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Dissertation

Fuzzy order relations and monotone mappings: categorical constructions and applications in aggregation process

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IEGULDĪJUMS TAVĀ NĀKOTNĒ

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Abstract

Fuzzy ordered relations, which are in the center of research of this work, play a crucial role in many theoretical and applied areas of Fuzzy Mathematics. In our Dissertation we develop the theory of fuzzy ordered relations in two different, but internally closely related, directions. First, we construct an L-valued category whose objects are L-E-ordered sets. To reach this goal, we construct a crisp category whose objects are L-E-ordered sets and whose morphisms are order-preserving mappings (in a certain fuzzy sense) and investigate basic properties of this category. Then we fuzzify the constructed category and investigate fundamental properties of the resulting L-valued category. In the second part of the work we investigate the role of fuzzy orders in aggregation processes. Here we study the three main topics: involving fuzzy orders in the definition of the degree of monotonicity, pointwise aggregation and A-T-aggregation of fuzzy relations, and finally, applications of the developed methods in multi-objective linear programming problems.

MSC: 03E72, 94D05, 18A05, 18B35

Keywords: fuzzy order relation, fuzzy equivalence relation, L-valued relation, L-E-order relation, category, fuzzy category, L-valued category, aggregation function, aggregation process, aggregation of fuzzy relations, monotonicity, multi-objective linear programming.

Anotācija

Nestrikta sakārtojuma koncepcija, kas ieņem centrālo vietu mūsu darbā, spēlē nozīmīgu lomu gan teorētiskās Nestriktās Matemātikas jomā gan tās lietojumos. Disertācijā mēs attīstām nestrikto sakārtojumu teoriju divos, iekšēji saistītos, virzienos. Sākumā mēs konstruējam L-vērtīgu kategoriju, kuras objekti ir L-E-sakārtotas kopas. Lai sasniegtu šo mērķi, mēs konstruējam klasisku kategoriju, kuras objekti ir L-E-sakārtotas kopas un morfismi ir sakārtojuma saglabājošas funkcijas. Darbā mēs pētām konstruētas kategorijas pamatīpašības. Tad mēs fazificējam konstruēto kategoriju un pētām iegūtas L-vērtīgas kategorijas fundamentālas īpašības. Tālāk mēs pētām nestrikta sakārtojuma lomu agregācijas procesos. Šeit tiek izpētīti trīs temati: nestrikta sakārtojuma ievešana monotonitātes pakāpes definīcijai; nestriktu attiecību agregācija un tās lietojumi daudzkriteriālajā lineārā programmēšanā; nestriktu attiecību A-T-agregācija.

MSC: 03E72, 94D05, 18A05, 18B35

Atslēgvārdi: nestrikta sakārtojuma attiecība, nestrikta ekvivalences attiecība, L-vērtīga attiecība, L-E-sakārtojuma attiecība, kategorija, nestrikta kategorija, L-vērtīga kategorija, agregācijas funkcija, agregācijas process, nestriktu attiecību agregācija, monotonitāte, daudzkriteriāla lineāra programmēšana.

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Contents

Chapter 1

Introduction

The aim of this work is to make a contribution to the theory of fuzzy relations. To be more precise we are interested in the study of fuzzy order relation[1] both from theoretical point of view and from its possible practical applications. Actually, first we study fuzzy order relations and order-preserving functions in the frame of category theory. Speaking about the "practical point of view" we are interested how the introduction of fuzzy orders can help us to solve real-world problems. In this part we consider two questions: involving fuzzy orders for the definition of degree of monotonicity for aggregation functions and the problem of aggregation of fuzzy order relations. Although the second part also consists mainly of theoretical results the necessity of these investigations is motivated by the real examples of decision making problems.

Since the introduction of the concept of a fuzzy set by L. A. Zadeh [48] and its generalization to L-fuzzy set by J. A. Goguen [16] in the second half of the 20th century, fuzzy analogues of basic concepts of classical mathematics were introduced and investigated, fuzzy relations among them. Fuzzy equivalence relation and fuzzy order relation are the most important and mostly applied of all fuzzy relations. In this work we focus on the fuzzy orders, but as we will see later the definition of fuzzy order without having a fuzzy equivalence relation in the base does not make much sense (see [23], [5], [6]). First time the definitions of a fuzzy order relation and a fuzzy equivalence relation were introduced by L.A. Zadeh in 1971 [49] under the name of a fuzzy partial ordering and a similarity relation respectively. Various properties of fuzzy equivalence relations and

[1] Actually in the first part of the work we consider a more general case than the fuzzy order relation - L-E-order relation, where L is a cl-monoid and E is an L-valued equivalence. However in the Introduction for simplicity we use the notion of "fuzzy order relation".

fuzzy orderings were investigated in this initial paper, decomposition of fuzzy relations and extended version of Szpilrajns theorem are among them. Fifteen years later U. Höhle and N. Blanchard in their paper [23] proposed to involve fuzzy equivalence relation for the definition of fuzzy order (partial ordering). In this paper the main purpose of the authors was ”to improve results on fuzzy partial orderings obtained by Zadeh” and the main motivation was that ”an axiom of antisymmetry without a reference to a concept of equality is meaningless”. U. Bodenhofer in his papers [5], [6] also explained that the reflexivity and antisymmetry should be defined with respect to fuzzy equivalence relation and illustrated this statement by examples from the fuzzy logic and fuzzy set theory. Many other authors during the last forty years have contributed to the development of fuzzy order relations (see e.g. [35], [24],[12], [42]).

In 80's years of the last century bipolar fuzzy relations[2] where defined and studied (see [2]). The notion of bipolar fuzzy relation is a generalization of the idea of fuzzy relation and is a pair of fuzzy relations, namely a membership and a non-membership function, which represent positive and negative aspects of the given information. Another (and may be the most popular) generalization of fuzzy relations - interval-valued fuzzy relations - was introduced independently. However bipolar fuzzy relations are isomorphic to interval-valued fuzzy relations (see e.g. [13]). Although in our work we do not investigate any generalization of fuzzy order relations some results could be generalized for the above described cases. This direction could be considered also as an idea for the future research.

In the last years theoretical results in the theory of fuzzy relations were involved for solving practical problems (see e.g.[8]). Significant applications of fuzzy order relations can be found in connection with decision making problems (see e.g. [24]). For the decision making one of the main questions related to fuzzy order relations is the question of aggregation of fuzzy relations. In this field fundamentals works are [39], [38], [15].

The **structure** of the thesis is the following: short abstracts in English and Latvian, the contents of the work, Introduction (you are reading it at present), Preliminaries, two chapters with main results, the list of the conferences where results of the Dissertation were presented, the list of references, and the list of publications of the author. The main part of the work consists of two independent for the first look parts (Chapter 3 and Chapter 4). However they are connected by the fundamental and important notion

[2]Bipolar fuzzy relations could be also called as intuitionistic fuzzy relations, Atanassovs intuitionistic fuzzy relations or bifuzzy relations.

of fuzzy order relation. The first part (Chapter 3), where we study the category whose objects are sets with fuzzy orders, was inspired by the course of Category theory at the University of Bremen by Prof. David Holgate and the papers by Prof. Aleksanders Šostaks [44],[47]. This part of the work is more theoretical, since we do not consider here any applications for the real-world problems, but construct and study the structure of categories from the pure mathematical point of view.

The main notion in this part of the work is the notion of an L-valued category. The concept of an L-valued category was introduced in [43] and later was studied in a series of papers see, e.g. [44], [46], [47] etc. These papers contain also many examples of L-valued categories which appear in the natural way, by "fuzzifying" classical categories. For example an L-valued category whose objects are fuzzy sets, an L-valued category whose objects are fuzzy topologies, etc. Actually an L-fuzzy category is a triple $(\mathbb{C}, \mu, \omega)$ where $\mathbb{C}$ is an ordinary category with the class of objects $Ob(\mathbb{C})$ and the class of morphisms $Mor(\mathbb{C})$, and $\omega : Ob(\mathbb{C}) \to L$, $\mu : Mor(C) \to L$ are respectively L-subclasses of the class of objects and the class of morphisms, subjected to certain properties. The intuitive meaning of the values $\omega(X)$ and $\mu(f)$ are the measures to which X and f are respectively the object and the morphism of the corresponding category. Interesting examples of L-valued categories were also constructed in [41].

Chapter 3 consists of three sections. The aim of Section 3.1 is to study L-E-ordered sets, where L is a cl-monoid, from the L-valued category theory point of view. This means that starting from the small categories we use already the concept of an L-valued category, defining the degree to which each morphism is indeed the morphism in this L-valued category. An alternative approach is to study generalized ordered structures in the frame of categories enriched over a quantale Ω (or simply, Ω-categories), see e.g. [28], [29], [27].

In our work we use the notion of an L-E-order relation and construct an analogue of $\mathbf{POS}$ category (the category of Partially Ordered Sets). In our category objects are L-E-ordered sets and morphisms are order-preserving mappings (in a fuzzy sense). By constructing the category from small categories (small L-valued categories), we show that the category is constructed in a natural way from the point of view of L-valued category theory. To realize this goal we follow the construction of the crisp $\mathbf{POS}$ category. We continue by studying the structure of the constructed category. Namely, we consider some special morphisms, objects and standard constructions such as product and coproduct.

To construct and study a fuzzy analogue of **POS** category is not a new idea, but in the literature we have found only the cases where the objects are fuzzy ordered sets defined in the sense of Zadeh. Also we haven't found the systematical study of any fuzzy analogue of **POS** category. However in the literature there are works related to the study of the categories where the objects are sets with fuzzy equivalence revelations on them and morphisms are equivalence-preserving mappings, see [21], [22], [41]. Here we want also to mention the work by H. Lai and D. Zhang [26] where they studied interrelationship between fuzzy preordered sets, topological spaces, and fuzzy topological spaces.

Section 3.2 is devoted to the fuzzyfication of the constructed category, where we introduce an L-valued subclass of the class of all its morphisms as a mapping from the class of morphisms to a cl-monoid L and thus we obtain an L-valued category. The intuitive meaning of the value of this mapping for a given morphism is the degree to which this morphism is an order-preserving mapping. Therefore we obtain the L-valued category whose objects are L-E-ordered sets and morphisms are "potential" order-preserving mappings. Further we study also the structure of this L-valued category and consider some operations such as products and coproducts. To study such operations in the frame of an L-valued category we have to define the above described notions in the context of L-valued category theory, by doing this we also contribute to the development of this theory.

In Section 3.3 we also consider an L-valued analogue of **POS** category but in this case we use another definition of fuzzy equivalence relation and this influences the nature of fuzzy order relation. In this section apart from "potential" morphisms we have also "potential" objects by introducing an L-valued subclass of the class of all its objects as a mapping from the class of objects to a cl-monoid L. Thus we generalized the category obtained in the previous section.

The second part of the work (Chapter 4) was inspired by the conference FSTA2008, where different aspects of aggregation functions (operators) were discussed. Chapter 4 consists of four sections. The idea for Section 4.1 "Degree of monotonicity" was born studying the above discussed categories: considering the L-valued categories we define the degree to which each morphism is an order-preserving mapping or, say it in a different way, monotone mapping. Thus we applied this idea to characterize aggregation functions. Namely for a function we define the degree to which this function is a monotone mapping. Further we show that this approach is not only a generalization of the monotonicity using the fuzzy language but this is a good tool for characterizing aggregation functions.

In Section 4.2 we recall the main definitions and results concerning the pointwise aggregation of fuzzy relations. We also investigate some properties which are necessary for our further investigations.

Section 4.3 presents an approach for solution of multi-objective linear programming problem. The idea is to involve fuzzy order relations and to use an aggregation for these relations. In our approach fuzzy order relations describe the objective functions while in ”classical” fuzzy approach ([50], [51]) the membership functions, which illustrate how far is the concrete point from the solution of an individual problem, are studied. Further, the global fuzzy order relation is constructed by aggregating the individual fuzzy order relations. Thus the global fuzzy relation contains the information about all objective functions and in the last step we find a maximum in the set of constrains with respect to the global fuzzy order relation. We illustrate this approach by an example.

Section 4.4 presents a new method for aggregation of fuzzy relations. The necessity of this method is explained by the examples. This tool employs a t-norm and an aggregation function to define the degree to which two elements are in relation when it is known (correspondingly pairwise) for all vectors aggregated into these elements. In the work we apply A-T-aggregation for fuzzy order relations and determine the necessary conditions for preservation of some relevant properties.

The results of this work were presented at twelve international conferences (see List of attended international conferences on page 95), eight local conferences, two international seminars (Workshop ”Algebra and its applications”, Joint Tartu-Riga seminar in pure mathematics). The main results are presented in seven papers (see List of Author's publications on page 100).

Chapter 2

Preliminaries

In this chapter we present results required for our further considerations. For fundamental results on fuzzy (or L-fuzzy) set theory we refer the reader to [14], [45].

2.1 Commutative cl-monoids

We start with the definition of a cl-monoid, which is a lattice with additional binary operation. For the basic notions and details on lattice theory see [4], [11].

Definition 2.1.1. (cf. [22]) *A commutative cl-monoid is a complete lattice* $(L, \leq, \wedge, \vee)$ *enriched with a further binary commutative operation* $*$*, satisfying the isotonicity condition:*

$$\forall \alpha, \beta, \gamma \in L \;\; \alpha \leq \beta \Rightarrow \alpha * \gamma \leq \beta * \gamma$$

and distributing over arbitrary joins:

$$\forall \alpha \in L \;\; and \;\; \forall \{ \beta_i : i \in I \} \subset L \quad \alpha * (\bigvee_i \beta_i) = \bigvee_i (\alpha * \beta_i).$$

A commutative cl-monoid is integral if and only if the unit element 1 *is also the universal upper bound in* L*. It is known that for the integral commutative cl-monoid with zero element* 0 *(i.e.* $\forall \alpha \in L \;\; \alpha * 0 = 0$*) the universal lower bound is zero element* 0.

Every integral commutative cl-monoid with zero is residuated, i.e., there exists a binary operation " $\mapsto$ *" (residuum) on* L *satisfying the following condition:*

$$\forall \alpha, \beta, \gamma \in L \quad \alpha * \beta \leq \gamma \Leftrightarrow \alpha \leq (\beta \mapsto \gamma).$$

Explicitly the residuum is given by

$$\alpha \mapsto \beta = \bigvee\{\lambda \in L : \alpha * \lambda \leq \beta\}.$$

For the given residuum " $\mapsto$ " *a biresiduum* " $\leftrightarrow$ " *is defined by:*

$$\alpha \leftrightarrow \beta = (\alpha \mapsto \beta) * (\beta \mapsto \alpha).$$

Let us mention some properties of integral commutative cl-monoids which will be useful in the sequel:

(i) If $\alpha \leq \beta$ and $\gamma \leq \delta$ then $\alpha * \gamma \leq \beta * \delta$;

(ii) $\alpha * \beta \leq \alpha \wedge \beta$;

(iii) $(\alpha \mapsto \gamma) * (\gamma \mapsto \beta) \leq \alpha \mapsto \beta$.

The proof of the above properties and some other useful properties of the integral, commutative cl-monoids can be found in [19], [10].

In our work we use an integral, commutative cl-monoid $(L, \leq, \wedge, \vee, *)$ with zero element but for simplicity in the sequel we will write simply cl-monoid.

2.2 T-norms

Here we propose the definition, basic examples and some properties of t-norms, which are important for our future consideration. The notion of a t-norm is fundamental in the framework of probabilistic metric spaces and in fuzzy logic. For us t-norms are important as a generalized conjunctions in fuzzy logic. Detailed information on t-norms can be found e.g. in [25], [45].

Definition 2.2.1. [25] *A triangular norm (t-norm for short) is a binary operation T on the unit interval $[0, 1]$, i.e. a function $T : [0, 1]^2 \to [0, 1]$ such that for all $x, y, z \in [0, 1]$ the following four axioms are satisfied:*

- $T(x, y) = T(y, x)$ *(commutativity);*
- $T(x, T(y, z)) = T(T(x, y), z)$ *(associativity);*
- $T(x, y) \leq T(x, z)$ *whenever* $y \leq z$ *(monotonicity);*
- $T(x, 1) = x$ *(boundary condition).*

It is well known that if T is a lower-semicontinuous, that is

$$\sup_{i\in I} T(x_i, y) = T(\sup_{i\in I} x_i, y),$$

then $([0,1], \leq, \wedge, \vee, T)$ is a particular choice of commutative cl-monoids.

Some of often used lower-semicontinuous (and, actually continuous) t-norms are mentioned below:

- $T_M(x,y) = \min(x,y)$ minimum t-norm;
- $T_P(x,y) = x \cdot y$ product t-norm;
- $T_L(x,y) = \max(x+y-1, 0)$ Łukasiewicz t-norm.

The corresponding residuums are:

- $x \mapsto_T y = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{otherwise} \end{cases}$ residuum coresponding to minimum t-norm;
- $x \mapsto_T y = \begin{cases} 1, & \text{if } x \leq y \\ \frac{y}{x}, & \text{otherwise} \end{cases}$ residuum coresponding to product t-norm;
- $x \mapsto_T y = \min(1-x+y, 1)$ residuum coresponding to Łukasiewicz t-norm.

A t-norm T is called Archimedean if and only if, for all pairs $(x,y) \in (0,1)^2$, there is $n \in \mathbb{N}$ such that $x_T^{(n)} < y$.

Product and Łukasiewicz t-norms are Archimedean while minimum t-norm is not.

We proceed with one of powerful tools for the construction of t-norms involving only one-place real function (additive generator) and addition. Furthermore, we use the same tool for constructing residuum and fuzzy equivalence.

Definition 2.2.2. [25] *Let $f : [a,b] \to [c,d]$ be a monotone function, where $[a,b]$ and $[c,d]$ are closed subintervals of the extended real line $[-\infty, \infty]$. The pseudo-inverse $f^{(-1)} : [c,d] \to [a,b]$ of f is defined by*

$$f^{(-1)}(y) = \begin{cases} sup\{x \in [a,b] \mid f(x) < y\} & \textit{if } f(a) < f(b), \\ sup\{x \in [a,b] \mid f(x) > y\} & \textit{if } f(a) > f(b), \\ a & \textit{if } f(a) = f(b). \end{cases}$$

Definition 2.2.3. [25] *An additive generator $t : [0,1] \to [0,\infty]$ of a t-norm T is a strictly decreasing function which is also right-continuous in 0 and satisfies $t(1) = 0$, such that for all $(x, y) \in [0,1]^2$ we have*

$$t(x) + t(y) \in Ran(t) \cup [t(0), \infty],$$

$$T(x, y) = t^{(-1)}(t(x) + t(y)).$$

Theorem 2.2.1. [25] *A function $T : [0,1]^2 \to [0,1]$ is a continuous Archimedean t-norm if and only if there exists a continuous additive generator t such that, for all $x, y \in [0,1]$, the following holds:*

$$T(x, y) = t^{(-1)}(\min(t(x) + t(y), t(0))).$$

Generator t is uniquely determined up to a positive multiplicative constant.

Theorem 2.2.2. [25] *If T is a continuous Archimedean t-norm and $t : [0,1] \to [0,\infty]$ an additive generator of T then the T-residuum $\mapsto_T$ can be obtained by the formula:*

$$x \mapsto_T y = t^{(-1)}(\max(t(y) - t(x), 0)).$$

2.3 Fuzzy relations

First time the definition of fuzzy order relation was introduced by L.A. Zadeh in 1971 under the name of a fuzzy partial ordering. Slightly modifying Zadeh's definition we present the following concept of L-valued order relation (we use the term "fuzzy" when $L = [0,1]$ and "L-valued" for an arbitrary cl-monoid L).

Let L be a fixed cl-monoid.

Definition 2.3.1. *(cf. e.g.*[49]*) Let X be a set. By an L-valued order relation we call an L-valued relation $P : X \times X \to L$ such that the following three axioms are fulfilled for all $x, y, z \in X$:*

1. $P(x, x) = 1$ - *reflexivity;*

2. $P(x, y) * P(y, z) \leq P(x, z)$ - *transitivity;*

3. $x \neq y \Rightarrow P(x, y) * P(y, x) = 0$ - *antisymmetry.*

Fifteen years later U. Höhle and N. Blanchard in their paper [23] proposed to involve L-valued equivalence relation[1] for the definition of L-valued order (partial ordering). For more resent results about fuzzy order defined with respect to the fuzzy equivalence relation see [5].

Let us first define an L-valued set and related category:

Definition 2.3.2. *(cf. e.g.* [23]*) By an L-valued set we call a pair (X, E) where X is a set and E is an L-valued equivalence relation, i.e. a mapping $E : X \times X \to L$ such that the following three axioms are fulfilled for all $x, y, z \in X$:*

1. *$E(x, x) = 1$ - reflexivity;*

2. *$E(x, y) * E(y, z) \leq E(x, z)$ - transitivity;*

3. *$E(x, y) = E(y, x)$ - symmetry.*

A mapping $f : (X, E_X) \to (Y, E_Y)$ is called extensional if

$$E_X(x_1, x_2) \leq E_Y(f(x_1), f(x_2))$$

for all $x_1, x_2 \in X$. L-valued sets and extensional mappings between them obviously form a category which is denoted L-$\mathbf{SET}$.

Remark 2.3.1. *In the paper [23] an L-valued set is called as an L-underdeterminate set and an L-valued equality as an L-equality relation. For the particular choices of cl-monoid an L-valued equality could be also called an M-equivalence relation (where $M = (M, \leq, \wedge, \vee, *)$) [12], an L-equivalence [10], an M-valued similarity relation or an M-similarity [21], a fuzzy equivalence relation w.r.t $*$ [5], [6], [7] and also a global M-valued equality or an $*$-equality.*

We continue with the definition of an L-E-order relation and an L-E-ordered set:

Definition 2.3.3. *(cf. e.g.* [23]*) Let (X, E) be an L-valued set. By an L-E-order relation on the L-valued set (X, E) we call an L-valued relation $P : X \times X \to L$ such that the following three axioms are fulfilled for all $x, y, z \in X$:*

1. *$E(x, y) \leq P(x, y)$ - E-reflexivity;*

2. *$P(x, y) * P(y, z) \leq P(x, z)$ - transitivity;*

[1] in the original work the term L-equality was used

3. $P(x, y) * P(y, x) \leq E(x, y)$ *-* E*-antisymmetry.*

A pair (X, P, E) *is called an* L*-*E*-ordered set.*

In the sequel we use Definition 2.3.3 of an L-E-order relation and we also give some comments how concrete results depend on the definition of reflexivity and antisymmetry.

Remark 2.3.2. *For particular choices of cl-monoid an* L*-*E*-order relation could be also called an* M*-*E*-partial ordering (where* $M = (M, \leq, \wedge, \vee, *)$*) [12] , a fuzzy ordering w.r.t* $*$ *or an* $*$*-*E*-ordering [5], [6].*

In the first part of the work, where we observe the categories we use the definition of L-E-order relation. In the second part we use fuzzy orders but still we use the definition which is based on the fuzzy equivalence relation. Let us here propose three examples where Zadeh's definition doesn't work and which justify the usage of fuzzy equivalence in the definition of fuzzy order relation. These examples were proposed in U.Bodenhofer work [5]:

1. It is a well-known and often-used fact in mathematical logic that there is a strong connection between implications and order relations. It is easy to see that the relation $\preceq$ which is defined as follows:

   $$\varphi \preceq \psi \Leftrightarrow (\varphi \rightarrow \psi \text{ is a tautology})$$

   where φ and ψ are formulas is an order relation on the set of formulas in a given language if we always consider two formulas as equal whenever their evaluations coincide for all interpretations. The same is true for t-norm-based logics on the unit interval. But if we consider a residuum $\mapsto_T$ (which corresponds to the concrete t-norm T and which is an analogue of implication in the t-norm-based logics) it is not a fuzzy order relation (antisymmetry is not fulfilled).

2. It is trivial that, for each non-empty crisp set X, the inclusion $\subseteq$ is an order relation on the power set $P(X)$. The same holds in the fuzzy case, i.e. the well-known inclusion

   $$A \subseteq B \Leftrightarrow (\forall x \in X : \; A(x) \leq B(x)),$$

   where $A : X \rightarrow [0, 1]$ and $B : X \rightarrow [0, 1]$ are fuzzy sets, defines an order relation on the fuzzy power set $F(X)$. But if we define a fuzzy inclusion as follows:

   $$INCL_T(A, B) = \inf_{x \in X} (A(x) \mapsto_T B(x))$$

   then it isn't a fuzzy order relation (antisymmetry is not fulfilled).

3. Let $\preceq$ be a crisp linear order relation on X. Then the relation $\preceq$ itself is the only fuzzy order relation which is $\preceq$-consistent and, as a consequence, the only fuzzification of $\preceq$, where "fuzzification" of a crisp relation defines as follows:

   Definition 2.3.4. [5] *Consider a crisp binary relation $\diamond$ on a set X. A fuzzy relation R is called $\diamond$- consistent if and only if the implication*

   $$y \diamond z \Rightarrow R(x, y) \leq R(x, z)$$

   holds for all $x, y, z \in X$. If additionally

   $$x \diamond y \Rightarrow R(x, y) = 1,$$

   R is called a fuzzification of $\diamond$ (R fuzzifies $\diamond$).

These three examples justify that the fuzzy order relation should be defined w.r.t. fuzzy equivalence relation.

Let us propose here construction methods for fuzzy equivalence relations and fuzzy order relations. These results will be important for the second part of the work.

Remark 2.3.3. *In the sequel, speaking about fuzzy relations, the fuzzy equivalence relation could be also called T-equivalence if we want to stress the t-norm T which is used in the definition of transitivity for correspondent fuzzy equivalence relation. By the same reasons fuzzy order relation could be called T-E-order relation, where E is a fuzzy equivalence relation.*

The following result establishes principles of construction of fuzzy equivalence relations from pseudo-metrics.

Theorem 2.3.1. *(see e.g.* [3]*) Let T be a continuous Archimedean t-norm with an additive generator t. For any pseudo-metric d, the mapping*

$$E_d(x, y) = t^{(-1)}(\min(d(x, y), t(0)))$$

is a T-equivalence.

Example 2.3.1. *Let us consider the set of real numbers $X = \mathbb{R}$ and metric $d(x, y) = |x - y|$ on it. Taking into account that $t_L(x) = 1 - x$ is an additive generator of T_L (Łukasiewicz t-norm) and that $t_P(x) = -ln(x)$ is an additive generator of T_P (product t-norm), we obtain two fuzzy equivalence relations:*

$$E_L(x, y) = \max(1 - |x - y|, 0);$$
$$E_P(x, y) = e^{-|x-y|}.$$

Definition 2.3.5. [5] *Let $\preceq$ be a crisp order on X and let E be a fuzzy equivalence relation on X. E is called compatible with $\preceq$ if and only if the following implication holds for all $x, y, z \in X : x \preceq y \preceq z \Rightarrow (E(x,z) \leq E(y,z)$ and $E(x,z) \leq E(x,y))$.*

Remark 2.3.4. *Let X be the set of real numbers and $\leq$ be a linear order on it. Then, for a fixed element x_0, $E(x, x_0)$ is non-decreasing in the interval $[-\infty, x_0]$ and non-increasing in the interval $[x_0, \infty]$, where E is a fuzzy equivalence relation which is compatible with $\leq$.*

Further, in our work we consider special type of fuzzy equivalence relation on interval $[0, 1]$:

$$E(x, y) = g(|x - y|), \tag{2.1}$$

where g is a non-increasing function. The necessary condition for the relation $E(x, y) = g(|x - y|)$ to be compatible with $\leq$ is that function g is non-increasing. Thus when we speak about fuzzy equivalence relation defined by formula (2.1) we require that function g should be chosen in such a way that E fulfill all the necessary conditions.

We continue with the construction of fuzzy order relations (namely, of strongly linear fuzzy order relations):

Definition 2.3.6. *(see e.g. [5]) A fuzzy order L is called strongly linear if and only if $\forall x, y \in X : \max(L(x, y), L(y, x)) = 1$.*

The following theorem states that strongly linear fuzzy order relations are uniquely characterized as fuzzifications of crisp linear orders.

Theorem 2.3.2. [5] *Let L be a binary fuzzy relation on X and let E be a T-equivalence on X. Then the following two statements are equivalent:*

1. *L is a strongly linear T-E-order on X.*
2. *There exists a linear order $\preceq$ the relation E is compatible with, such that L can be represented as follows:*

$$L(x, y) = \begin{cases} 1, & \text{if } x \preceq y \\ E(x, y), & \text{otherwise.} \end{cases}$$

This theorem shows that if we have a set X, a linear order $\preceq$ on it and a T-equivalence on X which is compatible with $\preceq$, then we can build a fuzzy linear order as

it was shown above.
Further, in some results we consider special type of fuzzy order relation on interval $[0,1]$:

$$L(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ g(|x-y|), & \text{otherwise,} \end{cases} \tag{2.2}$$

where g is a non-increasing function.
For more information regarding to fuzzy orders constructed on the base of fuzzy equivalence relations see [8],[6].
In general L defined by (2.2) is not a fuzzy order relation. The choice of a function g is important. The necessary condition for the relation $E(x,y) = g(|x-y|)$ to be compatible with $\leq$ is that function g is non-increasing. Therefore, if we prove a result for an arbitrary fuzzy relation L defined as above, the result will also hold for a fuzzy order relation

$$R(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ E(x,y) = g(|x-y|), & \text{otherwise,} \end{cases} \tag{2.3}$$

where E is a fuzzy equivalence relation compatible with $\leq$.
Fuzzy order relation defined as above can be widely used in practical applications, see e.g. [8].

2.4 L-valued categories

In this section, we recall the basic notions for L-valued categories. Main results on the classical (crisp) category theory can be found in [20], [1].

We start with the definition of an L-valued category.

Definition 2.4.1. [47] *An L-valued category $\mathbb{C}$ consists of:*

1. *A class $Ob(\mathbb{C})$ of potential objects.*
2. *An L-valued subclass ω of $Ob(\mathbb{C})$:*

$$\omega : Ob(\mathbb{C}) \to L.$$

3. *A class $Mor(\mathbb{C}) = \bigcup\{Mor_{\mathbb{C}}(X,Y) : X, Y \in Ob(\mathbb{C})\}$ of pairwise disjoint sets $Mor_{\mathbb{C}}(X,Y)$. For each pair of potential objects $X, Y \in Ob(\mathbb{C})$ the members of*

$Mor_{\mathbb{C}}(X,Y)$ are called potential morphisms from X to Y and the members of $Mor(\mathbb{C})$ are called potential morphisms of the category $\mathbb{C}$.

4. *An L-valued subclass μ of $Mor(\mathbb{C})$:*

$$\mu : Mor(\mathbb{C}) \to L,$$

such that if $f \in Mor_{\mathbb{C}}(X,Y)$, then $\mu(f) \leq \omega(X) \wedge \omega(Y)$.

5. *A composition $\circ$ of morphisms, i.e. for each triple $X, Y, Z \in Ob(\mathbb{C})$ there exists a map*

$$\circ : Mor_{\mathbb{C}}(X,Y) \times Mor_{\mathbb{C}}(Y,Z) \to Mor_{\mathbb{C}}(X,Z) \; ((f,g) \to g \circ f),$$

such that the following axioms are satisfied:

- *preservation of morphisms:*
  $\mu(g \circ f) \geq \mu(g) * \mu(f)$;
- *associativity:*
  if $f \in Mor_{\mathbb{C}}(X,Y)$, $g \in Mor_{\mathbb{C}}(Y,Z)$ and $h \in Mor_{\mathbb{C}}(Z,U)$,
  then $h \circ (g \circ f) = (h \circ g) \circ f$;
- *existence of identities:*
  for each $X \in Ob(\mathbb{C})$ there exists an identity $id_X \in Mor_{\mathbb{C}}(X,X)$ such that for all $X, Y, Z \in Ob(\mathbb{C})$, all $f \in Mor_{\mathbb{C}}(X,Y)$ and all $g \in Mor_{\mathbb{C}}(Z,X)$ it holds $f \circ id_X = f$ and $id_X \circ g = g$.
  Note that in [47] it was requested that for identities the following condition should fulfill:

  $$\mu(id_X) = \omega(X).$$

  In this case the definition of an L-valued class of objects ω is determined by the L-valued class μ of morphisms and hence this condition can be abandoned. We skip this condition and by this we generalize the notion of an L-valued category.
  Note that when applying the condition $\mu(f) \leq \omega(X) \wedge \omega(Y)$ for the identity $id_X : X \to X$ we have $\mu(id_X) \leq \omega(X)$, which is a very natural property saying that the identity could not exists if the object does not exist.

It is possible to create new categories from the existing ones:
Let $\mathbb{C} = (Ob(\mathbb{C}), \omega, Mor(\mathbb{C}), \mu, \circ)$ be an L-valued category and let another L-valued

subclass ω' of $Ob(\mathbb{C})$ is given: $\omega' : Ob(\mathbb{C}) \to L$. Then $\mathbb{C}' = (Ob(\mathbb{C}), \omega', Mor(\mathbb{C}), \mu', \circ)$ is an L-valued category, where $\mu'(f) = \omega'(X) * \mu(f) * \omega'(Y)$.
Proof:

- $\mu'(f) = \omega'(X) * \mu(f) * \omega'(Y) \leq \omega'(X) \wedge \omega'(Y)$.
- $\mu'(g \circ f) = \omega'(X) * \mu(g \circ f) * \omega'(Z) \geq \omega'(X) * \mu(g) * \mu(f) * \omega'(Z) \geq$
  $\geq (\omega'(X) * \mu(f) * \omega'(Y)) * (\omega'(Y) * \mu(g) * \omega'(Z)) = \mu'(g) * \mu'(f)$

It is natural to apply this construction when an L-valued category $\mathbb{C}$ is defined in such a way that $\omega(X) = 1$ for all $X \in Ob(\mathbb{C})$. Then we involve another subclass of objects ω' such that $\omega' < \omega$. This means that for at least one object X $\omega'(X) < 1$. Then $\mathbb{C}' = (Ob(\mathbb{C}), \omega', Mor(\mathbb{C}), \mu', \circ)$ is an L-valued category, where $\mu'(f) = \omega'(X)*\mu(f)*\omega'(Y)$. This construction will be applied in Section 3.3.

Remark 2.4.1. *Let $Ob_\alpha(\mathbb{C}) = \{X \in Ob(\mathbb{C}) : \omega(X) \geq \alpha\}$.*
The elements of $Ob_\alpha(\mathbb{C})$ will be referred as α-objects of the L-valued category $\mathbb{C}$, while the elements of $Ob(\mathbb{C})$ will be called potential objects or just objects of $\mathbb{C}$.

Similarly, the elements of $Mor_\alpha(\mathbb{C})$ ($Mor_\alpha(\mathbb{C}) = \{f \in Mor(\mathbb{C}) : \mu(f) \geq \alpha\}$) will be referred as α-morphisms of the L-valued category $\mathbb{C}$, while the elements of $Mor(\mathbb{C})$ will be called potential morphisms or just morphisms of $\mathbb{C}$.

Given an L-valued category $\mathbb{C} = (Ob(\mathbb{C}), \omega, Mor(\mathbb{C}), \mu, \circ)$ and $X \in Ob(\mathbb{C})$, intuitively we understand the value $\omega(X)$ as the degree to which a potential object X of the L-valued category $\mathbb{C}$ is indeed its objects; similarly, for $f \in Mor(\mathbb{C})$ the intuitive meaning of $\mu(f)$ is the degree to which a potential morphism f of $\mathbb{C}$ is indeed its morphism.

Definition 2.4.2. [44] *Given an L-valued category $\mathbb{C} = (Ob(\mathbb{C}), \omega, Mor(\mathbb{C}), \mu, \circ)$ one can construct a crisp category $\mathbb{C}_0 = (Ob(\mathbb{C}), Mor(\mathbb{C}), \circ)$ by taking all potential objects and potential morphisms of L-valued category $\mathbb{C}$ as objects and morphisms of crisp category $\mathbb{C}_0$ and leaving the composition law unchanged. $\mathbb{C}_0$ is called the bottom frame of L-valued category $\mathbb{C}$. We also denote the top frame $\mathbb{C}_1 = (Ob_1(\mathbb{C}), Mor_1(\mathbb{C}), \circ)$, where*
$Ob_1(\mathbb{C}) = \{X \in Ob(\mathbb{C}) : \omega(X) = 1\}$; $Mor_1(\mathbb{C}) = \{f \in Mor(\mathbb{C}) : \mu(f) = 1\}$ and where 1 is the universal upper bound in L.

A general scheme for fuzzification of classical categories [47].
Let $\mathbb{C} = (Ob(\mathbb{C}), Mor(\mathbb{C}), \circ)$ and $\mathbb{D} = (Ob(\mathbb{D}), Mor(\mathbb{D}), \circ)$ be two ordinary categories

and let $\Phi : \mathbb{C} \to \mathbb{D}$ be a functor. We define a new ordinary category Cat by setting $Ob(Cat) = Ob(\mathbb{C})$ and $Mor_{Cat}(X, Y) = Mor_{\mathbb{D}}(\Phi(X), \Phi(Y))$. Thus the morphisms from X to Y in Cat are the same as the morphisms from $\Phi(X)$ to $\Phi(Y)$ in $\mathbb{D}$. The composition law in Cat is naturally induced by the composition law in $\mathbb{D}$. Now, defining in a certain way an L-subclass of objects: $\omega : Ob(Cat) \to L$ and an L-subclass of morphisms $\mu : Mor(Cat) \to L$ satisfying Definition 2.4.1 we come to an L-valued category $(Ob(Cat), \omega, Mor(Cat), \mu)$. The category Cat could be also denoted as $\mathbb{C}_{\mathbb{D}\Phi}$ or $\mathbb{C}_{\mathbb{D}}$.

We continue with the definition of a functor between two L-valued categories.

Definition 2.4.3. *(see e.g.*[47] *) Let* $\mathbb{C} = (Ob(\mathbb{C}), \omega_{\mathbb{C}}, Mor(\mathbb{C}), \mu_{\mathbb{C}}, \circ)$ *and* $\mathbb{D} = (Ob(\mathbb{D}), \omega_{\mathbb{D}}, Mor(\mathbb{D}), \mu_{\mathbb{D}}, \circ)$ *be L-valued categories, then a functor F from $\mathbb{C}$ to $\mathbb{D}$ is a function that assigns to each $\mathbb{C}$-object A a $\mathbb{D}$-object $F(A)$, and to each $\mathbb{C}$-morphism $f : A \to A'$ a $\mathbb{D}$-morphism $F(f) : F(A) \to F(A')$, in such way that the following properties are satisfied:*

1. *F preserves composition, i.e. $F(g \circ f) = F(g) \circ F(f)$ provided the composition $g \circ f$ is defined;*
2. *F preserves identities, i.e. $F(id_X) = id_{F(X)}$ for any $X \in Ob(\mathbb{C})$;*
3. *$\mu_{\mathbb{C}}(f) \leq \mu_{\mathbb{D}}(F(f))$ for any $f \in Mor(\mathbb{C})$;*
4. *$\omega_{\mathbb{C}}(X) \leq \omega_{\mathbb{D}}(F(X))$ for any $X \in Ob(\mathbb{C})$.*

In [47] there was not the last condition since it follows from the third condition and the condition that $\mu(id_X) = \omega(X)$ for each $X \in Ob(\mathbb{C})$. Since we skip the condition $\mu(id_X) = \omega(X)$ we should add the forth condition for the definition of functor.

We now proceed with the definitions of some special objects and special morphisms for an L-valued category.

Let $X, Y \in Ob(\mathbb{C})$, where $\mathbb{C}$ is an L-valued category and let $\alpha, \beta \in L$.

Definition 2.4.4. [44] *An object I in the L-valued category $\mathbb{C}$ is called α-initial if and only if for every α-object X there exists a unique α-morphism $f : I \to X$. An object I is called initial if and only if it is α-initial for every α.*

Definition 2.4.5. [44] *An object T in the L-valued category $\mathbb{C}$ is called α-terminal if and only if for every α-object X there exists a unique α-morphism $f : X \to T$. An object T is called terminal if and only if it is α-terminal for every α.*

Definition 2.4.6. [44] *An object Z in the L-valued category $\mathbb{C}$ is called α-zero if and only if it is both α-initial and α-terminal. An object Z is called zero-object if and only if it is α-zero for every α.*

Definition 2.4.7. [44] *An α-morphism $f : X \to Y$ is called an α-section if and only if there exists an α-morphism $g : Y \to X$ such that $g \circ f = id_X$.*

Definition 2.4.8. [44] *An α-morphism $f : X \to Y$ is called an α-retraction if and only if there exists an α-morphism $g : Y \to X$ such that $f \circ g = id_Y$.*

Definition 2.4.9. [44] *An α-morphism $f : X \to Y$ is called an α-isomorphism if and only if it is both an α-section and an α-retraction.*

Definition 2.4.10. [44] *An α-morphism $f : X \to Y$ is called a β-mono-α-morphism, (or just a β-monomorphism for short) provided for all β-morphisms $h : Z \to X$ and $k : Z \to X$ such that $f \circ h = f \circ k$ it follows that $h = k$.*

Definition 2.4.11. [44] *An α-morphism $f : X \to Y$ is called a β-epi-α-morphism, (or just a β-epimorphism for short) provided for all β-morphisms $h : Y \to Z$ and $k : Y \to Z$ such that $h \circ f = k \circ f$ it follows that $h = k$.*

Definition 2.4.12. [44] *An α-morphism $f : X \to Y$ is called a β-bi-α-morphism, (or just a β-bimorphism for short) if and only if it is both β-monomorphisms and β-epimorphism.*

2.5 Aggregation functions

Aggregation functions play an important role in several areas, including fuzzy logic, decision making, expert systems, risk analysis and image processing. Recent books [33], [17] provide a comprehensive overview of aggregation functions, their properties and methods of their construction. The purpose of aggregation functions is to combine several input values into a single output value, which in some sense represents all the inputs. Typically the inputs and outputs are real numbers, often from $[0, 1]$, although other choices are possible, e.g.,discrete sets, intervals and linguistic labels.

Definition 2.5.1. [17] *An aggregation function is a mapping $A : [0, 1]^n \to [0, 1]$ which fulfills the following properties:*

- $A(x_1, ..., x_n) \leq A(y_1, ..., y_n)$ *whenever* $x_i \leq y_i$ *for all* $i \in \{1, ..., n\}$ *(monotonicity);*
- $A(0, ..., 0) = 0$ *and* $A(1, ..., 1) = 1$ *(boundary conditions).*

We shall make a distinction between aggregation function having a definite number of arguments and extended aggregation function (or aggregation operator) defined for any number of arguments:

Definition 2.5.2. [33] *An aggregation operator is a mapping* $A : \bigcup_{n \in \mathbb{N}} [0, 1]^n \to [0, 1]$ *which fulfills the following properties for every* $n \in \mathbb{N}$*:*

- $A(x_1, ..., x_n) \leq A(y_1, ..., y_n)$ *whenever* $x_i \leq y_i$ *for all* $i \in \{1, ..., n\}$ *(monotonicity);*
- $A(0, ..., 0) = 0$ *and* $A(1, ..., 1) = 1$ *(boundary conditions).*

In general, the number of the input values to be aggregated for the aggregation operator is unknown, and therefore an aggregation operator can be presented as a family $A = (A_{(n)})_{n \in \mathbb{N}}$, where $A_{(n)}$ is an n-ary aggregation function ($A_{(n)} = A \mid_{[0,1]^n}$).

We observe here some well-known examples of aggregation functions:

Example 2.5.1.

- *minimum* $MIN(\mathbf{x}) = \bigwedge_{i=1}^{n} x_i$*;*
- *maximum* $MAX(\mathbf{x}) = \bigvee_{i=1}^{n} x_i$*;*
- *arithmetic mean* $AM(\mathbf{x}) = \frac{1}{n} \sum_{i=1}^{n} x_i$,
- *weighted arithmetic means:* $WAM_w(\mathbf{x}) = \sum_{i=1}^{n} w_i x_i$, *where weights* $w_i \in [0, 1]$ *and* $\sum_{i=1}^{n} w_i = 1$*;*
- *partial minimum:* $MIN_K(\mathbf{x}) = \bigwedge_{i \in K} x_i$, *where* $K \subseteq \{1, ..., n\}$*;*
- *partial maximum:* $MAX_K(\mathbf{x}) = \bigvee_{i \in K} x_i$, *where* $K \subseteq \{1, ..., n\}$.

Chapter 3

L-valued category whose objects are L-E-ordered sets

3.1 Category whose objects are L-E-ordered sets

3.1.1 Construction of the category

Our aim is to construct an analogue of the category of partially ordered sets ($\mathbf{POS}$ or $\mathbf{POSET}$). As we know in classical mathematics POS category can be constructed from small categories (categories, where the class of objects is a set and the class of morphisms is also a set). Actually we can observe each ordered set $(X, \leq)$ as a small category where elements of a set are objects and morphism (unique) from an object x to an object y exists if and only if $x \leq y$. Then the functors between these small categories must be order-preserving mappings of the corresponding ordered sets. Thus category $\mathbf{POS}$ can be observed as a category whose objects are small categories and morphisms are functors between them.

Our aim now is to construct a category whose objects are L-E-ordered sets. To realize this construction we follow the construction of the crisp $\mathbf{POS}$ category. Thus, first we observe a small category whose objects are elements of an L-E-ordered set. Further we involve the degree to which each morphism is indeed the morphism of the category and the degree to which each object is indeed the object of the category, thus we get an L-valued category (a small L-valued category). Then we construct functors between small categories. Finally we observe the category whose objects are small categories and morphisms are functors between them. By the construction of category from small categories we justify that this category is constructed in a natural way from the point of view of L-valued categories.

For the construction of the following small category we have to use a cl-monoid without zero divisors ($\alpha * \beta = 0 \Rightarrow \alpha = 0 \;\; or \;\; \beta = 0$).
Let L be a cl-monoid without zero divisors, P be an L-E-order relation and (X, P, E) be an L-E-ordered set. Let us construct a small category $pos_{(X,P,E)}$ which has as objects elements of L-E-ordered set (X, P, E):

- Objects: $Ob(pos_{(X,P,E)}) = \{x : x \in X\}$;
- Morphisms: $f : x \to y$, $f \in Mor(pos_{(X,P,E)}) \Leftrightarrow P(x, y) > 0$.

Now we check that all properties of a category are fulfilled:

1. We first prove that if $f \in Mor(pos_{(X,P,E)})$ and $g \in Mor(pos_{(X,P,E)})$ then $f \circ g \in Mor(pos_{(X,P,E)})$.
   Let $g : x \to y$, $g \in Mor(pos_{(X,P,E)})$ and $f : y \to z$, $f \in Mor(pos_{(X,P,E)})$.
   Then $P(x, y) > 0$ and $P(y, z) > 0$. Since the cl-monoid L is without zero divisors we establish that $P(x, z) \geq P(x, y) * P(y, z) > 0$. Hence we obtain the existence of morphism $f \circ g \in Mor(pos_{(X,P,E)})$.

2. We prove the associativity condition for morphisms as follows:
   $f, g, h \in Mor(pos_{(X,P,E)}) \Rightarrow (f \circ g) \circ h = f \circ (g \circ h)$.

   Assume that $h : t \to x$, $g : x \to y$, $f : y \to z$. From the previous consideration we can conclude that compositions $(f \circ h) \circ h$ and $f \circ (g \circ h)$ exist. We know that from t to z exists only one morphism, hence $(f \circ g) \circ h = f \circ (g \circ h)$.

3. $P(x, x) = 1$, this means that there exists an identity morphism $id_x : x \to x$. Obviously $f \circ id_x = f$ for all $f : x \to y$ and $id_x \circ h = h$ for all $h : z \to x$.

Remark 3.1.1. *Since the antisymmetry is defined by means of L-valued equivalence, there could exist morphisms in the both directions between objects x and y: from x to y and from y to x. But in the case of antisymmetry from Definition 2.3.1, since L does not have zero divisors we conclude that for all $x \neq y$ either $P(x, y) = 0$ or $P(y, x) = 0$. This means that only one morphism between objects x and y could exist: either from x to y or from y to x.*

Now let us involve an L-valued subclass of the class of morphisms as a mapping from the class of morphisms to the cl-monoid L ($\mu : Mor(pos_{(X,P,E)}) \to L$) in the following way:

$$f : x \to y \Rightarrow \mu(f) = P(x, y)$$

and an L-valued subclass of the class of objects as a mapping from the class of objects to the cl-monoid L ($\omega : Ob(pos_{(X,P,E)}) \to L$):

$$x \in X \Rightarrow \omega(x) = 1.$$

We have constructed a small L-valued category

$$L\text{-}pos_{(X,P,E)} = (Ob(pos_{(X,P,E)}), \omega, Mor(pos_{(X,P,E)}), \mu, \circ).$$

Remark 3.1.2. *Intuitively the value $\mu(f)$ is the degree to which a potential morphism $f : x \to y$ of the category is indeed its morphism, this means the degree to which x is less or equal to y, what is actually characterized by the value $P(x, y)$.*

Next we verify all necessary properties for the L-valued category:

- The condition that $\mu(f) \leq \omega(x) \wedge \omega(y)$ for all $x, y \in Ob(L\text{-}pos_{(X,P,E)})$ and for all $f : x \to y$ is obvious since $\omega(x) = \omega(y) = 1$.
- We prove that $\mu(g \circ f) \geq \mu(g) * \mu(f)$, if the morphism $g \circ f$ exists:
  Let $f : x \to y$ and $g : y \to z$.
  Then by transitivity of the L-E-order P we have
  $\mu(g \circ f) = P(x, z) \geq P(x, y) * P(y, z) = \mu(f) * \mu(g)$.

Thus we have proven that the category $L\text{-}pos_{(X,P,E)}$ is constructed correctly.

Let $L\text{-}pos_{(X,P,E)}$ and $L\text{-}pos_{(Y,P',E')}$ be L-valued categories. We construct a functor F from the category $L\text{-}pos_{(X,P,E)}$ to the category $L\text{-}pos_{(Y,P',E')}$ such way that $E(x_1, x_2) \leq E'(F(x_1), F(x_2))$.

The necessary condition for F to be a functor is:

$$\forall f \in Mor(L\text{-}pos_{(X,P,E)}) \quad \mu_{L\text{-}pos_{(X,P,E)}}(f) \leq \mu_{L\text{-}pos_{(Y,P',E')}}(F(f)).$$

This means that $P(x_1, x_2) \leq P'(F(x_1), F(x_2))$.

$$\begin{array}{ccc} x_1 & \xrightarrow{F} & F(x_1) \\ f\Big\downarrow & & \Big\downarrow F(f) \\ x_2 & \xrightarrow{F} & F(x_2) \end{array}$$
$$P(x_1, x_2) \le P'(F(x_1), F(x_2))$$

Let us prove that this is also a sufficient condition for F to be a functor:

1. $f \in Mor_{L\text{-}pos_{(X,P,E)}}(x_1, x_2) \Rightarrow P(x_1, x_2) > 0 \Rightarrow$
   $\Rightarrow P'(F(x_1), F(x_2)) > 0 \Rightarrow F(f) \in Mor_{L\text{-}pos_{(Y,P',E')}}(F(x_1), F(x_2))$.
2. If the composition $g \circ f$ is defined when $f \in Mor_{L\text{-}pos_{(X,P,E)}}(x_1, x_2)$ and $g \in Mor_{L\text{-}pos_{(X,P,E)}}(x_2, x_3)$, then $F(g \circ f) = F(g) \circ F(f)$ because of the existence of only one morphism from $F(x_1)$ to $F(x_3)$.
3. $F(id_x) = id_{F(x)}$ for all $x \in Ob(L\text{-}pos_{(X,P,E)})$ since only one morphism from $F(x)$ to $F(x)$ exists.
4. Obviously $\omega(x) \le \omega(F(x))$ for any $x \in X$ since $\omega(x) = \omega(F(x)) = 1$.

We have proven that the functor was constructed correctly.

Next we define the category $\mathbf{POS}(L)$ (the analogue of crisp $\mathbf{POS}$ category) as a category of small categories. The objects of $\mathbf{POS}(L)$ will be L-E-ordered sets (categories of the type L-*pos*) and the morphisms will be functors between them:

- Objects: $Ob(\mathbf{POS}(L)) = \{(X, P, E)\}$ where (X, P, E) are L-E-ordered sets;
- Morphisms: $Mor(\mathbf{POS}(L)) = \{f : (X, P, E) \to (Y, P', E') \mid \forall x_1, x_2 \in X$ $E(x_1, x_2) \le E'(f(x_1), f(x_2));\ P(x_1, x_2) \le P'(f(x_1), f(x_2))\}$.

Remark 3.1.3. *If we use Definition 2.3.1 the constructions of the category L-pos and the category $\mathbf{POS}(L)$ are the same. We only have to skip the property $E(x_1, x_2) \le E'(f(x_1), f(x_2))$ for the morphisms of the category $\mathbf{POS}(L)$.*

Remark 3.1.4. *In the sequel we say that a mapping $f : (X, P, E) \to (Y, P', E')$ is order-preserving if $P(x_1, x_2) \le P'(f(x_1), f(x_2))$ for all $x_1, x_2 \in X$.*

Remark 3.1.5. *The property that a cl-monoid L is without zero divisors was useful only for the construction of the category of the type pos and, namely, for the existence of composition. If we are not interested in this particular construction we can define the category $\mathbf{POS}(L)$ straightway and skip the above mentioned condition for a cl-monoid.*

3.1.2 Properties of the category

Our next aim is to consider some properties of the category $\mathbf{POS}(L)$. We are going to study some special objects, morphisms and some standard constructions of $\mathbf{POS}(L)$ category.

We begin by studying some properties of $\mathbf{POS}(L)$ category.

Proposition 3.1.1. *The category* $\mathbf{POS}$ *is a full subcategory of the category* $\mathbf{POS}(L)$.

Proof. We have to prove that for all $(X, \leq)$ and $(Y, \leq')$, $Mor_{\mathbf{POS}}((X, \leq), (Y, \leq')) = Mor_{\mathbf{POS}(L)}((X, P_{\leq}, E_{=}), (Y, P_{\leq'}, E_{='}))$, where $P_{\leq} : X \times X \to L$ such that $P_{\leq}(x, y) = 1$ if $x \leq y$ and $P_{\leq}(x, y) = 0$ otherwise; $E_{=}(x, y) = 1$ if $x = y$ and $E_{=}(x, y) = 0$ otherwise. This is obvious since for crisp ordered sets $(X, \leq)$ and $(Y, \leq')$ the condition of preserving order for the function f is equivalent to the condition $P_{\leq}(x_1, x_2) \leq P_{\leq'}(f(x_1), f(x_2))$. □

Proposition 3.1.2. *Let L_1 and L_2 be two isomorphic cl-monoids (φ is an isomorphism) and $\mathbf{POS}(L_1)$, $\mathbf{POS}(L_2)$ be the correspondent categories. Then we can define the functor $F : \mathbf{POS}(L_1) \to \mathbf{POS}(L_2)$ by setting $F((X, P_1, E_1)) = (X, P_2, E_2)$, where $E_2(x_1, x_2) = \varphi(E_1(x_1, x_2))$, $P_2(x_1, x_2) = \varphi(P_1(x_1, x_2))$ and $F(f) = f$. Thus defined functor F is an isomorphism and categories $\mathbf{POS}(L_1)$ and $\mathbf{POS}(L_2)$ are isomorphic.*

Since the proof is straightforward we omit it.

Proposition 3.1.3. *If L_1 and L_2 are cl-monoids and $\varphi : L_1 \hookrightarrow L_2$ is an order-embedding and operation-preserving mapping then $\mathbf{POS}(L_1)$ is isomorphic to $\mathbf{POS}(\varphi(L_1))$, which is a full subcategory of the category $\mathbf{POS}(L_2)$.*

Proof. We define the functor $F : \mathbf{POS}(L_1) \to \mathbf{POS}(\varphi(L_1))$ as in the previous proposition. Thus defined functor F is an isomorphism and categories $\mathbf{POS}(L_1)$ and $\mathbf{POS}(\varphi(L_1))$ are isomorphic. It is easy to see that $Ob(\mathbf{POS}(\varphi(L_1))) \subseteq Ob(\mathbf{POS}(L_2))$ and that $Mor_{\mathbf{POS}(\varphi(L_1))}((X, P), (Y, P')) = Mor_{\mathbf{POS}(L_2)}((X, P), (Y, P'))$, where (X, P),$(Y, P') \in Ob(\mathbf{POS}(\varphi(L_1)))$. □

Proposition 3.1.4. *The category* $\mathbf{POS}(L)$ *is a connected category.*

Proof. To prove that the category $\mathbf{POS}(L)$ is a connected category we should show that for every two objects (X, P, E) and (Y, P', E') (X and Y are not empty sets) $Mor_{\mathbf{POS}(L)}((X, P, E), (Y, P', E')) \neq \emptyset$ which means that there exists a morphism

$f : (X, P, E) \to (Y, P', E')$.
Let us fix the objects (X, P, E) and (Y, P', E'). Further we construct a morphism $f : (X, P, E) \to (Y, P', E')$ in the following way: $f(x) = y_0$ for all $x \in X$ where y_0 is a fixed element from the set Y. Obviously $E(x_1, x_2) \leq E'(f(x_1), f(x_2))$ since $E'(f(x_1), f(x_2)) = E'(y_0, y_0) = 1$ and $P(x_1, x_2) \leq P'(f(x_1), f(x_2))$ since $P'(f(x_1), f(x_2)) = P'(y_0, y_0) = 1$. □

We continue by considering special objects in the category $\mathbf{POS}(L)$.

Proposition 3.1.5. *The empty set is the unique initial object in* $\mathbf{POS}(L)$.

Proposition 3.1.6. *The singleton set* (X, P, E) *where* $X = \{x\}$, $E(x, x) = 1$, $P(x, x) = 1$ *is the terminal object in* $\mathbf{POS}(L)$.

Corollary 3.1.1. *There are no zero objects in* $\mathbf{POS}(L)$.

We continue by considering special morphisms in the category $\mathbf{POS}(L)$.

Proposition 3.1.7. *A morphism* $f : (X, P, E) \to (Y, P', E')$ *is a monomorphism if and only if* f *is an injective mapping.*

Proof. The sufficiency is obvious. We continue by proving the necessity.
If the mapping $f : (X, P, E) \to (Y, P', E')$ is not an injection then there exist two elements x_1 and x_2 in the set X such that $x_1 \neq x_2$ but $f(x_1) = f(x_2)$.
Let (Z, P_Z, E_Z) be an L-E_Z-ordered set such that $Z = \{z\}$, $P_Z(z, z) = 1$, $E_Z(z, z) = 1$ and let $u, v : Z \to X$ be functions such that $u(z) = x_1$ but $v(z) = x_2$. Thus, obviously $f \circ u = f \circ v$ but $u \neq v$. Hence f is not a monomorphism. □

Proposition 3.1.8. *A morphism* $f : (X, P, E) \to (Y, P', E')$ *is an epimorphism if and only if* f *is a surjection.*

Proof. The sufficiency is obvious. We continue by proving the necessity.
If the mapping $f : (X, P, E) \to (Y, P', E')$ is not a surjection then there exists an element y_0 in the set Y such that $\forall x \in X\ f(x) \neq y_0$.
Let $Z = Y \bigcup \{z\}$ where $z \notin Y$, the L-valued equivalence E'' and L-E''-order P'' on the set Z we define in the following way:
$P''(y_1, y_2) = P'(y_1, y_2)$, $E''(y_1, y_2) = E'(y_1, y_2)$ if $y_1, y_2 \in Y$;
$P''(z, y) = P'(y_0, y)$, $E''(z, y) = E'(y_0, y)$ if $y \in Y$ and $y \neq y_0$;
$P''(y, z) = P'(y, y_0)$, $E''(y, z) = E'(y, y_0)$ if $y \in Y$ and $y \neq y_0$;

$P''(y_0, z) = 1$, $E''(y_0, z) = 1$;
$P''(z, y_0) = 1$, $E''(z, y_0) = 1$;
$P''(z, z) = 1$, $E''(z, z) = 1$.
It is easy to verify that E'' is an L-valued equivalence and P'' fulfills all necessary condition, that is P'' is E''-reflexive, transitive and E''-antisymmetric. Let us define now the functions $u : (Y, P', E') \to (Z, P'', E'')$ and $v : (Y, P', E') \to (Z, P'', E'')$ in the following way:
$u(y) = y$ for all $y \in Y$;

$$v(y) = \begin{cases} y, & \text{if } y \neq y_0 \\ z, & \text{otherwise} \end{cases}.$$

Obviously, functions u and v are extensional, order-preserving mappings and
$u \circ f = v \circ f$ but $u \neq v$. Hence f is not an epimorphism. □

Remark 3.1.6. *In case the order relation is realized in the sense of Definition 2.3.1., the value $P''(z, y_0)$ should be defined as $P''(z, y_0) = 0$. The rest of the proof need not be changed.*

From the previous two propositions and the definition of bimorphism we get the following proposition:

Proposition 3.1.9. *A morphism $f : (X, P, E) \to (Y, P', E')$ is a bimorphism if and only if f is a bijection.*

We know that in the category $\mathbf{POS}$ not every injection is a section, not every surjection is a retraction and not every bijection is an isomorphism. Provided that the category $\mathbf{POS}$ is a full subcategory of the category $\mathbf{POS}(L)$, in the category $\mathbf{POS}(L)$ we can find injections which are not sections, surjections which are not retractions and bijections which are not isomorphisms. Hence the category $\mathbf{POS}(L)$ is not balanced.

A morphism $f : (X, P, E) \to (Y, P', E')$ is an isomorphism in the category $\mathbf{POS}(L)$ if and only if it is a bijection, $E(x_1, x_2) = E'(f(x_1), f(x_2))$ and
$P(x_1, x_2) = P'(f(x_1), f(x_2))$ for all $x_1, x_2 \in X$.

We now turn to the study of special constructions in the category $\mathbf{POS}(L)$.

Theorem 3.1.1. *The product of a family $((X_i, P_i, E_i))_{i \in I}$ of $\mathbf{POS}(L)$ objects is a pair $((\prod_i X_i, P_\wedge, E_\wedge), (\pi_i)_{i \in I})$, where $\prod_i X_i = \{f : I \to \bigcup_i X_i \mid \forall i\ f(i) \in X_i\}$,
$E_\wedge(f, h) = \bigwedge_{i \in I} E_i(f(i), h(i))$, $P_\wedge(f, h) = \bigwedge_{i \in I} P_i(f(i), h(i))$ and
$\pi_i : (\prod_i X_i, P_\wedge, E_\wedge) \to (X_i, P_i, E_i)$ is defined by $\pi_i(f) = f(i)$.*

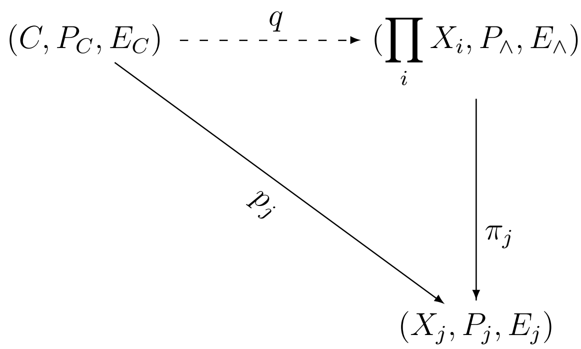
If I is a finite set then the Theorem 3.1.1. reduces to the following (more lucid) form:

Theorem 3.1.1.' *The product of a family $((X_i, P_i, E_i))_{i\in\{\overline{1,n}\}}$ of $\mathbf{POS}(L)$ objects is a pair $((X_1 \times X_2 \times \cdots \times X_n, P_\wedge, E_\wedge), (\pi_i)_{i\in\{\overline{1,n}\}})$, where*
$E_\wedge((a_1, a_2, ..., a_n), (b_1, b_2, ..., b_n)) = \bigwedge_i E_i(a_i, b_i)$,
$P_\wedge((a_1, a_2, ..., a_n), (b_1, b_2, ..., b_n)) = \bigwedge_i P_i(a_i, b_i)$ *and*
$\pi_i : (\prod\limits_i X_i, P_\wedge, E_\wedge) \to (X_i, P_i, E_i)$ *is defined by* $\pi_i((a_1, a_2, ..., a_n)) = a_i$.

Here we prove the general Theorem 3.1.1.

Proof.
- Let us prove that $(\prod\limits_i X_i, P_\wedge, E_\wedge)$ is an object in the category $\mathbf{POS}(L)$. This means we should prove that $E_\wedge$ is an L-valued equivalence and $P_\wedge$ is an L-$E_\wedge$-order relation. We skip the proof that $E_\wedge$ is an L-valued equivalence since it is similar to the proof that $P_\wedge$ is an L-$E_\wedge$-order relation. We now turn to prove $E_\wedge$-reflexivity, transitivity and $E_\wedge$-antisymmetry of the L-$E_\wedge$-order relation $P_\wedge$:
  - Since all relations P_i are E_i-reflexive $E_i(f(i), g(i)) \le P_i(f(i), g(i))$ for all $i \in I$. Therefore $\bigwedge\limits_i E_i(f(i), g(i)) \le \bigwedge\limits_i P_i(f(i), g(i))$ for all f, g .
    Thus $E_\wedge(f, g) \le P_\wedge(f, g)$ and hence L-$E_\wedge$-order relation $P_\wedge$ is $E_\wedge$-reflexive.
  - $P_\wedge(f, g) * P_\wedge(g, h) =$
    $= \bigwedge\limits_i P_i(f(i), g(i)) * \bigwedge\limits_i P_i(g(i), h(i)) \le \bigwedge\limits_i (P_i(f(i), g(i)) * P_i(g(i), h(i))) \le$
    $\le \bigwedge\limits_i P_i(f(i), h(i)) = P_\wedge(f, h)$. We have proven the transitivity of L-$E_\wedge$-order relation $P_\wedge$.
  - We have to prove that for all f, g it holds $P_\wedge(f, g) * P_\wedge(g, f) \le E_\wedge(f, g)$:
    $P_\wedge(f, g) * P_\wedge(g, f) =$
    $= \bigwedge\limits_i P_i(f(i), g(i)) * \bigwedge\limits_i P_i(g(i), f(i)) \le \bigwedge\limits_i (P_i(f(i), g(i)) * P_i(g(i), f(i))) \le$
    $\le \bigwedge\limits_i E_i(f(i), g(i)) = E_\wedge(f, g)$. We have proven the $E_\wedge$-antisymmetry of L-$E_\wedge$-order relation $P_\wedge$.
- We proceed to show that π_j are morphisms for all $j \in I$:
  $E_\wedge(f, h) = \bigwedge\limits_{i\in I} E_i(f(i), h(i)) \le E_j(f(j), h(j)) = E_j(\pi_j(f), \pi_j(h))$ for all $j \in I$;
  $P_\wedge(f, h) = \bigwedge\limits_{i\in I} P_i(f(i), h(i)) \le P_j(f(j), h(j)) = P_j(\pi_j(f), \pi_j(h))$ for all $j \in I$.
- The task is now to prove that for each pair $((C, P_C, E_C), (p_i)_{i\in I})$, where (C, P_C, E_C) is a $\mathbf{POS}(L)$ object and for each $j \in I$,
  $p_j : (C, P_C, E_C) \to (X_j, P_j, E_j)$ is a morphism there exists a unique $\mathbf{POS}(L)$

morphism $q : (C, P_C, E_C) \to (\prod_i X_i, P_\wedge, E_\wedge)$ such that for each $j \in I$, the triangle



commutes.

Let us first prove the existence of the morphism q.

We define $q : (C, P_C, E_C) \to (\prod_i X_i, P_\wedge, E_\wedge)$ in the following way:

$$\forall c \in C \; q(c) = f_c : \; f_c(j) = p_j(c) \; \forall j \in I.$$

We have to prove that q is an extensional, order-preserving mapping:

We know that p_j is an extensional, order-preserving mapping for every $j \in I$. This means that $E_C(c_1, c_2) \le E_j(p_j(c_1), p_j(c_2))$ and $P_C(c_1, c_2) \le P_j(p_j(c_1), p_j(c_2))$ for all $j \in I$.

Thus $E_C(c_1, c_2) \le \bigwedge_i E_i(p_i(c_1), p_i(c_2)) = E_\wedge(f_{c_1}, f_{c_2}) = E_\wedge(q(c_1), q(c_2))$;
$P_C(c_1, c_2) \le \bigwedge_i P_i(p_i(c_1), p_i(c_2)) = P_\wedge(f_{c_1}, f_{c_2}) = P_\wedge(q(c_1), q(c_2))$.

Now it is sufficient to prove that the above diagram commutes and that q is the unique morphism for which this diagram commutes. The proof is similar as in the case of product in the category SET.

□

Theorem 3.1.2. *A coproduct of a family $((X_i, P_i, E_i))_{i \in I}$ of $\mathbf{POS}(L)$ objects is a pair $((\mu_i)_{i \in I}, (\bigcup_i X_i, P_\cup, E_\cup))$ where $\bigcup_i X_i$ is disjoint union,*

$$E_\cup(a,b) = \begin{cases} E_i(a,b), & \text{if } a, b \in X_i \\ 0, & \text{otherwise} \end{cases}; \; P_\cup(a,b) = \begin{cases} P_i(a,b), & \text{if } a, b \in X_i \\ 0, & \text{otherwise} \end{cases}$$

and $\mu_i : (X_i, P_i, E_i) \to (\bigcup_i X_i, P_\cup, E_\cup)$ such that $\mu_i(a) = a$.

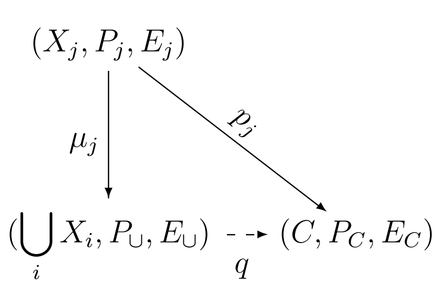
Proof. • Let us prove that $(\bigcup_i X_i, P_\cup, E_\cup)$ is an object in the category $\mathbf{POS}(L)$. This means we should prove that $E_\cup$ is an L-valued equivalence and $P_\cup$ is an

L-$E_{\cup}$-order relation. We skip the proof that $E_{\cup}$ is an L-valued equivalence since it is similar to the proof that $P_{\cup}$ is an L-$E_{\cup}$-order relation. We now turn to prove $E_{\cup}$-reflexivity, transitivity and $E_{\cup}$-antisymmetry of the L-$E_{\cup}$-order relation $P_{\cup}$:

- If $a, b \in \bigcup_i X_i$ and there exists an index j such that $a, b \in X_j$, then $E_{\cup}(a, b) = E_j(a, b) \leq P_j(a, b) = P_{\cup}(a, b)$ since all relations P_i are reflexive. Otherwise, $P_{\cup}(a, b) = 0$ and $E_{\cup}(a, b) = 0$ and then obviously $E_{\cup}(a, b) \leq P_{\cup}(a, b)$.
- We have to prove that $P_{\cup}(a, b) * P_{\cup}(b, c) \leq P_{\cup}(a, c)$. We consider the case when there exists an index j such that $a, b, c \in X_j$ (otherwise $P_{\cup}(a, b) = 0$ or $P_{\cup}(a, c) = 0$ and then obviously $P_{\cup}(a, b) * P_{\cup}(b, c) \leq P_{\cup}(a, c)$). Thus $P_{\cup}(a, b) * P_{\cup}(b, c) = P_j(a, b) * P_j(b, c) \leq P_j(a, c) = P_{\cup}(a, c)$ since all relations P_i are transitive.
- We have to prove that $P_{\cup}(a, b) * P_{\cup}(a, b) \leq E_{\cup}(a, b)$: We consider the case when there exists an index j such that $a, b \in X_j$ (otherwise $P_{\cup}(a, b) = 0$ and $P_{\cup}(b, a) = 0$ and then obviously $P_{\cup}(a, b) * P_{\cup}(b, a) \leq E_{\cup}(a, b)$). Thus $P_{\cup}(a, b) * P_{\cup}(b, a) = P_j(a, b) * P_j(b, a) \leq E_j(a, b) = E_{\cup}(a, b)$ since all P_i are antisymmetric L-valued relations.

- We are now in position to show that μ_j are morphisms for all $j \in I$. We have to prove that for all $j \in I$ and $\forall a, b \in X_j$ $P_j(a, b) \leq P_{\cup}(a, b)$, but we know even more, that $P_j(a, b) = P_{\cup}(a, b)$.

- Our next aim is to verify that for each pair $((p_i)_{i \in I}, (C, P_C, E_C))$, where (C, P_C, E_C) is a $\mathbf{POS}(L)$ object and $p_j : (X_j, P_j, E_j) \to (C, P_C, E_C)$ is a morphism for each $j \in I$, there exists a unique $\mathbf{POS}(L)$ morphism $q : (\bigcup_i X_i, P_{\cup}, E_{\cup}) \to (C, P_C, E_C)$ such that for each $j \in I$, the triangle



commutes.

Let us first prove the existence of the morphism q.

We define $q : (\bigcup_i X_i, P_\cup, E_\cup) \to (C, P_C, E_C)$ in the following way:

$$\forall x \in \bigcup_i X_i \;\; q(x) = p_j(x) : x \in X_j.$$

We have to prove that q is an order-preserving mapping:
$P_\cup(a,b) \leq P_C(q(a), q(b)) \; \forall a, b \in \bigcup_i X_i$.
We assume that there exists an index j such that $a, b \in X_j$, otherwise $P_\cup(a,b) = 0$ and obviously $P_\cup(a,b) \leq P_C(q(a), q(b))$.
Thus $a, b \in X_j \Rightarrow P_\cup(a,b) = P_j(a,b) \leq P_C(p_j(a), p_j(b)) = P_C(q(a), q(b))$.
Similarly we can prove that q is an extensional mapping.
Now it remains to prove that the above diagram commutes and that q is the unique morphism for which the diagram defined above commutes. The proof is similar as in the case of product in the category SET.

□

Theorem 3.1.3. *In the category $POS(L)$ the following diagram is a pullback:*

$$\begin{array}{ccc} (V, P_V, E_V) & \xrightarrow{\pi_X} & (X, P_X, E_X) \\ \downarrow{\scriptstyle \pi_Y} & & \downarrow{\scriptstyle g} \\ (Y, P_Y, E_Y) & \xrightarrow{f} & (Z, P_Z, E_Z) \end{array}$$

where $V = \{(x,y) \in X \times Y : g(x) = f(y)\}$*;*
$P_V((x_1,y_1),(x_2,y_2)) = P_X(x_1,x_2) \wedge P_Y(y_1,y_2)$*;*
$E_V((x_1,y_1),(x_2,y_2)) = E_X(x_1,x_2) \wedge E_Y(y_1,y_2)$*;*
$\pi_X(x,y) = x$*;* $\pi_Y(x,y) = y$*.*

The proof that E_V is an L-valued equivalence and P_V is an L-E_V-order relation is similar as in Theorem 3.1.1. The other part of the proof is similar as in the case of pullback in the category SET.

3.2 L-valued analogue of $\mathrm{POS}(L)$ category

We describe here three different L-valued categories whose objects are L-E-ordered sets:

1. Let us introduce the mapping μ for the category $\mathbf{POS}(L)$ described in the previous section in the following way:

   $$\mu(f) = \inf_{x_1,x_2 \in X} (P'(f(x_1), f(x_2)) \mapsto P(x_1, x_2))$$

   where $f : (X, P) \to (Y, P')$, $f \in Mor(\mathbf{POS}(L))$ and the mapping ω:
   $\omega((X, P, E)) = 1$ where (X, P, E) is an object of the category $\mathbf{POS}(L)$.
   In this case we obtain the category, where the intuitive meaning of the value $\mu(f)$ is the degree to which a morphism f is an order-reflecting mapping. Thus the obtained L-valued category is something more than just an L-valued analogue of $\mathbf{POS}$ category, because all morphisms are order-preserving mappings, but additionally we introduce the "order-reflecting" degrees for the morphisms. It is worth to mention that the bottom frame of this L-valued category is exactly the category $\mathbf{POS}(L)$. This approach was investigated in [Auth1] and we do not discuss it here.

2. The idea of the second approach is to omit the following order-preserving property: $P(x_1, x_2) \leq P'(f(x_1), f(x_2))$ for the morphism $f : (X, P) \to (Y, P')$, but to use the graded order-preserving property described by the mapping μ:

   $$\mu(f) = \inf_{x_1,x_2 \in X} (P(x_1, x_2) \mapsto P'(f(x_1), f(x_2))).$$

   In this case we obtain an L-valued analogue of the $\mathbf{POS}(L)$ category. To be more formal we describe this case by applying the scheme proposed on page 21. In our case the scheme can be described as follows.

   Let $\phi : \mathbf{POS}(L) \to L\text{-}\mathbf{SET}$ be the functor assigning to each $\mathbf{POS}(L)$ object (X, P, E) the support set (X, E) and leaving morphisms unchanged. Then according to the scheme we come to the category $\mathbf{POS}(L)_{L\text{-}\mathbf{SET}}$. Its objects are the same as in $\mathbf{POS}(L)$, but its morphisms are all mappings between the corresponding support sets. Starting from this category as the crisp bottom frame we define the L-valued category $L\text{-}\mathbf{POS}(L)$ by setting $\omega((X, P, E)) = 1$ for every $L\text{-}\mathbf{POS}(L)$ object (X, P, E) and define the mapping μ as above.

3. The idea of the third approach is to omit the following order-preserving property: $P(x_1, x_2) \leq P'(f(x_1), f(x_2))$ for the morphism $f : (X, P) \to (Y, P')$, but to use the graded order-preserving-and-reflecting property described by the mapping μ:

   $$\mu(f) = \inf_{x_1,x_2 \in X} (P(x_1, x_2) \leftrightarrow P'(f(x_1), f(x_2))).$$

In this case we obtain something more than just an L-valued analogue of the $\mathbf{POS}(L)$ category.

In the section below we consider only constructions 2 and 3. The main attention is paid to the properties of the category presented by construction 2 since we consider it as the direct L-valued analogue of $\mathbf{POS}(L)$ category.

3.2.1 Construction of the category

Let us observe the category L-$\mathbf{POS}(L)$.

L-$\mathbf{POS}(L)$-objects are L-E-ordered sets and L-$\mathbf{POS}(L)$-morphisms are extensional mappings between them.

$$
\begin{aligned}
&L\text{-}\mathbf{POS}(L) = (Ob(L\text{-}\mathbf{POS}(L)), \omega, Mor(L\text{-}\mathbf{POS}(L)), \mu, \circ), \ where\\
&Ob(L\text{-}\mathbf{POS}(L)) = \{(X,P,E) : (X,P,E) \text{ is an } L\text{-}E\text{-ordered set}\};\\
&Mor(L\text{-}\mathbf{POS}(L)) = \{f : (X,P,E) \to (Y,P',E') :\\
&\forall x_1, x_2 \in X \ E(x_1,x_2) \le E'(f(x_1), f(x_2))\};\\
&\mu(f) = \inf_{x_1,x_2 \in X} (P(x_1,x_2) \mapsto P'(f(x_1), f(x_2))), where\\
&f : (X,P,E) \to (Y,P',E');\\
&\omega((X,P,E)) = 1 \quad \forall (X,P,E) \in Ob(L\text{-}\mathbf{POS}(L)).
\end{aligned}
$$

Theorem 3.2.1.

$$L\text{-}\mathbf{POS}(L) = (Ob(L\text{-}\mathbf{POS}(L)), \omega, Mor(L\text{-}\mathbf{POS}(L)), \mu, \circ)$$

is an L-valued category.

Proof. It is obvious that $Ob(L\text{-}\mathbf{POS}(L))$ and $Mor(L\text{-}\mathbf{POS}(L))$ form a crisp category, thus we have to prove the conditions for the mappings μ and ω, the part which characterizes the L-valued case.

1. $\mu(f) \le \omega((X,P,E)) \wedge \omega((Y,P',E'))$ for all $(X,P,E), (Y,P',E') \in Ob(L\text{-}\mathbf{POS}(L))$ and for all $f \in Mor_{L\text{-}\mathbf{POS}(L)}((X,P,E),(Y,P',E'))$, since $\omega((X,P,E)) = 1$ and $\omega((Y,P',E')) = 1$.

2. Let us prove that $\mu(g \circ f) \ge \mu(g) * \mu(f)$ where $f : (X,P,E) \to (Y,P',E')$ and

$g : (Y, P', E') \to (Z, P'', E'')$:

$$
\begin{aligned}
\mu(g \circ f) &= \inf_{x_1,x_2} (P(x_1, x_2) \mapsto P''(g(f(x_1)), g(f(x_2)))) \geq \\
&\geq \inf_{x_1,x_2} ((P(x_1, x_2) \mapsto P'(f(x_1), f(x_2))) * \\
&* \ (P'(f(x_1), f(x_2)) \mapsto P''(g(f(x_1)), g(f(x_2))))) \geq \\
&\geq \inf_{x_1,x_2} (P(x_1, x_2) \mapsto P'(f(x_1), f(x_2))) * \\
&* \inf_{x_1,x_2} (P'(f(x_1), f(x_2)) \mapsto P''(g(f(x_1)), g(f(x_2)))) \geq \\
&\geq \inf_{x_1,x_2} (P(x_1, x_2) \mapsto P'(f(x_1), f(x_2))) * \\
&* \inf_{y_1,y_2} (P'(y_1, y_2) \mapsto P''(g(y_1), g(y_2))) = \\
&= \mu(f) * \mu(g).
\end{aligned}
$$

We obtain that $\mu(g \circ f) \geq \mu(g) * \mu(f)$.

We have used the properties of the cl-monoid and the following inequality in the proof :

$$
\begin{aligned}
&\inf_{x_1,x_2} (P'(f(x_1), f(x_2)) \mapsto P''(g(f(x_1)), g(f(x_2)))) \geq \\
&\qquad \geq \inf_{y_1,y_2} (P'(y_1, y_2) \mapsto P''(g(y_1), g(y_2)))
\end{aligned}
$$

This follows from the fact that:

$$
\begin{aligned}
&\{\, P'(f(x_1), f(x_2)) \mapsto P''(g(f(x_1)), g(f(x_2))) : x_1, x_2 \in X\} \ \subseteq \\
&\qquad \subseteq \{\, P'(y_1, y_2) \mapsto P''(g(y_1), g(y_2)) : y_1, y_2 \in Y\} \ \subset L.
\end{aligned}
$$

□

Now let us define the category FL-$\mathbf{POS}(L)$ which we have discussed in the third clause of the previous section. It is worth to mention that the only difference between L-valued category FL-$\mathbf{POS}(L)$ and L-valued category L-$\mathbf{POS}(L)$ is in the choice of mapping μ.

FL-$\mathbf{POS}(L)$-objects are L-E-ordered sets and FL-$\mathbf{POS}(L)$-morphisms are extensional mappings.

$$FL\text{-}\mathbf{POS}(L) = (Ob(FL\text{-}\mathbf{POS}(L)), \omega, Mor(FL\text{-}\mathbf{POS}(L)), \mu, \circ), \text{ where}$$
$$Ob(FL\text{-}\mathbf{POS}(L)) = \{(X, P, E) : (X, P, E) \text{ is an } L\text{-}E\text{-ordered set}\};$$
$$Mor(FL\text{-}\mathbf{POS}(L)) = \{f : (X, P, E) \to (Y, P', E') :$$
$$\forall x_1, x_2 \in X \; E(x_1, x_2) \leq E'(f(x_1), f(x_2))\};$$
$$\mu(f) = \inf_{x_1, x_2 \in X} (P(x_1, x_2) \leftrightarrow P'(f(x_1), f(x_2))), \text{ where}$$
$$f : (X, P, E) \to (Y, P', E');$$
$$\omega((X, P, E)) = 1 \quad \forall (X, P, E) \in Ob(FL\text{-}\mathbf{POS}(L)).$$

Theorem 3.2.2.

$$FL\text{-}\mathbf{POS}(L) = (Ob(FL\text{-}\mathbf{POS}(L)), \omega, Mor(FL\text{-}\mathbf{POS}(L)), \mu, \circ)$$

is an L-valued category.

Proof. All properties of an L-valued category (except of the property $\mu(g \circ f) \geq \mu(g) * \mu(f)$) are straightforward. It is only necessary to prove that

$$\begin{aligned}
& \inf_{x_1, x_2} (P(x_1, x_2) \leftrightarrow P''(g(f(x_1)), g(f(x_2)))) \geq \\
\geq \; & \inf_{x_1, x_2} (P(x_1, x_2) \leftrightarrow P'(f(x_1), f(x_2))) * \inf_{y_1, y_2} (P'(y_1, y_2) \leftrightarrow P''(g(y_1), g(y_2)))
\end{aligned}$$

where $f : (X, P, E) \to (Y, P', E')$, $g : (Y, P', E') \to (Z, P'', E'')$,

$x_1, x_2 \in X$ and $y_1, y_2 \in Y$:

$$\begin{aligned}
& \inf_{x_1, x_2} (P(x_1, x_2) \leftrightarrow P''(g(f(x_1)), g(f(x_2)))) = \\
= \; & \inf_{x_1, x_2} ((P(x_1, x_2) \mapsto P''(g(f(x_1)), g(f(x_2)))) * \\
& *(P''(g(f(x_1)), g(f(x_2))) \mapsto P(x_1, x_2))) \geq \\
\geq \; & \inf_{x_1, x_2} ((P(x_1, x_2) \mapsto P'(f(x_1), f(x_2))) * \\
& *(P'(f(x_1), f(x_2)) \mapsto P''(g(f(x_1)), g(f(x_2)))) * \\
& *(P''(g(f(x_1)), g(f(x_2))) \mapsto P'(f(x_1), f(x_2))) * \\
& *(P'(f(x_1), f(x_2)) \mapsto P(x_1, x_2))) \geq \\
\geq \; & \inf_{x_1, x_2} ((P(x_1, x_2) \mapsto P'(f(x_1), f(x_2))) * (P'(f(x_1), f(x_2)) \mapsto P(x_1, x_2))) * \\
& * \inf_{x_1, x_2} ((P'(f(x_1), f(x_2)) \mapsto P''(g(f(x_1)), g(f(x_2)))) * \\
& *(P''(g(f(x_1)), g(f(x_2))) \mapsto P'(f(x_1), f(x_2)))) \geq
\end{aligned}$$

$$\begin{aligned}
\geq & \inf_{x_1,x_2}((P(x_1,x_2) \mapsto P'(f(x_1),f(x_2))) * (P'(f(x_1),f(x_2)) \mapsto P(x_1,x_2)))) * \\
& * \inf_{y_1,y_2}((P'(y_1,y_2) \mapsto P''(g(y_1),g(y_2))) * (P''(g(y_1),g(y_2)) \mapsto P'(y_1,y_2)))) = \\
= & \inf_{x_1,x_2}(P(x_1,x_2) \leftrightarrow P'(f(x_1),f(x_2))) * \inf_{y_1,y_2}(P'(y_1,y_2) \leftrightarrow P''(g(y_1),g(y_2))).
\end{aligned}$$

□

Remark 3.2.1. *If the value of a mapping ω is equal to 1 for all objects of an L-valued category we do not write ω. For instance we will write*
L-$\mathbf{POS}(L) = (Ob(L\text{-}\mathbf{POS}(L)), Mor(L\text{-}\mathbf{POS}(L)), \mu, \circ)$.

3.2.2 Properties of the category

In this section we study properties of the category L-$\mathbf{POS}(L)$.

Proposition 3.2.1. *If we consider the crisp category $\mathbf{POS}$ as an L-valued category $\mathbf{POS} = (Ob(\mathbf{POS}), Mor(\mathbf{POS}), \mu_{\mathbf{POS}}, \circ)$, where $\mu_{\mathbf{POS}}(f)$ is equal to 1 if and only if f is an order-preserving mapping and 0 otherwise, then the category $\mathbf{POS}$ is an L-valued subcategory of the category L-$\mathbf{POS}(L)$.*

We continue by considering special objects and special morphisms in the category L-$\mathbf{POS}(L)$.

Proposition 3.2.2. *The empty set is the unique initial (1-initial) object in L-$\mathbf{POS}(L)$.*

Proposition 3.2.3. *The singleton set with the uniquely constructed L_1-E_1-order on it is the terminal (1-terminal) object in L-$\mathbf{POS}(L)$.*

Corollary 3.2.1. *There are no α-zero objects in L-$\mathbf{POS}(L)$.*

Proposition 3.2.4. *An α-morphism $f : (X,P) \to (Y,P')$ is a β-monomorphism (for any $\beta \in L$) if and only if f is an injective mapping.*

Proposition 3.2.5. *An α-morphism $f : (X,P) \to (Y,P')$ is a β-epimorphism (for any $\beta \in L$) if and only if f is a surjection.*

Proposition 3.2.6. *An α-morphism $f : (X,P) \to (Y,P')$ is a β-bimorphism (for any $\beta \in L$) if and only if f is a bijection.*

For the previous three proposition the proof is a straightforward generalization of the proofs of Propositions 3.1.6, 3.1.7 and 3.1.8.

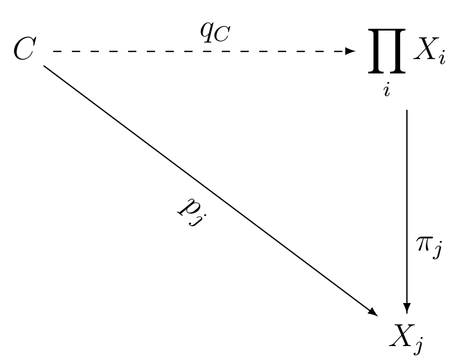
Remark 3.2.2. *From the above propositions it is easy to see that studying an L-valued category, if the property holds for the bottom and top frame of L-valued category, then this property holds for the L-valued category for each $\beta \in L$. For example, studying L-$\mathbf{POS}(L)$ category, the bottom frame is $\mathbf{SET}$ category, the top frame is $\mathbf{POS}(\mathbf{L})$ category. Knowing that "a morphism is an epimorphism if and only if f is a surjection" in $\mathbf{SET}$ and in $\mathbf{POS}(\mathbf{L})$ we conclude that an "α-morphism is a β-epimorphism (for any $\beta \in L$) if and only if f is a surjection".*

We continue by describing product in the category L-$\mathbf{POS}(L)$. To do this we define the product in the context of L-valued categories.

Let $\mathbb{C} = (Ob(\mathbb{C}), Mor(\mathbb{C}), \mu, \circ)$ be an L-valued category.

Definition 3.2.1. *A pair $(\prod_i X_i, (\pi_i)_{i\in I})$ is an α-product of a family $(X_i)_{i\in I}$ of $\mathbb{C}$-objects if and only if:*

- *$\prod_i X_i$ is a $\mathbb{C}$-object;*
- *π_i are $\mathbb{C}$-morphisms such that $\inf_i \mu(\pi_i) \geq \alpha$;*
- *for each pair $(C, (p_i)_{i\in I})$, where C is a $\mathbb{C}$-object and for each $j \in I$, $p_j : C \to X_j$ is a $\mathbb{C}$-morphism and $\mu(p_j) \geq \mu(\pi_j)$ there exists a unique $\mathbb{C}$-morphism $q_C : C \to \prod_i X_i$ such that $\mu(q_C) \geq \alpha$ and for each $j \in I$, the triangle*



commutes.

Now we propose an alternative definition where we try to separate the "crisp" and the "fuzzy parts", but first we should define the notion of an α-source.

Definition 3.2.2. *An α-source in an L-valued category $\mathbb{C}$ is a pair $(X, (f_i)_{i\in I})$, where X is a $\mathbb{C}$-object and $(f_i : X \to X_i)_{i\in I}$ is a family of $\mathbb{C}$-morphisms each with domain X and $\inf_i \mu(f_i) \geq \alpha$.*

Definition 3.2.3. *An α-source $(\prod_i X_i, (\pi_i)_{i\in I})$ in an L-valued category $\mathbb{C}$ is an α-product of a family $(X_i)_{i\in I}$ of $\mathbb{C}$-objects if and only if it is a product in a crisp category $\mathbb{C} = (Ob(\mathbb{C}), Mor(\mathbb{C}), \circ)$ and for each α-source $(C, (p_i)_{i\in I})$ such that $\mu(p_j) \geq \mu(\pi_j)$ for all $j \in I$, if q_C is a unique morphism $q_C : C \to \prod_i X_i$ then*

$$\mu(q_C) \geq \alpha.$$

Remark 3.2.3. *The idea of defining notions in the frame of L-valued categories is as follows: the product in the top frame of L-valued category should be 1-product in an L-valued category. The same idea is valid for other notions.*

Proposition 3.2.7. *If a pair $(\prod_i X_i, (\pi_i)_{i\in I})$ is an α-product of a family $(X_i)_{i\in I}$ of $\mathbb{C}$-objects then*

1. *for each β-isomorphism $h : A \to \prod_i X_i$ a pair $(A, (\pi_i \circ h)_{i\in I})$ is an $\alpha * \beta$-product of a family $(X_i)_{i\in I}$.*
2. *for each α-product $(A, (p_i)_{i\in I})$ of a family $(X_i)_{i\in I}$ of $\mathbb{C}$-objects there exists an α-isomorphism $h : A \to \prod_i X_i$ such that $p_i = \pi_i \circ h \ \forall i \in I$.*

Proposition 3.2.8. *If a pair $(A, (\pi_i)_{i\in I})$ is an α-product of a family $(A_i)_{i\in I}$ of $\mathbb{C}$-objects and for each $i \in I$ $(A_i, (p_{ij})_{j\in J_i})$ is an β-product of a family $(A_{ij})_{j\in J_i}$ then a pair $(A, (p_{ij} \circ \pi_i)_{i\in I})$ is an $\beta * \alpha$-product.*

The last two propositions are straightforward generalization of the classical case.

We observe some α-products of a family $((X_i, P_i, E_i))_{i\in\{\overline{1,n}\}}$ of L-$\mathbf{POS}(L)$ objects, where L is a concrete cl-monoid in the next examples.

Example 3.2.1. *Let L be a cl-monoid. A pair $((X_1 \times \cdots \times X_n, P_\wedge, E_\wedge), (\pi_i)_{i\in\{\overline{1,n}\}})$ is a 1-product of a family $((X_i, P_i, E_i))_{i\in\{\overline{1,n}\}}$ of L-$\mathbf{POS}(L)$ objects, where*
$E_\wedge((a_1, a_2, ..., a_n), (b_1, b_2, ..., b_n)) = \bigwedge_i E_i(a_i, b_i)$,
$P_\wedge((a_1, a_2, ..., a_n), (b_1, b_2, ..., b_n)) = \bigwedge_i P_i(a_i, b_i)$ and
$\pi_i : (\prod_i X_i, P_\wedge, E_\wedge) \to (X_i, P_i, E_i)$ are defined by $\pi_i((a_1, a_2, ..., a_n)) = a_i$.

In the next two examples for the brevity of explanation the relations E_i for all i are crisp equivalence relations: $E_i(a,b) = \begin{cases} 1, & \text{if } a = b \\ 0, & \text{otherwise} \end{cases}$
and the relation $E_\wedge$ will be defined as follows:
$E_\wedge((a_1, a_2, ..., a_n), (b_1, b_2, ..., b_n)) = \bigwedge_i E_i(a_i, b_i)$.

Example 3.2.2. *Let $L = ([0,1], \leq, \wedge, \vee, T, \mapsto_T)$ be a cl-monoid, where T is a t-norm without zero divisors, $\mapsto_T$ is a corresponding residuum and let A_ω be the weakest aggregation function defined by:*

$$A_\omega(x_1, x_2, ..., x_n) = \begin{cases} 1, & x_1 = x_2 = ... = x_n = 1 \\ 0, & \textit{otherwise} \end{cases}.$$

The requirement on a family $((X_i, P_i, E_i))_{i\in\{\overline{1,n}\}}$ of L-$\mathbf{POS}(L)$-objects is that sets X_i are non-empty sets and there exists an index j such that $\exists a_j, b_j \in X_j : P_j(a_j, b_j) \in (0, 1)$.

Then a pair $((X_1 \times \cdots \times X_n, P_{A_\omega}, E_\wedge), (\pi_i)_{i\in\{\overline{1,n}\}})$ is a 0-product of a family $((X_i, P_i, E_i))_{i\in\{\overline{1,n}\}}$ of L-$\mathbf{POS}(L)$ objects, where
$P_{A_\omega}((a_1, \ldots, a_n), (b_1, \ldots, b_n)) = A_\omega(P_1(a_1, b_1), \ldots, P_n(a_n, b_n))$ and
$\pi_i : (\prod_i X_i, P_{A_\omega}, E_\wedge) \to (X_i, P_i, E_i)$ are defined by $\pi_i((a_1, a_2, ..., a_n)) = a_i$.
For example for L-$\mathbf{POS}(L)$-morphism
$h : (X_1 \times \cdots \times X_n, P_\wedge, E_\wedge) \to (X_1 \times \cdots \times X_n, P_{A_\omega}, E_\wedge)$ $\mu(h)$ is equal to 0 if the above conditions are fulfilled and thus $((X_1 \times \cdots \times X_n, P_{A_\omega}, E_\wedge), (\pi_i)_{i\in\{\overline{1,n}\}})$ is a 0-product.

Example 3.2.3. *Let $L = ([0,1], \leq, \wedge, \vee, T_L, \mapsto_{T_L})$ be a cl-monoid, where T_L is Łukasiewicz t-norm and $\mapsto_{T_L}$ is the corresponding residuum.*
Then a pair $((X_1 \times X_2, P_{T_L}, E_\wedge), (\pi_i)_{i\in\{1,2\}})$ is a 0.5-product of a family $((X_i, P_i, E_i))_{i\in\{1,2\}}$ of L-$\mathbf{POS}(L)$ objects, where
$P_{T_L}((a_1, a_2), (b_1, b_2)) = T_L(P_1(a_1, b_1), P_2(a_2, b_2))$ and
$\pi_i : (\prod_i X_i, P_{T_L}, E_\wedge) \to (X_i, P_i, E_i)$ are defined by $\pi_i((a_1, a_2)) = a_i$.

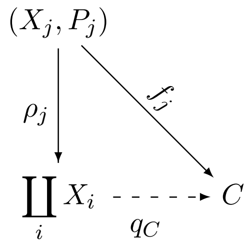
Proof:
We know that for the Łukasiewicz t-norm T_L (as it is a left-continuous t-norm) $x \mapsto_{T_L} y = 1 \Leftrightarrow x \leq y$. Obviously $T_L(P_1(a_1, b_1), P_2(a_2, b_2)) \leq P_i(a_i, b_i)$ for all i and for all $a_i, b_i \in X_i$. Thus $\mu(\pi_i) = 1$ for all i.
So for every other source $((C, P_C, E_C), (f_i)_{i\in\{1,2\}})$, where $\mu(\pi_i) \leq \mu(f_i)$ we know that $\mu(f_i) = 1$ and then $P_C(c_1, c_2) \leq P_i(f_i(c_1), f_i(c_2))$ for all $c_1, c_2 \in C$.
Thus there exists unique morphism $h_C : (C, P_C, E_C) \to (X_1 \times X_2, P_\wedge, E_\wedge)$ such that

$P_C(c_1, c_2) \leq P_\wedge(h_C(c_1), h_C(c_2))$ *(Theorem 3.1.1'), where* $P_\wedge((a_1, a_2), (b_1, b_2)) = P_1(a_1, b_1) \wedge P_2(a_2, b_2)$*. This gives* $\mu(h_C) = 1$*. Thus for every morphism* $q_C : (C, P_C, E_C) \to (X_1 \times X_2, P_{T_L}, E_\wedge)$ *the following inequality holds:* $\mu(q_C) \geq \mu(e_{X_1 \times X_2}) * \mu(h_C)$*, where* $e_{X_1 \times X_2} : (X_1 \times X_2, P_\wedge, E_\wedge) \to (X_1 \times X_2, P_{T_L}, E_\wedge)$ *is defined by* $e_{X_1 \times X_2}((x_1, x_2)) = (x_1, x_2)$*. It is easy to calculate that* $\mu(e_{X_1 \times X_2}) \geq 0.5$*, thus* $\mu(q_C) \geq 0.5$*.*

In the same way we could define and investigate other special constructions. For example:

Definition 3.2.4. *A pair* $((\rho_i)_{i \in I}, \coprod_i X_i,)$ *is an* α*-coproduct of a family* $(X_i)_{i \in I}$ *of* $\mathbb{C}$*-objects if and only if:*

- $\coprod_i X_i$ *is an object in the category* $\mathbb{C}$*;*
  ρ_i *are morphisms in the category* $\mathbb{C}$ *such that* $\inf_i \mu(\rho_i) \geq \alpha$*;*
- *for each pair* $((f_i)_{i \in I}, C)$*, such that* $\mu(f_j) \geq \mu(\rho_j)$ *there exists a unique* $\mathbb{C}$*-morphism* $q_C : \coprod_i X_i \to C$ *such that* $\mu(q_C) \geq \alpha$ *and for each* $j \in I$*, the following triangle commutes.*



3.3 Generalized L-$POS(L)$ category

Now let us replace the reflexivity property for fuzzy equivalence relation:

$$E(x, x) = 1 \; for \; all \; x \in X$$

with the following property of weak reflexivity:

$$E(x, x) \geq E(x, y) \; for \; all \; x, y \in X.$$

If there exists an element $x \in X$ such that $E(x, x) = 0$ then $E(x, y) = E(y, x) = 0$ for all elements y from the set X. Since it is not a natural situation we delete such elements from the set X. Further we observe only L-valued sets for which the following condition holds:

$$E(x, x) > 0 \; for \; all \; x \in X.$$

To distinguish from the previous case we call the relation which respects weak reflexivity, symmetry and transitivity as weak fuzzy equivalence and denote it by E_w. The intuitive meaning of the value $E_w(x,x)$ is the degree of the existence of an element x. Now we build a category $pos_{(X,P,E_w)}$ whose objects are elements of an L-E_w-ordered set:

- Objects: $Ob(pos_{(X,P,E_w)}) = \{x : x \in X\}$;
- Morphisms: $f : x \to y$, $f \in Mor(pos_{(X,P,E_w)}) \Leftrightarrow P(x,y) > 0$.

It is obviously a small category.

Now let us involve an L-valued subclass of the class of morphisms as a mapping from the class of morphisms to the cl-monoid L ($\mu : Mor(pos_{(X,P,E_w)}) \to L$) in the following way:

$$f : x \to y \Rightarrow \mu(f) = E_w(x,x) \wedge P(x,y) \wedge E_w(y,y)$$

and an L-valued subclass of the class of objects as a mapping from the class of objects to the cl-monoid L ($\omega : Ob(pos_{(X,P,E_w)}) \to L$)

$$x \in X \Rightarrow \omega(x) = E_w(x,x).$$

We have constructed a small L-valued category

$$L\text{-}pos_{(X,P,E_w)} = (Ob(pos_{(X,P,E_w)}), \omega, Mor(pos_{(X,P,E_w)}), \mu, \circ).$$

Let us verify all necessary properties for the L-valued category:

- Note that for a given morphism $f : x \to y$ we have
  $\mu(f) = E_w(x,x) \wedge P(x,y) \wedge E_w(y,y) \leq E_w(x,x) \wedge E_w(y,y) = \omega(x) \wedge \omega(y)$.
- The next property we have to prove is $\mu(g \circ f) \geq \mu(g) * \mu(f)$, if the morphism $g \circ f$ exists. Let $f : x \to y$ and $g : y \to z$. Then by transitivity of the fuzzy order P we have

  $$\mu(g \circ f) = E_w(x,x) \wedge P(x,z) \wedge E_w(z,z) \geq$$
  $$\geq E_w(x,x) \wedge (P(x,y) * P(y,z)) \wedge E_w(z,z).$$

  Let us denote $a := E_w(x,x)$;$b := P(x,y)$;$c := P(y,z)$;$d := E_w(z,z)$;
  $e := E_w(y,y)$. Then

  $a \wedge (b * c) \wedge d \geq (a * d) \wedge (b * c) \wedge (a * c) \geq (a * c) \wedge (a * d) \wedge (b * c) \wedge (b * d) \geq$

$$\geq (a \wedge b) * (c \wedge d) \geq (a \wedge b \wedge e) * (e \wedge c \wedge d).$$

Thus

$$\mu(g \circ f) = E_w(x,x) \wedge P(x,z) \wedge E_w(z,z) \geq$$

$$\geq (E_w(x,x) \wedge P(x,y) \wedge E_w(y,y)) * (E_w(y,y) \wedge P(y,z) \wedge E_w(z,z)) =$$

$$= \mu(f) * \mu(g).$$

Remark 3.3.1. *Obviously in the category $pos_{(X,P,E_w)}$ we have $\mu(id_x) = E_w(x,x) \wedge P(x,x) \wedge E_w(x,x) = E_w(x,x) = \omega(x)$, where id_x is the identity morphism. In this case the L-valued class of objects ω is determined by the L-valued class of morphisms μ.*

It is also possible to define the mapping $\mu : Mor(pos_{(X,P,E_w)}) \to L$ in the following way:

$$f : x \to y \Rightarrow \mu(f) = E_w(x,x) * P(x,y) * E_w(y,y).$$

In this situation $\mu(id_X) \leq \omega(X)$.

Then we build an L-valued category, whose objects are small categories, as in the previous sections. Thus we obtain the following L-valued category:

$$L\text{-}\mathbf{POS}(L) = (Ob(L\text{-}\mathbf{POS}(L)), \omega, Mor(L\text{-}\mathbf{POS}(L)), \mu, \circ), \text{ where}$$
$$Ob(L\text{-}\mathbf{POS}(L)) = \{(X,P,E_w) : (X,P,E_w) \text{ is an } L\text{-}E_w\text{-ordered set}\};$$
$$Mor(L\text{-}\mathbf{POS}(L)) = \{f : (X,P,E_w) \to (Y,P',E'_w) :$$
$$\forall x_1, x_2 \in X \; E_w(x_1,x_2) \leq E'_w(f(x_1), f(x_2))\};$$
$$\mu(f) = \inf_{x \in X}(E_w(x,x)) * \inf_{x_1,x_2 \in X}(P(x_1,x_2) \mapsto P'(f(x_1), f(x_2))) * \inf_{y \in Y}(E'_w(y,y)),$$
$$\text{where}$$
$$f : (X,P,E_w) \to (Y,P',E'_w);$$
$$\omega((X,P,E_w)) = \inf_{x \in X}(E_w(x,x)) \quad \forall (X,P,E_w) \in Ob(L\text{-}\mathbf{POS}(L)).$$

To prove that it is indeed an L-valued category we refer the reader to Theorem 3.2.1 and the construction of new L-valued categories from the existing ones (page 20).

3.4 Conclusion on L-valued categories

The main aim of this chapter was:

- To construct an L-valued analogue of $\mathbf{POS}$ category. The scheme of construction justify that this category is constructed in a natural way from the point of view of L-valued categories.
- To study the properties of the constructed L-valued category. What is not only important by itself but also helps to develop and investigate deeply the theory of L-valued categories.

In the future we are going to prepare the work about L-valued categories, what is now possible thanks to the studying of one concrete example. Studying L-$\mathbf{POS}(L)$ category we have made some correction in the definitions (see Definitions 2.4.1, 2.4.3), we defined some new notions and by this developed the scheme of defining new notions in the frame of L-valued category theory (e.g. Definition 3.2.1). The important is also the scheme of fuzzification and we are going to develop it (see also Remark 3.2.2 as one of the possible directions).

In this chapter we have constructed the L-valued category whose objects are L-E-ordered sets and morphisms are "potential" order-preserving mappings. To realize the construction we have introduced the degree (for the class of potential morphisms) to which each morphism is an order-preserving mapping, or, in other words, a monotone mapping. In the natural way we can use this idea for some practical applications. For example to construct an aggregation process in the context of L-valued $\mathbf{POS}$ categories and by this define the graded property of monotonicity for the aggregation function.

It is also possible to apply the above idea to the aggregation of fuzzy relations, in particular, to the aggregation of fuzzy orders. Our suggestion is to involve the degree to which aggregation operator preserves properties of fuzzy relations. So we will be able to calculate this degree for any aggregation function, not only for aggregation operators which preserve properties of fuzzy relations.

In the future we are going to apply the properties of constructed categories for the above mentioned practical applications.

Chapter 4

Involving fuzzy orders in aggregation processes

Representation of several input values by a single output value is the essence of the aggregation process. It is natural to require that the aggregation function should fulfill the boundary conditions and the condition of monotonicity. In the first part of this chapter we will focus on the condition of monotonicity in the process of aggregation and introduce the graded property of monotonicity. In the previous sections when constructing the L-valued analogue of $\mathbf{POS}$ category, we introduced the degree to which a morphism is an order preserving mapping, or, in other words, a monotone mapping. We use the same idea to define the graded property of monotonicity for the aggregation function. In the first section of this chapter the degree of monotonicity will be defined straightforward, but here we construct the aggregation process in the context of L-valued $\mathbf{POS}$ categories and obtain the same definition:

Let us observe the 1-product $((X_1 \times \cdots \times X_n, P_\wedge), (\pi_i)_{i\in\{\overline{1,n}\}})$ of a family $((X_i, P_i))_{i\in\{\overline{1,n}\}}$ of L-$POS(L)$ objects. Then if we construct a mapping
$f : (X_1 \times \cdots \times X_n, P_\wedge) \to (Y, P)$ in the category L-$POS(L)$, the order-preserving degree for this mapping is calculated as:
$\mu(f) = \inf\limits_{\mathbf{x},\mathbf{y}}(P_\wedge(\mathbf{x}, \mathbf{y}) \mapsto P(f(\mathbf{x}), f(\mathbf{y}))$ or
$\mu(f) = \inf\limits_{\mathbf{x},\mathbf{y}}(\bigwedge_i P_i(x_i, y_i) \mapsto P(f(\mathbf{x}), f(\mathbf{y})),$
where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$. We suggest to use this order-preserving degree for an arbitrary function f instead of the crisp definition of monotonicity for the aggregation process.

In the second part we focus on the aggregation of fuzzy relations, in particular, on the aggregation of fuzzy orders. In the literature there are works considering the problem, which aggregation functions preserve properties of fuzzy relations in the aggregation process (see e.g. [39],[38]). Our aim here is to involve the above mentioned concept of the degree μ in order to estimate to what extent do the aggregation functions preserve properties of fuzzy relations.

Our suggestion is to involve the degree to which aggregation function preserves properties of fuzzy relations. So we will be able to calculate this degree for any aggregation function, not only for aggregation functions which preserve properties of fuzzy relations.

Definition 4.0.1. *(cf.* [39] *) Let* $A : [0,1]^n \to [0,1]$ *be a mapping and let* $R_1, R_2, ..., R_n$ *be fuzzy relations* $(R_i : X \times X \to [0,1])$.
An aggregation fuzzy relation R_A $(R_A : X \times X \to [0,1])$ *is described by the formula*

$$R_A(x,y) = A(R_1(x,y), ..., R_n(x,y)),\ x, y \in X.$$

A function A *preserves a property of fuzzy relations if for every fuzzy relation* $R_1, R_2, ..., R_n$ *having this property,* R_A *also has this property.*

In order to calculate how well the function A preserves the properties of fuzzy relations $R_1, R_2, ..., R_n$ we use the mapping ξ: $\xi(A) = \inf\limits_i \mu_i$, where μ_i is the degree to which an aggregation function A preserves the property of the corresponding fuzzy relation R_i. The idea to define this degree lies in the construction of the fuzzy $\mathbf{POS}$ category. The degree of preservation the property of the corresponding fuzzy relation for aggregation functions is:

$$\xi(A) = \inf_i (\inf_{x_1,x_2} (R_i(x_1,x_2) \leftrightarrow R_A(x_1,x_2)))$$

Remark 4.0.1. *The mapping* ξ *could be also defined by the following equality:*

$$\xi(A) = \inf_i (\inf_{x_1,x_2} (R_i(x_1,x_2) \to R_A(x_1,x_2)))$$

In the paper [9] the graded property of monotonicity is involved and the graded notion of dominance is investigated. Although these notions are completely different from the described above, it could be interesting to study the connections between the notions defined in paper [9] and in our work.

The two ideas described above come from the construction of the L-valued $\mathbf{POS}$ category. In the first section of this chapter we precisely realize the first idea, that is defining the degree of monotonicity for an aggregation process. We do not develop here the second idea, but this is in our plans for the future research. In this work we are interested in the process of aggregation of fuzzy relations and in the last three sections we concentrate exactly in this question.

4.1 Degree of monotonicity

4.1.1 Motivation and definitions

The aim of this section is to introduce a fuzzy order relation in aggregation process, namely, to use a fuzzy order relation instead of the crisp order relation in the definition of monotonicity. Recall that an aggregation function is a mapping satisfying boundary conditions and the condition of monotonicity. In our work we focus only on the condition of monotonicity.

The next two examples illustrate our inspiration which led to the present research:

Alternat.	First attrib.	Second attrib.	Aggreg. result
a_1	0.1	0.3	0.2
a_2	0.01	0.31	0.29
a_3	0.2	0.4	0.3

Table 4.1: Motivating example 1.

Let us observe first the aggregation which is illustrated by the Table 4.1 and which is made by an expert. The expert has made the aggregation in his judgment (he did not apply any concrete aggregation function) and we don't know the motivation why he has chosen the following way to aggregate the information.

Now let us imagine that we have a task to analyze the result of aggregation. According to the definition of aggregation function, the mapping defined above is an aggregation function (monotonicity condition is fulfilled). But if we have our own look at this example and consider the aggregation results of alternatives a_1 and a_2 we will find out that results are rather strange (intuitively incorrect). The first attribute of the alternative a_2 is less than the first attribute of the alternative a_1, the second attribute of the alternative a_2 is greater than the second attribute of the alternative a_1, so we don't need to compare the aggregation results for these two alternatives since these alternatives are incomparable, hence crisp property of monotonicity is automatically fulfilled. On the other hand the second attribute of a_2 is greater than the second attribute of a_1 only a little (is equal to the second attribute of a_1 ”in fuzzy sense”), so, intuitively, we expect that if even the aggregation result of the alternative a_2 is greater than the aggregation result of the alternative a_1, then it should be greater only a little. But in our example aggregation result of the alternative a_1 is less than the aggregation result of the alternative a_2 and we have a big difference between the aggregation results. Actually, the element a_2 is less or equal than the element a_1 ”in a fuzzy sense” but for aggregation results we can't confirm the same. To avoid these situations we involve fuzzy order relation in order to define degree of monotonicity.

Another possible situation where fuzzy order could help is the problem when we have small mistakes in aggregation, what is actually illustrated by the ”Motivating example 2”.

Alternat.	First attrib.	Second attrib.	Aggreg. result
a_1	0.1	0.3	0.2
a_2	0.2	0.399	0.301
a_3	0.2	0.4	0.3

Table 4.2: Motivating example 2.

The small variation of data could change the result drastically. For the question ”Is this an aggregation function?” we could only answer ”Yes” or ”No”, thus a very small mistake or destroy of data could change the answer from ”Yes” to ”No”. Let us observe Example 2.

In this case it is not an aggregation function since the monotonicity condition for the pair (a_2, a_3) is not fulfilled. But the second row could be realized just as the damaged third one. Thus, in this case it would be useful to delete the second row or to involve the degree of monotonicity which not only says ”it is an aggregation function” or ”it is not an aggregation function” but gives us the degree to which a mapping is a monotone function.

Thus the aim of this section is to define the degree of monotonicity, to observe illustrating examples and to study the properties of the degree of monotonicity.

We continue with the definition of the degree of monotonicity.

Definition 4.1.1. *Let $f : [0,1]^n \to [0,1]$ be a function (aggregation function), $P : [0,1]^2 \to [0,1]$ be a fuzzy order relation and $\mapsto_T$ a residuum. We define the degree of monotonicity for a function (aggregation function) f w.r.t fuzzy relation P and residuum $\mapsto_T$ in the following way:*

$$M_{P,\mapsto_T}(f) = \inf_{\mathbf{x},\mathbf{y}}(\wedge_i P(x_i, y_i) \mapsto_T P(f(\mathbf{x}), f(\mathbf{y}))).$$

We also involve the degree of monotonicity for a function $f : \bigcup_{n\in N}[0,1]^n \to [0,1]$. In this case, we can introduce a function f as a family $f = (f_{(n)})_{n\in N}$, where $f_{(n)} : [0,1]^n \to [0,1]$ is the restriction of function f to $[0,1]^n$. Then we can calculate the degree of monotonicity for every f_n as

$$M_{P,\mapsto_T}(f_n) = \inf_{\mathbf{x},\mathbf{y}}(\wedge_i P(x_i, y_i) \mapsto_T P(f_n(\mathbf{x}), f_n(\mathbf{y})))$$

and the degree of monotonicity for function f will be

$$M_{P,\mapsto_T}(f) = \inf_{n}(M_{P,\mapsto_T}(f_n)).$$

In the sequel, we will often write $\mathbf{x}$ to denote an element $\mathbf{x} = (x_1, ..., x_n)$ and for simplicity of notation we write $\mathbf{x} \leq \mathbf{y}$ if $\mathbf{x} = (x_1, ..., x_n)$, $\mathbf{y} = (y_1, ..., y_n)$ and $x_i \leq y_i$ for all $i \in \{1, ..., n\}$.

Example 4.1.1. *Let us observe the examples which we have been presented in the introduction and let us calculate the degree of monotonicity for these aggregations. Let us denote by A the aggregation function, namely, $A(a_k)$ denotes the aggregation result for the alternative a_k. We calculate the degree of monotonicity with respect to the fuzzy order relation:*

$$P(a_i, b_i) = \begin{cases} 1, & \textit{if } a_i \leq b_i \\ \max(1 - |a_i - b_i|, 0), & \textit{otherwise} \end{cases},$$

based on Łukasiewicz T-norm (see Example 2.3.1 and Theorem 2.3.2) and the residuum corresponding to the same t-norm: $a \mapsto_T b = \min(1-a+b, 1)$ (Łukasiewicz residuum).

The preliminary results which we get calculating the value $\wedge_i P(a_{ki}, a_{ni}) \mapsto_T P(A(a_k), A(a_n))$ (let us denote this value by $\omega(a_k, a_n)$) for every two alternatives a_k and a_n are summarized in Table 4.3 and Table 4.4.

For the first example the degree of monotonicity is equal to 0.92. We note "deficiency" for the alternatives a_2 and a_1 - the result which we expected to get.

For the "Motivating example 2" the degree of monotonicity is equal to 0.999.

Table 4.3: Motivating example 1.

Alt. a_k and a_n	$\omega(a_k, a_n)$
a_1 and a_2	1
a_1 and a_3	1
a_2 and a_1	0.92
a_2 and a_3	1
a_3 and a_1	1
a_3 and a_2	1

Table 4.4: Motivating example 2.

Alt. a_k and a_n	$\omega(a_k, a_n)$
a_1 and a_2	1
a_1 and a_3	1
a_2 and a_1	0.999
a_2 and a_3	0.999
a_3 and a_1	1
a_3 and a_2	1

Further we study the properties of aggregation functions which have the degree of monotonicity equal to 1.

Proposition 4.1.1. *The degree of monotonicity for a function f with respect to a crisp linear order is equal to 1 if and only if f is a monotone function.*

Proof. First we prove that the degree of monotonicity for a monotone function with respect to a crisp linear order is equal to 1.
In the proof we distinguish the following two cases:

1. If $\mathbf{x} \leq \mathbf{y}$, then $\wedge_i P(x_i, y_i) = 1$. Provided that f is a monotone function, we conclude that, $f(x_1, ..., x_n) \leq f(y_1, ..., y_n)$ and hence $P(f(\mathbf{x}), f(\mathbf{y})) = 1$.
Finally $\wedge_i P(x_i, y_i) \mapsto_T P(f(\mathbf{x}), f(\mathbf{y})) = 1$.

2. If $\mathbf{x} \nleq \mathbf{y}$ then there exists an index k such that $x_k > y_k$. Therefore $P(x_k, y_k) = 0$ and $\wedge_i P(x_i, y_i) = 0$.
Thus $\wedge_i P(x_i, y_i) \mapsto_T P(f(\mathbf{x}), f(\mathbf{y})) = 0 \mapsto_T P(f(\mathbf{x}), f(\mathbf{y})) = 1$.

From the previous we can conclude that $\inf_{\mathbf{x},\mathbf{y}}(\wedge_i P(x_i, y_i) \mapsto_T P(f(\mathbf{x}), f(\mathbf{y}))) = 1$.
Now we prove that if the degree of monotonicity for a function with respect to a crisp linear order is equal to 1 then the initial function is monotone. Assume contrary that a function is not monotone, then there exist vectors $\mathbf{x}$ and $\mathbf{y}$ such that $\mathbf{x} \leq \mathbf{y}$ but $f(x_1, ..., x_n) > f(y_1, ..., y_n)$. This means that $\wedge_i P(x_i, y_i) = 1$ but $P(f(\mathbf{x}), f(\mathbf{y})) = 0$. Hence $\wedge_i P(x_i, y_i) \mapsto_T P(f(\mathbf{x}), f(\mathbf{y})) = 0$, which contradicts the fact that he degree monotonicity for the function with respect to a crisp linear order is equal to 1. By this we have finished the proof.

□

This also means that the degree of monotonicity for an aggregation function with respect to a crisp linear order is equal to 1.

We continue with the proposition stating that the degree of monotonicity for the weighted mean with respect to a certain fuzzy order relation and residuum, corresponding to the left-continuous t-norm is equal to 1.

Proposition 4.1.2. *Let f be the weighted mean: $f(x_1, x_2, ..., x_n) = \sum_{i=1}^{n} w_i x_i$ where weights w_i are non negative and $\sum_{i=1}^{n} w_i = 1$, and let g be a non-increasing function. Then the degree of monotonicity for function f with respect to the fuzzy order relation*

$$P(x_i, y_i) = \begin{cases} 1, & \text{if } x_i \leq y_i \\ g(|x_i - y_i|), & \text{otherwise} \end{cases}$$

and the residuum $\mapsto_T$, where T is a left-continuous t-norm, is equal to 1.

Proof. To find the value $\inf_{\mathbf{x},\mathbf{y}}(\wedge_i P(x_i, y_i) \mapsto_T P(f(\mathbf{x}), f(\mathbf{y})))$ we consider two cases:

1. If $x_i \leq y_i$, for all $i \in \{1, 2, ..., n\}$ then $P(x_1, y_1) = \cdots = P(x_n, y_n) = 1$ by the definition of fuzzy relation P.
   Since obviously $\sum_{i=1}^{n} w_i x_i \leq \sum_{i=1}^{n} w_i y_i$ it follows that $P(\sum_{i=1}^{n} w_i x_i, \sum_{i=1}^{n} w_i y_i) = 1$.
   Hence $(\wedge_i P(x_i, y_i) \mapsto_T P(f(\mathbf{x}), f(\mathbf{y}))) = (1 \mapsto_T 1) = 1$.

2. Now we consider the case when there exists a set $K \subseteq I$ $(I = \{1, 2, ..., n\})$ such that for all $k \in K$ $x_k > y_k$.
   Let $g(|x_l - y_l|) = \min_{k \in K} g(|x_k - y_k|) = \wedge_i P(x_i, y_i)$. Thus for all $k \in K$ $g(|x_l - y_l|) \leq$ $\leq g(|x_k - y_k|)$ and therefore $|x_k - y_k| \leq |x_l - y_l|$ since the function g is non-increasing.
   If $f(\mathbf{x}) \leq f(\mathbf{y})$ then $P(f(\mathbf{x}), f(\mathbf{y})) = 1$ and $(\wedge_i P(x_i, y_i) \mapsto_T P(f(\mathbf{x}), f(\mathbf{y}))) =$ $= (P(x_l, y_l) \mapsto_T 1) = 1$.
   Assume now that $f(\mathbf{x}) > f(\mathbf{y})$, i.e. $\sum_{i=1}^{n} w_i x_i > \sum_{i=1}^{n} w_i y_i$.
   Further $|f(\mathbf{x}) - f(\mathbf{y})| = |\sum_{i=1}^{n} w_i x_i - \sum_{i=1}^{n} w_i y_i| = \sum_{i=1}^{n} w_i x_i - \sum_{i=1}^{n} w_i y_i =$
   $= \sum_{i=1}^{n} w_i (x_i - y_i) \leq \sum_{k \in K} w_k (x_k - y_k) \leq (x_l - y_l) \cdot \sum_{k \in K} w_k \leq |x_l - y_l|$ and then $g(|x_l - y_l|) \leq g(|f(\mathbf{x}) - f(\mathbf{y})|)$. Finally, $(\wedge_i P(x_i, y_i) \mapsto_T P(f(\mathbf{x}), f(\mathbf{y}))) =$ $= (P(x_l, y_l) \mapsto_T P(f(\mathbf{x}), f(\mathbf{y}))) = 1$.

We have shown that $\wedge_i P(x_i, y_i) \mapsto_T P(f(\mathbf{x}), f(\mathbf{y})) = 1$ for all $\mathbf{x}$ and $\mathbf{y}$. Hence $M_{P,\mapsto_T}(f) = 1$. □

We continue with the more general case.

Theorem 4.1.1. *Let $f : [0,1]^n \to [0,1]$ be a monotone function, T be a continuous Archimedean t-norm with an additive generator t and let d be an arbitrary pseudo-metric in interval $[0,1]$ such that $d(a,b) \leq t(0)$ for all a and b from the unit interval. Then the degree of monotonicity for function f with respect to the residuum $\mapsto_T$ and the fuzzy order relation*

$$P(x_i, y_i) = \begin{cases} 1, & \text{if } x_i \leq y_i \\ E_d(x_i, y_i), & \text{otherwise} \end{cases}$$

is equal to 1 if and only if for all $\mathbf{x} = (x_1, ..., x_n)$, $\mathbf{y} = (y_1, ..., y_n)$

$$f(\mathbf{x}) > f(\mathbf{y}) \Rightarrow d(f(\mathbf{x}), f(\mathbf{y})) \leq \max_{x_i > y_i}(d(x_i, y_i)).$$

Proof. Let us prove the sufficiency:
To find the value $\wedge_i P(x_i, y_i) \mapsto_T P(f(\mathbf{x}), f(\mathbf{y}))$ for different $\mathbf{x}$ and $\mathbf{y}$ we consider the following three cases:

1. If $x_i \leq y_i$ for all $i \in \{1, .., n\}$ then $\wedge_i P(x_i, y_i) = 1$. According to the monotonicity of f, $f(\mathbf{x}) \leq f(\mathbf{y})$ and then $P(f(\mathbf{x}), f(\mathbf{y})) = 1$.
   Hence $\wedge_i P(x_i, y_i) \to P(f(\mathbf{x}), f(\mathbf{y})) = 1$.

2. If there exists j for which $x_j > y_j$ and $f(\mathbf{x}) \leq f(\mathbf{y})$ then $P(f(\mathbf{x}), f(\mathbf{y})) = 1$.
   Hence $\wedge_i P(x_i, y_i) \to P(f(\mathbf{x}), f(\mathbf{y})) = 1$.

3. If there exists j for which $x_j > y_j$ and $f(\mathbf{x}) > f(\mathbf{y})$, then let k be an integer for which $\wedge_i P(x_i, y_i) = P(x_k, y_k) = E_d(x_k, y_k) = t^{(-1)}(\min(d(x_k, y_k), t(0)))$.
   Since $f(\mathbf{x}) > f(\mathbf{y})$ we have $P(f(\mathbf{x}), f(\mathbf{y})) = t^{(-1)}(\min(d(f(\mathbf{x}), f(\mathbf{y})), t(0)))$.
   Then $\wedge_i P(x_i, y_i) \mapsto_T P(f(\mathbf{x}), f(\mathbf{y})) =$
   $= t^{(-1)}(\max(t(t^{(-1)}(\min(d(f(\mathbf{x}), f(\mathbf{y})), t(0)))) -$
   $- t(t^{(-1)}(\min(d(x_k, y_k), t(0)))), 0)) =$
   $= t^{(-1)}(\max(\min(d(f(\mathbf{x}), f(\mathbf{y})), t(0)) - \min(d(x_k, y_k), t(0)), 0))$.
   We consider two cases:

   - $\min(d(x_k, y_k), t(0)) = t(0) \Rightarrow \min(d(f(\mathbf{x}), f(\mathbf{y})), t(0)) \leq t(0) \Rightarrow$
     $t^{(-1)}(\max(\min(d(f(\mathbf{x}), f(\mathbf{y})), t(0)) - \min(d(x_k, y_k), t(0)), 0)) =$
     $= t^{(-1)}(0) = 1$

- $\min(d(x_k, y_k), t(0)) \neq t(0) \Rightarrow$
  $\min(d(x_k, y_k), t(0)) = d(x_k, y_k) \Rightarrow$
  $\min(d(f(\mathbf{x}), f(\mathbf{y})), t(0)) = d(f(\mathbf{x}), f(\mathbf{y})) \Rightarrow$
  $t^{(-1)}(\max(\min(d(f(\mathbf{x}), f(\mathbf{y})), t(0)) - \min(d(x_k, y_k), t(0)), 0)) =$
  $= t^{(-1)}(\max(d(f(\mathbf{x}), f(\mathbf{y})) - d(x_k, y_k), 0)) = t^{(-1)}(0) = 1$

Thus in both cases $\wedge_i P(x_i, y_i) \mapsto_T P(f(\mathbf{x}), f(\mathbf{y})) = 1$. Therefore the degree of monotonicity is equal to 1.
We continue by proving the necessity. Let us assume that there exist elements $\mathbf{x}$ and $\mathbf{y}$ such that $f(\mathbf{x}) > f(\mathbf{y})$ but $d(f(\mathbf{x}), f(\mathbf{y})) > \max\limits_{x_i > y_i}(d(x_i, y_i))$. Then we calculate the value $\wedge_i P(x_i, y_i) \rightarrow P(f(\mathbf{x}), f(\mathbf{y}))$. Let k be an integer for which
$\wedge_i P(x_i, y_i) = P(x_k, y_k) = E_d(x_k, y_k) = t^{(-1)}(\min(d(x_k, y_k), t(0)))$.
Since $f(\mathbf{x}) > f(\mathbf{y})$ we have $P(f(\mathbf{x}), f(\mathbf{y})) = t^{(-1)}(\min(d(f(\mathbf{x}), f(\mathbf{y})), t(0)))$.
Then $\wedge_i P(x_i, y_i) \mapsto_T P(f(\mathbf{x}), f(\mathbf{y})) =$
$= t^{(-1)}(\max(t(t^{(-1)}(\min(d(f(\mathbf{x}), f(\mathbf{y})), t(0)))) - t(t^{(-1)}(\min(d(x_k, y_k), t(0)))), 0)) =$
$= t^{(-1)}(\max(\min(d(f(\mathbf{x}), f(\mathbf{y})), t(0)) - \min(d(x_k, y_k), t(0)), 0)) < t^{(-1)}(0)$. The last inequality is true since $\min(d(f(\mathbf{x}), f(\mathbf{y})), t(0)) - \min(d(x_k, y_k), t(0)) > 0$ and $t^{(-1)}$ is a strictly decreasing mapping. Thus $\wedge_i P(x_i, y_i) \mapsto_T P(f(\mathbf{x}), f(\mathbf{y})) \neq 1$. □

We illustrate the theorem by the next two examples, where we apply it to minimum and maximum functions.

Example 4.1.2. *Let MIN be the minimum function:*

$$MIN(x_1, ..., x_n) = \min(x_1, ..., x_n),$$

T be a continuous Archimedean t-norm with an additive generator t and d be a pseudo-metric on interval $[0, 1]$ such that $d(a, b) = |a - b|$. Then the degree of monotonicity for function MIN with respect to the residuum $\mapsto_T$ and the fuzzy order relation

$$P(x_i, y_i) = \begin{cases} 1, & \text{if } x_i \leq y_i \\ E_d(x_i, y_i), & \text{otherwise} \end{cases}$$

is equal to 1:

$$M_{P,\mapsto_T}(MIN) = 1.$$

This follows from the previous theorem since MIN is obviously a monotone function and

$$MIN(\mathbf{x}) > MIN(\mathbf{y}) \Rightarrow$$

$$\Rightarrow d(MIN(\mathbf{x}), MIN(\mathbf{y})) \le \max_{x_i > y_i}(d(x_i, y_i)).$$

To prove the implication let $\min_i y_i = y_k$.
Then $MIN(\mathbf{x}) > MIN(\mathbf{y}) \Rightarrow d(MIN(\mathbf{x}), MIN(\mathbf{y})) =$
$= \min_i x_i - \min_i y_i = \min_i x_i - y_k \le x_k - y_k \le \max_{x_i > y_i}(d(x_i, y_i))$.
We can generalize this example for the case when the pseudo-metric is defined by $d(a, b) = \varphi(|a - b|)$, where φ is a non-decreasing function.

Example 4.1.3. *Let MAX be the maximum function:*

$$MAX(x_1, ..., x_n) = \max(x_1, ..., x_n),$$

T be a continuous Archimedean t-norm with an additive generator t and d be a pseudo-metric on interval $[0, 1]$ such that $d(a, b) = |a - b|$. Then the degree of monotonicity for function MAX with respect to the residuum $\mapsto_T$ and the fuzzy order relation

$$P(x_i, y_i) = \begin{cases} 1, & \text{if } x_i \le y_i \\ E_d(x_i, y_i), & \text{otherwise} \end{cases}$$

is equal to 1:

$$M_{P,\mapsto_T}(MAX) = 1.$$

This follows from the previous theorem since MAX is obviously a monotone function and

$$MAX(\mathbf{x}) > MAX(\mathbf{y}) \Rightarrow$$

$$\Rightarrow d(MAX(\mathbf{x}), MAX(\mathbf{y})) \le \max_{x_i > y_i}(d(x_i, y_i)).$$

To prove the implication let $\max_i x_i = x_k$.
Then $MAX(\mathbf{x}) > MAX(\mathbf{y}) \Rightarrow d(MAX(\mathbf{x}), MAX(\mathbf{y})) =$
$= \max_i x_i - \max_i y_i = x_k - \max_i y_i \le x_k - y_k \le \max_{x_i > y_i}(d(x_i, y_i))$.
We can generalize this example for the case when the pseudo-metric is defined by $d(a, b) = \varphi(|a - b|)$, where φ is a non-decreasing function.

4.1.2 Involving α-levels in the definition of the degree of monotonicity

Not for every monotone function f the degree of monotonicity $M_{P,\mapsto_T}(f)$ is equal to 1. We illustrate this by the following example:

Example 4.1.4. *Let us evaluate the degree of monotonicity for weak t-norm*

$$T_W(x_1, x_2) = \begin{cases} min(x_1, x_2), & \text{if } x_1 \vee x_2 = 1 \\ 0, & \text{otherwise} \end{cases},$$

(which is obviously a monotone function), with respect to the fuzzy order relation

$$P(x_i, y_i) = \begin{cases} 1, & \text{if } x_i \leq y_i \\ g(|x_i - y_i|), & \text{otherwise} \end{cases},$$

where g is a continuous non-increasing mapping, and the residuum $\mapsto_T$ corresponding to a left-continuous t-norm:

$M_{P,\mapsto_T}(T_W) \leq \inf\limits_{\substack{x=(1,1),\\ y=(y_0,y_0),\\ y_0\in[0,1)}} (\wedge_i P(x_i, y_i) \mapsto_T P(T_W(\mathbf{x}), T_W(\mathbf{y}))) =$

$= \inf\limits_{y_0\in[0,1)} (P(1, y_0) \mapsto_T P(1, 0)) = \sup\limits_{y_0\in[0,1)} P(1, y_0) \to_T P(1, 0) = 1 \to_T P(1, 0) =$

$= P(1, 0) = g(1)$. *It is not natural to take a mapping g in such way that $g(1) = 1$, since in this case g is equal to 1 for all arguments from the unit interval. So, for any mapping g such that $g(1) \neq 1$, it follows that $M_{P,\mapsto_T}(T_W)$ is not equal to 1.*

Calculating the degree of monotonicity of a monotone function f for every two elements $\mathbf{x}, \mathbf{y}$ such that $\mathbf{x} < \mathbf{y}$ we have to compute the value $\wedge_i P(y_i, x_i) \mapsto_T P(f(\mathbf{y}), f(\mathbf{x}))$ which is equal to $\wedge_i E(y_i, x_i) \mapsto_T E(f(\mathbf{y}), f(\mathbf{x}))$ in case when

$$P(x_i, y_i) = \begin{cases} 1, & \text{if } x_i \leq y_i \\ E(x_i, y_i), & \text{otherwise} \end{cases}.$$

Then if f is a monotone function the necessary condition for $M_{P,\mapsto_T}(f) = 1$ is

$\inf\limits_{\mathbf{x}<\mathbf{y}} (\wedge_i E(y_i, x_i) \mapsto_T E(f(\mathbf{y}), f(\mathbf{x}))) = 1.$

Intuitively this is the degree of the statement: "if $\mathbf{x}$ and $\mathbf{y}$ are indistinguishable then $f(\mathbf{x})$ and $f(\mathbf{y})$ are indistinguishable". This is something more than just generalization of monotonicity. But we think that it could be a useful condition for the study of aggregation processes.

Actually, if we want to be closer to the classical (crisp) definition of monotonicity, we can calculate the value $\wedge_i P(x_i, y_i) \mapsto_T P(f(\mathbf{x}), f(\mathbf{y}))$ only for those elements x, y which are in the relation $\mathbf{x} \leq \mathbf{y}$ in a certain fuzzy sense. By this we mean that the value $\wedge_i P(x_i, y_i)$ should be close to 1. One can choose a constant α from the interval $[0, 1]$ to define what "close to 1" does mean and calculate the degree of α-monotonicity:

Definition 4.1.2. *Let $f : [0, 1]^n \to [0, 1]$ be a function (aggregation function), $P : [0, 1]^2 \to [0, 1]$ be a fuzzy order relation and $\mapsto_T$ a residuum corresponding to the* *t-norm T. We define the degree of α-monotonicity for a function (aggregation function) f w.r.t fuzzy relation P and residuum $\mapsto_T$ in the following way:*

$$M^{\alpha}_{P,\mapsto_T}(f) = \inf_{\wedge_i P(x_i,y_i)\geq\alpha} (\wedge_i P(x_i, y_i) \mapsto_T P(f(\mathbf{x}), f(\mathbf{y}))).$$

It is easy to see that if a fuzzy order P and a t-norm T are fixed and $\alpha_1 \leq \alpha_2$ then $S_{\alpha_1} \subseteq S_{\alpha_2}$, where $S_{\alpha_1} = \{f : M^{\alpha_1}_{P,\to_T}(f) = 1\}$ and $S_{\alpha_2} = \{f : M^{\alpha_2}_{P,\to_T}(f) = 1\}$.

4.1.3 Example

Let us observe the following example, where f is the arithmetic mean destroyed at one point
$x_1 = x_2 = 0.5$:

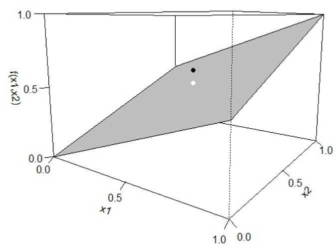


Figure 4.1: $f(x_1, x_2) = \begin{cases} \frac{x_1+x_2}{2}, x_1, x_2 \neq 0.5 \\ 0.6,\ x_1 = x_2 = 0.5 \end{cases}$

We involve the concrete fuzzy order relation:

$$P(x_i, y_i) = \begin{cases} 1, & \text{if } x_i \leq y_i \\ \max(1 - |x_i - y_i|, 0), & \text{otherwise} \end{cases},$$

based on Łukasiewicz T-norm (see Example 2.3.1 and Theorem 2.3.2) and the residuum corresponding to the same t-norm:
$a \mapsto_T b = \min(1 - a + b, 1)$ (Łukasiewicz residuum).

Function f is monotone everywhere except of point $(0.5, 0.5)$, so we define the defect of monotonicity of function f as

$$\text{def}(f) = f(0.5, 0.5) - \lim_{(x_1,x_2)\to(0.5,0.5)} f(x_1, x_2) = 0.1.$$

We calculate the degree of monotonicity for function f with respect to the fuzzy order relation P and the residuum $\mapsto_T$. According to Proposition 4.1.2, for every two elements x, y where $(x_1, x_2) \neq (0.5, 0.5)$ and $(y_1, y_2) \neq (0.5, 0.5)$
$\wedge_i P(x_i, y_i) \mapsto_T P(f(\mathbf{x}), f(\mathbf{y})) = 1$.
Thus we must find the value $\wedge_i P(x_i, y_i) \mapsto_T P(f(\mathbf{x}), f(\mathbf{y}))$ for all $\mathbf{x}, \mathbf{y}$ where $(x_1, x_2) = (0.5, 0.5)$ or $(y_1, y_2) = (0.5, 0.5)$. For the brevity of calculation we involve notation

$$\omega_{P,\mapsto_T}(\mathbf{x}, \mathbf{y}) = \wedge_i P(x_i, y_i) \mapsto_T P(f(\mathbf{x}), f(\mathbf{y})).$$

Then $M_{P,\mapsto_T}(f) = \inf_{\mathbf{x},\mathbf{y}} \omega_{P,\mapsto_T}(\mathbf{x}, \mathbf{y})$. We know that $\inf_{x,y \neq (0.5,0.5)} \omega_{P,\mapsto_T}(\mathbf{x}, \mathbf{y}) = 1$, so we have to find $\inf_{x=(0.5,0.5)} \omega_{P,\mapsto_T}(\mathbf{x}, \mathbf{y})$ and $\inf_{y=(0.5,0.5)} \omega_{P,\mapsto_T}(\mathbf{x}, \mathbf{y})$.
Throughout the calculation to shorten the notation we will write $\omega(\mathbf{x}, \mathbf{y})$ instead of $\omega_{P,\mapsto_T}(\mathbf{x}, \mathbf{y})$.

- $(x_1, x_2) = (0.5, 0.5)$
  If $f(\mathbf{x}) \leq f(\mathbf{y})$ then $\omega(\mathbf{x}, \mathbf{y}) = 1$, so we investigate the case when $f(\mathbf{x}) > f(\mathbf{y})$. This means that $\frac{y_1+y_2}{2} < 0.6$.
  We consider the following cases:

  1. If $y_1 < 0.5$ and $y_2 < 0.5$ then
     $\omega(\mathbf{x}, \mathbf{y}) = ((1 - |0.5 - y_1|) \wedge (1 - |0.5 - y_2|)) \mapsto_T (1 - |0.6 - \frac{y_1+y_2}{2}|) =$
     $= 0.4 + \frac{y_1+y_2}{2} + \max(0.5 - y_1, 0.5 - y_2)$. It is easy to verify, that the minimal value will be when $y_1 = y_2$.
     Thus $\omega(\mathbf{x}, \mathbf{y}) = 0.4 + \frac{y_1+y_1}{2} + 0.5 - y_1 = 0.9$.
  2. If $y_1 \geq 0.5$ and $y_2 < 0.5$ then
     $P(0.5, y_1) = 1$ and $P(0.5, y_1) \wedge P(0.5, y_2) = P(0.5, y_2)$.
     Thus $\omega(\mathbf{x}, \mathbf{y}) = P(0.5, y_2) \mapsto_T P(0.6, f(\mathbf{y})) =$
     $= (1 - |0.5 - y_2|) \mapsto_T (1 - |0.6 - \frac{y_1+y_2}{2}|) = \min(0.9 + \frac{y_1-y_2}{2}, 1) \geq 0.9$.
  3. The same considerations work for $y_1 \geq 0.5$ and $y_2 < 0.5$:
     $\omega(\mathbf{x}, \mathbf{y}) = \min(0.9 + \frac{y_2-y_1}{2}, 1) \geq 0.9$.
  4. If $y_1 \geq 0.5$ and $y_2 \geq 0.5$ then
     $\omega(\mathbf{x}, \mathbf{y}) = 1 \mapsto_T P(0.6, f(\mathbf{y})) = 1 \mapsto_T (1 - |0.6 - \frac{y_1+y_2}{2}|) =$
     $= 0.4 + \frac{y_1+y_2}{2} \geq 0.9$.

  The above investigations are visualized by the following illustration:

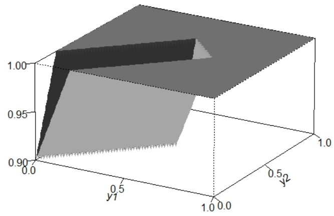


Figure 4.2: $\omega(\mathbf{x}, \mathbf{y})$, where $\mathbf{x} = (0.5, 0.5)$

- $(y_1, y_2) = (0.5, 0.5)$
  According to the previous proposition,
  $P(x_1, 0.5) \wedge P(x_2, 0.5) \mapsto_T P(f(\mathbf{x}), 0.5) = 1$. It is easy to verify (using the properties of the residuum $\mapsto_T$ and the properties of the fuzzy order relation P) that if $f(0.5, 0.5) \geq 0.5$ and $f(x_1, x_2) = \frac{x_1+x_2}{2}$ for all $(x_1, x_2) \neq (0.5, 0.5)$ then
  $P(x_1, 0.5) \wedge P(x_2, 0.5) \mapsto_T P(f(\mathbf{x}), f(0.5, 0.5)) = 1$.

From the previous investigations we conclude that $M_{P,\mapsto_T}(f) = \inf_{\mathbf{x},\mathbf{y}} \omega_{P,\mapsto_T}(\mathbf{x}, \mathbf{y})$ is equal to 0.9, which is well in accordance with our definition of the defect of monotonicity:

$$M_{P,\mapsto_T}(f) = 1 - \mathrm{def}(f) = 0.9.$$

We can generalize the previous example by considering function $h(x_1, x_2)$:

$$h(x_1, x_2) = \begin{cases} \frac{x_1+x_2}{2}, & x_1, x_2 \neq 0.5 \\ 0.5 + \mathrm{def}(h), & x_1 = x_2 = 0.5 \end{cases},$$

where $\mathrm{def}(h) \in [0, 0.5]$. In this case also

$$M_{P,\mapsto_T}(h) = 1 - \mathrm{def}(h).$$

Next we calculate the degree of monotonicity for the same function $f(x_1, x_2)$ with respect to the same fuzzy order relation P, but we use the residuum corresponding to the minimum t-norm :

$$a \mapsto_T b = \begin{cases} 1, & \text{if } a \leq b \\ b, & \text{otherwise} \end{cases}.$$

Reasoning similar to the previous situation we calculate the value $\omega(\mathbf{x}, \mathbf{y})$ as follows:

- $(x_1, x_2) = (0.5, 0.5)$
  As in the previous example we calculate the value $\omega(\mathbf{x}, \mathbf{y})$ if $\frac{y_1+y_2}{2} < 0.6$, because it is easy to see that $\omega(\mathbf{x}, \mathbf{y}) = 1$ for $\mathbf{y} : \frac{y_1+y_2}{2} > 0.6$.

  1. If $y_1 < 0.5$ and $y_2 < 0.5$ then
     $\omega(\mathbf{x}, \mathbf{y}) = 1$ if $\max(0.5 - y_1, 0.5 - y_2) \geq 0.6 - \frac{y_1+y_2}{2}$ and
     $\omega(\mathbf{x}, \mathbf{y}) = 0.4 + \frac{y_1+y_2}{2}$ otherwise.
     Thus if $y_1 = y_2 = 0$ then $\omega(\mathbf{x}, \mathbf{y}) = 0.4$.
  2. If $y_1 < 0.5$ and $y_2 \geq 0.5$ then
     $\omega(\mathbf{x}, \mathbf{y}) = 1$, if $\frac{y_2-y_1}{2} \geq 0.1$ and
     $\omega(\mathbf{x}, \mathbf{y}) = 0.4 + \frac{y_1+y_2}{2}$ otherwise. Which means that $\omega(\mathbf{x}, \mathbf{y}) \geq 0.4$.
  3. If $y_1 \geq 0.5$ and $y_2 < 0.5$ then
     $\omega(\mathbf{x}, \mathbf{y}) = 1$ if $\frac{y_1-y_2}{2} \geq 0.1$ and
     $\omega(\mathbf{x}, \mathbf{y}) = 0.4 + \frac{y_1+y_2}{2}$ otherwise. Which means that $\omega(\mathbf{x}, \mathbf{y}) \geq 0.4$.
  4. If $y_1 \geq 0.5$ and $y_2 \geq 0.5$ then $\omega(\mathbf{x}, \mathbf{y}) = 0.4 + \frac{y_1+y_2}{2} \geq 0.9$.

  The above investigations are visualised by the following illustration:

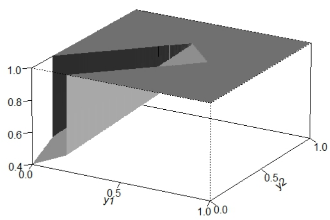


Figure 4.3: $\omega(\mathbf{x}, \mathbf{y})$, where $\inf\limits_{x,y} = (0.5, 0.5)$

- $(y_1, y_2) = (0.5, 0.5)$
  Analogously as in the case of Łukasiewicz residuum we get $\omega(\mathbf{x}, \mathbf{y}) = 1$.

From the previous investigations we conclude that $M_{P,\mapsto_T}(f) = 0.4$

Next we calculate the degree of monotonicity for the same function $f(x_1, x_2)$ with respect to the same fuzzy order relation P, but we use the residuum corresponding to the product t-norm :

$$a \mapsto_T b = \begin{cases} 1, & \text{if } a \leq b \\ \frac{b}{a}, & \text{otherwise} \end{cases}.$$

Reasoning similar to the previous situations we calculate the value $\omega(\mathbf{x}, \mathbf{y})$ as follows:

- $(x_1, x_2) = (0.5, 0.5)$
  As in the previous examples we calculate the value $\omega(\mathbf{x}, \mathbf{y})$ if $\frac{y_1+y_2}{2} < 0.6$, because it is easy to see that $\omega(\mathbf{x}, \mathbf{y}) = 1$ for $\mathbf{y}$: $\frac{y_1+y_2}{2} > 0.6$.

  1. If $y_1 < 0.5$ and $y_2 < 0.5$ then $\omega(\mathbf{x}, \mathbf{y}) = \frac{0.4+\frac{y_1+y_2}{2}}{1-\max(|0.5-y_1|,|0.5-y_2|)}$.
     Thus if $y_1 = y_2 = 0$ then $\omega(\mathbf{x}, \mathbf{y}) = 0.8$.
  2. If $y_1 < 0.5$ and $y_2 \geq 0.5$ then $\omega(\mathbf{x}, \mathbf{y}) = \frac{0.4+\frac{y_1+y_2}{2}}{0.5+y_1}$.
     Which means that $\omega(\mathbf{x}, \mathbf{y}) \geq 0.8$.
  3. If $y_1 \geq 0.5$ and $y_2 < 0.5$ then $\omega(\mathbf{x}, \mathbf{y}) = \frac{0.4+\frac{y_1+y_2}{2}}{0.5+y_2}$.
     Which means that $\omega(\mathbf{x}, \mathbf{y}) \geq 0.8$.
  4. If $y_1 \geq 0.5$ and $y_2 \geq 0.5$ then $\omega(\mathbf{x}, \mathbf{y}) = 0.4 + \frac{y_1+y_2}{2} \geq 0.9$.

  The above investigations are visualised by the illustration Fig.4.4.

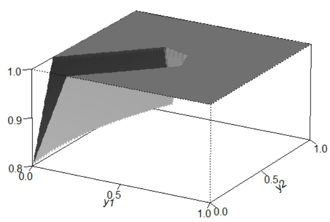


Figure 4.4: $\omega(\mathbf{x}, \mathbf{y})$, where $\mathbf{x} = (0.5, 0.5)$

- $(y_1, y_2) = (0.5, 0.5)$
  Analogously as in the case of Łukasiewicz residuum we get $\omega(\mathbf{x}, \mathbf{y}) = 1$.

From the previous investigations we conclude that $M_{P,\mapsto_T}(f) = 0.4$

Thus in these examples the best result is when both the fuzzy order P and the residuum correspond to the same, Łukasiewicz t-norm.
Now let observe the first case from the previous example, when the degree of monotonicity for the destroyed arithmetic mean is calculated with respect to the Łukasiewicz residuum. It is interesting to see what result we obtain if we suppose that two points are indistinguishable if the distance between them is less or equal to 0.1. Thus we would like "to get round" the deficiency of 0.1. Further we calculate the degree of monotonicity for the function

$$f(x_1, x_2) = \begin{cases} \frac{x_1+x_2}{2}, x_1, x_2 \neq 0.5 \\ 0.6,\ x_1 = x_2 = 0.5 \end{cases}$$

with respect to Łukasiewicz residuum and the following fuzzy order relation:

$$P_{Mod}(x_i, y_i) = \begin{cases} 1,\ if\ x_i \leq y_i + 0.1 \\ 1.1 - |x_i - y_i|,\ otherwise \end{cases} .$$

As in the previous example $\omega_{P_{Mod},\mapsto_{T_L}}(\mathbf{x}, \mathbf{y})$ is equal to 1 if $y = (0.5, 0.5)$. The results when $x = (0.5, 0.5)$ are visualized by the following illustration. To compare the results for $\omega_{P_{Mod},\mapsto_{T_L}}$ and $\omega_{P,\mapsto_{T_L}}(\mathbf{x}, \mathbf{y})$ we fix in the graph two points: $(0.5, 0.5, 0.9)$ and $(0.5, 0.5, 1)$.

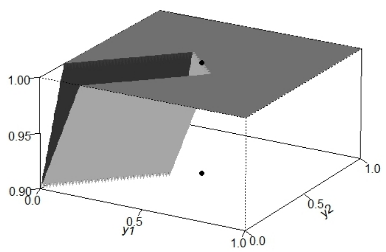


Figure 4.5: $\omega_{P_{Mod},\mapsto_{T_M}}(\mathbf{x}, \mathbf{y})$, where $x = (0.5, 0.5)$

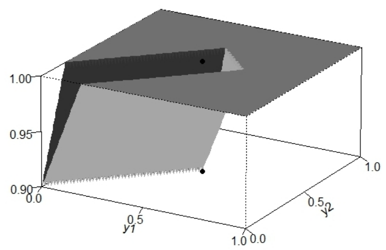


Figure 4.6: $\omega_{P,\mapsto_{T_M}}(\mathbf{x}, \mathbf{y})$, where $x = (0.5, 0.5)$

We see that using fuzzy order relation P_{Mod} where we tried "to get round" the deficiency of 0.1 we get the same result $M_{P_{Mod},\mapsto T_L}(f) = 1 - \mathrm{def}(f) = 0.9$. To improve the situation we can calculate the degree of α-monotonicity for $\alpha = 0.9$. In this case $M^{0.9}_{P,\mapsto T}(f) = 1$. So "to get round" the deficiency we should both modify the fuzzy order relation and use the definition of the degree of α-monotonicity.

4.1.4 Conclusion on fuzzy monotonicity

The degree of monotonicity is calculated for a concrete mapping and depends on this mapping, fuzzy order relation and residuum which are chosen by an expert. The degree of monotonicity takes its values from the interval [0,1]. In case of a crisp order relation the property of having the degree of monotonicity equal to 1 is equivalent to the property of being monotone in the crisp sense. We consider the behavior of the degree of monotonicity calculated with respect to the given fuzzy order relation and residuum, illustrating it with examples. Besides, we study how a deficiency of monotonicity influences the degree of monotonicity. In the work we also study necessary properties when the degree of monotonicity is equal to 1. Note that a more substantial theory is obtained by involving α levels in the definition of the degree of monotonicity.

4.2 Aggregation of fuzzy relations

Aggregations of fuzzy relations are important in fuzzy preference modeling, solving decision making problems and other problems having to do with imprecise information. The concept of aggregation of fuzzy relations has been studied in [24], [39],[15] et al. This topic is of high importance because it is a good reflection of some practical applications (see e.g. [8]).
It is necessary to investigate which aggregation operators are able to preserve properties of aggregated fuzzy relations during aggregation process. There are many works devoted to this topic and namely, to the concept of dominance of aggregation operators and principles of building dominating aggregation operators.

Let us consider the following example:
Assume that we have a query $\mathbf{q} = (q_1, ..., g_n)$, where each $q_i \in X_i$ is a value referring to the i-th field of the query. Given a data record $\mathbf{x} = (x_1, ..., x_n)$ such that $x_i \in X_i$ for all $i = 1, ..., n$. We denote the degrees to which a given record matches query $\mathbf{q}$ pointwice

$(R_i(x_i, q_i))$. If we model the aggregation by means of a mapping $A : [0,1]^n \to [0,1]$:

$$R(\mathbf{x}, \mathbf{q}) = A(R_1(x_1, q_1), ..., R_n(x_n, q_n)),$$

it is natural to require at least the following properties:

1. If $P_i(x_i, q_i) = 1$ for all i, then the global degree should be also 1. In other words: $A(1, ..., 1) = 1$.

2. If none of the single queries is matched at all, i.e. all degrees $R_i(x_i, q_i) = 0$, then the global degree of fulfillment should be 0, too: $A(0, ..., 0) = 0$.

3. If one degree $R_i(x_i, q_i)$ increases while the others are kept constant, the overall degree must not decrease, i.e. A should be non-increasing in each component.

That is exactly the definition of aggregation function.

Let us now investigate the problem of preservation of properties of fuzzy orders by an aggregation function.

Due to the fact that fuzzy order relations are based on the equivalence relations let us first focus on the aggregation of fuzzy equivalence relations E_i. The preservation of reflexivity is rather clear because of the boundary conditions of aggregation function. Preservation of symmetry is also obvious. The more interesting and complex question is about preservation of T-transitivity. Here we use the results about the preservation of T-transitivity studied in [39], where it is shown that preservation of T-transitivity is equivalent to the dominance of the t-norm T by the aggregation operator (or function) A.

Definition 4.2.1. [39] *Consider an n-argument aggregation function* $A : [0,1]^n \to [0,1]$ *and a t-norm T. We say that A dominates T if for all $x_i \in [0,1]$ with $i \in \{1, ..., n\}$ and $y_i \in [0,1]$ with $i \in \{1, ..., n\}$ the following property holds:*

$$T(A(x_1, ..., x_n), A(y_1, ..., y_n)) \leq A(T(x_1, y_1), ..., T(x_n, y_n)).$$

Theorem 4.2.1. [39] *Let $|X| > 3$ and let T be a t-norm. An aggregation function A preserves T-transitivity of fuzzy relations on X if and only if A belongs to the class of aggregation functions which dominate T.*

Corollary 4.2.1. *Let $|X| > 3$ and let T be a t-norm. If E_i are fuzzy equivalence relations (T-equivalences) for all $i \in \{1, ..., n\}$ then*

$$E(\mathbf{x}, \mathbf{y}) = A(E_1(x_1, y_1), ..., E_n(x_n, y_n))$$

is also a T-equivalence relation if A belongs to the class of aggregation functions which dominate T.

The following fact will be important in the next section:

Corollary 4.2.2. *Let $|X| > 3$ and let T be a t-norm. If E_i are fuzzy equivalence relations (T-equivalences) for all $i \in \{1, ..., n\}$ then*

$$E(x, y) = A(E_1(x, y), ..., E_n(x, y))$$

is also a T-equivalence relation in case when A belongs to the class of aggregation functions which dominate T.

We continue with the aggregation of fuzzy order relations.

Theorem 4.2.2. *Let $|X| > 3$ and let T be a t-norm. If E_i are fuzzy equivalence relations (T-equivalences) for all $i \in \{1, ..., n\}$; P_i are fuzzy order relations (T-E_i-orders) for all $i \in \{1, ..., n\}$ then $P(\mathbf{x}, \mathbf{y}) = A(P_1(x_1, y_1), ..., P_n(x_n, y_n))$ is a T-E-order relation in case when A belongs to the class of aggregation functions which dominate T and $E(\mathbf{x}, \mathbf{y}) = A(E_1(x_1, y_1), ..., E_n(x_n, y_n))$.*

Proof.

1. Since all P_i are E_i-reflexive ($E_i(x_i, y_i) \leq P_i(x_i, y_i)$) we have

$$A(E_1(x_1, y_1), ..., E_n(x_n, y_n)) \leq A(P_1(x_1, y_1), ..., P_n(x_n, y_n))$$

because of the monotonicity of the function A. Thus P is an E-reflexive fuzzy relation.

2. T-transitivity holds because of Theorem 4.2.1.

3. It remains to prove that $T(P(\mathbf{x}, \mathbf{y}), P(\mathbf{y}, \mathbf{x})) \leq E(\mathbf{x}, \mathbf{y})$:

$$T(P(\mathbf{x}, \mathbf{y}), P(\mathbf{y}, \mathbf{x})) =$$
$$= T(A(P_1(x_1, y_1), ..., P_n(x_n, y_n)), A(P_1(y_1, x_1), ..., P_n(y_n, x_n))) \leq$$
$$\leq A(T(P_1(x_1, y_1), P_1(y_1, x_1)), ..., T(P_n(x_n, y_n), P_n(y_n, x_n)))$$

because of the dominance of T by A. Further

$$A(T(P_1(x_1, y_1), P_1(y_n, x_n)), ..., T(P_n(x_n, y_n), P_n(y_n, x_n))) \leq$$
$$\leq A(E_1(x_1, y_1), ..., E_n(x_n, y_n))$$

since A is a monotone function and $T(P_i(x,y), P_i(y,x)) \leq E_i(x,y)$ fir every i. Thus we have proved the required inequality.

□

The following fact will be important in the next section:

Corollary 4.2.3. *Let $|X| > 3$ and let T be a t-norm. If E_i are fuzzy equivalence relations (T-equivalences) for all $i \in \{1, ..., n\}$; P_i are fuzzy order relations (T-E_i-orders) for all $i \in \{1, ..., n\}$ then $P(x,y) = A(P_1(x,y), ..., P_n(x,y))$ is T-E-order relation if A belongs to the class of aggregation functions which dominate T and $E(x,y) = A(E_1(x,y), ..., E_n(x,y))$.*

In the next two examples we observe aggregation functions which dominates Łukasievicz and product t-norms:

Example 4.2.1. [39] *For any $k > 2$ and any $p = (p_1, ..., p_k)$ with $\sum_{i=1}^{k} p_i \geq 1$ and $p_i \in [0, \infty]$ k-ary aggregation function*

$$A_p(x_1, ..., x_k) = \max(\sum_{i=1}^{k} x_i \cdot p_i + 1 - \sum_{i=1}^{k} p_i, 0)$$

dominates Łukasiewicz t-norm T_L.

Example 4.2.2. [39] *For any $k > 2$ and any $p = (p_1, ..., p_k)$ with $\sum_{i=1}^{k} p_i \geq 1$ and $p_i \in [0, \infty]$ k-ary aggregation function*

$$A_p(x_1, ..., x_k) = \prod_{i=1}^{k} x_i^{p_i}$$

dominates product t-norm T_P.

The above described results will be important in the next section.

Let us now introduce the notion of σ-aggregation but first we present the following motivating example:

Example 4.2.3. *Let consider the query "When I am in Latvia I usually visit Jurmala". If we apply pointwise aggregation it is not matched to the query "I usually visit Jurmala when I am in Latvia", since it disregards the order of words. Thus it is not a good reflection of the reality since both queries contain the same information. In this case the classical pointwise aggregation does not work.*

We proceed with the definition of σ-aggregation:

Definition 4.2.2. *Let $R_i : X_i \times X_i \to [0,1]$, $i \in \{1,...,n\}$ be binary fuzzy relations, A be an arbitrary aggregation function and $(y_{\sigma(1)},...,y_{\sigma(n)})$ be an arbitrary permutation of vector $\mathbf{y}$ (σ is a permutation from the set S_n of all permutations of the set $\{1,...,n\}$). Then*

$$\tilde{R}_\sigma(\mathbf{x},\mathbf{y}) = \max_{\sigma \in S_n}\{A(R_1(x_1, y_{\sigma(1)}),...,R_n(x_n, y_{\sigma(n)}))\}$$

is called σ-aggregation.

The domain and co-domain of $\tilde{R}_\sigma$ are the same like in the case of pointwise aggregation: $\tilde{R}_\sigma : X \times X \to [0,1]$, where $X = \prod_i X_i$.
The necessity of $\tilde{R}_\sigma$ is justified by similar practical needs as in the case of pointwise aggregation. Concerning the previous example $\tilde{R}_\sigma$ can measure the degree to which a query $\mathbf{y}$ matches a query $\mathbf{x}$, where all records are sentences and all elements of the record are words in the language where the order of words is not of the vital importance, e.g. Slavonic, Baltic languages or English in some particular cases. In this case we should use only one fuzzy relation R (not different R_i for each attribute i), because symmetry need to be preserved. Since we allow the arbitrary permutations of the elements of vectors preservation of symmetry is not possible when we have different R_i. Moreover to preserve the property of symmetry we should introduce a symmetric aggregation function.

Theorem 4.2.3. *If R is a T-transitive fuzzy relation, and A is a symmetric aggregation function which dominates T then $\tilde{R}_\sigma$ is also T-transitive, where*

$$\tilde{R}_\sigma(\mathbf{x},\mathbf{y}) = \max_{\sigma \in S_n}\{A(R(x_1, y_{\sigma(1)}),...,R(x_n, y_{\sigma(n)}))\}.$$

Proof. We need to show that

$$T(\tilde{R}_\sigma(\mathbf{x},\mathbf{y}), \tilde{R}_\sigma(\mathbf{y},\mathbf{z})) \leq \tilde{R}_\sigma(\mathbf{x},\mathbf{z})$$

for arbitrary $\mathbf{x},\mathbf{y},\mathbf{z}$.
One can see that if we use symmetric aggregation function and only one fuzzy relation R for all coordinates $i \in \{1,...,n\}$, it is not important which one of the vectors $\mathbf{x}$ or $\mathbf{y}$ or both of them we permute in Definition 4.2.2, result is the same. Therefore by Definition 4.2.2 there exists a permutation $\mathbf{x}^* = (x_1^*,...,x_n^*)$ of $\mathbf{x}$ such that

$$\tilde{R}_\sigma(\mathbf{x},\mathbf{y}) = A(R(x_1^*, y_1),...,R(x_n^*, y_n)).$$

Similarly there exists a permutation $\mathbf{z}^* = (z_1^*, ..., z_n^*)$ of $\mathbf{z}$ such that

$$\tilde{R}_\sigma(\mathbf{y}, \mathbf{z}) = A(R(y_1, z_1^*), ..., R(y_n, z_n^*)).$$

By the fact that A dominates T and that R is the T-transitive fuzzy relation we conclude:

$$T(\tilde{R}_\sigma(\mathbf{x}, \mathbf{y}), \tilde{R}_\sigma(\mathbf{y}, \mathbf{z})) =$$
$$T(A(R(x_1^*, y_1), ..., R(x_n^*, y_n)), A(R(y_1, z_1^*), ..., R(y_n, z_n^*))) \leq$$
$$\leq A(T(R(x_1^*, y_1), R(y_1, z_1^*)), ..., T(R(x_n^*, y_n), R(y_n, z_n^*))) \leq$$
$$\leq A(R(x_1^*, z_1^*), ..., R(x_n^*, z_n^*)) \leq \tilde{R}_\sigma(\mathbf{x}, \mathbf{z}).$$

□

Corollary 4.2.4. *If E is a T-transitive fuzzy equivalence relation, and A is a symmetric aggregation function which dominates T then $\tilde{E}_\sigma$ is also T-transitive fuzzy equivalence relation, where*

$$\tilde{E}_\sigma(\mathbf{x}, \mathbf{y}) = \max_{\sigma \in S_n} \{A(E(x_1, y_{\sigma(1)}), ..., E(x_n, y_{\sigma(n)}))\}.$$

4.3 Involving fuzzy orders for multi-objective linear programming

4.3.1 Problem formulation

In this section we work in the field of multi-objective (or Multiple Objective) linear programming (MOLP), which is an important tool for solving real-life optimization problems such as production planning, logistics, environment management, banking/finance planning etc. Our investigations are based on the fuzzy approach [50] where the membership functions are involved to prescribe how far the concrete point is from the solution of an individual problem. We propose to use fuzzy order relations instead of the membership functions described above. Further we describe the solution approach and investigate examples. Let us now focus on the formulation of the problem and the description of the scheme.

MOLP problem can be represented as follows:

$MAX\ Z$, where $Z = (z_1, ..., z_k)$ is a vector of objectives,

$z_i = \sum_{j=1}^{n} c_{ij}x_j$ where $i = 1, .., k,$

subject to $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$, $i = 1, ..., m.$ (1)

That is we must find a vector $\mathbf{x^o} = (x_1^o, ..., x_n^o)$ which maximizes k objective functions of n variables, and with m constraints. Let D denote a feasible region of the problem (1).

In problem (1), all objective functions can hardly reach their optima at the same time subject to the given constraints since usually the objective functions conflict one with another. Thus Pareto optimal solution (efficient solution) and optimal compromise solution are introduced:

Definition 4.3.1. [51] $\mathbf{x}^*$ *is called Pareto optimal solution if and only if there does not exist another* $\mathbf{x} \in D$ *such that* $z_i(\mathbf{x}^*) \leq z_i(\mathbf{x})$ *for all* i *and* $z_j(\mathbf{x}^*) \neq z_j(\mathbf{x})$ *for at least one* j.

Definition 4.3.2. [51] *An optimal compromise solution of a vector-maximum problem is a solution* $\mathbf{x} \in D$ *which is preferred by the decision maker to all other solutions, taking into consideration all criteria contained in the vector-valued objective function. It is generally accepted, that an optimal compromise solution has to be a Pareto optimal solution.*

Thus our main aim is to determine the optimal compromise solution. The fuzzy approach for solving MOLP proposed by Zimmermann [10] has given an effective way of measuring the satisfaction degree for MOLP. The idea is to identify the membership functions prescribing the fuzzy goals (solutions of individual problem) for the objective functions $z_i,\ i = 1, .., k$. The following linear function is an example of a membership function:

$$\mu_i(\mathbf{x}) = \begin{cases} 0, & \text{if } z_i(\mathbf{x}) < z_i^{min} \\ \dfrac{z_i(\mathbf{x}) - z_i^{min}}{z_i^{max} - z_i^{min}}, & z_i^{min} \leq z_i(\mathbf{x}) \leq z_i^{max} \\ 1, & z_i(\mathbf{x}) > z_i^{max} \end{cases},$$

where z_i^{max} is the solution of the individual problem

$MAX\ z_i$, s.t. $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$, $i = 1, ..., m$

and z_i^{min} is the solution of the individual problem

$MIN\ z_i$, s.t. $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$, $i = 1, ..., m.$

Usually the membership functions μ_i are linear functions and it is argued by the ”facilitation computation for obtaining solutions”. Further in the ”classical” fuzzy approach membership functions μ_i are aggregated. The main subject which is discussed in the majority of papers is the choice of an aggregation function.

Here we propose a completely different approach although we are still working in the fuzzy environment. We initiate involving fuzzy orders to solve the problem. To justify the choice of a fuzzy order let us first observe the classical linear programming problem when we must maximize the unique function $z = \sum_{j=1}^{n} c_j x_j$ where the vectors $(x_1, ..., x_n)$ belong to the set $D : \sum_{j=1}^{n} a_{ij} x_j \leq b_i$, $i = 1, ..., m$. In this case we can involve the relation $\preceq$:

$$\mathbf{x} \preceq \mathbf{y} \Leftrightarrow z(\mathbf{x}) \leq z(\mathbf{y})$$

which is obviously a crisp linear order with respect to the crisp equivalence relation

$$\mathbf{x} \doteq \mathbf{y} \Leftrightarrow z(\mathbf{x}) = z(\mathbf{y}).$$

Thus we can reformulate the problem in the following way: $MAX(D, \preceq)$. That is we should find a maximum in the set D which is ordered by the linear order $\preceq$. We use this idea to solve the multi-objective linear programming problem. Since we have more than one objective functions we should involve order relation for each objective function and they should be obviously fuzzy order relations to overcome the conflict of all objective functions. Further we aggregate fuzzy order relations to get one fuzzy order relation which includes the information about all objective functions and in the last step we must find a maximum in the set D with respect to the aggregated fuzzy order relation. Thus the scheme of the solution is as follows:

1. We define fuzzy order relations P_i which generalize the following crisp order relations:

   $$\mathbf{x} \preceq_i \mathbf{y} \Leftrightarrow z_i(\mathbf{x}) \leq z_i(\mathbf{y}),\ i = 1, .., k.$$

   Thus each fuzzy order relation describes corresponding objective function z_i.

2. We aggregate fuzzy orders using an aggregation function A which preserves the properties of initial fuzzy orders:

   $$P(\mathbf{x}, \mathbf{y}) = A(P_1(\mathbf{x}, \mathbf{y}), ..., P_k(\mathbf{x}, \mathbf{y})).$$

   Thus the aggregated fuzzy order relation P provides the information about all objective functions.

3. We find a maximum in the set D with respect to the aggregated fuzzy order relation P.

In our work we exactly realize the above described scheme. As we have seen above, solving the classical linear programming problem with one objective function there naturally arises a crisp linear order. This crisp linear order could be naturally generalized to fuzzy linear order solving multi-objective problem. Thus if we use fuzzy approach proposed by Zimmermann [50] and generalized by many others authors (see e.g. [36], [37], [18]) we do not take into account the information about these orders (which are reflective, transitive and antisymmetric relations), so this information is lost. Thus one of the advantages of our approach is that we take into account this information, and even more aggregating these fuzzy orders we use the aggregation function which preserves the properties of fuzzy orders. The other advantage is that in our approach we explain the "shape" of fuzzy order relation and choice of aggregation function (this is caused by the fuzzy environment (or t-norm) in which we are working). Moreover in our approach we can naturally use compensatory aggregation functions and even more we can use weights (see Section 4.3.3) to show the preference of objective functions.

As we wrote in Definition 4.3.2, an optimal compromise solution has to be a Pareto optimal solution. Although the "min" operator method, proposed by Zimmermann [50] has been proven to have several nice properties, the solution generated by this approach does not guarantee Pareto-optimality. As we will see later, in our approach we have found the properties which guarantee Pareto-optimality even regardless of the uniqueness of the optimal solution.

4.3.2 Solution approach

Let us now come back to the realization of our scheme. Our aim now is to involve fuzzy orders P_i which contain the information about objective functions z_i. Since we define fuzzy order relations it is necessary to define fuzzy equivalence relations first. To define the fuzzy equivalence relations we use the construction proposed in Theorem 2.3.1 where the relation are constructed on the base of pseudo-metrics. It is worth to mention that this approach is widely used in the literature for practical applications (see e.g. [8]).

Thus we build the following pseudo-metrics on the set D:

$$d_i(\mathbf{x}, \mathbf{y}) = \frac{|z_i(\mathbf{x}) - z_i(\mathbf{y})|}{z_i^{max} - z_i^{min}}.$$

Thus defined d_i are indeed pseudo-metrics and applying Theorem 2.3.1 we can build a T-equivalence relation:

$$E_i(\mathbf{x},\mathbf{y}) = t^{(-1)}(\min(\frac{|z_i(\mathbf{x}) - z_i(\mathbf{y})|}{z_i^{max} - z_i^{min}}, t(0))), \tag{4.1}$$

where t is an additive generator of a continuous Archimedean t-norm T.

Hence we should first choose a t-norm which plays a role of a generalized conjunction and further construct a T-equivalence using the correspondent additive generator t.

Example 4.3.1. *1.*

$$E_i(\mathbf{x},\mathbf{y}) = 1 - \frac{|z_i(\mathbf{x}) - z_i(\mathbf{y})|}{z_i^{max} - z_i^{min}} \tag{4.2}$$

are fuzzy T_L-equivalence relations.

2.

$$E_i(\mathbf{x},\mathbf{y}) = e^{-\frac{|z_i(\mathbf{x}) - z_i(\mathbf{y})|}{z_i^{max} - z_i^{min}}} \tag{4.3}$$

are fuzzy T_P-equivalence relations.

Remark 4.3.1. *Although the above defined pseudo-metrics are quite natural, other metrics can be also used. For example the following pseido-metrics can be chosen:*

$$d_i(\mathbf{x},\mathbf{y}) = C_i \cdot \frac{|z_i(\mathbf{x}) - z_i(\mathbf{y})|}{z_i^{max} - z_i^{min}},$$

where C_i is a real number greater than 0.
In this case $E_i(\mathbf{x},\mathbf{y}) = max(1 - C_i \cdot \frac{|z_i(\mathbf{x}) - z_i(\mathbf{y})|}{z_i^{max} - z_i^{min}}, 0)$ are fuzzy T_L-equivalence relations and $E_i(\mathbf{x},\mathbf{y}) = e^{-C_i \frac{|z_i(\mathbf{x}) - z_i(\mathbf{y})|}{z_i^{max} - z_i^{min}}}$ are fuzzy T_P-equivalence relations.

Further we build fuzzy order relations applying Theorem 2.3.2. Namely we construct T-E_i-orders where T is a chosen t-norm and E_i is the constructed fuzzy equivalence relation. To apply Theorem 2.3.2 we must also fix crisp order relations and in our case they are linear orders $\preceq_i$ on the set D:

$$\mathbf{x} \preceq_i \mathbf{y} \Leftrightarrow z_i(\mathbf{x}) \leq z_i(\mathbf{y}).$$

Let us show that fuzzy equivalence relation 4.1 is compatible with linear order $\preceq_i$:
$\mathbf{x} \preceq_i \mathbf{y} \preceq_i \mathbf{z} \Rightarrow (E_i(\mathbf{x},\mathbf{z}) \leq E_i(\mathbf{y},\mathbf{z})\ and\ E_i(\mathbf{x},\mathbf{z}) \leq E_i(\mathbf{x},\mathbf{y}))$.
If $\mathbf{x} \preceq_i \mathbf{y} \preceq_i \mathbf{z}$ then $z_i(\mathbf{x}) \leq z_i(\mathbf{y}) \leq z_i(\mathbf{z})$ and hence $|z_i(\mathbf{x}) - z_i(\mathbf{y})| \leq$

$\leq |z_i(\mathbf{x}) - z_i(\mathbf{z})|$. Furthermore $\min(\frac{|z_i(\mathbf{x})-z_i(\mathbf{y})|}{z_i^{max}-z_i^{min}}, t(0)) \leq \min(\frac{|z_i(\mathbf{x})-z_i(\mathbf{z})|}{z_i^{max}-z_i^{min}}, t(0))$. Hence by strictly decreasing monotonicity of $t^{(-1)}$ we get: $E_i(\mathbf{x}, \mathbf{z}) \leq E_i(\mathbf{x}, \mathbf{y})$. The same considerations are valid to show that $E_i(\mathbf{x}, \mathbf{z}) \leq E_i(\mathbf{y}, \mathbf{z})$.
Hence the following functions:

$$P_i(\mathbf{x}, \mathbf{y}) = \begin{cases} 1, & \text{if } \mathbf{x} \preceq_i \mathbf{y} \\ E_i(\mathbf{x}, \mathbf{y}), & \text{otherwise.} \end{cases} = \begin{cases} 1, & \text{if } z_i(\mathbf{x}) \leq z_i(\mathbf{y}) \\ E_i(\mathbf{x}, \mathbf{y}), & \text{otherwise.} \end{cases} \tag{4.4}$$

are T-E_i-orders, where E_i are defined by Equation 4.1.

Example 4.3.2. *1.*

$$P_i(\mathbf{x}, \mathbf{y}) = \begin{cases} 1, & \textit{if } z_i(\mathbf{x}) \leq z_i(\mathbf{y}) \\ 1 - \frac{|z_i(\mathbf{x})-z_i(\mathbf{y})|}{z_i^{max}-z_i^{min}}, & \textit{otherwise.} \end{cases}$$

are fuzzy order relations with respect to t-norm T_L and T_L-equivalence E_i defined by Equation 4.2.

2.

$$P_i(\mathbf{x}, \mathbf{y}) = \begin{cases} 1, & \textit{if } z_i(\mathbf{x}) \leq z_i(\mathbf{y}) \\ e^{-\frac{|z_i(\mathbf{x})-z_i(\mathbf{y})|}{z_i^{max}-z_i^{min}}}, & \textit{otherwise.} \end{cases}$$

are fuzzy order relations with respect to t-norm T_P and T_P-equivalence E_i defined by Equation 4.3.

The fuzzy order relations are constructed and we come to the next step where we aggregate corresponding relations. So we have to fuse the information about all fuzzy order relations P_i and get a global fuzzy order relation P which includes the information about all fuzzy order relations P_i and thereby also the information about all objective functions z_i. Let us introduce the aggregation function $A : [0, 1]^k \to [0, 1]$ which aggregates fuzzy order relations:

$$P(\mathbf{x}, \mathbf{y}) = A(P_1(\mathbf{x}, \mathbf{y}), ..., P_k(\mathbf{x}, \mathbf{y})).$$

It is also natural to require that the global fuzzy relation should fulfill the same properties as the individual fuzzy relations. Thus an aggregation function A should be chosen in such a way that it should preserve the properties of initial fuzzy order relations.

We show the importance of the requirement that aggregated fuzzy relation of fuzzy orders must be also a fuzzy order by the example of preservation of transitivity:
If $z_i(\mathbf{x}) \leq z_i(\mathbf{y})$ and $z_i(\mathbf{y}) \leq z_i(\mathbf{z})$ for all i it is natural that the element $\mathbf{z}$ is more

preferable for us than the element $\mathbf{x}$ in a global sense what is exactly guaranteed by the preservation of transitivity.

Further the multi-objective linear programming problem comes to the following problem:

$$\max_{\mathbf{y}} \min_{\mathbf{x}} P(\mathbf{x}, \mathbf{y}) \qquad (P)$$

Intuitively this means that we find for each $\mathbf{y} \in D$ the value $\min_{\mathbf{x}} P(\mathbf{x}, \mathbf{y})$, that is we find the degree to which $\mathbf{y}$ is greater (or better) than every $\mathbf{x} \in D$. In other words we find the degree to which $\mathbf{y}$ is a maximal element in the set D and later on we find $\mathbf{y}$ to which this satisfaction degree is the largest.

Theorem 4.3.1. *An optimal solution* $\mathbf{y}$ *to the problem* (P) *is a Pareto optimal solution if it is the unique optimal solution.*

Proof. Suppose that $\mathbf{y}$ is not a Pareto optimal solution. Then there exists another $\tilde{\mathbf{y}} \in D$ such that $z_i(\mathbf{y}) \leq z_i(\tilde{\mathbf{y}})$ for all i and $z_j(\mathbf{y}) \neq z_j(\tilde{\mathbf{y}})$ for at least one j. Let us now compare $P_i(\mathbf{x}, \mathbf{y})$ and $P_i(\mathbf{x}, \tilde{\mathbf{y}})$. Further we distinguish between the following three cases:

1. If $z_i(\mathbf{y}) \leq z_i(\tilde{\mathbf{y}}) \leq z_i(\mathbf{x})$ or $z_i(\mathbf{y}) < z_i(\tilde{\mathbf{y}}) \leq z_i(\mathbf{x})$ then $|z_i(\mathbf{x}) - z_i(\tilde{\mathbf{y}})| \leq |z_i(\mathbf{x}) - z_i(\mathbf{y})|$. Furthermore $\min(\frac{|z_i(\mathbf{x}) - z_i(\tilde{\mathbf{y}})|}{z_i^{max} - z_i^{min}}, t(0)) \leq$ $\leq \min(\frac{|z_i(\mathbf{x}) - z_i(\mathbf{y})|}{z_i^{max} - z_i^{min}}, t(0))$. Further since $t^{(-1)}$ is strictly decreasing we get: $E_i(\mathbf{x}, \mathbf{y}) \leq E_i(\mathbf{x}, \tilde{\mathbf{y}})$ and thus $P_i(\mathbf{x}, \mathbf{y}) \leq P_i(\mathbf{x}, \tilde{\mathbf{y}})$.

2. If $z_i(\mathbf{x}) \leq z_i(\mathbf{y}) \leq z_i(\tilde{\mathbf{y}})$ or $z_i(\mathbf{x}) \leq z_i(\mathbf{y}) < z_i(\tilde{\mathbf{y}})$ then $P_i(\mathbf{x}, \mathbf{y}) = P_i(\mathbf{x}, \tilde{\mathbf{y}}) = 1$ since $z_i(\mathbf{x}) \leq z_i(\mathbf{y})$ and $z_i(\mathbf{x}) < z_i(\tilde{\mathbf{y}})$ (or $z_i(\mathbf{x}) \leq z_i(\tilde{\mathbf{y}})$).

3. If $z_i(\mathbf{y}) < z_i(\tilde{\mathbf{y}})$ then there could be also the following situation: $z_i(\mathbf{y}) < z_i(\mathbf{x}) < z_i(\tilde{\mathbf{y}})$. Then $P_i(\mathbf{x}, \mathbf{y}) \leq P_i(\mathbf{x}, \tilde{\mathbf{y}})$ since $P_i(\mathbf{x}, \tilde{\mathbf{y}}) = 1$.

Thus for all $\mathbf{x} \in D$

$$A(P_1(\mathbf{x}, \mathbf{y}), P_2(\mathbf{x}, \mathbf{y}), ..., P_k(\mathbf{x}, \mathbf{y})) \leq A(P_1(\mathbf{x}, \tilde{\mathbf{y}}), P_2(\mathbf{x}, \tilde{\mathbf{y}}), ..., P_k(\mathbf{x}, \tilde{\mathbf{y}})).$$

Hence $\min_{\mathbf{x}} P(\mathbf{x}, \mathbf{y}) \leq \min_{\mathbf{x}} P(\mathbf{x}, \tilde{\mathbf{y}})$. This contradicts the fact that $\mathbf{y}$ is the unique optimal solution to the problem.

□

We can also prove the above theorem without demanding the "uniqueness of the optimal solution" but in this case we should require some specific properties:

Theorem 4.3.2. *An optimal solution* $\mathbf{y}$ *to the problem* (P) *is a Pareto optimal solution if* $z_i(\mathbf{x}) > z_i(\mathbf{y}) \Rightarrow P_i(\mathbf{x}, \mathbf{y}) < 1$ *and* A *is a strictly monotone function.*

Proof. Suppose that $\mathbf{y}$ is not a Pareto optimal solution. Then there exists another $\tilde{\mathbf{y}} \in D$ such that $z_i(\mathbf{y}) \leq z_i(\tilde{\mathbf{y}})$ for all i and $z_j(\mathbf{y}) < z_j(\tilde{\mathbf{y}})$ for at least one j.

We can follow considerations from the previous theorem and thus for every $\mathbf{x} \in D$ it holds

$$A(P_1(\mathbf{x}, \mathbf{y}), P_2(\mathbf{x}, \mathbf{y}), ..., P_k(\mathbf{x}, \mathbf{y})) \leq A(P_1(\mathbf{x}, \tilde{\mathbf{y}}), P_2(\mathbf{x}, \tilde{\mathbf{y}}), ..., P_k(\mathbf{x}, \tilde{\mathbf{y}})).$$

Moreover since D is linearly connected there exists such $\tilde{\mathbf{x}}$ that $z_j(\mathbf{y}) < z_j(\tilde{\mathbf{x}}) < z_j(\tilde{\mathbf{y}})$. Thus $P_j(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = 1$ since $z_j(\tilde{\mathbf{x}}) < z_j(\tilde{\mathbf{y}})$ and $P_j(\tilde{\mathbf{x}}, \mathbf{y}) < 1$ since

$$\forall i \text{ and } \forall\, \mathbf{x}, \mathbf{y} \in D\ z_i(\mathbf{x}) > z_i(\mathbf{y}) \Rightarrow P_i(\mathbf{x}, \mathbf{y}) < 1.$$

Thus, since A is a strictly monotone function

$$A(P_1(\tilde{\mathbf{x}}, \mathbf{y}), P_2(\tilde{\mathbf{x}}, \mathbf{y}), ..., P_k(\tilde{\mathbf{x}}, \mathbf{y})) < A(P_1(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}), P_2(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}), ..., P_k(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})).$$

Hence $\min_{\mathbf{x}} P(\mathbf{x}, \mathbf{y}) < \min_{\mathbf{x}} P(\mathbf{x}, \tilde{\mathbf{y}})$. This contradicts the fact that $\mathbf{y}$ is the optimal solution to the problem. □

The properties that $z_i(\mathbf{x}) > z_i(\mathbf{y}) \Rightarrow P_i(\mathbf{x}, \mathbf{y}) < 1$ and that A is a strictly monotone function are quite natural properties since by this we just require that the order P should react to any change of any of the functions z_i. Thus for practical applications we suggest to use fuzzy orders and aggregation functions respecting these properties.

4.3.3 Numerical example

Let us observe the following linear programming problem:
max $z_1 = x_1$,
max $z_2 = x_2$,
s.t. $x_1 + x_2 \leq 1$,
$x_1, x_2 \geq 0$.

Figure 4.7 shows the solution space of this problem where we colored in gray feasible region of the problem and dotted lines denote the level lines of the objective functions for which the corresponding objective reach its maximum. We have chosen the simple (in the sense of input data) problem in order not to pay attention at the

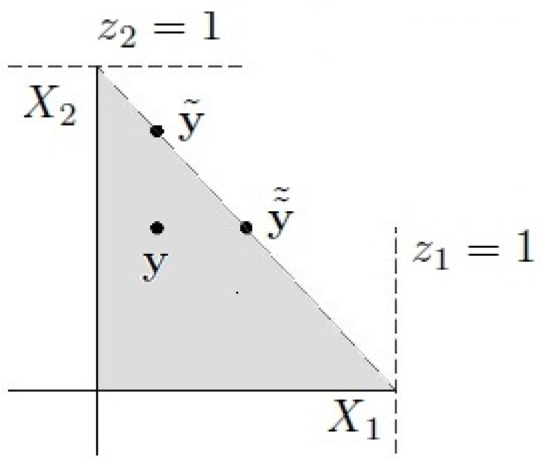


Figure 4.7:

details of computation but to illustrate the naturality of the proposed approach. Here we demonstrate the computation and how the result depends on the choice of an aggregation function and the base t-norm.

The point $(1, 0)$ is optimal solution with respect to the objective function z_1, the point $(0, 1)$ is the optimal solution with respect to the objective function z_2. Obviously the set $\{(x_1, x_2) : x_1 \in [0, 1], x_2 = 1 - x_1\}$ is the set of Pareto optimal solutions.

We follow the approach described above and apply the following fuzzy order relations based on Łukasiewicz t-norm (see Example 4.3.2):

$$P_1(\mathbf{x}, \mathbf{y}) = \begin{cases} 1, & \text{if } x_1 \leq y_1 \\ 1 - x_1 + y_1, & \text{otherwise.} \end{cases},$$

where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$;

$$P_2(\mathbf{x}, \mathbf{y}) = \begin{cases} 1, & \text{if } x_2 \leq y_2 \\ 1 - x_2 + y_2, & \text{otherwise.} \end{cases}.$$

Further we aggregate the corresponding fuzzy order relations with the help of the following aggregation function: $A(x, y) = \frac{x+y}{2}$ which is an aggregation function preserving T_L-transitivity. Thus:

$$P(\mathbf{x}, \mathbf{y}) = A(P_1(\mathbf{x}, \mathbf{y}), P_2(\mathbf{x}, \mathbf{y})) = \frac{P_1(\mathbf{x}, \mathbf{y}) + P_2(\mathbf{x}, \mathbf{y})}{2}$$

Further we must solve the following problem:

$$\max_{\mathbf{y}\in D}\min_{\mathbf{x}\in D} P(\mathbf{x},\mathbf{y}).$$

Let us contract the set D for the simplicity of calculations, that is we want to find a set B such that $B \subset D$ and

$$\max_{\mathbf{y}\in B}\min_{\mathbf{x}\in B} P(\mathbf{x},\mathbf{y}) = \max_{\mathbf{y}\in D}\min_{\mathbf{x}\in D} P(\mathbf{x},\mathbf{y}). \tag{4.5}$$

Let us prove that if $B = \{(x_1, x_2) : x_1 \in [0,1], x_2 = 1 - x_1\}$ then the equation 4.5 holds:

We start with the proof of the following equation:

$\max_{\mathbf{y}\in B}\min_{\mathbf{x}\in D} P(\mathbf{x},\mathbf{y}) = \max_{\mathbf{y}\in D}\min_{\mathbf{x}\in D} P(\mathbf{x},\mathbf{y}).$

Suppose contrary, that is we suppose that $\mathbf{y} = (y_1, y_2) \in D$ but $\mathbf{y} \notin B$. Then there exist points $\tilde{\mathbf{y}} = (y_1, 1 - y_1)$ and $\tilde{\tilde{\mathbf{y}}} = (1 - y_2, y_2)$ (see Figure 4.7) such that

$$\min_{\mathbf{x}\in D} P(\mathbf{x},\mathbf{y}) < \min_{\mathbf{x}\in D} P(\mathbf{x},\tilde{\mathbf{y}}) \text{ and}$$

$$\min_{\mathbf{x}\in D} P(\mathbf{x},\mathbf{y}) < \min_{\mathbf{x}\in D} P(\mathbf{x},\tilde{\tilde{\mathbf{y}}}).$$

Let us prove the first inequality:

We will prove that for all $\mathbf{x} \in D$ it holds

$$A(P_1(\mathbf{x},\mathbf{y}), P_2(\mathbf{x},\mathbf{y})) \leq A(P_1(\mathbf{x},\tilde{\mathbf{y}}), P_2(\mathbf{x},\tilde{\mathbf{y}}))$$

and there exists $\bar{\mathbf{x}} \in D$ such that $A(P_1(\bar{\mathbf{x}},\mathbf{y}), P_2(\bar{\mathbf{x}},\mathbf{y})) < A(P_1(\bar{\mathbf{x}},\tilde{\mathbf{y}}), P_2(\bar{\mathbf{x}},\tilde{\mathbf{y}}))$.

We know that $P_1(\mathbf{x},\mathbf{y}) = P_1(\mathbf{x},\tilde{\mathbf{y}})$ since $z_1(\mathbf{y}) = z_1(\tilde{\mathbf{y}})$.

Let us now compare $P_2(\mathbf{x},\mathbf{y})$ and $P_2(\mathbf{x},\tilde{\mathbf{y}})$. Obviously $z_2(\mathbf{y}) < z_2(\tilde{\mathbf{y}})$. To prove that $P_2(\mathbf{x},\mathbf{y}) \leq P_2(\mathbf{x},\tilde{\mathbf{y}})$ we distinguish between the following three cases:

1. If $z_2(\mathbf{y}) < z_2(\tilde{\mathbf{y}}) \leq z_2(\mathbf{x})$ then $P_2(\mathbf{x},\mathbf{y}) = 1 - |z_2(\mathbf{x}) - z_2(\mathbf{y})| <$ $< 1 - |z_2(\mathbf{x}) - z_2(\tilde{\mathbf{y}})| = P_2(\mathbf{x},\tilde{\mathbf{y}})$.

2. If $z_2(\mathbf{x}) \leq z_2(\mathbf{y}) < z_2(\tilde{\mathbf{y}})$ then $P_2(\mathbf{x},\mathbf{y}) = P_2(\mathbf{x},\tilde{\mathbf{y}}) = 1$ since $z_2(\mathbf{x}) \leq z_2(\mathbf{y})$ and $z_2(\mathbf{x}) < z_2(\tilde{\mathbf{y}})$.

3. If $z_2(\mathbf{y}) < z_2(\mathbf{x}) < z_2(\tilde{\mathbf{y}})$ then $P_2(\mathbf{x},\mathbf{y}) \leq P_2(\mathbf{x},\tilde{\mathbf{y}})$ since $P_2(\mathbf{x},\tilde{\mathbf{y}}) = 1$.

Thus for all $\mathbf{x} \in D$ $A(P_1(\mathbf{x},\mathbf{y}), P_2(\mathbf{x},\mathbf{y})) \leq A(P_1(\mathbf{x},\tilde{\mathbf{y}}), P_2(\mathbf{x},\tilde{\mathbf{y}}))$ Since there obviously exists $\bar{\mathbf{x}} \in D$ such that $z_2(\mathbf{y}) < z_2(\bar{\mathbf{x}}) < z_2(\tilde{\mathbf{y}})$ we have $P_2(\bar{\mathbf{x}},\tilde{\mathbf{y}}) = 1$ but $P_2(\bar{\mathbf{x}},\mathbf{y}) < 1$. Thus, because of the strict monotonicity of the function A

$$A(P_1(\bar{\mathbf{x}},\mathbf{y}), P_2(\bar{\mathbf{x}},\mathbf{y})) < A(P_1(\bar{\mathbf{x}},\tilde{\mathbf{y}}), P_2(\bar{\mathbf{x}},\tilde{\mathbf{y}})).$$

By this we have finished the proof that $\max_{\mathbf{y}\in B}\min_{\mathbf{x}\in D} P(\mathbf{x},\mathbf{y}) = \max_{\mathbf{y}\in D}\min_{\mathbf{x}\in D} P(\mathbf{x},\mathbf{y})$.
We continue with the proof of the following equation:
$\max_{\mathbf{y}\in B}\min_{\mathbf{x}\in B} P(\mathbf{x},\mathbf{y}) = \max_{\mathbf{y}\in B}\min_{\mathbf{x}\in D} P(\mathbf{x},\mathbf{y})$.
Assume contrary, that is we suppose that $\mathbf{x} = (x_1, x_2) \in D$ but $\mathbf{x} \notin B$. Then there exist points $\tilde{\mathbf{x}} = (x_1, 1 - x_1)$ and $\tilde{\tilde{\mathbf{x}}} = (1 - x_2, x_2)$ such that for the fixed $\mathbf{y} \in B$

$$P(\tilde{\mathbf{x}},\mathbf{y}) < P(\mathbf{x},\mathbf{y}) \ \text{ and}$$

$$P(\tilde{\tilde{\mathbf{x}}},\mathbf{y}) < P(\mathbf{x},\mathbf{y}).$$

Let us prove the first inequality:
We will prove that for all $\mathbf{y} \in B$ $A(P_1(\tilde{\mathbf{x}},\mathbf{y}), P_2(\tilde{\mathbf{x}},\mathbf{y})) \le A(P_1(\mathbf{x},\mathbf{y}), P_2(\mathbf{x},\mathbf{y}))$.
Obviously $P_1(\tilde{\mathbf{x}},\mathbf{y}) = P_1(\mathbf{x},\mathbf{y})$ since $z_1(\tilde{\mathbf{x}}) = z_1(\mathbf{x})$.
Let us now compare $P_2(\mathbf{x},\mathbf{y})$ and $P_2(\tilde{\mathbf{x}},\mathbf{y})$. Obviously $z_2(\mathbf{x}) < z_2(\tilde{\mathbf{x}})$. To prove that $P_2(\tilde{\mathbf{x}},\mathbf{y}) \le P_2(\mathbf{x},\mathbf{y})$ we distinguish between the following three cases:

1. If $z_2(\mathbf{x}) < z_2(\tilde{\mathbf{x}}) \le z_2(\mathbf{y})$ then $P_2(\mathbf{x},\mathbf{y}) = P_2(\tilde{\mathbf{x}},\mathbf{y}) = 1$ since $z_2(\mathbf{x}) < z_2(\mathbf{y})$ and $z_2(\tilde{\mathbf{x}}) \le z_2(\mathbf{y})$.
2. If $z_2(\mathbf{y}) \le z_2(\mathbf{x}) < z_2(\tilde{\mathbf{x}})$ then $P_2(\tilde{\mathbf{x}},\mathbf{y}) = 1 - |z_2(\tilde{\mathbf{x}}) - z_2(\mathbf{y})| <$ $< 1 - |z_2(\mathbf{x}) - z_2(\mathbf{y})| = P_2(\mathbf{x},\mathbf{y})$.
3. If $z_2(\mathbf{x}) < z_2(\mathbf{y}) < z_2(\tilde{\mathbf{x}})$ then $P_2(\tilde{\mathbf{x}},\mathbf{y}) \le P_2(\mathbf{x},\mathbf{y})$ since $P_2(\mathbf{x},\mathbf{y}) = 1$.

Thus for all $\mathbf{y} \in B$ it holds $A(P_1(\tilde{\mathbf{x}},\mathbf{y}), P_2(\tilde{\mathbf{x}},\mathbf{y})) \le A(P_1(\mathbf{x},\mathbf{y}), P_2(\mathbf{x},\mathbf{y}))$. Since there obviously exists $\bar{\mathbf{y}}$ such that $z_1(\mathbf{x}) < z_1(\bar{\mathbf{y}}) < z_1(\tilde{\mathbf{x}})$ we have $P_1(\mathbf{x},\bar{\mathbf{y}}) = 1$ but $P_1(\tilde{\mathbf{x}},\bar{\mathbf{y}}) < 1$. Thus, because of the strict monotonicity of the function A

$$A(P_1(\tilde{\mathbf{x}},\bar{\mathbf{y}}), P_2(\tilde{\mathbf{x}},\bar{\mathbf{y}})) < A(P_1(\mathbf{x},\bar{\mathbf{y}}), P_2(\mathbf{x},\bar{\mathbf{y}})).$$

Thus $\max_{\mathbf{y}\in B}\min_{\mathbf{x}\in B} P(\mathbf{x},\mathbf{y}) = \max_{\mathbf{y}\in D}\min_{\mathbf{x}\in D} P(\mathbf{x},\mathbf{y})$, where B is the set of Pareto optimal solutions. It is an important fact which makes calculations much easier.

Let us come back to our initial example and by the following figures we demonstrate the dependence of the value $\min_{\mathbf{x}\in B} P(\mathbf{x},\mathbf{y})$ on the choice of $\mathbf{y}$. The horizontal axes are the set B of Pareto optimal solutions:
$B = \{(y_1, y_2) : y_1 \in [0, 1], y_2 = 1 - y\}$, where the elements $\mathbf{y} = (y_1, y_2)$ of the set B are presented by its first coordinate:

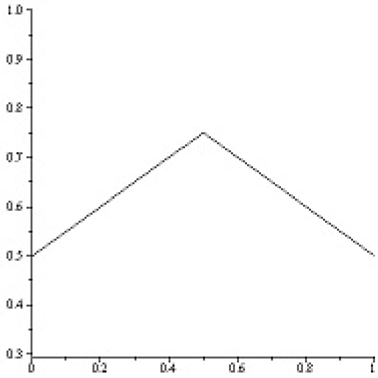


Figure 4.8:

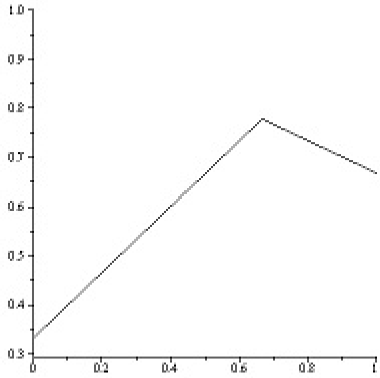


Figure 4.9:

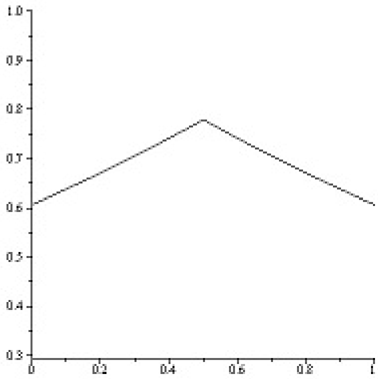


Figure 4.10:

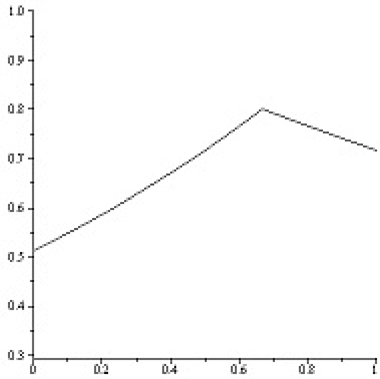


Figure 4.11:

Figure 4.8 and Figure 4.9 demonstrate the results when we use Łukasiewicz t-norm and $A(a_1, a_2) = \frac{a_1+a_2}{2}$ and $A(a_1, a_2) = \frac{2a_1+a_2}{3}$ respectively. The results are quite expected: when the weights are the same (1/2 and 1/2) the maximum point is exactly in the middle, but if the weights are 1/3 and 2/3 then the maximum point divides the unit interval respectively as 1/3 and 2/3. The results for the problem $\max_{\mathbf{y}\in B} \min_{\mathbf{x}\in B} P(\mathbf{x}, \mathbf{y})$ are the same when we use the product t-norm, but the shape of the function $f(\mathbf{y}) = \min_{\mathbf{x}\in B} P(\mathbf{x}, \mathbf{y})$ is slightly different, see Figure 4.10 and Figure 4.11.

4.3.4 Conclusions on MOLP

We proposed a solution approach for multi-objective linear programming problem where we have used fuzzy order relations instead of the membership functions prescribing the satisfaction degree of reaching the solution of individual problems. Further, to get an optimal compromise solution fuzzy order relations were aggregated and the "maximum" with respect to the aggregated fuzzy order relation has been found. Although the approach described in our work is more complicated in computations it has the following advantages:

1. This approach generalizes the classical linear programming approach which testifies to its naturality.

2. There is a reasonable explanation of the choice of the "shape" of fuzzy order relation. In classical fuzzy approach more often the choice of linear membership functions is not explained or is explained by "facilitation computation for obtaining solutions". The choice in our approach is caused by the fuzzy environment (determined by a t-norm) in which we are working.

3. There is a reasonable explanation of the choice of aggregation function. The choice in our approach is caused by the necessity to preserve the properties of initial fuzzy order relations.

We see the following two possible directions for future research:

1. We see that in our example the results do not depend on the choice of a t-norm. It is interesting to investigate how the choice of a t-norm affects the results in general.

2. The usage of fuzzy order relations are investigated only for the simplest fuzzy approach for solving multi-objective linear programming problems. It is interesting also to involve fuzzy order relations for two-level (multi-level) linear programming problems.

4.4 A-T-aggregation

This section is devoted to the concept of aggregation of fuzzy relations and, in particular, to the concept of A-T-aggregation. This tool employs a t-norm and an aggregation function to define the degree to which two elements are in relation when it is known (correspondingly pairwise) for all vectors aggregated into these elements. In our work we apply A-T-aggregation to fuzzy equivalence relations and fuzzy order relations and define necessary conditions for preservation of some relevant properties.

The concept of A-T-aggregation employs ideas of the extension principle. Let us recall the definition of T-extension of the n-argument function $\varphi : X_1 \times ... \times X_n \to Z$:

Definition 4.4.1. *(c.f. [45]) Let a mapping $\varphi : X_1 \times ... \times X_n \to Z$ be given, then the mapping $\tilde{\varphi} : F(X_1) \times ... \times F(X_n) \to F(Z)$ defined by the formula*

$$\tilde{\varphi}(M_1, ..., M_n)(z) = \sup_{(x_1,...,x_n)} \{T(M_1(x_1), ..., M_n(x_n)) :$$
$$x_1 \in X_1, ..., x_n \in X_n,\ \varphi(x_1, ..., x_n) = Z\},$$

where $M_1, ..., M_n$ are fuzzy sets is called the extension of the function φ defined on set $X_1 \times ... \times X_n$ to the function $\tilde{\varphi}$ defined on set $F(X_1) \times ... \times F(X_n)$.

However aggregation of fuzzy relations can't be considered just as a special case of generalization of aggregation process by its extension to the set of all fuzzy subsets (see e.g. [40],[31],[30],[34]) since a fuzzy relation is a function of two arguments. Although these two approaches have many common features they still differ. In this work we focus on aggregation of fuzzy equivalence relations and fuzzy order relations and we study necessary conditions for preservation of relevant properties.

4.4.1 Motivation and definitions

In this subsection we define A-T-aggregation.

Definition 4.4.2. *Let $R_i : [0,1] \times [0,1] \to [0,1]$, $i \in \{1, ..., n\}$ be binary fuzzy relations, $A : [0,1]^n \to [0,1]$ be an arbitrary aggregation function, T be an arbitrary t-norm. Then $\tilde{R}_{A,T} : [0,1] \times [0,1] \to [0,1]$ given by*

$$\tilde{R}_{A,T}(x, y) = \sup\{T(R_1(x_1, y_1), ..., R_n(x_n, y_n)) :$$
$$A(x_1, ..., x_n) = x, A(y_1, ..., y_n) = y\}$$

is called an A-T-aggregation of fuzzy relations.

We consider aggregation functions defined on $[0,1]^n$, thus we require the domain of R_i to be $[0,1] \times [0,1]$. Nevertheless extension to the class of aggregation functions defined on $\mathbb{I}^n$ ($A : \mathbb{I}^n \to \mathbb{I}$), where $\mathbb{I}$ is a nonempty real interval, is a matter of rescaling. In this case the domain of the relation $\tilde{R}_{A,T}$ would be $\mathbb{I} \times \mathbb{I}$ ($\tilde{R}_{A,T} : \mathbb{I} \times \mathbb{I} \to [0,1]$). To be more flexible on domains we propose the following generalized definition:

Definition 4.4.3. *Let $R_i : X_i \times X_i \to [0,1]$, $i \in \{1, ..., n\}$ be binary fuzzy relations, $f_i : X_i \to [0,1]$ be mappings, $A : [0,1]^n \to [0,1]$ be an arbitrary aggregation function, T be an arbitrary t-norm then $\tilde{R}_{A,T} : A([0,1]^n) \times A([0,1]^n) \to [0,1]$ given by*

$$\tilde{R}_{A,T}(x, y) = \sup\{T(R_1(x_1, y_1), ..., R_n(x_n, y_n)) :$$

$$A(f_1(x_1), ..., f_n(x_n)) = x, A(f_1(y_1), ..., f_n(y_n)) = y\}$$

is called A-T-aggregation of fuzzy relations.

The aim of A-T-aggregation is similar to the aim of pointwise aggregation (see e.g. [39]): to compare two elements which are described by n-ary vectors by fusing values of binary fuzzy relations acting on corresponding coordinates. The difference here is in the representation of comparable elements - these are n-ary vectors in the case of pointwise aggregation but we take into account the aggregation of corresponding vectors in the case of A-T-aggregation.

Remark 4.4.1. *In our work we will use the above definition of A-T-aggregation of fuzzy relations, but if the task is to compare elements which are given by vectors, the definition presented further is equivalent to the definition 4.4.2. Let $R_i : [0,1] \times [0,1] \to [0,1]$, $i \in \{1, ..., n\}$ be binary fuzzy relations, $A : [0,1]^n \to [0,1]$ be an arbitrary aggregation function and T be an arbitrary t-norm, then*

$$\tilde{R}_{A,T}(\mathbf{x}, \mathbf{y}) = \sup\{T(R_1(x'_1, y'_1), ..., R_n(x'_n, y'_n)) :$$

$$A(x'_1, ..., x'_n) = A(x_1, ..., x_n),$$

$$A(y'_1, ..., y'_n) = A(y_1, ..., y_n)\}$$

is called T-A-aggregation of fuzzy relations ($\mathbf{x} = (x_1, ..., x_n), \mathbf{y} = (y_1, ..., y_n)$).
In this case

$$\tilde{R}_{A,T} : [0,1]^n \times [0,1]^n \to [0,1].$$

This definition shows similarity of approaches and it could be useful if we want to compare the results of pointwise and T-A-aggregations. And now we explain the motivation for our definition of A-T-aggregation. The definition of monotonicity for an aggregation function A says that if for all $i \in \{1, ..., n\}$ it holds $x_i \leq y_i$, then $A(x_1, ..., x_n) \leq A(y_1, ..., y_n)$. Then for results of aggregation x and y we definitely know that $x \leq y$ whenever there exist vectors $(x_1, ..., x_n)$ and $(y_1, ..., y_n)$ such that $A(x_1, ..., x_n) = x$ and $A(y_1, ..., y_n) = y$ and $x_i \leq y_i$ for all $i \in \{1, ..., n\}$.

Now assume that for each coordinate we have different order relations:

$$x_1 \leq_1 y_1, \ldots, x_n \leq_n y_n.$$

Then if we continue by using "fuzzy language" we can rewrite:

$$x \leq y \text{ as } \tilde{R}(x, y)$$

and

$$x_i \leq_i y_i \text{ as } R_i(x_i, y_i) \ for \ all \ i \in \{1, ..., n\}.$$

As a connective we use a t-norm T, and, finally, as it is usually done in the context of fuzzy set theory we use supremum in the place where quantifier $\exists$ is used. Thus we get

$$\tilde{R}(x, y) = \sup\{T(R_1(x_1, y_1), ..., R_n(x_n, y_n)) :$$

$$A(x_1, ..., x_n) = x, A(y_1, ..., y_n) = y\},$$

what is exactly the definition 4.4.2. In this example we spoke about order relations but the same reasoning works also for equivalence relations.

Example 4.4.1. *We consider a data set describing progress in studies of students. For each student the estimations in tree main courses are given in a table. The data set is shown in Table 4.5.*

Table 4.5: Evaluation of students

Student	Course 1	Course 2	Course 3	Agg. result
s_1	0.8	0.7	0.6	0.7
s_2	0.8	0.9	0.6	0.77
s_3	0.6	0.7	0.5	0.6
s_4	0.8	0.5	1	0.77
s_5	0.9	0.5	0.8	0.73
s_6	0.7	0.9	0.6	0.73
s_7	0.7	0.8	0.8	0.77

In this table the students' results are given as well as the aggregated values of the results in three different courses, the arithmetic mean is used to aggregate the results. Assume now that we have to compare all students. We not only want to know for the pair of students s_i and s_j which student is better, but we want to know the degree to which student s_i is better than student s_j. Let us introduce the fuzzy order relations R_i to compare the marks of students in course i by the following way:

$$R_i(a_i, b_i) = \begin{cases} 1, & \text{if } a_i \leq b_i \\ \max(1 - |a_i - b_i|, 0), & \text{otherwise} \end{cases}.$$

In this example we introduce the same fuzzy order relations for all coordinates $i \in \{1, 2, 3\}$*, but generally they can differ. Let us compare the students* s_6 *and* s_7 *or, namely, let calculate the degree to which the student* s_6 *is worse than student* s_7 *(or, in other words, the degree to which student* s_7 *is better than student* s_6*). If we use pointwise aggregation and arithmetic mean as an aggregation function (the other aggregation function can be also used), the result will be* $\tilde{R}(s_6, s_7) = 0.97$*. However if we want to compare the results of students, taking into account the aggregation of the results, we should use* A-T*-aggregation (*A *- arithmetic mean). If we use minimum t-norm, then the result will be:* $\tilde{R}_{A,T}(s_6, s_7) = 1$ *since there exists a vector* $(0.8, 0.9, 0.6)$ *such that* $A(0.8, 0.9, 0.6) = A(0.7, 0.8, 0.8) = 0.77$ *and* $min(R_1(0.7, 0.8), R_2(0.9, 0.9), R_3(0.6, 0.6)) = 1$.

In the sequel we observe only fuzzy relations on interval $[0, 1]$ and we write simply R instead of $R : [0, 1] \times [0, 1] \to [0, 1]$.

4.4.2 Preservation of properties of fuzzy relations by A-T-aggregation

We study which properties of fuzzy relations are preserved in the process of aggregation. We focus only on the aggregation of fuzzy equivalence relations given by (2.1) and fuzzy order relations (2.3). Preservation of reflexivity and symmetry is trivial, so we leave these results without proofs.

Proposition 4.4.1. *Let* R_i *for all* $i \in \{1, ..., n\}$ *be reflexive fuzzy relations, then* $\tilde{R}_{A,T}$ *is also reflexive.*

Proposition 4.4.2. *Let* R_i *for all* $i \in \{1, ..., n\}$ *be symmetric fuzzy relations, then* $\tilde{R}_{A,T}$ *is also symmetric.*

More interesting question is about preservation of transitivity and compatibility for fuzzy equivalence relations. In the next results we show that for special aggregation functions and in the case when T_M is minimum t-norm

$$\tilde{R}_{A,T}(x, y) = T_M(R_1(x, y), ..., R_n(x, y)).$$

Obviously idempotence of the aggregation function is crucial here. Further we find a class of functions leading to this result. Let A_x denote the set of vectors $\mathbf{x}$ such that $A(\mathbf{x}) = x$, similarly we denote A_y.

Theorem 4.4.1. *If R_i for all $i \in \{1,...,n\}$ are fuzzy relations, $T = T_M$ (minimum t-norm), A is an idempotent aggregation function such that:*

$$R_i(x,y) \geq T_M(R_i(x_1,y_1), R_i(x_2,y_2), ..., R_i(x_n,y_n))$$

for all $(x_1,...,x_n) \in A_x, (y_1,...,y_n) \in A_y$ and for all $i \in \{1,...,n\}$, then

$$\tilde{R}_{A,T}(x,y) = T_M(R_1(x,y),...,R_n(x,y)).$$

Proof. Using

$$R_i(x,y) \geq T_M(R_i(x_1,y_1), R_i(x_2,y_2), ..., R_i(x_n,y_n))$$

by idempotence of T_M we get

$$T_M(R_i(x,y),...,R_i(x,y)) \geq$$

$$\geq T_M(R_i(x_1,y_1), R_i(x_2,y_2), ..., R_i(x_n,y_n))$$

for all relations $R_i, i \in \{1,...,n\}$.

Therefore, by monotonicity of T_M we get
$T_M(T_M(R_1(x,y),...,R_1(x,y)),...,T_M(R_n(x,y),...,R_n(x,y))) \geq$
$\geq T_M(T_M(R_1(x_1,y_1),...,R_1(x_n,y_n)),...,T_M(R_n(x_1,y_1),...,R_n(x_n,y_n)))$.

Using associativity of T_M we continue in the following way:
$T_M(T_M(R_1(x,y),...,R_n(x,y)),...,T_M(R_1(x,y),...,R_n(x,y))) \geq$
$\geq T_M(T_M(R_1(x_1,y_1),...,R_n(x_n,y_n)),...,T_M(R_1(x_1,y_1),...,R_n(x_n,y_n)))$.

Using idempotence of T_M from the previous we deduce that

$$T_M(R_1(x,y),...,R_n(x,y)) \geq T_M(R_1(x_1,y_1),...R_n(x_n,y_n)).$$

Since previous inequality holds for arbitrary vectors $(x_1,...,x_n) \in A_x, (y_1,...,y_n) \in A_y$ it also holds for supremum:

$$T_M(R_1(x,y),...,R_n(x,y)) \geq$$
$$\geq \sup_{(x_1,...,x_n) \in A_x, (y_1,...,y_n) \in A_y} \{T_M(R_1(x_1,y_1),...,R_n(x_n,y_n))\} = \tilde{R}_{A,T}(x,y)$$

But $(x,...,x) \in A_x, (y,...,y) \in A_y$ thus

$$\tilde{R}_{A,T}(x,y) = T_M(R_1(x,y),...,R_n(x,y)).$$

□

Remark 4.4.2. *We denote by $\tilde{E}_{A,T}$ the A-T-aggregation of fuzzy equivalence relations $E_1, E_2, ..., E_n$.*

It is easy to see, that if a fuzzy relation E is given by formula (2.1), then the property

$$E(x,y) \geq T_M(E(x_1,y_1), E(x_2,y_2), ..., E(x_n,y_n)) \tag{4.6}$$

is equivalent to the property:

$$|x-y| \leq \max_i(|x_i - y_i|).$$

Now we show that $\tilde{E}_{A,T}$ constructed by an additive aggregation function has property described in the previous theorem.

Proposition 4.4.3. *If E is a fuzzy relation given by formula (2.1), T_M is the minimum t-norm, A is an additive aggregation function, then*

$$E(x,y) \geq T_M(E(x_1,y_1), E(x_2,y_2), ..., E(x_n,y_n)),$$

for all $(x_1, ..., x_n) \in A_x, (y_1, ..., y_n) \in A_y$ and for all $i \in \{1, ..., n\}$.

Proof. We need to show for arbitrary $(x_1, ..., x_n) \in A_x$ and $(y_1, ..., y_n) \in A_y$ that $E(x,y) \geq T_M(E(x_1,y_1), E(x_2,y_2), ..., E(x_n,y_n))$. Equally, because of the nature of the fuzzy equivalence relation E (see formula (2.1)), it is sufficient to show that

$$|x-y| \leq \max_i(|x_i - y_i|).$$

Let us observe the case, when $x > y$: $|x-y| = x - y = A(x_1, ..., x_n) - A(y_1, ..., y_n)$. Now we substitute arguments of $A(x_1, ..., x_n)$ standing on the positions $l : x_l \leq y_l$ by the corresponding coordinate from the vector $(y_1, ..., y_n)$. So, by monotonicity of A we estimate:

$$x - y = A(x_1, ..., x_n) - A(y_1, ..., y_n) \leq$$
$$\leq A(x_1, ..., y_l, ..., x_n) - A(y_1, ..., y_l, ..., y_n).$$

Further by additivity of A we get:

$$x - y \leq A(x_1 - y_1, ..., 0, ..., x_n - y_n)$$

Since additivity implies idempotence we have:

$$x - y \leq \max_{k: x_k > y_k}(x_k - y_k) \leq \max_i(|x_i - y_i|).$$

Thus we have shown that the condition holds. The same considerations are valid when we observe the case when $x < y$:
$|x-y| = y - x \leq \max_{k: y_k > x_k}(y_k - x_k) \leq \max_i(|x_i - y_i|)$. □

But not only additivity of aggregation function can assure necessary property. Propositions 4.4.4 and 4.4.5 contain results for non additive aggregation functions.

Proposition 4.4.4. *If E is a fuzzy relation given by formula (2.1), T_M is the minimum t-norm, A is partial minimum:*

$$A(\mathbf{x}) = MIN_K(\mathbf{x}) = \bigwedge_{i \in K} x_i,$$

where $K \subseteq \{1, ..., n\}$, then

$$E(x, y) \geq T_M(E(x_1, y_1), E(x_2, y_2), ..., E(x_n, y_n)),$$

for all $(x_1, ..., x_n) \in A_x, (y_1, ..., y_n) \in A_y$.

Proof. If $E(x, y) = g(|x - y|)$, where g is a non-increasing mapping, then the condition

$$E(x, y) \geq T_M(E(x_1, y_1), E(x_2, y_2), ..., E(x_n, y_n))$$

is equivalent to the condition

$$|x - y| \leq \max_i(|x_i - y_i|).$$

Let us prove the last: Let $x > y$ and let $\bigwedge_{i \in K} y_i = y_k$. Then $|x - y| = \bigwedge_{i \in K} x_i - \bigwedge_{i \in K} y_i = \bigwedge_{i \in K} x_i - y_k \leq x_k - y_k \leq \max_{x_i > y_i, i \in K}(x_i - y_i) \leq \max_i(|x_i - y_i|)$. Now let $x \leq y$ and let $\bigwedge_{i \in K} x_i = x_k$. Then $|x - y| = \bigwedge_{i \in K} y_i - \bigwedge_{i \in K} x_i = \bigwedge_{i \in K} y_i - x_k \leq y_k - x_k \leq \max_{y_i > x_i, i \in K}(y_i - x_i) \leq \max_i(|x_i - y_i|)$. The both considerations (when $x < y$ and when $x \geq y$) lead to the fact that

$$|x - y| \leq \max_i(|x_i - y_i|).$$

By this we finish the proof. □

Proposition 4.4.5. *If E is a fuzzy relation given by formula (2.1), T_M is the minimum t-norm, A is partial maximum:*

$$A(\mathbf{x}) = MAX_K(\mathbf{x}) = \bigvee_{i \in K} x_i,$$

where $K \subseteq \{1, ..., n\}$, then

$$E(x, y) \geq T_M(E(x_1, y_1), E(x_2, y_2), ..., E(x_n, y_n)),$$

for all $(x_1, ..., x_n) \in A_x, (y_1, ..., y_n) \in A_y$.

Proof. □

As in the previous proof it is sufficient to prove that

$$|x - y| \leq \max_i(|x_i - y_i|).$$

Let $x > y$ *and let* $\bigvee\limits_{i \in K} x_i = x_k$*. Then* $|x-y| = \bigvee\limits_{i\in K} x_i - \bigvee\limits_{i\in K} y_i = x_k - \bigvee\limits_{i\in K} y_i \leq x_k - y_k \leq$ $\leq \max\limits_{x_i > y_i, i \in K}(x_i - y_i) \leq \max\limits_i(|x_i - y_i|)$*. If* $x \leq y$*, we denote* $\bigwedge\limits_{i\in K} y_i = y_k$ *and then* $|x-y| = \bigvee\limits_{i\in K} y_i - \bigvee\limits_{i\in K} x_i = y_k - \bigvee\limits_{i\in K} x_i \leq y_k - x_k \leq \max\limits_{y_i > x_i, i\in K}(y_i - x_i) \leq \max\limits_i(|x_i - y_i|)$*. By this the condition is proven.*

Additive aggregation functions are nothing else but the weighted arithmetic means WAM_w:

$$WAM_w(\mathbf{x}) = \sum_{i=1}^{n} w_i x_i,$$

where weights $w_i \in [0,1]$ and $\sum\limits_{i=1}^{n} w_i = 1$, for details see [17] Proposition 4.21. By the last three propositions we have proven that for every aggregation function A from the class $\mathcal{A} = \{MIN_K, MAX_K : K \subseteq \{1,...,n\}\} \cup$

$$\cup \{WAM_w : w_i \in [0,1] \ for \ all \ i, \sum_{i=1}^{n} w_i = 1\}$$

the following condition holds:

$$E(x,y) \geq T_M(E(x_1,y_1), E(x_2,y_2), ..., E(x_n,y_n)),$$

where $x = A(x_1,...,x_n)$, $y = A(y_1,...,y_n)$, E is a fuzzy relation given by formula (2.1) and T_M is the minimum t-norm. In [32] it is proven that the class $\mathcal{A}$ is the class of all increasing, bisymmetric and stable for positive linear transformations aggregation functions.

Theorem 4.4.1 implies:

Corollary 4.4.1. *If* E_i *for all* $i \in \{1,...,n\}$ *are fuzzy equivalence relations given by (2.1),* T_M *is the minimum t-norm,* $A \in \mathcal{A}$*, then*

$$\tilde{E}_{A,T_M}(x,y) = T_M(E_1(x,y),...,E_n(x,y)).$$

Now we study preservation of T-transitivity.

Theorem 4.4.2. *If E_i for all $i \in \{1, ..., n\}$ are T-transitive fuzzy equivalence relations, $A \in \mathcal{A}$ then $\tilde{E}_{A,T_M}$ is also a T-transitive fuzzy equivalence relation, where T_M is the minimum t-norm.*

Proof. Since T_M dominates every other t-norm T (for details see e.g. [39]) and all relations E_i are T-transitive we have:

$$T(\tilde{E}_{A,T_M}(x,y), \tilde{E}_{A,T_M}(y,z)) = T(T_M(E_1(x,y), ..., E_n(x,y)),$$
$$T_M(E_1(y,z), ..., E_n(y,z))) \leq T_M(T(E_1(x,y), E_1(y,z)), T(E_2(x,y), E_2(y,z)), ...,$$
$$..., T(E_n(x,y), E_n(y,z))) \leq T_M(E_1(x,z), ..., E_n(x,z)) = \tilde{E}_{A,T}(x,z).$$

□

Let $\leq$ be a crisp order. We consider the preservation of compatibility with $\leq$ when we aggregate fuzzy equivalence relations compatible with $\leq$.

Theorem 4.4.3. *If E_i for all $i \in \{1, ..., n\}$ are fuzzy equivalence relations compatible with $\leq$, T_M is the minimum t-norm, $A \in \mathcal{A}$, then $\tilde{E}_{A,T_M}$ is also compatible with $\leq$.*

Proof. We have to prove that if $x \leq y \leq z$, then $\tilde{E}_{A,T_M}(x,z) \leq \tilde{E}_{A,T_M}(y,z)$ and $\tilde{E}_{A,T_M}(x,z) \leq \tilde{E}_{A,T_M}(x,y)$ taking into account that all relations E_i are compatible with $\leq$: If $x \leq y \leq z$, then $\tilde{E}_{A,T_M}(x,z) = T_M(E_1(x,z), ..., E_n(x,z)) \leq$ $\leq T_M(E_1(x,y), ..., E_n(x,y)) = \tilde{E}_{A,T_M}(x,y)$. The same considerations work when we prove that $\tilde{E}_{A,T_M}(x,z) \leq \tilde{E}_{A,T_M}(x,y)$ if $x \leq y \leq z$. □

The choice of a t-norm is not important in the previous result, but it plays important role in other results and therefore we take it into considerations.

Corollary 4.4.2. *If E_i for all $i \in \{1, ..., n\}$ are fuzzy equivalence relations given by (2.1), T_M is the minimum t-norm, $A \in \mathcal{A}$, then $\tilde{E}_{A,T_M}$ is a fuzzy equivalence relation compatible with $\leq$.*

Consider aggregation of fuzzy order relations given by (2.3). Propositions 4.4.3-4.4.5 could be proven for fuzzy order relations defined by (2.3) and by that and Theorem 4.4.3 we have the following result:

Proposition 4.4.6. *If R_i for all $i \in \{1, ..., n\}$ are fuzzy order relations given by (2.3), T_M is the minimum t-norm, $A \in \mathcal{A}$, then*

$$\tilde{R}_{A,T_M}(x,y) = \begin{cases} 1, \ if \ x \leq y \\ T_M(R_1(x,y), ..., R_n(x,y)), \quad oth. \end{cases}.$$

Corollary 4.4.3. *If E_i for all $i \in \{1, ..., n\}$ are T-equivalences given by formula (2.1), R_i are T-E_i-orders given by formula (2.3), then $\tilde{R}_{A,T_M}$ is a T-$\tilde{E}_{A,T_M}$-order.*

4.4.3 Conclusion on A-T-aggregation

In this section we focused on aggregation of fuzzy equivalence relations and fuzzy order relations. Some results are obtained for special cases, when t-norm is minimum t-norm and aggregation function has some special properties. In the future we are going to expand obtained result to the more broad classes of t-norms, aggregation functions and fuzzy relations.

Chapter 5

Conclusion

In our work we have obtained the following results:

- An L-valued analogue of $\mathbf{POS}$ category has been constructed; the properties of the constructed L-valued category have been investigated;
- The degree of monotonicity and the degree of α-monotonicity have been defined and studied;
- The aggregation of fuzzy relation has been studied and applied for MOLP problems; new concepts of aggregation of fuzzy relations have been involved and investigated.

The motivation of the first part (categorical) of the work was to construct a new category using L-valued category theory and also develop further L-valued category theory working with the concrete examples. The ideas for the second part of the work came from categorical aspects of fuzzy order relation but each direction of the second part was also motivated by the real-world examples.

Each part of the research has some own concluding remarks, so here we just want to mention that, although each part is a accomplished research, it could be also further developed in the future and the ideas of future research are mentioned in the concluding remarks in each chapter or section.

List of the attended conferences

International conferences

FSTA 2012 (11th International Conference on Fuzzy Sets Theory and Applications), ”Fuzzy Orders for Solving MOLP Problems”, Liptovsky Jan, Slovakia

EUSFLAT-LFA 2011 (7th conference of the European Society for Fuzzy Logic and Technology and les rencontres francophones sur la Logique Floue et ses Applications), ”On another view of aggregation of fuzzy relations”, Aix-Les-Bains, France

AGOP 2011 (6th International Summer School on Aggregation Operators), ”A-T-aggregation of fuzzy relations”,Benevento, Italy

MMA 2011 (16th International Conference on Mathematical Modelling and Analysis), ”Involving fuzzy orders for multi-objective linear programming”, Sigulda, Latvia

APLIMAT 2011 (10th International Conference on Applied mathematics), ”Involving fuzzy order in the definition of monotonicity for aggregation function”, Bratislava, Slovakia

WCCI 2010 (World kongress on Computational Intelligence 2010), ”Degree of monotonicity in aggregation process”, Barcelona, Span

MMA 2010 (15th International Conference on Mathematical Modelling and Analysis), ”Involving fuzzy order in the definition of monotonicity for the aggregation process”, Druskininkai, Lithuania

FSTA 2010 (10th International Conference on Fuzzy Sets Theory and Applications), ”Categorical aspects of aggregation of fuzzy relations”, Liptovsky Jan, Slovakia

MMA 2009 (14th International Conference on Mathematical Modelling and Analysis), ”Fuzzy POS category and aggregation of fuzzy order relations”, Daugavpils, Latvia

FSTA 2008 (9th International Conference on Fuzzy Sets Theory and Applications), ”Generalized fuzzy POS category”, Liptovsky Jan, Slovakia

EUSFLAT 2007 (The 5th conference of the European Society for Fuzzy Logic and Technology), ”Fuzzy Order Relation and Fuzzy Ordered Set Category”, Ostrava, Czech Republic

Domestic conferences

9th Latvian Conference on Mathematics, ”On Applications of Fuzzy Order Relations”, Jelgava, Latvia

8th Latvian Conference of Mathematics, ”Some remarks on L-valued categories”, Valmiera, Latvia

7th Latvian Conference on Mathematics, ”Fuzzy Set Theory: Problems and Applications”, Rezekne, Latvia

6th Latvian Conference on Mathematics, ”Fuzzification of Ordered Set Theory”, Liepaja, Latvia

69th Annual scientific conference of University of Latvia, ”On Aggregation of Fuzzy Order Relations”, Riga, Latvia

68th Annual scientific conference of University of Latvia, ”Involving Fuzzy Order in the Definition of Monotonicity”, Riga, Latvia

67th Annual scientific conference of University of Latvia, ”Aggregation of Fuzzy Order Relations”, Riga, Latvia

66th Annual scientific conference of University of Latvia, ”Generalized fuzzy POS category”, Riga, Latvia

65th Annual scientific conference of University of Latvia, ”On L-$valued$ POS category”, Riga, Latvia

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