University of Latvia

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PhD Thesis

L-fuzzy valued measure and integral

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Abstract

In this thesis we develop the theory of measure and integral in the context of L-sets, when L is a complete, completely distributive lattice with the minimum t-norm. The main purpose of this thesis is to introduce the concept of measure and integral taking values in the L-fuzzy real line. We suggest the construction of L-fuzzy valued measure by extending the measure defined on σ-algebra of crisp sets to L-fuzzy valued measure defined on the tribe of L-sets. We introduce the L-fuzzy valued integral over the L-set with respect to an L-fuzzy valued measure, consider its properties and describe methods of L-fuzzy valued integration. We define an L-fuzzy valued norm using the L-fuzzy valued integral and apply it to estimate the error of approximation of real valued functions on L-set.

MSC: 03E72, 28E10, 26E50, 28A05, 28A12, 28A20, 28A25.

Key words and phrases: L-set, L-fuzzy real number, L-fuzzy real measure, L-fuzzy valued integral, L-fuzzy valued norm, approximation error.

Anotācija

Promocijas darbā mēra un integrāļa koncepcijas ir attīstītas L-kopu teorijas kontekstā, kur L ir pilns, pilnīgi distributīvs režģis ar minimuma t-normu. Darba mērķis ir izstrādāt mēra un integrāļa jēdzienus gadījumā, kad ne tikai kopas, bet arī mēra un integrāļa vērtības ir L-nestriktas. Darbā izstrādāta vispārīgā shēma parasto kopu mēra turpinājumam līdz L-kopu mēram ar vērtībām L-nestriktajā reālajā taisnē. Ir definēts L-nestrikti vērtīgs integrālis pa mērojamu L-kopu pēc L-nestrikti vērtīga mēra, izpētītas tā īpašības un aprakstītas integrēšanas metodes. Izmantojot integrāli ir uzdota L-nestrikti vērtīga norma, kura raksturo mērojamas funkcijas L-kopās. Ir aprakstīti daži L-nestrikti vērtīgas normas lietojumi funkciju aproksimāciju teorijā.

MSC: 03E72, 28E10, 26E50, 28A05, 28A12, 28A20, 28A25.

Atslēgvārdi: L-kopa, L-nestrikti reālā taisne, L-nestrikti vērtīgs mērs, L-nestrikti vērtīgs integrālis, aproksimācijas kļūda.

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IEGULDĪJUMS TAVĀ NĀKOTNĒ

Contents

Introduction

The history of measure theory starts in the late 19th and early 20th centuries by the works of E. Borel, H. Lebesgue, J. Radon and M. Frechet, among others. H. Lebesgue is known as a person that stands in the very beginning of the measure and integral theory foundation. In his "Lectures on integration and search for primitive functions" he challenged the goal to find a (non-negative) measure on the real line that would have existed for all bounded sets, and would satisfy three conditions: congruent sets have equal measure (i.e. the measure is invariant under the symmetry operations and transfer), measure is countably additive and measure of interval $(0, 1)$ is equal to 1. Lebesgue's studies have found a broad scientific response and were continued and developed by many mathematicians: E. Borel, M. Riess, G. Vitali, M. Frechet etc.

The main applications of measures are in the foundations of the Lebesgue integral, in A. Kolmogorov's axiomatisation of probability theory and in ergodic theory. In integration theory, specifying a measure allows one to define integrals on spaces more general than subsets of Euclidean space; moreover, the integral with respect to Lebesgue measure on Euclidean spaces is more general and has a richer theory than its predecessor, the Riemann integral. Probability theory considers measures that assign to the whole set the size 1, and considers measurable subsets to be events whose probability is given by the measure. Ergodic theory considers measures that are invariant under, or arise naturally from, a dynamical system.

Since the notion of measure was introduced the several ways to generalize this concept where attempted. In certain purposes it is useful to have a "measure" whose values are not restricted to the non-negative reals or infinity. For example, the countably additive set function with values in the (signed) real numbers is called a signed measure, while such a function with values in complex numbers is called a complex measure. Measures that take values in Banach spaces have been studied extensively. A measure that takes values in the set of self-adjoint projections on a Hilbert space is called a projection-valued measure; these are used in functional analysis for the spectral theorem. When it is necessary to distinguish the usual measures which take non-negative values from generalizations, the term positive measure is used. Positive measures are closed under conical combination but not general linear combination, while signed measures are the linear closure of positive measures. Another generalization is the finitely additive measure, which are sometimes called contents. This is the same as a measure, except that instead of requiring countable additivity we only require finite additivity.

As one can see all these ways to generalize the concept of measure in some sense is just manipulation with the properties of measure as a mapping (σ-additivity, additivity) and its domain (σ-algebra, algebra, semiring) or range ($\mathbb{R}_+, \mathbb{R}, \mathbb{R} \cup \{\infty\}$). Absolutely new possibilities for generalization of measure and integral have been discovered when the innovative theory of fuzzy sets were introduced.

Fuzzy mathematics forms a branch of mathematics related to fuzzy set theory and fuzzy logic. The actual history of the fuzzy sets theory starts in 1965 when L. A. Zadeh work [32] entitled "Fuzzy Sets" was published in the journal "Information and Control". Then in 1968 J. A. Gogen in [6] has developed and improved Zadeh ideas by implementing the concept of an L-set. L-set concept led to enormous interest, both among mathematicians - academics, as well as between the professionals who use mathematical ideas, concepts and results of various real process modelling.

A fuzzy subset A of a set X is a function (also called a membership function) $A : X \to L$, where L is the interval $[0,1]$. This function is also called a membership function. A membership function is a generalization of a characteristic function. More generally, instead of $[0,1]$ one can use a complete lattice L in a definition of L-fuzzy subset A. Usually, a fuzzification of mathematical concepts is based on a generalization of these concepts from characteristic functions to membership functions. For example, having two fuzzy subsets: the intersection and union can be defined as min and max of their membership functions. Instead of min and max one can use t-norm and t-conorm, respectively, for example, min can be replaced by multiplication. A straightforward fuzzification is usually based on min and max operations because in this case more properties of traditional mathematics can be extended to the fuzzy case.

The concept of an L-set creates applications in many other mathematical disciplines. Among them are also the measure and integral theory. In real processes there are often situations when the object is a set of "washed out" boundaries in the sense that a given element may belong to a given set to a greater or lower degree. Integral theory, which arose from this kind of practical tasks is rapidly evolving and increasingly is being successfully implemented. One can find plenty of papers devoted to the concept of measure and integral in the fuzzy context. Quite worthwhile approach to classify all types of measures in fuzzy sense was introduced by E. P. Klement and S. Weber in the joint paper called "Generalized measures" [17]. We just recall here some of the most important definitions to highlight all the diversity of the measure concept in fuzzy context and point out the area of possible widening to existing results.

We have already mentioned before that generalization process of measure by its nature reminds a "game" where we are "playing" with the mapping properties, its domain or range. As we see in [17] E. P. Klement and S. Weber put the "game rules" into the following frames: having a σ-complete lattice as domain and σ-complete, lattice ordered commutative semigroup as a range, the mapping satisfying boundary condition, valuation property and left-continuity can be considered as a generalized measure. Then for the different cases of domain and range the following examples of generalized measure can be described: probability measures of fuzzy events ([33]), possibility measures ([24], [34], [35]), fuzzy probability measures ([10], [14]), fuzzy-valued fuzzy measures ([12]), decomposable measures ([30]), measures of fuzzy sets ([29]).

Since the concept of integral comes along with the measure concept very closely, it is worth to mention that diversity of the integral in the fuzzy context does not inferior to the variety of measure concept. As examples of well known approaches to define fuzzy integral Sugeno, Choquet and Sipos integrals should be mentioned. The tremendous scope of papers devoted to the concept of fuzzy integral obviously requires proper classification. The concept of general and universal integrals that can be defined on arbitrary measurable spaces introduced in [16] by E. P. Klement, R. Mesiar, E. Pap, obviously, is a worthwhile attempt to clarify the overall picture in the entire diversity of multiple integral definitions. For instance, the Choquet and the Sugeno integrals are well-known examples of universal integrals.

Answering the question: "Should the measure of an L-set be a real number or it should have a fuzzy nature same as the set that it characterizes?" we give a preference to the second option. The same question can be addressed to the integral case. The fact that the majority of papers deals with the measure or integral defined as a real valued function we considered as a major lack.

To address this shortcoming, the work is intended to investigate measure and integral as an L-fuzzy valued structure. At that point we had to choose one of the several (quite different) approaches to the concept of an L-fuzzy real number. We decided in favour of fuzzy real numbers, as they were first defined by B. Hutton [9], and then thoroughly studied in a series of papers (see e.g. [21], [22], [23], [25]). The preference of using this approach for defining L-fuzzy real numbers is motivated by our intention to develop results on approximation from [1], [2].

Although, there are also some works where L-fuzzy valued measures and integral are involved, they used an alternative (essentially different) definition of a fuzzy real number (see e.g. [5], [31]). As an exception we can mention the E. P. Klement's approach (see e.g. [12], [17]) where fuzzy-valued measures take values in the collection of all probability distribution functions. The authors of the previously mentioned works devoted to a concept of fuzzy measure are mainly interested in finding appropriate definitions and studying the properties of this measures. As different from these works our main interest is in presenting the construction for a measure as well as calculation methods and possible applications for the integral.

I believe that this research work is very topical, because it combines the L-structure study, which is traditional to Latvian mathematicians (see e.g. [26]), as well as measure and integral theory further development, which opens a very wide range of package options. L-sets and the associated objects research in Latvia is conducted from the mid 80th leading by professor A. Šostak. These studies have gained international recognition. It is anticipated that the thesis results will actively continue traditional studies of Latvian mathematics in the theory of L-sets and also successfully develop this theoretical mathematics perspective direction.

Goal and objectives

The main objective of the doctoral thesis is to develop a theory of measure and integral in the L-fuzzy context, to generalize the constructions of measure and integral in the case when not only sets but also measure and integral take L-fuzzy values, where L is a complete, completely distributive lattice with the minimum t-norm.

The tasks of the thesis are strongly correlated with its goal:

1. to suggest a general scheme for extension of a crisp measure to an L-fuzzy valued measure taking values in the L-fuzzy real line,

2. to introduce the concept of an L-fuzzy valued integral of a real valued non-negative measurable function over an L-set with respect to the L-fuzzy valued measure,

3. to investigate the properties of an L-fuzzy valued integral and consider possible calculation methods,
4. to introduce the concept of an L-fuzzy valued norm by using the concept of an L-fuzzy valued integral,
5. to describe some possible applications of an L-fuzzy valued norm in the approximation theory.

Thesis structure

This thesis is structured in the following way.

Chapter 1 presents a general overview of necessary preliminaries. Starting with some concepts from the lattice theory continuing with the definition of an L-set and describing operations with L-sets by using concepts of triangular norm and conorm as well as involution. Then considering the notion of L-fuzzy real numbers and operations with them such as addition, countable addition and multiplication by a real number, as well as infimum and supremum of a set of L-fuzzy real numbers. Next the classes of L-sets introduced such as T-tribes, T-clans and T-semirings and also L-fuzzy valued functions defined on classes of L-sets and taking values in the range of non-negative L-fuzzy real numbers.

Chapter 2 contains the definitions of L-fuzzy valued measure, L-fuzzy valued elementary measure and L-fuzzy valued exterior measure. The main aim of the second chapter is to develop a construction of an L-fuzzy valued measure by extending a measure defined on a σ-algebra of crisp sets to an L-fuzzy valued measure defined on a T-tribe of L-sets in the case when T is the minimum t-norm. In order to realize the construction we consider the following steps: first, we describe the L-fuzzy valued elementary measured construction, second, we develop an approach to build the L-fuzzy valued exterior measure on the basis of the L-fuzzy valued elementary measure, then we describe the T-tribe of L-sets that are measurable with respect to the L-fuzzy valued exterior measure and, finally, we present the method of narrowing the L-fuzzy valued exterior measure to the L-fuzzy valued measure defined on the T-tribe.

Chapter 3 presents the concept of an L-fuzzy valued integral of a real valued function over an L-set with respect to an L-fuzzy valued measure. The definition of integral implemented stepwise: first considering the case when the integrand is a characteristic function, then extending it to the case when the integrand is a simple non-negative measurable function and, finally, the case with non-negative measurable function. Basic properties of an L-fuzzy valued integral are proved. Inspired by the fact from the classical measure theory stating that an integral of a given non-negative function can be considered as a measure we obtain two different approaches to define an L-fuzzy valued integral. The second approach shows that an L-fuzzy valued integral can be obtained by considering the measure ν_f using a crisp measure ν and then extending it to the L-fuzzy valued measure $\tilde{\mu}_f$ according to the construction described in Chapter 2. Finally we show that both approaches give us equivalent ways to define the L-fuzzy valued integral, i.e. $\mu_f = \tilde{\mu}_f$. This result can be described by the following diagram:

$$\begin{array}{ccc} \nu & \longrightarrow & \mu \\ \downarrow & & \downarrow \\ \nu_f & \rightarrow & \mu_f \end{array}$$

The second approach to the definition gives us an opportunity to simplify calculations of the L-fuzzy valued integral. Some special cases of integration for different types of L-sets are described.

Chapter 4 shows some possible applications of an L-fuzzy valued integral in approximation theory. For problems that can be solved only approximately, the notion of the error of a method of approximation plays the fundamental role. In order to estimate the quality of approximation on an L-fuzzy set, we need an appropriate L-fuzzy analogue of a norm. We introduce an L-fuzzy valued norm defined by an L-fuzzy valued integral with respect to an L-fuzzy valued measure μ. We describe the space $\mathscr{L}_1(E,\Sigma,\mu)$ of an L-fuzzy integrable over a measurable L-set $E \in \Sigma$ real valued functions. We also show how the introduced L-fuzzy valued norm can be applied to estimate on E the error of approximation of real valued functions $f \in \mathscr{L}_1(E,\Sigma,\mu)$. And, finally, some numerical examples are mentioned.

Approbation

The results obtained in the process of thesis writing have been presented at 12 international conferences: three EUSFLAT (European Society for Fuzzy Logic and Technology) conferences in 2007, 2009 and 2011 ([C02], [C08], [C12]); three FSTA (Fuzzy Set Theory and Applications) conferences in 2008, 2010 and 2012 ([C01], [C07], [C10]); three MMA (Mathematical Modelling and Analysis) conferences in 2009, 2010 and 2011 ([C04], [C06], [C09]); AGOP (Aggregation Operators) conference in 2011 [C03]; APLIMAT (Applied Mathematics) conference in 2011 [C05], ICTAA (International Conference on Topological Algebras and Applications) conference in 2008 [C11]; at 5 domestic conferences: three Conferences of University of Latvia in 2007, 2008 and 2009 ([C13], [C15], [C16]); two Conferences of Latvian Mathematical Society in 2006 and 2008 ([C14], [C17]); and at 4 international seminars ([C18]–[C21]).

The main results of the research have been reflected in 7 scientific publications [P1]–[P7] and 11 conference abstracts listed in p.72.

Chapter 1

Preliminaries

1.1 Lattices and t-norms

In matters where lattices are involved, our main sources of references are [3], [4]. However we reproduce here some definitions, notations and results.

Definition 1.1.1. *A poset $(L, \leq)$ is a set L in which a binary relation $\leq$ is defined, which satisfies for all $\alpha, \beta, \gamma \in L$ the following conditions:*

- $\alpha \leq \alpha$ *(reflexivity),*
- *if* $\alpha \leq \beta$ *and* $\beta \leq \alpha$*, then* $\alpha = \beta$ *(antisymmetry),*
- *if* $\alpha \leq \beta$ *and* $\beta \leq \gamma$*, then* $\alpha \leq \gamma$ *(transitivity).*

Definition 1.1.2. *Let $(L, \leq)$ be a poset and A be a subset of L. An element $\alpha \in L$ is called:*

- *upper bound of A iff* $\beta \leq \alpha$ *for all* $\beta \in A$*;*
- *join of A iff it is an upper bound of A and for every other upper bound* γ *of A it holds that* $\alpha \leq \gamma$*;*
- *lower bound of A iff* $\alpha \leq \beta$ *for all* $\beta \in A$*;*
- *meet of A iff it is a lower bound of A and for every other lower bound* γ *of A it holds that* $\gamma \leq \alpha$*.*

Note that the join and meet of an arbitrary subset (if they exist) are unique. We write $\bigvee A$ and $a \vee b$ to denote the joins, and $\bigwedge A$ and $a \wedge b$ to denote the meets.

Definition 1.1.3. *A poset* $(L,\leq)$ *is called a lattice iff it is closed under finite joins and meets.*

For convenience, going forward we write L meaning lattice $(L,\leq)$.

Definition 1.1.4. *A lattice L is called as a bounded lattice if it contains* $\bigvee L$ *(the greatest element or maximum) and* $\bigwedge L$ *(the least element or minimum), denoted* 1_L *and* 0_L *by convenience.*

Any lattice can be converted into a bounded lattice by adding a greatest and least elements, and every non-empty finite lattice is bounded, by taking the join (resp., meet) of all elements.

Definition 1.1.5. *A poset is called a complete lattice if all its subsets have both a join and a meet.*

In particular, every complete lattice is a bounded lattice.
Since lattices come with two binary operations, it is natural to ask whether one of them distributes over the other.

Definition 1.1.6. *A lattice L is called a distributive lattice if for any three elements* $\alpha,\beta,\gamma\in L$ *one of the following axioms is satisfied:*

- $\alpha\vee(\beta\wedge\gamma)=(\alpha\vee\beta)\wedge(\alpha\vee\gamma)$,
- $\alpha\wedge(\beta\vee\gamma)=(\alpha\wedge\beta)\vee(\alpha\wedge\gamma)$.

In the mathematical area of order theory, a completely distributive lattice is a lattice in which arbitrary joins distribute over arbitrary meets.

Definition 1.1.7. *A lattice L is said to be completely distributive if for any doubly indexed family* $\{\alpha_{jk}|j\in J,k\in K_j\}\subset L$*, we have*

$$\bigwedge_{j\in J}\bigvee_{k\in K_j}\alpha_{jk}=\bigvee_{f\in F}\bigwedge_{j\in J}\alpha_{jf(j)},$$

where F *is the set of choice functions* f *choosing for each index* $j\in J$ *some index* $f(j)\in K_j$*.*

Going forward we work only with lattices which are complete and completely distributive. Complete distributivity is a self-dual property, i.e. dualizing the above statement

yields the same class of complete lattices. Note that the unit interval $[0,1]$, ordered in the natural way, is a completely distributive lattice and the power set lattice for any set X is a completely distributive lattice.
In order to describe the operations with L-sets, such as intersection, union and difference, we consider functions defined in $L \times L$, which satisfy such important properties like associativity, commutativity and monotonicity. Such functions are known as triangular norms and triangular conorms. Also, to describe the complementary of an L-set and the difference between L-sets we need a concept of involution. Our sources for the references regarding t-norms and t-conorms are [15], [27].

Definition 1.1.8. *A function $T : L \times L \to L$ is called a triangular norm (t-norm for short) if it satisfies the following conditions for all $\alpha, \beta, \gamma \in L$:*

- $T(\alpha, 1_L) = 1_L$,
- $T(\alpha, \beta) \leq T(\alpha, \gamma)$ *whenever* $\beta \leq \gamma$,
- $T(\alpha, \beta) = T(\beta, \alpha)$,
- $T(\alpha, T(\beta, \gamma)) = T(T(\alpha, \beta), \gamma)$.

A function $S : L \times L \to L$ is called a triangular conorm (t-conorm for short) if it satisfies the following conditions for all $\alpha, \beta, \gamma \in L$:

- $S(\alpha, 0_L) = \alpha$,
- $S(\alpha, \beta) \leq S(\alpha, \gamma)$ *whenever* $\beta \leq \gamma$,
- $S(\alpha, \beta) = S(\beta, \alpha)$,
- $S(\alpha, S(\beta, \gamma)) = S(S(\alpha, \beta), \gamma)$.

A function $N : L \to L$ is called an order reversing involution if it satisfies the following conditions for all $\alpha, \beta \in L$:

- $N(N(\alpha)) = \alpha$,
- $N(\alpha) \geq N(\beta)$ *whenever* $\alpha \leq \beta$.

Note that $N(0_L) = 1_L$ and $N(1_L) = 0_L$.

If a lattice L provided with a t-norm T and an involution N, then the corresponding t-conorm is the function $S : L \times L \to L$ defined by

$$S(\alpha, \beta) = N(T(N(\alpha), N(\beta))).$$

It is easily seen that given a t-conorm S and an involution N, then $T : L \times L \to L$, defined by

$$T(\alpha, \beta) = N(S(N(\alpha), N(\beta)))$$

is a t-norm whose corresponding t-conorm is exactly the t-conorm S we started with.

Example 1.1.1.
Some of the most important pairs of t-norms and its corresponding t-conorms are the minimum T_M and the maximum S_M, the product T_P and the probabilistic sum S_P, the Lukasiewicz t-norm T_L and the Lukasievicz t-conorm S_L given by, respectively:

$T_M(\alpha, \beta) = min(\alpha, \beta)$, $\quad S_M(\alpha, \beta) = max(\alpha, \beta)$,
$T_P(\alpha, \beta) = \alpha \cdot \beta$, $\quad S_P(\alpha, \beta) = \alpha + \beta - \alpha \cdot \beta$,
$T_L(\alpha, \beta) = max(0, \alpha + \beta - 1)$, $\quad S_L(\alpha, \beta) = min(1, \alpha + \beta)$.

Note that the Lukaciewicz t-norm and product t-norm are defined in case when $L = [0, 1]$, but minimum t-norm we can define for arbitrary lattice.

1.2 L-sets

Given a (crisp) universe X and a complete, completely distributive lattice L, an L-fuzzy subset A of X (or, briefly, an L-set A) is characterized by its membership function

$$A : X \longrightarrow L,$$

where for $x \in X$ the value $A(x)$ is interpreted as the degree of membership of x in the L-set A. Thus, we do not make distinction between an L-set and its membership function. The class of all L-fuzzy subsets of X will be denoted L^X. It is obvious that each crisp subset of X is just a special case of an L-set.

Example 1.2.1. *A very special role is played in this survey by the following example of L-sets. For $M \subset X$, $\alpha \in L$ we define the L-set $A(M, \alpha) : X \to L$ by*

$$(A(M, \alpha))(x) = \begin{cases} \alpha, \ x \in M, \\ 0_L, \ x \notin M. \end{cases}$$

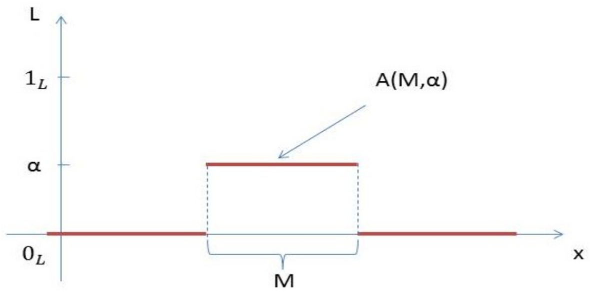

Figure 1.1: L-set $A(M,\alpha)$

The operations with L-sets $A,B \in L^X$ such as intersection, union and difference are defined by using a triangular norm T, its corresponding triangular conorm S and an involution N:

$$(A \mathsf{T} B)(x) = T(A(x), B(x)),$$

$$(A \mathsf{S} B)(x) = S(A(x), B(x)),$$

$$(A \mathsf{D} B)(x) = T(A(x), N(B(x))).$$

The operations for a sequence of L-sets $(A_n)_{n\in\mathbb{N}}$ such as intersection and union are defined in the following way:

$$(\mathop{\mathsf{T}}_{n=1}^{\infty} A_n)(x) = \bigwedge_{n\in\mathbb{N}} \mathop{\mathsf{T}}_{k=1}^{n} A_n(x) = \bigwedge_{n\in\mathbb{N}} (A_1(x) \mathsf{T} A_2(x) \mathsf{T} \ldots \mathsf{T} A_n(x)),$$

$$(\mathop{\mathsf{S}}_{n=1}^{\infty} A_n)(x) = \bigvee_{n\in\mathbb{N}} \mathop{\mathsf{S}}_{k=1}^{n} A_n(x) = \bigvee_{n\in\mathbb{N}} (A_1(x) \mathsf{S} A_2(x) \mathsf{S} \ldots \mathsf{S} A_n(x)).$$

Within this section all definitions are given for the case when T is an arbitrary t-norm. We also describe a concept of T-disjointness.

Definition 1.2.1. *A finite family of L-sets $A_1, A_2, \ldots, A_n$ is said to be T-disjoint (see e.g. [13]) iff for each $k \in \{1, \ldots, n\}$*

$$(\mathop{\mathsf{S}}_{j=1, j\neq k}^{n} A_j) \mathsf{T} A_k = \emptyset.$$

A countable family of L-sets is said to be T-disjoint iff every finite subfamily of this family is T-disjoint.

For the crisp sets such as $\emptyset$ or X we do not distinct the set and its characteristic function. For example, for all $x \in \mathbb{R}$ we have $\emptyset(x) = 0_L$.

1.3 Classes of L-sets

In order to consider L-fuzzy valued functions we describe such classes of L-sets as semirings, clans and tribes. We start with T-semirings (defined by analogy with the classical case, see e.g. [7]).

Definition 1.3.1. *A class $\wp \subset L^X$ is called a T-*semiring *on X iff the following properties are satisfied:*

- $\emptyset \in \wp$,
- *for all $A, B \in \wp$ we have $A \top B \in \wp$,*
- *for all $A, B \in \wp$ there exist such T-disjoint L-sets $A_1, A_2, \ldots, A_n \in \wp$ that* $A \,\mathsf{D}\, B = \mathop{\mathsf{S}}\limits_{i=1}^{n} A_i$.

In matters where T-clans and T-tribes are involved, our main source of references is [13].

Definition 1.3.2. *A class $\mathscr{A} \subset L^X$ is called a T-clan on X iff the following properties are satisfied:*

- $\emptyset \in \mathscr{A}$,
- *for all $A \in \mathscr{A}$ we have $N(A) \in \mathscr{A}$,*
- *for all $A, B \in \mathscr{A}$ we have $A \top B \in \mathscr{A}$.*

Definition 1.3.3. *A class $\Sigma \subset L^X$ is called a T-*tribe *on X iff the following properties are satisfied:*

- $\emptyset \in \Sigma$,
- *for all $A \in \Sigma$ we have $N(A) \in \Sigma$,*
- *for all sequences $(A_n)_{n \in N} \subset \Sigma$ we have $\mathop{\top}\limits_{n=1}^{\infty} A_n \in \Sigma$.*

Example 1.3.1. *The class of L-sets from Example 1.2.1 forms a T-semiring:*

$$\wp = \{A(M, \alpha) \mid M \subset X \text{ and } \alpha \in L\}.$$

1.4 L-fuzzy real line

For our purposes we use the L-fuzzy real numbers as they were first defined by B. Hutton [9] and then studied thoroughly in a series of papers (see e.g. [18], [21], [22], [23]).

Definition 1.4.1. *An L-fuzzy real number is a function $z : \mathbb{R} \to L$ such that*

- *z is non-increasing, i.e. $t_1 \le t_2 \Rightarrow z(t_1) \ge z(t_2)$,*
- *$\bigwedge_t z(t) = 0_L$, $\bigvee_t z(t) = 1_L$,*
- *z is left semi-continuous, i.e. for all $t_0 \in \mathbb{R}$ we have $\bigwedge_{t<t_0} z(t) = z(t_0)$.*

In the original papers on this subject (see [9], [18], [21]) the L-fuzzy real numbers were defined not as order reversing functions, but as equivalence classes of such functions. However each class of equivalence has a unique left semi-continuous representative and therefore an L-fuzzy real number can be identified with this representative.

The set of all L-fuzzy real numbers is called *the L-fuzzy real line* and it is denoted by $\mathbb{R}(L)$. An L-fuzzy real number z is called *non-negative* if $z(0) = 1_L$. We denote by $\mathbb{R}_+(L)$ the set of all non-negative L-fuzzy real numbers.

The ordinary real line $\mathbb{R}$ can be identified with the subspace $\{z_a \mid a \in \mathbb{R}\}$ of $\mathbb{R}(L)$ by assigning to a real number $a \in \mathbb{R}$ the fuzzy real number z_a defined by

$$z_a(t) = \begin{cases} 1_L, \text{ if } t \le a\,, \\ 0_L, \text{ if } t > a\,. \end{cases}$$

Operations with L-fuzzy real numbers such as addition $\oplus$ and multiplication by a real positive number are defined as follows:

$$(z_1 \oplus z_2)(t) = \bigvee_\tau \{z_1(\tau) \wedge z_2(t-\tau)\},\ (rz)(t) = z(\frac{t}{r}).$$

The supremum and the infimum of a set of non-negative L-fuzzy numbers $F \subset \mathbb{R}_+(L)$ are defined by the formulas (see e.g. [1], [2]):

$$(Inf\ F)(t) = \bigwedge\{z(t) \mid z \in F\},\ t \in \mathbb{R},$$

$$Sup\ F = Inf\{z \mid z \in \mathbb{R}(L), z \ge z' \text{ for all } z' \in F\}.$$

Due to F is bounded from below it is easy to see that $Inf\ F$ is an L-fuzzy real number. In case F is bounded from above (i.e. there exists $z_0 \in \mathbb{R}(L)$ such that $z \le z_0$ for all $z \in F$), $Sup\ F$ is an L-fuzzy real number, otherwise the condition

$$\bigwedge_t SupF(t) = 0_L$$

does not necessarily hold.

Going forward we will need also the countable addition of non-negative fuzzy real numbers. Given a sequence of non-negative fuzzy real numbers $(z_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+(L)$ we consider the countable sum

$$\bigoplus_{n=1}^{\infty} z_n = Sup\{z_1 \oplus z_2 \oplus ... \oplus z_n \mid n \in \mathbb{N}\}.$$

In the case when

$$\bigwedge_t (\bigoplus_{n=1}^{\infty} z_n)(t) \neq 0_L$$

the series $\bigoplus_{n=1}^{\infty} z_n$ diverges.

Example 1.4.1. *For $a \in \mathbb{R}_+$ and $\alpha \in L$ by $z(a,\alpha)$ we denote a special type of non-negative L-fuzzy real numbers*

$$(z(a,\alpha))(t) = \begin{cases} 1_L,\ t \leq 0, \\ \alpha,\ 0 < t \leq a, \\ 0_L,\ t > a, \end{cases}$$

that will play an important role in our work.

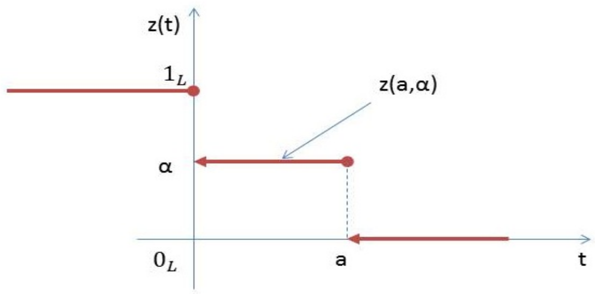

Figure 1.2: L-fuzzy real number $z(a,\alpha)$

Note that:

- $a_1, a_2 \in \mathbb{R}_+ \Rightarrow z(a_1,\alpha) \oplus z(a_2,\alpha) = z(a_1 + a_2, \alpha)$;
- $r \in \mathbb{R}_+ \Rightarrow rz(a,\alpha) = z(ra,\alpha)$;
- $a_i \in \mathbb{R}_+, i \in J$ and $\sup\{a_i \mid i \in J\} < +\infty \Rightarrow$
  $\Rightarrow Sup\{z(a_i,\alpha) \mid i \in J\} = z(\sup\{a_i \mid i \in J\}, \alpha)$.

1.5 L-fuzzy valued functions

Let $\mathscr{K}$ be a class of L-sets. Within this section some basic properties of an L-fuzzy valued function $\eta : \mathscr{K} \to \mathbb{R}_+(L)$ are considered.

Definition 1.5.1. *An L-fuzzy valued function* η *is called*

- *T-additive*
  if for all $A, B \in \mathscr{K}$ *such that* $A \top B = \emptyset$ *and* $A \,S\, B \in \mathscr{K}$ *it holds*

$$\eta(A \,S\, B) = \eta(A) \oplus \eta(B);$$

- *countably T-additive*
  if for all $(A_n)_{n\in\mathbb{N}} \subset \mathscr{K}$ *such that* $(\forall i, j \in \mathbb{N} : i \neq j \Rightarrow A_i \top A_j = \emptyset)$ *and* $\underset{n=1}{\overset{\infty}{S}} A_n \in \mathscr{K}$ *it holds*

$$\eta(\underset{n=1}{\overset{\infty}{S}} A_n) = \bigoplus_{n=1}^{\infty} \eta(A_n);$$

- *a T-valuation*
  if for all $A, B \in \mathscr{K}$ *such that* $A \top B$, $A \,S\, B \in \mathscr{K}$ *we have*

$$\eta(A \top B) \oplus \eta(A \,S\, B) = \eta(A) \oplus \eta(B);$$

- *left T-continuous*
  if for all $(A_n)_{n\in N} \subset \mathscr{K}$ *such that* $\underset{n=1}{\overset{\infty}{S}} A_n = A \in \mathscr{K}$ *and* $\forall n \in \mathbb{N} : A_n \leq A_{n+1}$ *we have*

$$Sup\{\eta(A_n) \mid n \in \mathbb{N}\} = \eta(A);$$

- *countably T-semiadditive*
  if for all $(A_n)_{n\in N} \subset \mathscr{K}$ *and for all* $A \in \mathscr{K}$ *such that* $A \leq \underset{n=1}{\overset{\infty}{S}} A_n$ *we have*

$$\eta(A) \leq \bigoplus_{n=1}^{\infty} \eta(A_n).$$

Remark 1.5.1. *If* η *is a T-valuation and* $\eta(\emptyset) = 0_L$*, then* η *is also T-additive, the converse not being generally true (see e.g. [13]). It is obvious that countably T-additive functions are always T-additive.*

Chapter 2

Construction of L-fuzzy valued measure

This section is devoted to an L-fuzzy valued generalization of the classical concept of measure. Referring to the paper [17] that type of generalization is called a triangular norm based measures.

We consider such L-fuzzy valued functions as T-measure, elementary T-measure and exterior T-measure. We also suggest a construction that allows us to extend a measure defined on a σ-algebra of crisp sets to the L-fuzzy valued T_M-measure defined on the T_M-tribe.

2.1 L-fuzzy valued measures

Definition 2.1.1. *An L-fuzzy valued function* $\eta : \mathscr{K} \to \mathbb{R}_+(L)$ *satisfying boundary condition* $\eta(\emptyset) = z(0, 1_L)$ *is called*

- *an L-fuzzy valued* T*-measure*
  if $\mathscr{K}$ *is a* T*-tribe and* η *is left* T*-continuous* T*-valuation;*

- *an L-fuzzy valued countably T-additive measure*
  if $\mathscr{K}$ *is a* T*-tribe and* η *is countably T-additive* T*-valuation;*

- *an L-fuzzy valued elementary* T*-measure*
  if $\mathscr{K}$ *is a* T*-semiring and* η *is* T*-additive;*

- *an L-fuzzy valued exterior* T*-measure*
  if $\mathscr{K} = L^X$ *and* η *is countably* T*-semiadditive.*

Remark 2.1.1. *L-fuzzy valued measures are countably T-additive functions.*

All main results of the thesis obtained in the case when T is the minimum t-norm, although, some of them are true in the case of other t-norms as well. Going forward we mean T_M every time the t-norm concept is involved and we use $\wedge, \vee$ instead of T, S when operations with L-sets are mentioned. Also for convenience, we say *L-fuzzy valued measure* meaning *L-fuzzy valued T_M-measure*.

2.2 Construction of L-fuzzy valued elementary measure

Let Φ be a σ-algebra of "crisp" subsets of X and ν is a finite measure $\nu : \Phi \to [0, +\infty)$. Our aim is to construct an L-fuzzy valued measure μ on a T_M-clan by extension a crisp measure ν. To achieve this we generalize the well known construction of "classic" measure theory (see e.g. [7]) to the L-fuzzy case.

To realize the construction we use the special type of L-sets with respect to σ-algebra Φ of crisp sets on X (see Example 1.2.1). For $M \in \Phi$, $\alpha \in L$ we consider L-sets $A(M, \alpha)$:

$$(A(M,\alpha))(x) = \begin{cases} \alpha, \; x \in M, \\ 0_L, \; x \notin M. \end{cases}$$

Proposition 2.2.1. *The class of L-sets*

$$\wp = \{A(M,\alpha) | M \in \Phi \textit{ and } \alpha \in L\}$$

is a T_M-semiring.

Proof.
For all $A(M,\alpha), B(K,\beta) \in \wp$ we have

$$A(M,\alpha) \wedge B(K,\beta) = C(M \cap K, \alpha \wedge \beta) \in \wp$$

and

$$A(M,\alpha) \setminus B(K,\beta) = A_1(M \setminus K, \alpha) \vee A_2(M \cap K, \alpha \wedge N(\beta)),$$

where

$$A_1(M \setminus K, \alpha),\; A_2(M \cap K, \alpha \wedge N(\beta)) \in \wp.$$

Taking into account that also $\emptyset = A(\emptyset, 0_L) \in \wp$, it is proved that $\wp$ is a T_M-semiring. □

We define an L-fuzzy valued function m on the T_M-semiring $\wp$ by the formula $m(A(M,\alpha)) = z(\nu(M),\alpha)$, where

$$(z(\nu(M),\alpha))(t) = \begin{cases} 1_L,\ t \le 0, \\ \alpha,\ 0 < t \le \nu(M), \\ 0_L,\ t > \nu(M), \end{cases}$$

is an L-fuzzy real number.

Proposition 2.2.2. *m is an L-fuzzy valued elementary measure.*

Proof.
It is easy to see that $m(\emptyset) = z(\nu(\emptyset), 0_L) = z_0$ and the equality

$$m(A(M,\alpha)) \oplus m(B(K,\beta)) = m(A(M,\alpha) \vee B(K,\beta))$$

is true if $A(M,\alpha) = \emptyset$ or $B(K,\beta) = \emptyset$.
Now let us consider $A(M,\alpha), B(K,\beta) \in \wp$ such that

$$A(M,\alpha) \wedge B(K,\beta) = \emptyset,\ A(M,\alpha) \cup B(K,\beta) \in \wp \text{ and } \alpha \neq 0_L, \beta \neq 0_L.$$

It follows that $M \cap K = \emptyset$ or $\alpha \wedge \beta = 0_L$. If $M \cap K = \emptyset$ then $\alpha = \beta$ and in this case

$$m(A(M,\alpha)) \oplus m(B(K,\alpha)) = z(\nu(M),\alpha) \oplus z(\nu(K),\alpha) =$$

$$= z(\nu(M) + \nu(K),\alpha) = z(\nu(M \cup K),\alpha) = m(A(M,\alpha) \vee B(K,\alpha)).$$

If $M \cap K \neq \emptyset$ then it is sufficient to consider the case when $\alpha \wedge \beta = 0_L$. In this case α and β are incomparable (due to the assumptions that $\alpha \neq 0_L, \beta \neq 0_L$). Because of

$$A(M,\alpha) \vee B(K,\beta) = C_1(M \cap K, \alpha \vee \beta) \vee C_2(M \setminus K, \alpha) \vee C_3(K \setminus M, \beta) \in \wp$$

we obtain that $K \setminus M = \emptyset$, $M \setminus K = \emptyset$ and hence $M = K$. Then

$$m(A(M,\alpha)) \oplus m(B(M,\beta)) = z(\nu(M),\alpha) \oplus z(\nu(M),\beta) =$$

$$= z(\nu(M), \alpha \vee \beta) = m(C(M, \alpha \vee \beta)) = m(A(M,\alpha) \vee B(M,\beta)).$$

By this we prove that m is T_M-additive. □

2.3 Measurable L-sets

For every $E \in L^X$ we define an L-fuzzy valued function $m^* : L^X \to \mathbb{R}_+(L)$ as follows

$$m^*(E) = Inf\{\bigoplus_{n=1}^{\infty} m(E_n) \mid (E_n)_{n\in\mathbb{N}} \subset \wp : E \leq \bigvee_{n=1}^{\infty} E_n\}.$$

Remark 2.3.1.

- *For every $E \in L^X$ there always exists a sequence $(E_n)_{n\in\mathbb{N}} \subset \wp : E \leq \bigvee_{n=1}^{\infty} E_n$. It is enough to take $E_1(X, 1_L)$. Thus, m^* is bounded from above in the following sense: $m^*(E) \leq z(\nu(X), 1_L)$ for all $E \in L^X$.*
- *For all $E \in \wp$ we have $m^*(E) = m(E)$.*
- *Let us note that defining $m^*(E)$ we can consider only T_M-disjoint sequences $(E_n)_{n\in\mathbb{N}}$.*

Proposition 2.3.1. *m^* is a countably T_M-semiadditive L-fuzzy valued function.*

Proof. Let us consider a sequence $(A_n)_{n\in\mathbb{N}} \subset L^X$ and an L-set $A \in L^X$ such that $A \leq \underset{n=1}{\overset{\infty}{S}} A_n$. To prove the inequality

$$m^*(A) \leq \bigoplus_{n=1}^{\infty} m^*(A_n)$$

we take sequences $(B_k^n)_{k\in\mathbb{N}} \in \wp$ such that
$A_n \leq \bigvee_{k=1}^{\infty} B_k^n, n \in \mathbb{N}$. Then

$$A \leq \bigvee_{n=1}^{\infty} A_n \leq \bigvee_{n=1}^{\infty} \bigvee_{k=1}^{\infty} B_k^n$$

and hence

$$m^*(A) \leq \bigoplus_{n=1}^{\infty} \bigoplus_{k=1}^{\infty} m(B_k^n).$$

Taking into account that this inequality holds independent of the choice of sequences $(B_k^n)_{k\in\mathbb{N}}$ we obtain

$$m^*(A) \leq \bigoplus_{n=1}^{\infty} m^*(A_n).$$

□

This proposition means that m^* is an L-fuzzy valued exterior measure.

Proposition 2.3.2. *For all $A, B \in L^X$ we have*

$$m^*(A) \oplus m^*(B) \geq m^*(A \wedge B) \oplus m^*(A \vee B).$$

Proof. For given L-sets A and B we consider two T_M-disjoint sequences:

$$(C_i(M_i, \alpha_i))_{i \in \mathbb{N}} \subset \wp : A \leq \bigvee_{i=1}^{\infty} C_i,$$

$$(D_j(K_j, \beta_j))_{j \in \mathbb{N}} \subset \wp : B \leq \bigvee_{j=1}^{\infty} D_j,$$

and define the following T_M-disjoint sequence

$$(H_{ij}(L_{ij}, \gamma_{ij}))_{i,j \in \mathbb{N}}, \text{ with } L_{ij} = M_i \cap K_j \text{ and } \gamma_{ij} = \alpha_i \wedge \beta_i.$$

Obviously, $A \wedge B \leq \bigvee_{i \in \mathbb{N}} \bigvee_{i \in \mathbb{N}} H_{ij}$.
As to L-set $A \vee B$ we can cover it with elements of three sequences:

$$(\tilde{H}_{ij}(L_{ij}, \lambda_{ij}))_{i,j \in \mathbb{N}} \quad \text{with} \quad L_{ij} = M_i \cap K_j,\, \lambda_{ij} = \alpha_i \vee \beta_i,$$

$$(\tilde{C}_i(\tilde{M}_i, \alpha_i))_{i \in \mathbb{N}} \quad \text{with} \quad \tilde{M}_i = M_i \setminus \bigcup_{j \in \mathbb{N}} K_j,$$

$$(\tilde{D}_j(\tilde{K}_j, \beta_j))_{j \in \mathbb{N}} \quad \text{with} \quad \tilde{K}_j = K_j \setminus \bigcup_{i \in \mathbb{N}} M_i.$$

Now let us transform the sum $\bigoplus_{i=1}^{\infty} m^*(C_i) \oplus \bigoplus_{j=1}^{\infty} m^*(D_j)$. We use the following notation:

$$F_{ij}^{\alpha}(L_{ij}, \alpha_i),\, F_{ij}^{\beta}(L_{ij}, \beta_j),\, i, j \in \mathbb{N}.$$

Because of σ-additivity of measure ν we get

$$\nu(M_i) = \nu(\tilde{M}_i) + \sum_{j=1}^{\infty} \nu(L_{ij}), i \in \mathbb{N},$$

$$\nu(K_j) = \nu(\tilde{K}_j) + \sum_{i=1}^{\infty} \nu(L_{ij}), j \in \mathbb{N}.$$

It follows

$$m^*(C_i) = z(\nu(M_i), \alpha_i) = z(\nu(\tilde{M}_i) + \sum_{j=1}^{\infty} \nu(L_{ij}), \alpha_i) =$$

$$= z(\nu(\tilde{M}_i), \alpha_i) \oplus \bigoplus_{j=1}^{\infty} z(\nu(L_{ij}), \alpha_i) = m^*(\tilde{C}_i) \oplus \bigoplus_{j=1}^{\infty} m^*(F_{ij}^{\alpha}),$$

$$m^*(D_j) = z(\nu(K_j), \beta_j) = z(\nu(\tilde{K}_j) + \sum_{i=1}^{\infty} \nu(L_{ij}), \beta_j) =$$

$$= z(\nu(\tilde{K}_j), \beta_j) \oplus \bigoplus_{i=1}^{\infty} z(\nu(L_{ij}), \beta_j) = m^*(\tilde{D}_j) \oplus \bigoplus_{i=1}^{\infty} m^*(F_{ij}^{\beta}).$$

Now we obtain

$$\bigoplus_{i=1}^{\infty} m^*(C_i) \oplus \bigoplus_{j=1}^{\infty} m^*(D_j) =$$

$$= \bigoplus_{i=1}^{\infty} (m^*(\tilde{C}_i) \oplus \bigoplus_{j=1}^{\infty} m^*(F_{ij}^{\alpha})) \oplus \bigoplus_{j=1}^{\infty} (m^*(\tilde{D}_j) \oplus \bigoplus_{i=1}^{\infty} m^*(F_{ij}^{\beta})) =$$

$$= \bigoplus_{i=1}^{\infty} m^*(\tilde{C}_i) \oplus \bigoplus_{j=1}^{\infty} m^*(\tilde{D}_j) \oplus \bigoplus_{i=1}^{\infty} \bigoplus_{j=1}^{\infty} m^*(F_{ij}^{\alpha}) \oplus \bigoplus_{j=1}^{\infty} \bigoplus_{i=1}^{\infty} m^*(F_{ij}^{\beta}) =$$

$$= \bigoplus_{i=1}^{\infty} m^*(\tilde{C}_i) \oplus \bigoplus_{j=1}^{\infty} m^*(\tilde{D}_j) \oplus \bigoplus_{i=1}^{\infty} \bigoplus_{j=1}^{\infty} (m^*(F_{ij}^{\alpha}) \oplus m^*(F_{ij}^{\beta})).$$

Since

$$m^*(F_{ij}^{\alpha}) \oplus m^*(F_{ij}^{\beta}) = m^*(H_{ij}) \oplus m^*(\tilde{H}_{ij}),$$

we continue

$$\bigoplus_{i=1}^{\infty} m^*(\tilde{C}_i) \oplus \bigoplus_{j=1}^{\infty} m^*(\tilde{D}_j) \oplus \bigoplus_{i=1}^{\infty} \bigoplus_{j=1}^{\infty} (m^*(F_{ij}^{\alpha}) \oplus m^*(F_{ij}^{\beta})) =$$

$$= \bigoplus_{i=1}^{\infty} m^*(\tilde{C}_i) \oplus \bigoplus_{j=1}^{\infty} m^*(\tilde{D}_j) \oplus \bigoplus_{i=1}^{\infty} \bigoplus_{j=1}^{\infty} m^*(\tilde{H}_{ij}) \oplus \bigoplus_{i=1}^{\infty} \bigoplus_{j=1}^{\infty} m^*(H_{ij}).$$

Now taking into account the fact that

$$A \wedge B \leq \bigvee_{i=1}^{\infty} \bigvee_{j=1}^{\infty} H_{ij} \text{ and } A \vee B \leq (\bigvee_{i=1}^{\infty} C_i) \vee (\bigvee_{j=1}^{\infty} D_j) \vee (\bigvee_{i=1}^{\infty} \bigvee_{j=1}^{\infty} \tilde{H}_{ij}),$$

we get

$$\bigoplus_{i=1}^{\infty} m^*(\tilde{C}_i) \oplus \bigoplus_{j=1}^{\infty} m^*(\tilde{D}_j) \oplus \bigoplus_{i=1}^{\infty} \bigoplus_{j=1}^{\infty} m^*(\tilde{H}_{ij}) \oplus \bigoplus_{i=1}^{\infty} \bigoplus_{j=1}^{\infty} m^*(H_{ij}) \geq$$

$$\geq m^*(A \wedge B) \oplus m^*(A \vee B).$$

So independent of the choice of sequences $(C_i)_{i\in\mathbb{N}}$, $(D_j)_{j\in\mathbb{N}}$ it holds

$$\bigoplus_{i=1}^{\infty} m^*(C_i) \oplus \bigoplus_{j=1}^{\infty} m^*(D_j) \geq m^*(A \wedge B) \oplus m^*(A \vee B).$$

Finally, by taking the infimum we obtain

$$m^*(A) \oplus m^*(B) \geq m^*(A \wedge B) \oplus m^*(A \vee B).$$

□

Next we define the concept of m^*-measurable L-sets. We generalize the concept of measurability in the sense of Caratheodory (see e.g. [7]).

Definition 2.3.1. *An L-set $A \in L^X$ is called a m^*-measurable L-set, if it satisfies the following conditions for all L-sets $E \in L^X$:*

(i) $m^*(A) \oplus m^*(E) = m^*(A \wedge E) \oplus m^*(A \vee E)$,

(ii) $m^*(N(A)) \oplus m^*(E) = m^*(N(A) \wedge E) \oplus m^*(N(A) \vee E)$.

We denote by Σ the class of all m^*-measurable L-sets. Some obvious properties of Σ:

- $\emptyset, X \in \Sigma$,
- $A \in \Sigma \Longrightarrow N(A) \in \Sigma$.

2.4 Construction of L-fuzzy valued measure

In this section we will prove that the class Σ of all m^*-measurable L-sets is a T_M-tribe and the restriction of m^* to Σ is an L-fuzzy valued measure.

Proposition 2.4.1. *Class Σ is a T_M-clan.*

Proof. For given L-sets $A_1, A_2 \in \Sigma$ and $E \in L^X$ we will prove the equality

$$m^*(E) \oplus m^*(A_1 \wedge A_2) = m^*(E \wedge (A_1 \wedge A_2)) \oplus m^*(E \vee (A_1 \wedge A_2)). \tag{2.1}$$

Since A_1 and A_2 are m^*-measurable we have

$$m^*(A_1) \oplus m^*(A_2) = m^*(A_1 \wedge A_2) \oplus m^*(A_1 \vee A_2),$$

$$m^*(A_1 \wedge E) \oplus m^*(A_1 \vee E) = m^*(A_1) \oplus m^*(E).$$

Now by summing up these two equalities we obtain

$$\begin{aligned} m^*(A_1) \oplus m^*(A_2) \oplus m^*(A_1 \wedge E) \oplus m^*(A_1 \vee E) = \\ = m^*(A_1 \wedge A_2) \oplus m^*(A_1 \vee A_2) \oplus m^*(A_1) \oplus m^*(E). \end{aligned} \tag{2.2}$$

Let us transform now the left part of (2.2). To do this we use (2.3) and (2.4):

$$\begin{aligned} m^*(A_1) \oplus [m^*(A_2) \oplus m^*(E \wedge A_1)] = \\ = m^*(A_1) \oplus [m^*(E \wedge A_1 \wedge A_2) \oplus m^*((E \wedge A_1) \vee A_2)] = \\ = m^*(E \wedge A_1 \wedge A_2) \oplus m^*((E \wedge A_1) \vee A_2 \vee A_1) \oplus m^*(((E \wedge A_1) \vee A_2) \wedge A_1) = \\ = m^*(E \wedge A_1 \wedge A_2) \oplus m^*(A_1 \vee A_2) \oplus m^*((E \vee A_2) \wedge A_1) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} m^*(E \vee A_1) \oplus m(A_1 \wedge (E \vee A_2)) = \\ = m^*(E \vee (A_1 \wedge A_2) \vee A_1) \oplus m^*(E \vee A_1 \wedge A_2 \wedge A_1) = \\ = m^*(E \vee (A_1 \wedge A_2)) \oplus m^*(A_1). \end{aligned} \tag{2.4}$$

Next we substitute (2.3) and (2.4) in (2.2):

$$m^*(E \wedge A_1 \wedge A_2) \oplus m^*(A_1 \vee A_2) \oplus m^*(E \vee (A_1 \wedge A_2)) \oplus m^*(A_1) =$$

$$= m^*(A_1 \wedge A_2) \oplus m^*(A_1 \vee A_2) \oplus m^*(A_1) \oplus m^*(E).$$

Finally, we obtain (2.1).
By analogy it can be proved that

$$m^*(E) \oplus m^*(A_1 \vee A_2) = m^*(E \wedge (A_1 \vee A_2)) \oplus m^*(E \vee (A_1 \vee A_2)).$$

Taking into account that $N(A_1 \vee A_2) = N(A_1) \wedge N(A_2)$ and $N(A_1 \wedge A_2) = N(A_1) \vee N(A_2)$, we get that $A_1 \vee A_2$ and $A_1 \wedge A_2$ are m^*-measurable. □

Proposition 2.4.2. *m^* is a T_M-additive L-fuzzy valued function on Σ.*

Proof. We will show that for T_M - disjoint L-sets $A_1, A_2, ..., A_n \in \Sigma$ it holds

$$m^*(\bigvee_{i=1}^{n} A_i) = \bigoplus_{i=1}^{n} m^*(A_i).$$

For $n = 2$ by using m^*-measurability and T_M - disjointness of A_1, A_2 we obtain

$$m^*(A_1) \oplus m^*(A_2) = m^*(A_1 \vee A_2) \oplus m^*(A_1 \wedge A_2) = m^*(A_1 \vee A_2) \oplus m^*(\mathbb{0}) = m^*(A_1 \vee A_2).$$

Now we assume that the equality holds for a given $n \in \mathbb{N}$ and we prove it for $n+1$ T_M-disjoint sets $A_1, A_2, ..., A_{n+1} \in \Sigma$

$$m^*(\bigvee_{k=1}^{n+1} A_k) = m^*((\bigvee_{k=1}^{n} A_k) \vee A_{n+1}) = m^*(\bigvee_{k=1}^{n} A_k) \oplus m^*(A_{n+1}) =$$

$$= \bigoplus_{k=1}^{n} m^*(A_k) + m^*(A_{n+1}) = \bigoplus_{k=1}^{n+1} m^*(A_k).$$

□

Theorem 2.4.1. *For all T_M-disjoint sequences of m^*-measurable L-sets $(A_n)_{n\in\mathbb{N}}$ it holds*

$$m^*(\bigvee_{n=1}^{\infty} A_n) = \bigoplus_{n=1}^{\infty} m^*(A_n).$$

Proof. To prove the equality we need to prove only one of the inequalities:

$$m^*(\bigvee_{n=1}^{\infty} A_n) \geq \bigoplus_{n=1}^{\infty} m^*(A_n).$$

For a given $n \in \mathbb{N}$ we have

$$m^*(\bigvee_{k=1}^{n} A_k) \leq m^*(\bigvee_{k=1}^{\infty} (A_k))$$

and

$$m^*(\bigvee_{k=1}^{n} A_k)) = \bigoplus_{k=1}^{n} m^*(A_k).$$

Hence

$$\bigoplus_{k=1}^{n} m^*(A_k) \leq m^*(\bigvee_{k=1}^{\infty} A_k).$$

Now by taking the supremum we obtain

$$\bigoplus_{k=1}^{\infty} m^*(A_k) \leq m^*(\bigvee_{k=1}^{\infty} A_k).$$

□

Proposition 2.4.3. $\wp \subset \Sigma$

Proof. First we will show that for a given $A(H,\alpha) \in \wp$ and all $E \in L^X$ it holds

$$m^*(A) \oplus m^*(E) = m^*(A \vee E) \oplus m^*(A \wedge E).$$

Because of Proposition 2.3.2 it is sufficient to prove the inequality

$$m^*(A) \oplus m^*(E) \leq m^*(A \vee E) \oplus m^*(A \wedge E).$$

We consider two T_M-disjoint sequences $(C_k(M_k,\beta_k))_{k\in\mathbb{N}} \subset \wp$ and $(D_k(K_k,\gamma_k))_{k\in\mathbb{N}} \subset \wp$ such that

$$A \vee E \leq \bigvee_{k=1}^{\infty} C_k \text{ and } A \wedge E \leq \bigvee_{k=1}^{\infty} D_k$$

and define two new sequences

$$(F_k(M_k \setminus H,\beta_k))_{k\in\mathbb{N}} \text{ and } (G_k(H \cap M_k,\beta_k))_{k\in\mathbb{N}}.$$

By T_M-additivity of m it holds

$$m(G_k) \oplus m(F_k) = m(C_k),\, k \in \mathbb{N}.$$

Taking into account

$$A \leq \bigvee_{k=1}^{\infty} G_k \text{ and } E \leq (\bigvee_{k=1}^{\infty} F_k) \vee (\bigvee_{k=1}^{\infty} G_k),$$

we obtain

$$m^*(A) = m(A) \leq \bigoplus_{k=1}^{\infty} m(G_k),$$

$$m^*(E) \leq \bigoplus_{k=1}^{\infty} m(F_k) \oplus \bigoplus_{k=1}^{\infty} m(D_k).$$

Summing up

$$m^*(A) \oplus m^*(E) \leq \bigoplus_{k=1}^{\infty} m(G_k) \oplus \bigoplus_{k=1}^{\infty} m(F_k) \oplus \bigoplus_{k=1}^{\infty} m(D_k) =$$

$$= \bigoplus_{k=1}^{\infty} (m(G_k) \oplus m(F_k)) \oplus \bigoplus_{k=1}^{\infty} m(D_k) = \bigoplus_{k=1}^{\infty} m(C_k) \oplus \bigoplus_{k=1}^{\infty} m(D_k).$$

So independent of a choice of sequences $(C_k)_{k\in\mathbb{N}}$ and $(D_k)_{k\in\mathbb{N}}$ it holds

$$\bigoplus_{k=1}^{\infty} m(C_k) \oplus \bigoplus_{k=1}^{\infty} m(D_k) \geq m^*(A) \oplus m^*(E).$$

Finally, by taking infimum we get

$$m^*(A) \oplus m^*(E) \leq m^*(A \vee E) \oplus m^*(A \wedge E).$$

Since

$$N(A) = B_1(H, N(\alpha)) \vee B_2(X \setminus H, 1_L),$$

where $B_1(H,N(\alpha)), B_2(X \setminus H, 1_L) \in \wp$, the equality

$$m^*(N(A)) \oplus m^*(E) = m^*(N(A) \wedge E) \oplus m^*(N(A) \vee E)$$

can be proved by analogy. □

Theorem 2.4.2. *For all sequences of m^*-measurable L-sets $(E_n)_{n\in\mathbb{N}}$ such that $\forall n \in \mathbb{N}:$ $E_n \leq E_{n+1}$ it holds*

$$Sup\{m^*(E_n) \mid \ n \in \mathbb{N}\} = m^*(\bigvee_{n=1}^{\infty} E_n).$$

Proof. It is obviously that

$$Sup\{m^*(E_n) \mid n \in \mathbb{N}\} \leq m^*(\bigvee_{n=1}^{\infty} E_n).$$

To prove the inverse inequality let us rewrite $Sup\{m^*(E_n) \mid n \in \mathbb{N}\}$ by using the definition of $m^*(E_n), n \in \mathbb{N}$:

$$Sup\{m^*(E_n) \mid n \in \mathbb{N}\} =$$

$$= Sup\{Inf\{\bigoplus_{k=1}^{\infty} m(B_k^n) \mid (B_k^n)_{k\in\mathbb{N}} \subset \wp : E_n \leq \bigvee_{k=1}^{\infty} B_k^n\} \mid n \in \mathbb{N}\} =$$

$$= Inf\{Sup\{\bigoplus_{k=1}^{\infty} m(B_k^n) \mid n \in \mathbb{N}\} \mid (B_k^n)_{k\in\mathbb{N}} \subset \wp : E_n \leq \bigvee_{k=1}^{\infty} B_k^n\}.$$

Let us note that in this formula you can take only T_M-disjoint sequences $(B_k^n)_{k\in\mathbb{N}}$, $n \in \mathbb{N}$, such that

$$\bigvee_{k=1}^{\infty} B_k^n \leq \bigvee_{k=1}^{\infty} B_k^{n+1}, \; n \in \mathbb{N}.$$

Let us denote

$$A_n = \bigvee_{k=1}^{\infty} B_k^n, \; n \in \mathbb{N}, \;\; A = \bigvee_{n=1}^{\infty} A_n = \bigvee_{n=1}^{\infty} \bigvee_{k=1}^{\infty} B_k^n.$$

Then

$$m^*(A_n) = \bigoplus_{k=1}^{\infty} m(B_k^n), \; n \in \mathbb{N},$$

and

$$Sup\{m^*(E_n) \mid n \in \mathbb{N}\} = Inf\{Sup\{m^*(A_n) \mid n \in \mathbb{N}\} \mid (B_k^n)_{k\in\mathbb{N}} \subset \wp : E_n \leq \bigvee_{k=1}^{\infty} B_k^n\}.$$

It is clear that

$$m^*(A) \geq m^*(\bigvee_{n=1}^{\infty} E_n).$$

Let us prove that

$$Sup\{m^*(A_n) \mid n \in \mathbb{N}\} = m^*(A).$$

For a given value $\alpha = m^*(A)(t_0)$ we consider sets

$$A^{\alpha} = \{x \in X \; : A(x) > \alpha\}, \; A_n^{\alpha} = \{x \in X \; : A_n(x) > \alpha\}, \; n \in \mathbb{N}.$$

Taking into account that $A^n = \bigvee_{k=1}^{\infty} B_k^n, n \in \mathbb{N}$, and $A^{\alpha} = \bigvee_{n=1}^{\infty} A_n^{\alpha}$, we obtain that

$$A_n^{\alpha}, A^{\alpha} \in \Phi, \; n \in \mathbb{N}, \;\; \text{and} \;\; t_0 = \nu(A^{\alpha}) = \lim \nu(A_n^{\alpha}).$$

Now the equality follows from the fact, that $Sup\{m^*(A_n) \mid n \in \mathbb{N}\}$ is left semi-continuous at $t_0 = \nu(A^{\alpha})$ and $m^*(A_n)(\nu(A_n^{\alpha})) \geq \alpha$, $n \in \mathbb{N}$. □

Theorem 2.4.3. *Class Σ is a T_M-tribe.*

Proof. Let us consider a sequence of m^*-measurable L-sets $(E_n)_{n\in\mathbb{N}} \subset \Sigma$. First we notice that

$$m^*(\bigvee_{n=1}^{\infty} E_n) = Sup\{m^*(\bigvee_{i=1}^{n} E_i) \mid n \in \mathbb{N}\}.$$

Now taking into account that $\bigvee_{i=1}^{n} E_i$ is m^*-measurable we obtain that for all L-sets $B \in L^X$ and for all $n \in \mathbb{N}$:

$$m^*(\bigvee_{i=1}^{n} E_i) \oplus m^*(B) = m^*((\bigvee_{i=1}^{n} E_i) \wedge B) \oplus m^*((\bigvee_{i=1}^{n} E_i) \vee B).$$

This means that for all $n \in \mathbb{N}$

$$m^*(\bigvee_{i=1}^{n} E_i) \oplus m^*(B) \leq m^*((\bigvee_{i=1}^{\infty} E_i) \wedge B) \oplus m^*((\bigvee_{i=1}^{\infty} E_i) \vee B),$$

and hence

$$Sup\{m^*(\bigvee_{i=1}^{n} E_i) \mid n \in \mathbb{N}\} \oplus m^*(B) \leq m^*((\bigvee_{i=1}^{\infty} E_i) \wedge B) \oplus m^*((\bigvee_{i=1}^{\infty} E_i) \vee B).$$

Finally we obtain

$$m^*(\bigvee_{n=1}^{\infty} E_n) \oplus m^*(B) = m^*((\bigvee_{n=1}^{\infty} E_n) \wedge B) \oplus m^*((\bigvee_{n=1}^{\infty} E_n) \vee B).$$

By analogy the result can be proved for $\bigwedge_{n=1}^{\infty} E_n$. □

Now let us denote by μ the restriction of m^* to T_M-tribe Σ:

$$\mu : \Sigma \to \mathbb{R}_+(L), \ \ \mu(E) = m^*(E) \ \text{ for all } E \in \Sigma.$$

As the final result, by extension of a crisp measure ν we obtain L-fuzzy valued measure $\mu : \Sigma \to \mathbb{R}_+(L)$ such that

- $\mu \mid_{\wp} = m$,
- $\mu \mid_{\Phi} = \nu$.

The last equality means that for every $M \in \Phi$ it holds $\mu(A(M, 1_L)) = z_{\nu(M)}$.

Chapter 3

L-fuzzy valued integral

In this section we define an L-fuzzy valued integral by analogy with the classical Lebesgue integral (see e.g. [7]). We consider the case when the integrand is a non-negative real valued function that is driven by applications.

3.1 Definition of L-fuzzy valued integral

We consider an L-fuzzy valued integral

$$\int_E f \, d\mu,$$

where

- E is a measurable L-set, i.e. $E \in \Sigma$,
- $f : X \to \mathbb{R}$ is a non-negative measurable function with respect to σ-algebra Φ,
- μ is an L-fuzzy valued measure obtained by the construction described above.

By analogy with the classical case we define an L-fuzzy valued integral stepwise, first, considering the case when integrand f is a characteristic function, then the case with f as a simple non-negative function and, finally, the case when f is a non-negative measurable function.

Case 1

f is a characteristic function:

$$f(x) = \chi_C(x) = \begin{cases} 1_L, \ x \in C \\ 0_L, \ x \notin C \end{cases}, \text{ where } C \in \Phi.$$

Integral defined as follows:

$$\int_E \chi_C d\mu = \mu(E \wedge C).$$

Note that $\mu(E \wedge C)$ is an L-fuzzy real number and in the case when $E \in 2^X$ we get

$$\mu(E \wedge C)(t) = \begin{cases} 1_L, \ t \leq \nu(E \cap C) \\ 0_L, \ t > \nu(E \cap C) \end{cases}.$$

Case 2

f is a simple non-negative measurable function (for short SNMF):

$$\int_E (\sum_{i=1}^{n} c_i \chi_{C_i}) \ d\mu = \bigoplus_{i=1}^{n} c_i \ \mu(C_i \wedge E),$$

whenever

- $c_i \in \mathbb{R}_+, \ C_i \in \Phi$ for all $i = 1, ..., n$,
- χ_{C_i} is the characteristic function of C_i , $i = 1, ..., n$,
- $C_1, ..., C_n$ are pairwise disjoint sets.

Case 3

f is a non-negative measurable function f (for short NMF):

$$\int_E f \ d\mu = Sup\{\int_E g \ d\mu \mid g \leq f \text{ and } g \text{ is SNMF}\}.$$

For $\mathbb{I}_f = \int_E f \ d\mu$ due to properties of the supremum of a set of L-fuzzy numbers, we have

- $\mathbb{I}_f$ is non-increasing, i.e. $t_1 \leq t_2 \Rightarrow \mathbb{I}_f(t_1) \geq \mathbb{I}_f(t_2)$,
- $\bigvee_t \mathbb{I}_f(t) = 1_L$,
- $\mathbb{I}_f$ is left semi-continuous, i.e. for all $t_0 \in \mathbb{R}$ we have $\bigwedge_{t<t_0} \mathbb{I}_f(t) = \mathbb{I}_f(t_0)$.

These are the properties of the L-fuzzy real numbers except one boundary condition that we need to require additionally. As a result we come up with the following definition.

Definition 3.1.1. *We say that a non-negative measurable function f is integrable over L-set E iff*

$$\bigwedge_t \mathbb{I}_f(t) = 0_L.$$

As known from the classical theory, a function considered as integrable when not only the integral exists but it also has a finite value and that is why the definition of an integrable function over an L-set looks natural to us.

Remark 3.1.1.

- *Let us note that if f is non-negative integrable over X with respect to measure ν then it is also L-fuzzy integrable over every $E \in \Sigma$.*
- *If a function f is integrable on the set SuppE with respect to measure ν then it is also integrable on the L-set E with respect to L-fuzzy valued measure μ.*

Proposition 3.1.1. *For every crisp set $M \in \Phi$ and $\alpha \in L$ it holds:*

$$\int_M f d\mu = z(\int_M f\, d\nu, 1_L),$$

$$\int_{A(M,\alpha)} f d\mu = z(\int_M f\, d\nu, \alpha).$$

Proof. The equality for the crisp set is obvious. We prove the equality for L-set $A(M,\alpha)$. For $f = \sum_{i=1}^{n} c_i \chi_{C_i}$ we get

$$\int_{A(M,\alpha)} \sum_{i=1}^{n} c_i \chi_{C_i} d\mu = \bigoplus_{i=1}^{n} c_i\ \mu(C_i \wedge A(M,\alpha)) = \bigoplus_{i=1}^{n} c_i\ z(\nu(M \cap C_i), \alpha) =$$

$$= z(\sum_{i=1}^{n} c_i\ \nu(M \cap C_i), \alpha) = z(\int_M f d\nu, \alpha).$$

In the case when f is NMF we have

$$\int_{A(M,\alpha)} f\ d\mu = Sup\{z(\int_M g\ d\nu, \alpha) \mid g \le f \text{ and } g \text{ is SNMF}\} =$$

$$= z(sup\{\int_M g\ d\nu, \alpha \mid g \le f \text{ and } g \text{ is SNMF}\}, \alpha) = z(\int_M f d\nu, \alpha).$$

□

3.2 Properties of L-fuzzy valued integral

In this section we consider properties of an L-fuzzy valued integral of L-fuzzy integrable functions. Here L-sets E, E_n $(n \in \mathbb{N})$ are measurable and functions f, f_n $(n \in \mathbb{N})$ are integrable.

(𝕀1) $r \in \mathbb{R}_+ \Rightarrow \int\limits_E rf d\mu = r \int\limits_E f d\mu$

Proof. In the case when integrand $f = \sum\limits_{i=1}^{n} c_i \chi_{C_i}$ is SNMF the equality follows from

$$\bigoplus_{i=1}^{n} (rc_i)\,\mu(C_i \wedge E) = r \bigoplus_{i=1}^{n} c_i\,\mu(C_i \wedge E).$$

Considering the case when f is NMF and $r > 0$ (if $r = 0$ then the equality is obvious) we have

$$\int\limits_E rf d\mu = Sup\{\int\limits_E g\,d\mu \mid g \leq rf, \text{ and } g \text{ is SNMF}\} =$$

$$= rSup\{\int\limits_E \frac{g}{r}\,d\mu \mid \frac{g}{r} \leq f, \text{ and } g \text{ is SNMF}\} = r\int\limits_E f d\mu.$$

□

(𝕀2) $f_1 \leq f_2 \Rightarrow \int\limits_E f_1 d\mu \leq \int\limits_E f_2 d\mu$

Proof. From

$$\{\int\limits_E g\,d\mu \mid g \leq f_1 \text{ and } g \text{ is SNMF}\} \subset \{\int\limits_E g\,d\mu \mid g \leq f_2 \text{ and } g \text{ is SNMF}\}$$

it follows

$$Sup\{\int\limits_E g\,d\mu \mid g \leq f_1 \text{ and } g \text{ is SNMF}\} \leq Sup\{\int\limits_E g\,d\mu \mid g \leq f_2 \text{ and } g \text{ is SNMF}\}.$$

□

(𝕀3) $E_1 \leq E_2 \Rightarrow \int\limits_{E_1} f\,d\mu \leq \int\limits_{E_2} f\,d\mu$

Proof. The inequality

$$Sup\{\int_{E_1} g\, d\nu \mid g \leq f \text{ and } g \text{ is SNMF}\} \leq Sup\{\int_{E_2} g\, d\nu \mid g \leq f \text{ and } g \text{ is SNMF}\}$$

holds due to

$$\int_{E_1} g d\mu \leq \int_{E_2} g d\mu, \text{ where } g = \sum_{i=1}^{n} c_i \chi_{C_i} \text{ is SNMF.}$$

The last inequality is equivalent to

$$\bigoplus_{i=1}^{n} c_i\, \mu(C_i \wedge E_1) \leq \bigoplus_{i=1}^{n} c_i\, \mu(C_i \wedge E_2),$$

that holds because of monotonicity of μ. □

($\mathbb{I}4$) $(E_k)_{k\in\mathbb{N}} : E_k \leq E_{k+1}$ and $\bigvee_{k\in\mathbb{N}} E_k = E \Rightarrow \int_E f d\mu = Sup\{\int_{E_k} f d\mu \mid k \in \mathbb{N}\}$

Proof. We start with the case when $f = \sum_{i=1}^{n} c_i \chi_{C_i}$ is SNMF:

$$\int_E f\, d\mu = \bigoplus_{i=1}^{n} c_i\, \mu(C_i \wedge E) = Sup\{\bigoplus_{i=1}^{n} c_i\, \mu(C_i \wedge E_k) \mid k \in \mathbb{N}\} = Sup\{\int_{E_k} f d\mu \mid k \in \mathbb{N}\}.$$

Now to prove the equality when f is NMF we show both inequalities "$\geq$" and "$\leq$".

The inequality $\int_E f\, d\mu \geq \int_{E_k} f\, d\mu$ for all $k \in \mathbb{N}$ implies the inequality

$$\int_E f\, d\mu \geq Sup\{\int_{E_k} f d\mu \mid k \in \mathbb{N}\}.$$

Taking into account that for all functions g (g is SNMF and $g \leq f$) we have

$$\int_E g\, d\mu = Sup\{\int_{E_k} g\, d\mu \mid k \in \mathbb{N}\} \leq Sup\{\int_{E_k} f\, d\mu \mid k \in \mathbb{N}\},$$

it follows that

$$\int_E f\, d\mu = Sup\{\int_E g\, d\mu \mid g \text{ is SNMF and } g \leq f\} \leq Sup\{\int_{E_k} f\, d\mu \mid k \in \mathbb{N}\}.$$

□

($\mathbb{I}5$) $(f_n)_{n\in\mathbb{N}} : f_n \leq f_{n+1}$ and $\lim_{n\to\infty} f_n = f \Rightarrow Sup\{\int_E f_n\, d\mu \mid n \in \mathbb{N}\} = \int_E f\, d\mu$

Proof. To prove the equality we show that both inequalities "≤" and "≥".
The inequality

$$Sup\{\int_E f_n\, d\mu \mid n \in \mathbb{N}\} \leq \int_E f\, d\mu$$

holds due to property ($\mathbb{I}2$). To show the opposite inequality by using a function g (g is SNMF and $g \leq f$) and a number $c \in (0,1)$ we define

$$M_n = \{x \in X \mid f_n(x) \geq cg(x),\ c \in (0,1)\} \text{ and } E_n = E \wedge M_n,\ n \in \mathbb{N}.$$

Obviously, $M_n \leq M_{n+1}$ and $\bigcup_{n\in\mathbb{N}} M_n = X$. Now we have

$$\int_E f_n d\mu \geq \int_{E\wedge M_n} f_n d\mu \geq \int_{E\wedge M_n} cg\, d\mu,$$

$$Sup\{\int_E f_n\, d\mu \mid n \in \mathbb{N}\} \geq c\, Sup\{\int_{E\wedge M_n} g\, d\mu \mid n \in \mathbb{N}\} = c \int_{\bigvee_{n\in\mathbb{N}} E_n} g\, d\mu.$$

It follows

$$Sup\{\int_E f_n\, d\mu \mid n \in \mathbb{N}\} \geq c \int_E f\, d\mu.$$

And finally,

$$Sup\{\int_E f_n\, d\mu \mid n \in \mathbb{N}\} \geq \int_E f\, d\mu.$$

□

($\mathbb{I}6$) $\int_E (f_1 + f_2) d\mu = \int_E f_1 d\mu \oplus \int_E f_2 d\mu$

Proof. Again we start with the case when integrands are SNMF:

$$f_1 = \sum_{i=1}^{n} c_i\, \chi_{C_i},\ f_2 = \sum_{j=1}^{k} b_j\, \chi_{B_j},\ f_1 + f_2 = \sum_{l=1}^{p} a_l\, \chi_{A_l}.$$

We suppose that $\bigcup_{i=1}^{n} C_i = \bigcup_{j=1}^{k} B_j = \bigcup_{l=1}^{p} A_l = X$. Then

$$\int_E (f_1 + f_2) d\mu = \bigoplus_{l=1}^{m} a_l\, \mu(A_l \wedge E) = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{k} \bigoplus_{l=1}^{m} a_l\, \mu(A_l \wedge C_i \wedge B_j \wedge E) =$$

$$= \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{k} \bigoplus_{l=1}^{m} (c_i + b_j)\, \mu(A_l \wedge C_i \wedge B_j \wedge E) =$$

$$= \bigoplus_{i=1}^{n} c_i \bigoplus_{j=1}^{k} \bigoplus_{l=1}^{m} \mu(A_l \wedge C_i \wedge B_j \wedge E)) \oplus \bigoplus_{j=1}^{k} b_j \bigoplus_{i=1}^{n} \bigoplus_{l=1}^{m} \mu(A_l \wedge C_i \wedge B_j \wedge E)) =$$

$$= \bigoplus_{l=1}^{n} (c_i \, \mu(C_i \wedge E)) \oplus \bigoplus_{j=1}^{k} (b_j \, \mu(B_j \wedge E)) = \int_E f_1 d\mu \oplus \int_E f_2 d\mu.$$

Now we consider the case when integrands are NMF. Then we can find non-increasing sequences of SNMF $(g_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$: $\lim_{n\to\infty} g_n = f_1$ and $\lim_{n\to\infty} h_n = f_2$.
Hence,

$$\int_E f_1 \, d\mu \oplus \int_E f_2 \, d\mu = Sup\{\int_E g_n \, d\mu \mid n \in \mathbb{N}\} \oplus Sup\{\int_E h_n \, d\mu \mid n \in \mathbb{N}\} =$$

$$= Sup\{\int_E (g_n + h_n) \, d\mu \mid n \in \mathbb{N}\} = \int_E (f_1 + f_2) \, d\mu.$$

□

($\mathbb{I}$7) $E_1 \wedge E_2 = \emptyset \Rightarrow \int\limits_{E_1 \vee E_2} f d\mu = \int\limits_{E_1} f d\mu \oplus \int\limits_{E_2} f d\mu$

Proof. Again we start with the case when integrand $f = \sum_{i=1}^{n} c_i \, \chi_{C_i}$ is SNMF:

$$\int_{E_1 \vee E_2} f \, d\mu = \bigoplus_{i=1}^{n} c_i \, \mu(C_i \wedge (E_1 \vee E_2)) = \bigoplus_{i=1}^{n} c_i \, (\mu(C_i \wedge E_1) \oplus \mu(C_i \wedge E_2)) =$$

$$= \bigoplus_{i=1}^{n} (c_i \, \mu(C_i \wedge E_1)) \oplus \bigoplus_{i=1}^{n} (c_i \, \mu(C_i \wedge E_2)) = \int_{E_1} f \, d\mu \oplus \int_{E_2} f \, d\mu.$$

To prove the equality when f is NMF we consider a non-increasing sequence of SNMF $(g_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} g_n = f$ and apply the technique described in the proof of the previous property. □

3.3 Alternative definition of L-fuzzy valued integral

From the classical measure theory (see e.g. [7]) it is known that having a measure ν and non-negative measurable function f we can get a new measure ν_f defined as following

$$\nu_f(M) = \int_M f \, d\nu, \; M \in \Phi.$$

In this section we apply the same approach in the fuzzy case. For a given real valued non-negative integrable over X function f we consider $\mu_f : \Sigma \to \mathbb{R}_+(L)$ defined as follows:

$$\mu_f(E) = \int_E f \, d\mu, \; E \in \Sigma.$$

Theorem 3.3.1. *μ_f is an L-fuzzy valued measure.*

Proof. Due to the integral property μ_f is left T_M-continuous. We need to prove that μ_f is a T_M-valuation, i.e.

$$\mu_f(E_1 \vee E_2) \oplus \mu_f(E_1 \wedge E_2) = \mu_f(E_1) \oplus \mu_f(E_1) \text{ for all } E_1, E_2 \in \Sigma.$$

Again we start with the case when integrand $f = \sum_{i=1}^{n} c_i \, \chi_{C_i}$ is SNMF and then continue for the case when f is NMF by taking a non-increasing sequence of SNMF $(g_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} g_n = f$. □

As we can see L-fuzzy valued measure μ_f was obtained by the following scheme:

- first extending crisp measure ν using the construction described in section 2.2 ;
- second defining a measure by using an integral with respect to L-fuzzy valued measure μ .

We can also show that μ_f can be obtained in another way described by the following diagram:

$$\begin{array}{ccc} \nu & \longrightarrow & \mu \\ \downarrow & & \downarrow \\ \nu_f & \rightarrow & \mu_f \end{array}$$

For a given σ-algebra $\Phi \subset 2^X$ and a finite measure $\nu : \Phi \to \mathbb{R}_+$ we define measure ν_f as following:

$$\nu_f(M) = \int_M f \, d\nu, \;\; M \in \Phi.$$

Then an L-fuzzy valued measure $\tilde{\mu}_f$ can be obtained by using the same scheme as in section 2.2.

- We define an L-fuzzy valued function $m_f : \wp \to \mathbb{R}_+(L)$ by

$$m_f(A(M,\alpha)) = z(\nu_f(M), \alpha),$$

and we extend it to the L-fuzzy valued function $m_f^* : L^X \to \mathbb{R}_+(L)$ as following:

$$m_f^*(E) = Inf\ \{\bigoplus_{n=1}^{\infty} m_f(E_n) \mid (E_n)_{n\in\mathbb{N}} \subset \wp : E \leq \bigvee_{n=1}^{\infty} E_n\}.$$

- We denote by Σ_f the class of all so called m_f^*-measurable L-sets $E \in L^X$ such that for all $B \in L^X$ it holds

$$m_f^*(B) \oplus m_f^*(E) = m_f^*(B \wedge E) \oplus m_f^*(B \vee E),$$
$$m_f^*(N(E)) \oplus m_f^*(B) = m_f^*(N(E) \wedge B) \oplus m_f^*(N(E) \vee B).$$

- Finally, considering $\tilde{\mu}_f$ as the restriction of m_f^* to Σ_f we obtain the L-fuzzy valued measure: $\tilde{\mu}_f(E) = m_f^*(E),\ E \in \Sigma_f$.

To show the equivalence of both definitions of an L-fuzzy valued integral described above we consider the following theorem.

Theorem 3.3.2. *For all integrable non-negative functions f it holds:*

(T1) $\Sigma \subset \Sigma_f$,

(T2) for all $E \in \Sigma$ we have $\mu_f(E) = \tilde{\mu}_f(E)$.

To prove Theorem 3.3.2 we use the following Lemma.

Lemma 3.3.1. *For all $E \in L^X$ it holds:*

(L1) if f is SNMF (i.e. $f = \sum_{i=1}^{n} c_i\chi_{C_i}$), then

$$m_f^*(E) = \bigoplus_{i=1}^{n} c_i\ m^*(E \wedge C_i);$$

(L2) if $(f_n)_{n\in\mathbb{N}}$ is a sequence of SNMF such that
$f_n \leq f_{n+1}$ and $\lim_{n\to\infty} f_n = f$, then

$$m_f^*(E) = Sup\{m_{f_n}^*(E) | n \in \mathbb{N}\}.$$

Proof. (L1) First we show that equality holds for an L-set $E = A(M,\alpha) \in \wp$. Taking into account the properties of L-fuzzy real numbers we get:

$$m_f(A(M,\alpha)) = z(\nu_f(M),\alpha) = z(\int_M f\, d\nu, \alpha) =$$

$$= z(\int_M \sum_{i=1}^{n} c_i \, \chi_{C_i} \, d\nu, \alpha) = z(\sum_{i=1}^{n} c_i \, \nu(M \wedge C_i), \alpha) =$$

$$= \bigoplus_{i=1}^{n} c_i \, z(\nu(M \wedge C_i), \alpha) = \bigoplus_{i=1}^{n} c_i \, m(A(M, \alpha) \wedge C_i).$$

Now for a T_M-disjoint sequence of L-sets $(E_k)_{k\in\mathbb{N}} \subset \wp$ such that $E \leq \bigvee_{k=1}^{\infty} E_k$ we obtain:

$$\bigoplus_{k=1}^{\infty} m_f(E_k) = \bigoplus_{k=1}^{\infty} \bigoplus_{i=1}^{n} c_i \, m(E_k \wedge C_i) = \bigoplus_{i=1}^{n} c_i \, \bigoplus_{k=1}^{\infty} m(E_k \wedge C_i).$$

Finally, recalling the definition of m_f^*:

$$m_f^*(E) = Inf \, \{\bigoplus_{k=1}^{\infty} m_f(E_k) \mid (E_k)_{k\in\mathbb{N}} \subset \wp : E \leq \bigvee_{k=1}^{\infty} E_k\} =$$

$$= \bigoplus_{i=1}^{n} c_i \, Inf \, \{\bigoplus_{k=1}^{\infty} m(E_k \wedge C_i) \mid (E_k)_{k\in\mathbb{N}} \subset \wp : E \leq \bigvee_{k=1}^{\infty} E_k\} = \bigoplus_{i=1}^{n} c_i \, m^*(E \wedge C_i).$$

(L2) We start with an L-set $E = A(M, \alpha) \in \wp$. Using the properties of L-fuzzy real numbers we get:

$$Sup\{m_{f_n}(A(M, \alpha)) | n \in \mathbb{N}\} = Sup\{z(\nu_{f_n}(M), \alpha) | n \in \mathbb{N}\} =$$

$$= z(\sup_n(\nu_{f_n}(M)), \alpha) = z(\nu_f(M), \alpha) = m_f(A(M, \alpha)).$$

Now taking any $E \in L^X$ and the fact that L if completely distributive lattice we obtain:

$$Sup \, \{m_{f_n}^*(E) | n \in \mathbb{N}\} =$$

$$= Sup\{ \, Inf \, \{\bigoplus_{k=1}^{\infty} m_{f_n}(E_k) \mid (E_k)_{k\in\mathbb{N}} \subset \wp : E \leq \bigvee_{k=1}^{\infty} E_k\} | n \in \mathbb{N}\} =$$

$$= Inf\{ \, Sup \, \{\bigoplus_{k=1}^{\infty} m_{f_n}(E_k) | n \in \mathbb{N}\} | (E_k)_{k\in\mathbb{N}} \subset \wp : E \leq \bigvee_{k=1}^{\infty} E_k\} =$$

$$= Inf\{\bigoplus_{k=1}^{\infty} m_f(E_k) | (E_k)_{k\in\mathbb{N}} \subset \wp : E \leq \bigvee_{k=1}^{\infty} E_k\} = m_f^*(E).$$

□

Now coming back to theorem 3.3.2.

Proof. (T1) We consider $E \in \Sigma$ and start with the case when f - SNMF or $f = \sum_{i=1}^{n} c_i \chi_{C_i}$. For every $i \in \{1,...,n\}$ if $E \in \Sigma$ then $E \wedge C_i \in \Sigma$ and following for all $B \in L^X$ we have

$$m^*(E \wedge C_i) \oplus m^*(B \wedge C_i) = m^*(E \wedge B \wedge C_i) \oplus m^*((E \wedge C_i) \vee (B \wedge C_i)).$$

Now multiplying the equality by c_i and summing up by i we obtain

$$\bigoplus_{i=1}^{\infty}(c_i\, m^*(E \wedge C_i)) \oplus \bigoplus_{i=1}^{\infty}(c_i\, m^*(B \wedge C_i)) =$$

$$= \bigoplus_{i=1}^{\infty}(c_i\, m^*(E \wedge B \wedge C_i)) \oplus \bigoplus_{i=1}^{\infty}(c_i\, m^*((E \wedge C_i) \vee (B \wedge C_i))).$$

And finally applying Lemma 3.3.1:

$$m^*_f(E) \oplus m^*_f(B) = m^*_f(E \wedge B) \oplus m^*_f(E \vee B).$$

By analogy we can prove the second equality

$$m^*_f(N(E)) \oplus m^*_f(B) = m^*_f(N(E) \wedge B) \oplus m^*_f(N(E) \vee B).$$

In the case when f is NMF there exists a sequence of SNMF

$$(f_n)_{n\in\mathbb{N}} : f_n \leq f_{n+1} \text{ and } \lim_{n\to\infty} f_n = f.$$

For all f_n it holds

$$m^*_{f_n}(E) \oplus m^*_{f_n}(B) = m^*_{f_n}(E \wedge B) \oplus m^*_{f_n}(E \vee B).$$

Now taking Sup we obtain

$$Sup\{m^*_{f_n}(E)|n \in \mathbb{N}\} \oplus Sup\{m^*_{f_n}(B)|n \in \mathbb{N}\} =$$

$$= Sup\{m^*_{f_n}(E \wedge B)|n \in \mathbb{N}\} \oplus Sup\{m^*_{f_n}(E \vee B)|n \in \mathbb{N}\}.$$

And due to Lemma 3.3.1

$$m^*_f(E) \oplus m^*_f(B) = m^*_f(E \wedge B) \oplus m^*_f(E \vee B).$$

The same logic can be used to prove the second equality.

(T2) We start again with SNMF f. Using Lemma 3.3.1 we see that

$$\mu_f(E) = \int_E f\, d\mu = \bigoplus_{i=1}^{n} c_i\, \mu(E \wedge C_i) = \bigoplus_{i=1}^{n} c_i\, m^*(E \wedge C_i) = m^*_f(E) = \tilde{\mu}_f(E).$$

In the case when f is NMF we use a sequence

$$(f_n)_{n\in\mathbb{N}} : f_n \leq f_{n+1} \text{ and } \lim_{n\to\infty} f_n = f.$$

By using Lemma 3.3.1 and the fact, that for SNMF the equality $\mu_{f_n} = \tilde{\mu}_{f_n}$ is true, we get:

$$\mu_f(E) = \int_E f\, d\mu = Sup\{\int_E f_n\, d\mu \mid n \in \mathbb{N}\} = Sup\{\mu_{f_n}(E) \mid n \in \mathbb{N}\} =$$

$$= Sup\{\tilde{\mu}_{f_n}(E) \mid n \in \mathbb{N}\} = m_f^*(E) = \tilde{\mu}_f(E).$$

□

This result gives us an opportunity to define an L-fuzzy valued integral by applying the following formula:

$$\int_E f\, d\mu = \tilde{\mu}_f(E),\ E \in \Sigma.$$

Summarizing, we want to mention that both ways of defining the L-fuzzy valued integral have their own motivation. The first approach has a theoretical foundation, while the second approach gives us an opportunity to simplify calculations of the L-fuzzy valued integral.

3.4 Integration over measurable fuzzy sets

In this section we suggest a method of calculation of the fuzzy valued integral over a measurable fuzzy set E in the case when $L = [0,1]$ and E is NMF (i.e. E is measurable with respect to σ-algebra Φ). The main idea of the method is based on the following reasoning. The fuzzy set we want to integrate over can be viewed as a non-negative function. Let us assume that this function is measurable with respect to σ-algebra Φ. It is known that every non-negative measurable function can be presented as a limit of a non-decreasing sequence of SNMF. Obviously, every fuzzy set that is SNMF can be presented as the union of T_M-disjoint fuzzy sets from the class $\wp$. And the L-fuzzy valued integral over an element from the class $\wp$ can be easily calculated.

This observation gives a reason for the following theorem.

Theorem 3.4.1. *If $E : X \to [0,1]$ is a measurable function with respect to σ-algebra Φ, then fuzzy set E is measurable with respect to T_M-tribe Σ.*

We describe the method gradually depending on the type of a fuzzy set E: first considering the case when E is an element of the class $\wp$, then extend it to the case when E is SNMF or a finite union of elements from the class $\wp$ and, finally, the case when E is NMF.

3.4.1 Integration over $A(M,\alpha)$

To show that for all $A(M,\alpha) \in \wp$ it holds

$$\int_{A(M,\alpha)} f d\mu = z(\int_M f\, d\nu, \alpha),$$

we use some special properties of the addition of fuzzy numbers $z(a,\alpha)$ described in subsection 1.4.1.
For $f = \sum_{i=1}^{n} c_i \chi_{C_i}$ we get

$$\int_{A(M,\alpha)} \sum_{i=1}^{n} c_i \chi_{C_i} d\mu = \bigoplus_{i=1}^{n} c_i\, \mu(C_i \wedge A(M,\alpha)) =$$

$$= \bigoplus_{i=1}^{n} c_i\, z(\nu(M \cap C_i), \alpha) = z(\sum_{i=1}^{n} c_i\, \nu(M \cap C_i), \alpha) = z(\int_M f d\nu, \alpha).$$

In the case when f is NMF we have

$$\int_{A(M,\alpha)} f\, d\mu = Sup\{z(\int_M g\, d\nu, \alpha) \mid g \leq f \text{ and } g \text{ is SNMF}\} =$$

$$= z(sup\{\int_M g\, d\nu, \alpha \mid g \leq f \text{ and } g \text{ is SNMF}\}, \alpha) = z(\int_M f d\nu, \alpha).$$

3.4.2 Integration over SNMF E

If E is SNMF then $E(\mathbb{R}) = \{\alpha_1, ..., \alpha_n\}$. We assume that

$$\alpha_1 > \alpha_2 > ... > \alpha_n \text{ and } M_i = E^{-1}(\alpha_i),\ i = 1, ..., n.$$

Then

- $i \neq j \Rightarrow M_i \cap M_j = \emptyset$;

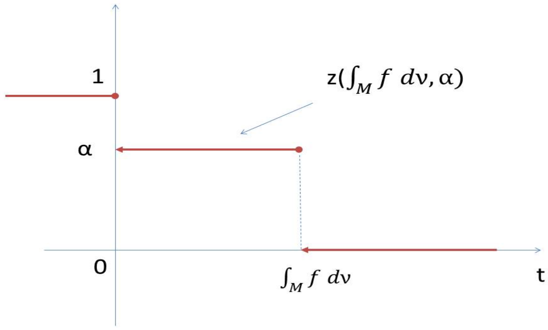

Figure 3.1: Integral $\int\limits_{A(M,\alpha)} f\,d\mu$

- $\bigcup\limits_{i=1}^{n} M_i = \mathbb{R}$;
- $E = \bigvee\limits_{i=1}^{n} A(M_i, \alpha_i)$;
- $E^{\alpha_i} = \bigcup\limits_{j=1}^{i} M_j$, where E^{α_i} is the α_i-cut of fuzzy set E .

Taking into account the property of addition of fuzzy numbers:

$$(\bigoplus_{i=1}^{n} z(a_i,\alpha_i))(t) = \begin{cases} 1, t \le 0, \\ \alpha_1,\ 0 < t \le a_1, \\ \ldots \\ \alpha_{i+1},\ a_1 + \ldots + a_i < t \le a_1 + \ldots + a_{i+1}, \\ \ldots \\ 0,\ t > a_1 + \ldots + a_n, \end{cases}$$

we obtain

$$\int_E f\,d\mu = \bigoplus_{i=1}^{n} \int\limits_{E(\alpha_i,M_i)} f\,d\mu = \bigoplus_{i=1}^{n} z(\int_{M_i} f d\nu, \alpha_i) =$$

$$= \begin{cases} 1, \ t \leq \int\limits_{M_1} f d\nu, \\ \dots \\ \alpha_i, \ \sum\limits_{j=1}^{i} \int\limits_{M_j} f d\nu < t \leq \sum\limits_{j=1}^{i+1} \int\limits_{M_j} f d\nu, \\ \dots \\ 0, \ t > \sum\limits_{j=1}^{n} \int\limits_{M_j} f d\nu, \end{cases} = \begin{cases} 1, \ t \leq \int\limits_{E^{\alpha_1}} f d\nu, \\ \dots \\ \alpha_i, \ \int\limits_{E^{\alpha_i}} f d\nu < t \leq \int\limits_{E^{\alpha_{i+1}}} f d\nu, \\ \dots \\ \alpha_n, \ \int\limits_{E^{\alpha_{n-1}}} f d\nu < t \leq \int\limits_{E^{\alpha_n}} f d\nu, \\ 0, \ otherwise. \end{cases}$$

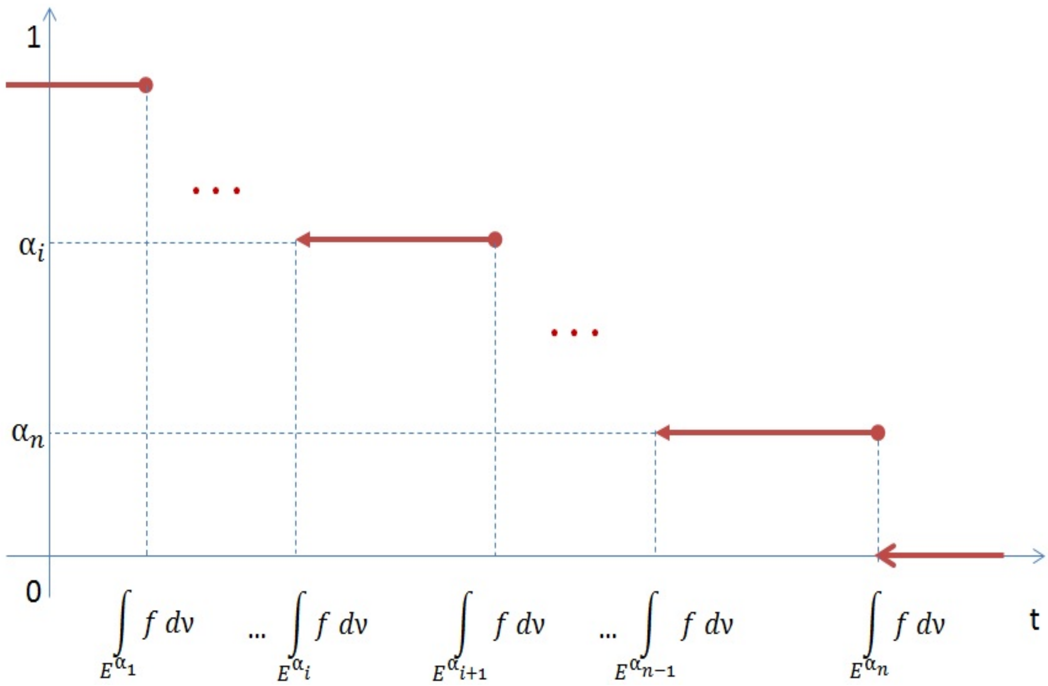

Figure 3.2: Integral $\int\limits_E f \, d\mu$, where E is SNMF

3.4.3 Integration over NMF E

As was already mentioned every NMF E can be presented as the limit of a non-decreasing sequence of SNMF. To describe this sequence we use the same logic as in the previous subsection. Let us take a sequence $(E_n)_{n\in\mathbb{N}}$ such as:

- $E_n(\mathbb{R}) = \{\alpha_1^n, ..., \alpha_{k_n}^n\}, \ n \in \mathbb{N};$
- $\alpha_i^n > \alpha_{i+1}^n, \ \ i = 1, ..., k_n - 1, \ n \in \mathbb{N};$

- $M_1^n = \{x \mid E(x) = \alpha_1^n\}$,
  $M_i^n = \{x \mid \alpha_i^n \leq E(x) < \alpha_{i-1}^n\},\ i = 2, ..., k_n,,\ n \in \mathbb{N}$;
- $E_n = \bigvee_{i=1}^{k_n} E(\alpha_i^n, M_i^n),\ n \in \mathbb{N}$;
- $E = \bigvee_n E_n$.

Denoting

$$I = \int_E f\, d\mu \text{ and } I_n = \int_{E_n} f\, d\mu$$

we get

$$I = Sup\{\int_{E_n} f\, d\mu \mid n \in \mathbb{N}\} = Sup\{I_n \mid n \in \mathbb{N}\}.$$

From the last equality we can get an approximate value of I by fixing n. Obviously, the integral accuracy in this case will be dependent on n .

Chapter 4

Applications of L-fuzzy valued integral in approximation theory

4.1 L-fuzzy valued norm

For a given linear space Y by the analogy with the classical case we consider the concept of a norm taking values in $R_+(L)$.

Definition 4.1.1.
An L-fuzzy valued norm on a linear space Y is a function $\|\cdot\| : Y \to \mathbb{R}_+(L)$ with the following properties: for all $r \in \mathbb{R}$ and all $y, y_1, y_2 \in Y$ it holds

- $\|y\| = z(0, 1_L) \Leftrightarrow y = 0_Y$,
- $\|ry\| = |r|\|y\|$,
- $\|y_1 + y_2\| \leq \|y_1\| \oplus \|y_2\|$.

For the space (A, Φ, ν) with a finite measure ν and $A \in \Phi$ we denote by F the linear space of all real valued functions that are integrable over A with respect to the measure ν. We suppose that F_0 is the subspace of F that contains all functions which are equal to 0 almost everywhere. Then $\mathscr{L}_1(A, \Phi, \nu)$ is the factor space F/F_0. We do not distinguish the functions of the space $\mathscr{L}_1(A, \Phi, \nu)$ that are equal almost everywhere with respect to the measure ν. The norm of a function $f \in \mathscr{L}_1(A, \Phi, \nu)$ is defined with

$$\|f\|_\nu = \int_A |f| \, d\nu.$$

Generalizing to the fuzzy case we use an L-set E instead of a set A and consider the space $\mathscr{L}_1(E,\Sigma,\mu)$ equipped with the L-fuzzy valued norm defined by the formula:

$$\|f\|_\mu = \int_E |f|\ d\mu,$$

where μ is an L-fuzzy valued measure and $E \in \Sigma$. We denote by $\mathscr{L}_1(E,\Sigma,\mu)$ the space of all L-fuzzy integrable over E real valued functions.
It is easy to show that the function

$$\|\cdot\|_\mu : \mathscr{L}_1(E,\Sigma,\mu) \to \mathbb{R}_+(L)$$

defined above satisfies the conditions of an L-fuzzy valued norm:

- $\|f\|_\mu = z(0,1_L)$ iff f is equal to 0 almost everywhere,
- $\|rf\|_\mu = \int_E |rf|\ d\nu = |r| \int_E |f|\ d\nu = |r|\|f\|_\mu$,
- $\|f_1+f_2\|_\mu = \int_E |f_1+f_2|\ d\mu \leq \int_E (|f_1|+|f_2|)\ d\mu =$
  $= \int_E |f_1|\ d\mu \oplus \int_E |f_2|\ d\mu = \|f_1\|_\mu \oplus \|f_2\|_\mu.$

4.2 Function approximation error on L-sets

For problems which can be solved only approximately the notion of the error of a method of approximation plays the fundamental role. A classical task of the approximation theory is to estimate the error of approximation for a given class of real valued functions defined on a crisp set. For us it seems natural to consider the case when the set we interpolate over is an L-set, meaning that our interest is more focused at some parts of the set we approximate over and maybe not so much important on the rest part of the set.

In order to estimate the quality of approximation on an L-fuzzy set, we need an appropriate L-fuzzy analogue of a norm. In this section we apply the L-fuzzy norm introduced above to investigate the error of approximation on an L-fuzzy set E of a real valued function f.

4.2.1 Theoretical background

Let us suppose that $E \in \Sigma$ and $f \in \mathscr{L}_1(suppE,\Phi,\nu)$. We consider a method of approximation described by

$$\mathscr{A} \colon \mathscr{L}_1(suppE,\Phi,\nu) \to \mathscr{U},$$

where $\mathscr{U} \subset \mathscr{L}_1(suppE, \Phi, \nu)$ is a finite-dimensional space of functions used for approximation. For example, it could be a space of polynomials or splines.

Definition 4.2.1. *The error of approximation $\mathscr{A}$ of a function f on an L-fuzzy set E is defined as follows:*

$$e(f, \mathscr{A}, E) = \|f - \mathscr{A}f\|_\mu.$$

Notice that the error of approximation in this case is characterized by an L-fuzzy real number which is obtained as the L-fuzzy valued integral over E. The introduced concept allows us to develop the well known idea of an optimal error method of approximation for this case. It is the method whose error is the infimum of the errors of all methods for a given problem characterized by L-fuzzy numbers.

Definition 4.2.2. *A method $\mathscr{A}_{OE}$ is called an optimal error method iff*

$$e(f, \mathscr{A}_{OE}, E) =$$

$$= Inf\ \{e(f, \mathscr{A}, E) \mid \mathscr{A} \colon \mathscr{L}_1(suppE, \Phi, \nu) \to \mathscr{U}\}.$$

This approach gives us a possibility to consider some extremal problems of approximation on L-fuzzy sets in the context of [1], [2], but it is not the principal aim of this paper. Our intention is to show how the most optimal decision can be taken when comparing several methods based on the error value.

4.2.2 Numerical example

In this subsection we illustrate with some numerical examples the dependence of the error $e(f, \mathscr{A}, E)$ on method $\mathscr{A}$ and set E. We suppose that $L = [0,1]$, $X = [0,1]$ and ν is the Lebesgue measure, and we consider the errors of approximation of the given function

$$f = \frac{1}{1+25x^2}$$

(the Runge example) by two methods:

- approximation $\mathscr{A}_1$ by the Lagrange interpolation polynomial of degree 10 with respect to the uniform mesh on $[0,1]$,
- approximation $\mathscr{A}_2$ by the interpolation natural cubic spline with respect to the same uniform mesh on $[0,1]$,

on two different L-sets E_1 and E_2:

$$E_1(x) = \begin{cases} 1, & x \in [0, 0.2], \\ 1.25(1-x), & x \in [0.2, 1], \end{cases} \qquad E_2(x) = \begin{cases} 1.25x, & x \in [0, 0.8], \\ 1, & x \in [0.8, 1]. \end{cases}$$

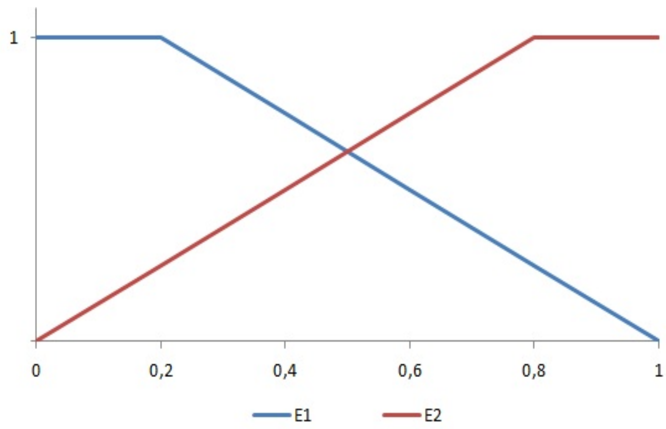

Figure 4.1: The graphs of L-sets E_1 and E_2

The errors $e(f, \mathscr{A}_j, E_i)$ of approximation of the function f on the L-set E_i by the method $\mathscr{A}_j$, $i = 1, 2$, $j = 1, 2$, are presented in the following table. Take into account that in the table we use the notation:

$$e(f, \mathscr{A}_j, E_i)(t) = \alpha.$$

$e(f, \mathscr{A}_1, E_1)$		$e(f, \mathscr{A}_1, E_2)$		$e(f, \mathscr{A}_2, E_1)$		$e(f, \mathscr{A}_2, E_2)$	
t	α	t	α	t	α	t	α
$6.3508 \cdot 10^{-4}$	0.0	$6.3508 \cdot 10^{-4}$	0.0	$28.2092 \cdot 10^{-4}$	0.0	$28.2092 \cdot 10^{-4}$	0.0
$6.2584 \cdot 10^{-4}$	0.1	$0.9613 \cdot 10^{-4}$	0.1	$28.1501 \cdot 10^{-4}$	0.1	$10.4002 \cdot 10^{-4}$	0.1
$6.2388 \cdot 10^{-4}$	0.2	$0.3556 \cdot 10^{-4}$	0.2	$28.1325 \cdot 10^{-4}$	0.2	$4.3225 \cdot 10^{-4}$	0.2
$6.2302 \cdot 10^{-4}$	0.3	$0.2124 \cdot 10^{-4}$	0.3	$28.1258 \cdot 10^{-4}$	0.3	$1.6781 \cdot 10^{-4}$	0.3
$6.2245 \cdot 10^{-4}$	0.4	$0.1658 \cdot 10^{-4}$	0.4	$28.1216 \cdot 10^{-4}$	0.4	$0.6252 \cdot 10^{-4}$	0.4
$6.2193 \cdot 10^{-4}$	0.5	$0.1467 \cdot 10^{-4}$	0.5	$28.1087 \cdot 10^{-4}$	0.5	$0.2681 \cdot 10^{-4}$	0.5
$6.2141 \cdot 10^{-4}$	0.6	$0.1381 \cdot 10^{-4}$	0.6	$28.0879 \cdot 10^{-4}$	0.6	$0.1353 \cdot 10^{-4}$	0.6
$6.2079 \cdot 10^{-4}$	0.7	$0.1334 \cdot 10^{-4}$	0.7	$28.0013 \cdot 10^{-4}$	0.7	$0.1059 \cdot 10^{-4}$	0.7
$6.1984 \cdot 10^{-4}$	0.8	$0.1296 \cdot 10^{-4}$	0.8	$27.8377 \cdot 10^{-4}$	0.8	$0.0953 \cdot 10^{-4}$	0.8
$6.1775 \cdot 10^{-4}$	0.9	$0.1251 \cdot 10^{-4}$	0.9	$27.4130 \cdot 10^{-4}$	0.9	$0.0860 \cdot 10^{-4}$	0.9
$6.0979 \cdot 10^{-4}$	1.0	$0.1173 \cdot 10^{-4}$	1.0	$25.6889 \cdot 10^{-4}$	1.0	$0.0816 \cdot 10^{-4}$	1.0

Let us note that

$$suppE_1 = suppE_2 = [0,1]$$

and

$$e(f, \mathscr{A}_1, [0,1]) = 6.3509 \cdot 10^{-4},$$

$$e(f, \mathscr{A}_2, [0,1]) = 28.2092 \cdot 10^{-4},$$

but it is easy to see that the error of approximation $\mathscr{A}_j$, $j = 1,2$, on the set E_2 is essentially less than the error on the set E_1.

To compare both L-fuzzy values one can use Figure 4.2 left chart for approximation $\mathscr{A}_1$ (i.e. by the Lagrange interpolation polynomial) and Figure 4.2 right chart for approximation $\mathscr{A}_2$ (i.e. by the interpolation natural cubic spline).

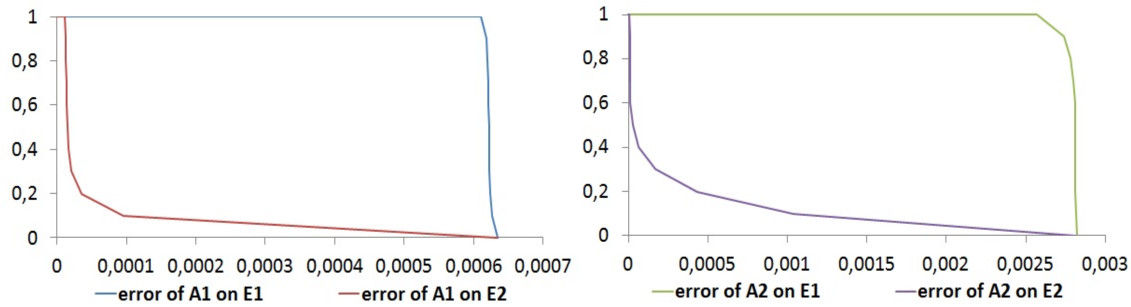

Figure 4.2: The graphs of errors $\alpha = e(f, \mathscr{A}_1, E_1)(t)$, $\alpha = e(f, \mathscr{A}_1, E_2)(t)$, $\alpha = e(f, \mathscr{A}_2, E_1)(t)$ and $\alpha = e(f, \mathscr{A}_2, E_2)(t)$

It is also interesting to see how both approximations $\mathscr{A}_1$ and $\mathscr{A}_2$ performing on the L-set E_2. As we can see from Figure 4.3 there is no obvious preference to any of two

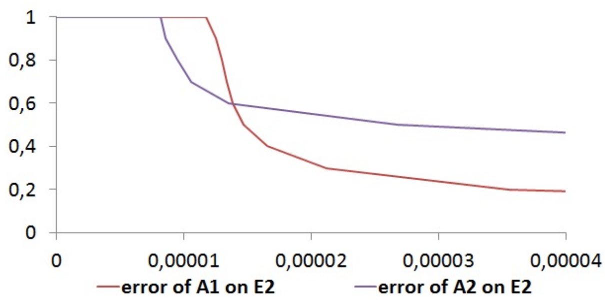

Figure 4.3: The graphs of errors $\alpha = e(f, \mathscr{A}_1, E_2)(t)$ and $\alpha = e(f, \mathscr{A}_2, E_2)(t)$

approximations, because they perform different on different levels.

Summarizing, we want to highlight that approximation error defined as an L-fuzzy number gives us opportunity to develop a strategy for choosing most optimal approximation algorithm on an L-set. As was illustrated in the previous example avoiding the "fuzziness" of the set we interpolate over and comparing the errors on the support of the L-set, we can fail with the choice of the best strategy.

4.3 Error of approximation on L-sets for classes of functions

As shown in the previous section, applying L-fuzzy valued integral to define the approximation error gives us a good tool to compare efficiency of different approximation methods and helps us to make more accurate decisions when choosing the most optimal approximation way on an L-set. Despite the numerical example 4.2.2 described above, when it comes to real life our knowledge about the function we want to approximate is rather limited, with some values maybe. That is why for us it seems natural to consider estimation of the approximation error on classes of functions.

4.3.1 Theoretical background

In 4.2.1 the space $\mathscr{L}_1$ was mentioned. Now let us consider space $\mathscr{L}_p = \mathscr{L}_p[0,1]$, where $1 \leq p \leq \infty$ with the norm $\|\cdot\|_p$ defined as follows:

$$\|f\|_p = \left(\int_0^1 |f(x)|^p dx\right)^{\frac{1}{p}} \text{ , where } 1 \leq p < \infty \text{ ,}$$

and

$$\|f\|_p = \sup_{x\in[0,1]} |f(x)| \text{ , where } p = \infty \text{ .}$$

Also we consider the space $\mathscr{L}_p^n$ of functions f that satisfy the following conditions:

- there exists $(n-1)$ derivative $f^{(n-1)}$ and it is absolutely continuous on $[0,1]$,
- $f^{(n)} \in \mathscr{L}_p$.

In other words, $\mathscr{L}_p^n$ is the set of n degree indefinite integrals from functions $f \in \mathscr{L}_p$.

Particularly, we observe the class containing functions $f \in \mathscr{L}_p^n$ with bounded derivatives. We denote by $KW_p^n (n = 1, 2, ...; 1 \leq p \leq \infty)$ the class of all functions $f \in \mathscr{L}_p^n$ satisfying

$$\|f^{(n)}\|_p \leq K,\ K \in \mathbb{R}_+.$$

We know that every function $f \in \mathscr{L}_p$ can be represented as follows

$$f(x) = \sum_{i=0}^{n-1} \frac{1}{i!}\, f^{(i)}(0)\, x^i + \frac{1}{(n-1)!} \int_0^1 f^{(n)}(u)(x-u)_+^{n-1} du.$$

Let us denote

$$p_{n-1}(x) = \sum_{i=0}^{n-1} \frac{1}{i!}\, f^{(i)}(0)\, x^i$$

and

$$r_{n-1}(x) = \frac{1}{(n-1)!} \int_0^1 f^{(n)}(u) \varphi_{n-1}(x,u) du,$$

where

$$\varphi_{n-1}(x,u) = (x-u)_+^{n-1} = \begin{cases} (x-u)^{n-1}, & x \geq u, \\ 0, & u > x. \end{cases}$$

Hence,

$$f(x) = p_{n-1}(x) + r_{n-1}(x),$$

where

- p_{n-1} is the $(n-1)$-degree Taylor polynomial of f centred at 0,
- r_{n-1} is an integral form of the remainder.

Now we want to estimate the error of approximation $\mathscr{A}$ of function f. We do the following assumptions regarding the choice of approximation method $\mathscr{A}$:

- for all $q \in P_{n-1}$ we have $\mathscr{A}q = q$, where P_{n-1} is the class of all polynomials with degree at most $(n-1)$;
- approximation $\mathscr{A}$ is linear;
- for $r(x) = \int_0^1 g(u)\, h(x,u)\, du$ it holds

  $$(\mathscr{A}r)(x) = \int_0^1 g(u)\, (\mathscr{A}h(x,u)) du,$$

  where approximation $\mathscr{A}$ applied only to argument x of function h.

Taking into account these assumptions we obtain:

$$f - \mathscr{A}f = (p_{n-1} + r_{n-1}) - \mathscr{A}(p_{n-1} + r_{n-1}) =$$

$$= (p_{n-1} - \mathscr{A}p_{n-1}) + (r_{n-1} - \mathscr{A}r_{n-1}) = r_{n-1} - \mathscr{A}r_{n-1}.$$

Coming back to initial notations:

$$f(x) - (\mathscr{A}f)(x) = r_{n-1}(x) - (\mathscr{A}r_{n-1})(x) =$$

$$= \frac{1}{(n-1)!}\int_0^1 f^{(n)}(u)\varphi_{n-1}(x,u)\,du - \frac{1}{(n-1)!}\int_0^1 f^{(n)}(u)(\mathscr{A}\varphi_{n-1}(x,u))\,du =$$

$$= \frac{1}{(n-1)!}\int_0^1 f^{(n)}(u)(\varphi_{n-1}(x,u) - \mathscr{A}\varphi_{n-1}(x,u))\,du$$

We denote by $U_{n-1}(x,u)$ the kernel of the integral representation of the error $f - \mathscr{A}f$ and define it as follows:

$$U_{n-1}(x,u) = \frac{\varphi_{n-1}(x,u) - \mathscr{A}\varphi_{n-1}(x,u)}{(n-1)!}.$$

Finally, we get

$$f(x) - (\mathscr{A}f)(x) = \int_0^1 f^{(r)}(u)U_{n-1}(x,u)\,du.$$

Now considering the error on an L-set E we obtain:

$$\int_E |f - \mathscr{A}f|\,d\mu \leq \int_E (\int_0^1 |f^{(n)}(u)||U_{n-1}(x,u)|\,du)\,d\mu.$$

Having this estimation for the approximation error we can observe how it will be calculated for particular classes of functions.

- In the case when $f \in KW_1^n$ the norm of $f^{(n)}$ is bounded as follows

  $$\|f^{(n)}\|_1 = \int_0^1 |f^{(n)}(u)|\,du \leq K$$

  and then

  $$\int_E |f - \mathscr{A}f|\,d\mu \leq \int_E (\int_0^1 |f^{(n)}(u)||U_{n-1}(x,u)|\,du)\,d\mu \leq$$

$$\leq \int_E \left(\int_0^1 |f^{(n)}(u)| \sup_{u\in[0,1]} |U_{n-1}(x,u)| \, du \right) d\mu =$$

$$= \int_E \left(\sup_{u\in[0,1]} |U_{n-1}(x,u)| \int_0^1 |f^{(n)}(u)| \, du \right) d\mu \leq K \int_E \sup_{u\in[0,1]} |U_{n-1}(x,u)| \, d\mu.$$

Hence,

$$\|f - \mathscr{A}f\|_\mu \leq K \int_E \sup_{u\in[0,1]} |U_{n-1}(x,u)| \, d\mu.$$

- In the case when $f \in KW_\infty^n$ the norm of $f^{(n)}$ is bounded as follows

$$\|f^{(n)}\|_\infty = \sup_{u\in[0,1]} |f^{(n)}(u)| \leq K$$

and then

$$\int_E |f - \mathscr{A}f| \, d\mu \leq \int_E \left(\int_0^1 |f^{(n)}(u)||U_{n-1}(x,u)| \, du \right) d\mu \leq$$

$$\leq K \int_E \left(\int_0^1 |U_{n-1}(x,u)| \, du \right) d\mu.$$

Hence,

$$\|f - \mathscr{A}f\|_\mu \leq K \int_E \left(\int_0^1 |U_{n-1}(x,u)| \, du \right) d\mu.$$

4.3.2 Numerical example

In this subsection we illustrate by some numerical examples how to use the technique described above. We consider classes KW_1^n and KW_∞^n when $n = 1$, i.e. the classes of functions such that the first derivative is bounded: KW_1^1 and KW_∞^1.

We examine the approximation $\mathscr{A}$ by polygons (i.e. the first degree splines) with respect to a mesh $\{x_0, x_1, ..., x_k\}$ on $[0,1]$ over L-set E defined as follows:

$$E(x) = \begin{cases} 1 - x, \; x \in [0,1], \\ 0, \; \text{otherwise}. \end{cases}$$

In this case

$$\varphi_0(x,u) = \begin{cases} 1, \; x \geq u, \\ 0, x < u. \end{cases}$$

And if $x_{i-1} < x < x_i$ then

$$U_0(x,u) = \begin{cases} \frac{x_i - x}{x_i - x_{i-1}}, & x_{i-1} < u < x, \\ -\frac{x - x_{i-1}}{x_i - x_{i-1}}, & x < u < x_i, \\ 0, & u \notin [x_{i-1}, x_i]. \end{cases}$$

Class KW^1_∞

Denoting

$$z_\infty = \int_E \left(\int_0^1 |U_0(x,u)|\, du \right) d\mu$$

for $f \in KW^1_\infty$ we have

$$\|f - \mathscr{A}f\|_\mu \leq K\, z_\infty.$$

Considering the case when $x_{i-1} < x < x_i$ we obtain

$$\int_0^1 |U_0(x,u)|\, du = \int_{x_{i-1}}^{x_i} |U_0(x,u)|\, du = \int_{x_{i-1}}^{x} \frac{x_i - x}{x_i - x_{i-1}}\, du + \int_x^{x_i} \frac{x - x_{i-1}}{x_i - x_{i-1}}\, du =$$

$$= 2\frac{(x_i - x)(x - x_{i-1})}{x_i - x_{i-1}}.$$

In the case of the uniform mesh $\{x_0, x_1, ..., x_k\}$ on $[0,1]$ for every $x \in [0,1]$ denoting $h = x_i - x_{i-1}$ we have

$$\int_0^1 |U_0(x,u)|\, du = 2h \left\{ \frac{x}{h} \right\} \left(1 - \left\{ \frac{x}{h} \right\} \right),$$

going forward we mean by $[r]$ the integer part and by $\{r\}$ the fractional part of $r \in \mathbb{R}$. This integral for the uniform mesh with 11 (i.e. $k = 10$) knots is plotted on the Figure 4.4.

The next step is to integrate this function over the L-set E. Applying the calculation method described in subsection 3.4.2 for all $\alpha \in [0,1]$ we obtain

$$z_\infty^{-1}(\alpha) = \frac{h^2}{3} \left(\left[\frac{1-\alpha}{h} \right] + \left\{ \frac{1-\alpha}{h} \right\}^2 (3 - 2\left\{ \frac{1-\alpha}{h} \right\}) \right).$$

Now by choosing α according to any mesh on the $[0,1]$ we can get the integral value as accurate as we want. For example, taking the uniform mesh of $[0,1]$ with step 0.01:

$$\alpha_0 = 1, \alpha_1 = 0.99, \alpha_2 = 0.98, ..., \alpha_{100} = 0,$$

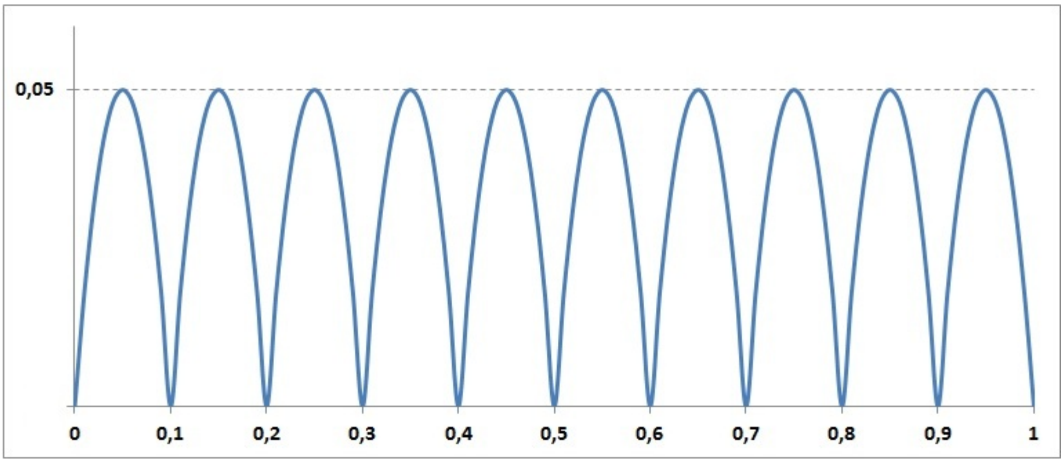

Figure 4.4: The graph of integral $\int_0^1 |U_0(x,u)|\,du$

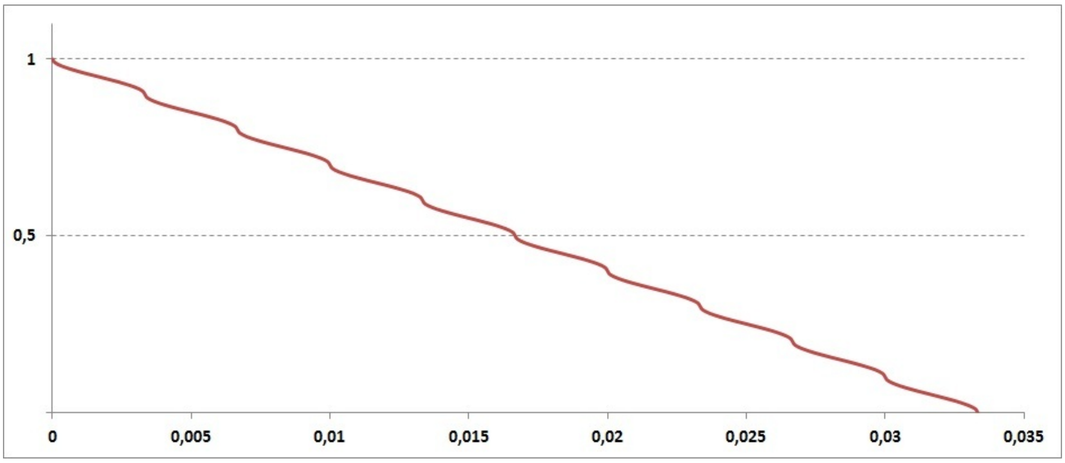

Figure 4.5: The graph of z_∞

we obtain fuzzy real number z_∞ plotted on Figure 4.5.
Changing from the uniform mesh to the following:
$x_0 = 0, x_1 = 0.02, x_2 = 0.03, x_3 = 0.05, x_4 = 0.1, x_5 = 0.15,$
$x_6 = 0.25, x_7 = 0.35, x_8 = 0.55, x_9 = 0.75, x_{10} = 1.00,$
we, obviously, obtain a different approximation error bound. On Figure 4.6 both approximation errors bounds are plotted with uniform and non-uniform meshes.

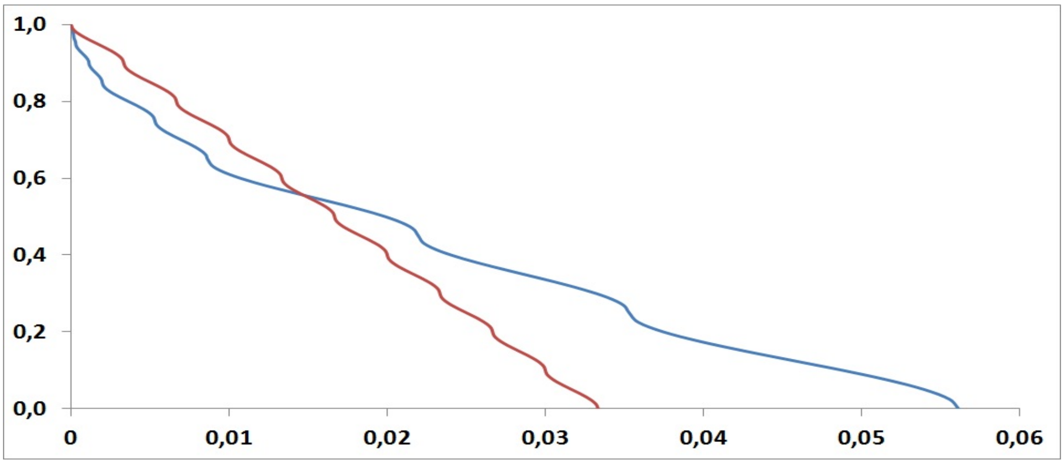

Figure 4.6: Comparison of the error bounds with the uniform and non-uniform meshes

In the crisp case we would give a preference for the method with the uniform mesh, while in the fuzzy case the decision is not that obvious, meaning that on the higher levels ($\alpha > 0.6$) the method with the non-uniform mesh performing better.

Class KW_1^1

Taking into consideration that $U_0(x,u) = 0$ when $x \in [x_{i-1}, x_i]$, $u \in [x_{j-1}, x_j]$ and $i \neq j$, for class KW_1^1 we can get more accurate error estimation by the following reasoning. Let us denote $E_i = E \wedge [x_{i-1}, x_i], i = 1, ..., k$. Then

$$\|f - \mathscr{A}f\|_\mu = \bigoplus_{i=1}^{k} \int_{E_i} \left(\int_0^1 |f'(u)||U_0(x,u)| \, du \right) d\mu =$$

$$= \bigoplus_{i=1}^{k} \int_{E_i} \left(\int_{x_{i-1}}^{x_i} |f'(u)||U_0(x,u)| \, du \right) d\mu \leq \bigoplus_{i=1}^{k} \int_{E_i} \left(\sup_{u \in [x_{i-1}, x_i]} |U_0(x,u)| \int_{x_{i-1}}^{x_i} |f'(u)| \, du \right) d\mu =$$

$$= \bigoplus_{i=1}^{k} \int\limits_{x_{i-1}}^{x_i} |f'(u)|du \int\limits_{E_i} \sup_{u\in[x_{i-1},x_i]} |U_0(x,u)|d\mu \leq$$

$$\leq Sup\{\int\limits_{E_j} \sup_{u\in[x_{j-1},x_j]} |U_0(x,u)|d\mu \mid j=1,..,k\} \sum_{i=1}^{k} \int\limits_{x_{i-1}}^{x_i} |f'(u)|du =$$

$$= Sup\{\int\limits_{E_j} \sup_{u\in[x_{j-1},x_j]} |U_0(x,u)|d\mu \mid j=1,..,k\} \int\limits_{0}^{1} |f'(u)|du \leq$$

$$\leq K\ Sup\{\int\limits_{E_j} \sup_{u\in[x_{j-1},x_j]} |U_0(x,u)|d\mu \mid j=1,..,k\}.$$

Now denoting

$$w_i = \int\limits_{E_i} \sup_{u\in[x_{j-1},x_j]} |U_0(x,u)|d\mu, i=1,...,k,$$

$$z_1 = Sup\{w_i | i=1,...,k\},$$

we have

$$\|f - \mathscr{A}f\|_\mu \leq K\ z_1.$$

Considering the case when $x_{i-1} < x < x_i$ we obtain

$$\sup_{u\in[x_{x-1},x_i]} |U_0(x,u)| = \begin{cases} \frac{x_i - x}{x_i - x_{i-1}}, & x_{i-1} < x < \frac{x_{i-1}+x_i}{2}, \\ -\frac{x - x_{i-1}}{x_i - x_{i-1}}, & \frac{x_{i-1}+x_i}{2} < x < x_i, \end{cases} .$$

This function is plotted on the following graph:

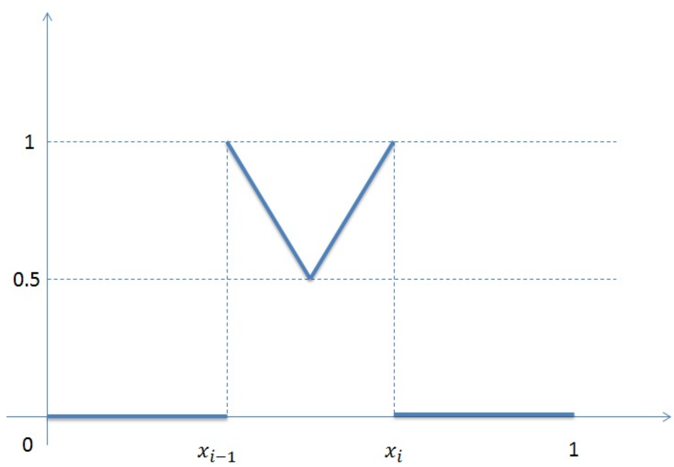

Figure 4.7: The graph of $\sup_{u\in[x_{i-1},x_i]} |U_0(x,u)|$

Taking integral over E_i we get:

$$w_i^{-1}(\alpha) = \int_{x_{i-1}}^{1-\alpha} \sup_{u \in [x_{i-1}, x_i]} |U_0(x,u)| dx, \text{ when } 1 - x_i \le \alpha \le 1 - x_{i-1}.$$

Hence,

$$w_i^{-1}(\alpha) = \frac{1-\alpha-x_{i-1}}{2}\left(2 - \frac{1-\alpha-x_{i-1}}{h_i}\right), \text{ when } x_{i-1} \le 1-\alpha \le \frac{x_{i-1}+x_i}{2},$$

and

$$w_i^{-1}(\alpha) = \frac{3}{4}h_i - \frac{x_i-(1-\alpha)}{2}\left(2 - \frac{x_i-(1-\alpha)}{h_i}\right), \text{ when } \frac{x_{i-1}+x_i}{2} \le 1-\alpha \le x_i.$$

It is easy to see that

$$w_i^{-1}(1-x_{i-1}) = 0 \text{ and } w_i^{-1}(1-x_i) = \frac{3}{4}h_i.$$

In the case when $t \notin [0, \frac{3}{4}h_i]$ we have

$$w_i(t) = \begin{cases} 1, & t \le 0, \\ 0, & t > \frac{3}{4}h_i. \end{cases}$$

To obtain value z_1 we take Sup of all fuzzy numbers $w_i,\ i = 1,...,k$.

These numbers as well as z_1 for uniform mesh with 11 knots are plotted on Figure 4.8 (the dashed lines for $w_i,\ i = 1,...,k$ and the even line for z_1).

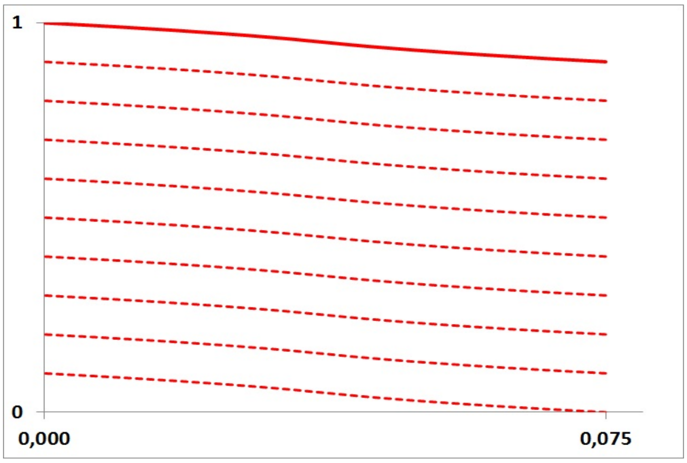

Figure 4.8: The graphs of w_i and z_1 in the case of the uniform mesh

Changing from the uniform mesh to the following:
$x_0 = 0, x_1 = 0.02, x_2 = 0.03, x_3 = 0.05, x_4 = 0.1, x_5 = 0.15,$

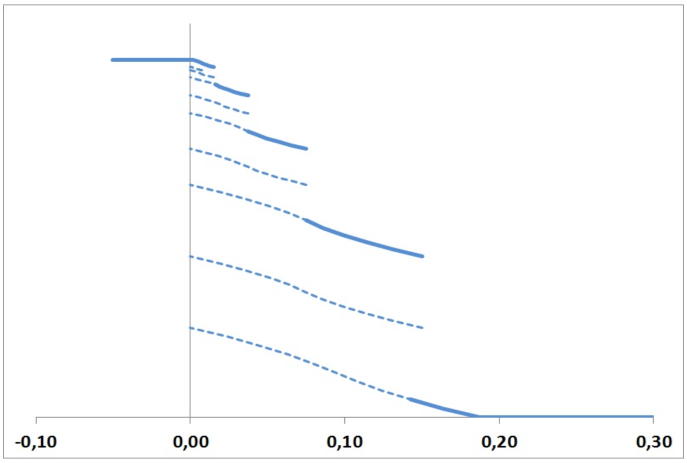

Figure 4.9: The graphs of w_i and z_1 in the case of non-uniform mesh

$x_6 = 0.25, x_7 = 0.35, x_8 = 0.55, x_9 = 0.75, x_{10} = 1.00,$
we obtain the integrals displayed on Figure 4.9.

To compare methods with the uniform and non-uniform meshes Figure 4.10 is presented. As we can see there is no obvious decision on what method is more preferable:

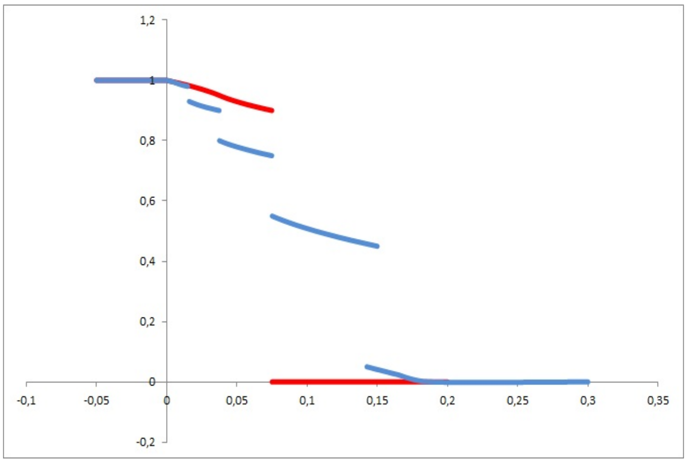

Figure 4.10: Comparison of the errors with the uniform and non uniform meshes

in the case when we are interested in points with the membership level close to 1 both methods have the same error bound, while on points with the membership degree in $[0.5, 0.98]$, obviously, the method with the non-uniform mesh is better, but on levels below 0.5 the error of approximation significantly lower for the method with the uniform mesh.

Conclusions

To conclude I would like to say that in my belief the overall goal of the thesis has been achieved:

- the concept of an L-fuzzy valued measure defined on T-tribe and taking values in the L-fuzzy real line is introduced, in the case when L is complete, completely distributive lattice,
- construction for an L-fuzzy valued measure is developed by extending the crisp finite measure,
- considered the concept of an L-fuzzy valued integral, the properties of integral are investigated and calculations methods described,
- some possible applications of an L-fuzzy valued integral in the approximation theory are suggested.

I believe that this research work is very relevant since it combines the L-structure study, which is traditional to Latvian mathematicians, as well as measure and integral theory further development. These studies could potentially contribute to Latvian researches and investigations of many valued structures and subsequently develop this perspective direction of theoretical mathematics.

List of conferences

International scientific conferences and congresses

C01 *11th International Conference on Fuzzy Set Theory and Applications (FSTA 2012)*, January 30 - February 3, 2012, Liptovsky Jan, Slovak Republic.

C02 *Joint 7th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT-2011) and Les Rencontres Francophones sur la Logique Floue et ses Applications (LFA-2011)*, July 18-22, 2011, Aix-Les-Bains, France.

C03 *6th International Summer School on Aggregation Operators (AGOP 2011)*, July 11-15, 2011, Benevento, Italy.

C04 *16th International Conference on Mathematical Modelling and Analysis (MMA 2011)*, May 25-28, 2011, Sigulda, Latvia.

C05 *10th International Conference on Applied mathematics (APLIMAT 2011)*, February 1-4, 2011, Bratislava, Slovak Republic.

C06 *15th International Conference on Mathematical Modelling and Analysis (MMA 2010)*, May 26-29, 2010, Druskininkai, Lithuania.

C07 *10th International Conference on Fuzzy Set Theory and Applications (FSTA 2010)*, February 1-5, 2010, Liptovsky Jan, Slovak Republic.

C08 *Joint 13th IFSA World Congress and 6th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT 2009)*, July 20-24, 2009, Lisbon, Portugal.

C09 *14th International Conference on Mathematical Modelling and Analysis (MMA 2009)*, May 27-30, 2009, Daugavpils, Latvia.

C10 *9th International Conference on Fuzzy Set Theory and Applications (FSTA 2008)*, February 4-8, 2008, Liptovsky Jan, Slovak Republic.

C11 *6th International Conference on Topological Algebras and Applications (ICTAA 2008)*, January 24-27, 2008, Tartu, Estonia.

C12 *5th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT 2007)*, Septemeber 11-14, 2007, Ostarva, Czech Republic.

Domestic scientific conferences

C13 *67th Conference of University of Latvia*, New scientist section, February 2, 2009, Riga, Latvia.

C14 *7th Conference of Latvian Mathematical Society*, April 18-19, 2008, Rezekne, Latvia.

C15 *66th Conference of University of Latvia*, New scientist section, February 21, 2008, Riga, Latvia.

C16 *65th Conference of University of Latvia*, New scientist section, February 8, 2007, Riga, Latvia.

C17 *6th Conference of Latvian Mathematical Society*, April 7-8, 2006, Liepaja, Latvia.

International seminars

C18 *A.Šostak Spring Workshop*, May 14, 2009, Riga, Latvia.

C19 *1st Czech-Latvian Seminar Advanced Methods in Soft Computing*, November 19-22, 2008, Trojanovice, Czech Republic.

C20 *International Seminar on Algebra and Applications*, May 4-6, 2007, Ratnieki, Latvia.

C21 *International Latvian-Finnish Seminar on Algebra and Topology*, November 8, 2006, Riga, Latvia.

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Author's publications

P1 V. Ruzha, S. Asmuss. A construction of a fuzzy valued measure based on minimum t-norm. In: *New Dimensions in Fuzzy Logic and Related Technologies. Proceedings of the 5th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT 2007)*, p. 175-178, Ostrava (Czech Republic), 2007. ISI CPCI-S

P2 V. Ruzha, S. Asmuss. A construction of an L-fuzzy valued measure of L-fuzzy sets. In: *Proceedings of the Joint 13th IFSA World Congress and 6th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT 2009)*, p. 1735-1739, Lisbon (Portugal), 2009. ISI CPCI-S

P3 V. Ruza. On an L-fuzzy valued integral with respect to an L-fuzzy valued T_M-measure. In: *Proceedings of the* 10^{th} *International Conference on Applied Mathematics (APLIMAT 2011)*, p. 605-614, Bratislava (Slovakia), 2011.

P4 V. Ruza. On an L-fuzzy valued integral with respect to an L-fuzzy valued T_M-measure. Paper is accepted for publication in the *Journal of Applied Mathematics*.

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