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AIGARS GEDROICS

Mathematical modelling of problems of mathematical physics with periodic boundary conditions

Doctoral Thesis

Thesis submitted for the Degree of Doctor of Science in Mathematics
Subbranch: Numerical Analysis

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Annotation

In this work new special algorithms are developed for ordinary and partial differential equation problems with periodic boundary conditions for numerical modeling. These algorithms are based on exact spectrum usage for spatial approximation of partial derivative and method of finite differences. Algorithms are shown for different types of two dimensional problems of mathematical physics, linear and nonlinear, basing on the method of lines and difference schemes with exact spectrum. Created algorithms are realized and results are compared using the program MATLAB. With the implemented algorithms several applied problems are solved, i.e. the 2D magnetohydrodynamic flow around cylinders placed periodically, 2D flow inside the cylinder depending on the external magnetic field and the metal distribution in peat layers.

Keywords: periodic boundary conditions, circulant matrix, finite difference, finite difference with exact spectrum.

Anotācija

Darbā izstrādāti jauni speciāli algoritmi parasto un parciālo diferenciālvienādojumu problēmu ar periodiskajiem nosacījumiem skaitliskai modelēšanai, kuri balstās uz precīzā spektra izmantošanu telpisko parciālo atvasinājuma aproksimēšanai ar galīgajām diferencēm. Algoritmi tiek veidoti dažādām divdimensiju matemātiskās fizikas problēmām (lineārām un nelineārām), balstoties uz taišņu metodes algoritmiem un precīzā spektra diferenču shēmām. Izveidotie algoritmi tiek realizēti un salīdzināti ar datorprogrammas MATLAB palīdzību. Ar iegūtajiem algoritmiem tiek risinātas vairākas lietišķas problēmas, t. sk. 2D magnetohidrodinamiska plūsma ap periodiski novietotiem cilindriem, 2D plūsma cilindrā ārējā magnētiskā lauka ietekmē un metāla koncentrācija kūdras slāņos.

Atslēgvārdi: periodiski robežnosacījumi, cikliskas matricas, galīgo diferenču shēma, galīgo diferenču shēma ar precīzo spektru.

Contents

Introduction

This work summarizes the research results of periodic boundary conditions in mathematical problems. There are known several classical types of boundary conditions – first, second, third type and periodic boundary conditions. Periodic boundary conditions (PBC) are commonly found in natural patterns. Examples could be a wave movement in the water, whirls in the air, water, and air flow behind obstacles. Also in cooling and heating devices with repeatedly placed elements PBC can be used in modeling them. Modeling of such processes and equipment in the numerical analysis requires periodic boundary condition usage. In numerical experiments, the physical model is replaced by a mathematical model, creating problems of mathematical physics for ordinary or partial differential equations with appropriate boundary conditions and initial conditions.

If in differential equation the value of unknown $u(x,\dots)$ is periodical in the direction x with period L, equation $u(x,\dots) = u(x+L,\dots)$ applies for all values of x. Thus the problem can be limited to the domain $[0, L]$ and in the boundaries of the segment conditions can be defined in the form of PBC $u(0,\dots) = u(L,\dots)$, $u_x(0,\dots) = u_x(L,\dots)$, where u_x is partial differential in the x direction.

Additionally the periodic boundary conditions appears while solving problems of mathematical physics in cylindrical and polar coordinates. These conditions apply to the interval $[0, 2\pi]$ of the angular argument ϕ.

Problem is discretized in homogenous grid in direction of variable x, $x_j = jh$, $j = \overline{0, N}$. The values $u(x_j,\dots)$ in the grid points are denoted with y_j. In the grid with 3-point stencil PBC are given with $y_0 = y_N$ and $y_1 = y_{N+1}$, creating a problem in points $[x_1,\dots,x_N]$.

Unlike other boundary condition types, PBC allows to freely increase approximation order by increasing the count of grid points. For example, when using $2m+1$ point stencil we need to use additional conditions of periodicity $y_k = y_{N+k}$, $k = \overline{-m, m}$. Thus we obtain algorithms with higher order precision.

The second advantage of PBC is the fact that approximation the differences with finite differences results in calculations with Nth order circulant matrix A which can be defined with the first row only. For such matrices it is easy to do arithmetic operations in shorter computation time. Also it is possible to get the inverse matrix analytically.

As another advantage one can mention the simple solution of the spectral problem. Orthonormal eigenvectors are $w_k = \sqrt{\frac{1}{N}}(1, z^k, z^{2k}, \cdots, z^{(N-1)k})^T$, where $z = e^{2\pi\imath/N}$, $\imath = \sqrt{-1}$ and they do not depend on the elements of matrix. By solving the spectral problem, we can express the matrix A in the form $A = WDW^*$, where W is matrix which consists of the eigenvectors in it's columns, W^* is the conjugate transpose of W and D – diagonal matrix with the eigenvalues of matrix A on the diagonal. The eigenvalues can be expressed in the form $\mu_k = \sum_{j=1}^{N} a_j z^{(j-1)k}$.

Additionally to the possibility to create finite difference schemes of higher precision (FDS), it is also possible to create difference schemes with precise spectrum (FDSES). It is obtained by replacing the discrete eigenvalues in matrix D with the eigenvalues of the continuous differential problem. The first works about FDSES construction for heat transfer problems with boundary conditions of the first type [10] and hyperbolic equation [1] were published in years 1975 and 1972. Similar algorithms for the boundary conditions of the first type were used in the works of A. Gedroics, H.Kalis, A.Buikis, I. Kangro, S.Rogovs, A. Cebers [2] [12] [3] [4] [17] [5] for solving heat transfer equations of hyperbolic type and for modeling dynamics of elongated magnetic droplet in a rotating magnetic field.

In the 2D problem algorithms the second argument t is not discretized. Method of lines is used to solve such problems [6]. Besides the 3D problems can be reduced to 2D problems by using the method of averaging with quadratic splines which was developed by professor A.Buikis. The results were presented in international conferences and published in scientific journals with co-authors A. Buikis, H. Kalis, I. Kangro, S. Rogovs, M. Marinaki. The new algorithms are programmed in MATLAB. Some applied problems were modeled numerically with their help.

General description of the thesis

Questions of research

- What is the history of FDS and FDSES?
- How to use use advantages of circulant matrix in numerical analysis?
- How to create FDS with higher precision order?
- How to create and implement FDSES for solving problems of mathematical physics?

Object of research: Numerical methods for solving problems of mathematical physics with PBC.

Aim of research: Improvement of existing numerical methods and creation of new ones for solving problems of mathematical physics with PBC.

Objectives of research

- Solve spectral problem for circulant matrices, create algorithms for operations with circulant matrices;
- Creation and approbation of FDS with higher precision order for the second order ordinary differential equations with PBC;
- FDS creation and approbation for linear and nonlinear heat transfer equation with PBC;
- FDS creation and approbation for wave equation with PBC;
- FDS creation and approbation for Poisson equation with PBC;
- FDS usage for solving some applied problems;
- Actualize the created algorithms through publications and reports in international conferences.

Bases of research: Computer program MATLAB, ESF project "Support for Doctoral Studies at University of Latvia" and ESF project No.2009/0223/1DP/1.1.1.2.0/09/APIA/VIAA/008 during years 2010—2012.

Methods of research: scientific and technological literature, analysis of publications and internet resources, consultations and discussions with scientists in Latvia and abroad, approbation of conclusions in practice and scientific contacts, creation and realization of numerical algorithms using MATLAB.

Scientific novelty and Main results:

- Identified and researched possible usage of numerical methods to effectively solve the problems of mathematical physics with PBC;
- Created effective algorithms for operations with circulant matrices basing on their properties. Algorithms implemented in MATLAB;
- Created multi-point stencil discrete models for derivatives in uniform grid. With these models obtained FDS with higher precision order for solving problems with PBC;
- In the scope of research for the first time FDSES advantages for increasing precision in modelling problems of mathematical physics with PBC numerically were described and proven;
- Basing on the research results implementation of discrete analogs for the classic linear problems of mathematical physics with PBC were obtained with the newly created methods (FDS and FDSES);
- The research results were used practically for solving several applied problems: modelling of metal particles in peat layers [32]; modelling of nonlinear heat transfer [11] [17]; creation, analysis and calculation of MHD liquid flow [22] [20] [25].

The thesis is created in the Faculty of Physics and Mathematics in supervision of professor Harijs Kalis with support by the ESF within the project "Support for Doctoral Studies at University of Latvia". Research was done during the years 2009—2013 in the subbranch of mathematics "Numerical analysis". The thesis also can be applicable to the subbranch "Mathematical modeling", because in the work several applied problems are modeled and solved, i.e. magnetohydrodynamic flows, modeling of temperature and electric fields, modeling layers of peat. Before choosing the subject of the research, existing theoretical literature was analysed and no known results for algorithm creation for problems of mathematical physics with PBC has not been found. Together with the supervisor qualitative and quantitative approach was used to ensure the validity of research for reaching the objectives. For approbation and public evaluation of results a learning aid for students of master degree "The finite difference schemes with the exact spectrum for solution some problems of mathematical physics" has been created.

Publications and reports

Publications

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1. Usage of circulant matrices in differential equation numerical analysis

1.1 Operations with circulant matrices

Circulant matrix with dimensions $N \times N$ is a matrix in the form

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_{N-2} & a_{N-1} & a_N \\ a_N & a_1 & a_2 & \cdots & a_{N-3} & a_{N-2} & a_{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_3 & a_4 & a_5 & \cdots & a_N & a_1 & a_2 \\ a_2 & a_3 & a_4 & \cdots & a_{N-1} & a_N & a_1 \end{pmatrix}$$

It can be expressed using it's first row:

$$A = [a_1;\ a_2;\ a_3; \cdots ;\ a_{N-1};\ a_N]$$

It can be expressed in another form

$$A = f(P),$$

where P is a permutation matrix in the form

$$P = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and function f is a polynomial

$$f(z) = \sum_{k=1}^{N} a_k z^{k-1}$$

These matrices has simple algorithms for their operations, like addition, multiplication, finding the inverse matrix. These can be made in less computational time comparing to standard matrices. Finding the eigenvalues and eigenvectors is also simpler.

Sum and product of circulant matrices is circulant matrix. When circulant matrix has an inverse matrix, it is also circulant.

MATLAB code for operations with circulant matrices can be found in the appendix A.1.

1.2 Spectral problem for circulant matrix

Eigenvalues of matrix A can be expressed in the form $f(z^k)$, $k = \overline{0, N-1}$ where z is primitive root of unity of the Nth order. We can choose the root $z = e^{2\pi\imath/N}$ ($\imath = \sqrt{-1}$) of the equation $x^N - 1 = 0$. Thus eigenvalues are in the form

$$\mu_k = \sum_{j=1}^{N} a_j e^{2\pi\imath(j-1)k/N}$$

Orthonormal eigenvectors of the matrix can be chosen in the form

$$w^k = \sqrt{\frac{1}{N}}(1, z^k, z^{2k}, \cdots, z^{(N-1)k})^T, \; k = \overline{0, N-1}$$

Also circulant matrices has property

$$A = WDW^* = W^*DW,$$

where W is matrix consisting of matrix A eigenvectors, matrix D – diagonal matrix with eigenvalues on the diagonal. Matrix W^* is conjugate transpose matrix of matrix W.

Additionally

$$WW^* = E$$

and for circulant matrices multiplication is commutative, i.e. A and B

$$AB = BA$$

1.3 Solving systems of algebraic equations with circulant matrices, examples

The linear problems of mathematical physics can be approximated with the linear system of algebraic equations in the following form

$$Au = f$$

where A, u, f are the corresponding quadratic matrix and column vectors of order N.

1.3.1 Sample problem

We look into such problem of differential equations with periodic boundary conditions

$$-u''(x) = f(x),\ u(0) = u(L),\ u'(0) = u'(L)$$

Problem is defined in the interval $x \in [0, L]$, f – fixed function. In the boundaries of the domain is periodic conditions, i.e. the function u values and values of derivative in the boundaries equal.

For any function $f(x)$ which is Riemann integrable the problem has unique solution if this conditions applies

$$\int_0^L f(x)\mathrm{d}x = 0$$

and a point for unknown function $u(x)$ must be fixed

$$u(x_0) = u_0, x_0 \in [0, L].$$

The proof of the first requirement can be shown by integrating the initial problem by x

$$\int_0^L f(x)\mathrm{d}x = \int_0^L -u''(x)\mathrm{d}x = -u'(L) + u'(0) = 0$$

The fixed point for function $u(x)$ is required because without it problem solution uniqueness couldn't be obtained. If function $v(x)$ would match as the solution of the problem, function $v(x) + \mathrm{C}$ would satisfy the problem as well.

The solution of this problem can be expressed analytically.

At first we find solution for the problem with $x_0 = 0$ and $u_0 = 0$. It is in the form

$$\bar{u}(x) = \int_0^L (t - x)f(t)\mathrm{d}t - \frac{x}{L}\int_0^L tf(t)\mathrm{d}t$$

Using simple transformation it's possible to find solution for the general problem as well.

$$u(x) = \bar{u}(x) - \bar{u}(x_0) + u_0$$

1.3.2 Problem discretization

We use 3 point stencil to discretize the problem by approximating operator $-u''$:

$$-u''(x_j) \approx \frac{-u(x_{j-1}) + 2u(x_j) - u(x_{j+1})}{h^2}$$

Initial problem can be rewritten in the form

$$Ay = f,$$

where $N \times N$ size matrix A approximates the operator $-u''$:

$$A = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ -1 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

Matrix A is circulant, i.e. it can be defined by its first row and each next row is copy of the previous one with element transition by one position. It can be defined in the circulant matrix form

$$A = \frac{1}{h^2}[2; -1; 0; \cdots; 0; 0; -1]$$

From the spectral problem for matrix A we know that its eigenvalues are

$$\mu_k = \frac{4}{h^2}\sin^2\frac{k\pi}{N}$$

and eigenvectors w^k are

$$w_j^k = \sqrt{\frac{1}{N}}\exp\left(\frac{2\pi\imath kj}{N}\right)$$

Using properties of circulant matrices, matrix A can be expressed in form $A = WDW^*$. Using this form, problem $Ay = f$ can be transformed in the form $Dv = W^*f$, where $y = Wv$. For indices $j = \overline{1, N-1}$ we can solve the corresponding values v_j:

$$v_j = \frac{1}{\mu_j}(W^*f)_j$$

For index $j = N$ we get equation, which is consistent with the problem condition $\int_0^L f(x)\mathrm{d}x = 0$. That's why it must be found from the fixed point condition $u(x_0) = u_0$. Stencil must be chosen so that x_0 is a point in the stencil. Let's suppose it is the point x_j. By inserting the known values in the equation $y = Wv$, we can find the unknown's v index N by formula $(Wv)_j = u_0$. Unknown y of the initial problem we find from transformation $y = Wv$.

1.3.3 Exact spectrum

The eigenvalues of the spectral problem's continuous model are

$$\lambda_k = \left(\frac{2\pi k}{L}\right)^2$$

and eigenvectors are

$$w^k = \sqrt{\frac{1}{L}}\exp\left(\frac{2\pi\imath kx}{L}\right)$$

By using function f in the form

$$f(x) = \sum_{k=-\infty}^{+\infty} b_k w^k(x),\ b_k = (w^{*k}, f),$$

we can express the precise solution in the form

$$u(x) = \sum_{k=-\infty}^{+\infty} a_k w^k(x),\ a_k = \frac{b_k}{\lambda_k} \tag{1.1}$$

Also in this case for index $k = 0$ when eigenvalue and b_0 tends to zero (because of integral condition on f), we cannot find the value of solution. That's why we need to use the fixed point to find it. It can be done by putting the fixed point condition in the equation 1.1.

$$a_0 = (u_0 - \sum_{k=-\infty,\neq 0}^{+\infty} a_k w^k(x_0))/w^0(x_0)$$

1.3.4 Real solutions

For periodical function $f(x)$ follows the complex expansion

$$f(x) = \sum_{k=-\infty}^{\infty} b_k w^k(x) = \sum_{k=1}^{\infty}(b_k w^k(x) + b_{-k} w^{-k}(x)) + \frac{b_0}{\sqrt{L}} =$$

$$\frac{1}{2}\sum_{k=1}^{\infty}((b_k + b_k)(w^k(x) + w_*^k(x)) + (b_k - b_{-k})(w^k(x) - w_*^k(x))) + \frac{b_0}{\sqrt{L}} =$$

$$\sum_{k=1}^{\infty}(b_{kc}\cos\frac{2\pi kx}{L} + b_{ks}\sin\frac{2\pi kx}{L}) + \frac{b_{0c}}{2}, b_k = (f, w_*^k),$$

where

$$b_{kc} = \frac{1}{\sqrt{L}}(b_k + b_{-k}) = \frac{1}{\sqrt{L}}\int_0^L f(x)(w_*^k(x) + w^k(x))dx) = \frac{2}{L}\int_0^L f(t)\cos\frac{2\pi kt}{L}dt,$$

$$b_{ks} = \frac{i}{\sqrt{L}}(b_k - b_{-k}) = \frac{i}{\sqrt{L}}\int_0^L f(x)(w_*^k(x) - w^k(x))dx) = \frac{2}{L}\int_0^L f(t)\sin\frac{2\pi kt}{L}dt$$

Therefore the solution of the problem can be also obtained in real form:

$$u(x) = \sum_{k=1}^{\infty}(a_{kc}\cos\frac{2\pi kx}{L} + a_{ks}\sin\frac{2\pi kx}{L}) + \frac{a_{0c}}{2},$$

where $a_{kc} = \frac{b_{kc}}{\lambda_k}$, $a_{ks} = \frac{b_{ks}}{\lambda_k}$. For vector f of N order with component f_j, $j = \overline{1,N}$ using $w_*^k = w^{N-k} = w^{-k}$, $w_j^N = w_j^0 = 1$, $j = \overline{1,N}$ we can similarly get the following expressions:

$$f = \sum_{k=1}^{N} b_k w^k = \sum_{k=1}^{N_2-1}(b_k w^k + b_{N-k} w^{N-k}) + b_{N_2} w^{N_2} + b_N w^N =$$

$$\frac{1}{2}\sum_{k=1}^{N_2-1}((b_k+b_{N-k})(w^k+w^{N-k})+(b_k-b_{N-k})(w^k-w^{N-k}))+b_{N_2}w^{N_2}+b_Nw^0, b_k=(f,w_*^k),$$

or

$$f_j=\sum_{k=1}^{*N_2}(b_{kc}\cos\frac{2\pi kj}{N}+b_{ks}\sin\frac{2\pi kj}{N})+\frac{b_{0c}}{2},$$

where

$$b_{kc}=\frac{1}{\sqrt{N}}(b_k+b_{N-k})=\frac{1}{\sqrt{N}}\sum_{j=1}^{N}f_j(w_{j*}^k+w_j^k)=\frac{2}{N}\sum_{j=1}^{N}f_j\cos\frac{2\pi kj}{N},$$

$$b_{ks}=\frac{i}{\sqrt{N}}(b_k-b_{N-k})=\frac{i}{\sqrt{N}}\sum_{j=1}^{N}f_j(w_{j*}^k-w_j^k)=$$

$$\frac{2}{N}\sum_{j=1}^{N}f_j\sin\frac{2\pi kj}{N}, k=\overline{1,N_2-1},$$

$$b_0=b_N=\frac{1}{\sqrt{N}}\sum_{j=1}^{N}f_j, b_{0c}=b_{Nc}=\frac{2}{\sqrt{N}}b_0,$$

$$b_{N_2c}=\frac{2}{\sqrt{N}}b_{N_2}=\frac{2}{N}\sum_{j=1}^{N}\cos(j\pi),$$

$$b_{N_2}w_k^{N_2}=\frac{1}{N}\sum_{j=1}^{N}\cos(j\pi)\cos(k\pi), N_2=\frac{N}{2}, b_{N_2s}=0,$$

$$\sum_{k=1}^{*N_2}\beta_k=\sum_{k=1}^{N_2-1}\beta_k+\frac{\beta_{N/2}}{2},$$

for periodic functions $b_0=b_N=0$.

This discrete Fourier expression can be represented in the following form [7]:

$$f_j=\sum_{k=1}^{N_2-1}(b_{kc}\cos\frac{2\pi kj}{N}+b_{ks}\sin\frac{2\pi kj}{N})+\frac{b_{N_2c}\cos(\pi j)}{2}+\frac{b_{0c}}{2}$$

Similarly the solution of the discrete problem can be represented in the following form:

$$y_j=\sum_{k=1}^{*N_2}(a_{kc}\cos\frac{2\pi kj}{N}+a_{ks}\sin\frac{2\pi kj}{N})+\frac{a_{0c}}{2},$$

where $a_{kc}=\frac{b_{kc}}{\mu_k}$, $a_{ks}=\frac{b_{ks}}{\mu_k}$, $a_{0c}=0$.

This solution can also be obtained from the orthonormal trigonometric functions. Using the relations [7]

$$\sum_{j=1}^{N}\sin(ax_j)=\frac{\sin(0.5a(L+h))\sin(0.5aL)}{\sin(0.5ah)},$$

$$\sum_{j=1}^{N} \cos(ax_j) = \frac{\sin(0.5a(L+h))\cos(0.5aL)}{\sin(0.5ah)} - 1$$

we obtain $\sum_{j=1}^{N} \sin_k \cos_s = 0$, $\sum_{j=1}^{N} \sin_k \sin_s = \sum_{j=1}^{N} \cos_k \cos_s = N_2 \delta_{k,s}$. $\sum_{j=1}^{N} \sin_k = \sum_{j=1}^{N} \cos_k = 0$, where $\sin_k, \cos_k$ are N-order column-vectors with the elements $\sin \frac{2\pi kj}{N}, \cos \frac{2\pi kj}{N}$, $k, s = \overline{1, N_2}, L = Nh$.

We have the relation

$$\sum_{j=1}^{N} f_j^2 = N_2 (\frac{b_{0c}^2}{2} + \frac{b_{N_2 c}^2}{2} + \sum_{k=1}^{N_2 - 1} (b_{kc}^2 + b_{ks}^2))$$

From

$$Ay = \sum_{k=1}^{*N_2} (a_{kc} A \cos_k + a_{ks} A \sin_k) + \frac{a_{0c}}{2} A \cos_0$$

and

$$A \cos_k = \mu_k \cos_k, A \sin_k = \mu_k \sin_k$$

follows $a_{kc}\mu_k = b_{kc}$, $a_{ks}\mu_k = b_{ks}$.

If in the discrete Fourier expression

$$f_j^M = \sum_{k=1}^{*M} (b_{kc} \cos \frac{2\pi kj}{N} + b_{ks} \sin \frac{2\pi kj}{N}) + \frac{b_{0c}}{2}, M < N_2$$

then using the least squares method we can prove that the Fourier coefficients b_{kc}, b_{ks}, $k = \overline{1, M}$ maintain own form and we can estimate the error

$$\sum_{j=1}^{N} (f_j - f_j^M)^2 = \sum_{j=1}^{N} f_j^2 - N_2 (\frac{b_{0c}^2}{2} + \sum_{k=1}^{M} (b_{kc}^2 + b_{ks}^2))$$

Compared the discrete Fourier coefficients B_{kc}, B_{ks} with to Fourier series coefficients b_{kc}, b_{ks} we have [7]

$$B_{0c} = b_{0c} + \sum_{m=1}^{\infty} b_{(Nm)c}, B_{kc} = b_{kc} + \sum_{m=1}^{\infty} (b_{(N(m-k))c} + b_{(N(m+k))c}),$$

$$B_{ks} = b_{ks} + \sum_{m=1}^{\infty} (-b_{(N(m-k))s} + b_{(N(m+k))s})$$

For $f(x) \in C^\nu(0, L)$ the coefficients decrease as $k^{-(\nu+1)}$.

We can obtain the solution of FDSES by replacing the discrete eigenvalues μ_k with the first N eigenvalues $\lambda_k, k = \overline{1, N}$.

1.3.5 Example

Let's examine example of problem with f in form

$$f(x) = x - 0.5,\ L = 1,\ x_0 = 0,\ u_0 = 0$$

Analytical solution for this problem is

$$u(x) = \frac{1}{12}(-2x^3 + 3x^2 - x)$$

MATLAB code for finding the approximated solution with 3 point stencil can be found in appendix A.2.

For $N = 50$ maximal error is $1.3 * 10^{-3}$.

1.4 Algorithms with precision of higher order

Algorithms with precision of higher order are useful for problems of this type. Because of periodic boundary conditions we can choose algorithm with arbitrary precision by adding more stencil points without a need for additional conditions. Such approach can be applied to derivative of arbitrary order. Here will be shown how to obtain approximation algorithms for the second, the first and the fourth derivative.

1.4.1 Second derivative

Here we will implement approximation with finite element method for the the second order derivative $-u''(x_j)$ for our problem. We take uniform grid with step h and grid points x_j. We build scheme using stencil with $2p+1$ points $(x_{j-p}, x_{j-p+1}, \ldots, x_j, \ldots, x_{j+p})$.

We use the method of unknown coefficients to find the constants C_k and E_p for the scheme

$$u''(x_j) = \frac{1}{h^2}\sum_{k=-p}^{p} C_k u(x_{j+k}) + E_p \frac{h^{2p}u^{(2p+2)}(\xi)}{(2p+2)!},\ x_{j-p} < \xi < x_{j+p}$$

Then we normalize the expression by substitution

$$t = \frac{x - x_j}{h}$$

$$u(x) = u(x_j + th) = \bar{u}(t)$$

$$u''(x) = \frac{1}{h^2}\bar{u}''(x)$$

$$u^{(2p+2)} = \frac{1}{h^{2p+2}}\bar{u}^{(2p+2)}$$

Thus we get simplified expression

$$\bar{u}''(0) = \frac{1}{h^2}\sum_{k=-p}^{p} C_k \bar{u}(k) + E_p \frac{\bar{u}^{(2p+2)}(\bar{\xi})}{(2p+2)!},\ -p < \bar{\xi} < p$$

It is known that for this type of schemes symmetry condition persists

$$C_m = C_{-m}$$

By using power functions $\bar{u}(t) = t^m$, $m = \overline{0, 2p+2}$ we can find the unknown coefficients.

- When $m = 0$ we get relation
$$C_0 = -2\sum_{m=1}^{p} C_m$$
- For odd coefficients $1 \leq m \leq 2p+1$ we get identity $0 = 0$.
- For even coefficients $2 \leq m \leq 2p$ we get algebraic equation system $Bc = e$ where $e = (1, 0, \cdots, 0)^T$ is column vector, $c = (C_1, C_2, \cdots, C_p)^T$. B is pth order Vandermonde matrix in the form
$$B = \begin{pmatrix} 1 & 4 & 9 & \cdots & (p-2)^2 & (p-1)^2 & p^2 \\ 1 & 4^2 & 9^2 & \cdots & (p-2)^4 & (p-1)^4 & p^4 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 4^{p-1} & 9^{p-1} & \cdots & (p-2)^{2p-2} & (p-1)^{2p-2} & p^{2p-2} \\ 1 & 4^p & 9^p & \cdots & (p-2)^{2p} & (p-1)^{2p} & p^{2p} \end{pmatrix}$$
- In the $m = 2p+2$ case we find the value of E_p in the form
$$E_p = -2(C_1 + 2^{2p+2}C_2 + \cdots + (p)^{2p+2}C_p) = -2\sum_{m=1}^{p} C_m m^{2p+2}$$

For the matrix of Vandermonde B with the elements of the first row $k_m = m^2$ the elements of the inverse matrix B^{-1} are known [8]:

$$b_{i,j} = \frac{(-1)^{j-1} \displaystyle\sum_{\substack{1 \leq q_1 < \cdots < q_{p-j} \leq p \\ q_1, \cdots, q_{p-j} \neq i}} k_{q_1} \times \cdots \times k_{q_{p-j}}}{k_i \displaystyle\prod_{1 \leq q \leq p, q \neq i} (k_q - k_i)}$$

From the equation $Bc = e$ follows that $c = B^{-1}e$. That's why we are interested in only the first column of the matrix B^{-1}, that is elements with $j = 1$:

$$b_{i,1} = \frac{\displaystyle\sum_{\substack{1 \leq q_1 < \cdots < q_{p-1} \leq p \\ q_1, \cdots, q_{p-1} \neq i}} k_{q_1} \times \cdots \times k_{q_{p-1}}}{k_i \displaystyle\prod_{1 \leq q \leq p, q \neq i} (k_q - k_i)} =$$

$$= \frac{\displaystyle\sum_{\substack{1 \leq q_1 < \cdots < q_{p-1} \leq p \\ q_1, \cdots, q_{p-1} \neq i}} q_1^2 \times \cdots \times q_{p-1}^2}{i^2 \displaystyle\prod_{1 \leq q \leq p, q \neq i} (q^2 - i^2)} =$$

$$= \frac{(p!)^2}{i^4 \prod\limits_{1 \le q \le p, q \ne i} (q^2 - i^2)} =$$

$$= \frac{2(-1)^{i-1}(p!)^2}{i^2(p-i)!(p+i)!}$$

Follows that

$$C_m = \frac{2(-1)^{m-1}(p!)^2}{m^2(p-m)!(p+m)!}$$

There are coefficients for some values of p:

1. $p = 1 : C_1 = 1, C_0 = -2, E_2 = -2,$
2. $p = 2 : C_1 = \frac{4}{3}, C_2 = -\frac{1}{12}, C_0 = -\frac{5}{2}, E_4 = 8,$
3. $p = 3 : C_1 = \frac{3}{2}, C_2 = -\frac{3}{20}, C_3 = \frac{1}{90}, C_0 = -\frac{49}{18}, E_4 = -72,$
4. $p = 4 : C_1 = \frac{8}{5}, C_2 = -\frac{1}{5}, C_3 = \frac{8}{315}, C_4 = -\frac{1}{560}, C_0 = -\frac{205}{72}, E_8 = 1152.$

In this case the finite difference matrix A approximated the second order derivative $-u''(x_j)$ is circulant with the form

$$A = -\frac{1}{h^2}[C_0, C_1, \cdots, C_p, 0, \cdots, 0, C_p, C_{p-1}, \cdots, C_2, C_1].$$

The eigenvalues of matrix A are

$$\begin{aligned}
\mu_k = -\frac{1}{h^2} \sum_{m=-p}^{p} C_m \exp(2\pi\imath km/N) = \\
= -\frac{1}{h^2} \left(C_0 + 2\sum_{m=1}^{p} C_m \cos\frac{2\pi km}{N} \right) = \\
= \frac{4}{h^2} \sum_{m=1}^{p} C_m \sin^2(\pi km/N) = \\
= \frac{8}{h^2}(p!)^2 \sum_{m=1}^{p} \frac{(-1)^{m-1}}{m^2(p-m)!(p+m)!} \sin^2(\pi km/N), \ k = \overline{1, N}
\end{aligned}$$

Our next goal is to simplify the μ_k formula. For this we will express $\sin^2(\pi km/N)$ using Spread polynomials. Spread polynomial S_n states that

$$\sin^2(m\theta) = S_m(\sin^2(\theta))$$

Spread polynomial has explicit formula [9]

$$S_m(s) = s \sum_{k=0}^{m-1} \frac{m}{m-k} \binom{2m-1-k}{k} (-4s)^{m-1-k}$$

Using transformation $u = m - k$ we get

$$S_m(s) = \sum_{u=1}^{m} \frac{m}{u} \binom{m+u-1}{m-u} (-4)^{u-1} s^u =$$
$$= \sum_{u=1}^{m} \frac{m}{u} \frac{(m+u-1)!}{(2u-1)!(m-u)!} (-4)^{u-1} s^u$$

Let's denote the coefficients of the polynomial with $\alpha_{m,u}$

$$\alpha_{m,u} = \frac{m}{u} \frac{(m+u-1)!}{(2u-1)!(m-u)!} (-4)^{u-1}$$

and

$$s = \sin(2\pi k/N)$$

We will also use simple transformation

$$\sum_{m=1}^{p} \sum_{u=1}^{m} \cdot = \sum_{u=1}^{p} \sum_{m=u}^{p} \cdot$$

Thus we can express previous formula for μ_k in such form

$$\mu_k = \frac{4}{h^2} \sum_{m=1}^{p} C_m S_m(s)$$
$$= \frac{4}{h^2} \sum_{m=1}^{p} C_m \sum_{u=1}^{m} \alpha_{m,u} s^u$$
$$= \frac{4}{h^2} \sum_{u=1}^{p} s^u \sum_{m=u}^{p} C_m \alpha_{m,u}$$

Next we denote with Q_u expression

$$Q_u = \sum_{m=u}^{p} C_m \alpha_{m,u}$$

This way we get formula $\mu_k = \frac{4}{h^2} \sum_{u=1}^{p} Q_u s^u$. What we want to show is that Q_u doesn't depend on stencil point count parameter p. That's why the eigenvalues of the next precision scheme for $p+1$ can be obtained by adding summand $Q_{p+1} s^{p+1}$ to the previous values.

We want to simplify it and show independence from p by proving that

$$Q_u = \frac{2((u-1)!)^2 4^{u-1}}{(2u)!}$$

From $C_m = \frac{1}{m^2} \prod_{k=1 \neq m}^{p} \frac{-k^2}{m^2-k^2}$ follows that we need to prove the expression

$$\sum_{m=u}^{n} \left(\prod_{k=1 \neq m}^{n} \frac{k^2}{k^2-m^2} \right) \frac{1}{m} \frac{(m+u-1)!}{(m-u)!} (-1)^{u-1} = ((u-1)!)^2$$

or

$$\sum_{m=v+1}^{n}\left(\prod_{k=1\neq m}^{n}\frac{k^2}{k^2-m^2}\right)\frac{1}{m}\frac{(m+v)!}{(m-v-1)!}(-1)^v=(v!)^2$$

where $v=u-1$.

This expression can be written in the following form:

$$\sum_{m=v+1}^{n}\left(\prod_{k=1\neq m}^{n}\frac{k^2}{k^2-m^2}\right)\left(\prod_{k=1}^{v}(k^2-m^2)\right)=(v!)^2$$

or

$$\sum_{m=v+1}^{n}\left(\prod_{k=1\neq m}^{n}\frac{k^2}{k^2-m^2}\right)\left(\prod_{k=1}^{v}\frac{k^2-m^2}{k^2}\right)=1$$

or

$$\sum_{m=u}^{n}\left(\prod_{k=u\neq m}^{n}\frac{k^2}{k^2-m^2}\right)=1$$

Proof. This can be proved using residue theorem for function

$$F(z)=\frac{1}{z}\prod_{k=v}^{n}\frac{k^2}{k^2-z^2}$$

Contour integral is used on function on circle C with radius $R>n$. Contour can be defined by formula $z=Re^{\imath\theta}$ with $\theta\in[0,2\pi]$. Integral value tends to 0 when $R\to\infty$. It can be proved by such estimation

$$\left|\oint_C F(z)\mathrm{d}z\right|=\left|\int_0^{2\pi}\frac{1}{Re^{\imath\theta}}\prod_{k=v}^{n}\frac{k^2}{k^2-R^2\exp(2\imath\theta)}\imath Re^{\imath\theta}\mathrm{d}\theta\right|\leq 2\pi\prod_{k=v}^{n}\frac{k^2}{R^2-k^2}$$

The circle contains $1+2(n-v+1)$ singularities inside, i.e. $z=0$ and $z=\pm v$, …, $z=\pm n$. From the residue theorem follows that sum of all residues multiplied by $2\pi\imath$ must be value of the contour integral around all singularities. In this case the sum must be 0. It's easy to see that residue in the point $z=0$ equals 1. Residue in the points $z=m\in[v,n]$ can be seen if $F(z)$ is rewritten in the form

$$F(z)=\frac{1}{z}\left(\prod_{k=v,k\neq m}^{n}\frac{k^2}{k^2-z^2}\right)\frac{m^2}{z+m}(-1)\frac{1}{z-m}$$

That means that residue in this point m is

$$-\frac{1}{2}\prod_{k=v,k\neq m}^{n}\frac{k^2}{k^2-m^2}$$

Also it's easy to see that residue in point $z=-m\in[-n,-v]$ is the same as in the respective positive valued points.

If all residues are summed up it leads us to the prove of the desired formula

$$\sum_{m=v}^{n} \prod_{k=v,k\neq m}^{n} \frac{k^2}{k^2 - m^2} = 1$$

1.4.2 First derivative

Similarly schemes with higher order for the first order derivative can be created. For the first order derivative approximation matrix A is modelled in the form

$$A = \frac{1}{h}[0, C_1, C_2, \ldots, C_p, 0, 0, \ldots, 0, -C_p, \cdots - C_2, -C_1]$$

With similar approach as it was done with the second order derivative, such values for constants C_k can be found:

$$C_m = \frac{(p!)^2(-1)^{m-1}}{m(p-m)!(p+m)!}$$

And, similarly, eigenvalue formula can be transformed to the form

$$\mu_k = \frac{2\imath}{h} \sum_{m=1}^{n} Q_m \cos\frac{\pi k}{N} \left(\sin\frac{\pi k}{N}\right)^{2m-1}$$

having

$$Q_m = \frac{2^{2m-2} \cdot ((m-1)!)^2}{(2m-1)!}$$

1.4.3 Fourth derivative

Approximation matrix for the fourth order derivative is built in form

$$A^{(4)} = \frac{1}{h^4}[C_0^{(4)}, C_1^{(4)}, C_2^{(4)}, \ldots, C_p^{(4)}, 0, 0, \ldots, C_p^{(4)}, C_{p-1}^{(4)}, \ldots, C_1^{(4)}]$$

Calculating C_m gives us

$$C_m^{(4)} = \frac{24(p!)^2(-1)^m}{m^2(p-m)!(p+m)!} \sum_{k=1\neq m}^{p} \frac{1}{k^2}, \quad C_0^{(4)} = -2\sum_{k=1}^{p} C_k^{(4)}$$

Eigenvalues can be expressed in the form

$$\mu_k^{(4)} = \frac{4}{h^4} \sum_{m=1}^{p} P_m^{(4)} \sin^{2m}(\pi k/N)$$

where

$$P_m^{(4)} = \frac{24(m!)^2 4^{m-1}}{m^2(2m)!} \sum_{s=1}^{m-1} \frac{1}{s^2}, \quad P_1^{(4)} = 0$$

1.5 Schemes with exact spectrum

Schemes with exact spectrum can be defined for all previously mentioned derivatives. Eigenvalues for approximation matrices are limit cases of approximations with higher order if the order tends to infinity.

To build schemes with exact spectrum, the eigenvalues μ_k of the method with higher precision must be replaced with eigenvalues of exact spectrum accordingly [10].

First derivative

For the first derivative eigenvalues must be replaced with values d_k in such manner

$$d_k = \begin{cases} \lambda_k, \ \text{when } k < N/2 \\ 0, \ \text{when } k = N/2 \\ -\lambda_{N-k}, \ \text{when } k > N/2 \end{cases}$$

where $\lambda_k = \frac{2\pi k \imath}{Nh}$.

Second derivative

For the second derivative $-u''$ eigenvalues must be replaced with values d_k in such manner

$$d_k = \begin{cases} \lambda_k, \ \text{when } k \leq N/2 \\ \lambda_{N-k}, \ \text{when } k > N/2 \end{cases}$$

where $\lambda_k = \left(\frac{2\pi k}{Nh}\right)^2$.

Fourth derivative

For the fourth derivative eigenvalues must be replaced with values d_k in such manner

$$d_k = \begin{cases} \lambda_k, \ \text{when } k \leq N/2 \\ \lambda_{N-k}, \ \text{when } k > N/2 \end{cases}$$

where $\lambda_k = \left(\frac{2\pi k}{Nh}\right)^4$.

To visualize the eigenvalues for schemes of higher order and schemes with exact spectrum, see joined list plots generated with $N = 100$ grid points in figures 1.1–1.3.

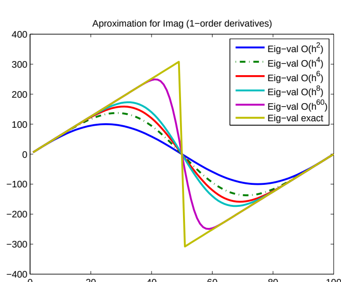

Fig. 1.1 Imaginary part of eigenvalues for first derivative schemes

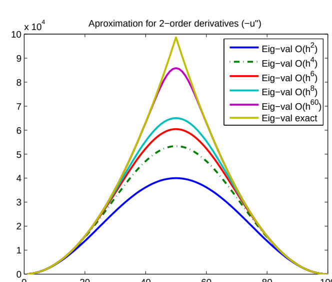

Fig. 1.2 Eigenvalues for $-u''$ schemes

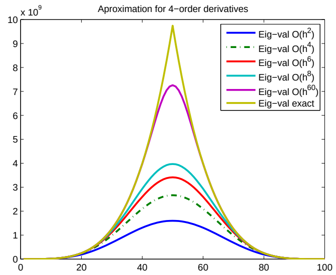

Fig. 1.3 Eigenvalues for fourth derivative schemes

2. Heat transfer equation with periodic BC

The solutions of the linear initial-boundary value problem for heat transfer equations are obtained analytically and numerically. We define the finite difference scheme with higher precision order and finite difference scheme with exact spectrum (FDSES), where the finite difference matrix A is represented in the form $A = WDW^*$. W, D is the matrices of finite difference matrix eigenvectors and eigenvalues correspondingly, W^* is the conjugate transpose matrix of matrix W. Problem is discretized for x and method of lines (MOL) is used.

2.1 Mathematical model

We consider a linear boundary heat transfer problem with periodic boundary conditions and initial condition in the following form [37]:

$$\begin{cases} \dfrac{\partial T(x,t)}{\partial t} = \dfrac{\partial}{\partial x}\left(\nu \dfrac{\partial T(x,t)}{\partial x}\right) + f(x,t), x \in (0,L), t \in (0,t_f), \\ T(0,t) = T(L,t), \dfrac{\partial T(0,t)}{\partial x} = \dfrac{\partial T(L,t)}{\partial x}, t \in (0,t_f), \\ T(x,0) = T_0(x), x \in (0,L), \end{cases} \tag{2.1}$$

where $\nu > 0$ is constant parameter, t_f is final time, T_0, f are given functions.

We consider uniform grid in the space $x_j = jh$, $j = \overline{0,N}$, $Nh = L$. Using the finite differences of the second order approximation for partial derivatives of second order respect to x we obtain the initial value problem for system of ordinary differential equations (ODEs) in the following matrix form

$$\dot{U}(t) + \nu A U(t) = F(t), \quad U(0) = U_0, \tag{2.2}$$

In 2.2 vector U is unknown variable and contains approximated values of temperature function T in the grid – $U(t) = (T(x_0,t), T(x_1,t), \ldots, T(x_N,t))$. Similarly $F(t) = (F(x_0,t), F(x_1,t), \ldots, F(x_N,t))$. Matrix A is one of the second derivative approximation matrix calculated in the section 1.4.1.

2.2 Analytical solution

We can consider the analytical solution of the discretized problem using the spectral representation of matrix $A = WDW^*$. From transformation $V = W^*U$ ($U = WV$)

follows such equation in matrix form

$$\dot{V}(t) + \nu D V(t) = G(t), \quad V(0) = W^* U_0, \tag{2.3}$$

where $V(t)$, $\dot{V}(t)$, $V(0)$, $G(t) = W^* F(t)$ are the column-vectors of Nth order with elements accordingly $v_k(t)$, $\dot{v}_k(t)$, $v_k(0)$, $g_k(t)$. The solution of this system is the function can be expressed in the form

$$v_k(t) = v_k(0)\exp(-\kappa_k t) + \int_0^t exp(-\kappa_k(t-\tau))g_k(\tau)d\tau, \tag{2.4}$$

where $\kappa_k = \nu\mu_k$.

The unknown U is found using reverse transformation formula $U = WV$.

To find continuous approximated solution $T(x,t)$ we can use Fourier method and express the solution in the form

$$T(x,t) = \sum_{k\in\mathbb{Z}} v_k(t) w^k(x)$$

where $w^k(x) = \sqrt{\frac{1}{L}} \exp\frac{2\pi\imath kx}{L}$ are the orthonormal eigenvectors, $v_k(t)$ is the solution (2.4), with $v_k(0) = (T_0, \bar{w}_k)$.

For the FDSES the matrix A is represented in the form $A = WDW^*$ and the diagonal matrix D contains the first N eigenvalues $d_k = \lambda_k$, $k=\overline{1,N}$ from the differential operator $(-\frac{\partial^2}{\partial x^2})$ in the following way:

1. $d_k = \lambda_k$ for $k = \overline{1, N_2}$, where $N_2 = N/2$.
2. $d_k = \lambda_{N-k}$ for $k = \overline{N_2, N}$.

If $d_k = \mu_k$, then we have the method of finite difference approximation with matrix A. The FDSES method is more stable as the method of finite difference by approximation with central difference (FDS), because the eigenvalues are larger $\lambda_k > \mu_k$. The results obtained with Fourier series contain on $x = 0$, $x = L$ oscillations (Gibbs phenomena). For FDSES method these oscillations disappear.

If the functions $f(x,t)$, $T_0(x)$ are proportional to the eigenvector $w_p(x) = \sqrt{1/L}\exp(2\pi ipx/L)$, $f(x,t) = g(t)w_p(x)$, $T_0(x) = a_0 w_p(x)$, then we can obtained the solution in the form $T(x,t) = y(t)w_p(x)$, where for function $y(t)$ follows the ODEs $\dot{y}(t) = -\nu\lambda_p y(t) + g(t)$ with $y(0) = a_0$, $\lambda_p = (\frac{2\pi p}{L})^2$.

We have the exact solution in this case with

$$y(t) = \exp(-\nu\lambda_p t)a_0 + \int_0^t \exp(-k\lambda_p(t-\xi))g(\xi)d\xi$$

The solution we can also obtain in real form:

$$
\begin{aligned}
T(x,t) =& \sum_{k=1}^{\infty}(a_{kc}(t)\cos\frac{2\pi kx}{L} + a_{ks}(t)\sin\frac{2\pi kx}{L}) + \frac{a_{0c}(t)}{2},\\
f(x,t) =& \sum_{k=1}^{\infty}(b_{kc}(t)\cos\frac{2\pi kx}{L} + b_{ks}(t)\sin\frac{2\pi kx}{L}) + \frac{b_{0c}(t)}{2},\\
b_{kc}(t) =& \frac{2}{L}\int_0^L f(\xi,t)\cos\frac{2\pi k\xi}{L}d\xi, b_{ks}(t) = \frac{2}{L}\int_0^L f(\xi,t)\sin\frac{2\pi k\xi}{L}d\xi,
\end{aligned}
\tag{2.5}
$$

where $a_{kc}(t), a_{ks}(t)$ are the corresponding solutions of (2.4) by

$$
a_{kc}(0) = \frac{2}{L}\int_0^L T_0(\xi)\cos\frac{2\pi k\xi}{L}d\xi, a_{ks}(0) = \frac{2}{L}\int_0^L T_0(\xi)\sin\frac{2\pi k\xi}{L}d\xi,
$$

$g_k(t) = b_{kc}(t)$ or $b_{ks}(t), \kappa_k = \nu\lambda_k$.

From Fourier series $T(x,t) = \sum_{k=-\infty}^{\infty} a_k(t)w^k(x)$, $f(x,t) = \sum_{k=-\infty}^{\infty} b_k(t)w^k(x)$, $b_k(t) = (f, w^k)_*$, follows ODEs $\dot{a}_k(t) = -\nu\lambda_k a_k(t) + b_k(t)$ with $a_k(0) = (T_0, w^k)_*, \lambda_k = (\frac{2\pi k}{L})^2$.

We have following solutions

$$
a_k(t) = \exp(-\nu\lambda_k t)a_k(0) + \int_0^t \exp(-\nu\lambda_k(t-\xi))b_k(\xi)d\xi.
$$

From orthonormal eigenvectors follows, that $b_p(t) = g(t), a_p(0) = a_0, b_k(t) = a_k(0) = 0, k \neq p$ and $a_p(t) = y(t)$.

From the discrete Fourier series

$$
\begin{aligned}
T(x_j,t) =& \sum_{k=-N/2}^{N/2} a_k(t)w^k(x_j),\\
f(x_j,t) =& \sum_{k=-N/2}^{N/2} b_k(t)w^k(x_j),\\
w^k(x_j) =& w_j^k = \sqrt{1/N}\exp(2\pi ikj/N)
\end{aligned}
$$

or in the vector form $U(t) = \sum_{k=-N/2}^{N/2} a_k(t)w^k$ we have ODEs

$$
\dot{a}_k(t) = -\nu\mu_k a_k(t) + b_k(t)
$$

with $a_k(0) = (U_0, w^{*k}), b_k(t) = (F(t), w^{*k})$.

We have following solutions

$$
a_k(t) = \exp(-\nu\mu_k t)a_k(0) + \int_0^t \exp(-\nu\mu_k(t-\xi))b_k(\xi)d\xi.
$$

From orthonormal eigenvectors follows, that $b_p(t) = g(t), a_p(0) = a_0, b_k(t) = a_k(0) = 0, k \neq p$ and $a_p(t) = y(t)$, if the eigenvalue μ_p are replaced with λ_p and $p \leq N/2$.

We can obtain the solution of the discrete problem also in the following real form

$$u_j(t)=\sum_{k=1}^{*N_2}(a_{kc}(t)\cos\frac{2\pi kj}{N}+a_{ks}(t)\sin\frac{2\pi kj}{N})+\frac{a_{0c}(t)}{2},$$

$$f_j(t)=\sum_{k=1}^{*N_2}(b_{kc}(t)\cos\frac{2\pi kj}{N}+b_{ks}(t)\sin\frac{2\pi kj}{N})+\frac{b_{0c}(t)}{2},$$

$$b_{kc}(t)=\frac{2}{N}\sum_{1}^{N}f_j(t)\cos\frac{2\pi kj}{N},b_{ks}(t)=\frac{2}{N}\sum_{1}^{N}f_j(t)\sin\frac{2\pi kj}{N},$$

where $a_{kc}(t),a_{ks}(t)$ are the corresponding solutions of (2.4) by

$$a_{kc}(0)=\frac{2}{N}\sum_{1}^{N}T_0(x_j)\cos\frac{2\pi kj}{N},a_{ks}(0)=\frac{2}{N}\sum_{1}^{N}T_0(x_j)\sin\frac{2\pi kj}{N},$$

$g_k(t)=b_{kc}(t)$ or $b_{ks}(t)$, $\kappa_k=\nu d_k$, $d_k=\mu_k$ for FDS, $d_k=\lambda_k$ for FDSES.

From the FDS the solution of the matrix equation (2.2) is $U(t)=\exp(-\nu tA)U(0)+\int_0^t\exp(-\nu A(t-\xi))F(\xi)d\xi$.

Using the matrix A representation $A=WDW^*$ and transformation $V=W^*U$ follows that for every matrix function $f(A)=Wf(D)W^*$ and $V=\exp(-\nu tD)V(0)+\int_0^t\exp(-\nu D(t-\xi))G(\xi)d\xi$.

Therefore we have the solution in the form (2.4). If $p\le N/2$ then the components $v_k(0)=(W^*U_0)_k=\sum_{j=1}^{N}w_j^{*k}T_0(x_j)=a_0\sum_{j=1}^{N}w_j^{*k}w_p(x_j)=\frac{a_0}{\sqrt{h}}(w^{*k},w^p)$, $g_k(t)=(W^*F)_k=\frac{g(t)}{\sqrt{h}}(w^{*k},w^p),k=\overline{1,N}$.

We get $v_p(0)=\frac{a_0}{\sqrt{h}},g_p(t)=\frac{g(t)}{\sqrt{h}},v_k(0)=g_k(t)=0,k\neq p$ and from (2.4) follows

$$v_p(t)=\frac{1}{\sqrt{h}}(\exp(-\nu\mu_pt)a_0+\int_0^t\exp(-\nu\mu_p(t-\xi)g(\xi)d\xi),v_k(t)=0,k\neq p.$$

For FDSES from $U=WV,w_j^k=\sqrt{h}w^k(x_j)$ and replaced the discrete eigenvalue μ_p with λ_p we obtain the exact solutions $T(x_j,t)=y(t)w_p(x_j),j=\overline{0,N}$.

If the functions $f(x,t),T_0(x)$ are proportional to the functions $f_1(x)=\sin(2\pi p_1x/L)$, $f_2(x)=\cos(2\pi p_2x/L)$, then using the expressions

$$f_1(x)=\frac{\sqrt{L}}{2i}(w_{p_1}(x)-w_{-p_1}(x)),f_2(x)=\frac{\sqrt{L}}{2}(w_{p_2}(x)+w_{-p_2}(x)),$$

$$w_{-p}(x_j)=w_{N-p}(x_j)=w_p^*(x_j),\mu_{-p}=\mu_p,\lambda_{-p}=\lambda_p$$

we have the preliminary results and the exact solution for $\max(p_1,p_2)\le N/2$.

Example

If $f(x,t)=g_1(t)f_1(x)+g_2(t)f_2(x),T_0(x)=\alpha_1f_1(x)+\alpha_2f_2(x)$ then we have $b_k(t)=\pm\frac{g_1(t)}{2i},k=\pm p_1$, $b_k(t)=\frac{g_2(t)}{2},k=\pm p_2,b_k(t)=0,k\neq(\pm p_1,\pm p_2)$, $a_k(0)=\pm\frac{\alpha_1}{2i},k=\pm p_1,a_k(0)=\frac{\alpha_2}{2},k=\pm p_2,a_k(0)=0,k\neq(\pm p_1,\pm p_2)$.

Therefore,

$$T(x,t)=a_{-p_1}(t)w_{-p_1}(t)+a_{p_1}(t)w_{p_1}(t)+a_{-p_2}(t)w_{-p_2}(t)+a_{p_2}(t)w_{p_2}(t)=$$

$$f_1(x)(\alpha_1 \exp(-\nu\lambda_{p_1}t) + \int_0^t \exp(-\nu\lambda_{p_1}(t-\xi))g_1(\xi)d\xi) +$$

$$f_2(x)(\alpha_2 \exp(-\nu\lambda_{p_2}t) + \int_0^t \exp(-\nu\lambda_{p_2}(t-\xi))g_2(\xi)d\xi).$$

In the discrete case we have the exact solution by replaced the discrete eigenvalues μ_{p_1}, μ_{p_2}, with $\lambda_{p_1}, \lambda_{p_2}$.

2.3 Equation of Conduction in Fourier's Ring

The equation

$$\frac{\partial V(x,t)}{\partial t} = \nu \frac{\partial^2 V(x,t)}{\partial x^2} \pm a^2 V(x,t) + g(x,t),$$

can be educed to the equation (2.1) using transformation $V(x,t) = \exp(\pm a^2 t)T(x,t)$, where a and ν are constant, $f(x,t) = g(x,t)\exp(\mp a^2 t)$. In this case the homogenous BCs of first kind and periodic BCs remains.

In book [13] is considered following interesting problem of heat transfer equation for short ring domain:

$$c\rho \frac{\partial V(x,t)}{\partial t} = k \frac{\partial^2 V(x,t)}{\partial t^2} - \frac{Hp}{\omega}(V(x,t) - V_0), x \in (-\pi, \pi),$$

where ρ, c are the density of the solid ring and heat capacity, k is heat conductivity, H is the heat transfer coefficient in external domain, p, ω are perimeter and cross section of the ring , V, V_0 are the temperature and constant external temperature. The book [13] is one of the first works where PBC are inspected. This problem for the first time consider Fourier for the mathematical and physical modelling.

We have the periodic BCs in the segment $[-\pi, \pi]$ and periodic initial function $T_0(x)$, $T_0(-\pi) = T_0(\pi)$. Using the transformation $V(x,t) = \exp(-a^2 t)T(x,t), a^2 = \frac{Hp}{\omega\rho c}$ we obtain by $V_0 = 0$ the problem (2.1) with $\nu = \frac{k}{c\rho}, f = 0, L = 2\pi$. The solution we can also obtained in the form:
$T(x,t) = \sum_{k=1}^{\infty}(a_{kc}(t)\cos(kx) + a_{ks}(t)\sin(kx)) + \frac{a_{0c}(t)}{2}$,
where $a_{kc}(t), a_{ks}(t)$ are the corresponding solutions:
$a_{kc}(t) = a_{kc}(0)\exp(-\kappa_k t), a_{ks}(t) = a_{ks}(0)\exp(-\kappa_k t)$,
$a_{kc}(0) = \frac{1}{\pi}\int_{-\pi}^{\pi} T_0(\xi)\cos(k\xi)d\xi$,
$a_{ks}(0) = \frac{1}{\pi}\int_{-\pi}^{\pi} T_0(\xi)\sin(k\xi)d\xi$, $\kappa_k = \nu\lambda_k, \lambda_k = k^2$.
If the ends of ring $x = \pm\pi$ is given constant temperature V_s then we have from the problem $V''(x) - \frac{a^2}{\nu}V(x) = 0, V(-\pi) = V(\pi) = V_s$ the stationary solution

$$V(x) = V_s \frac{\cosh(\bar{a}x)}{\cosh(\bar{a}\pi)},$$

where $\bar{a} = \sqrt{\frac{Hp}{k\omega}}$.
Using the stationary solution in [13] is determined the coefficient ν of heat conductivity. Assume that the source at the ends of ring is to cancel and in the ring the stationary temperature is obtained and that the ring begin to cool from the heat transfer in the external medium with the constant temperature equal to zero.

This process can be described with the following heat transfer equation $\frac{\partial V(x,t)}{\partial t} = \nu \frac{\partial^2 V(x,t)}{\partial x^2} - a^2 V(x,t)$, with the periodic BCs and initial conditions $V(x,0) = V_s \frac{\cosh(\bar{a}x)}{\cosh(\bar{a}\pi)}$.

Using the expressions [13] $\frac{\cosh(\bar{a}x)}{\cosh(\bar{a}\pi)} = \frac{2\bar{a}\tanh(\bar{a}\pi))}{\pi}[\frac{1}{2\bar{a}^2} + \sum_{j=1}^{\infty} \frac{\cos(j\pi)}{\bar{a}^2+j^2} \cos(jx)]$, we obtain the solution in the following form:

$$V(x,t) = \frac{2V_s\bar{a}}{\pi} \tanh(\bar{a}\pi) \exp(-a^2t)[\frac{1}{2\bar{a}^2} + \sum_{j=1}^{\infty} \frac{\cos(j\pi)}{\bar{a}^2 + j^2} \cos(jx) \exp(-\nu j^2 t)],$$

This series have quick convergence, because $V(x,t) \approx \frac{2V_s}{\pi\bar{a}} \tanh(\bar{a}\pi) \exp(-a^2 t)$ and it is possible estimate the coefficients $k.H$ to measure the temperature in two cross section $x = 0, x = \pi$ of the ring [13].

2.4 Sample nonlinear heat transfer equation

We shall consider the initial - boundary value problem for solving the following nonlinear heat transfer equation:

$$\frac{\partial T}{\partial t} = \frac{\partial^2 (g(T))}{\partial x^2} + f(T),$$

where $g(T) = T^{\sigma+1}, f(T) = aT^{\beta}$ is nonlinear functions with $a > 0, \beta \geq 1, \sigma \geq 0, T(x,t) \geq 0, T_0(x) \geq 0$.
In books [15], [16] by ($a = 1$) is proved with the first kind boundary conditions that
1) by $\beta < \sigma + 1$ exists global bounded solution for all t,
2) by $\beta \geq \sigma + 1$ exists global bounded solution for sufficient small $||T_0||$, but for larger $||T_0||$, exists finite value of time T_* when $u(x,t) \to \infty$ if $t \to T_*$. The initial value problem for ODEs (2.2) is in the form

$$\dot{U} + AG = F, U(0) = U_0,$$

where G, F are the vectors-column of N order with elements $g_k = g(u(x_k,t)), f_k = af(u(x_k,t)), k = \overline{1,N}$.
The numerical experiment with $L = 1$ and $T_0(x) = x(1-x) \geq 0$, is produced by MATLAB 7.4 solver "ode23s" [17]. For $a = 5$, $\sigma = \beta = 3$, ($\beta < \sigma + 1$), $t = 10$, $N = 6, 10, 20$ are obtained following maximal error using FDS and FDSES methods:
1) $N = 6$ – 0,0125 (FDS), 0.0011 (FDSES);
2) $N = 10$ – 0.0046 (FDS), 0.0003 (FDSES);
3) $N = 20$ – 0.0013 (FDS), 0.0001(FDSES).

In the fig. 2.1, 2.2, 2.3, 2.4 are represented 4 type solutions by $T_0(x) = \sin^{100}(\pi x)$, $N = 50, \sigma = 3$ for periodic boundary conditions obtained [11] [12]:
1) $\beta = 5, a = 100$, the solution is "blow up" locally by $T_* = 5.481136$,
2) $\beta = 4, a = 100$, the solution is "blow up" globally by $T_* = 0.2020261$,
3) $\beta = 5, a = 1$, the solutions tends to zero, if $t \to \infty$,
4) $\beta = 4, a = 0.01$, the solutions tends to the stationary limit.

The MATLAB program `nelper.m` can be found in appendix A.3.

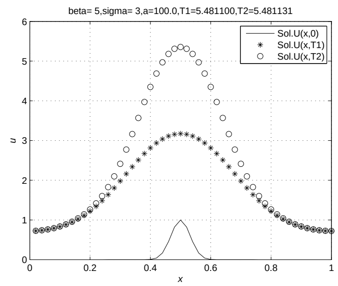

Fig. 2.1 $U \to \infty$ for $x = 0.5$, $\beta = 5, \sigma = 3, a = 100, T_* = 5.481136$

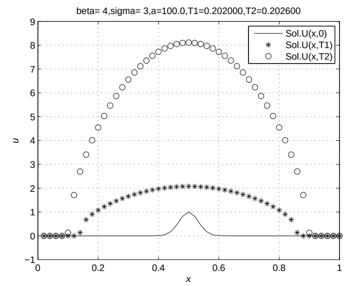

Fig. 2.2 $U \to \infty$ for $x \in (0,1)$, $\beta = 4, \sigma = 3, a = 100, T_* = 0.2020261$

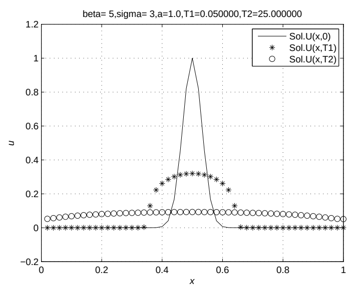

Fig. 2.3 $U \to 0$ if $t \to \infty$, for $\beta = 5, \sigma = 3, a = 1$

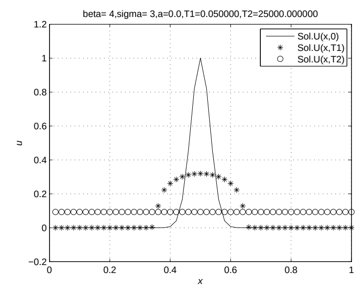

Fig. 2.4 $U \to$ stationary if $t \to \infty$, for $\beta = 4, \sigma = 3, a = 0.01$

We have following results by $N = 40$, $\beta = 5$, $\sigma = 3$, $a = 100$:
$T_* = 5.448350(FDS\ O(h^2))$, $T_* = 5.536841(FDS\ O(h^4))$, $T_* = 5.539480(FDS\ O(h^6))$, $T_* = 5.539669(FDS\ O(h^4))$, $T_* = 5.539397(FDSES)$.

Results of FDSES we can see in the fig. 2.5.

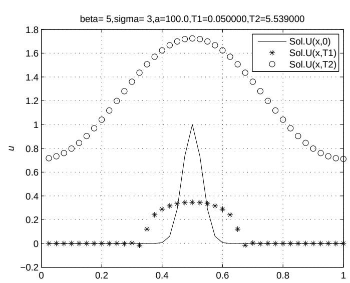

Fig. 2.5 The blow-up solution usind FDSES by $\beta = 5, \sigma = 3, a = 100$

2.5 Burger's equation

For numerical experiments we consider following nonlinear initial – boundary problem for Burger's equations in the following form [36]:

$$\frac{\partial T(x,t)}{\partial t} = \nu \frac{\partial^2 T(x,t)}{\partial x^2} - T(x,t)\frac{\partial T(x,t)}{\partial x},$$

where

1. $T_0(x) = 4\nu\pi \sin(2\pi x)/(2 + cos(2\pi x))$, or
2. $T_0(x) = \sin^{100}(\pi x), x \in (0,1)$.

Using the transformation $T = -2\nu \frac{\partial \ln(V)}{\partial x}$ we can obtain homogenous linear heat transfer equation

$$\frac{\partial V(x,t)}{\partial t} = \nu \frac{\partial^2 V(x,t)}{\partial x^2}.$$

For the first case of initial conduction we have the analytical solution:

$$V(x,t) = 2 + \exp(-4\pi^2 \nu t)\cos(2\pi x),$$

$$T(x,t) = 4\nu\pi \exp(-4\pi^2 \nu t)\sin(2\pi x)/V(x,t).$$

In this case the method of lines is in the form

$$\dot{U}(t) + \nu A U(t) = -0.5 A_1 U(t)^2, \quad U(0) = U_0,$$

where $A_1 = \frac{1}{4h}[0, 1, \cdots, 0, -1]$ is the 3-diagonal circulant matrix of N order. It approximates the first derivative with precision $O(h^2)$. Higher precision scheme can be chosen according to 1.4.2.

The numerical experiment with $\nu = 1, t = 0.2$, is produced by MATLAB solver "ode15s".

In the fig. 2.6 and fig. 2.7 we can see the results obtained by $N = 40$ for both type initial conditions by three moments of time $t = 0, t_1 = 0.01, t_2 > t_1$.

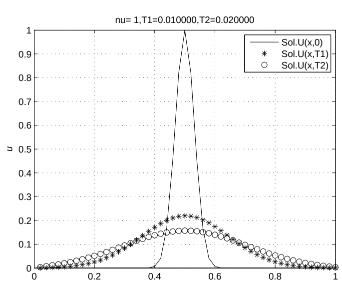

Fig. 2.6 Numerical solution by $N = 40, t_1 = 0.01, t_2 = 0.02$

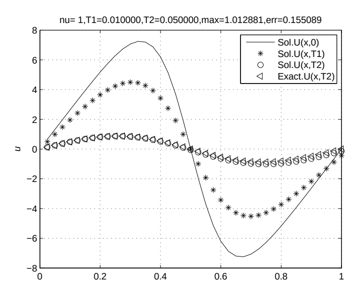

Fig. 2.7 Numerical and analytical solution by $N = 40, t_1 = 0.01, t_2 = 0.05, err = 0.155$

2.6 Example of heat transfer equation

We consider the initial boundary value problem (2.1) by $L = 1$, $\nu = 1$, $f = 0$, $T_0(x) = \sin(2\pi m x)$, where m is integer in $(1, N)$. Then the exact solution is $T(x,t) = \exp(-4\pi^2 m^2 t)\sin(2\pi m x)$.

The solution of (2.1) with the **Fourier method** can be obtained in the form $T(x,t) = \sum_{k=-\infty}^{\infty} v_k(t)w^k(x)$, where $v_k(t)$ is the solution of (2.4) in the form

$$v_k(t) = \exp(-\kappa_k t)v_k(0), \kappa_k = d_k = 4\pi^2 k^2,$$

$$v_k(0) = \int_0^1 T_0(x)w_{*k}(x)dx = 0,$$

$v_k(t) = 0$ for $k \neq \pm m$, $v_{\pm m}(0) = \frac{0.5}{\pm i}$, $v_{\pm m}(t) = \pm\frac{\exp(-(2\pi m)^2 t)}{2i}$,

$$T(x,t) = v_{-m}(t)w^{-m}(x) + v_m(t)w^m(x) = \exp(-4\pi^2 m^2 t)\sin(2\pi m x).$$

Therefore we have using the Fourier method the exact solution.
We can consider the **analytical solutions** for FDS of (2.2) using the spectral representation of matrix $A = WDW^*$.

From transformation $V = W^*U$ $(U = WV)$ follows the separate system of ODEs (2.3). The matrix solution of the system (2.3) is $V(t) = \exp(-Dt)V_0$, $D = diag(\mu_k)$ or in the form (2.4), where $v_k(t) = \exp(-\kappa_k t)v_k(0)$, $\kappa_k = \mu_k$, $v_k(0) = (W^*U_0)_k = 0$, $v_k(t) = 0$ for $k \neq m, k \neq N-m$.From $\mu_{N-k} = \mu_k$, $w^{N-k} = w^{*k}$ follows $v_m(0) = \frac{\sqrt{N}}{2i}$, $v_{N-m}(0) = -\frac{\sqrt{N}}{2i}$.

Therefore $U(t) = \exp(-\mu_m t)U_0$, where $U_0 = (\sin(2\pi m x_1), \cdots, \sin(2\pi m x_N))^T$ is the column-vector of the N order, $x_j = jh, j = \overline{1.N}, Nh = 1$.

The solution can be obtained in the matrix form $U(t) = W\exp(-Dt)W^*U_0$.
For the FDSES $\mu_m = d_m = (2\pi m)^2$ and we have also the exact solution.

Using the **discrete Fourier** transformation

$$U(t) = \sum_{k=1}^{N} a_k(t)w^k(Aw^k = \mu_k w^k),$$

we get $a_k(t) = \exp(-\mu_k t)a_k(0)$, where $a_k(0) = U_0.w^{*k} = 0$ for $k \neq m, k \neq N-m$, $a_m(0) = \frac{\sqrt{N}}{2i}, a_{N-m}(0) = -\frac{\sqrt{N}}{2i}$.

We have $U(t) = a_m(t)w^m + a_{N-m}(t)w_*^m = \frac{\sqrt{N}}{2i}\exp(-\mu_m t)(w^m - w_*^m) = \exp(-\mu_m t)U_0$.
For numerical calculation we consider the initial boundary value problem (2.1) with $t_f = 0.05, L = 1, f = 0, T_0 = \sin(2\pi m x)$, for $m = 1; 2; 3; 4, N = 10$.

The MATLAB code can be found in appendix A.5.

Using the operator **Siltm(10)** we obtain following maximal errors (see tab. 2.1):

Table 2.1 The FDS maximal error depending on order of approximation and m by $N = 10$

Method	m=1	m=2	m=3	m=4
$O(h^2)$	0.0115	0.0457	0.0899	0.0990
$O(h^4)$	5.10^{-4}	0.0084	0.0290	0.0410
$O(h^6)$	3.10^{-5}	0.0019	0.0126	0.0234
$O(h^8)$	2.10^{-6}	5.10^{-4}	0.0060	0.0153
FDSES	$1 \cdot 10^{-15}$	$6 \cdot 10^{-16}$	$1 \cdot 10^{-15}$	$7 \cdot 10^{-16}$

In the fig. 2.8, 2.9 we can see the FDSES exact solutions by $m = 4$ and $N = 10, N = 40$.

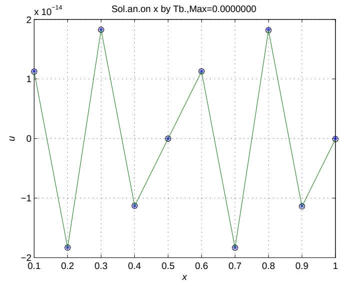

Fig. 2.8 FDSES solutions by $N = 10, m = 4, t_f = 0.05$

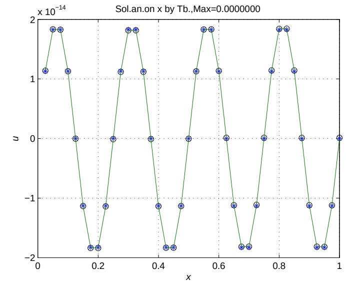

Fig. 2.9 FDSES solutions by $N = 40, m = 4, t_f = 0.05$

In the figs. 2.10 and 2.11 the error graphics for $m = 1$ can be observed.

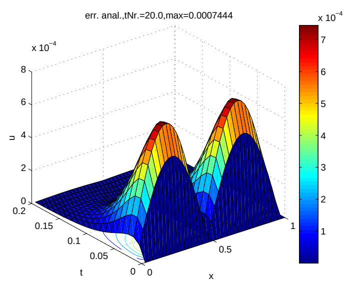

Fig. 2.10 Error with FDS by $N = 40, O(h^2)$

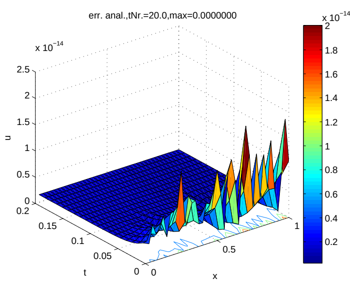

Fig. 2.11 Error with FDSES by $N = 40$

2.7 Mathematical model for heat transfer equations with convection

We consider the linear heat transfer equation in the following form [38]:

$$\frac{\partial T(x,t)}{\partial t} = \nu \frac{\partial^2 T(x,t)}{\partial x^2} + a \frac{\partial T(x,t)}{\partial x} + f(x,t) \tag{2.6}$$

with the periodic boundary conditions(2.1) (a=const).
We can use the **Fourier method** for solving the initial-boundary value problem in the form $T(x,t) = \sum_{k \in Z} a_k(t) w^k(x), f(x,t) = \sum_{k \in Z} b_k(t) w^k(x)$, where $w^k(x)$ are the orthonormal eigenvectors, $b_k(t) = (f, w^{*k}(x))$.
Then for the unknown functions $a_k(t)$ get the complex initial value problem for ODEs of first order:

$$\begin{cases} \dot{a}_k(t) + a_k(t)\lambda_k = b_k(t), \\ a_k(0) = \frac{1}{L}\int_0^L T_0(s) \exp \frac{-2i\pi ks}{L} ds, \\ b_k(t) = \frac{1}{L}\int_0^L f(s,t) \exp \frac{-2i\pi ks}{L} ds, \end{cases} \tag{2.7}$$

where $\lambda_k = \nu \big(\frac{2\pi k}{L}\big)^2 - ai\frac{2\pi k}{L}$.

The solution of (2.7) is

$$a_k(t) = \exp(-\lambda_k t)a_k(0) + \int_0^t \exp(\lambda_k)(t-s))b_k(s)ds.$$

The solution with the Fourier method can be obtained in real form with $f(x,t)$ in form (2.5) where $b_{kc}(t) = \frac{1}{\sqrt{L}}(b_k(t) + b_{-k}(t)) = \frac{1}{\sqrt{L}}\int_0^L f(x,t)(w^{*k}(x) + w^k(x))dx = \frac{2}{L}\int_0^L f(s,t)\cos\frac{2\pi ks}{L}ds$, $b_{ks}(t) = \frac{i}{\sqrt{L}}(b_k(t) - b_{-k}(t)) = \frac{i}{\sqrt{L}}\int_0^L f(x,t)(w^{*k}(x) - w^k(x))dx = \frac{2}{L}\int_0^L f(s,t)\sin\frac{2\pi ks}{L}ds$.
Therefore the solution we can obtain also in real form:
$T(x,t) = \sum_{k=1}^{\infty}(a_{kc}(t)\cos\frac{2\pi kx}{L} + a_{ks}(t)\sin\frac{2\pi kx}{L}) + \frac{a_{0c}(t)}{2}$,
where $a_{kc}(t), a_{ks}(t)$ are unknown functions.
From $f(x,t) = \frac{\partial T(x,t)}{\partial t} - (\frac{\partial^2 T(x,t)}{\partial x^2} + a\frac{\partial T(x,t)}{\partial x})$
follows $f(x,t) = \sum_{k=1}^{\infty}(\dot{a}_{kc}(t)\cos\frac{2\pi kx}{L} + \dot{a}_{ks}(t)\sin\frac{2\pi kx}{L}) + \frac{\dot{a}_{0c}(t)}{2} +$
$\sum_{k=1}^{\infty}((a_{kc}(t)Re(\lambda_k) + a_{ks}(t)Im(\lambda_k))\cos\frac{2\pi kx}{L} + (a_{ks}(t)Re(\lambda_k) - a_{kc}(t)Im(\lambda_k))\sin\frac{2\pi kx}{L})$,
because $(a_k(t)\lambda_k + a_{-k}(t)\lambda_{-k})/\sqrt{L} = a_{kc}(t)Re(\lambda_k) + a_{ks}(t)Im(\lambda_k)$,
$i(a_k(t)\lambda_k - a_{-k}(t)\lambda_{-k})/\sqrt{L} = a_{ks}(t)Re(\lambda_k) - a_{kc}(t)Im(\lambda_k)$, where $a_{kc}(t) = \frac{a_k(t)+a_{-k}(t)}{\sqrt{L}}$, $a_{ks}(t) = \frac{i(a_k(t)-a_{-k}(t)}{\sqrt{L}}$ are the coefficients in the expression from the solution $T(x,t)$.
Therefore we obtain the initial boundary value problem for the system of two ODEs:

$$\begin{cases} \dot{a}_{kc}(t) + a_{kc}(t)Re(\lambda_k) + a_{ks}(t)Im(\lambda_k) = b_{kc}(t), \\ \dot{a}_{ks}(t) + a_{ks}(t)Re(\lambda_k) - a_{kc}(t)Im(\lambda_k) = b_{ks}(t), \\ a_{kc}(0) = \frac{2}{L}\int_0^L T_0(s)\cos\frac{2\pi ks}{L}ds, a_{ks}(0) = \frac{2}{L}\int_0^L T_0(s)\sin\frac{2\pi ks}{L}ds. \end{cases} \tag{2.8}$$

2.7.1 Solutions of system of two ODEs

In the matrix form we have

$$\dot{A}_k(t) + \Lambda_k A_k(t) = B_k(t), A_k(0) = A_{k0}, \tag{2.9}$$

where

$$\Lambda_k = \begin{pmatrix} Re(\lambda_k) & Im(\lambda_k) \\ -Im(\lambda_k) & Re(\lambda_k) \end{pmatrix}$$

is the matrix of second order, $A_k(t)$, $B_k(t)$, A_{k0} are the column-vectors with elements $(a_{kc}(t), a_{ks}(t))$, $(b_{kc}(t), b_{ks}(t))$, $(a_{kc}(0), a_{ks}(0))$.

We can represent the matrix Λ_k in the form $\Lambda_k = PDP^{-1}$, where
$P = \begin{pmatrix} 0.5 & -i \\ -0.5i & 1 \end{pmatrix}$, $P^{-1} = \begin{pmatrix} 1 & i \\ 0.5i & 0.5 \end{pmatrix}$, $D = \begin{pmatrix} \lambda_k^* & 0 \\ 0 & \lambda_k \end{pmatrix}$,
where $\lambda_k^* = Re(\lambda_k) - iIm(\lambda_k)$, $Re(\lambda_k) = \nu\frac{4\pi^2 k^2}{L^2}$, $Im(\lambda_k) = -\frac{2\pi ka}{L}$.

Then the matrix solution of (2.9) $A_k(t) = \exp(-\Lambda t)A_{k0} + \int_0^t \exp(-\Lambda(t-s)B_k(s)ds$ with the transformations $\tilde{A}_k(t) = P^{-1}A_k(t), A_k(t) = P\tilde{A}_k(t)$ can be obtained in the form

$$\tilde{A}_k(t) = \exp(-Dt)\tilde{A}_{k0} + \int_0^t \exp(-D(t-s)\tilde{B}_k(s)ds,$$

where $\tilde{A}_{k0} = P^{-1}A_{k0}, \tilde{B}_k(t) = P^{-1}B_k(t)$.
For this separable we can determine the elements $\tilde{a}_{kc}(t), \tilde{a}_{ks}(t)$ of the column-vector $\tilde{A}_k$

depending on the elements
$\tilde{a}_{kc}(0) = a_{kc}(0) + ia_{ks}(0), \tilde{a}_{ks}(0) = 0.5ia_{kc}(0) + 0.5a_{ks}(0),$
$\tilde{b}_{kc}(t) = b_{kc}(t) + ib_{ks}(t), \tilde{b}_{ks}(t) = 0.5ib_{kc}(t) + 0.5b_{ks}(t)$
of the column-vectors $\tilde{A}_{k0}$, $\tilde{B}_k(t)$ we obtain the solution of the ODEs system (2.7) in the following form

$$\begin{cases} a_{kc}(t) = \exp(-Re(\lambda_k)t)(a_{kc}(0)\cos(Im(\lambda_k)t) - a_{ks}(0)\sin(Im(\lambda_k)t)) \\ \qquad + \int_0^t \exp(-Re(\lambda_k)(t-s))(b_{kc}(s)\cos(Im(\lambda_k)(t-s)) \\ \qquad - b_{ks}(s)\sin(Im(\lambda_k)(t-s))ds, \\ a_{ks}(t) = \exp(-Re(\lambda_k)t)(a_{kc}(0)\sin(Im(\lambda_k)t) + a_{ks}(0)\cos(Im(\lambda_k)t)) \\ \qquad + \int_0^t \exp(-Re(\lambda_k)(t-s))(b_{kc}(s)\sin(Im(\lambda_k)(t-s)) \\ \qquad + b_{ks}(s)\cos(Im(\lambda_k)(t-s))ds \end{cases} \tag{2.10}$$

For $k = 0$ we obtain $a_{0c}(t) = \int_0^t b_{0c}(s)ds + a_{0c}(0)$.

In the limit case $t \to \infty$ if the source function $f = f(x)$ not depending on t the we obtain from (2.9) the stationary solution.

If $a = 0$ then we have the expressions (2.8–2.10) with $Re(\lambda_k) = \nu\big(\frac{2k\pi}{L}\big)^2$, $Im(\lambda_k) = 0$.

2.7.2 Discrete problem with Iljin FDS

For the **discrete problem** we have the system of N ODEs in the form of (2.2)

$$\dot{U}(t) + \tilde{A}U(t) = F(t), \quad U(0) = U_0$$

where the circulant matrix $\tilde{A} = \frac{\nu}{h^2}[2\gamma, -(\gamma+\alpha), 0, 0, \ldots, 0, -(\gamma-\alpha)]$ [14], with the eigenvalues $\tilde{\mu}_k = \frac{4\nu}{h^2}(\sin(k\pi/N)^2(\gamma - i\alpha\cot\frac{k\pi}{N})$, $\gamma = \alpha\coth(\alpha)$, $\alpha = \frac{ah}{2\nu}$ (the order of approximation $O(h^2)$) and with the elements of the orthonormal eigenvectors $w_j^k = \sqrt{\frac{1}{N}}\exp(2\pi ikj/N)$, $w_{*j}^k = \sqrt{\frac{1}{N}}\exp(-2\pi ikj/N)$, $k, j = \overline{1, N}$. If $h \to 0$, then $\tilde{\mu}_k \to \lambda_k$. For the column-vector $F(t)$ elements $f_j(t)$ we obtain $f_j(t) = \sum_{k=1}^{*N_2}(b_{kc}(t)\cos\frac{2\pi kj}{N} + b_{ks}(t)\sin\frac{2\pi kj}{N}) + \frac{b_{0c}(t)}{2}$,
where
$b_{kc}(t) = \frac{2}{N}\sum_{j=1}^N f_j(t)\cos\frac{2\pi kj}{N}$,
$b_{ks}(t) = \frac{2}{N}\sum_{j=1}^N f_j(t)\sin\frac{2\pi kj}{N}, k = \overline{1, N_2}$,
$b_0(t) = b_N(t) = \frac{1}{\sqrt{N}}\sum_{j=1}^N f_j(t)$,
$b_{0c}(t) = b_{Nc}(t) = \frac{2}{\sqrt{N}}b_0(t)$,
$b_{N_2,c}(t) = \frac{2}{\sqrt{N}}$
$b_{N_2}(t) = \frac{2}{N}\sum_{j=1}^N \cos(j\pi)$,
$N_2 = \frac{N}{2}, b_{N_2s}(t) = b_{Ns}(t) = 0$,
$\sum_{k=1}^{*N_2}\beta_k = \sum_{k=1}^{N_2-1}\beta_k + \frac{\beta_{N/2}}{2}$.

For the solution $u_j(t) = \sum_{k=1}^{*N_2}(a_{kc}(t)\cos\frac{2\pi kj}{N} + a_{ks}(t)\sin\frac{2\pi kj}{N}) + \frac{a_{0c}(t)}{2}$
and $u_j(0) = \sum_{k=1}^{*N_2}(a_{kc}(0)\cos\frac{2\pi kj}{N} + a_{ks}(0)\sin\frac{2\pi kj}{N}) + \frac{a_{0c}(0)}{2}$
with $a_{kc}(0) = \frac{2}{N}\sum_{j=1}^N u_j(0)\cos\frac{2\pi kj}{N}$, $a_{ks}(0) = \frac{2}{N}\sum_{j=1}^N u_j(0)\sin\frac{2\pi kj}{N}$
we need to determine the unknown functions $a_{kc}(t)$, $a_{ks}(t)$ of the following expressions

$f_j(t) = \dot{u}_j + (\tilde{A}u)_j = \sum_{k=1}^{*N_2}(\dot{a}_{kc}(t)\cos\frac{2\pi kj}{N} + \dot{a}_{ks}(t)\sin\frac{2\pi kj}{N}) + \frac{\dot{a}_{Nc}(t)}{2}+$
$\sum_{k=1}^{*N_2}(a_{kc}(t)Re(\tilde{\mu}_k) + a_{ks}(t)Im(\tilde{\mu}_k))\cos\frac{2\pi kj}{N} + (a_{ks}(t)Re(\tilde{\mu}_k) - a_{kc}(t)Im(\tilde{\mu}_k))\sin\frac{2\pi kj}{N})$.

Therefore, for the determine the functions $a_{kc}(t), a_{ks}(t)$ we obtain the systems of ODEs (2.8,2.9) and the solution (2.10), where the eigenvalues λ_k are replaced with the discrete eigenvalues $\tilde{\mu}_k$, $k = \overline{1, N}$.

If $a = 0$ then $Re(\tilde{\mu}_k) = \frac{4\nu}{h^2}(\sin(k\pi/N)^2, Im(\tilde{\mu}_k) = 0$.

2.7.3 Discrete problem in multi-point stencil

Using the **multi-point stencil** with the order of approximation $O(h^{2n}), n = 1, 2, 3, \dots$ we have $\tilde{A} = \nu A - aA^0$, $\tilde{\mu}_k = \nu\mu_k - ai|\mu_k^0|$, where μ_k, μ_k^0 are the eigenvalues from circulant matrices A, A^0.

The **complex** expressions we can obtain from following matrices representation $A = WDW^*$, $A^0 = WD^0W^*$, $D = diag(\mu_k)$, $D^0 = diag(\mu_k^0)$. Then the system of ODEs is

$$\dot{V}(t) + (\nu D - aD^0)V(t) = \tilde{F}(t),$$

where $V(t) = W^*U(t)$, $U(t) = WV(t)$, $\tilde{F}(t) = W^*F(t)$ or $\dot{v}_k(t) + (\nu\mu_k - a\mu_k^0)v_k(t) = \tilde{f}_k(t)$ ($v_k(t)$, $\tilde{f}_k(t)$ are the elements of column-vectors $V(t)$, $\tilde{F}(t)$). We have following solution:

$$v_k(t) = \exp(-\tilde{\mu}_k t)v_k(0) + \int_0^t \exp(-\tilde{\mu}_k(t-\tau)\tilde{f}_k(\tau)d\tau, k = \overline{1, N-1}$$

where $\tilde{\mu}_k = \nu\mu_k - a\mu_k^0$, $v_N(t) = v_N(0) + \int_0^t \tilde{f}_N(\tau)d\tau$, $v_k(0) = (W^*U(0))_k$.
The **real** solutions can be easily obtained from following expressions:

$$Aw^k = \mu_k w^k, Aw^{*k} = \mu_k w_*^k, A^0 w^k = i|\mu_k^0|w^k, A^0 w^{*k} = -i|\mu_k^0|w^{*k}$$

or

$$A\cos_k = \mu_k\cos_k, A\sin_k = \mu_k\sin_k, A^0\cos_k = -|\mu_k^0|\sin_k, A^0\sin_k = |\mu_k^0|\cos_k,$$

where $\cos_k$, $\sin_k$ are N-order column-vectors with the elements $\cos\frac{2\pi kj}{N}$, $\sin\frac{2\pi kj}{N}$, $j = \overline{1, N}$.

We have following properties for the scalar products:
$\frac{2}{N}\cos_k\cos_m = \delta_{k,m}$, $\frac{2}{N}\sin_k\sin_m = \delta_{k,m}$, $\frac{2}{N}\cos_k\sin_m = 0$.
Then

$$\dot{U}(t)=\sum_{k=1}^{*N_2}(\dot{a}_{kc}(t)\cos_k+\dot{a}_{ks}(t)\sin_k)+\frac{\dot{a}_{0c}(t)}{2}\cos_0,$$
$$U(t)=\sum_{k=1}^{*N_2}(a_{kc}(t)\cos_k+a_{ks}(t)\sin_k)+\frac{a_{0c}(t)}{2}\cos_0,$$
$$AU(t)=\sum_{k=1}^{*N_2}\mu_k(a_{kc}(t)\cos_k+a_{ks}(t)\sin_k), A\cos_0=0,$$
$$A^0U(t)=\sum_{k=1}^{*N_2}|\mu_k^0|(-a_{kc}(t)\sin_k+a_{ks}(t)\cos_k), A^0\cos_0=0,$$
$$F(t)=\sum_{k=1}^{*N_2}(b_{kc}(t)\cos_k+b_{ks}(t)\sin_k)+\frac{b_{0c}(t)}{2}\cos_0,$$
$$b_{kc}(t)=\frac{2}{N}F(t)\cos_k, b_{ks}(t)=\frac{2}{N}F(t)\sin_k,$$
$$U(0)=\sum_{k=1}^{*N_2}(a_{kc}(0)\cos_k+b_{ks}(0)\sin_k)+\frac{b_{0c}(0)}{2}\cos_0,$$

where the unknown functions $a_{kc}(t)$, $a_{ks}(t)$ are obtained from the following ODEs

$$\begin{cases}\dot{a}_{kc}(t)+\nu\mu_k a_{kc}(t)-a|\mu_k^0|a_{ks}(t)=b_{kc}(t),\\ \dot{a}_{ks}(t)+\nu\mu_k a_{ks}(t)+a|\mu_k^0|a_{kc}(t)=b_{ks}(t),\\ a_{kc}(0)=\frac{2}{N}U_0\cos_k, a_{ks}(0)=\frac{2}{N}U_0\sin_k.\end{cases} \tag{2.11}$$

This problem is equal to (2.8), where $Re(\lambda_k)=\nu\mu_k$, $Im(\lambda_k)=-a|\mu_k^0|$. Using the eigenvalues λ_k in (2.11) we have FDSES.

For **initial data** proportional to fixed frequency $k_0\le N/2$:

$$U_0=a_c\cos_{k_0}+a_s\sin_{k_0}, F(t)=b_c(t)\cos_{k_0}+b_s(t)\sin_{k_0}$$

we have $a_{kc}(0)=a_c\delta_{k_0,k}$, $a_{ks}(0)=a_s\delta_{k_0,k}$, $b_{kc}(t)=b_c(t)\delta_{k_0,k}$, $b_{ks}(t)=b_s(t)\delta_{k_0,k}$, and $U(t)=a_{k_0c}(t)\cos_{k_0}+a_{k_0s}(t)\sin_{k_0}$, where $a_{k_0c}(t)$, $a_{k_0s}(t)$ are the solutions of ODEs (2.11) in the form (2.10), where $Re(\lambda_k)=\nu\mu_k$, $Im(\lambda_k)=-a|\mu_k^0|$.

2.7.4 Euler-Newton FDS for solving Cauchy problem

We consider the discrete homogenous heat transfer problem in multi-point stencil with the periodic boundary conditions (2.2). Using Euler-Newton FDS [39] we have in every time step ($t_n=n\tau, n=1,2,\dots$)

$$(U^{n+1}-U^n)/\tau=-\mathit{\Gamma} AU^n, n\ge 0, U^0=U_0, \mathit{\Gamma}=(\exp(-A\tau)-E)(-A\tau)^{-1}, \tag{2.12}$$

where $U^n=U(t_n)$, A,E are the N-order circular and unit matrices. From (2.12) follows $U^{n+1}=\exp(-A\tau)U^n$ or $U^n=\exp(-At_n)U_0$. For matrix A representation follows $\exp(-A\tau)W=\exp(-D\tau)W$ or $\exp(-A\tau)w^k=\exp(-\mu_k\tau)w^k, k=\overline{1,N}$. Then

$V^n = W^*U^n, (U^n = WV^n), U^{n+1} = WV^{n+1}, V^{n+1} = \exp(-D\tau)V^n, V^0 = W^*U_0$.
For real solutions we have $\exp(-A\tau)\cos_k = \exp(-\mu_k\tau)\cos_k$,
$\exp(-A\tau)\sin_k = \exp(-\mu_k\tau)\sin_k$ and $U^{n+1} = \sum_{k=1}^{*N_2}(a_{kc}^{n+1}\cos_k + a_{ks}^{n+1}\sin_k) + \frac{a_{0c}^{n+1}}{2}\cos_0$,
$U^n = \sum_{k=1}^{*N_2}(a_{kc}^n\cos_k + a_{ks}^n\sin_k) + \frac{a_{0c}^n}{2}\cos_0$,
$\exp(-A\tau)U^n = \sum_{k=1}^{*N_2}\exp(-\mu_k\tau)(a_{kc}^n\cos_k + a_{ks}^n\sin_k) + \frac{a_{0c}^n}{2}\cos_0$,
$U_0 = \sum_{k=1}^{*N_2}(a_{kc}^0\cos_k + a_{ks}^0\sin_k) + \frac{a_{0c}^0}{2}\cos_0$,
$a_{kc}^0 = \frac{2}{N}U_0\cos_k, a_{ks}^0 = \frac{2}{N}U_0\sin_k$.
Therefore, we get
$a_{kc}^{n+1} = exp(-\mu_k\tau)a_{kc}^n, a_{ks}^{n+1} = exp(-\mu_k\tau)a_{ks}^n, n = 0, 1, 2, ...,$
or $a_{kc}^n = exp(-\mu_k t_n)a_{kc}^0, a_{ks}^n = exp(-\mu_k t_n)a_{ks}^0$,
$U^n = \sum_{k=1}^{*N_2} exp(-\mu_k t_n)(a_{kc}^0\cos_k + a_{ks}^0\sin_k) + \frac{a_{0c}^0}{2}\cos_0$.

2.8 Heat transfer problem with periodically placed heat source

We consider the heat transfer problem with the periodically placed heat sources at the xz-planes in the y-axes direction. The temperature depends only on y-coordinate and time t. We have the energy equation

$$\rho C_p\frac{\partial T(y,t)}{\partial t} = k\frac{\partial^2 T(y,t)}{\partial y^2} + Q(y,t),$$

where $Q(y,t)$ is the source term.

Let us introduce the following non-dimensional variables:

$$y' = \frac{y}{L_0},\ t = \frac{L_0^2}{\nu}t',\ Pr = \frac{\rho C_p\nu}{k},\ T' = \frac{T - T_\infty}{T_0},\ Q' = \frac{L_0^2}{kT_0}Q.$$

Here L_0, ρ, C_p, k, Q, ν are the half period, density, specific heat at constant pressure, thermal conductivity, intensity of the applied heat source, kinematic viscosity. T_∞, T_w, $T_0 = T_w - T_\infty$ are temperature of the fluid away from the plane surface, temperature of the plane surface, reference temperature, Pr is Prandtl number. We shall consider a heat source of the form $Q = Q_0\delta(y)H(t)$, where δ, H are Dirac δ-function and Heaviside function.

The dimensionless problem for heat transfer equation is in the form (2.1) with $\nu = 1/Pr$, $f(y,t) = Q_0\delta(y - L_0)H(t)/Pr$, $y \in [0, L_0]$, $L = 2L_0$, $T_0 = 0$. The method of lines is in the form (2.2), where $F = \frac{Q_0}{Prh}$, $U(0) = 0$ and A is replaced with $\frac{1}{Pr}A$.

To consider that
$T(y,t) = \sum_{k=1}^{\infty}(a_{kc}(t)\cos\frac{2\pi ky}{L} + a_{ks}(t)\sin\frac{2\pi ky}{L}) + \frac{a_{0c}(t)}{2}$,
$f(y,t) = \sum_{k=1}^{\infty}(b_{kc}(t)\cos\frac{2\pi ky}{L} + b_{ks}(t)\sin\frac{2\pi ky}{L}) + \frac{b_{0c}(t)}{2}$,
$b_{kc}(t) = \frac{2}{L}\int_0^L Q_0\delta(\xi - L_0)/Pr\cos\frac{2\pi k\xi}{L}d\xi = \frac{2Q_0}{LPr}\cos(\pi k),\ b_{ks}(t) = 0$,
where $a_{kc}(t), a_{ks}(t)$ are the corresponding solutions of (2.4) by
$a_{kc}(0) = a_{ks}(0) = 0$, $g_k(t) = b_{kc}(t), \kappa_k = \lambda_k$ or
$a_{kc}(t) = \int_0^t \frac{2Q_0}{LPr}\cos(\pi k)\exp(-\frac{\lambda_k}{Pr}(t-\xi))d\xi = \frac{2Q_0}{LPr}\cos(\pi k)\frac{Pr}{\lambda_k}(1 - \exp(-\frac{\lambda_k}{Pr}t))$, $a_{0c}(t) = \frac{2Q_0}{LPr}t$.

Then the exact solution is in the following form:

$$T(y,t) = \frac{2Q_0}{LPr}\left(\sum_{k=1}^{\infty}\frac{Pr}{\lambda_k}\left(1-\exp\left(-\frac{\lambda_k t}{Pr}\right)\right)\cos(\pi k)\cos\frac{2\pi k y}{L}+\frac{t}{2}\right).$$

where $\lambda_k = (\frac{2\pi k}{L})^2$.

For $L = 4$, $N = 100$, $Q_0 = 1$, $Pr = 0.71; 7.0$, $t = 1$ using Matlab we have following figs. 2.12-2.15. MATLAB code can be found in appendix A.6.

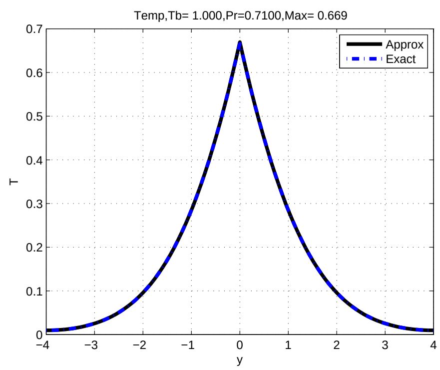

Fig. 2.12 Temperature $T(y,1)$ depending on y by $Pr = 0.71$

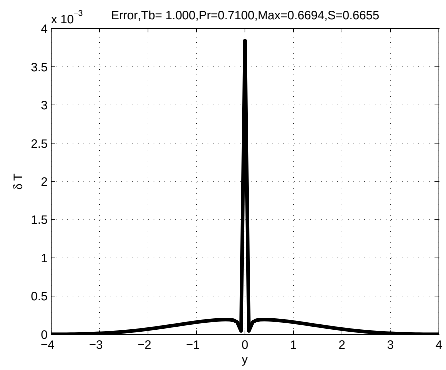

Fig. 2.13 Error depending on y at $t = 1, Pr = 0.71$

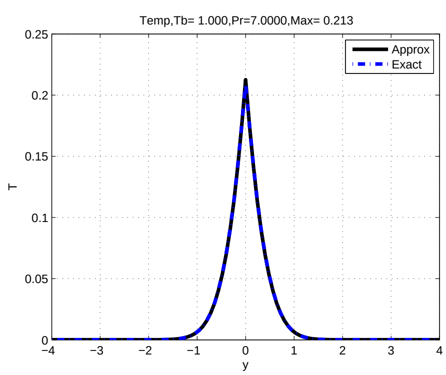

Fig. 2.14 Temperature $T(y,1)$ depending on y by $Pr = 7$

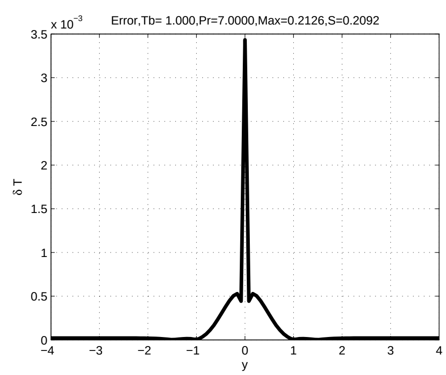

Fig. 2.15 Error depending on y at $t = 1, Pr = 7$

2.9 MHD problem with convectively driven flow past an infinite periodically placed planes

The laminar 1D unsteady flow of an viscous incompressible conducting fluid past a periodically placed xz-planes is considered. Initially, it is assumed that the xz-planes and the fluid are at same temperature T_∞. At $t > 0$ in the xz-planes is raised with the periodically placed heat sources $Q(y,t) = Q_0\delta(y-(2n+1)L_0)H(t)$, $n \in Z$, where δ, H are Dirac δ-function and Heaviside function, L_0 is the half period. An uniform magnetic field is applied in the z-axes direction perpendicular to the velocity vector. The fluid is assumed to be slightly conducting, so that the magnetic Reynolds number is much less than unity and hence the induced magnetic field is negligible in comparison to the applied magnetic field. The motion is due to free convection only. Under these assumptions the governing boundary layer equations for momentum, and energy for free convective flow with Boussinesq approximation are as follows [40]:

$$\begin{cases} \frac{\partial u(y,t)}{\partial t} = \nu \frac{\partial^2 u(y,t)}{\partial y^2} + g\beta(T(y,t) - T_\infty) - \\ \quad - \frac{\sigma B_0^2 u(y,t)}{\rho}, y \in [-L_0, L_0], t > 0, \\ \rho Cp \frac{\partial T(y,t)}{\partial t} = k \frac{\partial^2 T(y,t)}{\partial y^2} + Q(y,t), \\ T(y,0) = T_\infty, u(y,0) = 0, u(-L_0,t) = u(L_0,t) = 0, \\ T(-2L_0,t) = T(0,t), \frac{\partial T(-2L_0,t)}{\partial y} = \frac{\partial T(0,t)}{\partial y}. \end{cases} \tag{2.13}$$

The equations (2.13) are reduced to the following nondimensional form:

$$\begin{cases} \frac{\partial u(y,t)}{\partial t} = \frac{\partial^2 u(y,t)}{\partial y^2} + GrT(y,t) - Mu(y,t), y \in (-L,L), t > 0, \\ Pr \frac{\partial T(y,t)}{\partial t} = \frac{\partial^2 T(y,t)}{\partial y^2} + Q'(y,t), \\ T(y,0) = u(y,0) = 0, u(-L,t) = u(L,t) = 0, \\ T(-2L,t) = T(0,t), \frac{\partial T(-2L,t)}{\partial y} = \frac{\partial T(0,t)}{\partial y}. \end{cases} \tag{2.14}$$

Here the following non dimensional quantities are introduced:

$$\begin{array}{l} y' = \frac{y}{L_0}, t = \frac{L_0^2}{\nu} t', u' = \frac{u}{U}, Pr = \frac{\nu \rho Cp}{k}, \\ M = \frac{\sigma B_0^2 L_0^2}{\rho \nu}, T' = \frac{T - T_\infty}{T_0}, \\ Q' = \frac{L_0^2}{kT_0, Q_0} Q, Gr = \frac{\beta g L_0^2 (T_0 - T\infty)}{\nu U}. \end{array}$$

Here L_0, L, B_0, ρ, σ, β, Cp, k, Q_0, ν, g, u, T_0 are the dimensional and non dimensional values of the half period, transverse magnetic field strength, fluid density, electrical conductivity of fluid, volumetric coefficient of thermal expansion, specific heat at constant pressure, thermal conductivity, intensity of the applied heat source, kinematics viscosity, acceleration due to gravity, x-component of velocity, temperature of the plane surface, Pr, Gr, M are Prandtl, thermal Grashof, Magnetic (Hartman^2) numbers.

We can determine at $y = -L$ the following dimensionless parameters: $\tau = -\frac{\partial u(-L,t)}{\partial y}$ – skin friction, $Nu = -\frac{\partial T(-L,t)}{\partial y}$ – Nusselt number. We shall consider the heat source of the form $Q = Q_0 \delta(y) H(t)$, where δ, H are Dirac δ-function and Heaviside function.

Using the finite differences of second order approximation for partial derivatives respect to y we obtain from (2.14) the initial value problem for system of ODEs in the following matrix form

$$\begin{cases} \dot{U}(t) + AU(t) + MU(t) = GrV(t), U(0) = 0, \\ \dot{V}(t) + \frac{1}{Pr} BV(t) = \frac{1}{Prh} F, V(0) = 0, \end{cases} \tag{2.15}$$

where A is the 3-diagonal matrix of $2N-1$ order in the standard form with elements $\frac{1}{h^2}(-1,2,-1)$ on diagonals, $A(1,2N-1) = A(2N-1,1) = 0$, $y_j = -L + jh$, $y_{2N} = L$, $h = \frac{L}{N}$, $j = \overline{0, 2N}$.

B is the circulant 3-diagonal matrix of $2N$ order in the form $\frac{1}{h^2}[-1, 2, -1]$ with the elements $B(1, 2N) = B(2N, 1) = -1$, $F = (1, 0, 0, \dots)$ is the unit column-vector of the $2N$ order, $U(t)$, $V(t)$, $\dot{U}(t)$, $\dot{V}(t)$ are the column-vectors of $2N-1$ and $2N$ orders with elements $u_j(t) \approx u(y_j, t))$, $v_j(t) \approx T(y_j - L, t)$, $\dot{u}_j(t) \approx \frac{\partial u(y_j,t)}{\partial t}$, $\dot{v}_j(t) \approx \frac{\partial T(y_j - L,t)}{\partial t}$.

For air $Pr = 0.71$ and $N = 100$, $Gr = 100$, $M = 0; 1000$, $L = 2; 4$, $t = 0.5; 1.6; 4.8; 6.4; 8.0$ using Matlab we have following Figs. 2.16—2.19. In Figs. 2.20—2.23 are represented the results obtained by modelling the water for $Pr = 7$.

The maximal values of the temperature by $t = 8$, $L = 2$ is: $\max(T) = 2.15$ by $Pr = 0.71$ and $\max(T) = 0.60$ by $Pr = 7$. For $L = 4$ we have: $\max(T) = 2.067$ by $Pr = 0.71$ and $\max(T) = 0.59$ by $Pr = 7$. For the Nusselt number we have constant value $Nu = 0.32$ undepending on the time.

MATLAB code can be found in appendix A.7.

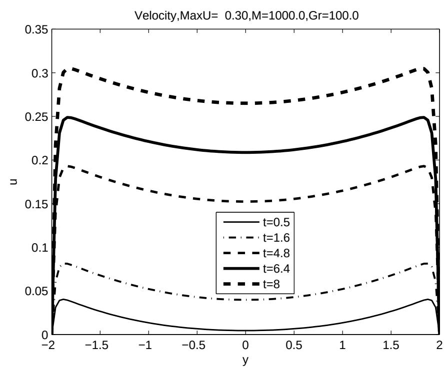

Fig. 2.16 The solutions $u(y, t)$ depending on $y \in [-L, L]$ at different time moments at $L = 2, M = 1000, Pr = 0.71, (\max(u) = 0.31, \tau \in [-7.35, -0.45])$

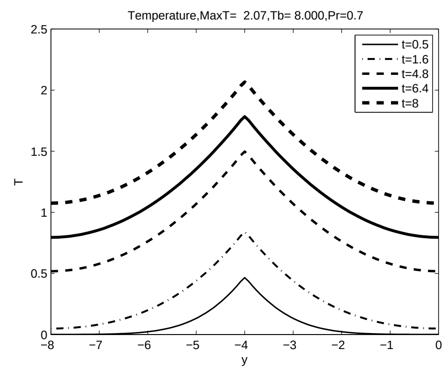

Fig. 2.17 The solutions $T(y, t)$ depending on $y \in [-2L, 0]$ at different time moments$L = 4, Pr = 0.71, (\max(T) = 2.067)$

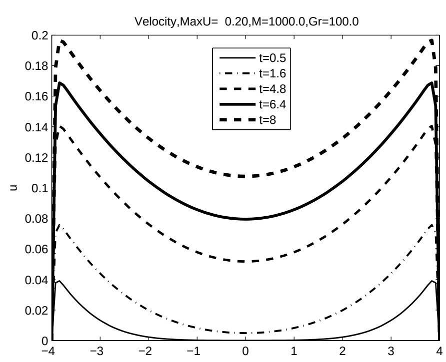

Fig. 2.18 The solutions $u(y, t)$ depending on $y \in [-L, L]$ at different time moments $L = 4, M = 1000, Pr = 0.71, (\max(u) = 0.20, \tau \in [-3.23, -0.3])$

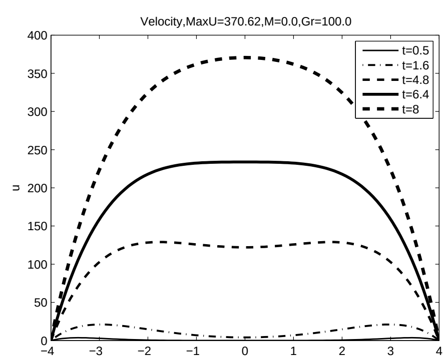

Fig. 2.19 The solutions $u(y, t)$ depending on $y \in [-L, L]$ at different time moments $L = 4, M = 0, Pr = 0.71, (\max(u) = 370.6, \tau \in [-311.6, -3.1])$

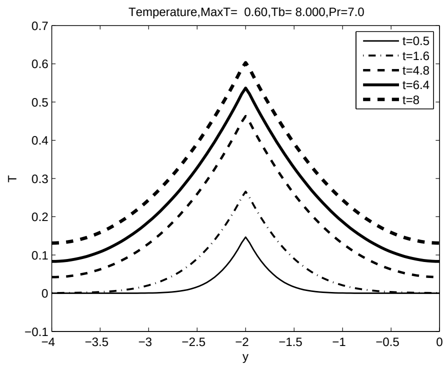

Fig. 2.20 The solutions $T(y,t)$ depending on $y \in [-2L, 0]$ at $L = 2, Pr = 7, (\max(T) = 0.595)$

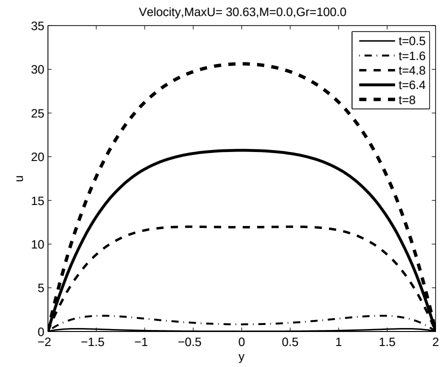

Fig. 2.21 The solutions $u(y,t)$ depending on $y \in [-L, L]$ at $L = 2, M = 0, Pr = 7, (\max(u) = 30.6, \tau \in [-48.4, -0.49])$

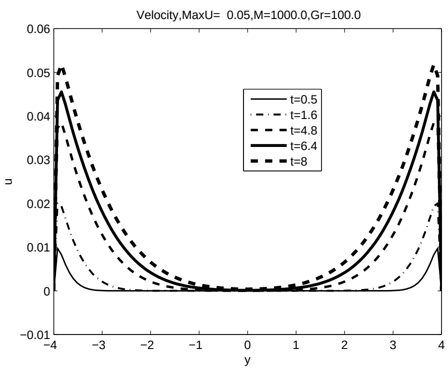

Fig. 2.22 The solutions $u(y,t)$ depending on y at different time moments $L = 4, M = 1000, Pr = 7, (\max(u) = 0.05, \tau \in [-0.9, -0.06])$

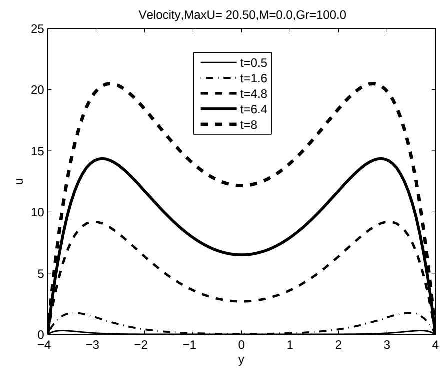

Fig. 2.23 The solutions $u(y,t)$ depending on y at different time moments $L = 4, M = 0, Pr = 7, (\max(u) = 20.49, \tau \in [-41.9, -0.4])$

2.10 System of parabolic type equations

We consider the initial-boundary problem of linear M-order system in the following form:

$$\begin{cases} \frac{\partial T_m(x,t)}{\partial t} = \sum_{s=1}^{M} \frac{\partial}{\partial x}(k_{m,s}\frac{\partial T_m(x,t)}{\partial x}) + f_m(x,t), x \in (0,L), t \in (0,t_f), \\ T_m(0,t) = T_m(L,t), \frac{\partial T_m(0,t)}{\partial x} = \frac{\partial T_m(L,t)}{\partial x}, t \in (0,t_f), \\ T_m(x,0) = T_{m,0}(x), x \in (0,L), m = \overline{1,M}, \end{cases} \tag{2.16}$$

where $\tilde{K}$ is the positive definite matrix with the elements $k_{m,s}$,$T_{m,0}, f_m(x,t)$ are given functions.
This system we can rewritten in the matrix form

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}(\tilde{K}\frac{\partial u(x,t)}{\partial x}) + f(x,t), x \in (0,L), t \in (0,t_f), \\ u(0,t) = u(L,t), \frac{\partial u(0,t)}{\partial x} = \frac{\partial u(L,t)}{\partial x}, t \in (0,t_f), \\ u(x,0) = u_0(x), x \in (0,L), \end{cases} \tag{2.17}$$

where u, f are column-vectors with elements$T_m, f_m, m = \overline{1, M}$.
Using the Fourier series we can obtain the solution in the following form similar to (2.5):
$u(x,t) = \sum_{k=1}^{\infty}(a_{kc}(t)\cos\frac{2\pi kx}{L} + a_{ks}(t)\sin\frac{2\pi kx}{L}) + \frac{a_{0c}(t)}{2}$,
$f(x,t) = \sum_{k=1}^{\infty}(b_{kc}(t)\cos\frac{2\pi kx}{L} + b_{ks}(t)\sin\frac{2\pi kx}{L}) + \frac{b_{0c}(t)}{2}$,
$b_{kc}(t) = \frac{2}{L}\int_0^L f(\xi,t)\cos\frac{2\pi k\xi}{L}d\xi, b_{ks}(t) = \frac{2}{L}\int_0^L f(\xi,t)\sin\frac{2\pi k\xi}{L}d\xi$,
where the column-vectors $a_{kc}(t), a_{ks}(t)$ of the M order are the corresponding solutions of the following differential equations

$$\begin{cases} \dot{a}_{kc}(t) + \lambda_k\tilde{K}a_{kc}(t) = b_{kc}(t), a_{kc}(0) = \frac{2}{L}\int_0^L u_0\xi)\cos\frac{2\pi k\xi}{L}d\xi, \\ \dot{a}_{ks}(t) + \lambda_k\tilde{K}a_{ks}(t) = b_{ks}(t), a_{ks}(0) = \frac{2}{L}\int_0^L u_0\xi)\sin\frac{2\pi k\xi}{L}d\xi, \end{cases} \tag{2.18}$$

where $b_{kc}(t), b_{ks}(t)$ are the column-vectors of M order.
The solution of this system is the vector functions

$$\begin{cases} a_{kc}(t) = \exp(-\lambda_k\tilde{K}t)a_{kc}(0) + \int_0^t exp(-\lambda_k\tilde{K}(t-\tau))b_{kc}(\tau)d\tau, \\ a_{ks}(t) = \exp(-\lambda_k\tilde{K}t)a_{ks}(0) + \int_0^t exp(-\lambda_k\tilde{K}(t-\tau))b_{ks}(\tau)d\tau. \end{cases} \tag{2.19}$$

For the **discrete problem** ($O(h^{2n})$ order of approximation) we have the system of ODEs in the following form:

$$\begin{cases} \dot{v}_j(t) = \tilde{K}\Lambda v(t) + f_j(t), t \in [0, t_f], \\ v_j(0) = u_0(x_j), x_j = jh, Nh = L, j = \overline{1, N}, \end{cases} \tag{2.20}$$

where the column-vectors of the M order $v_j(t) \approx u(x_j,t), f_j(t) = f(x_j,t)$,
the expression of the finite difference operator in multi-point stencil with 2n+1-points $\Lambda v_j = \frac{1}{h^2}(C_n(v_{j-n} + v_{j+n}) + ... + C_1(v_{j-1} + v_{j+1}) + C_0 v_j), C_0 = -\sum_{p=1}^n C_p$, $C_p = \frac{2(n!)^2(-1)^{p-1}}{p^2(n-p)!(n+p)!}, p = \overline{1, n}$. We have following matrix representation for circulant matrix $(A = -\Lambda) = -\frac{1}{h^2}[C_0, C_1, \ldots, C_n, 0, \ldots, 0, C_n, \ldots, C_1]$, with the eigenvalues $\mu_k = \frac{4}{h^2}\sum_{m=1}^n P_m \sin^{2m}(\pi k/N), P_m = \frac{2((m-1!))^2 4^{m-1}}{(2m)!}$.
Using the discrete Fourier method we can obtain the solution in the following form:
$v_j(t) = \sum_{k=1}^{*N_2}(a_{kc}(t)\cos\frac{2\pi kj}{N} + a_{ks}(t)\sin\frac{2\pi kj}{N}) + \frac{a_{0c}(t)}{2}$,
$f_j(t) = \sum_{k=1}^{*N_2}(b_{kc}(t)\cos\frac{2\pi kj}{N} + b_{ks}(t)\sin\frac{2\pi kj}{N}) + \frac{b_{0c}(t)}{2}$,
$b_{kc}(t) = \frac{2}{N}\sum_{j=1}^N f_j(t)\cos\frac{2\pi kj}{N}, b_{ks}(t) = \frac{2}{N}\sum_{j=1}^N f_j(t)\sin\frac{2\pi kj}{N}$,
where $a_{kc}(t), a_{ks}(t)$ are the corresponding solutions of (2.18, 2.19, λ_k is replaced with μ_k) with $a_{kc}(0) = \frac{2}{N}\sum_{j=1}^N u_0(x_j)\cos\frac{2\pi kj}{N}, a_{ks}(0) = \frac{2}{N}\sum_{j=1}^N u_0(x_j)\sin\frac{2\pi kj}{N}$.
For FDSES, by replacing the discrete eigenvalue μ_k with λ_k we obtain the exact solutions for initial data with the frequency $\leq N/2$.

2.11 System of parabolic type equations with convection

We consider the initial - boundary problem of linear M-order system in the following form:

$$\begin{cases} \frac{\partial T_m}{\partial t} = \sum_{s=1}^M(\frac{\partial}{\partial x}(k_{m,s}\frac{\partial T_m}{\partial x}) + p_{m,s}\frac{\partial T_m}{\partial x}) + f_m, \\ T_m(0,t) = T_m(L,t), \frac{\partial T_m(0,t)}{\partial x} = \frac{\partial T_m(L,t)}{\partial x}, \\ T_m(x,0) = T_{m,0}(x), x \in (0,L), m = \overline{1, M}, \end{cases} \tag{2.21}$$

where $\tilde{K}$ is the positive definite M-order matrix with different positive eigenvalues $\mu_K > 0$ and the elements $k_{m,s}$, P is the real M-order matrix with different real eigenvalues μ_P and the elements $p_{m,s}, m, s = \overline{1, M}$..
This system can be rewritten in the matrix form

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}(\tilde{K}\frac{\partial u(x,t)}{\partial x}) + P\frac{\partial u(x,t)}{\partial x} + f(x,t), \\ u(0,t) = u(L,t), \frac{\partial u(0,t)}{\partial x} = \frac{\partial u(L,t)}{\partial x}, t \in (0, t_f), \\ u(x,0) = u_0(x), x \in (0, L), \end{cases} \tag{2.22}$$

where u, f are column-vectors with elements$T_m, f_m, m = \overline{1, M}$.
Using the Fourier series the solution can be obtained in the following form:
$u(x,t) = \sum_{k=1}^{\infty}(a_{kc}(t)\cos\frac{2\pi kx}{L} + a_{ks}(t)\sin\frac{2\pi kx}{L}) + \frac{a_{0c}(t)}{2}$,
$f(x,t) = \sum_{k=1}^{\infty}(b_{kc}(t)\cos\frac{2\pi kx}{L} + b_{ks}(t)\sin\frac{2\pi kx}{L}) + \frac{b_{0c}(t)}{2}$,
$b_{kc}(t) = \frac{2}{L}\int_0^L f(\xi,t)\cos\frac{2\pi k\xi}{L}d\xi, b_{ks}(t) = \frac{2}{L}\int_0^L f(\xi,t)\sin\frac{2\pi k\xi}{L}d\xi$,
where the column-vectors $a_{kc}(t), a_{ks}(t)$ of the M order are the corresponding solutions of the following differential equations similar to (2.8)

$$\begin{cases} \dot{a}_{kc}(t) + \lambda_k\tilde{K}a_{kc}(t) - \frac{2\pi k}{L}Pa_{ks}(t) = b_{kc}(t), \\ \dot{a}_{ks}(t) + \lambda_k\tilde{K}a_{ks}(t) + \frac{2\pi k}{L}Pa_{kc}(t) = b_{ks}(t), \end{cases} \tag{2.23}$$

where $b_{kc}(t), b_{ks}(t)$ are the column-vectors of M order, $a_{kc}(0) = \frac{2}{L}\int_0^L u_0\xi)\cos\frac{2\pi k\xi}{L}d\xi$,, $a_{ks}(0) = \frac{2}{L}\int_0^L u_0\xi)\sin\frac{2\pi k\xi}{L}d\xi$. The solution of this system we can obtain in the following form:

$$a_k(t) = \exp(-Rt)a_k(0) + \int_0^t exp(-R(t-\tau))b_k(\tau)d\tau, \tag{2.24}$$

where a_k, b_k are the column-vectors and $R = \begin{pmatrix} \lambda_k\tilde{K} & -\frac{2\pi k}{L}P \\ \frac{2\pi k}{L}P & \lambda_k\tilde{K} \end{pmatrix}$ the matrix of the $2M$ order.
For the **discrete problem** ($O(h^{2n})$ order of approximation) we have the system of ODEs in the following form:

$$\begin{cases} \dot{v}_j(t) = \tilde{K}\Lambda v_j(t) + P\Lambda^0 v_j(t) + f_j(t), t \in [0, t_f], \\ v_j(0) = u_0(x_j), x_j = jh, Nh = L, j = \overline{1, N}, \end{cases} \tag{2.25}$$

where the column-vectors of the M order $v_j(t) \approx u(x_j,t), f_j(t) = f(x_j,t)$,
the expressions of the finite difference operators in multi-point stencil with 2n+1-points ($N \geq 2n+1$) $\Lambda v_j = \frac{1}{h^2}(C_n(v_{j-n}+v_{j+n})+\cdots+C_1(v_{j-1}+v_{j+1})+C_0 v_j), C_0 = -\sum_{p=1}^n C_p$,
$C_p = \frac{2(n!)^2(-1)^{p-1}}{p^2(n-p)!(n+p)!}, p = \overline{1, n}$
$\Lambda^0 v_j = \frac{1}{h}(c_n(v_{j+n} - v_{j-n}) + \cdots + c_1(v_{j+1} - v_{j-1}))$, $c_p = \frac{(n!)^2(-1)^{p-1}}{p(n-p)!(n+p)!}, p = \overline{1, n}$.
We have the following matrix representation for circulant matrices $A = -\Lambda = -\frac{1}{h^2}[C_0, C_1, \ldots, C_n, 0, \ldots, 0, C_n, \ldots, C_1]$, with the eigenvalues $\mu_k = \frac{4}{h^2}\sum_{p=1}^n Q_p \sin^{2p}(\pi k/N), Q_p = \frac{2((p-1!))^2 4^{p-1}}{(2p)!}$ and $A^0 = \Lambda^0 = \frac{1}{h}[0, c_1, \ldots, c_n, 0, \ldots, 0, -c_n, \ldots, -c_1]$, with the eigenvalues $\mu_k^0 = \frac{2i}{h}\sum_{p=1}^n c_p \sin\frac{2\pi pk}{N}$.
Using the discrete Fourier method the solution can be obtained in the following form:
$v_j(t) = \sum_{k=1}^{*N_2}(a_{kc}(t)\cos\frac{2\pi kj}{N} + a_{ks}(t)\sin\frac{2\pi kj}{N}) + \frac{a_{0c}(t)}{2}$,

$f_j(t) = \sum_{k=1}^{*N_2} (b_{kc}(t) \cos \frac{2\pi kj}{N} + b_{ks}(t) \sin \frac{2\pi kj}{N}) + \frac{b_{0c}(t)}{2}$,
$b_{kc}(t) = \frac{2}{N} \sum_{j=1}^{N} f_j(t) \cos \frac{2\pi kj}{N}, b_{ks}(t) = \frac{2}{N} \sum_{j=1}^{N} f_j(t) \sin \frac{2\pi kj}{N}$,
where $a_{kc}(t), a_{ks}(t)$ are the corresponding solutions of (2.23,2.24 and $\lambda_k, \frac{2\pi k}{L}$ are replaced with $\mu_k, Im(\mu_k^0)$, $a_{kc}(0) = \frac{2}{N} \sum_{j=1}^{N} u_0(x_j) \cos \frac{2\pi kj}{N}, a_{ks}(0) = \frac{2}{N} \sum_{j=1}^{N} u_0(x_j) \sin \frac{2\pi kj}{N}$.
We can also obtain this real form from the following expressions
$A \cos_k = \mu_k \cos_k, A^0 \cos_k = -|\mu_k^0| \sin_k, A \sin_k = \mu_k \sin_k, A^0 \sin_k = |\mu_k^0| \cos_k$, where $\sin_k, \cos_k$ are N-order column-vectors with the elements $\sin \frac{2\pi kj}{N}, \cos \frac{2\pi kj}{N}$ and using the orthonormal conditions
$\sum_{j=1}^{N} \sin_k \cos_s = \sum_{j=1}^{N} \sin_k \sin_s = \sum_{j=1}^{N} \cos_k \cos_s = \frac{N}{2} \delta_{k,s}$.
Then for fixed frequency k in the initial data the solution can be written in the form $u(t) = d_s(t) \sin_k + d_c(t) \cos_k$, vector-functions where $d_s(t), d_c(t)$ are unknown the time-depending vector-functions. Then $\dot{u}(t) = \dot{d}_s(t) \sin_k + \dot{d}_c(t) \cos_k$,
$Au(t) = \mu_k(d_s(t) \sin_k + d_c(t) \cos_k), A^0 u(t) = |\mu_k^0|(d_s(t) \cos_k - d_c(t) \sin_k)$.

For FDSES, replaced the discrete eigenvalue $\mu_k, Im(\mu_k^0)$ with $\lambda_k, \frac{2\pi k}{L}$ we obtain the exact solutions for initial data with the frequency $\leq N/2$.

2.12 Stability of approximations for time-dependent problems

The time-dependent difference equation (2.25) using the MN-order column-vector $v(t)$ with the elements $v_j^m(t), j = \overline{1, N}, m = \overline{1, M}$ in the form

$$\dot{v}(t) = (-\tilde{K} \bigotimes A + P \bigotimes A^0) v(t) + f(t) \tag{2.26}$$

serves as an approximation to the differential problem (2.22), in the sense that any smooth solution $u(t)$ satisfies the approximation (2.25) modulo a small local truncation error $\Psi(h,t) = O(h^{2n})$:

$$\frac{\partial u(t)}{\partial t} = (-\tilde{K} \bigotimes A) u(t) + (P \bigotimes B) u(t) + f(t) + \Psi(h,t), \tag{2.27}$$

where MN-order matrices

$$\tilde{K} \bigotimes A = \begin{pmatrix} k_{1,1}A & \cdots & k_{1,M}A \\ \cdots & \cdots & \cdots \\ k_{M,1}A & \cdots & k_{M,M}A \end{pmatrix}, P \bigotimes A^0 = \begin{pmatrix} p_{1,1}A^0 & \cdots & p_{1,M}A^0 \\ \cdots & \cdots & \cdots \\ p_{M,1}A^0 & \cdots & p_{M,M}A^0 \end{pmatrix},$$

are Kronecker tensor product, $u(t), u(0), f(t)$ are MN column-vectors with the elements $u_j^m(t), u_j^m(0), f_j^m, m = \overline{1, M}, j = \overline{1, N}$.
Matrices can be defined with the representation $A = WDW^*, A^0 = WD^0W^*$, and solved numerically with the Matlab operator "kron",
In order to link the local order of accuracy with he desired global convergence rate of the approximation, one has to verify stability. We say that approximation (2.25) is stable, if for all sufficiently small h the following estimate holds

$$||\exp(Bt)|| \leq C_{t_f}, 0 \leq t \leq t_f, C_{t_f}, \tag{2.28}$$

where $B = -\tilde{K} \bigotimes A + P \bigotimes A^0, C_{t_f} > 0-$ is constant.
If eigenvalues of matrices $\tilde{K}, P$ are $\lambda_s(\tilde{K}) > 0, \lambda_s(P), s = \overline{1, M}$ then the transformation of M-order matrices W_K, W_P, and the representation $\tilde{K} = W_K D_K W_K^{-1}, P = W_P D_P W_P^{-1}$, exist where $D_K = diag(\lambda_s(\tilde{K})), D_P = diag(\lambda_s(P))$ are the diagonal matrices. From properties of Kroneker tensor product follows $(W^* = W^{-1})$:
$B = -\tilde{K} \bigotimes A + P \bigotimes A^0 = -(W_K D_K W_K^{-1} \bigotimes W D W^*) + (W_P D_P W_P^{-1} \bigotimes W D^0 W^*) =$
$(W_K \bigotimes W)(-D_K \bigotimes D)(W_K^{-1} \bigotimes W^*) + (W_P \bigotimes W)(-D_P \bigotimes D^0)(W_P^{-1} \bigotimes W^*)=$
$(W_K \bigotimes W)(-D_K \bigotimes D)(W_K \bigotimes W)^{-1} + (W_P \bigotimes W)(-D_P \bigotimes D^0)(W_P \bigotimes W)^{-1}.$
The eigenvalues $\lambda(B)$ of matrix B are $-\mu_k \lambda_s(\tilde{K}) + \mu_k^0 \lambda_s(P), k = \overline{1, N}, s = \overline{1, M}$ with the $Re(\lambda(B)) \leq 0$ and the system of ODEs is stable.
For the approximation, if $v_j^m(t) = T_m(x_j, t), m = \overline{1, M}$ and for every time moment t
$u_j'' = \Lambda u_j + E_{2n} \frac{h^{2n} u^{(2n+2)}(\xi_j)}{2n+2)!}, E_{2n} = -2 \sum_{k=1}^n C_k k^{2n+2}$
$u_j' = \Lambda^0 u_j + e_{2n} \frac{h^{2n} u^{(2n+1)}(\xi_j)}{2n+1)!}, e_{2n} = -2 \sum_{k=1}^n c_k k^{2n+1}, x_{n-j} < \xi_j < x_{n+j},$
then $\dot{u}(t) = Bv(t) + f(t) + \Psi(h,t)$,where v, f, Ψ are MN-order column- vectors and $\Psi(h,t) = O(h^{2n})$,or $||\Psi(h,t)|| \leq h^{2n}(\frac{|E_{2n}|}{(2n+2)!}||\tilde{K}||M_{2n+2}(t) + \frac{|e_{2n}|}{(2n+1)!}||P||M_{2n+1}(t))$.
$M_s = \max |\frac{\partial^s T(x,t)}{\partial x^s}$ is the maximal estimate for corresponding derivatives.
Given stability we can now estimate the global error $e(t) = v(t) - u(t)$ and find, that the error $e(t)$ is governed by the error equation

$$\frac{\partial e(t)}{\partial t} = Be(t) + \Psi(h,t).$$

The solution of this equation is given by $e(t) = \exp(Bt)e(0) + \int_0^t \exp(B(t-\xi))\Psi(h,\xi)d\xi$.
From $Re(\lambda(B)) \leq 0$ follows that $||\exp(Bt)|| \leq C_{t_f}$ and

$$||e(t)|| \leq C_{t_f}(||e(0)|| + \sup_{0<\tau<t_f} ||\Psi(h,\tau)||) = O(h^{2n}).$$

Thus, if both $||\Psi||$ and$||e(0)||$ are of order $O(h^{2n})$, then stability will retain the 2nd order of convergence rate later on $||e(t)|| = O(h^{2n})$.

Example: $M = 2, L = 1, T_b = 0.1, T_0^1(x) = B_{1,1}\sin(2q\pi x) + B_{1,2}\cos(2q\pi x), T_0^2(x) = B_{2,1}\sin(2q\pi x) + B_{2,2}\cos(2q\pi x),$
$f^1(x,t) = A_{1,1}(t)\sin(2q\pi x) + A_{1,2}(t)\cos(2q\pi x), f^2(x,t) = A_{2,1}(t)\sin(2q\pi x) + A_{2,2}(t)\cos(2q\pi x), q = \frac{k_0}{L},$
$B_{1,1} = 0, B_{1,2} = 1, B_{2,1} = -1, B_{2,2} = 0, A_{1,1} = 5, A_{1,2} = 10, A_{2,1} = -10, A_{2,2} = -5$. We can consider two 2nd order matrices

$$\tilde{B} = \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix}, \tilde{A} = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}$$

Matrices

$$\tilde{K} = \begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix}, P = \begin{pmatrix} 2 & -5 \\ 1 & -4 \end{pmatrix}$$

are with eigenvalues $\lambda_{\tilde{K}} = (1; 5), \lambda_P = (1; -3)$
. Considered the solution in the form $T_1(x,t) = d_{1,2}(t)\sin(2q\pi x) + d_{1,2}(t)\cos(2q\pi x)$,
$T_2(x,t) = d_{2,1}(t)\sin(2q\pi x) + d_{2,2}(t)\cos(2q\pi x)$, for fixed frequency $q = \frac{k_0}{L}$ we can determine the functions $d_{1,1}, d_{2,1}, d_{2,1}, d_{2,2}$ from 4 ODEs:

$$\dot{d}(t) = A_2 d(t) + F, d(0) = d_0, \tag{2.29}$$

where the 4-order matrix

$$A_2 = \begin{pmatrix} -4\pi^2 q^2 \tilde{K} & -2\pi q P \\ 2\pi q P & -4\pi^2 q^2 \tilde{K} \end{pmatrix}$$

and 4th order column-vectors $d = (d_{1,1}, d_{2,1}, d_{1,2}, d_{2,2})^T, d(0) = (B_{1,1}, B_{2,1}, B_{1,2}, B_{2,2})^T, F = (A_{1,1}, A_{2,1}, A_{1,2}, A_{2,2})^T$.
This system can be solved with Matlab solver "ode15s" and the solutions are compared with the approximate solutions.
Discrete solution is obtained from (2.26), where $u(0), f(t)$ are 2N-order column-vectors, where determine the column of Nx2 matrices $A_{sc}\tilde{B}^T$ and $A_{sc}\tilde{A}^T$. Here A_{sc} is Nx2 matrix with the two column of the elements $\sin(2\pi x_j), \cos(2\pi x_j), j = \overline{1, N}, q = 1$.

See Matlab program `PDSper2` in appendix A.8.

For maximal errors for $u^1(t), u^2(t)$ depending on the order of approximation we have:
$1)N = 20, 0.0017; 0.0008(O(h^2)), 0.0028; 0.0011(O(h^4)), 0.0016; 0.0006(O(h^6)),$
$2.110^{-7}; 2.610^{-7}(O(h^8)), 9.910^{-8}; 9.610^{-8}(FDSES),$
$2)N = 10, 0.0071; 0.0032(O(h^2)), 0.0113; 0.0043(O(h^4)), 8.310^{-6}; 8.510^{-6}(O(h^6)),$
$5.610^{-8}; 6.110^{-7}(O(h^8)), 5.310^{-8}; 2.010^{-8}(FDSES),$
$3)N = 80, 1.110^{-4}; 5.010^{-5}(O(h^2)), 1.810^{-4}; 6.910^{-5}(O(h^4)), 0.0022; 9.010^{-4}(O(h^6)),$
$2.710^{-7}; 1.710^{-7}(O(h^8)), 1.010^{-7}; 2.310^{-8}(FDSES).$
In the Figs. the behavior of coefficients $d(t)$ in the time, the form of the solutions by $t = t_f = 0.1$ are represented.

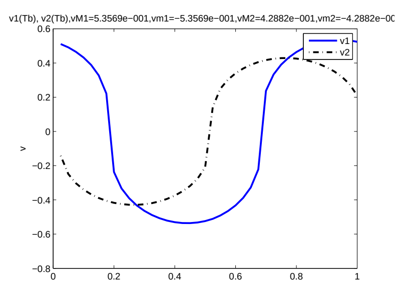

Fig. 2.24 Solutions by $T_b = 0.1, N = 40$ depending on x

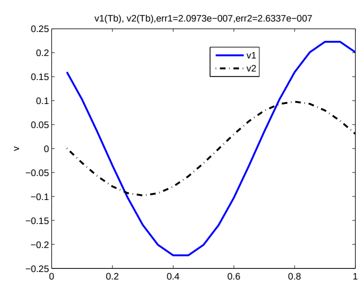

Fig. 2.25 Solutions by $t = 0.1$

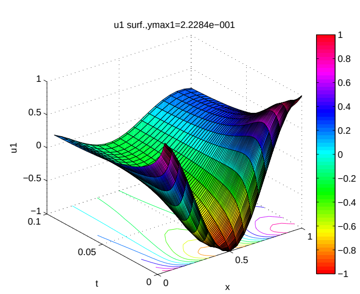

Fig. 2.26 Solution $u_1(x, t)$

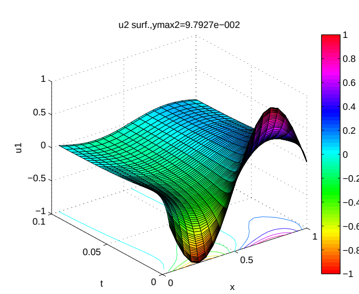

Fig. 2.27 Solution $u_2(x, t)$

The Matlab program in appendix A.9 calculates the real form for data with fixed frequency q of oscillations $L = 1; 10, k_0 = 1; 8; 10; 15$ (the element $A_{2,1} = -1000$ in the matrix $\tilde{A}$ is changed).

For maximal errors for $u^1(t), u^2(t), N = 20$ (the stationary solutions) depending on the order of approximation we have:
1)$L = 1, k_0 = 1, t_f = 0.2,\ 0.0017; 0.0007(O(h^2)), 8.3 10^{-6}; 7.9 10^{-6}(O(h^4)),$
$3.9 10^{-8}; 1.3 10^{-7}(O(h^6)), 1.8 10^{-8}; 6.4 10^{-9}(O(h^8)), 0; 0(FDSES),$
2)$L = 1, k_0 = 8, t_f = 0.05,\ 0.0023; 0.0011(O(h^2)), 0.00074; 0.00040(O(h^4)),$
$4.1 10^{-4}; 2.3 10^{-4}(O(h^6)), 2.6 10^{-4}; 1.6 10^{-4}(O(h^8)), 0; 0(FDSES),$
3)$L = 10, k_0 = 8, t_f = 0.3,\ 0.1277; 0.0861(O(h^2)), 0.033; 0.039(O(h^4)),$
$0.0077; 0.0235(O(h^6)), 0.0059; 0.0180(O(h^8)), 1.1 10^{-16}; 2.8 10^{-17}(FDSES).$
4)$L = 10, k_0 = 10, t_f = 0.3,\ 0.1308; 0.0240(O(h^2)), 0.049; 0.025(O(h^4)),$
$0.0212; 0.025(O(h^6)), 0.0065; 0.025(O(h^8)), 0; 0(FDSES).$
5)$L = 1, k_0 = 15, t_f = 0.5,\ 0.3247; 0.2874(O(h^2)), 0.2179; 0.2208(O(h^4)),$
$0.1841; 0.2017(O(h^6)), 0.1704; 0.1945(O(h^8)), 2.7 10^{-17}; 6.9 10^{-8}(FDSES).$
Interesting results are obtained for hyperbolic system ($\tilde{K} = 0$), see Figs. 2.28, 2.29 for $N = 20, L = 1, k_0 = 10, t_f = 1.5; 2$. We have periodic solution in the time with the period $T_p = \frac{1}{q} = \frac{L}{k_0}$.
In this case the matrix A_2 has 4 imaginary eigenvalues $\pm i\frac{2\pi k_0}{L}, \pm 3i\frac{2\pi k_0}{L}$ (the characteristic polynomial is $z^4 + 10z^2 + 9, \lambda = \frac{2\pi k_0}{L}$). The analytical solution of ODEs (2.29) is in the form $d(t) = \exp(A_2 t)d(0) + A_2^{-1}(\exp(A_2 t) - E)F$, where E is 4th order unit matrix. For the periodical solution $d(T) = d(0)$ follows that the matrix function $\exp(A_2 T_p) = E$ does not depend on F. Using the eigenvalues of matrix A_2 we can obtain $\exp(A_2 T) = a_0 E + a_1 A_2 + a_2 A_2^2 + a_3 A_2^3$, where the unknown coefficients $a_k, k = 0, 1, 2, 3$ are determined from the following equations:
$\cos(2\tilde{q}T_p) = a_0 - 4a_2\tilde{q}^2, \sin(2\tilde{q}T_p) = 2a_1\tilde{q} - 8a_3\tilde{q}^3, \cos(6\tilde{q}T_p) = a_0 - 36a_2\tilde{q}^2, \sin(6\tilde{q}T_p) = 6a_1\tilde{q} - 108a_3\tilde{q}^3,$
where $\tilde{q} = \pi q = \frac{\pi k_0}{L}$. From $a_0 = 1, a_1 = a_2 = a_3 = 0$ follows that $\cos(2\tilde{q}T_p) = 1, \sin(2\tilde{q}T_p) = 0$ or $T_p = \frac{2\pi}{2\tilde{q}} = \frac{L}{k_0}$. For $N = 20, L = 1, k_0 = 1, T_p = t_f = 1$ depending on the order of approximation we have the following maximal errors:
$0.630; 0.402(O(h^2)), 0.0134; 0.0083(O(h^4)), 2.8 10^{-4}; 1.7 10^{-4}(O(h^6)),$
$5.9 10^{-6}; 3.4 10^{-6}(O(h^8)), 1.6 10^{-7}; 1.5 10^{-7}(O(h^{20})), 1.1 10^{-16}; 4.4 10^{-16}(FDSES).$
For the complex expressions (Matlab program **PDSper2**) these errors are greater:
$2.9 10^{-4}; 1.8 10^{-4}(O(h^6)), 2.1 10^{-5}; 1.8 10^{-5}(O(h^8)),$
$1.5 10^{-5}; 1.5 10^{-5}(O(h^{20})), 1.4 10^{-5}; 1.3 10^{-5}(FDSES).$
For approximation of the first order derivative with the order $O(h^{20})$ we use the following Matlab code:

```
lk1=2/h*sin(2*pi*h*NT).*(0.5  +1/3*(sin(pi*h*NT)).^2+4/15*(sin(
    pi*h*NT)).^4  +8/35*(sin(pi*h*NT)).^6+(prod(1:5))^2*4^4/(5*
    prod(1:10))*(sin(pi*h*NT)).^8  +(prod(1:6))^2*4^5/(6*prod
    (1:12))*(sin(pi*h*NT)).^(10)+(prod(1:7))^2*4^6/(7*prod(1:14)
    )*(sin(pi*h*NT)).^(12)+  (prod(1:8))^2*4^7/(8*prod(1:16))*(
    sin(pi*h*NT)).^(14)+(prod(1:9))^2*4^8/(9*prod(1:18))*(sin(pi
    *h*NT)).^(16)+(prod(1:10))^2*4^9/(10*prod(1:20))*(sin(pi*h*
    NT)).^(18));
```

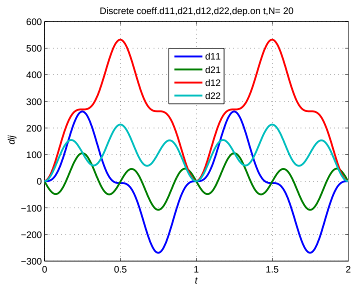

Fig. 2.28 Solutions of vector d(t) by $t_f = 2, \tilde{K} = 0, k_0 = L = 10$ depending on t

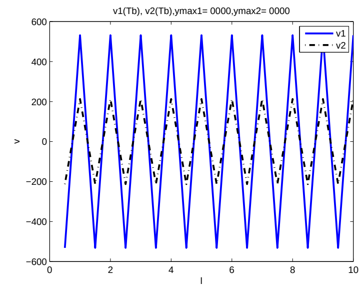

Fig. 2.29 Solutions by $t_f = 1.5, \tilde{K} = 0, k_0 = L = 10$ depending on x

2.13 System of nonlinear parabolic type equations

We consider the nonlinear system of M-heat transfer equation with periodical BCs in the following form:

$$\begin{cases} \frac{\partial T_m}{\partial t} = \sum_{s=1}^{M} \left(\frac{\partial}{\partial x}\left(k_{m,s}\frac{\partial g_{1,m}(T_m)}{\partial x}\right) + p_{m,s}\frac{\partial g_{2,m}(T_m)}{\partial x}\right) + f_m g_{3,m}(T_m), \\ T_m(x,0) = T_{m,0}(x), x \in (0,L), m = \overline{1,M} \end{cases} \tag{2.30}$$

or

$$\frac{\partial T(x,t)}{\partial t} = \tilde{K}\frac{\partial^2 g_1(T(x,t))}{\partial x^2} + P\frac{\partial g_2(T(x,t))}{\partial x} + f(x)g_3(T(x,t)), \tag{2.31}$$

where $g_1(T), g_2(T), g_3(T)$ are the M-order column-vectors with the elements $g_{1,m}(T_m), g_{2,m}(T_m), g_{3,m}(T_m)$ $\overline{1,M}$,
We have following discrete form

$$\dot{u}(t) = (-\tilde{K} \bigotimes A)g_1(u(t)) + (P \bigotimes A^0)g_2(u(t)) + f g_3(u(t)). \tag{2.32}$$

As an example for $M = 2$ we consider nonlinear power functions $g_{1,1} = T^{\alpha_1}, g_{1,2} = T^{\alpha_2}, g_{2,1} = T^{\beta_1}, g_{2,2} = T^{\beta_2}$,
$g_{3,1} = T^{\gamma_1}, g_{3,2} = T^{\gamma_2}$.
See MATLAB program in appendix A.10.

1) If $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 3, \gamma_1 = \gamma_2 = 2$ (see figs. with $t_f = 10$, we have stationary symmetric, periodic oscillations in the space), then for the maximal and minimal values of solutions u^1, u^2 (the minimal value is equal to maximal with opposite sign) depending on t and the values of the solutions depending on x by $t = t_f$, we have:
$N = 40$:
$0.17752; 0.10843(O(h^2)), 0.17755; 0.10847(O(h^4)), 0.17686; 0.10791(O(h^6))$,
$0.17715; 0.10822(O(h^8)), 0.17715; 0.10822(FDSES)$,
if N=80, then for FDSES and $O(h^8)$: 0.17723; 0.10827, but for $O(h^2)$ 0.17732; 0.10832.

2) In the next figs. for $\gamma_1 = \gamma_2 = 0, t_f = 0.1$ (solutions tend fast to stationary).
If $P = 0$ (without convection), we obtain fig. 2.24.

3) In the next Figs. 2.34–2.37 for $\gamma_1 = \gamma_2 = 2, T_b = 10, L = 2; 3; 4$ (for L=3 solutions tends slowly to stationary only by $t_f = 20$.)

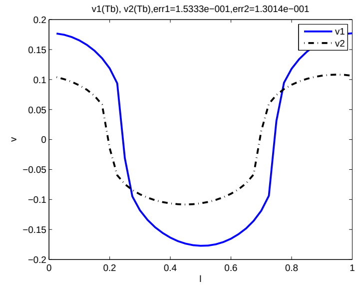

Fig. 2.30 Solutions by $t_f = 10, N = 40$ depending on x

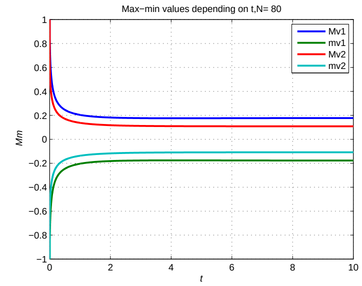

Fig. 2.31 Maximal and minimal values depending on t

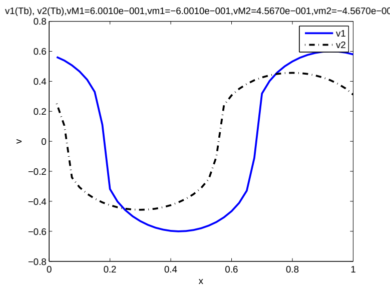

Fig. 2.32 Solutions by $t_f = 0.1, N = 40$ depending on x

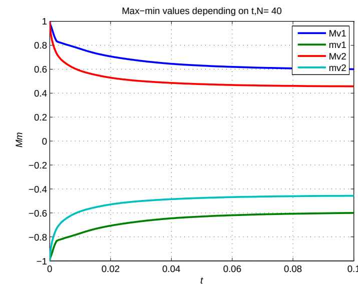

Fig. 2.33 Maximal and minimal values depending on t

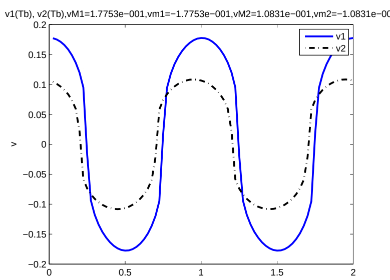

Fig. 2.34 Solutions by $t_f = 10, N = 80, L = 2$ depending on x

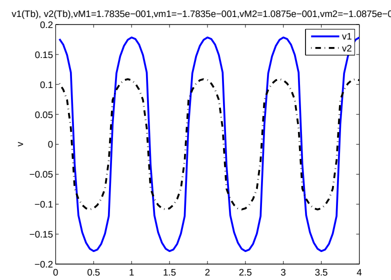

Fig. 2.35 Solutions by $t_f = 10, N = 80, L = 4$ depending on x

4) For different values of $\alpha_1 = 3, \alpha_2 = 5, \beta_1 = \beta_2 = 3, \gamma_1 = \gamma_2 = 2, t_f = 10, L = 2$ results are represented in fig. 2.38.
If $\alpha_1 = 5, \alpha_2 = 3$, then we have fig. 2.39. In this case the Matlab operators in appendix A.8 need to be changed with code:

```
[T1,Y1]=ode15s(@SIST1,[0,Tb],yy0,options,A1,A2,K,P,F2,N);
function F=SIST1(t,yy,A1,A2,K,P,F1,N)
F=-kron(K,A1)*[yy(1:N).^5; yy(N+1:2*N).^3]+kron(P,A2)*[yy(1:N)
    .^3; yy(N+1:2*N).^3]+F1.*[yy(1:N).^2; yy(N+1:2*N).^2];
```

In Figs. 2.40, 2.41 the results by $\alpha_1 = 5, \alpha_2 = 3$ for matrix $0.1\tilde{K}$ with the eigenvalues $0.1, 0.5$ are represented. We can see the oscillations in time ($N = 40, t_f = 5$). We obtain the following maximal values depending on $O(h^{2n}), n = 1, 2, 3, 4$ and for FDSES:

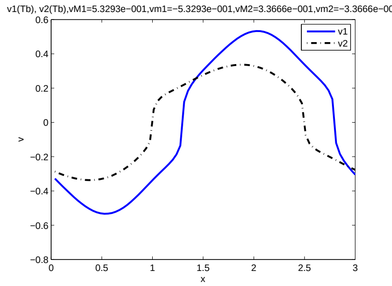

Fig. 2.36 Solutions by $t_f = 20, N = 80, L = 3$ depending on x

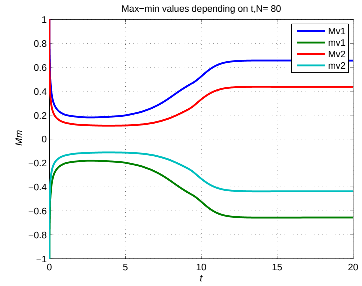

Fig. 2.37 Maximal and minimal values depending on t

$1.5461; 0.9288(O(h^2)), 1.4914; 0.8911(O(h^4)), 1.5061; 0.8969(O(h^6)), 1.4919; 0.8894(O(h^8)), 1.5081; 0.89$

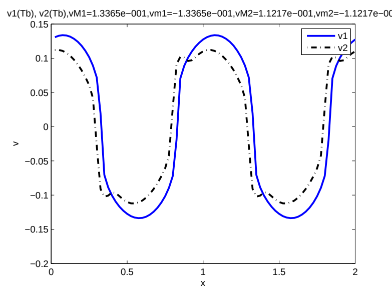

Fig. 2.38 Solutions by $t_f = 10, N = 80, L = 2, \alpha = 3; 5$ depending on x

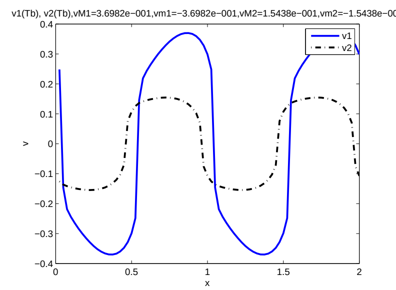

Fig. 2.39 Solutions by $t_f = 10, N = 80, L = 2, \alpha = 5; 3$ depending on x

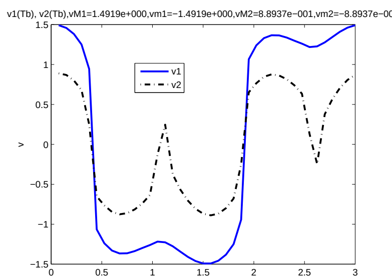

Fig. 2.40 Solutions by $t_f = 5, N = 40, L = 3, \alpha = 5; 3$ depending on x

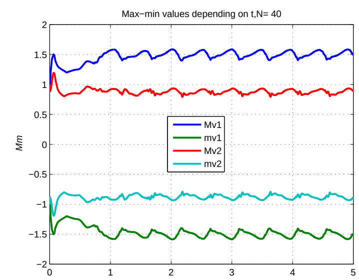

Fig. 2.41 Maximal and minimal values depending on t by $L = 3$

5) Interesting results are obtained if $\tilde{K} = 0, \beta_1 = \beta_2 = 1$ and when the functions $g_{3,1}(T), g_{3,2}(T)$ are trigonometric functions $\sin(T)$ or $\cos(T)$.
For $g_{3,1}(T) = g_{3,2}(T) = \sin(T), N = 80, L = t_f = 1$ we have the following maximal and minimal values of solution depending on the order of approximation:
$2.71, -1.96; 2.14, -1.28(O(h^2)), 2.40, -1.93; 1.76, -1.33(O(h^4)), 2.39, -1.92; 1.73, -1.34(O(h^6)),$
$2.38, -1.88; 1.73, -1.32(O(h^8)), 2.38, -1.86; 1.72, -1.32(FDSES)$ and $(O(h^{20}))$ (see Figs. 2.42,2.43). If $N = 20$ then for FDSES we have:
$2.37, -1.83; 1.73, -1.31.$

For $g_{3,1}(T) = g_{3,2}(T) = \cos(T)$ we have symmetric oscillations in the space (see Figs. 2.44,2.45). In the Figs. 2.46,2.47 and 2.48,2.49 the solutions obtained by $g_{3,1}(T) =$

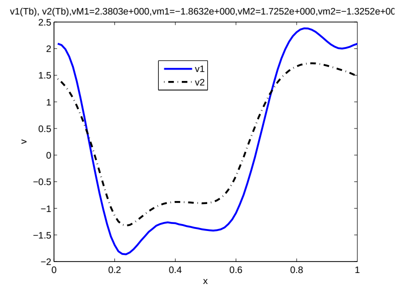

Fig. 2.42 Solutions by $t_f = 1, N = 80, L = 1, g_{3,1} = g_{3,2} = \sin(T)$ depending on x

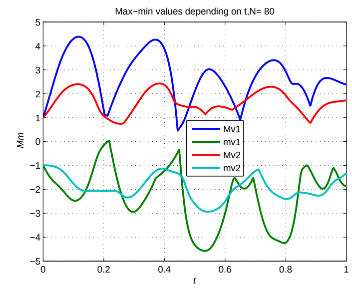

Fig. 2.43 Maximal and minimal values depending on t

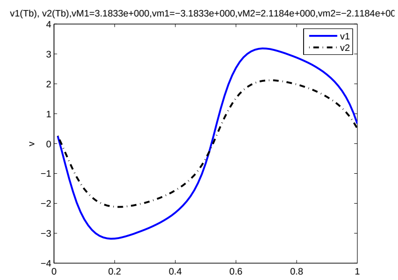

Fig. 2.44 Solutions by $t_f = 1, N = 80, L = 1, g_{3,1} = g_{3,2} = \cos(T)$ depending on x

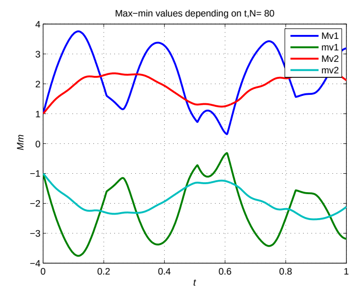

Fig. 2.45 Maximal and minimal values depending on t

$\sin(T), g_{3,2}(T) = \cos(T)$ and $g_{3,1}(T) = \cos(T), g_{3,2}(T) = \sin(T)$ are represented.

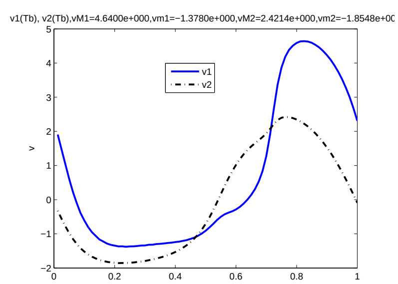

Fig. 2.46 Solutions by $t_f = 1, N = 80, L = 1, g_{3,1} = \sin(T), g_{3,2} = \cos(T)$ depending on x

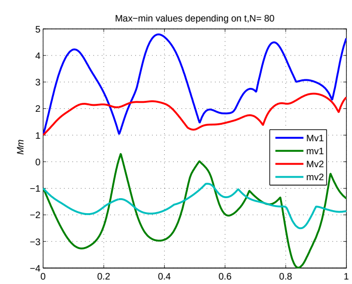

Fig. 2.47 Maximal and minimal values depending on t

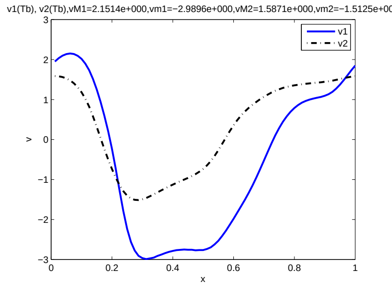

Fig. 2.48 Solutions by $t_f = 1, N = 80, L = 1, g_{3,1} = \cos(T), g_{3,2} = \sin(T)$ depending on x

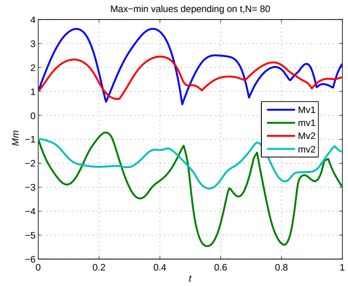

Fig. 2.49 Maximal and minimal values depending on t

3. Poisson equation with periodic BC

The solutions of the linear boundary value problem for Poisson equations are obtained analytically and numerically. Using the method of lines (lines are parallel to the y axis) for periodic BC we define the FDSES, where the finite difference matrix A is represented in the form $A = WDW^*$ (W, D are the matrices of finite difference eigenvectors and eigenvalues correspondingly, W^* is the conjugate transpose matrix) and the elements of diagonal matrix D are replaced with the first eigenvalues from the differential operator.

3.1 Mathematical model

We consider the boundary value problem for Poisson equation with the periodic BCs in the x direction:

$$\begin{cases} \frac{\partial^2 T(x,y)}{\partial y^2} + \frac{\partial^2 T(x,y)}{\partial x^2} = f(x,y), x \in (0,L), y \in (0,H) \\ T(0,y) = T(L,y), \frac{\partial T(0,y)}{\partial x} = \frac{\partial T(L,y)}{\partial x}, y \in (0,H) \\ T(x,0) = T_l(x), T(x,H) = T_r(x), x \in (0,L) \end{cases} \tag{3.1}$$

where $T_l(x)$, $T_r(x)$ are given BC function in the y direction. Similarly we can consider the problem with the periodic BCs in the both x and y directions:

$$\begin{cases} \frac{\partial^2 T(x,y)}{\partial y^2} + \frac{\partial^2 T(x,y)}{\partial x^2} = f(x,y), x \in (0,L), y \in (0,H) \\ T(0,y) = T(L,y), \frac{\partial T(0,y)}{\partial x} = \frac{\partial T(L,y)}{\partial x}, y \in (0,H) \\ T(x,0) = T(x,H), \frac{\partial T(x,0)}{\partial y} = \frac{\partial T(x,H)}{\partial y}, x \in (0,L) \end{cases} \tag{3.2}$$

This problem has unique solutions by $\int_0^H \int_0^L f(x,y)dxdy = 0$, $T(x_0,y_0) = T_0$, where $x_0 \in [0,L]$, $y_0 \in [0,H]$, T_0 are fixed constants.
We consider uniform grid in the space $x_j = jh$, $j = \overline{0,N}$, $Nh = L$, where N is even number.

Using the finite differences for partial derivatives of the second order with respect to x (see section 1.4.1) we obtain the boundary value problem for system of ordinary differential equations (ODEs) in the following matrix form

$$\ddot{U}(y) - AU(y) = F(y), U(0) = U_l, U(H) = U_r, \tag{3.3}$$

$$\ddot{U}(y) - AU(y) = F(y), U(0) = U(H), \dot{U}(0) = \dot{U}(H), \tag{3.4}$$

where A is the circulant matrix of N order, $U(y)$, $\ddot{U}(y)$, $F(y)$, U_l, U_r are the column-vectors of N order with the elements $u_j(y) \approx T(x_j, y))$, $\ddot{u}_j(t) \approx \frac{\partial^2 T(x_j,y)}{\partial y^2}$, $f_j(y) = f(x_j, y)$, $u_j(0) = T_l(x_j), u_j(H) = T_r(x_j), j = \overline{0, N}$.

The corresponding discrete spectral problem $Aw^n = \mu_n w^n$, $n = \overline{1, N}$ with circulant matrix has the following solution:

$$\begin{cases} w^n = \sqrt{1/N}(w_1^n, w_2^n, \dots, w_N^n)^T, \\ \mu_n = \frac{4}{h^2}\sin^2(\pi n h/L), \end{cases} \tag{3.5}$$

where $w_j^n = \phi_n(x_j) = \exp(2\pi i n x_j/L)$, $j = \overline{1, N}, i = \sqrt{-1}$ are the components of orthonormal eigenvector w^n. The eigenvalues of matrix A for different order of approximation $O(h^k)$, $k \geq 2$ are:

1. $k = 4$, $h^2 A = [-\frac{5}{2}, \frac{4}{3}, -\frac{1}{12}, 0, \dots, 0, -\frac{1}{12}, \frac{4}{3}]$,
   $h^2\mu_n = -4(\sin^2(\pi n h/L) + \frac{1}{3}\sin^4(\pi n h/L))$,
2. $k = 6$, $h^2 A = [-\frac{49}{18}, \frac{3}{2}, -\frac{3}{20}, \frac{1}{90}, 0, \dots, 0, \frac{1}{90}, -\frac{3}{20}, \frac{3}{2}]$,
   $h^2\mu_n = -4(\sin^2(\pi n h/L) + \frac{1}{3}\sin^4(\pi n h/L) + \frac{8}{45}\sin^6(\pi n h/L))$,
3. $k = 8$, $h^2 A = [-\frac{205}{72}, \frac{8}{5}, -\frac{1}{5}, \frac{8}{315}, -\frac{1}{560}, 0, \dots, 0, -\frac{1}{560}, \frac{8}{315}, -\frac{1}{5}, \frac{8}{5}]$.
   $h^2\mu_n = -4(\sin^2(\pi n h/L) + \frac{1}{3}\sin^4(\pi n h/L) + \frac{8}{45}\sin^6(\pi n h/L) +$
   $\frac{4}{35}\sin^8(\pi n h/L))$,

Therefore the matrix A can be represented in form $A = WDW^*$, where the column of the matrix W and the diagonal matrix D contains N orthonormal eigenvectors w^n and eigenvalues μ_n, $n = \overline{1, N}$ correspondingly ($W^*W = E$, $W^{-1} = W^*$).
The solution of the spectral problem for differential equations

$$-w''(x) = \lambda w(x), x \in (0, L), w(0) = w(L), w'(0) = w'(L),$$

is in the following form:

$$w_n(x) = L^{-1}\phi_n(x) = L^{-1}\exp(2\pi i n x/L), \lambda_n = (2\pi n/L)^2, n > 0$$

3.2 Analytical solution

We can consider the **analytical solutions** of the system of ODEs (3.3, 3.4) using the spectral representation of matrix $A = WDW^*$. From transformation $V = W^*U(U = WV)$ follows the separate system of ODEs

$$\ddot{V}(y) - DV(y) = G(y), V(0) = W^*U_l, V(H) = W^*U_r, \tag{3.6}$$

$$\ddot{V}(y) - DV(y) = G(y), V(0) = V(H), \dot{V}(0) = \dot{V}(H), \tag{3.7}$$

where $V(y)$, $\ddot{V}(y)$, $\dot{V}(0)$, $\dot{V}(H)$, $V(0)$, $V(H)$, $G(y) = W^*F(y)$ are the column-vectors of N order with elements $v_k(y)$, $\ddot{v}_k(y)$, $\dot{v}_k(0)$, $\dot{v}_k(H)$, $v_k(0)$, $v_k(H)$, $g_k(y)$, $k = \overline{1, N}$.

The solution of the system (3.6) is the sum of two solutions function (with homogenous equation and with homogenous BC)

$$\begin{aligned} &v_k(y) = (\sinh(\kappa_k H))^{-1}[v_k(0)\sinh(\kappa_k(H - y)) + v_k(H)\sinh(\kappa_k y)] - \\ &\int_0^H G_k(\xi, y) g_k(\xi) d\xi, \end{aligned} \tag{3.8}$$

where $\kappa_k = \sqrt{\mu_k}, G_k(\xi, y)$ is the Green function in following way:

$$G_k(\xi, y) = \begin{cases} \frac{\sinh(\kappa_k(H-y))\sinh(\kappa_k\xi)}{\kappa_k \sinh(\kappa_k H)}, \ 0 \le \xi \le y, \\ \frac{\sinh(\kappa_k(H-\xi))\sinh(\kappa_k y)}{\kappa_k \sinh(\kappa_k H)}, \ y \le \xi \le H. \end{cases}$$

For $\mu_N = 0$ from (3.8) follows

$$v_N(y) = v_N(0) + \frac{y}{H}(v_N(H) - v_N(0)) - \int_0^H G_N(\xi, y) g_N(\xi) d\xi,$$

where

$$G_N(\xi, y) = \begin{cases} \frac{(H-y)\xi}{H}, \ 0 \le \xi \le y, \\ \frac{(H-\xi)y}{H}, \ y \le \xi \le H. \end{cases}$$

The general solution of (3.7) is following

$$v_k(y) = C_k \sinh(\kappa_k y) + B_k \cosh(\kappa_k y) + \frac{1}{\kappa_k}\int_0^y g_k(\xi)\sinh(\kappa_k(y-\xi))d\xi,$$

where C_k, B_k are constant. Using the BCs (3.7) we have

$$C_k = \frac{0.5}{\kappa_k \sinh(0.5\kappa_k H)} \int_0^H g_k(\xi)\sinh(\kappa_k(\xi - 0.5H))d\xi,$$

$$B_k = -\frac{0.5}{\kappa_k \sinh(0.5\kappa_k H)} \int_0^H g_k(\xi)\cosh(\kappa_k(\xi - 0.5H))d\xi.$$

In this case the analytical solution (3.8) is

$$v_k(y) = -0.5(\kappa_k \sinh(0.5\kappa_k H))^{-1} \int_0^H G_k(\xi, y) g_k(\xi) d\xi, \tag{3.9}$$

where $G_k(\xi, y)$ is the Green function in the following way:

$$G_k(\xi, y) = \begin{cases} \cosh(\kappa_k(H/2 - y + \xi)), \ 0 \le \xi \le y, \\ \cosh(\kappa_k(H/2 - \xi + y)), \ y \le \xi \le H. \end{cases}$$

We can also use the **Fourier method** for solving (3.1) in the form $T(x, y) = \sum_{k \in Z} v_k(t) w^k(x)$, where $w^k(x)$ are the orthonormal eigenvectors, $v_k(y)$ is the solution (3.8) with $v_k(0) = (T_l, w_k^*), v_k(H) = (T_r, w_k^*)$.

The solution can be also obtained in the real form:

$$T(x, y) = \sum_{k=1}^{\infty}(a_{kc}(y)\cos\frac{2\pi kx}{L} + a_{ks}(y)\sin\frac{2\pi kx}{L}) + \frac{a_{0c}(y)}{2},$$

$$f(x, y) = \sum_{k=1}^{\infty}(b_{kc}(y)\cos\frac{2\pi kx}{L} + b_{ks}(y)\sin\frac{2\pi kx}{L}) + \frac{b_{0c}(y)}{2},$$

$$b_{kc}(y) = \frac{2}{L}\int_0^L f(\xi, y)\cos\frac{2\pi k\xi}{L}d\xi, b_{ks}(y) = \frac{2}{L}\int_0^L f(\xi, y)\sin\frac{2\pi k\xi}{L}d\xi,$$

where $a_{kc}(y)$, $a_{ks}(y)$ are the corresponding solutions of (3.8, 3.9) by

$$a_{kc}(0) = \frac{2}{L}\int_0^L T_l(\xi)\cos\frac{2\pi k\xi}{L}d\xi, a_{ks}(0) = \frac{2}{L}\int_0^L T_l(\xi)\sin\frac{2\pi k\xi}{L}d\xi,$$
$$a_{kc}(H) = \frac{2}{L}\int_0^L T_r(\xi)\cos\frac{2\pi k\xi}{L}d\xi, a_{ks}(H) = \frac{2}{L}\int_0^L T_r(\xi)\sin\frac{2\pi k\xi}{L}d\xi,$$
$$g_k(y) = b_{kc}(y) \text{ or } b_{ks}(y).$$

Similarly we can obtain the solution of the discrete problem also in the real form:

$$u_j(y) = \sum_{k=1}^{*N_2}(a_{kc}(y)\cos\frac{2\pi kj}{N} + a_{ks}(y)\sin\frac{2\pi kj}{N}) + \frac{a_{0c}(y)}{2},$$
$$f_j(y) = \sum_{k=1}^{*N_2}(b_{kc}(y)\cos\frac{2\pi kj}{N} + b_{ks}(y)\sin\frac{2\pi kj}{N}) + \frac{b_{0c}(y)}{2},$$
$$b_{kc}(y) = \frac{2}{N}\sum_1^N f_j(y)\cos\frac{2\pi kj}{N}, b_{ks}(y) = \frac{2}{N}\sum_1^N f_j(y)\sin\frac{2\pi kj}{N},$$

where $a_{kc}(y)$, $a_{ks}(y)$ are the corresponding solutions of (3.8, 3.9) by

$$a_{kc}(0) = \frac{2}{N}\sum_1^N T_l(x_j)\cos\frac{2\pi kj}{N}, a_{ks}(0) = \frac{2}{N}\sum_1^N T_l(x_j)\sin\frac{2\pi kj}{N},$$
$$a_{kc}(H) = \frac{2}{N}\sum_1^N T_r(x_j)\cos\frac{2\pi kj}{N}, a_{ks}(H) = \frac{2}{N}\sum_1^N T_r(x_j)\sin\frac{2\pi kj}{N},$$
$$g_k(y) = b_{kc}(y) \text{ or } b_{ks}(y)$$

For FDSES μ_k are replaced with λ_k.

For the FDSES the matrix A is represented in the form $A = WDW^*$ and the diagonal matrix D contain the first N eigenvalues $d_k = \lambda_k$, $k=\overline{1,N}$ from the differential operator $(-\frac{\partial^2}{\partial x^2})$ in following way:

1. $d_k = \lambda_k$ for $k = \overline{1, N_2}$, where $N_2 = N/2$.
2. $d_k = \lambda_{N-k}$ for $k = \overline{N_2, N}$.

If $d_k = \mu_k$, then we have the method of FDS.

The FDSES method is more stable as the method of finite difference by approximation with central difference (FDS), because the eigenvalues are larger $\lambda_k > \mu_k$.

3.3 Analytical solution in the matrix form

For homogenous equations ($F = 0$) (3.3) we can obtain the solution in the form

$$U_1(y) = \sinh^{-1}(A_1 H)[\sinh(A_1 y)U_r + \sinh(A_1(H - y))U_l],$$

where $A_1 = \sqrt{A}$.
For homogenous BC ($U_l = U_r = 0$) the solution is $U_2(y) = -\int_0^H G(\xi, y)F(\xi)d\xi$, where

$$G(\xi, y) = \begin{cases} \sinh(A_1(H-y))\sinh(A_1\xi)(A_1\sinh(A_1H))^{-1}, \ 0 \le \xi \le y, \\ \sinh(A_1(H-\xi))\sinh(A_1y)(A_1\sinh(A_1H))^{-1}, \ y \le \xi \le H \end{cases}$$

is Green matrix-function.

The solution of the problem (3.3) is $U(y) = U_1(y) + U_2(y)$. Multiply this solution left with $W^{-1} = W^*$, and using the expressions $A = WDW^*$, $f(A) = Wf(D)W^*$, $W^*U_l = V(0)$, $W^*U_r = V(H)$, $W^*F = G(y)$ (f is every function) we obtain the column-vector $V(y)$ with components (3.8).

If we consider that $det(A) = det(A_1) = 0$ we can not directly obtained the solution in the matrix form.

If the periodic BCs are given also in the y direction, from (3.4) we have the following vector-solution

$$U(y) = -0.5A_1^{-1}\sinh^{-1}(0.5A_1H)\int_0^H G(\xi, y)F(\xi)d\xi,$$

where

$$G(\xi, y) = \begin{cases} \cosh(A_1(H/2 - y + \xi)), \ 0 \le \xi \le y, \\ \cosh(A_1(H/2 + y - \xi)), \ y \le \xi \le H \end{cases}$$

For $\mu_N = 0$ from (3.8) follows

$$v_N(y) = v_N(0) + \frac{y}{H}(v_N(H) - v_N(0)) - \int_0^H G_N(\xi, y)g_N(\xi)d\xi,$$

where

$$G_N(\xi, y) = \begin{cases} \frac{(H-y)\xi}{H}, \ 0 \le \xi \le y, \\ \frac{(H-\xi)y}{H}, \ y \le \xi \le H. \end{cases}$$

For $\mu_N = 0$ from (3.9) follows $v_N(y) = 0$.

3.4 Some examples and numerical results

3.4.1 Boundary value problem with periodic BC in one direction

For numerical calculation we consider the boundary value problem (3.1) with $H = L = 1, f = 0$, $T_l = 0$, $T_r(x) = \sinh(2\pi)\cos(2\pi x)$, $T(x, y) = \sinh(2\pi y)\cos(2\pi x)$. Using the Fourier method we obtain $v_k(0) = 0$, $v_k(H) = 0$ for $k \ne \pm 1$, $v_{\pm 1}(H) = \frac{\sinh(2\pi)}{2i}$, $v_{\pm 1}(y) = \frac{\sinh(2\pi y)}{2}$, $T(x, y) = \cos(2\pi x)\sinh(2\pi y)$.

MATLAB program code can be found in appendix A.11.

Using the operator **PuasPer(40,10)** we obtain the following maximal errors:

0.0956 (FDS $O(h^2)$), see fig. 3.1
0.00031 (FDS $O(h^4)$),
$1.2 \cdot 10^{-6}$ (FDS $O(h^6)$),

$5 \cdot 10^{-9}$ (FDS $O(h^8)$),
$6 \cdot 10^{-12}$ (FDSES), see fig. 3.2.

By $N = M = 10$ we obtain

$3 \cdot 10^{-4}$ (FDS-$O(h^8)$),
10^{-12} (FDSES).

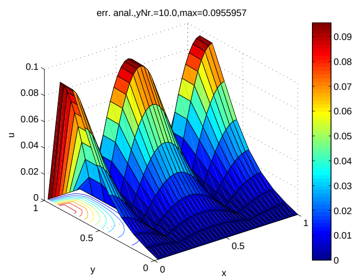

Fig. 3.1 Error with FDS by $N = 40, M = 10, O(h^2)$

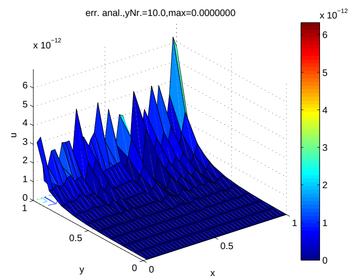

Fig. 3.2 Error with FDSES by $N = 40, M = 10$

3.4.2 Matrix solution of boundary value problem with periodic BC in two directions

Using the periodic BCs in both directions with right sides function $f(x, y) = -8\pi^2 \cos(2\pi x) \cos(2\pi y)$ by $L = H = 1$ we have the exact solution $T(x, y) = \cos(2\pi x) \cos(2\pi y)$. From the approximated solution follows that

$$U(y) = 8\pi^2 \cos(2\pi y)(A_1^2 + 4\pi^2 E)^{-1} g,$$

where E is the unit matrix of the N order,

$$g = (\cos(2\pi x_1), \cos(2\pi x_2), \cdots, \cos(2\pi x_N))^T$$

is the column-vector of the N order.
From (3.9) follows that
$g_k(\xi) = -8\pi^2 \frac{cos(2\pi\xi)}{\sqrt{N}} \sum_{j=1}^{N} \exp(-2\pi j x_k) \cos(2\pi x_j) = -8\pi^2 \frac{\sqrt{N}}{2} \cos(2\pi\xi)$ for $k = 1$ and $k = N - 1$. For other numbers of k we have $g_k(\xi) = 0$. We use the integrals $\int \cosh(a\xi + b) \cos(c\xi) d\xi =$
$\frac{a}{a^2+c^2} \sinh(a\xi+b) \cos(c\xi) + \frac{c}{a^2+c^2} \cosh(a\xi+b) \sin(c\xi)$, where $a = \pm\kappa_k, b = H/2 \pm y, c = 2\pi$. Then $v_1(y) = v_{N-1} = 4\pi^2\sqrt{N} \cos(2\pi y)/(\kappa^2 + 4\pi^2)$, where $\kappa = \kappa_1 = \kappa_{N-1}$.
Therefore the components of the approximated solution U is
$u_j(y) = \frac{1}{\sqrt{N}} v_1(y)(\exp(2\pi x_j) + \exp(2\pi x_j(N - 1))) = \frac{8\pi^2}{\kappa_1^2+4\pi^2} \cos(2\pi y) \cos(2\pi x_j)$.
MATLAB code with operator `puas4` can be found in appendix A.12.

Using the operator `puas4(10)` we obtain following maximal errors:
0.016502 (FDS $O(h^2)$),

0.000837 (FDS $O(h^4)$),
$52 \cdot 10^{-6}$ (FDS $O(h^6)$),
$4 \cdot 10^{-6}$ (FDS $O(h^8)$),
$5 \cdot 10^{-15}$ (FDSES).

3.4.3 Analytical solution of boundary value problem with periodic BC in two directions

The file `PuasPer2.m` in appendix A.13 is used for the analytical solution (3.9) with the quadrature trapezoid formula to calculate the integrals
$\int_0^{y_j} F_1(y_j,t)dt \approx \frac{h_1}{2}(F_1(y_j,0) + F_1(y_j,y_j)) + h_1 \sum_{m=1}^{j-1} F_1(y_j,y_m), j = \overline{1,M}$,
$\int_{y_j}^{H} F_2(y_j,t)dt \approx \frac{h_1}{2}(F_2)(y_j,H) + F_2(y_j,y_j)) + h_1 \sum_{m=j+1}^{M-1} F_2(y_j,y_m), j = \overline{1,M-1}$,
$\int_{y_M}^{H} F_2(y_j,t)dt = 0$,
where $F_1(y,t) = \cosh(\kappa_k(0.5 * H - y + t))g_k(t)$, $F_2(y,t) = \cosh(\kappa_k(0.5 * H + y - t))g_k(t), y_j = j * h_1, h_1 = H/M$.

Using the operator **PuaPer2(10,200)** we obtain following maximal errors:
0.0333 (FDS $O(h^2)$),
0.0020 (FDS $O(h^4)$),
0.00043 (FDS $O(h^6)$),
0.00034 (FDS $O(h^8)$),
0.00033 (FDSES).

For $N = 10, FDS - O(h^2)$ and different M follows:
0.0351 ($M = 80$),
0.0335 ($M = 160$),
0.0333 ($M = 200$).

3.4.4 Kronecker-tensor solution of problem with periodic BC in two directions

In the m.file `PuasTen2` (see appendix A.14) is used the Kronecker-tensor method for the solution (3.2). We consider also uniform grid in the y direction $y_m = mh_1$, $m = \overline{0,M}$, $Mh_1 = H$, where M is even number.
Using the finite differences of second order approximation for partial derivatives of second order respect to x, y we obtain the system of linear algebraical equations of the $N \times M$ order in the following matrix form

$$Au = -g, \tag{3.10}$$

where $A = E_2 \otimes A_1 + A_2 \otimes E_1$ is the block wise matrix of the $N.M$ order determined with the Kronecker tensor product in following form $C = B^1 \otimes B^2$, where B^1, B^2 are the square matrices correspondingly of N, M orders and the matrix C of $N \times M$ order is following

$$C = \begin{pmatrix} b^1_{1,1}B^2 & b^1_{1,2}B^2 & b^1_{1,3}B^2 & ... & b^1_{1,N-2}B^2 & b^1_{1,N-1}B^2 & b^1_{1,N}B^2 \\ b^1_{2,1}B^2 & b^1_{2,2}B^2 & b^1_{2,3}B^2 & ... & b^1_{2,N-2}B^2 & b^1_{2,N-1}B^2 & b^1_{2,N}B^2 \\ .. & .. & .. & ... & .. & .. \\ b^1_{N,1}B^2 & b^1_{N,2}B^2 & b^1_{N,3}B^2 & ... & b^1_{N,N-2}B^2 & b^1_{N,N-1}B^2 & b^1_{N,N}B^2 \end{pmatrix}$$

A_1, A_2 are circulant matrices $A_1 = \frac{1}{h^2}[2\ -1\ 0\ ...\ 0\ 0\ -1]$,
$A_2 = \frac{1}{h_1^2}[2\ -1\ 0\ ...\ 0\ 0\ -1]$; E_1, E_2 are the unit matrices with the N, M order correspondingly, u, g are the column-vectors of $N.M$ order with following elements $u_{j,m} \approx T(x_j, y_m), g_{j,m} = f(x_j, y_m), j = \overline{1, N}, m = \overline{1, M}$,
Using the matrices spectral representation $A_k = W_k D_k W_k^*, k = 1; 2$ and the properties of Kronecker product $AC \otimes BD = (A \otimes B)(C \otimes D)$,
$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ we get

$$A = (W_2 \otimes W_1)(E_2 \otimes D_1 + D_2 \otimes E_1)(W_2^* \otimes W_1^*)$$

and $u = -A^{-1}g, (W_k^{-1} = W_k^*, (W_k^*)^{-1} = W_k)$ where

$$A^{-1} = (W_2 \otimes W_1)(E_2 \otimes D_1 + D_2 \otimes E_1)^{-1}(W_2^* \otimes W_1^*)$$

For the solution of the problem $Au = g$ analytically we use the transformation $W^*u = v$ or $u = Wv$, where $W = W_2 \otimes W_1, W^* = W_2^* \otimes W_1^*$. Then $Dv = -W^*g$ or $d_{j,m}v_{j,m} = -(W^*g)_{j,m}, j = \overline{1, N}, m = \overline{1, M}$, where $D = E_2 \otimes D_1 + D_2 \otimes E_1$.
For $j = N, M = M$ we have $d_{N,M} = 0$, the value $v_{N,M}$ is indeterminable and we can take $v_{N,M} = 0$. The solution is in the form $u = Wv$.

Using the operator `PuaTen2(10,10)` we obtain by different order of FDS and of FDSES following maximal errors:

1. $O(h_1^2)$: 0.06711 (FDS $O(h^2)$), 0.0347 (FDS $O(h^4)$), 0.0331 (FDS $O(h^6)$), 0.03301 (FDS $O(h^8)$), 0.03300 (FDSES),
2. $O(h_1^4)$: 0.0347 (FDS $O(h^2)$), 0.00335 (FDS $O(h^4)$), 0.00178 (FDS $O(h^6)$), 0.00168 (FDS $O(h^8)$), 0.00167 (FDSES),
3. $O(h_1^6)$: 0.0331 (FDS $O(h^2)$), 0.00178 (FDS $O(h^4)$), 0.00021 (FDS $O(h^6)$), 0.00011 (FDS $O(h^8)$), 0.00010 (FDSES),
4. $O(h_1^8)$: 0.03301 (FDS $O(h^2)$), 0.00168 (FDS $O(h^4)$), 0.00011 (FDS $O(h^6)$), 0.00001 (FDS $O(h^8)$), 7.10^{-6} (FDSES),
5. FDSES: 0.03300 (FDS $O(h^2)$), 0.00167 (FDS $O(h^4)$), 0.00010 (FDS $O(h^6)$), 7.10^{-6} (FDS $O(h^8)$), 10^{-15} (FDSES).

For $FDS - O(h^2) + O(h_1^2)$ and FDSES in both direction the errors are represented in the fig. 3.3, 3.4.

3.5 Poisson equation with BCs of the first kind

For BCs of first kind we consider special boundary value problem (3.1) with the homogenous BC in the x direction

$$T(0, y) = T(L, y) = 0, y \in [0, H].$$

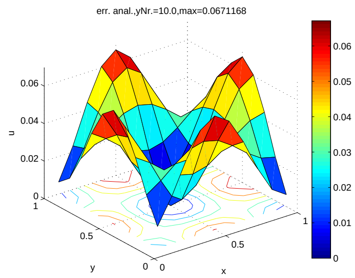

Fig. 3.3 Error with FDS of $O(h^2+h_1^2)$ by $N = M = 10$.

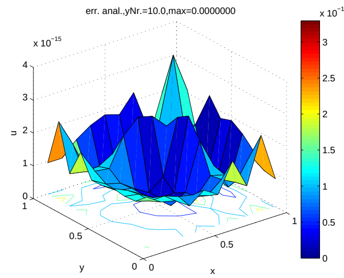

Fig. 3.4 Error with FDSES in (x,y) direction by $N = M = 10$.

We have the discrete problem (3.3) with the standard 3-diagonal matrix A of the $M = N-1$ order. We use the matrix representation $A = WDW, W^* = W$ where the diagonal matrix D contain the discrete eigenvalues $\mu_k = \frac{4}{h^2}\sin^2\frac{\pi k}{2N}$ and the column of the matrix W is equal to the orthonormal eigenvectors w^k with the elements $w_j^k = w^k(x_j) = \sqrt{\frac{2}{N}}\sin\frac{\pi kj}{N},\ k,j=\overline{1,M}$. Then we have the separate system of ODEs (3.6) with the solution (3.8). For the FDSES the matrix A is represented in the form $A = WDW$ and the diagonal matrix D contain the first M eigenvalues $d_k = \lambda_k = (\frac{\pi k}{L})^2,\ k=\overline{1,M}$ from the differential operator $(-\frac{\partial^2}{\partial x^2})$. Then in the solution (3.8) needs replaced μ_k with λ_k.

For special data
$T_l(x) = a_1 w_{p_1}(x), T_r(x) = a_2 w_{p_2}(x), f(x,t) = g(t) w_{p_3}(x)$ we have the exact solution for Fourier method and for FDSES by $M \geq \max(p_1,p_2,p_3)$ in the form

$$T(x,y) = a_1 w_{p_1}(x) v_{p_1}(y) + a_2 w_{p_2}(x) v_{p_2}(y) + w_{p_3} v_{p_3},$$

where
$v_{p_1}(y) = \sinh(\kappa_{p_1}H))^{-1}\sinh(\kappa_{p_1}(H-y)), v_{p_2}(y) = \sinh(\kappa_{p_2}H))^{-1}\sinh(\kappa_{p_2}y)$,
$v_{p_3}(y) = -\int_0^H G_{p_3}(\xi,y)g(\xi)d\xi$, $G_p(\xi,y)$ is the Green function in following way:

$$G_p(\xi,y) = \begin{cases} \frac{\sinh(\kappa_p(H-y))\sinh(\kappa_p\xi)}{\kappa_p\sinh(\kappa_p H)},\ 0 \leq \xi \leq y, \\ \frac{\sinh(\kappa_p(H-\xi))\sinh(\kappa_p y)}{\kappa_p\sinh(\kappa_p H)},\ y \leq \xi \leq H. \end{cases}$$

For FDS $(x = x_j, j = \overline{1,M})$ $\kappa_k = \sqrt{\mu_k}$,but for FDSES $\kappa_k = \sqrt{\lambda_k}, k = (p_1,p_2,p_3)$.

4. Wave equation with periodic BC

4.1 Mathematical model

We consider the initial boundary value problem with periodic BCs

$$\begin{cases} \dfrac{\partial^2 T(x,t)}{\partial t^2} = a^2 \dfrac{\partial^2 T(x,t)}{\partial x^2} + f(x,t), x \in (0,L), t \in (0,t_f), \\ T(0,t) = T(L,t), \dfrac{\partial T(0,t)}{\partial x} = \dfrac{\partial T(L,t)}{\partial x}, t \in (0,t_f), \\ T(x,0) = T_0(x), \dfrac{\partial T(x,0)}{\partial t} = \bar{T}_0(x), x \in (0,L). \end{cases} \tag{4.1}$$

Using uniform grid $x_j = jh$, $j = \overline{0,N}$, $Nh = L$, (N is even number) we obtain the system of ODEs

$$\begin{cases} \ddot{U}(t) + a^2 A U(t) = F(t), \\ U(0) = U_0, \dot{U}(0) = \bar{U}_0 \end{cases} \tag{4.2}$$

where A is the approximation matrix of N order for second derivative (see 1.4.1) for which eigenvalues μ_k, eigenvectors w_k are known. The eigenvalues λ_k and eigenvectors for spectral problem also are known (see 1.5).

The solution of (4.1) with the Fourier method can be obtained in the following form: $f(x,t) = \sum_{k=-\infty}^{\infty} g_k(t) w^k(x)$, $g_k(t) = (w^{*k}, f)$, $T(x,t) = \sum_{k=-\infty}^{\infty} v_k(t) w^k(x)$, where $v_k(t)$ is the solution of

$$v_k(t) = \frac{\dot{v}_k(0)}{\kappa_k} \sin(\kappa_k t) + v_k(0)\cos(\kappa_k t) + \frac{1}{\kappa_k} \int_0^t \sin(\kappa_k (t-\tau)) g_k(\tau) d\tau, \tag{4.3}$$

with $\kappa_k = \sqrt{a^2 d_k}$ (d_k is μ_k or λ_k depending on the method selected), by $k \neq 0$. For $k = 0$ we obtain

$$v_0(t) = \dot{v}_0(0)t + v_0(0) + \int_0^t (t-\tau) g_0(\tau) d\tau.$$

The solution can be also obtained in the real form (2.5). $a_{kc}(t)$, $a_{ks}(t)$ are the corresponding solutions of (4.3) by

$$a_{kc}(0) = \frac{2}{L} \int_0^L T_0(\xi) \cos\frac{2\pi k \xi}{L} d\xi, \dot{a}_{kc}(0) = \frac{2}{L} \int_0^L \bar{T}_0(\xi) \cos\frac{2\pi k \xi}{L} d\xi,$$
$$a_{ks}(0) = \frac{2}{L} \int_0^L T_0(\xi) \sin\frac{2\pi k \xi}{L} d\xi, \dot{a}_{ks}(0) = \frac{2}{L} \int_0^L \bar{T}_0(\xi) \sin\frac{2\pi k \xi}{L} d\xi,$$
$$g_k(t) = b_{kc}(t) \text{ or } b_{ks}(t).$$

4.2 Analytical solution

We can consider the **analytical solutions** of (4.2) using the spectral representation of matrix $A = WDW^*$. From transformation $V = W^*U$ ($U = WV$) follows the separate system of ODEs

$$\begin{cases} \ddot{V}(t) + a^2 DV(t) = G(t), \\ V(0) = WU_0, \dot{V}(0) = W\bar{U}_0 \end{cases} \tag{4.4}$$

where the column-vectors are of N order.

The solution of the system (4.4) is in the form (4.3), where $k = \overline{1, N-1}$, $d_k = \mu_k$. For $k = N$ the solution is $v_N(t) = \dot{v}_N(0)t + v_N(0) + \int_0^t (t-\tau) g_k(\tau) d\tau$. The solution of (4.2) is in the form $U = WV$.

If $d_k = \lambda_k$ then we can obtain the solution of FDSES in following way:

1. $d_k = \lambda_k$ for $k = \overline{1, N_2}$, where $N_2 = N/2$.
2. $d_k = \lambda_{N-k}$ for $k = \overline{N_2, N-1}, d_N = 0$.

Similarly to (2.5) we can obtain the solution of the discrete problem in the following real form

$$u_j(t) = \sum_{k=1}^{*N_2} (a_{kc}(t) \cos \frac{2\pi kj}{N} + a_{ks}(t) \sin \frac{2\pi kj}{N}) + \frac{a_{0c}}{2},$$

$$f_j(t) = \sum_{k=1}^{*N_2} (b_{kc}(t) \cos \frac{2\pi kj}{N} + b_{ks}(t) \sin \frac{2\pi kj}{N}) + \frac{b_{0c}(t)}{2},$$

$$b_{kc}(t) = \frac{2}{N} \sum_{j=1}^{N} f_j(t) \cos \frac{2\pi kj}{N}, b_{ks}(t) = \frac{2}{N} \sum_{j=1}^{N} f_j(t) \sin \frac{2\pi kj}{N},$$

where $a_{kc}(t)$, $a_{ks}(t)$ are the corresponding solutions of (4.3) by

$$a_{kc}(0) = \frac{2}{N} \sum_{1}^{N} T_0(x_j) \cos \frac{2\pi kj}{N}, \dot{a}_{kc}(0) = \frac{2}{N} \sum_{1}^{N} \bar{T}_0(x_j) \cos \frac{2\pi kj}{N},$$

$$a_{ks}(0) = \frac{2}{N} \sum_{1}^{N} T_0(x_j) \sin \frac{2\pi kj}{N}, \dot{a}_{ks}(0) = \frac{2}{N} \sum_{1}^{N} \bar{T}_0(x_j) \sin \frac{2\pi kj}{N},$$

$$g_k(t) = b_{kc}(t) \text{ or } b_{ks}(t).$$

4.3 Example of wave equation for one wave number

For numerical calculation we consider the initial boundary value problem (4.1) with $f = 0$, $T_0 = \sin(2\pi x)$, $\bar{T}_0 = 0$, $T(x,t) = \sin(2\pi x)\cos(2\pi t)$.

Using the Fourier method we obtain $v_k(0) = 0$, $v_k(t) = 0$ for $k \neq \pm 1$, $v_{\pm 1}(0) = \frac{1}{\pm i}$, $v_{\pm 1}(t) = \pm \frac{\cos(2\pi t)}{2i}$, $T(x,t) = \cos(2\pi t)\sin(2\pi x)$.

MATLAB code can be found in appendix A.15.

Using the operator **Wave2(10)** we obtain following maximal errors ($t_f = 1$):
0.0755 (FDS $O(h^2)$),

0.0038 (FDS $O(h^4)$),
0.00024 (FDS $O(h^6)$),
0.00002 (FDS $O(h^8)$),
10^{-12} (FDSES).

By $N = 40$ the results are:
0.0049 (FDS $O(h^2)$),
0.000016 (FDS $O(h^4)$),
10^{-7} (FDS $O(h^6)$),
2.10^{-10} (FDS $O(h^8)$),
10^{-14} (FDSES), (see fig. 4.1, 4.2)

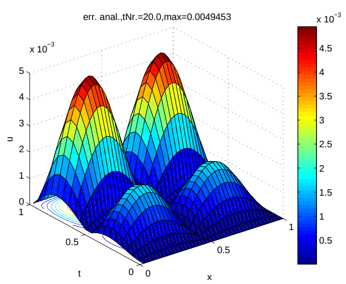

Fig. 4.1 Error with FDS by $N = 40, O(h^2)$

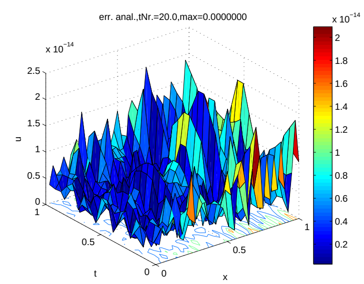

Fig. 4.2 Error with FDSES by $N = 40$

4.4 Example of wave equation for different wave number

We consider the initial boundary value problem (4.1) by $L = 1$, $a = 1$, $f = 0$, $T_0(x) = \sin(2\pi mx)$, $\bar{T_0} = 0$, where m is integer in $(1, N)$ with $m \le N/2$. Then the exact solution is $T(x,t) = \cos(2\pi mt)\sin(2\pi mx)$.

The solution of (4.1) with the **Fourier method** can be obtained in the following form:
$T(x,t) = \sum_{k=-\infty}^{\infty} v_k(t)w^k(x)$, where $v_k(t)$ is the solution of (4.3) in the form $v_k(t) = \cos(\kappa_k t)v_k(0)$, $\kappa_k = \sqrt{d_k} = 2\pi k$, $v_k(0) = \int_0^1 T_0(x)w^{*k}(x)dx = 0$, $v_k(t) = 0$ for $k \ne \pm m$, $v_{\pm m}(0) = \frac{0.5}{\pm i}$, $v_{\pm m}(t) = \pm\frac{\cos(2\pi mt)}{2i}$, $T(x,t) = \cos(2\pi mt)\sin(2\pi mx)$. Therefore we have used the Fourier method for the exact solution.

We can consider the **analytical solutions** for FDS of (4.2) using the spectral representation of matrix $A = WDW^*$. From transformation $V = W^*U$ $(U = WV)$ follows the separate system of ODEs (4.4).

The solution of the system (4.4) is $V(t) = \cos(\sqrt{D}t)V_0$, $D = diag(\mu_k)$ or in the form (4.3), where $v_k(t) = \cos(\kappa_k t)v_k(0), \kappa_k = \sqrt{\mu_k}$, $v_k(0) = (W^*U_0)_k = 0$, $v_k(t) = 0$ for $k \ne m$, $k \ne N - m$. From $\mu_{N-m} = \mu_k$, $w^{N-k} = w^{*k}$ follows $v_m(0) = \frac{\sqrt{N}}{2i}$, $v_{N-m}(0) = -\frac{\sqrt{N}}{2i}$.

Therefore $U(t) = \cos(\sqrt{\mu_m}t)U_0$, where $U_0 = (\sin(2\pi mx_1), \ldots, \sin(2\pi mx_N))^T$ is the column-vector of the N order, $x_j = jh$, $j = \overline{1.N}$, $Nh = 1$.

The solution can be obtained in the matrix form $U(t) = W\cos(\sqrt{D}t)W^*U_0$. For the FDSES $\sqrt{\mu_m} = \sqrt{d_m} = 2\pi m$ and we have also the exact solution.

Using the **discrete Fourier** transformation $U(t) = \sum_{k=1}^{N} a_k(t) w^k (A w^k = \mu_k w^k)$, we get $a_k(t) = \cos(\sqrt{\mu_k} t) a_k(0)$, where $a_k(0) = U_0 w^{*k} = 0$ for $k \neq m, k \neq N-m$, $a_m(0) = \frac{\sqrt{N}}{2i}, a_{N-m}(0) = -\frac{\sqrt{N}}{2i}$.

We have $U(t) = a_m(t) w^m + a_{N-m}(t) w_*^m = \frac{\sqrt{N}}{2i} \cos(\sqrt{\mu_m} t)(w^m - w_*^m) = \cos(\sqrt{\mu_m} t) U_0$. For numerical calculation we consider the initial boundary value problem (4.1) with $t_f = L = 1, f = 0, T_0 = \sin(2\pi m x)$, $\bar{T}_0 = 0$, $T(x,t) = \sin(2\pi m x)\cos(2\pi m t)$, for $m = 1; 2; 3; 4$, $N = 10$.

MATLAB code can be found in appendix A.16.

Using the operator **Wave2m(10)** we obtain the following maximal errors ($t_f = 1$) (see tab. 4.1):

Table 4.1 The FDS maximal error depending on order of approximation and m by $N = 10$

Method	m=1	m=2	m=3	m=4
$O(h^2)$	0.0050	0.2958	1.797	–
$O(h^4)$	$1.2 10^{-7}$	0.0110	0.4301	–
$O(h^6)$	3.10^{-8}	6.10^{-4}	0.0911	1.550
$O(h^8)$	2.10^{-10}	3.10^{-5}	0.0219	0.9724
FDSES	2.10^{-15}	1.10^{-15}	3.10^{-15}	1.10^{-15}

In the fig. 4.3, 4.4 we can see the FDSES exact solutions by $m = 4$ and $N = 10, N = 20$.

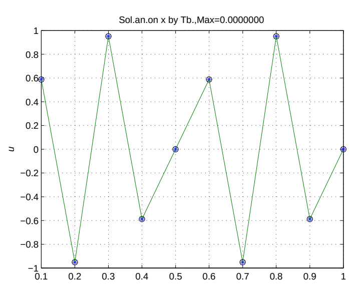

Fig. 4.3 FDSES solutions by $N = 10, m = 4, t_f = 1$

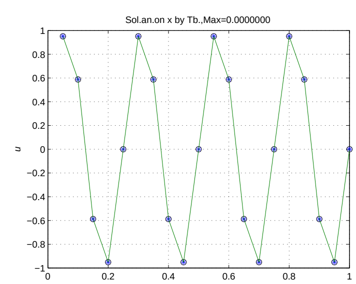

Fig. 4.4 FDSES solutions by $N = 20, m = 4, t_f = 1$

4.5 Example of nonlinear wave equation

We shall consider the initial - boundary value problem for solving the following nonlinear wave equation:

$$
\begin{cases}
\dfrac{\partial^2 T(x,t)}{\partial t^2} = \dfrac{\partial^2 (g(T))}{\partial x^2} + f(T), x \in (0,L), t \in (0,t_f), \\
T(0,t) = T(L,t), \dfrac{\partial T(0,t)}{\partial x} = \dfrac{\partial T(L,t)}{\partial x}, t \in (0,t_f), \\
T(x,0) = T_0(x), \dfrac{\partial T(x,0)}{\partial t} = \bar{T}_0(x), x \in (0,L).
\end{cases}
\tag{4.5}
$$

where $g(T)$, $f(T)$ is nonlinear given functions. Using the FDS we obtain from (4.5) the initial value problem for system of nonlinear ODEs of the second order in the following matrix form

$$
\begin{cases}
\ddot{U}(t) = -AG(U) + F(U), \\
U(0) = U_0, \dot{U}(0) = \bar{U}_0,
\end{cases}
\tag{4.6}
$$

where G, F are the vectors-column of N order with elements $g_k = g(u(x_k,t))$, $f_k = f(u(x_k,t))$, $k = \overline{1,N}$.

To use the Matlab solvers we need to write the system of ODEs in the normal form

$$
\begin{cases}
\dot{y}_1(t) = y_2(t), \\
\dot{y}_2(t) = -AG(y_1) + F(y_1), \\
y_1(0) = U_0, y_2(0) = \bar{U}_0,
\end{cases}
\tag{4.7}
$$

$U(t) = y_1(t)$ or

$$
\dot{y}(t) = A_1 G(y) + BF(y),
$$

where y is the the column-vectors of $2N$ order with elements (y_1, y_2), A_1, B are the matrices of $2N$ order in the following form:

$$
A_1 = \begin{pmatrix} 0 & E \\ -A & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ E & 0 \end{pmatrix}
$$

The numerical experiment with $L = 1$, $t_f = 0.1$ and $F = aT^\beta$, $g(T) = T^{\sigma+1}$, $\sigma = 2$, $T_0 = \sin(2\pi x)$, $\bar{T}_0 = 0$, $\beta = a = 0$ is produced by MATLAB 7.4 solver "ode15s".

MATLAB code can be found in appendix A.17.

In table 4.2 are shown maximal values of solution $\max |y_1(t_f)|$ depending on N and order of approximation.

Table 4.2 The maximal values $\max |y_1(t_f)|$ depending on order of approximation and N

Method	N=10	N=20	N=40
$O(h^2)$	0.6622	0.5515	0.5735
$O(h^4)$	0.6078	0.5883	0.5928
$O(h^6)$	0.5579	0.5947	0.5957
$O(h^8)$	0.5322	0.5961	0.5964
FDSES	0.5058	0.5950	0.5928

In the figs. 4.5, 4.6 we can see the FDSES solutions by $N = 40$.

The numerical experiment with $L = 1$, $t_f = 0.8$ and $F = a(sin(T))^\beta$, $g(T) = T^{\sigma+1}$, $\sigma = 0$, $T_0 = \sin^{100}(\pi x)$, $\bar{T}_0 = 0$, $\beta = a = 1$ is produced also by MATLAB 7.4 solver "ode15s" with operator $F = A * y.^\wedge sigma1 + a * B * sin(y).^\wedge beta$.

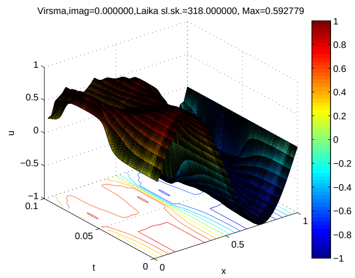

Fig. 4.5 FDSES solutions-surface by $N = 40$

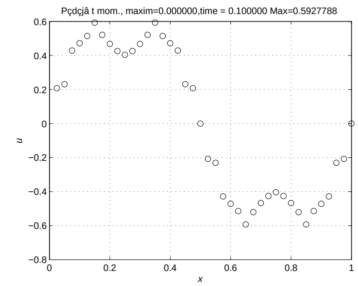

Fig. 4.6 FDSES solutions by $N = 20, t = t_f = 0.1$

In table 4.3 are shown maximal values of solution $\max|y_1(t_f)|$ depending on N and order of approximation.

Table 4.3 The maximal values $\max|y_1(t_f)|$ depending on order of approximation and N

Method	N=10	N=20	N=40	N=80
$O(h^2)$	0.4789	0.3076	0.4325	0.5038
$O(h^4)$	0.4097	0.3890	0.4806	0.5227
$O(h^6)$	0.4936	0.4124	0.5073	0.5242
$O(h^8)$	0.4593	0.3859	0.5163	0.5243
FDSES	0.5306	0.5243	0.5243	0.5243

In the figs. 4.7, 4.8 we can see the FDSES solutions by $N = 80$.

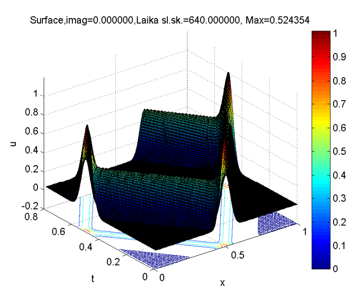

Fig. 4.7 FDSES solutions-surface by $N = 80$

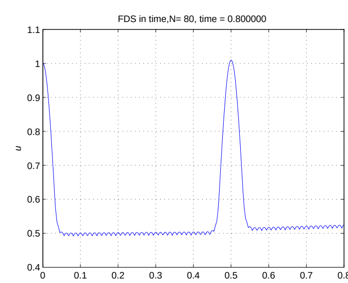

Fig. 4.8 FDSES max solutions depending on t by $N = 80$

In the figs. 4.9, 4.10 we can see the FDSES and FDS $O(h^2)$ solutions by $N = 80$, $t_f = 0.8$.

We can see, that FDS methods give the solutions with oscillations. FDSES method is without oscillations and the solution is positive even if $N = 10$.

4.6 Mathematical model for wave equation with convection

We consider the linear wave equation in the following form:

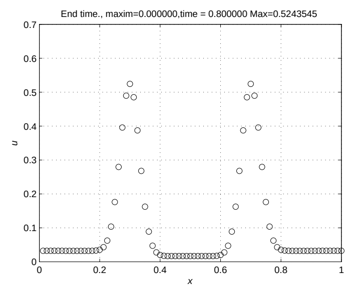

Fig. 4.9 FDSES solutions by $N = 80, t_f = 0.8$

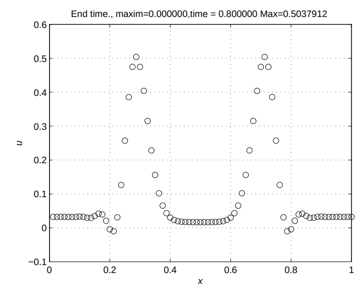

Fig. 4.10 FDS $-O(h^2)$ solutions by $N = 80, t_f = 0.8$

$$\frac{\partial^2 T(x,t)}{\partial t^2} = \frac{\partial^2 T(x,t)}{\partial x^2} + a\frac{\partial T(x,t)}{\partial x} + f(xt) \tag{4.8}$$

with the periodic boundary conditions (4.1) (a=const).
We can use the **Fourier method** for solving the initial-boundary value problem in the form $T(x,t) = \sum_{k\in Z} a_k(t)w^k(x), f(x,t) = \sum_{k\in Z} b_k(t)w^k(x)$, where $w^k(x)$ are the orthonormal eigenvectors, $b_k(t) = (f, w^{*k}(x))$.
Then for the unknown functions $a_k(t)$ get the complex initial value problem for ODEs of the second order:

$$\begin{cases} \ddot{a}_k(t) + a_k(t)\lambda_k = b_k(t), \\ a_k(0) = \dfrac{1}{L}\displaystyle\int_0^L T_0(s)\exp\frac{-2i\pi ks}{L}ds, \\ \dot{a}_k(0) = \dfrac{1}{L}\displaystyle\int_0^L \bar{T}_0(s)\exp\frac{-2i\pi ks}{L}ds, \\ b_k(t) = \dfrac{1}{L}\displaystyle\int_0^L f(s,t)\exp\frac{-2i\pi ks}{L}ds. \end{cases} \tag{4.9}$$

The solution of (4.9) is

$$a_k(t) = \cos(\sqrt{\lambda_k}t)a_k(0) + \frac{\sin(\sqrt{\lambda_k}t))}{\sqrt{\lambda_k}}\dot{a}_k(0) + \int_0^t \frac{\sin(\sqrt{\lambda_k}(t-s))}{\sqrt{\lambda_k}}b_k(s)ds.$$

The solution with the Fourier method can also obtain in real form. Functions $f(x,t)$ and $T(x,t)$ can be expressed in form (2.5).

From $f(x,t) = \frac{\partial^2 T(x,t)}{\partial t^2} - (\frac{\partial^2 T(x,t)}{\partial x^2} + a\frac{\partial T(x,t)}{\partial x})$
follows $f(x,t) = \sum_{k=1}^{\infty}(\ddot{a}_{kc}(t)\cos\frac{2\pi kx}{L} + \ddot{a}_{ks}(t)\sin\frac{2\pi kx}{L}) + \frac{\ddot{a}_{0c}(t)}{2} +$
$\sum_{k=1}^{\infty}((a_{kc}(t)Re(\lambda_k) + a_{ks}(t)Im(\lambda_k))\cos\frac{2\pi kx}{L} + (a_{ks}(t)Re(\lambda_k) - a_{kc}(t)Im(\lambda_k))\sin\frac{2\pi kx}{L})$,
because $(a_k(t)\lambda_k + a_{-k}(t)\lambda_{-k})/sqrtN = a_{kc}(t)Re(\lambda_k) + a_{ks}(t)Im(\lambda_k)$, $i(a_k(t)\lambda_k - a_{-k}(t)\lambda_{-k})/\sqrt{L} = a_{ks}(t)Re(\lambda_k) - a_{kc}(t)Im(\lambda_k)$, where $a_{kc}(t) = \frac{a_k(t)+a_{-k}(t)}{\sqrt{L}}, a_{ks}(t) = \frac{i(a_k(t)-a_{-k}(t)}{\sqrt{N}}$ are the coefficients in the expression from the solution $T(x,t)$.
Therefore we obtain the initial boundary value problem for the system of two ODEs:

$$\begin{cases} \ddot{a}_{kc}(t) + a_{kc}(t)Re(\lambda_k) + a_{ks}(t)Im(\lambda_k) = b_{kc}(t), \\ \ddot{a}_{ks}(t) + a_{ks}(t)Re(\lambda_k) - a_{kc}(t)Im(\lambda_k) = b_{ks}(t), \\ a_{kc}(0) = \frac{2}{L}\int_0^L T_0(s)\cos\frac{2\pi ks}{L}ds, a_{ks}(0) = \frac{2}{L}\int_0^L T_0(s)\sin\frac{2\pi ks}{L}ds, \\ \dot{a}_{kc}(0) = \frac{2}{L}\int_0^L \bar{T}_0(s)\cos\frac{2\pi ks}{L}ds, \dot{a}_{ks}(0) = \frac{2}{L}\int_0^L \bar{T}_0(s)\sin\frac{2\pi ks}{L}ds. \end{cases} \quad (4.10)$$

In the matrix form we have

$$\ddot{A}_k(t) + \Lambda_k A_k(t) = B_k(t), A_k(0) = A_{k0}, \dot{A}_k(0) = \dot{A}_{k0} \quad (4.11)$$

where $\Lambda_k = \begin{pmatrix} Re(\lambda_k) & Im(\lambda_k) \\ -Im(\lambda_k) & Re(\lambda_k) \end{pmatrix}$ is the matrix of the second order, $A_k(t), B_k(t), A_{k0}$, $\dot{A}_{k0}$ are the column-vectors with elements $(a_{kc}(t), a_{ks}(t))$, $(b_{kc}(t), b_{ks}(t))$, $(a_{kc}(0), a_{ks}(0))$, $(\dot{a}_{kc}(0), \dot{a}_{ks}(0))$.

We can represented the matrix Λ_k in the form $\Lambda_k = PDP^{-1}$, where $P = \begin{pmatrix} 0.5 & -i \\ -0.5i & 1 \end{pmatrix}$, $P^{-1} = \begin{pmatrix} 1 & i \\ 0.5i & 0.5 \end{pmatrix}$, $D = \begin{pmatrix} \lambda_k^* & 0 \\ 0 & \lambda_k \end{pmatrix}$, where $\lambda_k^* = Re(\lambda_k) - iIm(\lambda_k)$, $Re(\lambda_k) = \frac{4\pi^2 k^2}{L^2}$, $Im(\lambda_k) = -\frac{2\pi ka}{L}$.
Then the matrix solution of (4.11)

$$A_k(t) = \cos(\sqrt{\Lambda_k}t)A_{k0} + \Lambda_k^{-0.5}\sin(\sqrt{\Lambda_k}t)\dot{A}_{k0} + \Lambda_k^{-0.5}\int_0^t \sin(\sqrt{\Lambda_k}(t-s)B_k(s)ds$$

with the transformations $\tilde{A}_k(t) = P^{-1}A_k(t), A_k(t) = P\tilde{A}_k(t)$ can obtain in the form

$$A_k(t) = \cos(\sqrt{D}t)\tilde{A}_{k0} + D^{-0.5}\sin(\sqrt{D}t)\tilde{\dot{A}}_{k0} + D^{-0.5}\int_0^t \sin(\sqrt{D}(t-s)\tilde{B}_k(s)ds$$

where $\tilde{A}_{k0} = P^{-1}A_{k0}$, $\tilde{\dot{A}}_{k0} = P^{-1}\dot{A}_{k0}$, $\tilde{B}_k(t) = P^{-1}B_k(t)$.
For this separable we can determine the elements $\tilde{a}_{kc}(t)$, $\tilde{a}_{ks}(t)$ of the column-vector $\tilde{A}_k$ depending on the elements $\tilde{a}_{kc}(0) = a_{kc}(0) + ia_{ks}(0)$, $\tilde{a}_{ks}(0) = 0.5ia_{kc}(0) + 0.5a_{ks}(0)$, $\tilde{\dot{a}}_{kc}(0) = \dot{a}_{kc}(0) + i\dot{a}_{ks}(0)$, $\tilde{\dot{a}}_{ks}(0) = 0.5i\dot{a}_{kc}(0) + 0.5\dot{a}_{ks}(0)$,
$\tilde{b}_{kc}(t) = b_{kc}(t) + ib_{ks}(t)$, $\tilde{b}_{ks}(t) = 0.5ib_{kc}(t) + 0.5b_{ks}(t)$
of the column-vectors $\tilde{A}_{k0}$, $\tilde{\dot{A}}_{k0}$, $\tilde{B}_k(t)$ we obtain the solution of the ODEs system (2.7) in the following form

$$
\begin{cases}
a_{kc}(t) = & Re(\cos(\sqrt{\lambda_k}t))a_{kc}(0) + a_{ks}(0)Im(\cos(\sqrt{\lambda_k}t)) \\
& + Re(\frac{\sin(\sqrt{\lambda_k}t))}{\sqrt{\lambda_k}})\dot{a}_{kc}(0) + \dot{a}_{ks}(0)Im(\frac{\sin(\sqrt{\lambda_k}t))}{\sqrt{\lambda_k}}) \\
& + \int_0^t (Re(\frac{\sin(\sqrt{\lambda_k}(t-s))}{\sqrt{\lambda_k}})b_{kc}(s) + b_{ks}(s)Im(\frac{\sin(\sqrt{\lambda_k}(t-s))}{\sqrt{\lambda_k}}))ds, \\
a_{ks}(t) = & -Im(\cos(\sqrt{\lambda_k}t))a_{kc}(0) + a_{ks}(0)Re(\cos(\sqrt{\lambda_k}t)) \\
& - Im(\frac{\sin(\sqrt{\lambda_k}t)}{\sqrt{\lambda_k}})\dot{a}_{kc}(0) + \dot{a}_{ks}(0)Re(\frac{\sin(\sqrt{\lambda_k}t))}{\sqrt{\lambda_k}}) \\
& - \int_0^t (Im(\frac{\sin(\sqrt{\lambda_k}(t-s))}{\sqrt{\lambda_k}})b_{kc}(s) + b_{ks}(s)Re(\frac{\sin(\sqrt{\lambda_k}(t-s))}{\sqrt{\lambda_k}}))ds,
\end{cases}
\tag{4.12}
$$

For $k = 0$ we obtain $a_{0c}(t) = \int_0^t b_{0c}(s)ds + a_{0c}(0) + t\dot{a}_{0c}(0)$.

For numerical calculation we consider the initial boundary value problem with $f(x,t) = -2\pi a \cos(2\pi x)\cos(2\pi t)$, $T_0 = \sin(\pi x)$, $\bar{T}_0 = 0$, $T(x,t) = \sin(\pi x)cos(\pi t)$.
From (4.9) follows:
$T(x,t) = a_1(t)\exp(2\pi i x) + a_{-1}(t)\exp(-2\pi i x)$, $a_1(0) = \frac{1}{2i}$, $a_{-1}(0) = -\frac{1}{2i}$, $b_1(t) = b_{-1}(t) = -\pi a \cos(2\pi t)$,
$\dot{a}_1(0) = \dot{a}_{-1}(0) = 0$, $\lambda_1 = 4\pi^2 - 2\pi a i$, $\lambda_{-1} = 4\pi^2 + 2\pi a i$, $a_1(t) = \cos(\sqrt{\lambda_1}t)/(2i) + (\cos(2\pi t) - cos(\sqrt{\lambda_1}t))/(2i)$,
$a_{-1}(t) = -\cos(\sqrt{\lambda_{-1}}t)/(2i) - (\cos(2\pi t) - cos(\sqrt{\lambda_{-1}}t))/(2i)$.
Therefore $T(x,t) = (\cos(\sqrt{\lambda_1}t)\exp(2\pi i x) - \cos(\sqrt{\lambda_{-1}}t)\exp(-2\pi i x))/(2i)+ ((\cos(2\pi t) - \cos(\sqrt{\lambda_1}t)\exp(2\pi i x))\exp(2\pi i x) - (\cos(2\pi t) - \cos(\sqrt{\lambda_{-1}}t))\exp(-2\pi i x))/(2i) = \cos(2\pi t)sin(2\pi x)$. From (4.12) we have:
$T(x,t) = a_{1c}\cos(2\pi x) + a_{1s}\sin(2\pi x)$, $a_{1c}(0) = 0, a_{1s}(0) = 1$, $\dot{a}_{1c}(0) = \dot{a}_{1s}(0) = 0$,
$b_{1s}(0) = 0, b_{1c}(0) = -2\pi a \cos(2\pi t)$, $a_{1c}(t) = Im(\cos(\sqrt{\lambda_1}t)) + Re((\cos(2\pi t) - cos(\sqrt{\lambda_1}t))/i$,
$a_{1s}(t) = Re(\cos(\sqrt{\lambda_1}t))$, $T(x,t) = Re(\cos(\sqrt{\lambda_1}t))\sin(2\pi x)$.

We have for $a = L = 1$ the following MATLAB operators:

$\mathbf{t = 0:0.01:1; l = 4*\pi^2 - 2*\pi*i; v = real}(\cos(\sqrt{(}\mathbf{l})*\mathbf{t})); \mathbf{v1} = \cos(\mathbf{2}*\pi*\mathbf{t});$
$\mathbf{plot}(\mathbf{t}, \mathbf{v}, '*', \mathbf{t}, \mathbf{v1}, '\mathbf{o}')$.

In the fig. 4.11 are represented the exact and approximate solutions.

For the **discrete problem** we have the system of N ODEs in the form of (4.2), where $a^2 = 1$ and the circulant matrix
$A = \frac{1}{h^2}[2\gamma, -(\gamma+\alpha), 0, 0, \ldots 0, -(\gamma-\alpha)]$,
with the eigenvalues $\mu_k = \frac{4}{h^2}(\sin(k\pi/N)^2(\gamma - i\alpha\cot\frac{k\pi}{N})$,
and with the elements of the orthonormal eigenvectors
$w_j^k = \sqrt{\frac{1}{N}}\exp(2\pi i kj/N)$, $w_{*j}^k = \sqrt{\frac{1}{N}}\exp(-2\pi i kj/N)$, $k, j = \overline{1,N}$. Here $\gamma = \alpha\coth(\alpha)$, $\alpha = ah/2$.
For the column-vector $F(t)$ elements $f_j(t)$ we obtain $f_j(t) = \sum_{k=1}^{*N_2}(b_{kc}(t)\cos\frac{2\pi kj}{N} + b_{ks}(t)\sin\frac{2\pi kj}{N}) + \frac{b_{0c}(t)}{2}$,
where
$b_{kc}(t) = \frac{2}{N}\sum_{j=1}^N f_j(t)\cos\frac{2\pi kj}{N}$, $b_{ks}(t) = \frac{2}{N}\sum_{j=1}^N f_j(t)\sin\frac{2\pi kj}{N}, k = \overline{1,N_2}$,
$b_0(t) = b_N(t) = \frac{1}{\sqrt{N}}\sum_{j=1}^N f_j(t)$, $b_{0c}(t) = b_{Nc}(t) = \frac{2}{\sqrt{N}}b_0(t), b_{N_2,c}(t) = \frac{2}{\sqrt{N}}$
$b_{N_2}(t) = \frac{2}{N}\sum_{j=1}^N \cos(j\pi)$, $N_2 = \frac{N}{2}, b_{N_2 s}(t) = b_{Ns}(t) = 0$,

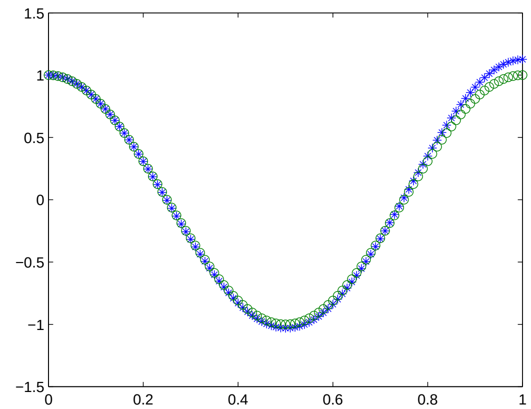

Fig. 4.11 Wave solutions $\cos(2\pi t)\sin(2\pi x)$ depending on t by $a = 1, L = 1, N = 100, x = \pi/4, t_f = 1$

$\sum_{k=1}^{*N_2} \beta_k = \sum_{k=1}^{N_2-1} \beta_k + \frac{\beta_{N/2}}{2}$.
For the solution $u_j(t) = \sum_{k=1}^{*N_2}(a_{kc}(t)\cos\frac{2\pi kj}{N} + a_{ks}(t)\sin\frac{2\pi kj}{N}) + \frac{a_{0c}(t)}{2}$,
$u_j(0) = \sum_{k=1}^{*N_2}(a_{kc}(0)\cos\frac{2\pi kj}{N} + a_{ks}(0)\sin\frac{2\pi kj}{N}) + \frac{a_{0c}(0)}{2}$
$\dot{u}_j(0) = \sum_{k=1}^{*N_2}(\dot{a}_{kc}(0)\cos\frac{2\pi kj}{N} + \dot{a}_{ks}(0)\sin\frac{2\pi kj}{N}) + \frac{\dot{a}_{0c}(0)}{2}$
with $a_{kc}(0) = \frac{2}{N}\sum_{j=1}^{N} u_j(0)\cos\frac{2\pi kj}{N}$, $a_{ks}(0) = \frac{2}{N}\sum_{j=1}^{N} u_j(0)\sin\frac{2\pi kj}{N}$,
$\dot{a}_{kc}(0) = \frac{2}{N}\sum_{j=1}^{N} \dot{u}_j(0)\cos\frac{2\pi kj}{N}$, $\dot{a}_{ks}(0) = \frac{2}{N}\sum_{j=1}^{N} \dot{u}_j(0)\sin\frac{2\pi kj}{N}$
we need to determine the unknown functions $a_{kc}(t)$, $a_{ks}(t)$ of the following expressions
$f_j(t) = \ddot{u}_j + (Au)_j = \sum_{k=1}^{*N_2}(\ddot{a}_{kc}(t)\cos\frac{2\pi kj}{N} + \ddot{a}_{ks}(t)\sin\frac{2\pi kj}{N}) + \frac{\ddot{a}_{Nc}(t)}{2} +$
$\sum_{k=1}^{*N_2}(a_{kc}(t)Re(\mu_k) + a_{ks}(t)Im(\mu_k))\cos\frac{2\pi kj}{N} + (a_{ks}(t)Re(\mu_k) - a_{kc}(t)Im(\mu_k))\sin\frac{2\pi kj}{N})$.

Therefore, for the determine the functions $a_{kc}(t)$, $a_{ks}(t)$ we obtain the systems of ODEs (4.10, 4.11) and the solution (4.12), where the eigenvalues λ_k are replaced with the discrete eigenvalues μ_k, $k = \overline{1,N}$.

4.7 System of hyperbolic type equations with periodic BCs

We consider the initial boundary problem of linear M-order system in the following form:

$$\begin{cases} \dfrac{\partial^2 T_m}{\partial t^2} = \sum\limits_{s=1}^{M}(\dfrac{\partial}{\partial x}(k_{m,s}\dfrac{\partial T_m}{\partial x}) + p_{m,s}\dfrac{\partial T_m}{\partial x}) + f_m, \\ T_m(0,t) = T_m(L,t), \dfrac{\partial T_m(0,t)}{\partial x} = \dfrac{\partial T_m(L,t)}{\partial x}, \\ T_m(x,0) = T_{m,0}(x), \dfrac{\partial T_m(x,0)}{\partial t} = \tilde{T}_m(x), x \in (0,L), m = \overline{1,M}, \end{cases} \tag{4.13}$$

where $\tilde{K}$ is the positive definite M-order matrix with different positive eigenvalues $\mu_K > 0$ and the elements $k_{m,s}$, P is the real M-order matrix with different real eigenvalues μ_P and the elements $p_{m,s}$, $m, s = \overline{1,M}$.

This system can be rewritten in the matrix form

$$\begin{cases} \dfrac{\partial^2 u(x,t)}{\partial t^2} = \dfrac{\partial}{\partial x}(\tilde{K}\dfrac{\partial u(x,t)}{\partial x}) + P\dfrac{\partial u(x,t)}{\partial x} + f(x,t), \\ u(0,t) = u(L,t), \dfrac{\partial u(0,t)}{\partial x} = \dfrac{\partial u(L,t)}{\partial x}, t \in (0,t_f), \\ u(x,0) = u_0(x), \dfrac{\partial u(x,0)}{\partial t} = \tilde{u}_0(x), x \in (0,L), \end{cases} \tag{4.14}$$

where u, f are column-vectors with elements T_m, f_m, $m = \overline{1,M}$.
Using the Fourier series the solution we can obtain problem in the following form:
$u(x,t) = \sum_{k=1}^{\infty}(a_{kc}(t)\cos\frac{2\pi kx}{L} + a_{ks}(t)\sin\frac{2\pi kx}{L}) + \frac{a_{0c}(t)}{2}$,
$f(x,t) = \sum_{k=1}^{\infty}(b_{kc}(t)\cos\frac{2\pi kx}{L} + b_{ks}(t)\sin\frac{2\pi kx}{L}) + \frac{b_{0c}(t)}{2}$,
$b_{kc}(t) = \frac{2}{L}\int_0^L f(\xi,t)\cos\frac{2\pi k\xi}{L}d\xi, b_{ks}(t) = \frac{2}{L}\int_0^L f(\xi,t)\sin\frac{2\pi k\xi}{L}d\xi$,
where the column-vectors $a_{kc}(t)$, $a_{ks}(t)$ of the M order are the corresponding solutions of the following differential equations

$$\begin{cases} \ddot{a}_{kc}(t) + \lambda_k\tilde{K}a_{kc}(t) - \dfrac{2\pi k}{L}Pa_{ks}(t) = b_{kc}(t), \\ \ddot{a}_{ks}(t) + \lambda_k\tilde{K}a_{ks}(t) + \dfrac{2\pi k}{L}Pa_{kc}(t) = b_{ks}(t), \end{cases} \tag{4.15}$$

where $b_{kc}(t)$, $b_{ks}(t)$ are the column-vectors of M order, $a_{kc}(0) = \frac{2}{L}\int_0^L u_0\xi)\cos\frac{2\pi k\xi}{L}d\xi,$, $a_{ks}(0) = \frac{2}{L}\int_0^L u_0\xi)\sin\frac{2\pi k\xi}{L}d\xi$, $\dot{a}_{kc}(0) = \frac{2}{L}\int_0^L \tilde{u}_0\xi)\cos\frac{2\pi k\xi}{L}d\xi,$, $\dot{a}_{ks}(0) = \frac{2}{L}\int_0^L \tilde{u}_0\xi)\sin\frac{2\pi k\xi}{L}d\xi$.
For the **discrete problem** ($O(h^{2n})$ order of approximation) we have the system of ODEs in the following form:

$$\begin{cases} \ddot{v}_j(t) = \tilde{K}\Lambda v_j(t) + P\Lambda^0 v_j(t) + f_j(t), t \in [0,t_f], \\ v_j(0) = u_0(x_j), \dot{v}_j(0) = \tilde{u}_0(x_j), x_j = jh, Nh = L, j = \overline{1,N}, \end{cases} \tag{4.16}$$

where the column-vectors of the M order $v_j(t) \approx u(x_j,t), f_j(t) = f(x_j,t)$,
the expressions of the finite difference operators in multi-point stencil with 2n+1-points ($N \geq 2n+1$)
$\Lambda v_j = \frac{1}{h^2}(C_n(v_{j-n} + v_{j+n}) + \cdots + C_1(v_{j-1} + v_{j+1}) + C_0 v_j), C_0 = -\sum_{p=1}^n C_p,\ C_p = \frac{2(n!)^2(-1)^{p-1}}{p^2(n-p)!(n+p)!}, p = \overline{1,n}$
$\Lambda^0 v_j = \frac{1}{h}(c_n(v_{j+n} - v_{j-n}) + \cdots + c_1(v_{j+1} - v_{j-1}))$, $c_p = \frac{(n!)^2(-1)^{p-1}}{p(n-p)!(n+p)!}, p = \overline{1,n}$.
We have following matrix representation for circulant matrices
$A = -\Lambda = -\frac{1}{h^2}[C_0, C_1, \ldots, C_n, 0, \ldots, 0, C_n, \ldots, C_1]$,
with the eigenvalues $\mu_k = \frac{4}{h^2}\sum_{p=1}^n Q_p\sin^{2p}(\pi k/N)$, $Q_p = \frac{2((p-1!))^2 4^{p-1}}{(2p)!}$ and $A^0 = \Lambda^0 = \frac{1}{h}[0, c_1, \ldots, c_n, 0, \ldots, 0, -c_n, \ldots, -c_1]$, with the eigenvalues $\mu_k^0 = \frac{2i}{h}\sum_{p=1}^n c_p\sin\frac{2\pi pk}{N} = \frac{2i}{h}\sin\frac{2\pi k}{N}\sum_{p=1}^n q_p\sin^{2p-2}\frac{\pi k}{N}$, where $q_p = \frac{(p!)^2 4^{p-1}}{p(2p)!}$.
Using the discrete Fourier method the solution we can obtain the problem in the following form:
$v_j(t) = \sum_{k=1}^{*N_2}(a_{kc}(t)\cos\frac{2\pi kj}{N} + a_{ks}(t)\sin\frac{2\pi kj}{N}) + \frac{a_{0c}(t)}{2}$,
$f_j(t) = \sum_{k=1}^{*N_2}(b_{kc}(t)\cos\frac{2\pi kj}{N} + b_{ks}(t)\sin\frac{2\pi kj}{N}) + \frac{b_{0c}(t)}{2}$,
$b_{kc}(t) = \frac{2}{N}\sum_{j=1}^N f_j(t)\cos\frac{2\pi kj}{N}, b_{ks}(t) = \frac{2}{N}\sum_{j=1}^N f_j(t)\sin\frac{2\pi kj}{N}$,

where $a_{kc}(t)$, $a_{ks}(t)$ are the corresponding solutions of (4.15) and λ_k, $\frac{2\pi k}{L}$ are replaced with μ_k, $Im(\mu_k^0)$, $a_{kc}(0) = \frac{2}{N}\sum_{j=1}^{N} u_0(x_j)\cos\frac{2\pi kj}{N}$, $a_{ks}(0) = \frac{2}{N}\sum_{j=1}^{N} u_0(x_j)\sin\frac{2\pi kj}{N}$.
$\dot{a}_{kc}(0) = \frac{2}{N}\sum_{j=1}^{N} \tilde{u}_0(x_j)\cos\frac{2\pi kj}{N}$, $\dot{a}_{ks}(0) = \frac{2}{N}\sum_{j=1}^{N} \tilde{u}_0(x_j)\sin\frac{2\pi kj}{N}$.

This real form we can obtain also from the following expressions
$A\cos_k = \mu_k \cos_k, A^0\cos_k = -|\mu_k^0|\sin_k, A\sin_k = \mu_k\sin_k, A^0\sin_k = |\mu_k^0|\cos_k$, where $\sin_k$, $\cos_k$ are N-order column-vectors with the elements $\sin\frac{2\pi kj}{N}$, $\cos\frac{2\pi kj}{N}$ and using the orthonormal conditions
$\sum_{j=1}^{N}\sin_k\cos_s = \sum_{j=1}^{N}\sin_k\sin_s = \sum_{j=1}^{N}\cos_k\cos_s = 0.$
Then for fixed frequency k in the initial data the solution can be written in the form $u(t) = d_s(t)\sin_k + d_c(t)\cos_k$, where $d_s(t)$, $d_c(t)$ are unknown the time-depending vector-functions. Then $\ddot{u}(t) = \ddot{d}_s(t)\sin_k + \ddot{d}_c(t)\cos_k$,
$Au(t) = \mu_k(d_s(t)\sin_k + d_c(t)\cos_k),\ A^0u(t) = |\mu_k^0|(d_s(t)\cos_k - d_c(t)\sin_k)$.
For FDSES, replaced the discrete eigenvalue μ_k, $Im(\mu_k^0)$ with λ_k, $\frac{2\pi k}{L}$ we obtain the exact solutions for initial data with the frequency $\leq N/2$.

The time-dependent difference equation (4.16) using the MN-order column-vector $v(t)$ with the elements $v_j^m(t)$, $j=\overline{1,N}, m=\overline{1,M}$ in the form

$$\ddot{v}(t) = (-\tilde{K}\bigotimes A + P\bigotimes A^0)v(t) + f(t) \tag{4.17}$$

serves as an approximation to the differential problem, in the sense that any smooth solution $u(t)$ satisfies the approximation (4.16) modulo a small local truncation error $\Psi(h,t) = O(h^{2n})$:

$$\frac{\partial^2 u(t)}{\partial t^2} = (-\tilde{K}\bigotimes A)u(t) + (P\bigotimes B)u(t) + f(t) + \Psi(h,t), \tag{4.18}$$

where MN-order matrices

$$\tilde{K}\bigotimes A = \begin{pmatrix} k_{1,1}A & \cdots & k_{1,M}A \\ \cdots & \cdots & \cdots \\ k_{M,1}A & \cdots & k_{M,M}A \end{pmatrix}, P\bigotimes A^0 = \begin{pmatrix} p_{1,1}A^0 & \cdots & p_{1,M}A^0 \\ \cdots & \cdots & \cdots \\ p_{M,1}A^0 & \cdots & p_{M,M}A^0 \end{pmatrix},$$

are Kronecker tensor products, $u(t)$, $u(0)$, $\tilde{u}(0)$, $f(t)$ are MN column-vectors with the elements $u_j^m(t), u_j^m(0)$, $\tilde{u}_j^m(0)$, f_j^m, $m=\overline{1,M}$, $j=\overline{1,N}$.
Matrices also can be defined with the representation $A = WDW^*$, $A^0 = WD^0W^*$, and solved numerically with the Matlab using the operator "kron",
If eigenvalues of matrices $\tilde{K}$, P are $\lambda_s(K) > 0$, $\lambda_s(P)$, $s=\overline{1,M}$ then exist the transformation of M-order matrices W_K, W_P, and the representation $K = W_K D_K W_K^{-1}, P = W_P D_P W_P^{-1}$, where $D_K = diag(\lambda_s(K)), D_P = diag(\lambda_s(P))$ are the diagonal matrices. From properties of Kronecker tensor product follows $(W^* = W^{-1})$:
$B = -K\bigotimes A + P\bigotimes A^0 = -(W_K D_K W_K^{-1}\bigotimes WDW^*) + (W_P D_P W_P^{-1}\bigotimes WD^0W^*) =$
$(W_K\bigotimes W)(-D_K\bigotimes D)(W_K^{-1}\bigotimes W^*) + (W_P\bigotimes W)(-D_P\bigotimes D^0)(W_P^{-1}\bigotimes W^*)=$
$(W_K\bigotimes W)(-D_K\bigotimes D)(W_K\bigotimes W)^{-1} + (W_P\bigotimes W)(-D_P\bigotimes D^0)(W_P\bigotimes W)^{-1}$.
The eigenvalues $\lambda(B)$ of matrix B are $-\mu_k\lambda_s(K) + \mu_k^0\lambda_s(P), k=\overline{1,N}, s=\overline{1,M}$ with the $Re(\lambda(B)) \leq 0$ and the system of ODEs is stable.
For the approximation, if $v_j^m(t) = T_m(x_j,t), m=\overline{1,M}$ and for every time moment t
$u_j'' = \Lambda u_j + E_{2n}\frac{h^{2n}u^{(2n+2)}(\xi_j)}{2n+2)!}, E_{2n} = -2\sum_{k=1}^{n} C_k k^{2n+2}$
$u_j' = \Lambda^0 u_j + e_{2n}\frac{h^{2n}u^{(2n+1)}(\xi_j)}{2n+1)!}, e_{2n} = -2\sum_{k=1}^{n} c_k k^{2n+1}, x_{n-j} < \xi_j < x_{n+j},$

then $\ddot{u}(t) = Bv(t) + f(t) + \Psi(h,t)$, where v, f, Ψ are MN-order column-vectors and $\Psi(h,t) = O(h^{2n})$,or $||\Psi(h,t)|| \leq h^{2n}(\frac{|E_{2n}|}{(2n+2)!}||K||M_{2n+2}(t) + \frac{|e_{2n}|}{(2n+1)!}||P||M_{2n+1}(t))$.
$M_s = \max|\frac{\partial^s T(x,t)}{\partial x^s}|$ is the maximal estimate for corresponding derivatives.
Given stability we can now estimate the global error $e(t) = v(t) - u(t)$ and to find, that the error $e(t)$ is governed by the error equation

$$\frac{\partial^2 e(t)}{\partial t^2} = Be(t) + \Psi(h,t).$$

The solution of this equation is given by $e(t) = \cos(\sqrt{B}t)e(0) + \sin(\sqrt{B}t)(\sqrt{B})^{-1}\dot{e}(0) + (\sqrt{B})^{-1}\int_0^t \sin(\sqrt{B}(t-\xi))\Psi(h,\xi)d\xi$.
From $Re(\lambda(B)) \leq 0$ and that $||\Psi||$ and $||e(0)||$, $||\dot{e}(0)||$ are of order $O(h^{2n})$ follows the $2n$-order of convergence rate later on $||e(t)|| = O(h^{2n})$.

5. Numerical modeling of applied problems

In this chapter we will investigate modeling of several problems which arise in magnetohydrodynamics, process of heat transfer, also modeling in multilayer domain. These models were created in the works [22], [25], [20], [32], and [35].

5.1 Mathematical modeling of the 2D MHD flow around infinite cylinders with square-section placed periodically

The external 2D magnetic field has two components of the induction is in the following dimensionless form:
$\mathbf{B}_x = \cos(\alpha)$, $\mathbf{B}_y = \sin(\alpha)$, where α is the angle between the x-axes and direction of the induction vector. We analyze the flow depending on three way of homogeneous magnetic field: the field parallel to x-axis ($\alpha = 0$), transverse field ($\alpha = \frac{\pi}{2}$) and sloping fields ($\alpha = \frac{\pi}{4}$ or $\alpha = \frac{3\pi}{4}$).

The magnetic field creates the $\mathbf{F}_x(x, y)$, $\mathbf{F}_y(x, y)$ components of the Lorentz force $\mathbf{F}$. From the vector of Lorenz force $\mathbf{F} = \sigma(\mathbf{E} + V \times \mathbf{B}) \times \mathbf{B}$ for the 2D magnetic field we obtain
$\mathbf{F}_x = -\sigma \mathbf{B}_x(V_x \mathbf{B}_y - V_y \mathbf{B}_x + \mathbf{E}_z), \quad \mathbf{F}_y = \sigma \mathbf{B}_x(V_x \mathbf{B}_y - V_y \mathbf{B}_x + \mathbf{E}_z)$,
where $\mathbf{E}_z = const$ is the axial component of the electric field $\mathbf{E}$, σ is the electric conductivity, V_x, V_y are the components of velocity vector V.
The z-component of the vector's $curl\mathbf{F}$ has on influence to a liquid motion, which can be described by the dimensionless stationary Navier–Stokes equation in Cartesian coordinates (x, y) [20]:

$$\begin{cases} -\zeta V_y = -\frac{\partial \bar{p}}{\partial x} + Re^{-1} \Delta V_x + S\mathbf{F}_x, \\ \zeta V_x = -\frac{\partial \bar{p}}{\partial y} + Re^{-1} \Delta V_y + S\mathbf{F}_y, \\ \frac{\partial(V_x)}{\partial x} + \frac{\partial(V_y)}{\partial \phi} = 0. \end{cases} \tag{5.1}$$

where Δ is Laplace operator, $\bar{p} = p + 0.5\mathbf{V^2} - S\mathbf{E}_z A_z$, p is the pressure, A_z is the magnetic stream function, $\mathbf{B}_x = \frac{\partial A_z}{\partial y}$, $\mathbf{B}_y = -\frac{\partial A_z}{\partial x}$
or $A_z = y\cos(\alpha) - x\sin(\alpha)$, $\zeta = \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}$ is the vorticity function,
$Re = \frac{U_0 L_0}{\nu}, S = \frac{\sigma \mathbf{B}_0^2 L_0}{\rho U_0}$ are the Reynolds and Stuart numbers,
ρ, ν are the density and kinematic viscosity.
The cylinders are electrically non-conducting and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} j_z dxdy = 0$, where
$j_z = \sigma(\mathbf{E}_z + V_x \sin(\alpha) + V_y \cos(\alpha))$ is the axial component for the density of the electric current [21].
The equations (5.1) were put in the dimensionless form by scaling all the lengths to L_0 (the side of the square), U_0 (velocity), $\mathbf{B}_0$ (magnetic field), $P_0 = U_0^2\rho$ (pressure). The hydrodynamic stream function ψ can be determined by the formulas

$V_x = \frac{\partial \psi}{\partial y}, V_y = -\frac{\partial \psi}{\partial x}$. Eliminating pressure from the (5.1) one obtains

$$0 = J(\psi, \zeta) + Re^{-1}\Delta\zeta + Sf, \zeta = -\Delta\psi, \tag{5.2}$$

where $f = \sin(2\alpha)\frac{\partial^2 \psi}{\partial x \partial y} + \cos^2(\alpha)\frac{\partial^2 \psi}{\partial x^2} + \sin^2(\alpha)\frac{\partial^2 \psi}{\partial y^2}$
is the z- component of the vector's $curl\mathbf{F}$,
$J(\psi, \zeta) = \frac{\partial \psi}{\partial x}\frac{\partial \zeta}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \zeta}{\partial x}$ is the Jacobian of the functions ψ and ζ.
Using the boundary conditions (BCs) of symmetry and periodical ones we can consider only the domain that contains quarter of two cylinders. We can consider two situations, when cylinders are placed in the parallel series in-line arrangements and in the parallel series series-staggered arrangements. In this work we look into the case when cylinders are placed in the parallel series – in-line arrangements. We consider the domain $\Omega = \Omega_1 \bigcup \Omega_2$ (see figs. 5.1, 5.2),
where $\Omega_1 = \{(x, y) : l_1 \le x \le l_2, 0 \le y \le L_1\}, \Omega_2 = \{(x, y) : 0 \le x \le l, L_1 \le y \le L\}, 0 < l_1 < l_2 < l, 0 < L_1 < L$.
Here $C_1 = \{(x, y) : 0 < x < l_1, 0 < y < L_1\}$ and $C_2 = \{(x, y) : l_2 < x < l, 0 < y < L_1\}$ are the quarter of cylinders,
$L^1 = \{(x, L) : 0 \le x \le l\}, L^2 = \{(x, 0) : l_1 \le x \le l_2\}$ are the plane of symmetry with BCs $V_y = 0, \zeta = 0, \psi = \psi_0 \in L^1, \psi = 0 \in L^2$,
$W^1 = \{(x, L_1) : 0 < x \le l_1\}, W^2 = \{(x, L_1) : l_2 \le x < l\}, W^3 = \{(l_1, y) : 0 < y \le L_1\}$ and $W^4 = \{(l_2, y) : 0 < y \le L_1\}$ are the walls of the cylinders with the non slip BCs $V_x = V_y = \psi = 0$,
$I_n = \{(0, y) : L_1 < y \le L\}$ is the inlet and $O_t = \{(l, y) : L_1 < y \le L\}$ is the outlet with the periodical BCs for ψ, ζ, U_x, U_y.

From the conditions of electrically non-conductivity of the cylinders

$$\int_\Omega \int j_z dxdy = \int_\Omega \int (\mathbf{E}_z + \frac{\partial \psi}{\partial \mathbf{B}})dxdy = 0$$

follows $\mathbf{E}_z Ll = -\oint_{\partial\Omega} \psi \cos(\mathbf{n}, \mathbf{B})ds = -(l\sin(\alpha) + \cos(\alpha)(\int_{L_1}^{L}(\psi(l, y) - \psi(0, y))dy) = -l\sin(\alpha)$, or $\mathbf{E}_z = -\sin(\alpha)/L$.

On the walls we use the following BCs [19] $\zeta^k = \beta\frac{\partial \psi}{\partial n} + \zeta^{k-1}, k = 1, 2, \cdots,$
where k is the number of iterations with $\zeta^0 = 0$, $\beta > 0$ is the parameter, n is the direction of the external normal vector on the wall.

For numerical calculations we consider an uniform quadratic grid $((N + 1) \times M)$.

For cylinders **placed in the parallel series**
1) $\Omega_1^h = \{(x_i, y_j), \; x_i = (i-1)h, \; y_j = (j-1)h, \}i = \overline{N_1, N_2}, \; j = \overline{1, M_1}, \; (N_1 - 1)h = l_1, (M_1 - 1)h = L_1,$
2) $\Omega_2^h = \{(x_i, y_j), \; x_i = (i-1)h, \; y_j = (j-1)h, \}i = \overline{1, N+1}, \; j = \overline{M_1, M}, \; (N - 1)h = l_1, (M - 1)h = L,$
where $h = \frac{l_1}{N_1 - 1} = \frac{l_2}{N_2 - 1} = \frac{l}{N - 1} = \frac{L}{M - 1} = \frac{L_1}{M_1 - 1}$.

Subscripts (i, j) refer to x, y indices with the mesh spacing h.
The equations (5.2) in the uniform grid (x_i, y_j) are replaced with difference equations of the second order approximation in 5- point stencil and the numerical calculations are carried out by using Seidel iterations with under relaxation for vorticity.

Numerical results are obtained for dimensionless values $L = 1, L_1 = 0.5, l_1 = 0.5, l_2 = 1.5, l_3 = 2.5, l_4 = 3.5, l = 2$ or
$l = 4$(for enlarged domain) $Re = 40, S = 0, 2.5, 5, 25.0, 250.0,$

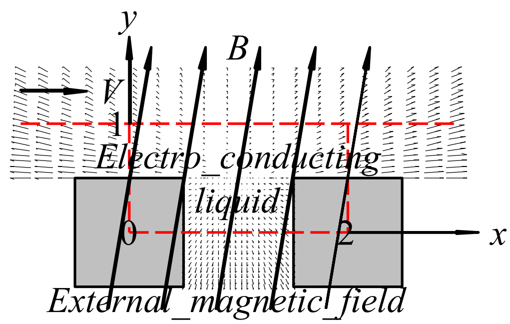

Fig. 5.1 Domain for parallel placed cylinders(2 cylinders, $L_1 = 0.5, L = 1, l_1 = 0.5, l_2 = 1.5, l = 2$)

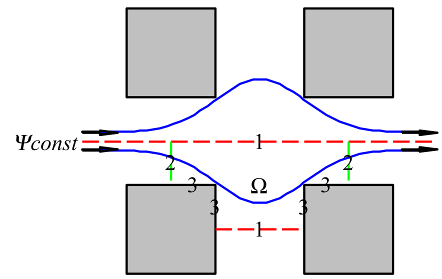

Fig. 5.2 Domain for parallel placed cylinders(4 cylinders, BCs: 1-symmetry,2- periodical, 3-walls)

$Ha^2 = 0, 100, 200, 1000, 10000,\ \alpha = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4},\ \beta \in [5, 15], \omega \in [0.1, 0.8]$.
For physical model we consider the liquid steel with following parameters:
$\rho = 7.10^3[\frac{kg}{m^3}], \nu = 10^{-6}[\frac{m^2}{s}], \sigma = 7.10^5\frac{1}{\Omega m}$.
The characteristic length scale $L_0 = 4.10^{-3}[m]$, the magnitude of the uniform flow velocity $U_0 = 10^{-2}[\frac{m}{s}]$ and pressure $P_0 = 0.7[\frac{kg}{ms^2}]$.The applied magnetic field $\mathbf{B}_0 = \sqrt{\frac{S}{40}} \in [0, 2.5][T]$.
For inductionless approximation we have that the magnitude of the magnetic Reynolds number $Re_m = \mu_0\sigma U_0 L_0 \approx 3.10^{-5}$ is small ($\mu_0 = 4\pi 10^{-7}[\frac{mkg}{s^2A^2}]$ is the magnetic permeability in vacuum).
For obtaining the dimension values we need multiplied the dimensionless values with following scalar factors:
1) the maximal and minimal value of velocity with $U_0 = 10^{-2}$,
2) the vorticity with $\zeta_0 = \frac{U_0}{L_0} = 2.5$,
3) the stream function and the fluid volume with $\Psi_0 = U_0 L_0 = 4.10^{-5}$,
4) the pressure with $P_0 = \rho U_0 = 0.7$.

The calculation and their graphical visualization were made by means of the MATLAB software for 4 different grids:
1)$h = 0.05, N_1 = 11, N_2 = 31, N = 41, M_1 = 11, M = 21$,
2)$h = 0.025, , N_1 = 21, N_2 = 61, N = 81, M_1 = 21, M = 41$,
3)$h = 0.0125, N_1 = 41, N_2 = 121, N = 161, M_1 = 41, M = 81$,
4)$h = 0.00625, N_1 = 81, N_2 = 241, N = 321, M_1 = 81, M = 161$.
For the iteration process with maximal errors $\leq 10^{-6}$ for Ψ and $\leq 10^{-4}$ for ζ the numbers of iterations $K \in [1000, 20000]$ are depending on the parameters.

If $S = 0, \omega = 0.5$ then depending on the numbers of iterations K we have corresponding minimal value of $\Psi = -0.048(K = 1000), -0.051(K = 2000), -0.052(K = 5000), -0.052(K = 10000)$.
If $Re = 40, S = 250, B_0 = 2.5, \alpha = \frac{\pi}{4}, \omega = 0.4$ then depending on the grid numbers N, M we have corresponding dimensionless values:

1) min $(\Psi) = -0.0034(N = 161, M = 81), -0.0033(N = 321, M = 161)$,
2) max $(V) = 5.65(N = 161, M = 81), 5.60(N = 321, M = 161)$,
3) max $(\zeta) = 35.65(N = 161, M = 81), 35.70(N = 321, M = 161)$.

In the corresponding figs. 5.3 and 5.4 we can see the levels of the dimensionless stream function $\Psi = const$ for $S = 0$ and $S = 2.5$. We see that in the second case vortices are disappearing. In the publication [22] also numerical results for pressure and velocity can be seen. We use for the calculation the grid Nr. 3 with $N = 161, M = 81$.

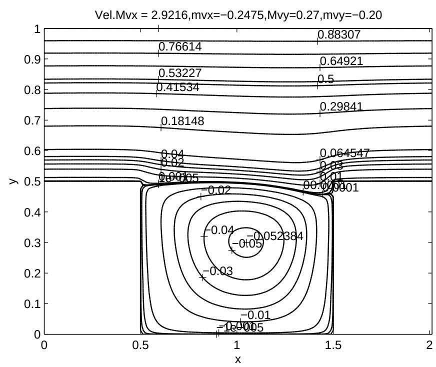

Fig. 5.3 Levels of stream function for $Re = 40, S = 0$

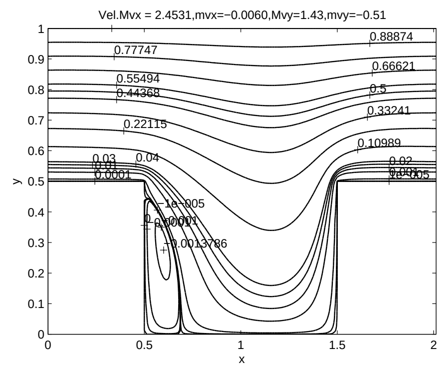

Fig. 5.4 Levels of stream function for $\alpha = \frac{\pi}{2}, S = 2.5$

By $S = 25$ the vortices disappears. Similarly we can look into situations when magnetic field direction is $\alpha = \frac{\pi}{4}$.

From numerical results follows that the vortex formation and MHD flow are depending on the form of external magnetic fields and on the values of Reynolds and Stuart numbers. In the strong transverse magnetic field the vortexes are deleted and on the walls of the cylinders Hartmann boundary layers developed.

5.2 Mathematical modeling of 2D magnetohydrodynamics and temperature fields, risen by electromagnetic forces between two infinite coaxial cylinders

In the papers [25] a new type of heat generator is modeled. Previously in papers ([26], [27], [28], [29]) cylinder form electrical heat generators with six or nine circular conductors were modelled – electrodes placed on the surfaces of the cylinder. Here conductors have forms of six bars placed parallel to the cylinder axis in the central part of the cylinder. In the (fig. 5.5) we can see the real electrical heat generator.

The alternating current is fed to $\tilde{N}$ infinite discrete conductors in forms of bars, which are placed parallelly to the cylinder axis in the domain $r < r_0 < R$. In the fig. 5.6 we can see the mathematical model with 6 conductors ($\tilde{N} = 6$.)

Let the cylindrical domain between two infinite cylinders $\Omega = \{(r, \phi, z) : r_0 < r < R, 0 \le \phi \le 2\pi, -\infty < z < \infty\}$ contain viscous electrically conducting incompressible liquid, where r_0, R are the radii of the coaxial cylinders.

In the weakly conductive liquid-electrolyte the current creates the radial $\mathbf{B}_r(t, r, \phi)$ and the azimuthal $\mathbf{B}_\phi(t, r, \phi)$ components of the magnetic field as well the axial component of the induced electric field $\mathbf{E}_z(t,r,\phi)$ depending on the time t. For calculating the electromagnetic fields outside the electrodes, the averaging method over the time interval $2\pi/\omega$ is used (ω is the angular frequency of the alternating current).

Magnetic fields creates the radial $\mathbf{F}_r(r,\phi)$ and azimuthal $\mathbf{F}_\phi(r,\phi)$ components of the Lorentz' force $\mathbf{F}$.

The axial component of the vector $curl\mathbf{F}$ gives rise to a liquid motion. The stationary 2D flow of incompressible viscous liquid between the cylinders is described by the system of the Navier–Stokes equations in the polar coordinates (r,ϕ), $r_0 < r < R$ in the following form [24]:

$$\begin{cases} M(V_r) - r^{-1}V_\phi^2 = -\tilde{\rho}^{-1}\frac{\partial p}{\partial r} + \nu(\Delta V_r - r^{-2}V_r - 2r^{-2}\frac{\partial V_\phi}{\partial \phi}) + \tilde{\rho}^{-1}\mathbf{F}_r \\ M(V_\phi) + r^{-1}V_r V\phi = -(\tilde{\rho}r)^{-1}\frac{\partial p}{\partial \phi} + \nu(\Delta V_\phi - r^{-2}V_\phi + 2r^{-2}\frac{\partial V_r}{\partial \phi}) + \tilde{\rho}^{-1}\mathbf{F}_\phi \\ \frac{\partial(rV_r)}{\partial r} + \frac{\partial(V_\phi)}{\partial \phi} = 0. \end{cases} \tag{5.3}$$

Here V_r, V_ϕ are the radial and azimuthal components of velocity vector $\mathbf{V}$, depending on the coordinates r,ϕ; Δ is Laplace operator, $\Delta g = r^{-1}\frac{\partial}{\partial r}(r\frac{\partial g}{\partial r}) + r^{-2}\frac{\partial^2 g}{\partial \phi^2}$, $M(g) = V_r\frac{\partial g}{\partial r} + r^{-1}V_\phi\frac{\partial g}{\partial \phi}$ are the convective parts of the equations,
$\tilde{\rho}$, ν are the density and kinematic viscosity, p is the pressure, $g = V_r; V_\phi$.

Determined the vorticity functions or the axial component of the vector $curl\mathbf{V}$ with formulas $\tilde{\omega} = r^{-1}(\partial(rV_\phi)/\partial r - \partial V_r/\partial\phi)$ we obtain

$$\begin{cases} -V_\phi\tilde{\omega} = -\tilde{\rho}^{-1}\frac{\partial \tilde{p}}{\partial r} - \nu r^{-1}\frac{\partial \tilde{\omega}}{\partial \phi} + \tilde{\rho}^{-1}\mathbf{F}_r, \\ V_r\tilde{\omega} = -\tilde{\rho}^{-1}r^{-1}\frac{\partial \tilde{p}}{\partial \phi} + \nu\frac{\partial \tilde{\omega}}{\partial r} + \tilde{\rho}^{-1}\mathbf{F}_\phi, \\ \frac{\partial(rV_r)}{\partial r} + \frac{\partial(V_\phi)}{\partial \phi} = 0, \end{cases} \tag{5.4}$$

where $\tilde{p} = p + 0.5\mathbf{V^2}$.

For alternating current the averaged values of Lorenz force $\mathbf{F}_r$, $\mathbf{F}_\phi$ are obtained, applying the Biot–Savart and Ohm's laws in the following form [23]:

$$\begin{cases} \mathbf{F}_r(r,\phi) = 0.5K_0 S_{\tilde{N}}^{\alpha}, \\ \mathbf{F}_\phi(r,\phi) = 0.5K_0 S_{\tilde{N}}^{\beta}, \end{cases} \tag{5.5}$$

where

$$S_{\tilde{N}}^{\alpha} = \sum_{i,j=1}^{\tilde{N}} \sin((j-i)\theta)\alpha_{i,j}, S_{\tilde{N}}^{\beta} = \sum_{i,j=1}^{\tilde{N}} \sin((j-i)\theta)\beta_{i,j},$$

$$\alpha_{i,j} = -\frac{\ln(\rho_i)(r_j\cos(\phi-\phi_j) - r)}{\rho_j^2}, \beta_{i,j} = \frac{\ln(\rho_i)r_j\sin(\phi-\phi_j)}{\rho_j^2},$$

$K_0 = (\frac{a^2\mu j_0}{2})^2\sigma\omega$, $\mu = 4\pi 10^{-7}\frac{mkg}{s^2A^2}$ is the magnetic permeability in vacuum, σ is the electric conductivity,
j_0 is the amplitude of alternating current density

$$j_i = j_0\cos(\omega t) + (i-1)\theta, i = \overline{1,\tilde{N}}, \tag{5.6}$$

$\theta = const$ is the phase (usually $\theta = 120^0$ and the frequency of the alternating current is $50Hz$), (r_i,ϕ_i) is the polar coordinate of the center for wires

$L_i = \{r - r_i \le a < r_0, \phi_i - \alpha_i \le \phi \le \phi_i + \alpha_i, -\infty \le z \le +\infty\}$ with radius a, $\rho_i = \sqrt{(r_i^2 + r^2 - 2rr_i\cos(\phi - \phi_i))}, \alpha_i = \arcsin(a/r_i)$.

Similarly for averaged values of source term in heat transport equation [23]

$$j_z^2(r,\phi) = 0.5K_0\sigma\omega S_{\tilde{N}}^{\gamma}, \tag{5.7}$$

where $S_{\tilde{N}}^{\gamma} = \sum_{i,j=1}^{\tilde{N}} \cos((j-i)\theta)\gamma_{i,j}$, $\gamma_{i,j} = \ln(\rho_i).\ln(\rho_j)$.

Having calculated the axial component of the curl for force vector $f = rot_z\mathbf{F}$, [23] the average value is

$$f(r,\phi) = 0.5K_0 S_{\tilde{N}}^{\delta}, \tag{5.8}$$

where $S_{\tilde{N}}^{\delta} = \sum_{i,j}^{\tilde{N}} \sin((j-i)\theta)\delta_{i,j}$,
$\delta_{i,j} = \frac{1}{r}[\frac{\partial}{\partial r}(r\beta_{i,j}) - \frac{\partial}{\partial\phi}(\alpha_{i,j})] = g_{i,j} - g_{j,i}$,
$g_{i,j} = \frac{r_i\sin(\phi-\phi_i)}{\rho_i^2}\frac{(r_j\cos(\phi-\phi_j)-r)}{\rho_j^2}$.
On the walls (the surfaces of the cylinders $r = R$ and $r = r_0$) we have the non-slipping conditions $\mathbf{V} = 0$.

By eliminating the pressure $\tilde{p}$ from the first two equations of the system of PDEs (5.4) one obtains

$$M(\tilde{\omega}) = \nu\Delta\tilde{\omega} + \tilde{\rho}^{-1}f, \tag{5.9}$$

where f is the axial component of the vector $curl\mathbf{F}$.
The hydrodynamic stream function ψ can be determined with formulas $V_r = r^{-1}\frac{\partial\psi}{\partial\phi}, V_\phi = -\frac{\partial\psi}{\partial r}$.
Then from the equation of continuity and from vorticity function it follows, that $\tilde{\omega} = -\Delta\psi$.
From (5.9) follows the system of two PDEs for solving the vorticity function $\tilde{\omega}$ and stream function ψ [33]:

$$\begin{cases} \Delta\psi = -\tilde{\omega} \\ r^{-1}J(\tilde{\omega},\psi) = \nu\Delta\tilde{\omega} + \tilde{\rho}^{-1}f, \end{cases} \tag{5.10}$$

where $J(\tilde{\omega},\psi) = (\partial\tilde{\omega}/\partial r)(\partial\psi/\partial\phi) - (\partial\tilde{\omega}/\partial\phi)(\partial\psi/\partial r)$ is the Jacobian of the functions ψ and $\tilde{\omega}$.
The use of the stream function and the vorticity as the dependent variables is the fundamental reason for the difficulty in implementing the boundary conditions for vorticity [33].
By eliminating the functions $\tilde{\omega}$ from (5.10), we obtain the PDEs of the fourth order

$$r^{-1}J(\Delta\psi,\psi) = \nu\Delta^2\psi - \tilde{\rho}^{-1}f, \tag{5.11}$$

where $J(\Delta\psi,\psi)$ is the Jacobian of the functions ψ and $\Delta\psi$.
In the azimuthal direction we have the periodic conditions: $\psi(r,0) = \psi(r,2\pi)$, $\frac{\partial\psi(r,0)}{\partial\phi} = \frac{\partial\psi(r,2\pi)}{\partial\phi}$.

The steady energy equation reduces to the heat transport equation for incompressible flow with source terms and with constant properties. The stationary distribution of temperature field $T(r,\phi)$ in a conducting ring is described by the following boundary-value problem for the heat transport equation:

$$\begin{cases} \tilde{\rho} c r^{-1} J(T, \psi) = k \Delta T + \sigma^{-1} j_z^2, \\ \frac{\partial T(R,\phi)}{\partial r} = 0, T(r_0, \phi) = T_a, \end{cases} \tag{5.12}$$

where c, k are a corresponding constants of specific thermal capacity ($c = 4000 \frac{J}{kgK}$) and coefficient of heat conductivity ($k = 0.6 \frac{W}{mK}$),
j_z^2 is the source term, T_a is the given constant fixed temperature.

The equations (5.11, 5.12) were put in the dimensionless form scaling all the lengths to $L = R$ (the radius of the tube), the velocities V_r, V_ϕ to U_0, stream function ψ to $\psi_0 = U_0 R$, the induction $\mathbf{B}_r, \mathbf{B}_\phi$ of magnetic field to $\mathbf{B}_0$, the pressure p to $P_0 = U_0^2 \tilde{\rho}$ and temperature T to T_a, where $U_0 = \nu / R$. Further the denotes of all variables are unchanged.

In the publication [25] it is shown how to obtain the source function using the magnetic and electric fields provided.

The procedure of iterations

$$\begin{cases} \Delta^2 \psi^{(m+1)} = a_0 r^{-1} J(\Delta \psi^{(m)}, \psi^{(m)}) + b_0 f, \\ \Delta T^{(m+1)} = Pr\, r^{-1} J(T^{(m)}, \psi^{(m+1)}) - K_T j_z^2, \end{cases} \tag{5.13}$$

together with the boundary conditions is realized using finite difference approximation with central differences.
In the linear case ($J = 0$) we have only one iteration. If the nonlinear terms (convective terms J) is dominant $a_0 >> 1$, then the method of under-relaxation is used with the parameter $\omega_r < 1$.

The stream function equation (5.13) in the uniform grid (r_i, ϕ_j) is replaced by the vector difference equations of the second order approximation on 5- point stencil:

$$A_i \Psi_{i-2} + B_i \Psi_{i-1} + C_i \Psi_i + D_i \Psi_{i+1} + E_i \Psi_{i+2} + F_i^H = 0, \tag{5.14}$$

where Ψ_i are column-vectors of the M-order with components $\psi_{i,j} \approx \psi^{(m+1)}(r_i, \phi_j), j = \overline{1, M}, i = \overline{3, N-1}$,
A_i, B_i, C_i, D_i, E_i are the circulant symmetric matrices of M-order.

The heat transport equation (5.13) is replaced by vector difference equations of second order approximation in 3- point stencil:

$$A1_i T_{i-1} - C1_i T_i + B1_i T_{i+1} + F_i^T = 0, \tag{5.15}$$

where T_i are column-vectors with components $T_{i,j} \approx T^{(m+1)}(r_i, \phi_j), j = \overline{1, M}$,
$A1_i, B1_i, C1_i$ are the circulant symmetric matrices of the M-order.

The boundary conditions for external magnetic fields are replaced by difference equations from the second order of approximation. For the vector function Ψ_i using the 3-point $(r_1, r_2, r_3), (r_{N+1}, r_N, r_{N-1})$ stencils by $r = \eta, r = 1$ we obtain the second order approximations.

The vector difference schemes (5.14, 5.15) are solved by the Gauss elimination method using the calculations of circulant matrices.

Calculations and their graphic visualizations were made by means of the computer programs MATLAB with $\eta = 0.2, N = M = 80$.
The exactness of numerical results are testing with different numbers of grid points N, M. From numerical experiment follows that for N=M=40 and N=M=80 the results are

coincident with 4 decimal places.
The number M_{it} of iterations and the under-relaxation parameter ω_r are depending on the parameters Re, S, Ha, K^H, K^T. These values are determined by the following inequalities:

$$\psi_{er} = \frac{\max|\psi^{m+1} - \psi^m|}{\max|\psi^{m+1}|} < 10^{-4}, T_{er} = \frac{\max|T^{m+1} - T^m|}{\max|T^{m+1}|} < 10^{-4}.$$

For the alternating field induced by six electrodes are used $\theta = \frac{2\pi}{3}$ and $\theta = \frac{\pi}{3}$.
The liquid has following parameters:
kinematic viscosity $\nu \approx 10^{-6}\frac{m^2}{s}$, density of liquid $\tilde{\rho} \approx 1000\frac{kg}{m^3}$ and the electric conductivity $\sigma \approx 100\Omega^{-1}m^{-1}$. The parameter $K_0 = \pi 10^{-10}I^2$, radius R of the cylinder is $0.10m$, the density of the current amplitude $j_0 \approx 10^4 I\frac{A}{m^2}$, the radius a of the electrodes is $0.005m$, where $I = 100A$.
We have the following parameters: $K^H \approx 31, Pr = 6.7, K_T \approx 50$. For given values of the parameters are $\psi_{er} < 10^{-4}, T_{er} < 10^{-4}$ by $M_{it} = 200, \omega_r = 0.5$.

We consider different connections of the conductors $[L_1, L_2, L_3, L_4, L_5, L_6]$. This connections were denoted with $[1, 2, 3, 4, 5, 6]$.

By $\theta = \pi/3$ and with following coordinates of conductors centers (fig. 5.6)
$L_1(r_1, \phi_1) = (r_*, 0^0)$, $L_2(r_2, \phi_2) = (r_*, 60^0)$, $L_3(r_3, \phi_3) = (r_*, 120^0)$, $L_4(r_4, \phi_4) = (r_*, 180^0)$, $L_5(r_5, \phi_5) = (r_*, 240^0)$, $L_6(r_6, \phi_6) = (r_*, 300^0)$, $r_* = 0.015m$.

In (fig. 5.7, fig. 5.8) are the distributions of stream function and temperature for connections $[1, 2, 3, 4, 5, 6]$ of conductors by maximal temperature $\max(T) = \mathbf{10.56}$. In the (fig. 5.9, fig. 5.10) are the results for connections $[2, 3, 5, 6, 1, 4]$ by $\max(T) = 6.57$ and by two vortices.

We have 2 vortices by connections $[2, 3, 5, 6, 1, 4]$, $[1, 3, 5, 2, 4, 6]$, $[1, 4, 5, 6, 2, 3]$, 3 vortices by connections $[1, 6, 2, 5, 3, 4]$, $[1, 4, 2, 5, 3, 6]$ and 4 vortices by connections $[2, 4, 6, 1, 3, 5]$, $[2, 6, 5, 3, 1, 4]$.

Similar results can be obtained for $\theta = 2\pi/3$.

From numerical results we conclude that the vortex formation is strongly depending from the connection of electrodes, the temperature field is strongly depending on vortex field.

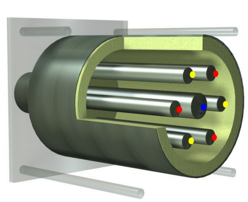

Fig. 5.5 The real heat generator

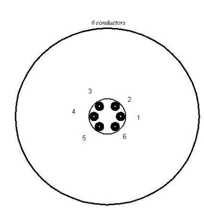

Fig. 5.6 The 2D mathematical model of the heat generator

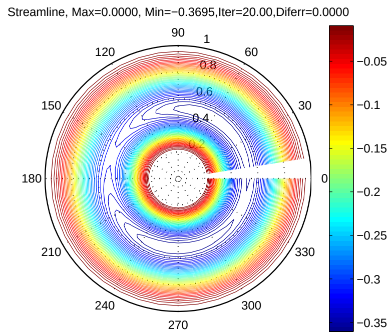

Fig. 5.7 Stream function for [1,2,3,4,5,6], $\theta = \pi/3$

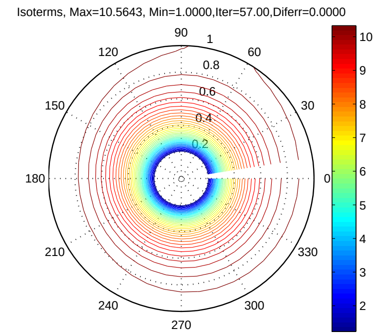

Fig. 5.8 Temperature for [1,2,3,4,5,6],$\theta = \pi/3, T_{max} = 10.56$

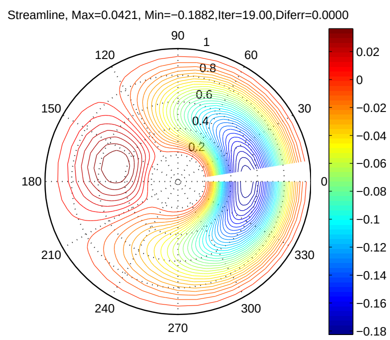

Fig. 5.9 Stream function for [2,3,5,6,1,4], $\theta = \pi/3$

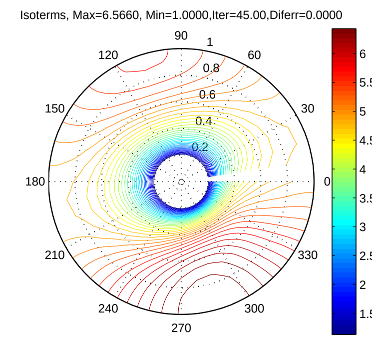

Fig. 5.10 Temperature for [2,3,5,6,1,4],$\theta = \pi/3, T_{max} = 6.57$

5.3 Mathematical modeling of 2D magnetohydrodynamics flow in the ring by external magnetic field

Let the cylindrical domain $\{(r, \phi, z) : r_0 < r < R, 0 \leq \phi \leq 2\pi, -\infty < z < \infty\}$ contain viscous electrically conducting incompressible liquid, where r_0, R are the radii of the coaxial cylinders. The surfaces of these cylinders can rotate with corresponding angular velocities – Ω_0, Ω_1. Different types of external 2D magnetic fields can be considered – uniform, radial, axial, bipolar and sum of axial and uniform fields.

In this work uniform external 2D magnetic field is considered. Field is added with the radial $\mathbf{B}_r(r, \phi) = \mathbf{B}_0(1 - r^{-2}a_\mu)\sin(\phi)$ and the azimuthal $\mathbf{B}_\phi(r, \phi) = \mathbf{B}_0(1 + r^{-2}a_\mu)\cos(\phi)$ components of the induction for magnetic field, where $a_\mu = \frac{(\mu-1)r_0^2}{\mu+1}, \mu = \frac{\mu_1}{\mu_0}, \mu_1, \mu_0$ – are the corresponding magnetic permeability in the liquid $(R > r > r_0)$ and in the internal cylinder $(r \leq r_0)$ (if $\mu = 1$ then this field is homogeneous and parallel to Oy axis [30], if $\mu = 0, \mu_0 = \infty$ then the internal cylinder is ferromagnetic, if $\mu = \infty, \mu_1 = \infty$ then the liquid is ferromagnetic).

Here $\mathbf{B}_0$ is the scale of the induction for magnetic field. These magnetic fields with the vector of induction $\mathbf{B}$ are solutions of the following homogenous Maxwell's equations $div\mathbf{B} = curl\mathbf{B} = 0$.

This components of the external magnetic fields creates the radial $F_r(r, \phi)$ and azimuthal $F_\phi(r, \phi)$ components of the Lorentz' force $\mathbf{F}$.
The axial component of the vector's $curl\mathbf{F}$ give rise to a liquid motion. The stationary 2D flow of incompressible viscous liquid in a cylinder is described by the system of the Navier–Stokes equations in the polar coordinates $(r, \phi), r_0 < r < R$.

The model is similar to the model in the previous section equation 5.3, where

$$F_r = -\sigma \mathbf{B}_\phi (V_r \mathbf{B}_\phi - V_\phi \mathbf{B}_r + \mathbf{E}_z), \quad F_\phi = \sigma \mathbf{B}_r (V_r \mathbf{B}_\phi - V_\phi \mathbf{B}_r + \mathbf{E}_z),$$

$\mathbf{E}_z = const$ is the azimuthal component of the electric field $\mathbf{E}$, σ is the electric conductivity. The walls (the surfaces of the cylinders $r = R$ and $r = r_0$) rotate with the velocities $V_\phi = r_0 \Omega_0$ and $V_\phi = R\Omega_1$ corresponding ($V_r = 0$). Equation is transformed in to nondimensional form and approximated using finite differences and uses the same numerical methods used in section 5.2.

Calculation and their graphic visualization were made by means of the computer programs MATLAB for $Re \in [100, 1000]; S \in [0.01, 10], \eta = 0.2, \omega_0 \in [-5, 5], \omega_1 \in [-1, 1], N = 80, M = 30, \omega_r \in [0.05, 0.5], M_{it} \in [80, 400]$.

By the **uniform magnetic field** with different values of μ in the rotated flows the different form of vortexes developed. In the figs. 5.11, 5.12; 5.13; 5.14 are the distributions of stream function by $Re = 100, S = 10, \omega_0 = -5; 0; 5, \omega_1 = -1; 0; 1, \mu = 1; 0$.

An original method was used to calculate the circular matrices. With the finite difference method the distributions of magnetohydrodynamics flows are calculated.
In the publication [20] can see that from numerical results follows that the distributions of MHD flow is depending on the velocity of walls rotation and of the form of the external magnetic fields:
1) for the radial magnetic field we have the radial symmetry of the flow and on the walls of the cylinder the Hartman boundary layers developed,
2) for the uniform magnetic field we can see different form of vortex formation in the fluid depending on the level for ferromagnetic,
3) in the bipolar magnetic field the flow to get off from inner cylinder which stay stationary.

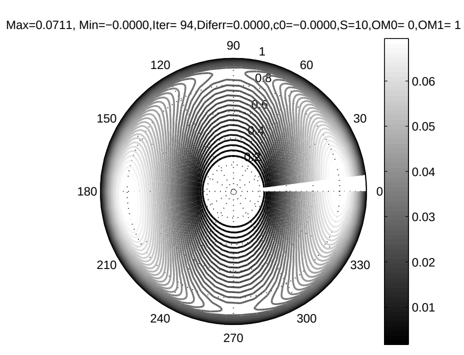

Fig. 5.11 Stream function for $\mu = 1, S = 10, \omega_0 = 0, \omega_1 = 1$

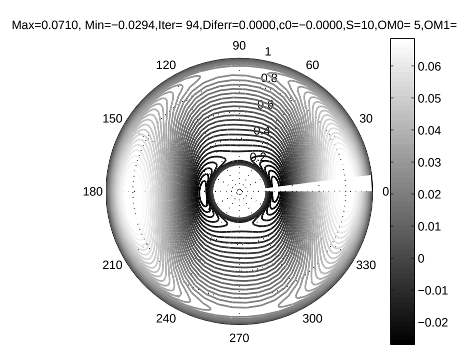

Fig. 5.12 Stream function for $\mu = 0, S = 10, \omega_0 = 0, \omega_1 = 1$

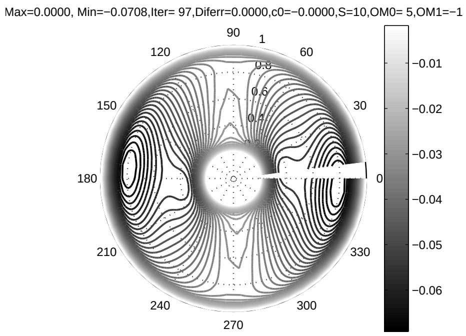

Fig. 5.13 Stream function for $\mu = 1, S = 10, \omega_0 = 5, \omega_1 = -1$

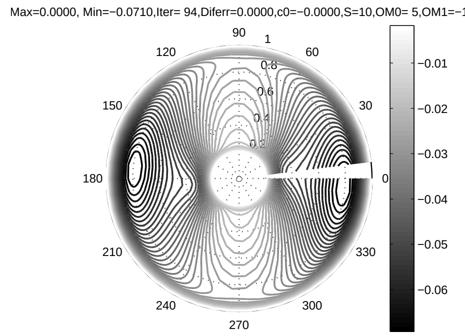

Fig. 5.14 Stream function for $\mu = 0, S = 10, \omega_0 = -5, \omega_1 = 1$

5.4 The mathematical modeling of Ca and Fe distribution in peat layers

We consider averaging and finite difference methods for solving the 3-D boundary-value problem in multilayer domain. We consider the metals **Fe** and **Ca** concentration in the layered peat blocks. Using experimental data the mathematical model for calculation of concentration of metals in different points in peat layers is developed. A specific feature of these problems is that it is necessary to solve the 3-D boundary-value problems for elliptic type partial differential equations (PDEs) of second order with piece-wise diffusion coefficients in the layered domain. We develop here a finite-difference method for solving of a problem of one, two and three peat blocks with periodical boundary condition in x direction. This procedure allows to reduce the 3-D problem to a system of 2-D problems by using circulant matrix.

The process of diffusion the metal in the peat block is consider in 3-D parallelepiped

$$\Omega = \{(x, y, z) : 0 \leq x \leq l, 0 \leq y \leq L, 0 \leq z \leq Z\}.$$

The domain Ω consist of multilayer medium. We will consider the stationary 3-D problem of the linear diffusion theory for multilayer piece-wise homogenous materials of N layers in the form

$$\Omega_i = \{(x, y, z) : x \in (0, l), y \in (0, L), z \in (z_{i-1}, z_i)\}, i = \overline{1, N},$$

where $H_i = z_i - z_{i-1}$ is the height of layer $\Omega_i, z_0 = 0, z_N = Z$. We will find the distribution of concentrations $c_i = c_i(x, y, z)$ in every layer Ω_i at the point $(x, y, z) \in \Omega_i$ by solving the following partial differential equation (PDE):

$$D_{ix}\partial^2 c_i/\partial x^2 + D_{iy}\partial^2 c_i/\partial y^2 + D_{iz}\partial^2 c_i/\partial z^2 + f_i(x, y, z) = 0, \qquad (5.16)$$

where D_{ix}, D_{iy}, D_{iz}, are constant diffusion coefficients, $c_i = c_i(x, y, z)$ – the concentrations functions in every layer, $f_i(x, y, z)$ - the fixed source function.
The values c_i and the flux functions $D_{iz}\partial c_i/\partial z$ must be continues on the contact lines between the layers $z = z_i, i = \overline{1, N-1}$:

$$\begin{aligned} &c_i|_{z_i} = c_{i+1}|_{z_i}, \\ &D_{iz}\partial c_i/\partial z|_{z_i} = D_{(i+1)z}\partial c_{i+1}/\partial z|_{z_i}, \end{aligned} \tag{5.17}$$

where $i = \overline{1, N-1}$.

We assume that the layered material is bounded above and below with the plane surfaces $z = 0, z = Z$ with fixed boundary conditions in following form:

$$c_1(x, y, 0) = C_0(x, y), \quad c_N(x, y, Z) = C_a(x, y) \tag{5.18}$$

where C_0, C_a are given concentration-functions.
We have two forms of fixed boundary conditions in the x, y directions:
1) the periodical conditions by $x = 0, x = l$ in the form
$c_i(0, y, z) = c_i(l, y, z), \partial c_i(0, y, z)/\partial x = \partial c_i(l, y, z)/\partial x$,
2) the symmetrical conditions by $y = 0, y = L$
$\partial c_i(x, 0, z)/\partial y = \partial c_i(x, L, z)/\partial y = 0$.
For solving the problem (5.16)-(5.18) we will consider conservative averaging (AV) and finite difference (FD) methods. These procedures allow to reduce the 3-D problem to some 2D boundary value problem for the system of partial differential equations with circulant matrix in the x-directions.

The equation of (5.16) are averaged along the heights H_i of the layers Ω_i and quadratic integral splines along z coordinate in following form one used [31]

$$c_i(x, y, z) = C_i(x, y) + m_i(x, y)(z - \overline{z_i}) + e_i(x, y)G_i((z - \overline{z_i})^2/H_i^2 - 1/12), \tag{5.19}$$

where $G_i = H_i/D_{iz}, \overline{z_i} = (z_{i-1} + z_i)/2, z \in [z_{i-1}, z_i]$,
m_i, e_i, C_i are the unknown coefficients of the spline-function,
$C_i(x, y) = H_i^{-1} \int_{z_{i-1}}^{z_i} c_i(x, y, z)dz$ are the average values of c_i, $i = \overline{1, N}$.
After averaging the system (5.16) along every layer Ω_i, we obtain N system of 2-D PDE

$$D_{ix}\partial^2 C_i/\partial x^2 + D_{iy}\partial^2 C_i/\partial y^2 + 2H_i^{-1}e_i + F_i(x, y) = 0, \tag{5.20}$$

The calculation of unknown functions m_i, e_i using the conditions and other calculations can be found in the work [32].

In the case $N = 3$ (three layers) we have equations for m_i, c_i

$$\begin{aligned} &e_i = e_{i,1}C_1 + e_{i,2}C_2 + e_{i,3}C_3 + e_{i,0}, \\ &m_i = m_{i,1}C_1 + m_{i,2}C_2 + m_{i,3}C_3 + m_{i,0}, i = 1; 2; 3, \end{aligned} \tag{5.21}$$

Unknown functions $m_{i,j}$, $c_{i,j}$ can be found also using the conditions provided.

For solving 2-D problems we consider an uniform grid $(N_x \times (N_y + 1))$:
$\omega_h = \{(x_i, y_j),\ x_i = ih_x,\ y_j = (j-1)h_y, i = \overline{1, N_x},\ j = \overline{1, N_y + 1},\ N_xh_x = l, N_yh_y = L.\}$
Subscripts (i, j) refer to x, y indices, the mesh spacing in the x_i, y_j directions are h_x and h_y.
We can have the PDEs (5.20) rewritten in following vector form:

$$D_x\partial^2 C/\partial x^2 + D_y\partial^2 C/\partial y^2 - AC + \tilde{F} = 0, \tag{5.22}$$

where D_x, D_y are the 3 order diagonal matrices with elements $D_{1x},\ D_{2x},\ D_{3x}$ and $D_{1y},\ D_{2y},\ D_{3y}$,

C is the 3 order vectors-column with elements $C_1,\ C_2,\ C_3$, $\tilde{F}$ is also the vectors-column with elements $\tilde{F}_1,\ \tilde{F}_2,\ \tilde{F}_3$,
and A is the block matrix in following form:

$$A = -2 \begin{pmatrix} e_{1,1}/H_1 & e_{1,2}/H_1 & e_{1,3}/H_1 \\ e_{2,1}/H_2 & e_{2,2}/H_2 & e_{2,3}/H_2 \\ e_{3,1}/H_3 & e_{3,2}/H_3 & e_{3,3}/H_3 \end{pmatrix}$$

The equation (5.22) with periodical conditions for vector function C in the uniform grid (x_i, y_j) is replaced by vector difference equations of second order approximation:

$$AA\, W_{j-1} - CC\, W_j + BB\, W_{j+1} + \bar{F}_j = 0, \tag{5.23}$$

where $W_j, \bar{F}_j, j = \overline{2, N_y}$ are the $M \times N, (M = N_x)$ order vectors -column with elements $C_{k,i,j} \approx C_k(x_i, y_j), \bar{F}_{k,i,j} = \tilde{F}_k(x_i, y_j), i = \overline{1, M}, k = 1; 2; 3,$
$AA, CC, BB = AA$ are the 3 block- matrices of M order circulant symmetric matrix.

As described in the chapter 1, the circulant matrix can to give with the first rows and the calculation (matrix inversion and multiplication) can be carried out with MATLAB using simple formula for obtaining the first M elements of matrix.

The vectors-column W_j from (5.23) is calculated by Thomas algorithm ([34]) in the matrix form using MATLAB. Diffusion coefficients are found by solving 1D problem in the z-direction using data from experiments.

For numerical results we consider the metals **Fe** and **Ca** concentration in the 3 layer peat blocks Ω with $L = l = 1m, Z = 3$. $L = l = 1m, Z = H_1 + H_2 + H_3 = 3m, H_1 = 1m, H_2 = 1.5m, H_3 = 0.5m$.
On the top of earth $(z = Z)$ the concentration $c\frac{mg}{kg}$ of metals is measured in following nine points in the (x, y) plane:
1) for Fe:
$c(0.1, 0.2) = 1.69, c(0.5, 0.2) = 1.83, c(0.9, 0.2) = 1.72, c(0.1, 0.5) = 1.70, c(0.5, 0.5) = 1.88, c(0.9, 0.5) = 1.71, c(0.1, 0.8) = 1.71, c(0.5, 0.8) = 1.82, c(0.9, 0.8) = 1.73,$
2)for Ca:
$c(0.1, 0.2) = 3.69, c(0.5, 0.2) = 4.43, c(0.9, 0.2) = 3.72, c(0.1, 0.5) = 4.00, c(0.5, 0.5) = 4.63, c(0.9, 0.5) = 4.11, c(0.1, 0.8) = 3.71, c(0.5, 0.8) = 4.50, c(0.9, 0.8) = 3.73.$
The data are smoothing in matrix C_a by 2D interpolation with MATLAB operator, using the spline function. In figs. 5.15-5.16 we can see the distribution of concentration c for Fe and for Ca in the (x, y) plane by $z = Z$.

For the peat block corresponding experimental (c_{exp}) and numerical (c_{num}) results by $x = 0.5m, y = 0.5m$ depending on z are obtained in the table 5.1.

5.5 On the mathematical modeling of the diffusion equation with piecewise constant coefficients in multilayer domain

We consider the 2D stationary boundary value problem for diffusion equation with piecewise constant coefficients in multilayer domain. In one direction (in x-axes direction)

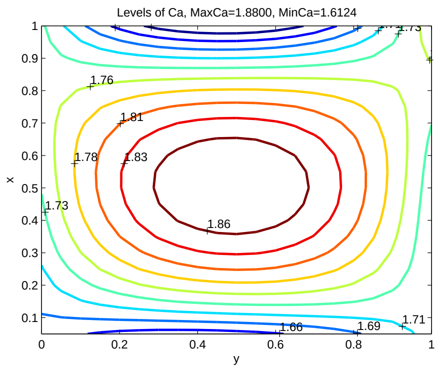

Fig. 5.15 Levels of c by $z = Z$ for Fe

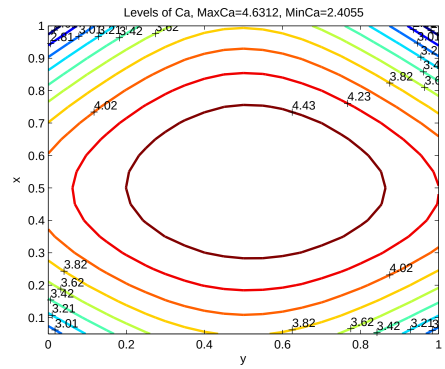

Fig. 5.16 Levels of c by $z = Z$ for Ca

Table 5.1 The experimental and numerical results by $x = 0.5m, y = 0.5m$ depending on z

z	$c_{num} - Fe$	$c_{exp} - Fe$	$c_{num} - Ca$	$c_{exp} - Ca$
0.000	0.681	0.660	1.267	1.300
0.250	0.719		1.395	
0.500	0.756		1.522	
0.750	0.793		1.649	
1.000	0.829	0.830	1.780	1.900
1.250	0.926		1. 844	
1.500	1.025		1.914	
1.750	1.127	1.150	1.985	1.980
2.000	1.231		2.057	
2.250	1.337		2.132	
2.500	1.4458	1.500	2.207	2.380
2.750	1.649		3.351	
3.000	1.880	1.880	4.630	4.630

we have the homogenous periodical boundary conditions (PBC). In the publication [35] also boundary conditions of the first type are used.

The process of diffusion is consider in 2-D domain

$$\Omega = \{(x, y) : 0 \leq x \leq l, 0 \leq y \leq L\}.$$

The domain Ω consists of multilayer medium. We will consider the stationary 2-D problem of the linear diffusion theory for multilayer piecewise homogenous materials of M layers in the form

$$\Omega_j = \{(x, y) : x \in (0, l), y \in (y_{j-1}, y_j)\}, j = \overline{1, M},$$

where $h_j = y_j - y_{j-1}$ is the height of layer $\Omega_j, y_0 = 0, y_M = L$. We will find the distribution of concentrations $u_j = u_j(x, y)$ in every layer Ω_j at the point $(x, y) \in \Omega_j$ by solving the following partial differential equation (PDE):

$$k_j \partial^2 u_j / \partial x^2 + k_j \partial^2 u_j / \partial y^2 + f_j(x, y) = 0, \tag{5.24}$$

where k_j are constant diffusion coefficients, $u_j = u_j(x, y)$ – the concentrations functions in every layer, $f_j(x, y)$ – the fixed source function.
The values u_j and the flux functions $k_j \partial u_j / \partial y$ must be continuous on the contact lines between the layers $y = y_j, j = \overline{1, M-1}$:

$$u_j(x, y_j) = u_{j+1}(x, y_j),$$
$$k_j \partial u_j(x, y_j)/\partial y = k_{j+1} \partial u_{j+1}(x, y_j)/\partial y. \tag{5.25}$$

We assume that the layered material is bounded above and below with the plane surfaces $y = 0, y = L$ with fixed boundary conditions of the third kind in the following form:

$$\gamma_1 k_1 \partial u_1(x, 0)/\partial y - \alpha_1(u_1(x, 0) - T_1(x)) = 0,$$
$$\gamma_2 k_M \partial u_M(x, L)/\partial y + \alpha_2(u_M(x, L) - T_2(x)) = 0, \tag{5.26}$$

where $\gamma_1^2 + \alpha_1^2 \neq 0, \gamma_2^2 + \alpha_2^2 \neq 0$, T_1, T_2 are given functions. For $\gamma_1 = \gamma_2 = 0$ we have the BC of first kind. We have periodical conditions in the x, y directions by $x = 0, x = l$ in the form

$$u_j(0, y) = u_j(l, y), \partial u_j(0, y)/\partial x = \partial u_j(l, y)/\partial x, \tag{5.27}$$

For the 1D model equation $k_j u_j''(y) + f_j(y) = 0$ the solution is undepending on $x, u_j''(y) = \frac{d^2 u_j(y)}{d^2 y}$.

Analytical solution for this problem is found using Fourier method. Averaging method is used in the same manner as in the section 5.4. We reduce 2D problem to 1D problem.

Finite difference approximation can be also used for modelling the problem and using circulant matrices and their properties. As example we look into 2 layer model. AV method is compared with the method of FDS and FDSES. In the figs. 5.17–5.20 the numerical differences of solutions using different types of numerical models can be observed.

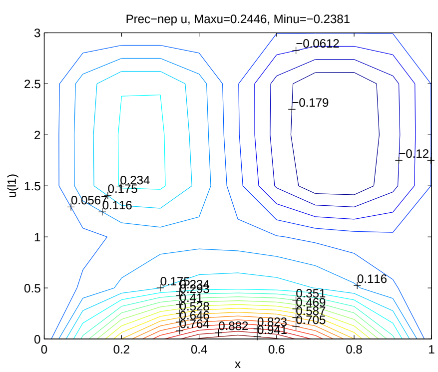

Fig. 5.17 Exact solution

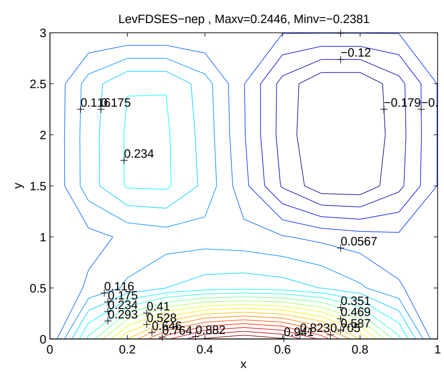

Fig. 5.18 FDSES solution

The 2D diffusion problem in M layered domain described by a boundary value problem of the system of PDEs with piecewise constant diffusion coefficients are approximate on the 1D boundary value problem of a system of M ODEs . This algorithm can be used for solving the problem of metal concentration in the layered peat blocks. The total cost of an averaged method for engineering calculations is determined from the number of grid points in every of two layers. The FDSES method is exact method.

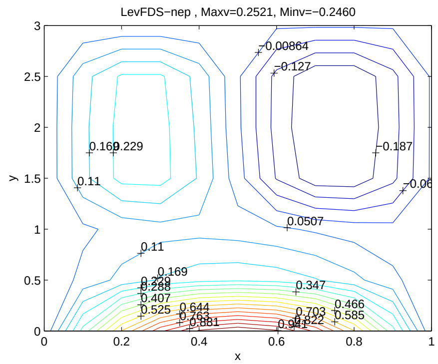

Fig. 5.19 FDS solution

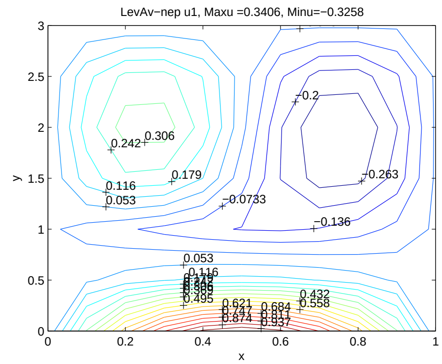

Fig. 5.20 Averaged solution

Conclusions

In the process of this research the author improved his research skills and significantly expanded his knowledge base on usage of solving analytical and numerical methods in linear and nonlinear problems of mathematical physics with periodic boundary conditions (PBC).

The most important conclusions:

- Usage of correct FDS with higher precision order and FDSES is researcher's instrument to create improved and new numerical methods in modelling problems of mathematical physics and numerically analyze them;
- Created algorithms for operations with circulant matrices and implemented in MATLAB;
- Solved spectral problem for circulant matrices which arises when the first, the second and the fourth derivatives are approximated with finite differences in uniform grid with multi-point stencil;
- Solved differential equation boundary problems of the second order with implemented FDS with higher precision order and FDSES methods;
- By modelling classic linear problems of mathematical physics, they were solved effectively using FDSES method. Created algorithms were implemented in MATLAB.
- It is shown that using the FDS and FDSES methods it is possible to model numerically mathematical physics problem linear systems as well;
- It is shown that created methods and algorithms can be applied to nonlinear problems of mathematical physics of concrete type;
- By solving different problems numerically and analytically FDSES advantage over FDS with higher order was shown;
- Created methods and algorithms were used for modelling applied problems: mathematical modelling of power appliances of new type based on principles of the vortex effect; mathematical modelling of metal concentration in peat layers; modelling of nonlinear heat transfer; creation, analysis and numerical calculations of MHD liquid flow.

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A. Appendix, MATLAB code

A.1 Working with circulant matrices

Code of MATLAB for working with circulant matrices:

```
classdef CirculantMatrix
    % Circulant matrix operations

    properties
        vector = [];
        size = 0;
    end

    methods
        % constructor
        function obj = CirculantMatrix(vector)
            if exist('vector', 'var')
                obj.vector = vector;
                obj.size = length(vector);
            end
        end

        function size = checkSize(A, B)
            if B.size ~= A.size
                error 'Matrix dimensions do not match';
            end
            size = A.size;
        end

        % multiplication of two circulant matrix objects
        function res = mult(A, B)
            M = A.checkSize(B);
            a = A.vector;
            b = B.vector;
            c = zeros(1, M);
            b(M+1:2*M) = b(1:M);
            for s = 1:M
                c(s) = a(1:M) * b(M+s:-1:s+1)';
            end
            res = CirculantMatrix(c);
```

```
        end

        % multiplication with vector
        function res = multWithVector(A, b)
            M = A.size;
            c = zeros(1, M);
            c(1) = A.vector * b;
            for s = 1:M
                c(s) = A(M-s+2:M)*b(1:s-1)+A(1:M-s+1)*b(s:M)
            end
            res = c';
        end

        % sum of two circulant matrix objects
        function res = plus(A, B)
            A.checkSize(B);
            res = CirculantMatrix(A.vector + B.vector);
        end

        function res = inverse(A)
            n = A.size;
            x = (0:n-1)' * (0:n-1);
            x = x * (-2 * pi * 1i / n);
            x = exp(x);
            u = x;

            x = obj.vector * x;
            x = 1./x;

            b = (u(1, :) .* x) * u' / n;
            res = CirculantMatrix(b);
        end

        function A = display(A)
            m = A.matrix();
            disp(m);
        end

        function m = matrix(A)
            v = A.vector;
            m = [];
            for i = 1:A.size
                m = [m; v];
                v = [v(end), v(1:end-1)];
            end
        end
    end
end
```

For blockwise circulant matrices $A_p = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$, $A_b = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ and vector $C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$ we have following MATLAB programs:

1) **X=**$apg(A_p)$ for matrix A_p inversion:

```
function [X1,X2,X3,X4]= apg(A1,A2,A3,A4)
%Inverse matrix (X1,X2;X3,X4)=(A1,A2;A3,A4)^(-1)
a1=CMR(A2,Cikl(A4));X1=Cikl(A1-CMR(a1,A3));
X3=-CMR(CMR(Cikl(A4),A3),X1);
a2=CMR(A3,Cikl(A1));X4=Cikl(A4-CMR(a2,A2));
X2=-CMR(CMR(Cikl(A1),A2),X4);
```

2) **X=br**(A_b, C_p) for two matrices multiplication:

```
function [X1,X2,X3,X4]=br(A1,A2,C1,C2,C3,C4)
% blocmatrix (A1,0; 0,A2)multipl.with matrix (C1,C2;C3,C4)
X1=CMR(A1,C1);X3=CMR(A2,C3);X4=CMR(A2,C4);X2=CMR(A1,C2);
```

3) **X=rb**(C_p, A_b) for two matrices multiplication:

```
function [X1,X2,X3,X4]=rb(C1,C2,C3,C4,A1,A2)
% matrix (C1,C2;C3,C4)multipl.with blocmatr. (A1,0;0,A2)
X1=CMR(C1,A1);X3=CMR(C3,A1);X4=CMR(C4,A2);X2=CMR(C2,A2);
```

4) **X=brv**(A_b, C) for matrix multiplication with vector C:

```
function [X1,X2]=brv(A1,A2,C1,C2)
% blocmatrix (A1,0;0,A2)multipl.with vector(C1,C2)
X1=CMRV(A1,C1);X2=CMRV(A2,C2);
```

5) **c=rv**(A_p, C) for matrix multiplication with vector C:

```
function [X1,X2]=rv(A1,A2,A3,A4,C1,C2)
% matrix (A1,A2;A3,A4)multipl. with vector(C1,C2)
X1=CMRV(A1,C1)+ CMRV(A2,C2);X2=CMRV(A3,C1)+ CMRV(A4,C2);
```

6) **Xp=mm2v**(A_p, C_p) for 2 matrices multiplication:

```
function [X11,X12,X21,X22]=mm2(A11,A12,A21,A22,C11,C12,C21,C22
    )
% matrix (A11,A12;A21,A22)multipl. with matrix (C11,C12, C21,
    C22)
X11=CMR(A11,C11)+CMR(A12,C21);X12=CMR(A11,C12)+CMR(A12,C22);
X21=CMR(A21,C11)+CMR(A22,C21);X22=CMR(A21,C12)+CMR(A22,C22);
```

For blockwise circulant matrices $A_p = \begin{pmatrix} A_1 & A_2 & A_3 \\ A_4 & A_5 & A_6 \\ A_7 & A_8 & A_9 \end{pmatrix}$, $A_b = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{pmatrix}$ and vector $C = \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}$ we have following MATLAB programs:

1) **X=**$apg33(A_p)$ for matrix A_p inversion:

```
function [X11,X12,X13,X21,X22,X23,X31,X32,X33]=apg33(A11,A12,
    A13,A21,A22,A23,A31,A32,A33)
%Inverse matrix X=A^-1
a11=Cikl(A11);
a71=CMR(A31,a11);a13=CMR(a71,A13);
a18=CMR(a71,A12)-A32;a41=CMR(A21,a11);
a16=CMR(a41,A13)-A23;a12=CMR(a41,A12);
a15= A22-a12;a19= Cikl(A33-a13);a20=CMR(a19,a18);
a44=Cikl(a15-CMR(a16,a20));a69=CMR(a16,a19);
X21=-CMR(a44,CMR(a69,a71)+a41);X22=a44;X23=CMR(a44,a69);
X31=CMR(a20,X21)-CMR(a19,a71);X32=CMR(a20, X22);
 X33=CMR(a20,X23) +a19;
X11= a11-CMR(a11,CMR(A12,X21) +CMR(A13,X31));
 X12=-CMR(a11,CMR(A12,X22)+CMR(A13,X32));
X13=-CMR(a11,CMR(A12,X23) +CMR(A13,X33));
```

2) **X=bdm**(A_b, C_p) for two matrices multiplication:

```
function [X11,X12,X13,X21,X22,X23,X31,X32,X33]=bdm(A1,A2,A3,
    C11,C12,C13,C21,C22,C23,C31,C32,C33)
% bloc-matrix A multiply with matrix C
X11=CMR(A1,C11);X12=CMR(A1,C12);X13=CMR(A1,C13);
X21=CMR(A2,C21);X22=CMR(A2,C22);X23=CMR(A2,C23);
X31=CMR(A3,C31);X32=CMR(A3,C32);X33=CMR(A3,C33);
```

3) **X=bmd**(C_p, A_b) for two matrices multiplication:

```
function [X11,X12,X13,X21,X22,X23,X31,X32,X33]=bmd(C11,C12,C13
    ,C21,C22,C23,C31,C32,C33,A1,A2,A3)
% matrix C multipl.with bloc-matrix A
X11=CMR(C11,A1);X21=CMR(C21,A1);X31=CMR(C31,A1);
X12=CMR(C12,A2);X22=CMR(C22,A2);X32=CMR(C32,A2);
X13=CMR(C13,A3);X23=CMR(C23,A3);X33=CMR(C33,A3);
```

4) **X=bdv**(A_b, C) for matrix multiplication with vector C:

```
function [X1,X2,X3]=bdv(A1,A2,A3,C1,C2,C3)
% bloc-matrix A multipl.with vector C
X1=CMRV(A1,C1);X2=CMRV(A2,C2);X3=CMRV(A3,C3);
```

5) **c=bmv**(A_p, C) for matrix multiplication with vector C:

```
function [X1,X2,X3]=bmv(A11,A12,A13,A21,A22,A23,A31,A32,A33,C1
    ,C2,C3)
%   matrix A multipl.with vector C
X1=CMRV(A11,C1)+ CMRV(A12,C2)+CMRV(A13,C3);
X2=CMRV(A21,C1)+ CMRV(A22,C2)+CMRV(A23,C3);
X3=CMRV(A31,C1)+ CMRV(A32,C2)+CMRV(A33,C3);
```

6) **X=mm**(A_p, C_p) for 2 matrices multiplication:

```
function [X11,X12,X13,X21,X22,X23,X31,X32,X33]= mm(A11,A12,A13
    ,A21,A22,A23,A31,A32,A33,C11,C12,C13,C21,C22,C23,C31,C32,C33
    )
% matrix A multipl.with matrix C
X11=CMR(A11,C11)+CMR(A12,C21)+CMR(A13,C31);
X12=CMR(A11,C12)+CMR(A12,C22)+CMR(A13,C32);
X13=CMR(A11,C13)+CMR(A12,C23)+CMR(A13,C33);
X21=CMR(A21,C11)+CMR(A22,C21)+CMR(A23,C31);
X22=CMR(A21,C12)+CMR(A22,C22)+CMR(A23,C32);
X23=CMR(A21,C13)+CMR(A22,C23)+CMR(A23,C33);
X31=CMR(A31,C11)+CMR(A32,C21)+CMR(A33,C31);
X32=CMR(A31,C12)+CMR(A32,C22)+CMR(A33,C32);
X33=CMR(A31,C13)+CMR(A32,C23)+CMR(A33,C33);
```

A.2 Spectral problem, 3 point stencil example

Here is MATLAB code for finding the approximated solution with 3 point stencil:

```
% Puasona problēmas aproksimācija ar cikliskajiem
    sākumnosacījumiem
function maxError = puasons()

    % Nosacījumi
    fFunkcija = @(x) x-0.5;
    precAtr = @(x) 1/12*(-2*x^3+3*x^2-x);
    L = 1;
    N = 50;
    h = 1/N;
    j_x0 = N;
    u0 = 0;

    sablons = 1/N:1/N:L;
    f = arrayfun(fFunkcija, sablons);
    sablons = [0, sablons];
    precY = arrayfun(precAtr, sablons);

    w = exp(2*pi*1i/N);
    W = zeros(N, N);
    for j=1:N
        for k=1:N
            W(j, k) = 1/sqrt(N) * w^(j * k);
        end
    end

    m = zeros(1, N);
    for k=1:N
```

```
        m(k) = 4/h^2 * sin(k * pi / N)^2;
    end

    v = zeros(N, 1);
    Wf = W' * f';
    for k=1:N-1
        v(k, 1) = Wf(k) / m(k);
    end

    v(N, 1) = (u0 - W(j_x0, :) * v) / W(j_x0, N);

    y = real(W * v);
    y = [y(N, 1); y];

    plot(sablons, precY, sablons, y, '--rs');

    maxError = max(abs(precY - y'));

end
```

A.3 Nonlinear heat transfer equation

The following MATLAB program is in m.file `nelper.m`

```
%ODE system dU/dt=AU^sigm1+ a U^beta,with solv.,period.BC
%t=Tb, A =WDW ,FDS and FDSES,sigma1=sigma+1
function nelper(N)
  sigma=3;sigma1=sigma+1;beta=5;a=100;
  N1=N+1; Tb1=0.05;Tb2=6.4811;L=1;x=linspace(0,L,N1)';
  h=L/N;N2=N-1;NT=[1:N];
  lk0=(2*pi/L*NT).^2; % exact eig-val.
  %FDS O(h^2)
  lk2=4/h^2*(sin(pi*h*NT)).^2;
  %FDS O(h^4)
  lk4=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4);
  %FDS O(h^6)
  lk6=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+
      8/45*(sin(pi*h*NT)).^6);
  %FDS O(h^8)
  lk8=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+
      8/45*(sin(pi*h*NT)).^6+4/35*(sin(pi*h*NT)).^8);
  %FDS
  d=lk2;
  %NH=N/2; d(1:NH)=lk0(1:NH);
  %d(NH:N2)=lk0(NH:-1:1);d(N)=0;%FDSES
```

```
    figure,plot(NT,lk2,'-',NT,lk4,'-.', NT,lk6,'*',NT,lk8,'o',NT,d,'d')
    legend('Eig-val O(h^2)','Eig-val O(h^4)','Eig-val O(h^6)', 'Eig-val O(h^8)','Eig-val exact')
   W=exp(2*pi*h*i*[1:N]'*[1:N]);x=x(2:N1);
    A2=zeros(N,N);
    %A2=A2+diag(ones(N2,1),1)+diag(ones(N2,1),-1)-2*diag(ones(N,1));
    %A2(1,N)=1; A2(N,1)=1;A2=A2/(h^2):
    %D=-h*(1./W)*A2*W;% control
    A=real(-h*W*diag(d)*conj(W)); %FDS and FDSES
    y0=sin(pi*x).^100;
    options=odeset('RelTol', 1.0e-7);
    [T1,Y1]=ode15s(@SIST,[0 Tb1],y0,options,A,sigma1,beta,a);
    im=max(imag(Y1(end,:)));
    figure,plot(x,y0,'k-')
    hold on
    plot(x,real(Y1(end,:)'),'k*')
    [T2,Y2]=ode15s(@SIST,[Tb1 Tb2],Y1(end,:),options,A,sigma1,beta,a);
    plot(x,real(Y2(end,:)'),'ko')
    grid on
    title(sprintf('beta=%2.0f,sigma=%2.0f,a=%3.1f, T1=%8.6f,T2=%8.6f',beta, sigma,a,T1(end),T2(end)))
    xlabel('\itx'), ylabel('\itu')
    legend('Sol.U(x,0)','Sol.U(x,T1)', 'Sol.U(x,T2)')
    figure
   T=[T1;T2];Y=[max(real(Y1(:,:)'))';max(real(Y2(:,:)'))'];
    plot(T,Y)
    grid on
    title(sprintf('DV lab.aproks.DS,N=%3.0f, time = %8.6f ',N,T2(end)))
    xlabel('\itt'), ylabel('\itu')
function F=SIST(t,y,A,sigma1,beta,a)
   F=A*y.^sigma1+a*y.^beta;
```

A.4 Linear heat transfer equation

We have following MATLAB `SiltPer.m`:

```
%system ODE U_t+k AU=f with periodical BC
%t=Tb,u(x,t)=sin(2 pi x)exp(-4 pi^2 t),k=1,f=0,N-even
function SiltPer(N)
N1=N+1;MK=20; Tb=0.2;L=1;
x=linspace(0,L,N1)';t=linspace(0,Tb,MK);
h=L/N;N2=N-1;k=1;x=x(2:N1);
```

```
%A2=A2-diag(ones(N2,1),1)-diag(ones(N2,1),-1)+2*diag(ones(N,1))
    ;
%A2(1,N)=-1; A2(N,1)=-1; A2=A2/h^2; %matrix A, control
NT=(1:N)'/L;
lk=4/h^2*(sin(pi*h*NT)).^2; %O(h^2)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4);%O(h^4)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+8/45*(sin(pi*
    h*NT)).^6);%O(h^6)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+8/45*(sin(pi*
    h*NT)).^6+4/35*(sin(pi*h*NT)).^8);%O(h^8)
Ck=sqrt(h/L);
lk0=(2*(1:N)'*pi/L).^2;
d=lk; %FDS
NH=N/2; d(1:NH)=lk0(1:NH);
d(NH:N2)=lk0(NH:-1:1);d(N)=0;%FDSES
W=Ck*exp(2*pi*i*(1:N)'*x'/L)';
W1=Ck*exp(-2*pi*i*(1:N)'*x'/L)';
A2=W*diag(d)*W1; %FDS or FDSES
y1=sin(2*pi*x); % init-cond
P=W1*y1;P1=zeros(MK,N);
for k=1:N
    b=d(k); %FDS or FDSES
    P1(:,k)=P(k)*exp(-b*t');
end
P2=(W*P1')';
prec=-sin(2*pi*x)*exp(-4*pi^2*t);% exact
Ma1=max(max(abs(P2-prec')));%max error an.
X1=ones(MK,1)*x';Y1=t'*ones(1,N);
figure,plot(t',max(abs(P2(:,1:N)'-prec)),'k*')% max error on t
title(sprintf('err. Max-sol.an.on t, Max=%9.7f ',Ma1))
xlabel('\itt'), ylabel('\itu')
figure,plot(x,P2(end,1:N)','ko')
grid on
title(sprintf('Sol.an.on x by Tb.,Max=%9.7f ',Ma1))
xlabel('\itx'), ylabel('\itu')
figure, surfc(X1,Y1,abs(P2-prec'))% error anl.
colorbar
xlabel('x'), ylabel('t'), zlabel('u')
title(sprintf('err. anal.,tNr.=%4.1f,max=%9.7f',MK,Ma1))
```

A.5 Heat transfer equation with periodic BC

We have following MATLAB `Silt.m`:

```
%t=Tb,u(x,t)=sin(2 pi m x)exp(-(2 pi m)^2 t),m<N-even
function Siltm(N)
```

```
N1=N+1;MK=10;m=4; Tb=0.05;L=1;x=linspace(0,L,N1)';t=linspace(0,Tb,MK);
h=L/N;N2=N-1;x=x(2:N1);
%A2=A2-diag(ones(N2,1),1)-diag(ones(N2,1),-1)+2*diag(ones(N,1));
%A2(1,N)=-1; A2(N,1)=-1; A2=A2/h^2; %matrix A, control
NT=(1:N)'/L;
lk=4/h^2*(sin(pi*h*NT)).^2; %O(h^2}
%lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4);%O(h^4)
%lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+%8/45*(sin(pi*h*NT)).^6);%O(h^6)
%lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+%8/45*(sin(pi*h*NT)).^6+4/35*(sin(pi*h*NT)).^8);%O(h^8)
Ck=sqrt(h/L);
lk0=(2*(1:N)'*pi/L).^2;
d=lk; %FDS
%NH=N/2; d(1:NH)=lk0(1:NH);
%d(NH:N2)=lk0(NH:-1:1);d(N)=0;%FDSES
W=Ck*exp(2*pi*i*(1:N)'*x'/L);
W1=Ck*exp(-2*pi*i*(1:N)'*x'/L);
A2=W*diag(d)*W1; %FDS or FDSES
y1=sin(2*pi*m*x); % init-cond
P=zeros(N,1);P=W1*y1;P1=zeros(MK,N);
for k=1:N
    b=d(k); %FDS or FDSES
    P1(:,k)=exp(-b*t')*P(k);
    end
P2=(W*P1.').';% this is transponation operator
P21=W*diag(exp(-d*t(end)))*W1*y1;%okei !
prec=sin(2*pi*m*x)*exp(-(2*pi*m)^2*t);% exact
Ma1=max(max(abs(P2-prec')));%max error an.
X1=ones(MK,1)*x';Y1=t'*ones(1,N);
figure,plot(t',max(abs(P2(:,1:N).'-prec)),'k*')% max error on t
title(sprintf('err. Max-sol.an.on t, Max=%9.7f ',Ma1))
xlabel('\itt'), ylabel('\itu')
figure,plot(x,P21,'ko',x,prec(1:N,end),'*',x,P2(end,1:N),'-')
%figure,plot(x,P2(end,1:N)','ko')
grid on
title(sprintf('Sol.an.on x by Tb.,Max=%9.7f ',Ma1))
xlabel('\itx'), ylabel('\itu')
figure, surfc(X1,Y1,abs(P2-prec'))% error anl.
colorbar
xlabel('x'), ylabel('t'), zlabel('u')
title(sprintf('err. anal.,tNr.=%4.1f,max=%9.7f',MK,Ma1))
```

A.6 Heat transfer problem with periodically placed heat source

We have following Matlab program:

```
%ODE Dekarta PrT_t=T''+Q_0 delta(x)H(t), u_t= GrT+u''-Mu, u(0,t)=0=,
%x \in[-L,L], L=period, Exact
function DalveidaP(NN)
L=4;L1=L*2;Q0=1; N=NN/2;N1=N+1;Pr=7;Tb=1;NN1=NN+1;NN2=NN-1;
x=linspace(0,L1,NN1)';x=x(2:NN1);h=L1/NN;
NT=(1:NN)'/NN;TA=zeros(NN1,1);
lk=4/h^2*(sin(pi*NT)).^2; %2.order FDS eigenvalues
lk=4/h^2*((sin(pi*NT)).^2+1/3*(sin(pi*NT)).^4);%4.order
lk=4/h^2*((sin(pi*NT)).^2+1/3*(sin(pi*NT)).^4+8/45*(sin(pi*NT)).^6);%6.order FDS
lk=4/h^2*((sin(pi*NT)).^2+1/3*(sin(pi*NT)).^4+8/45*(sin(pi*NT)).^6+4/35*(sin(pi*NT)).^8);%8.order FDS
Ck=sqrt(1/NN);NNN=(1:NN)';
lk0=(2*(1:NN)'*pi/(L1)).^2; %exact eigenvalues
d=lk; %FDS
d(1:N)=lk0(1:N);
d(N:NN2)=lk0(N:-1:1);d(NN)=0;%FDSES
B1=zeros(NN,NN);Q=zeros(NN,1); Q(N)=Q0/(Pr*h);
W=Ck*exp(2*pi*i*(1:NN)'*x'/L1)'; % Eigen vectors
W1=Ck*exp(-2*pi*i*(1:NN)'*x'/L1)';% Eigen vectors-conjugate
B1=-W*diag(d)*W1/Pr;
%B1=B1-2*diag(ones(NN,1))+diag(ones(NN-1,1),-1)+diag(ones(NN-1,1),1);
%B1(1,NN)=1; B1(NN,1)=1; B1=B1/(Pr*h^2);%3-diag. matrix
options=odeset('RelTol', 1.0e-7);y0=zeros(NN,1);
[T1,Y]=ode15s(@SIST,[0,Tb],y0,options,B1,Q);% Matlab
X=Y(end,:)';T=zeros(NN1,1);T(2:NN1,1)=X;
T(1,1)=T(NN1,1);
y1=[-L;x-L];Max=T(N1,1), max(T),KF=(2*Q0)/(L1*Pr);
S=KF*(sum(Pr*(1-exp(-lk0*Tb/Pr))./lk0)+0.5*Tb)
for j=1:NN
TA(j+1)=KF*(sum(Pr*(-1).^NNN.*cos(2*pi*NNN*x(j)/L1).*(1-exp(-lk0*Tb/Pr))./lk0)+0.5*Tb);
end
TA(1)=TA(NN1);
figure, plot(y1,T,'-k',y1,TA,'-.', 'LineWidth',3)
grid on,set(gca,'XTick',-L:1:L)
title(sprintf('Temp,Tb=%6.3f,Pr=%6.4f,Max=%6.3f',Tb,Pr,Max))
xlabel('y'), ylabel('T')
legend('Approx','Exact')
figure, plot(y1,abs(T-TA),'-k','LineWidth',3)
grid on,set(gca,'XTick',-L:1:L)
```

```
title(sprintf('Error,Tb=%6.3f,Pr=%6.4f,Max=%6.4f,S=%6.4f',Tb,Pr,Max,S))
xlabel('y'), ylabel('\delta T')
```

A.7 MHD problem with convectively driven flow past an infinite periodically placed planes

We have following Matlab program `DalveidaP.m`:

```
%ODE Dekarta PrT_t=T''+Q_0 delta(x)H(t), u_t= GrT+u''-Mu,
%u(-L,t)=u(L,t)=0,
%x \in[-L,L], 2L=period, Exact T \in [-2L,0]
function DalveidaP(NN)
L=2;L1=L*2;Q0=1; N=NN/2;N1=N+1;Gr=100;M=1000;Pr=0.71;
Tb=1;NN1=NN+1;NN2=NN-1;
x=linspace(0,L1,NN1)';x=x(2:NN1);h=L1/NN;
NT=(1:NN)'/NN;TA=zeros(NN1,1);
lk=4/h^2*(sin(pi*NT)).^2; %2.order FDS eigenvalues
lk=4/h^2*((sin(pi*NT)).^2+1/3*(sin(pi*NT)).^4);%4.order FDS
lk=4/h^2*((sin(pi*NT)).^2+1/3*(sin(pi*NT)).^4+8/45*(sin(pi*NT)).^6);%6.order
lk=4/h^2*((sin(pi*NT)).^2+1/3*(sin(pi*NT)).^4+8/45*(sin(pi*NT)).^6+4/35*(sin(pi*NT)).^8);%8.order FDS
Ck=sqrt(1/NN);NNN=(1:NN)';
lk0=(2*(1:NN)'*pi/(L1)).^2; %exact eigenvalues
d=lk; %FDS
d(1:N)=lk0(1:N);
d(N:NN2)=lk0(N:-1:1);d(NN)=0;%FDSES
B1=zeros(NN,NN);Q=zeros(NN,1); Q(N)=Q0/(Pr*h);
W=Ck*exp(2*pi*i*(1:NN)'*x'/L1)'; % Eigen vectors
W1=Ck*exp(-2*pi*i*(1:NN)'*x'/L1)';% Eigen vectors-conjugate
B1=-W*diag(d)*W1/Pr;
B2=zeros(NN2,NN2);
B2=B2-2*diag(ones(NN2,1))+diag(ones(NN2-1,1),-1)+diag(ones(NN2-1,1),1);
B2=B2/(h^2);%3-diag. matrix
M11=81;M1=M11-1;t1=linspace(0,Tb,M11);u0=zeros(NN2,1);
T0=zeros(NN,1);options=odeset('RelTol', 1.0e-7);
tau=zeros(M1,1);Nu=zeros(M1,1);
for ii=1:M1
[T1,Y1]=ode15s(@SIST,[t1(ii),t1(ii+1)],T0,options,B1,Q);
T0=Y1(end,:);
Nu(ii)=real(3* Y1(end,N)-4*Y1(end,N1) +Y1(end,N1+1))/(2*h);
XT(ii,:)=real([Y1(end,NN),Y1(end,:)]);
for k=1:NN
    cp(k)=spline(T1(:),Y1(:,k));
```

```
end
[T2,Y2]=ode15s(@SIST1,[t1(ii),t1(ii+1)],u0,options,B2,M,Gr,cp,NN2,N);
u0=Y2(end,:);XU(ii,:)=real([0,Y2(end,:),0]);
tau(ii)=real(-4*Y2(end,1) +Y2(end,2))/(2*h);
end
MaxT=max(XT(end,:)),MaxU=max(XU(end,:))
 y1=[-L;x-L];KF=(2*Q0)/(L1*Pr);
S=KF*(sum(Pr*(1-exp(-lk0*Tb/Pr))./lk0)+0.5*Tb)
for j=1:NN
TA(j+1)=KF*(sum(Pr*(-1).^NNN.*cos(2*pi*NNN*x(j)/L1).*(1-exp(-lk0*Tb/Pr))./lk0)+0.5*Tb);
end
TA(1)=TA(NN1);
figure
plot([-L;x-L],XU(5,:)','k-','LineWidth',1.2)
hold on
plot([-L;x-L],XU(16,:)','k-.','LineWidth',1.6)
hold on
plot([-L;x-L],XU(48,:)', 'k--','LineWidth',2.0)
hold on
plot([-L;x-L],XU(64,:)', 'k-','LineWidth',2.4)
hold on
plot([-L;x-L],XU(80,:)', 'k--','LineWidth',2.8)
%axis([0 1 -0.5 0.5])
xlabel('y'), ylabel('u')
legend('t=0.5','t=1.6','t=4.8','t=6.4','t=8')
title(sprintf('Velocity,MaxU=%6.2f,M=%3.1f,Gr=%3.1f',MaxU,M,Gr))
figure
plot([-L;x-L],XT(5,:)', 'k-','LineWidth',1.2)
hold on
plot([-L;x-L],XT(16,:)', 'k-.','LineWidth',1.6)
hold on
plot([-L;x-L],XT(48,:)', 'k--','LineWidth',2.0)
hold on
plot([-L;x-L],XT(64,:)', 'k-','LineWidth',2.4)
hold on
plot([-L;x-L],XT(80,:)', 'k--','LineWidth',2.8)
%axis([0 1 -0.5 0.5])
xlabel('y'), ylabel('T')
legend('t=0.5','t=1.6','t=4.8','t=6.4','t=8')
%legend('t=2','t=4','t=6','t=8','t=10')
title(sprintf('Temperature,MaxT=%6.2f,Tb=%6.3f,Pr=%3.1f',MaxT,Tb,Pr))
figure, plot(y1,XT(M1,:),'-k',y1,TA,'-.', 'LineWidth',3)
grid on,set(gca,'XTick',-L:1:L)
```

```
title(sprintf('T by Tb,Tb=%6.3f,Pr=%6.4f,Max=%6.3f',Tb,Pr,MaxT))
xlabel('y'), ylabel('T')
legend('Approx','Exact')
figure, plot(y1,abs([Y1(end,NN),Y1(end,:)]'-TA),'-k','LineWidth',3)
grid on,set(gca,'XTick',-L:1:L)
title(sprintf('Error,Tb=%6.3f,Pr=%6.4f,MaxT=%6.4f,S=%6.4f',Tb,Pr,MaxT,S))
xlabel('y'), ylabel('\delta T')
figure, plot(t1(2:M11),tau','-k','LineWidth',3)
hold on, plot(t1(2:M11),Nu','-.k','LineWidth',3)
xlabel('t'), ylabel('\tau,Nu')
legend('Skin friction','Nusselt')
title(sprintf('u,T,M=%3.1f,Gr=%3.1f,Pr=%3.1f',M,Gr,Pr))
Mt=max(tau),MNu=max(Nu),mt=min(tau),mNu=min(Nu)
function F=SIST(t,y,A,Q)
F=A*y+ Q;
function F=SIST1(t,y,A,M,Gr,cp,NN2,N)
E=eye(NN2);
F=(A-M*E)*y;
for k=1:N
F(k)=F(k)+ Gr*ppval(cp(k+N),t);
end
for k=N+1:NN2
F(k)=F(k)+ Gr*ppval(cp(k-N),t);
end
```

A.8 System of parabolic type equations

We have following Matlab program `PDSper2.m`:

```
%ODE Sist U_t= KU''+P U'+F with periodical BC U(0)=U(L)=0,
%U'(0)=U'(L) examp. sin(2 pi x), cos(2pix/L),2 equa.
function PDSper2(N)% N-even
N1=N+1;NN=2*N;NH=N/2;N2=N-1; L=1;x=linspace(0,L,N1)';
x=x(2:N1);h=L/N;k0=1;
Tb=0.1; K=[2, -3; -1, 4];P= [2,-5; 1,-4];
d=zeros(N,1);d1=zeros(N,1);
%P=[0,0;0,0];
A=[-4*pi^2*K, -2*pi*P; 2*pi*P, -4*pi^2*K];
y00=[0,1;-1,0],y0=[y00(1,1);y00(2,1);y00(1,2);y00(2,2)]
f00=[5,10;-10,-5];F=[f00(1,1);f00(2,1);f00(1,2);f00(2,2)]
NT=(1:N)'/L;
options=odeset('RelTol', 1.0e-7);
[T,Y]=ode15s(@SIST,[0,Tb],y0,options,A,F);% Matlab solveri
```

```
K0=length(T);
figure,plot(T,Y(:,1),T,Y(:,2),T,Y(:,3),T,Y(:,4), 'LineWidth',2)
grid on
title(sprintf('Coeff.d11,d21,d12,d22,dep.on t,N=%3.0d',N))
legend('d11','d21','d12','d22')
xlabel('\itt'), ylabel('\it dij')
u1=Y(:,1)*sin(2*pi*x)' +Y(:,3)*cos(2*pi*x)';
ym1=max(abs(u1(end,:)))%exact
u2=Y(:,2)*sin(2*pi*x)' +Y(:,4)*cos(2*pi*x)';
ym2=max(abs(u2(end,:)))%exact
figure,plot(x',u1(end,:),x',u2(end,:),'k-.','LineWidth',2)
xlabel('l'), ylabel('u')
title(sprintf('u1(Tb), u2(Tb),ymax1=%5.4d,ymax2=%5.4d',ym1,ym2))
legend('u1','u2')
X11=ones(K0,1)*x';Y11=T*ones(1,N);
figure,surfc(X11,Y11,u1) % telpiskā bilde
colormap(hsv)
colorbar
xlabel('x'), ylabel('t'), zlabel('u1')
title(sprintf('u1 surf.,ymax1=%5.4d',ym1))
X11=ones(K0,1)*x';Y11=T*ones(1,N);
figure,surfc(X11,Y11,u2) % telpiskā bilde
colormap(hsv)
colorbar
xlabel('x'), ylabel('t'), zlabel('u1')
title(sprintf('u2 surf.,ymax2=%5.4d',ym2))
%lk=4/h^2*(sin(pi*h*NT)).^2 ; %2.deriv. O(h^2)
%lk1=1/h*sin(2*pi*h*NT); %1.deriv. O(h^2)
%lk=4/h^2*(sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4 ;
%lk1=1/h*(4/3*sin(2*pi*h*NT)-1/6*sin(4*pi*h*NT)); %4.order
%lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4 +8/45*(sin(
    pi*h*NT)).^6+4/35*(sin(pi*h*NT)).^8);
%lk1=1/h*(24/15*sin(2*pi*h*NT)-2/5*sin(4*pi*h*NT)+8/105*sin(6*
    pi*h*NT)-1/140*sin(8*pi*h*NT)); %8.order
%lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+8/45*(sin(pi
    *h*NT)).^6);
%lk1=1/h*(3/2*sin(2*pi*h*NT)-3/10*sin(4*pi*h*NT)+1/30*sin(4*pi*
    h*NT)); %6.order
Ck=sqrt(h/L);
lk0=(2*(1:N)'*pi/L).^2 ;
lk01=2*(1:N)'*pi/L; %exact eigenvalues
%d=lk; d1=-i*lk1; %FDS
d(1:NH)=(lk0(1:NH));
d(NH:N2)=(lk0(NH:-1:1));%FDSES
d1(1:NH)=-i*(lk01(1:NH));
d1(NH:N2)=i*(lk01(NH:-1:1));
W=Ck*exp(2*pi*i*(1:N)'*x'/L)'; % Eigen vectors
```

```
W1=Ck*exp(-2*pi*i*(1:N)'*x'/L)';% Eigen vectors-conjugate
A1=W*diag(d)*W1;A2=W*diag(d1)*W1;
y01=[sin(2*pi*x),cos(2*pi*x)]*y00';
F1=[sin(2*pi*x),cos(2*pi*x)]*f00';
F2=[F1(:,1); F1(:,2)];
yy0=[y01(:,1); y01(:,2)];
[T1,Y1]=ode15s(@SIST1,[0,Tb],yy0,options,A1,A2,K,P,F2);
im1=max(abs(imag(Y1(end,:))))
v1=Y1(end,1:N);v2=Y1(end,N1:NN);
%v1=Y1(end,1,:);v2=Y1(end,2,:);
sta1=max(abs(u1(end,:)-v1)), sta2=max(abs(u2(end,:)-v2))
figure,plot(x',v1,x',v2,'k-.','LineWidth',2)
xlabel('l'), ylabel('v')
title(sprintf('v1(Tb), v2(Tb),err1=%5.4d,err2=%5.4d',sta1,sta2))
legend('v1','v2')
function F=SIST(t,y,A,F1)
F=A*y +F1;
function F=SIST1(t,yy,A1,A2,K,P,F1)
F=-kron(K,A1)*yy+kron(P,A2)*yy+F1;
```

A.9 Stability of approximations for time-dependent problems

We have following Matlab program `PDSper1R.m`:

```
%ODE systU_t=K U''+P U'+F with periodical BC U(0)=U(L)=0, U'(0)=U'(L)
% example
function PDSper1R(N)% N-even
N1=N+1; L=10;x=linspace(0,L,N1)';x=x(2:N1);h=L/N;
NN=2*N;NH=N/2;NH4=N/4;N2=N-1; Tb=1;K=[2, -3; -1, 4];P= [2,-5; 1,-4];
A=[-4*pi^2*100*K/L^2, -2*pi*P*10/L; 2*pi*P*10/L, -4*pi^2*100*K/L^2];
d=zeros(N,1);d1=zeros(N,1);d2=zeros(N,1);k0=10;
y00=[0,1;-1,0],y0=[y00(1,1);y00(2,1);y00(1,2);y00(2,2)]
f00=[5,10;-1000,-5];F=[f00(1,1);f00(2,1);f00(1,2);f00(2,2)];
NT=(1:N)'/L;
options=odeset('RelTol', 1.0e-7);
[T,Y]=ode15s(@SIST,[0,Tb],y0,options,A,F);% Matlab solveri
K0=length(T);
figure,plot(T,Y(:,1),T,Y(:,2),T,Y(:,3),T,Y(:,4), 'LineWidth',2)
grid on
title(sprintf('Coeff.d11,d21,d12,d22,dep.on t,N=%3.0d',N))
legend('d11','d21','d12','d22')
xlabel('\itt'), ylabel('\it dij')
```

```
u1=Y(:,1)*sin(2*pi*k0*x/L)' +Y(:,3)*cos(2*pi*k0*x/L)';
ym1=max(max(abs(u1)));%exact
u2=Y(:,2)*sin(2*pi*k0*x/L)' +Y(:,4)*cos(2*pi*k0*x/L)';
ym2=max(max(abs(u2)));%exact
figure,plot(x',u1(end,:),x',u2(end,:),'k-.','LineWidth',2)
xlabel('l'), ylabel('u')
title(sprintf('u1(Tb), u2(Tb),ymax1=%5.4d,ymax2=%5.4d',ym1,ym2)
    )
legend('u1','u2')
lk=4/h^2*(sin(pi*h*NT)).^2 ; %2.deriv. O(h^2)
lk1=1/h*sin(2*pi*h*NT); %1.deriv. O(h^2)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4) ;
lk1=2/h*sin(2*pi*h*NT).*(0.5 +1/3*(sin(pi*h*NT)).^2); %4.order
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4 + 8/45*(sin(
    pi*h*NT)).^6+4/35*(sin(pi*h*NT)).^8);
lk1=2/h*sin(2*pi*h*NT).*(0.5 +1/3*(sin(pi*h*NT)).^2 + 4/15*(sin
    (pi*h*NT)).^4 +8/35*(sin(pi*h*NT)).^6);%8. order
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+8/45*(sin(pi*
    h*NT)).^6);
lk1=2/h*sin(2*pi*h*NT).*(0.5 +1/3*(sin(pi*h*NT)).^2 +4/15*(sin(
    pi*h*NT)).^4);
Ck=sqrt(h/L);
lk0=(2*(1:N)'*pi/L).^2 ;
lk01=2*(1:N)'*pi/L; %exact eigenvalues
d=lk;d1=lk1;
d=lk0;d1=lk01;
A1=[-d(k0)*K, -d1(k0)*P; d1(k0)*P, -d(k0)*K];
[T1,Y1]=ode15s(@SIST,[0,Tb],y0,options,A1,F);
K0=length(T1);
figure,plot(T1,Y1(:,1),T1,Y1(:,2),T1,Y1(:,3),T1,Y1(:,4), '
    LineWidth',2)
grid on
title(sprintf('Discrete coeff.d11,d21,d12,d22,dep.on t,N=%3.0d'
    ,N))
legend('d11','d21','d12','d22')
xlabel('\itt'), ylabel('\it dij')
v1=Y1(:,1)*sin(2*pi*k0*x/L)' +Y1(:,3)*cos(2*pi*k0*x/L)';
ym1=max(max(abs(v1)));%appr
v2=Y1(:,2)*sin(2*pi*k0*x/L)' +Y1(:,4)*cos(2*pi*k0*x/L)';
ym2=max(max(abs(v2)));%appr
im1=max(abs(imag(Y1(end,:))))
ma1=max(abs(u1(end,:)-v1(end,:))), ma2=max(abs(u2(end,:)-v2(end
    ,:)))
figure,plot(x',v1(end,:),x',v2(end,:),'k-.','LineWidth',2)
xlabel('l'), ylabel('v')
title(sprintf('v1(Tb), v2(Tb),ymax1=%5.4d,ymax2=%5.4d',ma1,ma2)
    )
legend('v1','v2')
```

```
function F=SIST(t,y,A,F1)
F=A*y +F1;
```

A.10 System of nonlinear parabolic type equations

We have following Matlab program `2286.m`:

```
[T1,Y1]=ode15s(@SIST1,[0,Tb],yy0,options,A1,A2,K,P,F2);
K0=length(T1)
im1=max(abs(imag(Y1(end,:))))
v1=Y1(end,1:N);v2=Y1(end,N1:NN);vM1=max(real(v1));vm1=min(real(v1));
vM2=max(real(v2));vm2=min(real(v2));
v1M=max(real(Y1(:,1:N)'));v1m=min(real(Y1(:,1:N)'));
v2M=max(real(Y1(:,N1:NN)'));v2m=min(real(Y1(:,N1:NN)'));
figure,plot(T1,v1M,T1,v1m,T1,v2M,T1,v2m, 'LineWidth',2)
grid on
title(sprintf('Max-min values depending on t,N=%3.0d',N))
legend('Mv1','mv1','Mv2','mv2')
xlabel('\itt'), ylabel('\it Mm')
%sta1=max(abs(u1(end,:)-v1)), sta2=max(abs(u2(end,:)-v2))
figure,plot(x',v1,x',v2,'k-.','LineWidth',2)
xlabel('x'), ylabel('v')
title(sprintf('v1(Tb), v2(Tb),vM1=%5.4d,vm1=%5.4d,vM2=%5.4d,...
vm2=%5.4d',vM1,vm1,vM2,vm2))
legend('v1','v2')
function F=SIST(t,y,A,F1)
F=A*y +F1;
function F=SIST1(t,yy,A1,A2,K,P,F1)
F=-kron(K,A1)*yy.^3+kron(P,A2)*yy.^3+F1.*yy.^2;
```

A.11 Boundary value problem with periodic BC in one direction

We have following MATLAB file `PuasPer.m`:

```
%system ODE U_yy-AU=f with periodical BC
%t=Tb,u(x,y)=cos(2 pi x)sinh(2 pi y),f=0,N-even
function PuasPer(N,M)
N1=N+1; H=1;L=1;x=linspace(0,L,N1)';y=linspace(0,H,M);
h=L/N;N2=N-1;x=x(2:N1);
%A2=A2-diag(ones(N2,1),1)-diag(ones(N2,1),-1)+2*diag(ones(N,1));
%A2(1,N)=-1; A2(N,1)=-1; A2=A2/h^2; %matrix A, control
NT=(1:N)'/L;
```

```
lk=4/h^2*(sin(pi*h*NT)).^2; %O(h^2)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4);%O(h^4)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+8/45*(sin(pi*h*NT)).^6);%O(h^6)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+8/45*(sin(pi*h*NT)).^6+4/35*(sin(pi*h*NT)).^8);%O(h^8)
Ck=sqrt(h/L);
lk0=(2*(1:N)'*pi/L).^2;
d=lk; %FDS
%NH=N/2; d(1:NH)=lk0(1:NH);
%d(NH:N2)=lk0(NH:-1:1);d(N)=0;%FDSES
W=Ck*exp(2*pi*i*(1:N)'*x'/L)';
W1=Ck*exp(-2*pi*i*(1:N)'*x'/L)';
A2=W*diag(d)*W1; %FDS or FDSES
yr=sinh(2*pi*L)*cos(2*pi*x);
yl=zeros(N,1); % bound-cond
P=W1*yl;P1=zeros(M,N);P0=W1*yr;
for k=1:N2
    b=sqrt(d(k)); %FDS or FDSES
P1(:,k)=P(k)*cosh(b*y')+(P0(k)-P(k)*cosh(b*H))/sinh(b*H)*sinh(b*y');
end
P1(:,N)=P(N)+(P0(N)-P(N))*y'/H;
P2=(W*P1')';
prec=cos(2*pi*x)*sinh(2*pi*y);% exact
Ma1=max(max(abs(P2-prec')));%max error an.
X1=ones(M,1)*x';Y1=y'*ones(1,N);
figure,plot(y',max(abs(P2(:,1:N)'-prec)),'k*')% max error on y
title(sprintf('err. Max-sol.an.on y, Max=%9.7f ',Ma1))
xlabel('\ity'), ylabel('\itu')
figure, surfc(X1,Y1,abs(P2-prec'))% error anl.
colorbar
xlabel('x'), ylabel('y'), zlabel('u')
title(sprintf('err. anal.,yNr.=%4.1f,max=%9.7f',M,Ma1))
```

A.12 Matrix solution of boundary value problem with periodic BC in two directions

We have following MATLAB file `puas4.m` for matrix solution when $N = M$:

```
% u_yy= -u_xx+f, period. BC in x and y direc.
%u(x,y)=cos(2px)cos(2py)- exact sol.
function puas4(N)
N2=N-1;N1=N+1;h=1/N;L=1;H=1;
x=linspace(0,L,N+1);NT=(1:N)'/L;
y=linspace(0,H,N+1);
```

```
x=x(2:N1);y=y(2:N1)';
lk=4/h^2*(sin(pi*h*NT)).^2; %O(h^2)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4);%O(h^4)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+8/45*(sin(pi*
    h*NT)).^6);%O(h^6)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+8/45*(sin(pi*
    h*NT)).^6+4/35*(sin(pi*h*NT)).^8);%O(h^8)
Ck=sqrt(h/L);
lk0=(2*(1:N)'*pi/L).^2; %exact eig-val.
d=lk; %FDS
%NH=N/2; d(1:NH)=lk0(1:NH);% FDSES
%d(NH:N2)=lk0(NH:-1:1);d(N)=0;%FDSES
W=Ck*exp(2*pi*i*(1:N)'*x/L)';
W1=Ck*exp(-2*pi*i*(1:N)'*x/L)';
B1=W*diag(-d)*W1; %FDS or FDSES
%B1=zeros(N,N);
%B1=B1-1/12*diag(ones(N2-1,1),2)+4/3*diag(ones(N2,1),1)-1/12*
    diag(ones(N2-1,1),-2)+4/3*diag(ones(N2,1),-1)-5/2*diag(ones(
    N,1));
%B1(1,N)=4/3; B1(N,1)=4/3;B1(1,N2)=-1/12; B1(2,N)=-1/12;
%B1(N,2)=-1/12; B1(N2,1)=-1/12;% control O(h^4)
%B1=B1-2*diag(ones(N,1))+diag(ones(N2,1),-1)+diag(ones(N2,1),1)
    ;
%B1(1,N)=1;B1(N,1)=1; %control O(h^2)
%B1=B1/(h^2);
e1=eye(N);
B2=sqrtm(-B1);B3=inv(B2^2+4*pi^2*e1);
B=8*pi^2*B3;prec=cos(2*pi*x')*cos(2*pi*y');
for iy=1:N
y1=y(iy,1);
g0=cos(2*pi*x)*cos(2*pi*y1);
u(:,iy)=B*g0';
end
Amax=max(max(abs(u'-prec')));
X=ones(N,1)* x; Y=y*ones(1,N);
surfc(X,Y,abs(u'-prec'))
colormap(gray),
colorbar
xlabel('x'), ylabel('y'), zlabel('u')
view(135,45)
title(sprintf('Period.BC in both direc. FDS O(h^2),h=%4.2f,
    Error=%8.6e',h,Amax))
```

A.13 Analytical solution of boundary value problem with periodic BC in two directions

```
%system ODE U_yy-AU=f with periodical BC in 2 direct,.an. sol.
%u(x,y)=cos(2 pi x)cos(2 pi y),f=-8 \pi^2 u(x,y),N-even
function PuasPer2(N,M)
N1=N+1; M1=M+1;H=1;L=1;x=linspace(0,L,N1)';y=linspace(0,H,M1);
h=L/N;N2=N-1;M2=M-1;x=x(2:N1);y=y(2:M1)';h1=H/M;
%A2=A2-diag(ones(N2,1),1)-diag(ones(N2,1),-1)+2*diag(ones(N,1))
    ;
%A2(1,N)=-1; A2(N,1)=-1; A2=A2/h^2; %matrix A,O(h^2), control
NT=(1:N)'/L;
lk=4/h^2*(sin(pi*h*NT)).^2; %O(h^2)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4);%O(h^4)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+...
8/45*(sin(pi*h*NT)).^6);%O(h^6)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+...
8/45*(sin(pi*h*NT)).^6+4/35*(sin(pi*h*NT)).^8);%O(h^8)
Ck=sqrt(h/L);
lk0=(2*(1:N)'*pi/L).^2;
d=lk; %FDS
NH=N/2; d(1:NH)=lk0(1:NH);
d(NH:N2)=lk0(NH:-1:1);d(N)=0;%FDSES
W=Ck*exp(2*pi*i*(1:N)'*x'/L)';
W1=Ck*exp(-2*pi*i*(1:N)'*x'/L)';
A2=W*diag(d)*W1; %FDS or FDSES
f=-8*pi^2*cos(2*pi*y)*cos(2*pi*x');g=W1*f';
gk=zeros(M,1);
P1=zeros(M,N);
for k=1:N
     gk(:)=g(k,:);
     b=sqrt(d(k)); %FDS or FDSES
     for j=1:M
     if j==M v2(j)=0;end
     s1=0; for j1=1:j-1
s1=s1+F1(j*h1,j1*h1,b,H)*gk(j1); end ;
v1(j)=0.5*h1*(F1(j*h1,0,b,H)*gk(M)+F1(j*h1,j*h1,b,H)*gk(j)). .
    .
+h1*s1;
     s2=0;for j1=j+1:M-1
s2=s2+F2(j*h1,j1*h1,b,H)*gk(j1); end
     if j~=M
v2(j)=0.5*h1*(F2(j*h1,H,b,H)*gk(M)+F2(j*h1,j*h1,b,H)*gk(j)). .
    .
+h1*s2;end
if k ~= N  P1(j,k)=-0.5 /(b*sinh(0.5*H*b))*(v1(j)+v2(j));end
     if k==N P1(j,k)=0; end
     end; end
P2=(W*P1')';
P2=P2-P2(M,N)+1;
```

```
prec=cos(2*pi*y)*cos(2*pi*x');im =max(max(abs(imag(P2))))
Ma1=max(max(abs(P2-prec)));%max error an.
X1=ones(M,1)*x';Y1=y*ones(1,N);
figure,plot(y,max(abs(P2(:,1:N)'-prec')),'k*')% max error on y
title(sprintf('err. Max-sol.an.on y, Max=%9.7f ',Ma1))
xlabel('\ity'), ylabel('\itu')
figure, surfc(X1,Y1,abs(P2-prec))% error anl.
colorbar
xlabel('x'), ylabel('y'), zlabel('u')
title(sprintf('err. anal.,yNr.=%4.1f,max=%9.7f',M,Ma1))
function f=F1(y,t,b,H)
f=cosh(b*(0.5*H -y +t));
function f=F2(y,t,b,H)
f=cosh(b*(0.5*H +y -t));
```

A.14 Kronecker-tensor solution of problem with periodic BC in two directions

```
%system ODE U_yy-AU=f with periodical BC in 2 direct,
%u(x,y)=cos(2 pi x)cos(2 pi y),f=-8 \pi^2 u(x,y),N,M-even
%Kroneker-Tensor algorithm
function PuasTen2(N,M)
N1=N+1; M1=M+1;H=1;L=1;x=linspace(0,L,N1)';y=linspace(0,H,M1);
h=L/N;N2=N-1;M2=M-1;x=x(2:N1);y=y(2:M1)';h1=H/M;NM=N*M;NM2=NM
    -1;
%A2=A2-diag(ones(N2,1),1)-diag(ones(N2,1),-1)+2*diag(ones(N,1))
    ;
%A2(1,N)=-1; A2(N,1)=-1; A2=A2/h^2; %matrix A,O(h^2), control
NT=(1:N)'/L;MT=(1:M)'/H;
lk=4/h^2*(sin(pi*h*NT)).^2; %O(h^2)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4);%O(h^4)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+8/45*(sin(pi*
    h*NT)).^6);%O(h^6)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+8/45*(sin(pi*
    h*NT)).^6+4/35*(sin(pi*h*NT)).^8);%O(h^8)
lk1=4/h1^2*(sin(pi*h1*MT)).^2; %O(h1^2)
lk1=4/h1^2*((sin(pi*h1*MT)).^2+1/3*(sin(pi*h1*MT)).^4);%O(h1^4)
lk1=4/h1^2*((sin(pi*h1*MT)).^2+1/3*(sin(pi*h1*MT)).^4+8/45*(sin
    (pi*h1*MT)).^6);%O(h1^6)
lk1=4/h1^2*((sin(pi*h1*MT)).^2+1/3*(sin(pi*h1*MT)).^4+8/45*(sin
    (pi*h1*MT)).^6+4/35*(sin(pi*h1*MT)).^8);%O(h1^8)
Ck=sqrt(h/L);
Ck1=sqrt(h1/H);
lk0=(2*(1:N)'*pi/L).^2;
lk01=(2*(1:M)'*pi/H).^2;
```

```
d=lk; %FDS-x
d1=lk1; %FDS-y
NH=N/2; d(1:NH)=lk0(1:NH);
d(NH:N2)=lk0(NH:-1:1);d(N)=0;%FDSES-x
MH=M/2; d1(1:MH)=lk01(1:MH);
d1(MH:M2)=lk01(MH:-1:1);d1(M)=0;%FDSES-y
W=Ck*exp(2*pi*i*(1:N)'*x'/L)';
W1=Ck*exp(-2*pi*i*(1:N)'*x'/L)';
Wy=Ck1*exp(2*pi*i*(1:M)'*y'/H)';
Wy1=Ck1*exp(-2*pi*i*(1:M)'*y'/H)';
Wxy=kron(Wy,W);Wxy1=kron(Wy1,W1);
A1=W*diag(d)*W1; %FDS or FDSES, control
A2=Wy*diag(d1)*Wy1; %FDS or FDSES
f=8*pi^2*cos(2*pi*y)*cos(2*pi*x');
f=reshape(f',NM,1);g=Wxy1*f;
dd=zeros(NM,1);gg=zeros(NM,1);P2=zeros(NM,1);
for j=1:M j1=1+(j-1)*N;j2=j1+N2; dd(j1:j2)=d(:)+d1(j);end
for ji=1:NM2 gg(ji)=g(ji)/dd(ji); end
gg(NM)=0;
P2=Wxy*gg;P2=reshape(P2,N,M)';
P2=P2-P2(M,N)+1;
prec=cos(2*pi*y)*cos(2*pi*x');im =max(max(abs(imag(P2))))
Ma1=max(max(abs(P2-prec)));%max error an.
X1=ones(M,1)*x';Y1=y*ones(1,N);
figure,plot(y,max(abs(P2(:,1:N)'-prec')),'k*')% max error on y
title(sprintf('err. Max-sol.an.on y, Max=%9.7f ',Ma1))
xlabel('\ity'), ylabel('\itu')
figure, surfc(X1,Y1,abs(P2-prec))% error anl.
colorbar
xlabel('x'), ylabel('y'), zlabel('u')
title(sprintf('err. anal.,yNr.=%4.1f,max=%9.7f',M,Ma1))
```

A.15 Example of wave equation with periodic BC for one wave number

We have the following MATLAB file `Wave2.m`:

```
%system ODE U_tt+a^2 AU=f with periodical BC
%t=Tb,u(x,t)=sin(2 pi x)cos(2 pi t),a=1,f=0,N-even
function Wave2(N)
N1=N+1;MK=20; Tb=1;L=1;x=linspace(0,L,N1)';t=linspace(0,Tb,MK);
h=L/N;N2=N-1;a=1;a2=a^2;x=x(2:N1);
%A2=A2-diag(ones(N2,1),1)-diag(ones(N2,1),-1)+2*diag(ones(N,1))
    ;
%A2(1,N)=-1; A2(N,1)=-1; A2=A2/h^2; %matrix A, control
```

```
NT=(1:N)'/L;
lk=4/h^2*(sin(pi*h*NT)).^2; %O(h^2)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4);%O(h^4)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+8/45*(sin(pi*h*NT)).^6);%O(h^6)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+8/45*(sin(pi*h*NT)).^6+4/35*(sin(pi*h*NT)).^8);%O(h^8)
Ck=sqrt(h/L);
lk0=(2*(1:N)'*pi/L).^2;
d=lk; %FDS
NH=N/2; d(1:NH)=lk0(1:NH);
d(NH:N2)=lk0(NH:-1:1);d(N)=0;%FDSES
W=Ck*exp(2*pi*i*(1:N)'*x'/L);
W1=Ck*exp(-2*pi*i*(1:N)'*x'/L);
A2=W*diag(d)*W1; %FDS or FDSES
y2=zeros(N,1);
y1=sin(2*pi*x); % init-cond
P=W1*y1;P1=zeros(MK,N);P0=W1*y2;
for k=1:N2
    b=sqrt(a2*d(k)); %FDS or FDSES
    P1(:,k)=P(k)*cos(b*t')+P0(k)/b*sin(b*t');
end
P1(:,N)=P(N)+P0(N)*t';
P2=(W*P1.').';% operator of transponation
prec=sin(2*pi*x)*cos(2*pi*t);% exact
Ma1=max(max(abs(P2-prec')));%max error an.
X1=ones(MK,1)*x';Y1=t'*ones(1,N);
figure,plot(t',max(abs(P2(:,1:N).'-prec)),'k*')% max error on t
title(sprintf('err. Max-sol.an.on t, Max=%9.7f ',Ma1))
xlabel('\itt'), ylabel('\itu')
figure,plot(x,P2(end,1:N).','ko')
grid on
title(sprintf('Sol.an.on x by Tb.,Max=%9.7f ',Ma1))
xlabel('\itx'), ylabel('\itu')
figure, surfc(X1,Y1,abs(P2-prec'))% error anl.
colorbar
xlabel('x'), ylabel('t'), zlabel('u')
title(sprintf('err. anal.,tNr.=%4.1f,max=%9.7f',MK,Ma1))
```

A.16 Example of wave equation with periodic BC for different wave number

We have following MATLAB file `Wave2m.m`:

```
%system ODE U_tt+ AU=0 with periodical BC
%t=Tb,u(x,t)=sin(2 pi m x)cos(2 pi m t),m<N-even
```

```
function Wave2m(N)
N1=N+1;MK=3;m=2; Tb=1;L=1;x=linspace(0,L,N1)';t=linspace(0,Tb,MK);
h=L/N;N2=N-1;a=1;a2=a^2;x=x(2:N1);
%A2=A2-diag(ones(N2,1),1)-diag(ones(N2,1),-1)+2*diag(ones(N,1));
%A2(1,N)=-1; A2(N,1)=-1; A2=A2/h^2; %matrix A, control
NT=(1:N)'/L;
lk=4/h^2*(sin(pi*h*NT)).^2; %O(h^2)
%lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4);%O(h^4)
%lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+8/45*(sin(pi*h*NT)).^6);%O(h^6)
%lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+8/45*(sin(pi*h*NT)).^6+4/35*(sin(pi*h*NT)).^8);%O(h^8)
Ck=sqrt(h/L);
lk0=(2*(1:N)'*pi/L).^2;
d=lk; %FDS
%NH=N/2; d(1:NH)=lk0(1:NH);
%d(NH:N2)=lk0(NH:-1:1);d(N)=0;%FDSES
W=Ck*exp(2*pi*i*(1:N)'*x'/L);
W1=Ck*exp(-2*pi*i*(1:N)'*x'/L);
A2=W*diag(d)*W1; %FDS or FDSES
y1=sin(2*pi*m*x); % init-cond
P=zeros(N,1);P=W1*y1;P1=zeros(MK,N);
for k=1:N
      b=sqrt(a2*d(k)); %FDS or FDSES
      P1(:,k)=cos(b*t')*P(k);
      end
P2=(W*P1.').';% this is transponation operator
P21=W*diag(cos(sqrt(d)*t(end)))*W1*y1;%okei !
prec=sin(2*pi*m*x)*cos(2*pi*m*t);% exact
Ma1=max(max(abs(P2-prec')));%max error an.
X1=ones(MK,1)*x';Y1=t'*ones(1,N);
figure,plot(t',max(abs(P2(:,1:N).'-prec)),'k*')% max error on t
title(sprintf('err. Max-sol.an.on t, Max=%9.7f ',Ma1))
xlabel('\itt'), ylabel('\itu')
figure,plot(x,P21,'ko',x,prec(1:N,end),'*',x,P2(end,1:N),'-')
%figure,plot(x,P2(end,1:N)','ko')
grid on
title(sprintf('Sol.an.on x by Tb.,Max=%9.7f ',Ma1))
xlabel('\itx'), ylabel('\itu')
figure, surfc(X1,Y1,abs(P2-prec'))% error anl.
colorbar
xlabel('x'), ylabel('t'), zlabel('u')
title(sprintf('err. anal.,tNr.=%4.1f,max=%9.7f',MK,Ma1))
```

A.17 Nonlinear wave equation with periodic BC

We have the following MATLAB file `svarst3per.m`:

```
%PDE U_tt=AG +F with periodic BC
%t=Tb, A =WDW* with different aproksimation
function svarst3per(N)
sigma=2;sigma1=sigma+1;beta=0;a=0;
N1=N+1; Tb=0.1;L=1;x=linspace(0,L,N1)';
h=L/N;N2=N-1;NT=[1:N]/L;NN=2*N;NH=N/2;
lk0=(2*pi/L*(1:N)').^2; % precīzās īpašv.
lk2=4/h^2*(sin(pi*h*NT)).^2; %O^2
lk4=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4);%O^4
lk6=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+8/45*(sin(pi
    *h*NT)).^6);%O^6
lk8=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+8/45*(sin(pi
    *h*NT)).^6+ 4/35*(sin(pi*h*NT)).^8);%O^8
W=exp(2*pi*h*i*[1:N]'*[1:N]/L);x=x(2:N1);lk=zeros(N,1);
%a=lk2(1)
%A2=zeros(N,N);
%A2=A2+diag(ones(N2,1),1)+diag(ones(N2,1),-1)-2*diag(ones(N,1))
    ;
%A2(1,N)=1; A2(N,1)=1;A2=A2/h^2;
%D=-h*(1./W)*B2*W;
lk(1:NH)=lk0(1:NH);
lk(NH:N2)=lk0(NH:-1:1);%FDSES
A2=-h*W*diag(lk8)*conj(W);
A=zeros(NN,NN);y1=(sin(2*pi*x));y2=zeros(N,1);
y0=[y1;y2];Z1=zeros(N,N);A=[Z1,eye(N,N);A2,Z1];
B=[Z1,Z1; eye(N,N),Z1];
options=odeset('RelTol', 1.0e-7);
[T,Y]=ode15s(@SIST,[0 Tb],y0,options,A,sigma1,beta,a,B);
im=max(abs(imag(Y(end,:))));
MA=max(abs(real(Y(end,1:N))));
figure,plot(x,real(Y(end,1:N)'),'ko')
grid on
title(sprintf('End time., maxim=%8.6f,time = %8.6f Max=%9.7f ',
    im,T(end),MA))
xlabel('\itx'), ylabel('\itu')
figure
plot(T(:),max(real(Y(:,1:N)')))
grid on
title(sprintf('FDS in time,N=%3.0f, time = %8.6f ',N,T(end)))
xlabel('\itt'), ylabel('\itu')
K=length(T);X1=ones(K,1)*x';Y1=T*ones(1,N);
figure, surfc(X1,Y1,real(Y(:,1:N)))
colorbar
xlabel('x'), ylabel('t'), zlabel('u')
```

```
title(sprintf('Surface,imag=%8.6f,Laika  sl.sk.=%3.06f,  Max=%8.6
    f',im,K,MA))
function F=SIST(t,y,A,sigma1,beta,a,B)
F=A*y.^sigma1+a*B*(y).^beta;
```