P_1, ..., P_n) = R.$$

The absorbing element like the neutral element is unique if it exists:

**Proposition 3.8.** If $\tilde{A} : \cup_{n\in\mathbb{N}} F(\mathbb{R})^n \to F(\mathbb{R})$ is a gagop w.r.t. $\prec$ and $R \in F(\mathbb{R})$ is the absorbing element of $\tilde{A}$, then it is unique.

*Proof.* Let's assume that $R$ and $R^*$ are absorbing elements of $\tilde{A}$ and $R \neq R^*$. We consider an arbitrary $n \in \mathbb{N}$ and vector $(P_1, ..., P_n)$ s.t. $R, R^* \in \{P_1, ..., P_n\}$. $R$ is the absorbing element therefore according to definition 58

$$\tilde{A}(P_1, ..., P_n) = R. \tag{55}$$

$R^*$ as well is the absorbing element of $\tilde{A}$, therefore:

$$\tilde{A}(P_1, ..., P_n) = R^*. \tag{56}$$

We have obtained contradiction, thus our assumption on existence of $R^*$ is incorrect. $\square$

**Definition 59** (OTHER)**.** A gagop $\tilde{A} : \cup_{n\in\mathbb{N}} F(\mathbb{R})^n \to F(\mathbb{R})$ is said to be:
(1) shift-invariant if

$$\forall n \in \mathbb{N}, \forall B \in F(\mathbb{R}), \forall P_1, ..., P_n \in F(\mathbb{R}) :$$
$$\tilde{A}(P_1 + B, ..., P_n + B) = \tilde{A}(P_1, ..., P_n) + B$$

(2) homogeneous if

$$\forall n \in \mathbb{N}, \forall B \in F(\mathbb{R}), \forall P_1, ..., P_n \in F(\mathbb{R}) :$$
$$\tilde{A}(BP_1, ..., BP_n) = B\tilde{A}(P_1, ..., P_n)$$

(3) linear if it homogeneous and shift-invariant.

We note that addition and multiplication in definition 59 are extended by means of extension principle based on an arbitrary continuous t-norm.
We have restricted ourselves to the most interesting properties, which characterize gagops in the most broad sense. Other properties of gagops can be defined in the same manner, just carrying over crisp definitions.

## 3.6 Generalized aggregation: Pointwise extension

We start the study of a pointwise extension of an arbitrary agop $A$ in the next subsections. At the beginning we shortly review possible sets of inputs, later we consider properties and we finalize the section considering a pointwise extension of an arbitrary agop and an arbitrary $\gamma$-agop w.r.t. to order relations defined in section 3.3.
We recall that

$$\tilde{A} : \cup_{n\in\mathbb{N}} F(\mathbb{R})^n \to F(\mathbb{R})$$

is a pointwise extension of an arbitrary agop $A$ provided that:

$$\forall x \in \mathbb{R}\ \tilde{A}(P_1, ..., P_n)(x) = A(P_1(x), ..., P_n(x)). \tag{57}$$

Hereinafter $F(\mathbb{R}) = \{P | P : \mathbb{R} \to [0, 1]\}$ will denote the set of all upper semicontinuous fuzzy sets with bounded $\alpha$-cuts for every $\alpha > 0$.
Further when we speak about particular agops $A$ we use notations introduced in section 3.1, for the notations of t-norms we refer to section 1.5.

### 3.6.1 The set of inputs

When $A$ is a continuous agop, the result of formula (57) (for the finite number of inputs) is an upper semicontinuous fuzzy set with bounded $\alpha$-cuts $\forall\alpha > 0$, so the output belongs to $F(\mathbb{R})$. If $A$ is not continuous then upper semicontinuity of the output is lost, but we can apply construction introduced in section 3.4.2 and thus we harmonize the set of inputs $F(\mathbb{R})$ and definition of pointwise extension given by formula (57).

If we take inputs from $FQ(\mathbb{R})$ then applying formula (57) we can lose the convexity of the aggregated result. The following example illustrates this: let's $P_1, P_2 \in FQ(\mathbb{R})$ and $P_1, P_2$ have disjunctive supports then $\tilde{A}(P_1, P_2)$, where $A > min$ is not a convex fuzzy set.

The convexity is the necessary characteristic of the elements of the classes $FQ(\mathbb{R}), FI(\mathbb{R}), FN(\mathbb{R}), FTN(\mathbb{R}), FTI(\mathbb{R})$, therefore none of them can be taken in the role of the set of inputs of a pointwise extension of an arbitrary agop without serious restrictions.

Thus $F(\mathbb{R})$ is the only possible set of inputs among the sets defined previously, and $\mathbb{R}$ is the only thing open for changes, e.g. we can take $[a, b] \subseteq \mathbb{R}$ instead of $\mathbb{R}$.

### 3.6.2 Properties of a pointwise extension

We assume in this section that $\prec$ is the order relation defined on $F(\mathbb{R})$ and $\tilde{A} : \cup_{n\in\mathbb{N}} F(\mathbb{R})^n \to F(\mathbb{R})$ defined by formula (57) is a gagop w.r.t. $\prec$.

Further in this section we determine and prove necessary conditions for $\tilde{A}$ to be symmetric, associative, bisymmetric and idempotent gagop, also conditions for the existence of neutral and absorbing elements are considered.

**Proposition 3.9.** Let $\tilde{A}$ be a pointwise extension of $A$, then the following assertions hold:
(1) if $A$ is symmetric then $\tilde{A}$ is symmetric,
(2) if $A$ is associative then $\tilde{A}$ is associative,
(3) if $A$ is bisymmetric then $\tilde{A}$ is bisymmetric,
(4) if $A$ is idempotent then $\tilde{A}$ is idempotent.

*Proof.* In order to prove (1) we need to show that $\forall P_1, ..., P_n \in F(\mathbb{R}), \forall x \in \mathbb{R}, \forall n \in \mathbb{N}$ :

$$\tilde{A}(P_1, ..., P_n)(x) = \tilde{A}(P_{\pi(1)}, ..., P_{\pi(n)})(x) \tag{58}$$

for all permutations $\pi = (\pi(1), ..., \pi(n))$ of $(1, ..., n)$.

Due to definition of $\tilde{A}$ and symmetry of $A$ we can write:

$$\tilde{A}(P_1, ..., P_n)(x) = A(P_1(x), ..., P_n(x)) =$$

$$= A(P_{\pi(1)}(x), ..., P_{\pi(n)}(x)) \tag{59}$$

for an arbitrary permutation $\pi = (\pi(1), ..., \pi(n))$ of $(1, ..., n)$.

We again recall definition of $\tilde{A}$ and continue (59):

$$A(P_{\pi(1)}(x), ..., P_{\pi(n)}(x)) = \tilde{A}(P_{\pi(1)}, ..., P_{\pi(n)})(x). \tag{60}$$

The proof of (2) and (3) is analogous to (1) and we skip it here.
In order to prove (4) we need to show that $\forall P_i \in F(\mathbb{R}), \forall x \in \mathbb{R}$, $\forall n \in \mathbb{N}$:

$$\tilde{A}_{(n)}(P_i, ...P_i)(x) = P_i(x). \tag{61}$$

It immediately follows from the definition of $\tilde{A}$ and idempotence of $A$:

$$\tilde{A}_{(n)}(P_i, ...P_i)(x) = A_{(n)}(P_i(x), ..., P_i(x)) = P_i(x). \tag{62}$$

□

**Proposition 3.10.** If $\tilde{A}$ is a pointwise extension of $A$, $a$ and $e$ are correspondingly absorbing and neutral elements of $A$, then the following assertions hold:
(1) $R(x) = a,\ \forall x \in \mathbb{R}$ is the absorbing element of $\tilde{A}$,
(2) $E(x) = e,\ \forall x \in \mathbb{R}$ is the neutral element of $\tilde{A}$.

*Proof.* In order to prove (1) we need to show that

$$\forall P_1, ..., P_n \in F(\mathbb{R}), \forall x \in \mathbb{R}, \forall n \in \mathbb{N}$$
$$\tilde{A}(P_1, ..., P_n)(x) = R(x) \tag{63}$$

if $R \in \{P_1, ..., P_n\}$.
Let's consider an arbitrary $x \in \mathbb{R}$, then according to the definition of $\tilde{A}$ we can write:

$$\tilde{A}(P_1, ..., P_n)(x) = A(P_1(x), ..., P_n(x)). \tag{64}$$

Since $R \in \{P_1, ..., P_n\}$ and $R(x) = a, \forall x \in \mathbb{R}$, one of $A(P_1(x), ..., P_n(x))$ arguments is $a$.
Since $a$ is the absorbing element of $A$, then (64) can be continued in the following way:

$$A(P_1(x), ..., P_n(x)) = a \tag{65}$$

for an arbitrary chosen $x \in \mathbb{R}$.
Thus, we have received that

$$\forall P_1, ..., P_n \in F(\mathbb{R}), \forall x \in \mathbb{R}$$
$$\tilde{A}(P_1, ..., P_n)(x) = a \tag{66}$$

if $R \in \{P_1, ..., P_n\}$.
And according to proposition 3.8 $R$ is the unique absorbing element.
In order to prove (2) we need to show that

$$\forall P_1, ..., P_n \in F(\mathbb{R}), \forall x \in \mathbb{R}, \forall n \in \mathbb{N}$$

$$\tilde{A}_{(n)}(P_1,...P_i,...,P_n)(x) = \tilde{A}_{(n-1)}(P_1,...P_{i-1},P_{i+1}...,P_n)(x) \quad (67)$$

if $P_i = E$ for some $i \in \{1,...,n\}$.
We consider an arbitrary $x \in \mathbb{R}$ and assume that $P_i(x) = E(x) = e, \forall x$ for some $i \in \{1,...,n\}$, then exploiting definition of $\tilde{A}$ we can write:

$$\tilde{A}_{(n)}(P_1,...P_i,...,P_n)(x) = A_{(n)}(P_1(x),...,P_i(x),...,P_n(x)). \quad (68)$$

Since $i$-th argument of $A_{(n)}(P_1(x),...,P_i(x),...,P_n(x))$ is $e$ and can be omitted as it is the neutral element of $A$.
Thus we continue (68) in the following way:

$$A_{(n)}(P_1(x),...,P_i(x),...,P_n(x)) = A_{(n-1)}(P_1(x),...,P_{i-1}(x),P_{i+1}(x)...,P_n(x)) =$$

$$= \tilde{A}_{(n-1)}(P_1,...P_{i-1},P_{i+1},...,P_n)(x). \quad (69)$$

We remind that throughout the work we follow the convention $A_{(1)}(x) = x$, thus $\tilde{A}_{(1)}(P) = P$ and according to proposition 3.7 $E$ is the unique neutral element. $\square$

### 3.6.3 Shift - invariance of a pointwise extension

We explore shift-invariance property of a pointwise extension

$$\tilde{A} : \cup_{n\in\mathbb{N}} F(\mathbb{R})^n \to F(\mathbb{R})$$

which is a gagop w.r.t. some order relation $\prec$ defined on $F(\mathbb{R})$.
$\tilde{A}$ is a shift invariant if

$$\forall n \in \mathbb{N}, \forall B \in F(\mathbb{R}), \forall P_1,...,P_n \in F(\mathbb{R}) :$$

$$\tilde{A}(P_1 + B,...,P_n + B) = \tilde{A}(P_1,...,P_n) + B. \quad (70)$$

We have already established that $F(\mathbb{R})$ is the only possible set of inputs, but similarly like before we assume that all intermediary operations do not lead out of $F(\mathbb{R})$, i.e. $P_1,...,P_n$ and $B$ are chosen appropriately.
We consider a pointwise extension of shift-invariant agops. Shift-invariance of an agop implies its idempotence and as a result compensation property, thus, for the agops observed in this section the following holds:

$$(\min(x_1,...,x_n) \le A(x_1,...,x_n) \le \max(x_1,...,x_n)) \Rightarrow (T \le A), \quad (71)$$

where $T$ is a contiuous t-norm used for the extension of an addition operation. Further we specify $A$, $T$ and figure out cases where shift-invariance holds or does not hold.
First we consider a case when $A = T$. Evidently, the equality is possible only when $A = T = min$. The following result holds:

**Proposition 3.11.** Let $\tilde{A} : \cup_{n\in\mathbb{N}} F(\mathbb{R})^n \to F(\mathbb{R})$ be the pointwise extension of $A = min$ and $T = T_M$, then the folowing inequaliy holds:

$$(\tilde{A}(P_1,...,P_n) + B)(z) \le \tilde{A}(P_1 + B,...,P_n + B)(z), \forall z \in \mathbb{R}, \forall n \in \mathbb{N}.$$

*Proof.* We consider an arbitrary $n \in \mathbb{N}$ and $z \in \mathbb{R}$ such that $(\tilde{A}(P_1, ..., P_n) + B)(z) > 0$ and $\tilde{A}(P_1 + B, ..., P_n + B)(z) > 0$.
Consider $(\tilde{A}(P_1, ..., P_n) + B)(z)$:
according to theorem 3.4 $\exists x^*, y^* : x^* + y^* = z$ and

$$(\tilde{A}(P_1, ..., P_n) + B)(z) = T(\tilde{A}(P_1, ..., P_n)(x^*), B(y^*)). \tag{72}$$

We use definition of a pointwise extension and its idempotence implied by shift-invariance of A and continue formula (72):

$$\begin{aligned}(\tilde{A}(P_1, ..., P_n) + B)(z) &= T(\tilde{A}(P_1, ..., P_n)(x^*), B(y^*)) = \\ &= T(A(P_1(x^*), ..., P_n(x^*)), B(y^*)) = \\ &= T(A(P_1(x^*), ..., P_n(x^*)), A_{(n)}(B(y^*), ..., B(y^*))).\end{aligned} \tag{73}$$

Since $A = T$, we substitute $A$ by $T$ and apply t-norm associativity and symmetry:

$$\begin{aligned}(\tilde{A}(P_1, ..., P_n) + B)(z) &= T(T(P_1(x^*), ..., P_n(x^*)), T_{(n)}(B(y^*), ..., B(y^*))) = \\ &= T(T(P_1(x^*), B(y^*)), ..., T(P_n(x^*), B(y^*))).\end{aligned} \tag{74}$$

Now we consider $\tilde{A}(P_1 + B, ..., P_n + B)(z)$:
according to the definition of a pointwise extension :

$$\tilde{A}(P_1 + B, ..., P_n + B)(z) = A((P_1 + B)(z), ..., (P_n + B))(z). \tag{75}$$

We refer to theorem 3.4 and according to it $\forall i = \overline{1, n}\ \exists x_i, y_i : x_i + y_i = z$ and

$$(P_i + B)(z) = T(P_i(x_i), B(y_i)), \tag{76}$$

thus

$$A((P_1 + B)(z), ..., (P_n + B))(z) = A(T(P_1(x_1), B(y_1)), ..., T(P_n(x_n), B(y_n))). \tag{77}$$

Since $A = T$ it allows us to continue formula (77):

$$A((P_1 + B)(z), ..., (P_n + B))(z) = T(T(P_1(x_1), B(y_1)), ..., T(P_n(x_n), B(y_n))). \tag{78}$$

According to the definition of $x_i, y_i$ $\forall i = 1, ..., n$

$$T(P_i(x_i), B(y_i)) \geq T(P_i(x^*), B(y^*)), \forall i = \overline{1, n}, \tag{79}$$

therefore according to the monotonicity of $T$ we obtain that (78) $\geq$ (74) and thus

$$\tilde{A}(P_1 + B, ..., P_n + B)(z) \geq (\tilde{A}(P_1, ..., P_n) + B)(z). \tag{80}$$

□

The graphical example provided below shows that the strict inequality observed in proposition 3.11 is possible.

**Example 10.** Let $\tilde{A}$ be a pointwise extension of $min(x_1, x_2) = \min(x_1, x_2)$, $T_M(x_1, x_2) = \min(x_1, x_2)$, $P_1 = (10, 20, 30), P_2 = (30, 40, 50), B = (40, 50, 60)$ - triangular numbers.
Figure below ($\tilde{A}(P_1 + B, P_2 + B)$ is marked in red and $\tilde{A}(P_1, P_2) + B$ is marked in black on the graph) shows that $\tilde{A}(P_1 + B, P_2 + B) > \tilde{A}(P_1, P_2) + B$.

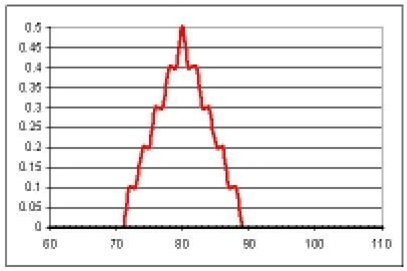


Figure 1: Pointwise extension of $min(x_1, x_2) = \min(x_1, x_2)$

Thus $\tilde{A}$ being the pointwise extension of $min$ is not a shift-invariant gagop, when $T = T_M$.

**Proposition 3.12.** Let $\tilde{A} : \cup_{n\in\mathbb{N}} F(\mathbb{R})^n \to F(\mathbb{R})$ be the pointwise extension of $A = max$ and $T$ is an arbitrary continuous t-norm, then $\tilde{A}$ is a shift-invariant gagop.

*Proof.* We need to show that

$$\tilde{A}(P_1 + B, ..., P_n + B)(z) = (\tilde{A}(P_1, ..., P_n) + B)(z), \forall z \in \mathbb{R}. \tag{81}$$

We consider an arbitrary $z \in \mathbb{R}$: $\tilde{A}(P_1+B, ..., P_n+B)(z) > 0$ and $(\tilde{A}(P_1, ..., P_n)+B)(z) > 0$.
According to the definition of pointwise-extension

$$\tilde{A}(P_1 + B, ..., P_n + B)(z) = \max((P_1 + B)(z), ..., (P_n + B)(z)). \tag{82}$$

According to theorem 3.4 $\forall i = 1, ..., n\ \exists\ x_i, y_i : x_i + y_i = z$ and

$$(P_i + B)(z) = T(P_i(x_i), B(y_i)), \tag{83}$$

we put the results of formula (83) into (82) and obtain the following result:

$$\tilde{A}(P_1 + B, ..., P_n + B)(z) = \max(T(P_1(x_1), B(y_1)), ..., T(P_n(x_n), B(y_n))) =$$

$$= T(P_s(x_s), B(y_s)), \tag{84}$$

where $1 \le s \le n$.
Now we consider $(\tilde{A}(P_1, ..., P_n) + B)(z)$, according to theorem 3.4 $\exists x^*, y^*$ : $x^* + y^* = z$ and

$$(\tilde{A}(P_1, ..., P_n) + B)(z) = T(\tilde{A}(P_1, ..., P_n)(x^*), B(y^*)). \tag{85}$$

According to the definition of pointwise extension:

$$\tilde{A}(P_1, ..., P_n)(x^*) = \max(P_1(x^*), ..., P_n(x^*)) = P_k(x^*), \tag{86}$$

where $1 \leq k \leq n$.
Thus we obtain:

$$(\tilde{A}(P_1, ..., P_n) + B)(z) = T(P_k(x^*), B(y^*)). \tag{87}$$

It is obvious that $\tilde{A}(P_1 + B, ..., P_n + B)(z) \geq (\tilde{A}(P_1, ..., P_n) + B)(z)$, because:

$$T(P_s(x_s), B(y_s))) \geq T(P_i(x_i), B(y_i)) \forall i = \overline{1, n} \tag{88}$$

and in particular

$$T(P_s(x_s), B(y_s)) \geq T(P_k(x_k), B(y_k)). \tag{89}$$

But according to the definition of $x_k, y_k$:

$$T(P_k(x_k), B(y_k)) \geq T(P_k(x^*), B(y^*)). \tag{90}$$

If we assume, that $\tilde{A}(P_1 + B, ..., P_n + B)(z) > (\tilde{A}(P_1, ..., P_n) + B)(z)$ and return to formulas (84), (87) we get:

$$T(P_s(x_s), B(y_s)) > T(P_k(x^*), B(y^*)). \tag{91}$$

But then $\exists x_s, y_s : x_s + y_s = z$ and

$$T(\tilde{A}(P_1, ..., P_n)(x_s), B(y_s)) = T(\max(P_1(x_s), ..., P_n(x_s)), B(y_s))$$

$$\geq T(P_s(x_s), B(y_s)) > T(P_k(x^*), B(y^*)) = (\tilde{A}(P_1, ..., P_n) + B)(z) \tag{92}$$

but this impossible due to the definition of $x^*, y^*$. $\square$

The results of this section allow us to state that in general, i.e. for an arbitrary shift-invariant agop and an arbitrary continuous t-norm pointwise extension is not a shift-invariant gagop.
The proof of propositions 3.11 and 3.12 can be redone in the same manner without any changes for another continuous operations e.g. multiplication. Thus we can state that $\tilde{A}$ is a homogeneous and linear gagop when $A = max$ and $T$ is an arbitrary continuous t-norm and it is not homogeneous and linear in general.
All properties of $\tilde{A}$ observed in the preceding section showed good correspondence with analogous properties of an agop $A$, i.e. symmetry of $A$ implied symmetry of $\tilde{A}$ and so on. Shift-invariance does not comply with this scheme, because it employs extension of an addition via extension principle, and this is not harmonized with the definition of $\tilde{A}$. Thus shift-invariance of $\tilde{A}$ is not obvious and even uncommon. We face similar problems when consider homogeneity and linearity.

### 3.6.4 Pointwise extension w.r.t. vertical order relations

Studing properties of $\tilde{A}$ we assumed that it is a gagop w.r.t. some order relation. Now we study $\tilde{A}$ from the prospective to preserve boundaries and monotonicity, i.e. we show that there exist order relations such that $\tilde{A}$ is a gagop w.r.t. them. We start with vertical orders defined previously in the work. It is proved (see [47]) that pointwise extension is a gagop w.r.t. $\subseteq_{F1}$.
The central matter of this section is the order relation $\subseteq^{\alpha}_{F_1}$.
We remind that speaking about boundary condition ($\tilde{A}1$) w.r.t.$\subseteq^{\alpha}_{F_1}$ we require that $\forall n \in \mathbb{N}$ $n$-ary aggregation of arbitrary elements from the class of minimal elements should be equal to an element from the same class, then we say that the boundary condition ($\tilde{A}1$) in definition 49 is satisfied.
More accurately:

$$\forall n \in \mathbb{N}, \forall \tilde{0}_1, ..., \tilde{0}_n \in \Theta \; \exists \tilde{0}_k \in \Theta :$$
$$\tilde{A}_{(n)}(\tilde{0}_1, ..., \tilde{0}_n) = \tilde{0}_k \tag{93}$$

The definition of $\gamma$-agop agrees with the definition of a pointwise extension, therefore at the beginning we formulate conditions under which a pointwise extension of a $\gamma$-agop $A$ will be a gagop w.r.t. $\subseteq^{\alpha}_{F_1}$. Later we consider the same problem for an arbitrary agop.

**Theorem 3.13.** *If $\tilde{A}$ is a pointwise extension of a $\gamma$-agop $A$, and $\gamma > \alpha$, then it is a gagop w.r.t. order relation $\subseteq^{\alpha}_{F1}$.*

*Proof.* We need to show $\tilde{A}1 - \tilde{A}3$ from definition 49.
$\tilde{A}1$:
According to definitions of pointwise extension and formula (49) for an arbitrary $x \in \mathbb{R}$, $n \in \mathbb{N}$ and $\tilde{0}_1, ..., \tilde{0}_n \in \Theta$ we can write:

$$\tilde{A}_{(n)}(\tilde{0}_1, ..., \tilde{0}_n)(x) = A_{(n)}(\tilde{0}_1(x), ..., \tilde{0}_n(x)) = A_{(n)}(x_1, ..., x_n), \tag{94}$$

where $x_i \leq \alpha, \forall i = 1, ..., n$.
Since $x_i \leq \alpha < \gamma$, then according to proposition 3.3 we can continue (94):

$$A_{(n)}(x_1, ..., x_n) = A_{(n)}(0, ..., 0) = 0. \tag{95}$$

So, $\forall x \in \mathbb{R}$ for an arbitrary $n \in \mathbb{N}$ and $\tilde{0}_1, ..., \tilde{0}_n \in \Theta$

$$\tilde{A}_{(n)}(\tilde{0}_1, ..., \tilde{0}_n)(x) = 0 \leq \alpha$$

and this means that formula (93) holds.
$\tilde{A}2$ :
According to definitions of the pointwise extension and $\tilde{1}$ for an arbitrary $x \in \mathbb{R}$ and $n \in \mathbb{N}$ we can write:

$$\tilde{A}_{(n)}(\tilde{1}, ..., \tilde{1})(x) = A_{(n)}(\tilde{1}(x), ..., \tilde{1}(x)) = A_{(n)}(1, ..., 1). \tag{96}$$

The equality $A_{(n)}(1, .., 1) = 1$ follows from the boundary condition for $\gamma$-agop. Thus,

$$\tilde{A}_{(n)}(\tilde{1}, ..., \tilde{1})(x) = 1 = \tilde{1}(x), \; \forall x \in \mathbb{R}. \tag{97}$$

$\tilde{A}3$ :
It is given that $(\forall i = \overline{1,n})$ $(P_i \subseteq^{\alpha}_{F1} Q_i)$ and we need to show that

$$\tilde{A}(P_1, ..., P_n) \subseteq^{\alpha}_{F1} \tilde{A}(Q_1, ..., Q_n). \tag{98}$$

According to definition of pointwise extension $\forall x \in \mathbb{R}$ we can write :

$$\tilde{A}(P_1, ..., P_n)(x) = A(P_1(x), ..., P_n(x)) \tag{99}$$

$$\tilde{A}(Q_1, ..., Q_n)(x) = A(Q_1(x), ..., Q_n(x)). \tag{100}$$

If $P_i(x) \geq \alpha$ then according to the inclussion $P_i(x) \subseteq^{\alpha}_{F_1} Q_i(x)$ $P_i(x) \leq Q_i(x)$.
If $P_i(x) < \alpha$ and thus $P_i(x) < \gamma$, then $\varphi_\gamma(P_i(x)) = 0$ and according to proposition 3.3 formula (99) can be continued in the following way:

$$\tilde{A}(P_1, ..., P_n)(x) = A(P_1(x), ...0..., P_n(x)), \tag{101}$$

where 0 stands on the positions, which belong to the index set $I_1 = \{i_1, ..., i_k\} \subseteq \{1, ..., n\} : P_i(x) < \alpha, \forall i \in I_1$.
Anyhow for $i \in I_1$ $Q_i(x) \geq 0$ and therefore $P_i(x) \leq Q_i(x)$.
Thus

$$P_i(x) \leq Q_i(x) \forall i$$

and hence monotonicity of $A$ provides the following inequality:

$$A(P_1(x), ..., P_n(x)) \leq A(Q_1(x), ..., Q_n(x)). \tag{102}$$

So, we have shown (98) $\square$

We have mentioned at the beginning of the section that pointwise extension of an arbitrary agop is a gagop w.r.t. $\subseteq_{F_1}$, and it also follows as a corollary from theorem 3.13 when $\alpha = 0$ and $\gamma = 0$. In this case condition $\gamma > \alpha$ is not needed, because $x_i = 0$ $\forall i$ and formula (95) holds due to the boundary condition. The proof of the second boundary condition and monotonicity does not require condition $\gamma > \alpha$.
Following the proof of theorem 3.13 we see that idempotence of an agop is necessary for modified condition $(\tilde{A}1)$. Compensation property implied by idempotence of $A$ provides the following implications:

$$A(P_1(x), ..., P_n(x)) \geq \alpha \Rightarrow^{compensation} \forall i \ P_i(x) \geq \alpha \Rightarrow^{\subseteq^{\alpha}_{F_1}}$$

$$\Rightarrow^{\subseteq^{\alpha}_{F_1}} \forall i \ P_i(x) \leq Q_i(x) \Rightarrow^{monotonicity}$$

$$\Rightarrow^{monotonicity} A(P_1(x), ..., P_n(x)) \leq A(Q_1(x), ..., Q_n(x))$$

and thus the following assertion holds:

**Theorem 3.14.** *If $\tilde{A}$ is a pointwise extension of an idempotent agop $A$, then it is a gagop w.r.t. order relation $\subseteq^{\alpha}_{F1}$.*

We have shown that $\gamma$-agops are not idempotent, but on the other hand theorem 3.13 illustrates that property $(A_\gamma)$ can compensate idempotence in some cases.

### 3.6.5 Pointwise extension w.r.t. horizontal order relations

We again refer to [47], where we can find that $\tilde{A}$ is not a gagop w.r.t. $\prec_I$.
Further in this section we study $\subseteq^{\alpha}_{F_2}$.
We declared before that elements of $\Theta$ are equivalent, thus speaking about boundary conditions $(\tilde{A}1)$ w.r.t. $\subseteq^{\alpha}_{F_2}$ we require that $\forall n \in \mathbb{N}$ $n$-ary aggregation of arbitrary elements from $\Theta$ should be equal to an element from the same class, then we say that the boundary condition $(\tilde{A}1)$ in definition 49 is satisfied. The same we require for the second boundary condition $(\tilde{A}2)$ w.r.t. $\subseteq^{\alpha}_{F_2}$.
More accurately:

$$\begin{aligned} &\forall n \in \mathbb{N}, \forall \tilde{0}_1, ..., \tilde{0}_n \in \Theta \ \exists \tilde{0}_k \in \Theta : \\ &\qquad \tilde{A}_{(n)}(\tilde{0}_1, ..., \tilde{0}_n) = \tilde{0}_k \end{aligned} \tag{103}$$

$$\begin{aligned} &\forall n \in \mathbb{N}, \forall \tilde{1}_1, ..., \tilde{1}_n \in \Sigma \ \exists \tilde{1}_k \in \Sigma : \\ &\qquad \tilde{A}_{(n)}(\tilde{1}_1, ..., \tilde{1}_n) = \tilde{1}_k \end{aligned} \tag{104}$$

At first we construct example showing that monotonicity does not hold (in different situations), then we study boundary conditions $(\tilde{A}1)$ and $(\tilde{A}2)$ w.r.t $\subseteq^{\alpha}_{F_2}$.

**Example 11.** Let's consider triangular numbers

$$P_1 = (1, 2, 3),\ P_2 = (5, 6, 7),\ Q_1 = (3, 4, 5),\ Q_2 = (7, 8, 9)$$

and pointwise extension of $max(x_1, x_2) = \max(x_1, x_2)$:

$$P_1 \subseteq^{\alpha}_{F_2} Q_1 \text{ and } P_2 \subseteq^{\alpha}_{F_2} Q_2,\ \forall \alpha \in (0, 1].$$

$$\tilde{A}(P_1, P_2)(x) = \max(P_1(x), P_2(x)) =$$

$$= \begin{cases} x - 1, & \text{if } x \in [1; 2] \\ 3 - x, & \text{if } x \in (2; 3] \\ x - 5, & \text{if } x \in [5; 6] \\ 7 - x, & \text{if } x \in (6; 7] \\ 0, & \text{otherwise.} \end{cases} \tag{105}$$

$$\tilde{A}(Q_1, Q_2)(x) = \max(Q_1(x), Q_2(x)) =$$

$$= \begin{cases} x - 3, & \text{if } x \in [3; 4] \\ 5 - x, & \text{if } x \in (4; 5] \\ x - 7, & \text{if } x \in [7; 8] \\ 9 - x, & \text{if } x \in (8; 9] \\ 0, & \text{otherwise.} \end{cases} \tag{106}$$

According to formulas (105) and (106)

$$\tilde{A}(P_1, P_2)(6) = 1,\ \tilde{A}(Q_1, Q_2)(4) = 1, \tag{107}$$

thus for an arbitrary $\alpha^* \in (0, 1]$

$$\max\{x : \tilde{A}(P_1, P_2) \geq \alpha^*\} \geq 6 \tag{108}$$

and

$$\min\{x : \tilde{A}(Q_1, Q_2) \geq \alpha^*\} \leq 4 \tag{109}$$

and therefore $\tilde{A}(P_1, P_2) \neg \subseteq^{\alpha}_{F_2} \tilde{A}(Q_1, Q_2), \forall \alpha \in (0, 1]$.
Pointwise extension of *max* is not a gagop w.r.t. $\subseteq^{\alpha}_{F_2}$ for an arbitrary $\alpha \in (0, 1]$. Thus we have shown that in general monotonicity does not hold when we consider a pointwise extension of an idempotent agop.

Pointwise extension of non idempotent agops is not a gagop w.r.t. $\subseteq^{\alpha}_{F_2}$. It can be shown if we substitute *max* with the strongest agop $A_s$ in example 11.
Application of $\gamma$-agop, $\gamma \in [0; 1]$:

$$A_\gamma = \max(\omega_1 x_1, \omega_2 x_2),$$

where
$\omega_i = \begin{cases} 0, & \text{if } x < \gamma \\ 1 & \text{if } x \geq \gamma \end{cases}$
in example 11 shows that pointwise extension of $\gamma$-agops need not be a gagop w.r.t $\subseteq^{\alpha}_{F_2}$.
Below formulated results show that pointwise extension of a $\gamma$-agop and pointwise extension of an idempotent agop preserves boundaries w.r.t. $\subseteq^{\alpha}_{F_2}$:

**Proposition 3.15.** Let $\tilde{A} : \cup_{n\in\mathbb{N}} F(\mathbb{R})^n \to F(\mathbb{R})$ be a pointwise extension of $\gamma$-agop $A$, and $\gamma \geq \alpha$, then for an arbitrary $n \in \mathbb{N}$ and arbitrary $\tilde{0}_1, ..., \tilde{0}_n \in \Theta$

$$\tilde{A}_{(n)}(\tilde{0}_1, ..., \tilde{0}_n) \in \Theta.$$

*Proof.* Let's take arbitrary elements $\tilde{0}_i \in \Theta, i = 1, ..., n$:

$$\tilde{0}_i(x) = \begin{cases} 1, & \text{if } x = a \\ \alpha_x < \alpha, & \text{otherwise.} \end{cases} \tag{110}$$

We consider $x \neq a$.
According to the definitions of pointwise extension and $\tilde{0}_i$ and using the fact that $\alpha_x < \alpha \leq \gamma$ we can write:

$$\tilde{A}_{(n)}(\tilde{0}_1, ..., \tilde{0}_n)(x) = A_{(n)}(\tilde{0}_1(x), ..., \tilde{0}_n(x)) =$$

$$= A_{(n)}(\alpha_x, ..., \alpha_x) = A_{(n)}(0, ..., 0) = 0 < \alpha. \tag{111}$$

We consider $x = a$. According to the definitions of a pointwise extension and $\tilde{0}_i$ we can write:

$$\tilde{A}_{(n)}(\tilde{0}_1, ..., \tilde{0}_n)(x) = A_{(n)}(\tilde{0}_1(x), ..., \tilde{0}_n(x)) = A_{(n)}(1, ..., 1) = 1. \tag{112}$$

Formulas (111),(112) show that the assertion holds. □

**Proposition 3.16.** Let $\tilde{A} : \cup_{n\in\mathbb{N}} F(\mathbb{R})^n \to F(\mathbb{R})$ be a pointwise extension of $\gamma$-agop $A$, and $\gamma \geq \alpha$, then for an arbitrary $n \in \mathbb{N}$ and arbitrary $\tilde{1}_1, ..., \tilde{1}_n \in \Sigma$

$$\tilde{A}_{(n)}(\tilde{1}_1, ..., \tilde{1}_n) \in \Sigma.$$

*Proof.* Let's take arbitrary elements $\tilde{1}_i \in \Sigma, i = 1, ..., n$:

$$\tilde{1}_i(x) = \begin{cases} 1, & \text{if } x = b \\ \alpha_x < \alpha, & \text{otherwise.} \end{cases} \tag{113}$$

We consider $x \neq b$. According to the definitions of a pointwise extension and $\tilde{1}_i$ and given the fact that $\alpha_x < \alpha \leq \gamma$, we can write:

$$\tilde{A}_{(n)}(\tilde{1}_1, ..., \tilde{1}_n)(x) = A_{(n)}(\tilde{1}_1(x), ..., \tilde{1}_n(x)) =$$

$$= A_{(n)}(\alpha_x, ..., \alpha_x) = A_{(n)}(0, ..., 0) = 0 < \alpha. \tag{114}$$

Now we consider $x = b$ and apply definitions of a pointwise extension and $\tilde{1}_i$:

$$\tilde{A}_{(n)}(\tilde{1}_1, ..., \tilde{1}_n)(x) = A_{(n)}(\tilde{1}_1(x), ..., \tilde{1}_n(x)) = A_{(n)}(1, ..., 1) = 1 \tag{115}$$

Formulas (114),(115) show that the assertion holds. □

**Remark 8.** Result of propositions 3.15 and 3.16 hold for idempotent agops.

## 3.7 Generalized aggregation: $T$-extension of an agop $A$

We study a $T$-extension of an arbitrary agop in this section. The extension of $\gamma$-agop is not considered here as its definition is not in accordance with the definition of $T$-extension. In order to distinguish between pointwise extension and $T$-extension we will denote the latest $\widehat{A}$.
We recall that

$$\widehat{A}(P_1, ..., P_n)(x) = \sup\{T(P_1(x_1), ..., P_n(x_n))|$$

$$(x_1, ..., x_n) \in \mathbb{R}^n : A(x_1, ..., x_n) = x\} \tag{116}$$

is called a $T$-extension of an agop $A$, where $T$ is an arbitrary t-norm. We consider the case when $T$ is a continuous t-norm in the work.
We consider agops defined on a closed interval, and definition of an agop on an arbitrary interval $[a, b] \subseteq \mathbb{R}$ is a matter of the rescaling, if compared with agop defined on $[0, 1]$, therefore we study $T$-extension of $A$ defined over the unit interval:

$$\widehat{A}(P_1, ..., P_n)(x) = \sup\{T(P_1(x_1), ..., P_n(x_n))|$$

$$(x_1, ..., x_n) \in [0, 1]^n : A(x_1, ..., x_n) = x\}. \tag{117}$$

By $F([0, 1]) = \{P|P : [0, 1] \to [0, 1]\}$ we will denote the set of upper semicontinuous fuzzy sets. The set of fuzzy quantities, intervals and numbers defined over the unit interval will be denoted $FQ([0, 1])$, $FN([0, 1])$, $FI([0, 1])$, $FTI([0, 1])$ and $FTN([0, 1])$.

### 3.7.1 The set of inputs

We determine possible sets of inputs of a $T$-extension $\widehat{A}$ of an agop $A$.
It is obvious that upper semicontinuity maybe lost when we aggregate the finite number of upper semicontinuous fuzzy sets, e.g. when $A$ is not continuous. But we treat this problem similarly like we did in the case of a pointwise extension and thus we consider that the set of inputs $F([0,1])$ is coherent with the definition of $\widehat{A}$. Additionally, definition of input values over $[0,1]$ provides that $\alpha$-cuts are bounded for all $\alpha \geq 0$.
In the sequent sections we examine $FQ([0,1])$, $FI([0,1])$, $FN([0,1])$, $FTI([0,1])$ and $FTN([0,1])$ and define necessary conditions for the corresponding set to be the set of inputs.

**Fuzzy quantities**

We explore $FQ([0,1])$ in the role of the set of inputs of $\widehat{A}$.
The following results for $T$-extension by means of any continuous t-norm immediately follows as a corollary from theorem 3.6:

**Corollary 4.** If $\widehat{A}$ is a $T$-extension of a continuous agop $A$ and $P_1, ..., P_n \in FQ([0,1])$ then

$$\widehat{A}(P_1, ..., P_n) \in FQ([0,1]).$$

**Fuzzy intervals**

We explore $FI([0,1])$ (definition 10) in the role of the set of inputs of $T$-extension.
The proposition formulated below shows that the result of aggregation of fuzzy intervals in the case of $T$-extension is a fuzzy interval.

**Proposition 3.17.** If $P_1, ..., P_n \in FI([0,1])$, $I_1, ..., I_n$ are their corresponding vertices and $A$ is a continuous agop then $\widehat{A}(P_1, ..., P_n) \in FI([0,1])$ and its vertex is $I = \{A(x_1, ..., x_n) | (x_1, ..., x_n) : x_i \in I_i, i = 1...n\}$.

*Proof.* The class of fuzzy intervals is a subclass of fuzzy quantities, therefore according to the results of corollary 4 $\widehat{A}(P_1, ..., P_n)$ will be a fuzzy quantity at least, when $P_1, ..., P_n \in FI([0,1])$. This implies, that if $\widehat{A}(P_1, ..., P_n)$ has a vertex, then it is in the form of continuous interval, otherwise $\widehat{A}(P_1, ..., P_n)$ will not be convex.
Now we show that $\widehat{A}(P_1, ..., P_n)(x)$=1 iff $x \in I$.
For an arbitrary $(x_1, ..., x_n) : x_1 \in I_1, ..., x_n \in I_n$ we have:

$$\widehat{A}(P_1, ..., P_n)(A(x_1, ..., x_n)) = T(P_1(x_1), ..., P_n(x_n)) = T(1, ..., 1) = 1. \quad (118)$$

Now we show that if $(x_1, ..., x_n)$ is not of the form $x_1 \in I_1, ..., x_n \in I_n$ then $\widehat{A}(P_1, ..., P_n)(A(x_1, ..., x_n)) \neq 1$.
If we consider $(y_1, ..., y_n) : \exists y_{i_1}, ..., y_{i_k} : y_{i_s} \notin I_{i_s}, s = 1, ..., k$ then $P_{i_s}(y_{i_s}) < 1$ and by neutrality of $T$ versus 1 we obtain:

$$\widehat{A}(P_1, ..., P_n)(A(y_1, ..., y_n)) = T(P_1(y_1), ..., P_n(y_n)) < 1. \quad (119)$$

Since $A$ is a continuous agop and $I_1 \times ... \times I_n$ is a compact and connected set its image is an interval, thus $I$ is the vertex of $\widehat{A}(P_1, ..., P_n)$. $\square$

### Fuzzy numbers

We consider $FN([0,1])$ in this section. We show that continuity of $A$ is a sufficient condition for the aggregated result to be a fuzzy number in the case when input values are in the form of fuzzy numbers.

**Proposition 3.18.** If $P_1, ..., P_n \in FN([0,1])$, $x_1^*, ..., x_n^*$ are their corresponding vertices and $A$ is continuous then $\widehat{A}(P_1, ..., P_n) \in FN([0,1])$ and $A(x_1^*, ..., x_n^*)$ is its vertex.

*Proof.* immediately follows from proposition 3.17. $\square$

### Triangular fuzzy numbers

We are interested in the preservation of triangular form of the elements of $FTN([0,1])$. We say that the form is preserved if aggregated result is in the form of a triangular number when the input information is in the form of triangular numbers.
Preservation of the form is more sensitive to the choice of $A$ and $T$. Continuity of $A$ may be deficient. E.g. it is a known fact that addition of triangular numbers and multiplication with constant (when extension principle employs $T_M$, see e.g. [46]) is a triangular number again, therefore $T_M$-extension of an arithmetic mean or weighted mean should be a triangular number. On the other hand multiplication of triangular numbers and raise to the n-th power do not preserve the shape, and as a result $T_M$-extension of a geometric mean is not a triangular number any more. The following graphical examples illustrates this. We aggregate triangular numbers $P_1 = (0.5, 0.65, 0.8), P_2 = (0, 0.5, 1)$ in all following examples. Throughout this section $P_1$ is coloured in black, $P_2$ in red and aggregated result in blue colour on the charts.

**Example 12.** Let

$$\widehat{A}(P_1, ..., P_n)(x) = \sup\{T_M(P_1(x_1), P_2(x_2)) | (x_1, x_2) \in [0,1]^n : \frac{x_1 + x_2}{2} = x\}$$

in the first case and

$$\widehat{A}(P_1, ..., P_n)(x) = \sup\{T_M(P_1(x_1), P_2(x_2)) | (x_1, x_2) \in [0,1]^n : 0.3x_1 + 0.7x_2 = x\}$$

in the second case.
Figure below shows that $T_M$-extension of arithmetic mean and weighted mean preserves the shape:

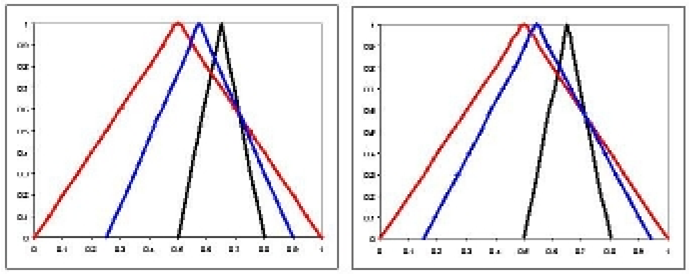

Figure 2: Left graph: $T_M$-extension of $M(x_1, x_2) = \frac{x_1+x_2}{2}$; Right graph: $T_M$ extension of $W(x_1, x_2) = 0.3x_1 + 0.7x_2$

**Example 13.** Let

$$\widehat{A}(P_1, ..., P_n)(x) = \sup\{T_M(P_1(x_1), P_2(x_2))|(x_1, x_2) \in [0, 1]^n : \sqrt{x_1 x_2} = x\}.$$

The next figure indicates that $T_M$-extension of $G(x_1, x_2) = \sqrt{x_1 x_2}$ does not preserve the shape:

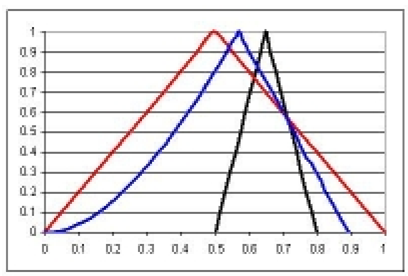


Figure 3: $T_M$-extension of $G(x_1, x_2) = \sqrt{x_1 x_2}$

So, we can conclude that $T_M$-extension not always preserves the shape. When $A$ is a combination of operations, which preserve triangular shape, then $T_M$-extension preserves the shape, otherwise the shape can be lost.
Now we consider another continuous t-norms: $T_P$ and $T_L$. $T_P$ is not a linear and $T_L$ is a piecewise linear t-norm.
$T_P$-extension of $\frac{x_1+x_2}{2}$ does not preserve the shape as different to $T_M$-extension, as we can see from the example below.

**Example 14.** Let

$$\widehat{A}(P_1, ..., P_n)(x) = \sup\{T_P(P_1(x_1), P_2(x_2))|(x_1, x_2) \in [0, 1]^n : \frac{x_1 + x_2}{2} = x\}.$$

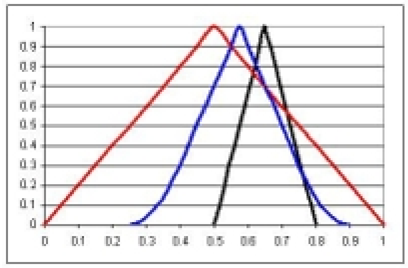


Figure 4: $T_P$-extension of $M(x_1, x_2) = \frac{x_1+x_2}{2}$

But $T_L$- extension of $M(x_1, x_2) = \frac{x_1+x_2}{2}$ preserves the shape:

**Example 15.** Let

$$\widehat{A}(P_1, ..., P_n)(x) = \sup\{T_L(P_1(x_1), P_2(x_2))|(x_1, x_2) \in [0, 1]^n : \frac{x_1 + x_2}{2} = x\}.$$

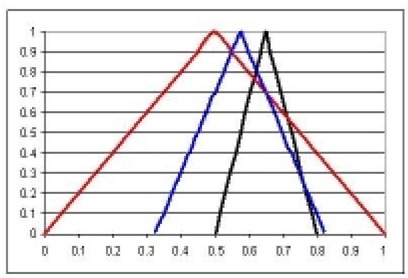


Figure 5: $T_L$-extension of $M(x_1, x_2) = \frac{x_1+x_2}{2}$

Presentating these results at the conference FSTA 2010 the author was suggested by prof. R.Mesiar to investigate the additive generator of the t-norm $T$ ([22]) and the generator of the agop $A$ ([6]). When they both have a linear form (e.g. in the example 15) the form of an aggregated result should be again triangular. In a more broad sense: when additive generator of a t-norm and generator of an agop are harmonized, (e.g. log for the first and exp for the last) then we can expect good accordance of the input and output values in terms of the shape. We did not get any particular results in this area yet, but this is an open problem for the author and the study is in the process.

### 3.7.2 Symmetry, associativity and bisymmetry of a $T$-extension

This and the three subsequent sections are devoted to the properties of $T$-extension. We assume that there exists an order relation such that $T$-extension

is a gagop w.r.t. this order relation and we explore properties of the gagop. This assumption is justifiable because further in sections 3.7.7 and 3.7.8 we show that such order relations exist.
Hereinafter we assume continuity of a t-norm and normally we use a continuous agop. We show that it is easy to obtain symmetric, associative or bisymmetric $\widehat{A}$, and the corresponding property is implied by the same property of $A$. Properly, the same property of t-norm is essential, but any t-norm is symmetric and associative (and as a result bisymmetric) by definition, therefore additional conditions for t-norm are not required.

**Proposition 3.19.** If $A$ is a continuous and symmetric agop then

$$\widehat{A} : \cup_{n \in \mathbb{N}} F([0,1])^n \to F([0,1])$$

is a symmetric gagop.

*Proof.* We need to show that:

$$\forall P_1, ..., P_n \in F([0,1]), \forall x \in [0,1]$$
$$\widehat{A}(P_1, ..., P_n)(x) = \widehat{A}(P_{(\pi_1)}, ..., P_{(\pi_n)})(x), \tag{120}$$

where $\pi = (\pi(1), ..., \pi(n))$ is an arbitrary permutation of $(1, ..., n)$.
First we consider $\widehat{A}(P_1, ..., P_n)(x)$:
A is continuous therefore according to theorem 3.4
$\exists\ x_1^*, ..., x_n^*$ :

$$A(x_1^*, ..., x_n^*) = x \tag{121}$$

and

$$\begin{aligned} T(P_1(x_1^*), ..., P_n(x_n^*)) = \sup\{T(P_1(x_1), ..., P_n(x_n))| \\ (x_1, ..., x_n) \in [0,1]^n : A(x_1, ..., x_n) = x\}. \end{aligned} \tag{122}$$

Now we consider $\widehat{A}(P_{(\pi_1)}, ..., P_{(\pi_n)})(x)$:
similarly: $\exists\ x_1^{'}, ..., x_n^{'}$ :

$$A(x_1^{'}, ..., x_n^{'}) = x \tag{123}$$

and

$$\begin{aligned} T(P_{(\pi_1)}(x_1^{'}), ..., P_{(\pi_n)}(x_n^{'})) = \sup\{T(P_{(\pi_1)}(x_1), ..., P_{(\pi_n)}(x_n))| \\ (x_1, ..., x_n) \in [0,1]^n : A(x_1, ..., x_n) = x\}. \end{aligned} \tag{124}$$

We assume that

$$\widehat{A}(P_1, ..., P_n)(x) > \widehat{A}(P_{(\pi_1)}, ..., P_{(\pi_n)})(x)$$

thus we obtain

$$T(P_1(x_1^*), ..., P_n(x_n^*)) > T(P_{(\pi_1)}(x_1^{'}), ..., P_{(\pi_n)}(x_n^{'})). \tag{125}$$

Symmetry of t-norm allows us to write

$$T(P_1(x_1^*), ..., P_n(x_n^*)) = T(P_{(\pi_1)}(x^*_{(\pi_1)}), ..., P_{(\pi_n)}(x^*_{(\pi_n)})) >$$
$$> T(P_{(\pi_1)}(x_1^{'}), ..., P_{(\pi_n)}(x_n^{'})). \tag{126}$$

Now we consider vector $(x^*_{(\pi_1)}, ..., x^*_{(\pi_n)})$ and apply symmetry of $A$

$$A(x^*_{(\pi_1)}, ..., x^*_{(\pi_n)}) = A(x_1^*, ..., x_n^*) = x. \tag{127}$$

But according to formula (126)

$$T(P_{(\pi_1)}(x^*_{(\pi_1)}), ..., P_{(\pi_n)}(x^*_{(\pi_n)})) > T(P_{(\pi_1)}(x_1^{'}), ..., P_{(\pi_n)}(x_n^{'})),$$

however this is impossible, thus we have obtained a contradiction.
Assumption on $\widehat{A}(P_1, ..., P_n)(x) < \widehat{A}(P_{(\pi_1)}, ..., P_{(\pi_n)})(x)$ will lead to the similar contradiction. □

**Proposition 3.20.** If $A$ is a continuous and associative agop then

$$\widehat{A} : \cup_{n\in\mathbb{N}} F([0,1])^n \to F([0,1])$$

is an associative gagop.

*Proof.* We need to show that

$$\forall n, m \in \mathbb{N}\ \forall P_1, ..., P_n, Q_1, ..., Q_m \in F([0,1])$$
$$\widehat{A}(P_1, ..., P_n, Q_1, ..., Q_m) = \widehat{A}(\widehat{A}(P_1, ..., P_n), \widehat{A}(Q_1, ..., Q_m)). \tag{128}$$

We consider the left and the right parts of (128) in an arbitrary $z \in [0,1]$.
Consider $\widehat{A}(P_1, ..., P_n, Q_1, ..., Q_m)(z)$ :
According to theorem 3.4 $\exists\ s_1^*, ..., s_n^*, t_1^*, ..., t_m^*$:

$$A(s_1^*, ..., s_n^*, t_1^*, ..., t_m^*) = z \tag{129}$$

and

$$\widehat{A}(P_1, ..., P_n, Q_1, ..., Q_m)(z) = T(P_1(s_1^*), ..., P_n(s_n^*), Q_1(t_1^*), ..., Q_m(t_m^*)). \tag{130}$$

Consider $\widehat{A}(\widehat{A}(P_1, ..., P_n), \widehat{A}(Q_1, ..., Q_m))(z)$ :
Let's assume that

$$\widehat{A}(P_1, ..., P_n) = P^*$$

and

$$\widehat{A}(Q_1, ..., Q_m) = Q^*,$$

where $P^*, Q^* \in F([0,1])$.
Thus we consider $\widehat{A}(P^*, Q^*)(z)$ :
according to theorem 3.4 $\exists\ x^*, y^*$:

$$A(x^*, y^*) = z \tag{131}$$

and

$$\widehat{A}(P^*, Q^*)(z) = T(P^*(x^*), Q^*(y^*)). \tag{132}$$

Following the definition of $P^*, Q^*$ and employing theorem 3.4 for an arbitrary $x^*, y^* \in [0,1]$ we can write:

$$P^*(x^*) = T(P_1(x_1^*), ..., P_n(x_n^*)), \tag{133}$$

where $x_1^*, ..., x_n^* : A(x_1^*, ..., x_n^*) = x^*$ and

$$Q^*(y^*) = T(Q_1(y_1^*), ..., Q_m(y_n^*)), \tag{134}$$

where $y_1^*, ..., y_m^* : A(y_1^*, ..., y_m^*) = y^*$.
We put (133) and (134) into (132) and obtain:

$$\widehat{A}(P^*, Q^*)(z) = T(T(P_1(x_1^*), ..., P_n(x_n^*)), T(Q_1(y_1^*), ..., Q_m(y_n^*))), \tag{135}$$

where

$$A(A(x_1^*, ..., x_n^*), A(y_1^*, ..., y_m^*)) = z. \tag{136}$$

Using associativity of $T$ and $A$ we continue (135) and (136) in the following way:

$$\widehat{A}(P^*, Q^*)(z) = T(P_1(x_1^*), ..., P_n(x_n^*), Q_1(y_1^*), ..., Q_m(y_n^*)) \tag{137}$$

$$A(x_1^*, ..., x_n^*, y_1^*, ..., y_m^*) = z. \tag{138}$$

If we assume that $\widehat{A}(P_1, ..., P_n, Q_1, ..., Q_m)(z) > \widehat{A}(\widehat{A}(P_1, ..., P_n), \widehat{A}(Q_1, ..., Q_m))(z)$, then we take vector $(s_1^*, ..., s_n^*, t_1^*, ..., t_m^*)$ in the role of $(x_1^*, ..., x_n^*, y_1^*, ..., y_m^*)$ and we obtain a higher value of $\widehat{A}(\widehat{A}(P_1, ..., P_n), \widehat{A}(Q_1, ..., Q_m))(z)$, but this contradicts the definition of $(x_1^*, ..., x_n^*, y_1^*, ..., y_m^*)$.
Similar contradiction we obtain if we assume that $\widehat{A}(P_1, ..., P_n, Q_1, ..., Q_m)(z) < \widehat{A}(\widehat{A}(P_1, ..., P_n), \widehat{A}(Q_1, ..., Q_m))(z)$. □

The result related to bisymmetry of $T$-extension we state without proof, because it can be performed combining ideas used in the proofs of the previous results:

**Proposition 3.21.** If $A$ is a continuous and bisymmetric agop then

$$\widehat{A} : \cup_{n\in\mathbb{N}} F([0,1])^n \to F([0,1])$$

is a bisymmetric gagop.

We have shown that in the case of a continuous agop $A$ in the role of the set of inputs we can take $FQ([0,1]), FI([0,1])$ and $FN([0,1])$. Thus the results of this section hold for other sets of inputs:

**Corollary 5.** If $A$ is a continuous and symmetric agop then the following assertations hold:
(1) $\widehat{A} : \cup_{n\in\mathbb{N}} FQ([0,1])^n \to FQ([0,1])$ is a symmetric gagop;
(2) $\widehat{A} : \cup_{n\in\mathbb{N}} FI([0,1])^n \to FI([0,1])$ is a symmetric gagop;
(3) $\widehat{A} : \cup_{n\in\mathbb{N}} FN([0,1])^n \to FN([0,1])$ is a symmetric gagop.

**Corollary 6.** If $A$ is a continuous and associative agop then the following assertions hold:
(1) $\widehat{A} : \cup_{n\in\mathbb{N}} FQ([0,1])^n \to FQ([0,1])$ is an associative gagop;
(2) $\widehat{A} : \cup_{n\in\mathbb{N}} FI([0,1])^n \to FI([0,1])$ is an associative gagop;
(3) $\widehat{A} : \cup_{n\in\mathbb{N}} FN([0,1])^n \to FN([0,1])$ is an associative gagop.

**Corollary 7.** If $A$ is a continuous and bisymmetric agop then the following assertions hold:
(1) $\widehat{A} : \cup_{n\in\mathbb{N}} FQ([0,1])^n \to FQ([0,1])$ is a bisymmetric gagop;
(2) $\widehat{A} : \cup_{n\in\mathbb{N}} FI([0,1])^n \to FI([0,1])$ is a bisymmetric gagop;
(3) $\widehat{A} : \cup_{n\in\mathbb{N}} FN([0,1])^n \to FN([0,1])$ is a bisymmetric gagop.

### 3.7.3 Idempotence of a $T$-extension

We consider idempotence property of $\widehat{A}$ in this section. Similarly like previously we expect that idempotence of $A$ and $T$ should imply idempotence of $\widehat{A}$, but as the following graphical example shows this implication does not hold. In general, i.e, if we take an idempotent $T$ and an idempotent $A$, $T$-extension is not idempotent:

**Example 16.**

$$\widehat{A}(P,P)(x) = \sup\{T_M(P(x_1),P(x_2)) | (x_1,x_2) \in [0,1]^n : \frac{x_1+x_2}{2} = x\}$$

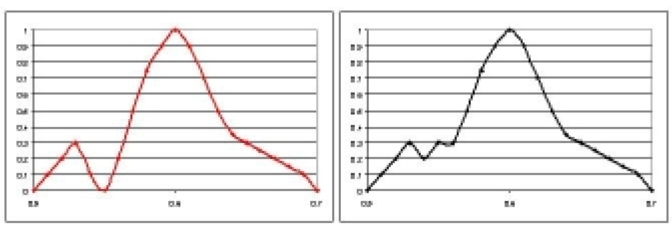

Figure 6: Left graph: not convex fuzzy set $P(x)$; Right graph: $\widehat{A}(P,P)(x)$

P is not a convex fuzzy set in example 16, and the following result shows that convexity is crucial for the idempotence of $\widehat{A}$:

**Proposition 3.22.** If $\widehat{A} : \cup_{n\in\mathbb{N}} FQ([0,1])^n \to FQ([0,1])$ is $T_M$-extension of an arbitrary continuous, idempotent agop $A$ then it is an idempotent gagop.

*Proof.* We need to show that

$$\tilde{A}_{(n)}(P,...,P)(x) = P(x), \forall n \in \mathbb{N}, \forall x \in [0,1]. \tag{139}$$

We consider an arbitrary $P(x) \in FQ([0,1])$ and $x^* \in [0,1]$, then according to the definition of a $T$-extension and theorem 3.4 $\exists\ x_1^*, ..., x_n^*$ s.t.:

$$A(x_1^*, ..., x_n^*) = x^*$$

and

$$\widehat{A}_{(n)}(P, ..., P)(x^*) = T_M(P(x_1^*), ..., P(x_n^*)). \tag{140}$$

We denote $S = \{(x_1, ..., x_n) : A(x_1, ..., x_n) = x^*\}$ and $(x_1^*, ..., x_n^*) \in S$.
$A$ is an idempotent agop, therefore $(x^*, ..., x^*) \in S$ and $T_M(P(x^*), ..., P(x^*)) = P(x^*)$.
Idempotence of $A$ implies compensation property, i.e. for an arbitrary $(x_1, ..., x_n) \in S$ the following holds:

$$\min_{i=1,...,n} x_i \leq A(x_1, ..., x_n) \leq \max_{i=1,...,n} x_i \tag{141}$$

$$\min_{i=1,...,n} x_i \leq x^* \leq \max_{i=1,...,n} x_i. \tag{142}$$

For an arbitrary $(x_1, ..., x_n) \in S$ s.t. $(x_1, ..., x_n) \neq (x^*, ..., x^*)$ one of the following properties holds:
(i): $\exists\ x_{i_1}, ..., x_{i_k}, 1 \leq k \leq n-1$: $x_{i_j} < x^*\ \forall j = 1, ..., k$ and for the rest $x_{i_s}\ s \notin \{1, ..., k\}$: $x_{i_s} \geq x^*$
(ii): $\exists\ x_{i_1}, ..., x_{i_k}, 1 \leq k \leq n-1$: $x_{i_j} > x^*\ \forall j = 1, ..., k$ and for the rest $x_{i_s}\ s \notin \{1, ..., k\}$: $x_{i_s} \leq x^*$.
If neither (i) no (ii) hold then $(x_1, ..., x_n)$ s.t.

$$x_i < x^*\ \forall i$$

or

$$x_i > x^*\ \forall i.$$

But in the first case according to the compensation property ((141), (142)) we obtain

$$A(x_1, ..., x_n) \leq \max(x_1, ..., x_n) < x^*$$

in the second case we obtain

$$x^* < \min(x_1, ..., x_n) \leq A(x_1, ..., x_n)$$

but then $(x_1, ..., x_n) \notin S$.
Now we take an arbitrary $(x_1, ..., x_n) \in S$ and assume that (i) holds:
for an arbitrary $x_l < x^*$ and arbitrary $x_k \geq x^*$ convexity of $P$ implies that

$$P(x^*) \geq T_M(P(x_l), P(x_k)). \tag{143}$$

If we add the rest coordinates of the vector we can only reduce the minimum thus we can continue formula (143) in the following way:

$$P(x^*) \geq T_M(P(x_1), ..., P(x_n)). \tag{144}$$

So, we have obtained that for an arbitrary vector $(x_1, ..., x_n) \in S$

$$T_M(P(x^*), ..., P(x^*)) = P(x^*) \geq T_M(P(x_1), ..., P(x_n)),$$

and this means that

$$\widehat{A}_{(n)}(P, ..., P)(x^*) = T_M(P(x^*), ..., P(x^*)) = P(x^*).$$

The assumption that (ii) holds will lead us to the same result. □

As a corollary from the previous proposition we obtain the next result:

**Corollary 8.** If $A$ is a continuous, idempotent agop then for $T_M$-extension $\widehat{A}$ the following assertions hold:
(1) $\widehat{A} : \cup_{n\in\mathbb{N}} FI([0,1])^n \to FI([0,1])$ is an idempotent gagop;
(2) $\widehat{A} : \cup_{n\in\mathbb{N}} FN([0,1])^n \to FN([0,1])$ is an idempotent gagop.

**Remark 9.** Recall that $T_M$ is the only idempotent t-norm. Now we show that only $T_M$-extension ensures idempotence of $\widehat{A}$ (given conditions of proposition 3.22).
If we take an arbitrary t-norm $T < T_M$ then according to the proof of proposition 3.22

$$\begin{aligned} \forall (x_1, ..., x_n) \in S,\ P(x^*) = T_M(P(x^*), ..., P(x^*)) = \\ = \max\{T_M(P(x_1), ..., P(x_n)) | (x_1, ..., x_n) \in S\}. \end{aligned} \tag{145}$$

Now applying the upper bound of the class of t-norms we obtain

$$\max\{T_M(P(x_1), ..., P(x_n)) | (x_1, ..., x_n) \in S\} >$$

$$\max\{T(P(x_1), ..., P(x_n)) | (x_1, ..., x_n) \in S\} = \widehat{A}_{(n)}(P, ..., P)(x^*). \tag{146}$$

Combining the result of formulas (145) and (146) we obtain:

$$P(x^*) > \widehat{A}_{(n)}(P, ..., P)(x^*). \tag{147}$$

Similarly, idempotence of $\widehat{A}$ does not hold when t-norm differs from $T_M$ and the set of inputs is a subset of $FQ([0,1])$.

### 3.7.4 Neutral and absorbing elements of a $T$-extension

Further formulated results outline the nature of neutral and absorbing elements of $\widehat{A}$.

**Proposition 3.23.** Let $\widehat{A} : \cup_{n\in\mathbb{N}} F([0,1])^n \to F([0,1])$ be an arbitrary $T$-extension of a continuous agop $A$, and $e$ is the neutral element of $A$, then

$$E(x) = \begin{cases} 1, & \text{if } x = e \\ 0, & \text{if } x \neq e \end{cases}$$

is the neutral element of $\widehat{A}$.

*Proof.* $P_1, ..., P_n$ are given and $P_i = E$ for some $i$, then we need to show that for an arbitrary $x \in [0,1]$

$$\widehat{A}(P_1, ..., P_n)(x) = \widehat{A}_{n-1}(P_1, ..., P_{i-1}, P_{i+1}, ..., P_n)(x). \tag{148}$$

First we consider $\widehat{A}(P_1, ..., P_n)(x)$:
According to theorem 3.4 $\exists\ x_1^*, ..., x_n^*$ :

$$A(x_1^*, ..., x_n^*) = x \tag{149}$$

and

$$\widehat{A}(P_1, ..., P_n)(x) = T(P_1(x_1^*), ..., P_n(x_n^*)). \tag{150}$$

Since $x_i^* = e$ in the formula (149), therefore applying neutrality of $A$ versus $e$ we can continue (149):

$$A(x_1^*, ..., x_n^*) = A(x_1^*, ..., x_{i-1}^*, e, x_{i+1}^*, ...x_n^*) =$$
$$= A_{(n-1)}(x_1^*, ..., x_{i-1}^*, x_{i+1}^*, ...x_n^*) = x. \tag{151}$$

As $P_i(x_i^*) = P_i(e) = 1$, applying neutrality of t-norm $T$ versus 1 we continue formula (150) in the following way:

$$\widehat{A}(P_1, ..., P_n)(x) = T(P_1(x_1^*), ..., P_n(x_n^*)) =$$
$$= T(P_1(x_1^*), ..., P_{i-1}(x_{i-1}^*), 1, P_{i+1}(x_{i+1}^*), ..., P_n(x_n^*)) =$$
$$= T_{(n-1)}(P_1(x_1^*), ..., P_{i-1}(x_{i-1}^*), P_{i+1}(x_{i+1}^*), ..., P_n(x_n^*)). \tag{152}$$

Now we consider $\widehat{A}_{n-1}(P_1, ..., P_{i-1}, P_{i+1}, ..., P_n)(x)$:
according to theorem 3.4 $\exists\ y_1^*, ..., y_{i-1}^*, y_{i+1}^*, ..., y_{n-1}^*$ :

$$A(y_1^*, ..., y_{i-1}^*, y_{i+1}^*, ..., y_{n-1}^*) = x \tag{153}$$

and

$$\widehat{A}_{n-1}(P_1, ..., P_{i-1}, P_{i+1}, ..., P_n)(x) =$$
$$= T(P_1(y_1^*), ..., P_{i-1}(y_{i-1}^*), P_{i+1}(y_{i+1}^*), ..., P_n(y_{n-1}^*)). \tag{154}$$

If we assume that $\widehat{A}(P_1, ..., P_n)(x) > \widehat{A}_{n-1}(P_1, ..., P_{i-1}, P_{i+1}, ..., P_n)(x)$, then according to formulas (152) and (154) we can take vector $(x_1^*, ..., x_n^*), x_i^* = e$ instead of $(y_1^*, ..., y_{i-1}^*, y_{i+1}^*, ..., y_{n-1}^*)$ and we obtain a higher value of $\widehat{A}_{n-1}(P_1, ..., P_{i-1}, P_{i+1}, ..., P_n)(x)$ than obtained previously, but this contradicts the definition of vector $(y_1^*, ..., y_{i-1}^*, y_{i+1}^*, ..., y_{n-1}^*)$.
To the similar contradiction will lead us the assumption $\widehat{A}(P_1, ..., P_n)(x) < \widehat{A}_{n-1}(P_1, ..., P_{i-1}, P_{i+1}, ..., P_n)(x)$. □

According to proposition 3.7 $E(x)$ is the unique neutral element.
Since $FN([0,1]) \subset FI([0,1]) \subset FQ([0,1])$ and $E(x) \in FN([0,1])$ then defining $T$-extension on the classes $FN([0,1]), FI([0,1])$ or $FQ([0,1])$ we obtain a gagop with the neutral element given in proposition 3.23.
Now we consider the absorbing element of an arbitrary $T$-extension.

**Proposition 3.24.** Let $\widehat{A} : \cup_{n\in\mathbb{N}} F([0,1])^n \to F([0,1])$ be an arbitrary $T$-extension of a continuous agop $A$, then

$$R(x) = 0 \ \forall x \in [0,1]$$

is the absorbing element of $\widehat{A}$.

*Proof.* $P_1, ..., P_n$ are given and $P_i = R$ for some $i$, then we need to show that for an arbitrary $x \in [0,1]$

$$\widehat{A}(P_1, ..., P_n)(x) = R(x) = 0. \tag{155}$$

According to theorem 3.4 $\exists\ x_1^*, ..., x_n^*$ s.t.:

$$A(x_1^*, ..., x_n^*) = x \tag{156}$$

and

$$\widehat{A}(P_1, ..., P_n)(x) = T(P_1(x_1^*), ..., R(x_i^*), ..., P_n(x_n^*)). \tag{157}$$

For an arbitrary $x_i^* \in [0,1]$ $R(x_i^*) = 0$, using this fact and applying absorbing property of 0 for an arbitrary t-norm $T$ we continue (157):

$$\widehat{A}(P_1, ..., P_n)(x) = 0. \tag{158}$$

We have shown that for an arbitrary $x \in [0,1]$ $\widehat{A}(P_1, ..., P_n)(x) = 0$ and thus the assertion holds.

□

There is no other absorbing element as the uniqueness of $R(x)$ is ensured by proposition 3.8.
The question how to interpret $R(x)$ arises. On the one hand element $R$ belongs to the class $F([0,1])$, but on the other hand it does not have any real value, i.e. in any point its value 0. Thus question on the nature of $R$ is rather philosophical. We skip the philosophical part of this question and consider that the absorbing element of $T$-extension exists, it is from the class $F([0,1])$ and it is given in proposition 3.24.
Since $R(x) \in FQ([0,1])$, $T$-extension defined on the class $FQ([0,1])$ has the same absorbing element.
Element $R(x) \notin FN([0,1])$ and there is no other absorbing element in $FN([0,1])$. If we assume that there exists $R^*(x) \in FN([0,1])$ and it differs from $R(x)$ then $R^*(x) \in F([0,1])$, but this contradicts the result of proposition 3.8. Thus defining a $T$-extension on the class $FN([0,1])$ or $FI([0,1])$ we deal with a gagop without the absorbing element.
The interesting fact should be noticed here: $T$-extension of an agop without the absorbing element can result in a gagop with the absorbing element.

### 3.7.5 Shift-invariance of a $T$-extension

We consider a shift-invariance of a $T$-extension in this section. As we agreed before $\widehat{A}$ acts on $F([0,1])$ therefore we should give clarifications to the definition of shift-invariance regarding possible values of $B, P_1, ..., P_n$ in order to stay within the same class:

**Definition 60.** A gagop $\widehat{A} : \cup_{n\in\mathbb{N}} F([0,1])^n \to F([0,1])$ is said to be shift-invariant if

$$\forall n \in \mathbb{N}, \forall B, P_1, ..., P_n \in F([0,1]):$$
$$\forall i = \overline{1,n}\ P_i + B \in F([0,1]),$$
$$\widehat{A}(P_1 + B, ..., P_n + B), \widehat{A}(P_1, ..., P_n), \widehat{A}(P_1, ..., P_n) + B \in F([0,1])$$
$$\widehat{A}(P_1 + B, ..., P_n + B) = \widehat{A}(P_1, ..., P_n) + B$$

When we consider shift-invariance of a $T$-extension we understand it in the sense of definition 60.
Further we denote $T_1$ a continuous t-norm, which is used for the extension of an addition operation and $T_2$ will denote a continuous t-norm, which is used for the extension of a continuous agop $A$.
In general shift-invariance of an agop $A$ does not imply shift invariance of $\widehat{A}$. This property depends not only on $A$, $T_1, T_2$ but also on the properties of the set of inputs.
Further we indicate special conditions, which preserve shift-invariance of a $T$-extension.

**Proposition 3.25.** If $T_1 = T_2 = T_M$, $A$ is a continuous shift-invariant agop defined by means of operations of addition and multiplcation with $c \in \mathbb{R}$, then

$$\widehat{A} : \cup_{n\in\mathbb{N}} FTN([0,1])^n \to FTN([0,1])$$

is a shift-invariant gagop.

*Proof.* We use the following notations:
$P_i = (p_1^i, p_2^i, p_3^i), i = \overline{1,n}$, $B = (b_1, b_2, b_3)$, then $T_1 = T_2 = T_M$ and properties of $A$ allow us write the same form for fuzzy triangular number $\widehat{A}(P_1, ..., P_n)$:

$$\widehat{A}(P_1, ..., P_n) = (A(p_1^1, ..., p_1^n), A(p_2^1, ..., p_2^n), A(p_3^1, ..., p_3^n)).$$

And then $\widehat{A}(P_1, ..., P_n) + B \in FTN([0,1])$:

$$\widehat{A}(P_1, ..., P_n)+B = (A(p_1^1, ..., p_1^n)+b_1, A(p_2^1, ..., p_2^n)+b_2, A(p_3^1, ..., p_3^n)+b_3). \quad (159)$$

On the other hand $P_i + B = (p_1^i + b_1, p_2^i + b_2, p_3^i + b_3) \in FTN([0,1])$ and hence $\widehat{A}(P_1 + B, ..., P_n + B) \in FTN([0,1])$ has the following form:

$$\widehat{A}(P_1 + B, ..., P_n + B) =$$

$$= (A(p_1^1 + b_1, ..., p_1^n + b_1), A(p_2^1 + b_2, ..., p_2^n + b_2), A(p_3^1 + b_3, ..., p_3^n + b_3)). \quad (160)$$

Given shift-invariance of A we can continue formula (160) in the following way:

$$\widehat{A}(P_1 + B, ..., P_n + B) =$$

$$= (A(p_1^1, ..., p_1^n) + b_1, A(p_2^1, ..., p_2^n) + b_2, A(p_3^1, ..., p_3^n) + b_3). \quad (161)$$

We see that formulas (159) and (161) give us the same result. □

The next result does not allow us to state that $\widehat{A}$ is a shift-invariant in the sense of definition 60, because it requires a special form of $B$. We consider this special case as shift-invariance of $\widehat{A}$ w.r.t. $B$:

**Proposition 3.26.** If $T_1 = T_2 = T$ is an arbitrary t-norm, $A$ is a continuous, additive agop, $B$ is a crisp interval and $\widehat{A} : \cup_{n\in\mathbb{N}} F([0,1])^n \to F([0,1])$, then

$$\widehat{A}(P_1, ..., P_n) + B = \widehat{A}(P_1 + B, ..., P_n + B).$$

*Proof.* We consider an arbitrary z:

$$(\widehat{A}(P_1, ..., P_n) + B)(z) > 0,$$

then according to theorem 3.4 $\exists\ x^*, y^* : x^* + y^* = z$ and

$$(\widehat{A}(P_1, ..., P_n) + B)(z) = \sup\{T_1(\widehat{A}(P_1, ..., P_n)(x), B(y))|\ (x, y) : x + y = z\} =$$

$$= T_1(\widehat{A}(P_1, ..., P_n)(x^*), B(y^*)). \quad (162)$$

It follows that $B(y^*) = 1$ otherwise $T_1(\widehat{A}(P_1, ..., P_n)(x^*), B(y^*)) =$
$= T_1(\widehat{A}(P_1, ..., P_n)(x^*), 0) = 0$, but then $(\widehat{A}(P_1, ..., P_n) + B)(z) = 0$.
Applying $T_1$ neutrality versus 1 we can continue formula (162) in the following way:

$$(\widehat{A}(P_1, ..., P_n) + B)(z) = \widehat{A}(P_1, ..., P_n)(x^*). \quad (163)$$

According to the definition of $T$-extension and theorem 3.4 $\exists\ (x_1^*, ..., x_n^*)$ : $A(x_1^*, ..., x_n^*) = x^*$ and

$$\widehat{A}(P_1, ..., P_n)(x^*) = \sup\{T_2(P_1(x_1), ..., P_n(x_n))|\ (x_1, ..., x_n) : A(x_1, ..., x_n) = x^*\} =$$

$$= T_2(P_1(x_1^*), ..., P_n(x_n^*)). \quad (164)$$

We summarize the nature of the vector $(x_1^*, ..., x_n^*)$:

$$\exists y^* :\ A(x_1^*, ..., x_n^*) + y^* = z \quad (165)$$

and the above reasonings give us the following result:

$$(\widehat{A}(P_1, ..., P_n) + B)(z) = T_2(P_1(x_1^*), ..., P_n(x_n^*)). \quad (166)$$

Now in the similar manner we consider

$$\widehat{A}(P_1 + B, ..., P_n + B)(z) > 0.$$

According to the definition of $T$-extension and theorem 3.4 $\exists$ $(z_1, ..., z_n) : A(z_1, ..., z_n) = z$ and

$$\widehat{A}(P_1 + B, ..., P_n + B)(z) =$$
$$= \sup\{T_2((P_1 + B)(x_1), ..., (P_n + B)(x_n))|\ (x_1, ..., x_n) : A(x_1, ..., x_n) = z\} =$$
$$T_2((P_1 + B)(z_1), ..., (P_n + B)(z_n)). \tag{167}$$

On the other hand $\forall i = \overline{1, n}$ $\exists$ $(s_i^*, t_i^*)$: $s_i^* + t_i^* = z_i$

$$(P_i + B)(z_i) = \sup\{T_1(P_i(s_i), B(t_i))|\ (s_i, t_i) : s_i + t_i = z_i\} =$$
$$= T_1(P_i(s_i^*), B(t_i^*)). \tag{168}$$

It follows that $B(t_i^*) = 1$ $\forall i = \overline{1, n}$ otherwise $T_1(P_i(s_i^*), B(t_i^*)) = T_1(P_i(s_i^*), 0) = 0$ and thus $\widehat{A}(P_1 + B, ..., P_n + B)(z) = 0$.
We use $T_1$ neutrality versus 1 and continue (168) in the following way:

$$(P_i + B)(z_i) = P_i(s_i^*). \tag{169}$$

We apply formula (169) to (167):

$$\widehat{A}(P_1 + B, ..., P_n + B)(z) = T_2(P_1(s_1^*), ..., P_n(s_n^*)), \tag{170}$$

where $(s_1^*, ..., s_n^*)$:

$$\exists\ (t_1^*, ..., t_n^*) :\ A(t_1^* + s_1^*, ..., t_n^* + s_n^*) = z. \tag{171}$$

Now if we assume that $(\widehat{A}(P_1, ..., P_n) + B)(z) > \widehat{A}(P_1 + B, ..., P_n + B)(z)$ then using aditivity of $A$ (and its idempotence implied by aditivity) and comparing formulas (165),(171) we conclude that if we take:

$$(s_1^*, ..., s_n^*) = (x_1^*, ..., x_n^*)$$

and

$$(t_1^*, ..., t_n^*) = (y^*, ..., y^*)$$

we obtain value of $\widehat{A}(P_1+B, ..., P_n+B)(z)$ higher than its value given by formula (170), but this contradicts definition of vector $(s_1^*, ..., s_n^*)$.
Similarly if we assume that $(\widehat{A}(P_1, ..., P_n) + B)(z) < \widehat{A}(P_1 + B, ..., P_n + B)(z)$ we obtain contradiction to the definition of vector $(x_1^*, ..., x_n^*)$. $\square$

**Example 17.** Let $P_1 = (0.01, 0.02, 0.03), P_2 = (0.03, 0.04, 0.05)$ be triangular numbers and $B = [0.05, 0.07]$ a crisp interval.
1st case:

$$\widehat{A}(P_1, P_2)(x) = \sup\{T_M(P_1(x_1), P_2(x_2))|(x_1, x_2) \in [0, 1]^n : \frac{x_1 + x_2}{2} = x\}$$

and $T_1 = T_M$.
2nd case:

$$\widehat{A}(P_1, P_2)(x) = \sup\{T_P(P_1(x_1), P_2(x_2))|(x_1, x_2) \in [0, 1]^n : \frac{x_1 + x_2}{2} = x\}$$

and $T_1 = T_P$.
On the figures below the black dashed line denote $\widehat{A}(P_1, P_2) + B$ and the red continuous line denotes $\widehat{A}(P_1 + B, P_2 + B)$.

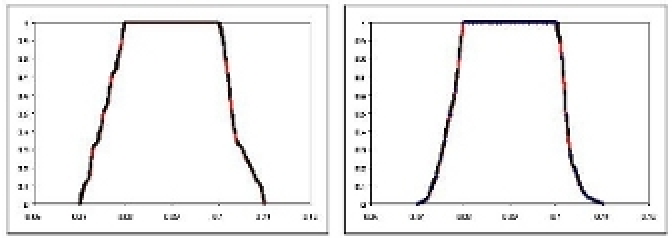

Figure 7: Left graph: $T_1 = T_2 = T_M$; Right graph: $T_1 = T_2 = T_P$

We can see that on the both graphs values of $\widehat{A}(P_1, P_2) + B$ match values of $\widehat{A}(P_1 + B, P_2 + B)$.

**Proposition 3.27.** If $T_1 = T_2 = T$ is an arbitrary t-norm, $A$ is a continuous and additive agop, $\widehat{A} : \cup_{n \in \mathbb{N}} FQ([0,1])^n \to FQ([0,1])$ is an idempotent gagop then $\widehat{A}$ is a shift-invariant gagop.

*Proof.* We again consider $\widehat{A}(P_1 + B, ..., P_n + B)(z)$ and $(\widehat{A}(P_1, ..., P_n) + B)(z)$ and z is such that both values are greater than 0.
According to theorem 3.4 $\exists\ (z_1, ..., z_n) : A(z_1, ..., z_n) = z$ and

$$\widehat{A}(P_1 + B, ..., P_n + B)(z) = T((P_1 + B)(z_1), ..., (P_n + B)(z_n)). \tag{172}$$

On the other hand $\forall i = \overline{1, n}\ \exists\ (s_i^*, t_i^*) : s_i^* + t_i^* = z_i$ and

$$(P_i + B)(z_i) = T(P_i(s_i^*), B(t_i^*)). \tag{173}$$

Summarizing the above considerations we have

$$\exists\ (s_1^*, ..., s_n^*), (t_1^*, ..., t_n^*) :\ A(s_1^* + t_1^*, ..., s_n^* + t_n^*) = z \tag{174}$$

and

$$\widehat{A}(P_1 + B, ..., P_n + B)(z) = T(T(P_1(s_1^*), B(t_1^*)), ..., T(P_n(s_n^*), B(t_n^*))). \tag{175}$$

Now we use the idempotence of $\widehat{A}$ and consider $(\widehat{A}(P_1, ..., P_n) + B)(z)$:

$$(\widehat{A}(P_1, ..., P_n) + B)(z) = (\widehat{A}(P_1, ..., P_n) + \widehat{A}_{(n)}(B, ..., B))(z). \tag{176}$$

Again we use theorem 3.4 and find $(x^*, y^*) : x^* + y^* = z$ and

$$(\widehat{A}(P_1, ..., P_n) + \widehat{A}(B, ..., B))(z) = T(\widehat{A}(P_1, ..., P_n)(x^*), \widehat{A}(B, ..., B)(y^*)). \tag{177}$$

Similarly like previously referring to the definition of $T$-extension and theorem 3.4 we get $(x_1^*, ..., x_n^*)$, $(y_1^*, ..., y_n^*) : A(x_1^*, ..., x_n^*) = x^*$, $A(y_1^*, ..., y_n^*) = y^*$ and

$$\widehat{A}(P_1, ..., P_n)(x^*) = T(P_1(x_1^*), ..., P_n(x_n^*)) \tag{178}$$

$$\widehat{A}(B,...,B)(y^*) = T(B(y_1^*),...,B(y_n^*)). \quad (179)$$

Thus we obtain:

$$\exists\ (x_1^*,...,x_n^*),(y_1^*,...,y_n^*) : A(x_1^*,...,x_n^*) + A(y_1^*,...,y_n^*) = z \quad (180)$$

and

$$(\widehat{A}(P_1,...,P_n) + B)(z) = T(T(P_1(x_1^*),...,P_n(x_n^*)),T(B(y_1^*),...,B(y_n^*))). \quad (181)$$

Further we apply T associativity and symmetry and continue formula (181) in the following way:

$$(\widehat{A}(P_1,...,P_n) + B)(z) = T(T(P_1(x_1^*),B(y_1^*)),...,T(P_n(x_n^*),B(y_n^*))). \quad (182)$$

Considering the nature of vectors $(s_1^*,...,s_n^*),(t_1^*,...,t_n^*)$ and $(x_1^*,...,x_n^*),(y_1^*,...,y_n^*)$ given by formulas (174) and (180) and using the aditivity of $A$ we conclude that

$$\widehat{A}(P_1 + B,...,P_n + B)(z) = (\widehat{A}(P_1,...,P_n) + B)(z).$$

□

As we have shown in section 3.7.3 only in the case when $T_2 = T_M$ we can obtain an idempotent gagop, therefore $T_1 = T_2 = T_M$ in proposition 3.27 and it can not be considered as a generalization of proposition 3.26.
We go over example 17 again with the only change - in the role of $B$ we take the triangular number $B = (0.05, 0.06, 0.07)$, then we can see that in the case of $T_1 = T_2 = T_P$ shift-invariance does not hold.

**Example 18.** Let $P_1 = (0.01, 0.02, 0.03), P_2 = (0.03, 0.04, 0.05)$,
$B = (0.05, 0.06, 0.07)$ be triangular numbers.
1st case:

$$\widehat{A}(P_1,P_2)(x) = \sup\{T_M(P_1(x_1),P_2(x_2))|(x_1,x_2) \in [0,1]^n : \frac{x_1 + x_2}{2} = x\}$$

and $T_1 = T_M$.
2nd case:

$$\widehat{A}(P_1,P_2)(x) = \sup\{T_P(P_1(x_1),P_2(x_2))|(x_1,x_2) \in [0,1]^n : \frac{x_1 + x_2}{2} = x\}$$

and $T_1 = T_P$.
On the figures below the black dashed line denote $\widehat{A}(P_1,P_2) + B$ and the red continuous line denotes $\widehat{A}(P_1 + B,P_2 + B)$.

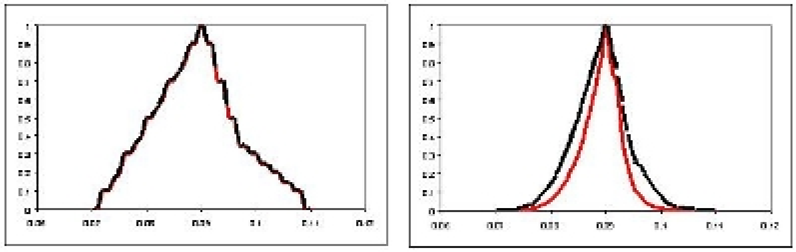

Figure 8: Left graph: $T_1 = T_2 = T_M$; Right graph: $T_1 = T_2 = T_P$

So, $T_P$ does not preserve shift-invarinace in a more general case, however it does in a special case, when $B$ is a crisp interval.

Idempotence is a necessary condition for an agop (and a t-norm as well) to be shift-invariant, but only $T_M$ is an idempotent t-norm, therefore it is the only shift-invariant t-norm. Thus only in the case of $T_M$-extension we can speak about shift-invariant gagop in a general sense.

### 3.7.6 Concluding remarks on properties of a $T$-extension

All observed properties of a $T$-extension (apart from the absorbing element) are determined by the same property of the agop $A$ and the t-norm $T$. Absorbing element stands alone in this context as an arbitrary $T$-extension will have absorbing element regardless $A$ has it or not.

Properties of $T$-extension are tightly related to properties of the t-norm and sometimes we can manage a desired property by choosing an appropriate t-norm, but not always it is possible. For an example in the case of absorbing element, the form of absorbing element can not be changed, because 0 is the absorbing element of any t-norm. Thus such cases can not be managed by changing t-norm. But substituting t-norm in the formula (116) with e.g. a nullnorm ([6]) $N$ we obtain $N$-extension:

$$\widehat{A}(P_1, ..., P_n)(x) = \sup\{N(P_1(x_1), ..., P_n(x_n)|(A(x_1, ..., x_n) = x)\}. \qquad (183)$$

Performing $U$-extension, i.e. extension via uninorm $U$ ([6]) we can obtain neutral element with different properties and also we can use compensation property of uninorms.

If we consider shift-invariance of a $T$-extension then we can see that substitution of a t-norm with another operator can bring benefits, because we can extend the list of idempotent gagops.

The definition of a $T$-extension can be generalized via an arbitrary agop $A^*$ (with desired properties):

$$\widehat{A}(P_1, ..., P_n)(x) = \sup\{A^*(P_1(x_1), ..., P_n(x_n)|A(x_1, ..., x_n) = x\}. \qquad (184)$$

Such flexibility of the definition gives many advantages related to properties of a $T$-extension.

### 3.7.7 $T$-extension w.r.t. vertical order relations

In the preceding sections we considered properties of a $T$-extension with assumption that it is a gagop. Now we show accuracy of this assumption, i.e. we prove that there exist order relations such that $T$-extension is a gagop w.r.t. these order relations.
We start with the class of vertical order relations introduced in section 3.3.1 and namely $\subseteq_{F_1}^{\alpha}$. The following result shows that $\widehat{A}$ is a gagop w.r.t. $\subseteq_{F_1}^{\alpha}$. Idempotence of $A$ is not required regardless that it was required in the case of pointwise extension. Similarly like previously we consider continuous t-norms.

**Theorem 3.28.** *An arbitrary $T$-extension $\widehat{A} : \cup_{n\in\mathbb{N}} F([0,1])^n \to F([0,1])$ of an arbitrary continuous agop $A$ is a gagop w.r.t. $\subseteq_{F_1}^{\alpha}$.*

*Proof.* We need to show modified $(\tilde{A}1)$ (see formula (93)) and $(\tilde{A}2), (\tilde{A}3)$ from definition 49.
In order to prove boundary conditions $(\tilde{A}1)$ and $(\tilde{A}2)$ we need to show, that:

$$\widehat{A}_{(n)}(\tilde{0}_1, ..., \tilde{0}_n)(x) = \sup\{T(\tilde{0}_1(x_1), ..., \tilde{0}_n(x_n)) | A(x_1, ..., x_n) = x\} \in \Theta \quad (185)$$

and

$$\widehat{A}_{(n)}(\tilde{1}, ..., \tilde{1})(x) = \sup\{T(\tilde{1}(x_1), ..., \tilde{1}(x_n)) | A(x_1, ..., x_n) = x\} = \tilde{1}(x) \quad (186)$$

for an arbitrary $n \in \mathbb{N}$ and arbitrary $x \in [0,1]$.
If we consider an arbitrary vector $(x_1, ..., x_n) \in [0,1]^n$ then applying the restriction from above of an arbitrary t-norm by $T_M$ we obtain:

$$T(\tilde{0}_1(x_1), ..., \tilde{0}_n(x_n)) \leq T_M(\tilde{0}_1(x_1), ..., \tilde{0}_n(x_n)). \quad (187)$$

According to the definition of $\tilde{0}_i$, $\forall x_i \in [0,1]$ $\tilde{0}_i(x_i) \leq \alpha$ thus the same is true for the minimum, i.e.:

$$T_M(\tilde{0}_1(x_1), ..., \tilde{0}_n(x_n)) \leq \alpha. \quad (188)$$

Evidently using formulas (187), (188) for an arbitrary $x \in [0,1]$ we obtain $\widehat{A}_{(n)}(\tilde{0}, ..., \tilde{0})(x) \leq \alpha$ and therefore according to the definition of $\tilde{0}_i$ (formula (49))

$$\widehat{A}_{(n)}(\tilde{0}_1, ..., \tilde{0}_n)(x) \in \Theta. \quad (189)$$

According to the definition of $\tilde{1}$ (formula 48) for an arbitrary vector $(x_1, ..., x_n) \in [0,1]^n$ the following holds:

$$T(\tilde{1}(x_1), ..., \tilde{1}(x_n)) = T(1, ..., 1) = 1 = \tilde{1}(x) \text{ for an arbitrary } x \in [0,1] \quad (190)$$

and thus formula (186) is straight forward.
The proof of $(\tilde{A}3)$ requires the following implication:

$$(\forall i = 1, ..., n, P_i \subseteq_{F_1}^{\alpha} Q_i) \Rightarrow (\widehat{A}(P_1, ..., P_n) \subseteq_{F_1}^{\alpha} \widehat{A}(Q_1, ..., Q_n)). \quad (191)$$

We take an arbitrary $x \in [0,1] : \widehat{A}(P_1, ..., P_n)(x) \geq \alpha$ and consider $\widehat{A}(P_1, ..., P_n)(x)$ and $\widehat{A}(Q_1, ..., Q_n)(x)$:
according to theorem 3.4 $\exists\ (x_1^*, ..., x_n^*)$ :

$$A(x_1^*, ..., x_n^*) = x \tag{192}$$

and

$$\widehat{A}(P_1, ..., P_n)(x) = T(P_1(x_1^*), ..., P_n(x_n^*)). \tag{193}$$

Similarly $\exists\ (x_1^{'}, ..., x_n^{'})$ :

$$A(x_1^{'}, ..., x_n^{'}) = x \tag{194}$$

and

$$\widehat{A}(Q_1, ..., Q_n)(x) = T(Q_1(x_1^{'}), ..., Q_n(x_n^{'})). \tag{195}$$

We remind that

$$\widehat{A}(P_1, ..., P_n)(x) \geq \alpha \tag{196}$$

and using $\forall i\ P_i \subseteq_{F_1}^{\alpha} Q_i$, i.e.:

$$(P_i(x_i^*) \geq \alpha) \Rightarrow (P_i(x_i^*) \leq Q_i(x_i^*)) \tag{197}$$

and formula (193) we can write

$$\alpha \leq T(P_1(x_1^*), ..., P_n(x_n^*)) \leq P_i(x_i^*) \leq Q_i(x_i^*),\ \forall i. \tag{198}$$

Using the monotonicity of t-norm we can continue in the following way:

$$T(P_1(x_1^*), ..., P_n(x_n^*)) \leq T(Q_1(x_1^*), ..., Q_n(x_n^*)). \tag{199}$$

But according to the definition of vector $(x_1^{'}, ..., x_n^{'})$

$$T(Q_1(x_1^*), ..., Q_n(x_n^*)) \leq T(Q_1(x_1^{'}), ..., Q_n(x_n^{'})) \tag{200}$$

and thus

$$\widehat{A}(P_1, ..., P_n)(x) \leq \widehat{A}(Q_1, ..., Q_n)(x). \tag{201}$$

Point $x$ was chosen according to the formula (196) and we obtained inequality (201), thus we have shown that $(\tilde{A}3)$ holds. □

$\subseteq_{F_1}$ is a special case of the class of order relations $\subseteq_{F_1}^{\alpha}$ thus as a corollary from theorem 3.28 we obtain:

**Corollary 9.** An arbitrary $T$-extension $\widehat{A} : \cup_{n\in\mathbb{N}} F([0,1])^n \to F([0,1])$ of an arbitrary continuous agop $A$ is a gagop w.r.t. $\subseteq_{F_1}$.

Evidently that results of theorem 3.28 and corollary 9 hold for other sets of inputs: $FQ([0,1])$, $FI([0,1])$ and $FN([0,1])$.

### 3.7.8 $T$-extension w.r.t. horizontal order relations

It is proven in [47] that $T_M$-extension is a gagop w.r.t. $\prec_I$ defined in section 3.3.2.
Further in this section we study an arbitrary $T$-extension w.r.t. $\subseteq_{F_2}^{\alpha}$, defined in section 3.3.2. Again we use an arbitrary lower continuous t-norm and an arbitrary continuous agop $A$. We prove that $T$-extension is a gagop w.r.t. $\subseteq_{F_2}^{\alpha}$.
Throughout the work $\widehat{A}$ aggregates fuzzy sets defined on the interval $[0,1]$, therefore we introduce clarifications to definition 48 of order relation $\subseteq_{F_2}^{\alpha}$:

**Definition 61.** Let $\alpha \in (0,1]$, $P, Q \in F([0,1])$

$$P \subseteq_{F_2}^{\alpha} Q \Leftrightarrow \overline{P}^{\alpha} \leq \underline{Q}^{\alpha},$$

where

$$P^{\alpha} = \{x : P(x) \geq \alpha\}, \qquad \min P^{\alpha} = \underline{P}^{\alpha}, \qquad \max P^{\alpha} = \overline{P}^{\alpha}$$
$$Q^{\alpha} = \{x : Q(x) \geq \alpha\}, \qquad \min Q^{\alpha} = \underline{Q}^{\alpha}, \qquad \max Q^{\alpha} = \overline{Q}^{\alpha}.$$

The classes

$$\Theta = \{\tilde{0}(x) | \tilde{0}(x) = 1, \text{ if } x = 0 \text{ and } \tilde{0}(x) < \alpha \text{ if } x \in (0,1]\},$$

$$\Sigma = \{\tilde{1}(x) | \tilde{1}(x) = 1, \text{ if } x = 1 \text{ and } \tilde{1}(x) < \alpha \text{ if } x \in [0,1)\}$$

we will call correspondingly the class of minimal and maximal elements.

**Theorem 3.29.** *An arbitrary $T$-extension $\widehat{A} : \cup_{n \in \mathbb{N}} F([0,1])^n \to F([0,1])$ of an arbitrary continuous agop $A$ is a gagop w.r.t. $\subseteq_{F_2}^{\alpha}$.*

*Proof.* First we show that the modified border condition ($\tilde{A}1$) (see formula (103)) holds.
We consider $\widehat{A}_{(n)}(\tilde{0}_1, ..., \tilde{0}_n)(x)$ in an arbitrary point $x \in [0,1]$ and for arbitrary $n \in \mathbb{N}$ and arbitrary $\tilde{0}_1, .., \tilde{0}_n \in \Theta$. Two different cases $x = 0$ and $x \neq 0$ will be considered separately.
1st case $x = 0$:
according to theorem 3.4 and definition of $T$-extension $\exists\ (x_1^*, ..., x_n^*)$:
$A(x_1^*, ..., x_n^*) = 0$ and

$$\widehat{A}_{(n)}(\tilde{0}_1, ..., \tilde{0}_n)(0) = T(\tilde{0}_1(x_1^*), ..., \tilde{0}_n(x_n^*)). \tag{202}$$

Evidently $(x_1^*, ..., x_n^*) = (0, ..., 0)$. In such case $A(x_1^*, ..., x_n^*) = A(0, ..., 0) = 0$ and

$$\widehat{A}_{(n)}(\tilde{0}_1, ..., \tilde{0}_n)(0) = T(\tilde{0}_1(0), ..., \tilde{0}_n(0)) = T(1, ..., 1) = 1 \tag{203}$$

as according to definition 61 $\tilde{0}_i(0) = 1$.
If $(x_1^*, ..., x_n^*) \neq (0, ..., 0)$ then among $x_i^*, i = 1, .., n$ exists at least one $x_k^*$ such that $x_k^* \neq 0$ and as a result according to definition 61 $\tilde{0}_k(x_k^*) < \alpha$ and thus

$$T(\tilde{0}_1(x_1^*), ..., \tilde{0}_n(x_n^*)) < \alpha \leq 1. \tag{204}$$

2nd case $x \neq 0$:
according to theorem 3.4 and definition of $T$-extension $\exists$ $(x_1^{'}, ..., x_n^{'})$: $A(x_1^{'}, ..., x_n^{'}) = x$ and

$$\widehat{A}_{(n)}(\tilde{0}_1, ..., \tilde{0}_n)(x) = T(\tilde{0}_1(x_1^{'}), ..., \tilde{0}_n(x_n^{'})) \tag{205}$$

$(x_1^{'}, ..., x_n^{'}) \neq (0, ..., 0)$ otherwise $A(x_1^{'}, ..., x_n^{'}) = A(0, ..., 0) = 0$, but we consider $x \neq 0$. Thus among $x_i^{'}, i = 1, .., n$ exists at least one $x_k^{'}$ such that $x_k^{'} \neq 0$ and according to definition of $\tilde{0}$, $\tilde{0}_k(x_k^{'}) < \alpha$. Evidently using t-norm neutrality versus 1 formula (205) can be continued in the following way:

$$\widehat{A}_{(n)}(\tilde{0}_1, ..., \tilde{0}_n)(x) = T(\tilde{0}_1(x_1^{'}), ..., \tilde{0}_n(x_n^{'})) \leq \tilde{0}(x_k^{'}) < \alpha. \tag{206}$$

Thus we have obtained that

$$\widehat{A}_{(n)}(\tilde{0}_1, ..., \tilde{0}_n) \in \Theta \tag{207}$$

and this means that modified $(\tilde{A}1)$ holds. Similarly we show that modified $(\tilde{A}2)$ holds.
In order to prove the monotonicity $(\tilde{A}3)$ we should show the following implication:

$$(\forall i = 1, ..., n, P_i \subseteq_{F_2}^{\alpha} Q_i) \Rightarrow (\widehat{A}(P_1, ..., P_n) \subseteq_{F_2}^{\alpha} \widehat{A}(Q_1, ..., Q_n)). \tag{208}$$

We denote $A_P^{\alpha}$ $\alpha$-cut of $\widehat{A}(P_1, ..., P_n)$, i.e.

$$A_P^{\alpha} = \{x : \widehat{A}(P_1, ..., P_n)(x) \geq \alpha\}. \tag{209}$$

We take an arbitrary $x \in A_P^{\alpha}$ according to the definition of $T$-extension and theorem 3.4 $\exists$ $(x_1^*, ..., x_n^*)$: $A(x_1^*, ..., x_n^*) = x$ and

$$\widehat{A}(P_1, ..., P_n)(x) = T(P_1(x_1^*), ..., P_n(x_n^*)) \geq \alpha. \tag{210}$$

Formula (210) give us the following result:

$$P_i(x_i^*) \geq \alpha \ \forall i = 1, ..., n \tag{211}$$

and this means, that

$$x_i^* \in \{x : P_i(x) \geq \alpha\} \tag{212}$$

that is $x_i^*$ belong to the $\alpha$-cut of $P_i$.
Similarly $A_Q^{\alpha}$ denotes $\alpha$-cut of $\widehat{A}(Q_1, ..., Q_n)$:

$$A_Q^{\alpha} = \{y : \widehat{A}(Q_1, ..., Q_n)(y) \geq \alpha\} \tag{213}$$

and for arbitrary $y \in A_Q^{\alpha}$ $\exists$ $(y_1^*, ..., y_n^*)$: $A(y_1^*, ..., y_n^*) = y$ and

$$\widehat{A}(Q_1, ..., Q_n)(y) = T(Q_1(y_1^*), ..., Q_n(y_n^*)) \geq \alpha. \tag{214}$$

The same reasoning like above leads us to the following result:

$$y_i^* \in \{y : Q_i(y) \geq \alpha\}. \tag{215}$$

If we translate the left part of the implication (208) into language of $\alpha$-cuts we get:

$$\overline{P}_i^\alpha \leq \underline{Q}_i^\alpha, \tag{216}$$

where $\overline{P}_i^\alpha = \max_x\{x : P_i(x) \geq \alpha\}$ and $\underline{Q}_i^\alpha = \min_y\{y : Q_i(y) \geq \alpha\}$. Given this fact we refer to formulas (212) and (216) and get the following result:

$$x_i^* \leq y_i^* \ \forall i = 1, ..., n. \tag{217}$$

Applying the monotonicity of agop $A$ we get:

$$A(x_1^*, ..., x_n^*) \leq A(y_1^*, ..., y_n^*), \tag{218}$$

thus for an arbitrary $x \in A_P^\alpha$ and an arbitrary $y \in A_Q^\alpha$ we get inequality

$$x \leq y \tag{219}$$

and as a result:

$$\max A_P^\alpha \leq \min A_Q^\alpha. \tag{220}$$

□

Evidently results of theorem 3.29 can be extended to another sets of inputs: $FQ([0,1])$, $FI([0,1])$ and $FN([0,1])$.

## 3.8 Outline of practical applications of gagops

The notion of a gagop, which is widely studied in this chapter, is not a mathematical abstraction although it is a very interesting mathematical object. We outline possible applications of gagops in this section.
To make more clear potential of gagops in practical applications we start with agops. Agops are an indispensable tool for different communities dealing with procession of information coming from different sources. Production of more accurate and often more reliable and stable estimation is an important task for many real-world problems, which can be considered within the framework of aggregation of information.
This final evaluation is a base for conclusion or decision, therefore agops are widely used in multi-criteria decision making and multi-attributes classification. An example of application of information aggregation in financial decision making can be found e.g. in [30].
More sophisticated decision and classification problems based on interacting criteria or attributes can be solved by means of fuzzy integrals, which are a special class of agops ([15],[16]).
Aggregation of information represented by fuzzy sets is a central matter in intelligent systems where fuzzy rule base and reasoning mechanism are applied.

Application of generalized aggregation operators in intelligent systems is shown e.g. in [43].
The list of areas where agops play an important role can be continued, and this is important motivation for the development of this theory.
Gagops are fuzzified analogue of agops, therefore fuzzified analogue of the initial problem, where agops are applied, can be solved by means of gagops. Further we briefly outline some problems, where processing of inexact data can be done by means of gagops.

### Multi-criteria decision making and multi-attributes classification

Problems of multi-criteria decision making and multi-attributes classification usually are associated with a class of attributes and data coming from different sources and describing these attributes. Production of a single value associated with each alternative and characterizing it as much as possible is the important stage of task. Afterwards values are compared or put into classification algorithms and thus decision is made.
Let's assume we have $m$ alternatives, which are characterized by $n$ attributes. Data are vague and imprecise therefore the set of values of each attribute can be modelled by means of fuzzy sets. Thus an arbitrary alternative is characterized by the fuzzy vector:

$$a_i = (a_1^i, ..., a_n^i), i = 1, ..., m,$$

where $a_j^i$ is a fuzzy set.
Gagops deal with aggregation of a fuzzy vector into a single fuzzy set, which is a basis for decision makers.
If we have a matrix of coefficients, which is a fuzzy matrix (due to vagueness of available data)

$$\mathbf{M}_F = (M_{ij})_{n \times n}$$

then the minimax decision rule (used for minimizing the maximum possible loss in decision theory or game theory) in the fuzzy framework has the following form:

$$\mathbf{min}_i \mathbf{max}_j (M_{ij}),$$

where $\mathbf{min}, \mathbf{max}$ are generalized min and max operators, and thus the gagops apparatus can be used.
The maximin and other decision rules are generalized in the same manner.

### Aggregation of fuzzy relations

Aggregation of fuzzy relations is a popular topic in the community of researches, who deal with fuzzy sets and practical applications (see e.g. [9], [17]). One of the important aims in the aggregation is preservation of desired properties of fuzzy relations. Apriori known properties of aggregation function can help to determine if an aggregated fuzzy relation has the desired property or not. The pointwise extension and the $T$-extension whose properties are studied in this chapter can be an appropriate tool.

**Fuzzy cognitive maps**

The notion of fuzzy cognitive map (FCM) was introduced by Bart Kosko in 1986 ([23]). More on FCMs can be found e.g. in [23],[24],[50].
The definition of FCM is the following:

**Definition 62** ([50])**.** An FCM is a directed graph with concepts like policies, events etc. as nodes and causalities as edges. It represents causal relationship between concepts

Causalities show relation between concepts, and usually have values from the interval $[0, 1]$. But to deal with uncertainty of the real world it is more appropriate to model them by means of fuzzy sets, thus the apparatus of gagops is applicable again. Application of gagops can be useful in so called fuzzy casual algebra ([23]) for estimation of indirect effect of a concept $C_i$ to a concept $C_j$ and for estimation of the total effect ([23]). Also application of other agops apart from min and max can be considered in this framework.
FCMs play major role when concerned data is unsupervised, therefore experts opinions are crucial. When many experts provide a FCM related to some process then aggregation of these FCM (e.g. by means of gagops) give more reliable FCM.
We have outlined some possible areas, where theoretical results provided in this chapter can be used. Evidently that provided list is not limited to these areas. The only price what we have to pay for processing of inexact data by means of gagops is fuzzy output.

## 3.9 Concluding remarks on generalized aggregation

We summarize results of the third chapter and outline directions for further study in this section. Notion of a $\gamma$-agop which is introduced at the very beginning is generalization of the class of agops if we consider it for an arbitrary $\gamma$. Although $\gamma$-agops have some disadvantages they simplify aggregation process and yield to be studied in more details. Different modifications of transformation $\varphi_\gamma$ can be considered, e.g. we can take:

$$\varphi_\gamma(x) = \begin{cases} 0, \text{ if } x < \gamma_1, \\ \gamma_1, \text{ if } \gamma_1 \leq x \leq \gamma_2 \\ ... \\ \gamma_{n-1}, \text{ if } \gamma_{n-1} \leq x \leq \gamma_n, \end{cases}$$

where $0 < \gamma_1 \leq \gamma_2 \leq .... \leq \gamma_n \leq 1$, in the role of transformation. This transformation partitions $[0, 1]^n$ into finite number of classes of equivalences. Applying different partitions $\gamma = \{\gamma_1, ..., \gamma_n\}$ , $\beta = \{\beta_1, ..., \beta_m\}$ we can employ ideas of the theory of rough sets and build upper and lower estimation of an arbitrary set $S$, such that $S \subseteq [0, 1]^n$ and study agop behaviour on it. Ability of $\gamma$-agops to join ideas of rough sets and fuzzy sets is charming and serves as motivation for further study.
Generalized aggregation, which forms the biggest part of author's contribution is

performed for two generalization methods (pointwise extension and $T$-extension) and is studied according to the joint scheme. It is shown that pointwise extension does not preserve convexity of fuzzy sets, e.g. aggregating fuzzy numbers as an output we can obtain more general fuzzy set without necessary properties. And this is an obvious disadvantage of pointwise extension. $T$-extension preserves convexity and even straight lines in some particular cases, therefore it is more applicable when identity of input and output information is important. From the prospective of symmetry, associatively and bisymmetry the both generalization methods have the same position: none special requirements for $A$ is necessary.

The idempotence property, which is straight forward for pointwise extension, is a more complicated in the case of $T$-extension, only $T_M$ and convexity of the input information can ensure idempotence of $T$-extension.

Neutral and absorbing elements are straight forward for pointwise extension and are implied by the corresponding element of agop $A$. Neutral element of $T$-extension is quite specific, but absorbing element is constant zero.

Shift-invariance (and in the same manner can be considered homogeneity and linearity properties) turned out to be non trivial property in both cases. And this is obviously due to additional operation of addition, which needs to be performed (apart from the extension of an agop $A$) according to the extension principle. This additional action narrows the list of cases when the property holds and it puts restrictions on an agop $A$ and a t-norm.

Pointwise extension preserves boundary conditions and monotonicity w.r.t. vertical order relations considered in the work. But it does not work w.r.t. horizontal order relations. Although it is not carefully proved, but it seems that pointwise extension is not a gagop w.r.t. order relations which act on the horizontal axis. $T$-extension is a gagop w.r.t. vertical and horizontal order relations observed in the work. And it seems that it will be a gagop w.r.t. organically defined order relations. From this prospective $T$-extension has more advantages than pointwise extension. In the author's opinion $T$-extension is more natural than pointwise extension in the framework of generalized aggregation.

Concluding on properties of $T$-extension we have already outlined possible directions for future study of gagops. Namely we can consider an $A^*$-extension as an another method of generalization:

$$\widehat{A}(P_1, ..., P_n)(x) = \sup\{A^*(P_1(x_1), ..., P_n(x_n)) : A(x_1, ..., x_n) = x\},$$

where $A^*$ is an arbitrary agop or another specific monotone function (e.g. any generator of a special class of agops). Appropriately chosen $A^*$ determines the properties of $A^*$-extension, and in such manner we can manage desired properties and avoid unnecessary difficulties.

Opulence of generalized aggregation with new results and urgency of practical applications make this theory charming for us for further development.

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