UNIVERSITY OF LATVIA

FACULTY OF PHYSICS AND MATHEMATICS

Tabita Bobinska

MATHEMATICAL MODELS AND THEIR SOLUTIONS FOR DOMAINS OF COMPLEX FORM

PhD Thesis

Field: Mathematics
Subfield: Mathematical modelling

Scientific supervisor:
Emer. prof., Dr. habil. math. Andris Buiķis

Riga, 2014

Abstract

In the PhD thesis original mathematical models and their solutions for domains of complex form are considered. Intensive steel quenching process for elements with fins is described by 3D hyperbolic and parabolic heat conduction equation. For deriving an analytical solution for the problem, Green's function method and its modification is used. For systems with double wall and double fins that are employed in modern computers, 2D stationary and transient heat transfer problems are given. Our approximate solution is constructed using conservative averaging method, finite difference scheme and its modifications for boundary conditions. A new mathematical model for a willow flute is presented which considers 1D linear wave equation with a source function and third type boundary conditions.

Keywords: heat conduction equations, wave equation, Green's function, conservative averaging, finite difference method, method of separation of variables, L-shape sample, double wall with double fins, willow flute

Contents

.

Introduction

There are many practical applications and important uses for both single layer systems with fins, and for systems with double wall and double fins. They are used as heat transfer equipment, for intensive quenching etc. To describe these processes we formulate the problems in 3D and 2D, and present precise and numerical methods for obtaining solutions. The case of non-linear boundary conditions is also considered. (See the first three chapters of the thesis.)

The last chapter is dedicated to constructing a mathematical model for a willow flute. When constructing the model for this instrument one can stumble upon a few questions – is it enough to use a linear wave equation, how to describe the airstream that impinges the edged opening and mass flow in the passageway etc. These facts are considered in our model.

Together with co-authors the author has **8** publications on the subject of the PhD thesis. The author has reported on **6** international and **3** local conferences.

Importance of the Subject

Intensive quenching processes that are considered in the first chapter of the thesis, have a significant impact in heat treating industry. Originated by Dr. N. Kobasko, intensive water-quenching technologies are broadly developed. Since his discovery, there have been numerous experiments and studies conducted by Dr. Kobasko in cooperation with scientists from USA, Ukraine, Mexico. One of those international joint research projects is "Database for cooling capacities of various quenchants to be developed with the modern computational and experimental techniques", WSEAS.

Systems with double wall and double fins are used in modern computers' cooling systems. But here to help to remove excess heat from computer components, cooling with air is replaced by liquid cooling. Based on boiling heat transfer in microchannel passages it provides an efficient way to remove high heat flux in small areas. This results in a non-linear problem, that we solve using different kind of mathematical methods.

Normally when describing musical instruments, the mathematical model contains a wave equation that is solved by the method of separation of variables. In the thesis we use this method when modelling a willow flute.

Aim of the Thesis

The broader goal of this thesis is to develop mathematical models for domains of complex form. It includes the following objectives:

- describe intensive steel quenching for 3D L-shape sample by both hyperbolic and parabolic heat conduction equation;
- propose a new approach for deriving solutions for stationary and transient problems in 2D systems with double and double fin;
- develop a mathematical model for a willow flute.

Research Methodology

Analytical and numerical methods:

- conservative averaging method;
- Green's function method and its modification for regular, non-canonical domains;
- separation of variables;
- finite difference method and its modification.

Scientific Novelty and Main Results

- New mathematical models are proposed for intensive quenching when choosing hyperbolic heat equation to describe the process in thin 3D L-shape samples. Exact solutions for hyperbolic and parabolic heat equations are constructed.

- Models for 2D stationary and transient problems for systems with double wall and double fins are given. Both linear and nonlinear case (i.e., when assigning boiling conditions) has been considered.

- We propose a mathematical model for a Norwegian flute, using a linear wave equation. We use a source function to describe the process where the airstream tends to form vortices when impinging the edged opening. But the mass flow in the passageway is approximated by 3rd type boundary condition.

Publications, Reports on Conferences

Publications

1. T. Bobinska, M. Buike, A. Buikis, Heat Transfer with Full Boiling in System with Double Wall and Double Fins, *Proceedings of the 11th WSEAS International Conference on Heat Transfer, Thermal Engineering and Environment, Vouliagmeni, Greece*, WSEAS Press, 2013, pp. 148-153. ISBN: 978-1-61804-183-8. Available from Internet: http://www.wseas.us/e-library/conferences/2013/Vouliagmeni/FLUHE/FLUHE-24.pdf (last checked 28/10/2013).

2. T. Bobinska, M. Buike, A. Buikis, Heat Transfer with Partial Boiling in System with Double Wall and Double Fins, *Proceedings of the 11th WSEAS International Conference on Heat Transfer, Thermal Engineering and Environment, Vouliagmeni, Greece*, WSEAS Press, 2013, pp. 132-137. ISBN: 978-1-61804-183-8. Available from Internet: http://www.wseas.us/e-library/conferences/2013/Vouliagmeni/FLUHE/FLUHE-22.pdf (last checked 28/10/2013).

3. T. Bobinska, M. Buike, A. Buikis, H.H. Cho, Stationary Heat Transfer in System with Double Wall and Double Fins, *Proceedings of the 11th WSEAS Int. Conf. on Data Networks, Communications, Computers (DNCOCO'12) and the 5th WSEAS Int. Conf. on Materials Science (MATERIALS'12), Sliema, Malta*, WSEAS Press, 2012, pp. 260-265. ISBN 978-1-61804-118-0. Available from Internet: http://www.wseas.us/e-library/conferences/2012/Sliema/DNMAT/DNMAT-43.pdf (last checked 28/10/2013).

4. T. Bobinska, A. Buikis, A Mathematical Model for a Willow Flute, *Proceedings of the 9th IASME/WSEAS Int. Conf. on HEAT TRANSFER, THERMAL ENGINEERING and ENVIRONMENT (HTE'11), Florence, Italy, 2011*, WSEAS Press, 2011, pp. 188-192. ISBN 978-1-61804-026-8. Available from Internet: http://www.wseas.us/e-library/conferences/2011/Florence/HEAFLU/HEAFLU-29.pdf (last checked 28/10/2013).

5. T. Bobinska, M. Buike, A. Buikis, Comparing Solutions of Hyperbolic and Parabolic Heat Conduction Equations for L-shape Samples, *Proceedings of the 9th IASME/WSEAS Int. Conf. on HEAT TRANSFER, THERMAL ENGINEERING and ENVIRONMENT (HTE'11), Florence, Italy, 2011*, WSEAS Press, 2011, pp. 384-389. ISBN 978-1-61804-026-8. Available from Internet: http://www.wseas.us/e-library/conferences/2011/Florence/HEAFLU/HEAFLU-66.pdf (last checked 28/10/2013).

6. T. Bobinska, M. Buike, A. Buikis, Hyperbolic Heat Equation as Mathematical Model for Steel Quenching of L-shape and T-shape Samples, Direct and Inverse Problems. *WSEAS TRANSACTIONS on HEAT and MASS TRANSFER*, **5**(3): 63-72, 2010. ISSN 1790-5044.

Available from Internet: http://www.wseas.us/e-library/transactions/heat/2010/89-626.pdf (last checked 28/10/2013).

7. T. Bobinska, M. Buike, A. Buikis, Hyperbolic Heat Equation as Mathematical Model for Steel Quenching of L-Shape Samples, Part 2 (Inverse Problem), In S.H. Sohrab, H.J. Catrakis and N. Kobasko (Eds.), *Proceedings of the 5th IASME/WSEAS Int. Conf. on CONTINUUM MECHANICS (CM'10), Cambridge, UK, 2010,*. WSEAS Press, 2010, pp. 21-26. ISBN 978-960-474-158-8. Available from Internet: http://www.wseas.us/e-library/conferences/2010/Cambridge/CFH/CFH-01.pdf (last checked 28/10/2013).

8. A. Buikis, M. Buike, T. Bobinska, Solution of hyperbolic heat equation reduction to integral equation for a right corner, *Proceedings of the Int. Conf. Differential Equations and Their Applications (DETA'09) Panevezys, Lithuania, 2009*, KTU, 2009, pp. 14-16.

Reports on International Scientific Conferences

1. T. Bobinska, M. Buiķe, A. Buikis, H.H. Cho. Stationary Heat Transfer in System with Double Wall and Double Fins. *11th WSEAS Int. Conf. on Data Networks, Communications, Computers (DNCOCO'12) and the 5th WSEAS Int. Conf. on Materials Science (MATERIALS'12*. Sliema, Malta, September 7-9, 2012. Proceedings, pp. 260-265.

2. T. Bobinska, M. Buike, A. Buikis. Double Wall with Double Fins. *17th International Conference Mathematical Modelling and Analysis*. Tallinn, Estonia, June 6-9, 2012. Abstracts, p. 9.

3. T. Bobinska, A. Buikis. A Mathematical Model for a Willow Flute. *9th IASME/WSEAS International Conference on HEAT TRANSFER, THERMAL ENGINEERING AND ENVIRONMENT*. Florence, Italy, August 23–25, 2011. Proceedings, pp. 188-192.

4. T. Bobinska, M. Buike, A. Buikis. Comparing Solutions of Hyperbolic and Parabolic Heat Conduction Equations for L-shape Samples. *9th IASME/WSEAS International Conference on HEAT TRANSFER, THERMAL ENGINEERING AND ENVIRONMENT*. Florence, Italy, August 23–25, 2011. Proceedings, pp. 384-389.

5. T. Bobinska, A. Buikis. A Mathematical Model for a Norwegian Flute. *16th International Conference Mathematical Modelling and Analysis*. Sigulda, Latvia, May 25-28, 2011. Abstracts, p. 17.

6. T. Bobinska, M. Buike, A. Buikis. Comparing Solutions of Hyperbolic and Parabolic Heat Conduction Equations for L-shape Samples. *15th International Conference Mathematical Modelling and Analysis*. Druskininkai, Lithuania, May 26-29, 2010. Abstracts, p. 8.

7. T. Bobinska, M. Buike, A. Buikis. Hyperbolic Heat Equation as Mathematical Model for Steel Quenching of L-Shape Samples. *5th IASME/WSEAS International Conference on CONTINUUM MECHANICS*. Cambridge, UK, February 23–25, 2010. Proceedings, pp. 21-26.

Reports on Local Scientific Conferences

1. T. Bobinska, M. Buike, A. Buikis. A Mathematical Model for Double Wall with Double Fins. *9th Latvian Mathematical Conference*. Jelgava, Latvia, March 30-31, 2012. Abstracts, p. 9.

2. T. Bobinska, M. Buiķe. Precīzais un tuvinātais atrisinājums stacionārai un nestacionārai problēmai L-veida apgabalam. *The 69th Scientific Conference of the University of Latvia*, Section of Mathematical Modelling and Numerical Analysis. Riga, February 17, 2011.

3. T. Bobinska, M. Buike, A. Buikis. On Direct and Inverse Problems for Hyperbolic Heat Equation for L-shape and T-shape Samples. *8th Latvian Mathematical Conference*. Valmiera, Latvia, April 9-10, 2010. Abstracts, p. 11.

1 Hyperbolic and Parabolic Heat Conduction Equations for L-shape Samples

One of the most important heat treatment processes in metallurgy is quenching, where it is used for metal hardening to improve mechanical properties of the material and produce a much tougher and durable item with stronger structure. In this process metal parts are heated to the required temperature and then cooled rapidly in air, oil, water, or another liquid. Many of these methods for hardening steel parts are widely known as intensive quenching (IQ). Although there have been various, often little-known, IQ processes designed since the 1920's, the IQ technology was greatly influenced and developed by Dr. N. Kobasko (Ukraine). In contrast with conventional quenching, the IQ process is conducted in highly agitated water or brine with a cooling rate that is significantly faster than quenching in oil, air, etc., [2], [3], [38].

The mathematical models developed to conduct analysis on IQ mostly include transient heat conduction equations. But as IQ is a process where the temperature of the sample is changing very quickly, different kind of models should be considered. In [28] Prof. Buikis proposed to describe the process by a hyperbolic heat conduction equation. As it has been shown in several publications, these new mathematical models give good results and are reasonable in the real-world situation context, e.g., [4], [18], [27], [28], etc.

According to the previous studies and research [5], [8], [9], [23], here we consider initial-boundary value problems (IBVPs) for both parabolic and hyperbolic heat conduction equations describing IQ process for a thin L-shape sample. Using Green's function method and conservative averaging we construct analytical solutions of inverse and direct problems in the form of an integral equation of the 2nd kind. After that, if necessary, it is possible to compare the rate of change of the temperatures in a small neighbourhood of the initial time $t = 0$ after determining solutions for these problems.

1.1 The physical problem

As it is well known, heat conduction in a solid body can be described by the well-known Fourier equation

$$\frac{\partial T}{\partial t} = a^2 \nabla^2 T + \frac{f}{\tilde{c}\rho}, \ a^2 = \frac{k}{\tilde{c}\rho}, \tag{1.1}$$

where T denotes the temperature of the body with thermal conductivity k, and f is the density of heat sources, t is time, $\tilde{c}$ specific heat capacity, ρ density of the body. And a^2 is the thermal diffusivity of the material. The equation can be derived by combining the law of conservation of energy (the 1st law of thermodynamics)

$$\tilde{c}\rho \frac{\partial T}{\partial t} = -\nabla \cdot \mathbf{q} + f \tag{1.2}$$

with Fourier's law

$$\mathbf{q}(\mathbf{x}, t) = -k\nabla T(\mathbf{x}, t). \tag{1.3}$$

In a number of physical situations equation (1.1) implies an arbitrarily high thermal propagation speed. One of such cases is IQ. When immersing the heated part into a quenchant, the initial speed of propagation of the heat tends to infinity but actually is finite (see [39]). So Fourier's law at the initial time is no longer suited to describe heat propagation. In these cases (1.3) can be replaced by a more general law proposed by Cattaneo and Vernotte:

$$\tau \frac{\partial \mathbf{q}}{\partial t}(\mathbf{x}, t) + \mathbf{q}(\mathbf{x}, t) = -k\nabla T(\mathbf{x}, t) \cdot \tag{1.4}$$

Here $\tau > 0$ is a material property and is called the relaxation time. Introducing equation (1.4) into (1.2) leads us to a hyperbolic heat conduction equation

$$\frac{1}{C^2}\frac{\partial^2 T}{\partial t^2} + \frac{1}{a^2}\frac{\partial T}{\partial t} = \nabla^2 T + \frac{1}{k}\left(f + \tau\frac{\partial f}{\partial t}\right)$$

or

$$\tau\frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = a^2\nabla^2 T + \frac{1}{\tilde{c}\rho}\left(f + \tau\frac{\partial f}{\partial t}\right). \tag{1.5}$$

This so-called Telegrapher's equation or a damped wave equation admits a finite speed of propagation for T. Here C is the characteristic speed with $C^2 = a^2/\tau$. (More information on hyperbolic heat equations can be found here: [51], [31], and [32].)

k, $\tilde{c}$, ρ are generally dependant on T and on the material too, but throughout the paper we assume that these are constants.

1.2 Non-dimensionalization

Let's image that we have an element with rectangular fins (see Fig. 1.1, from [17]) that is heated and then cooled rapidly in a suitable fluid, e.g., water or brine. Since the figure can be divided into several symmetrical L-shaped parts, we can use equations (1.1), (1.5) to describe IQ process for one part only and get the same results as if we had quenched the entire figure.
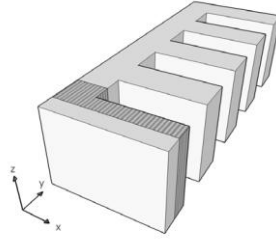
Fig. 1.1 Element with fins

In order to simplify the problem, we'll use non-dimensionalization. The time and spatial variables are non-dimensionalized by setting

$$\hat{x} = \frac{x}{B+R}, \ \hat{y} = \frac{y}{B+R}, \ \hat{z} = \frac{z}{B+R}, \ \hat{t} = \frac{t}{T^*}, \tag{1.6}$$

and

$$\delta = \frac{\Delta}{B+R}, \ l = \frac{L}{B+R}, \ b = \frac{B}{B+R}, \ \omega = \frac{\Omega}{B+R},$$

where $2B$ is the width of the fin, $2R$ the space between the fins, Ω the height of the wall, Δ the length of the wall, L the length of the fin.

If $V^0(x, y, z, t)$ denotes the temperature distribution in the wall and the temperature distribution in the fin is designed by $V(x, y, z, t)$, then we can define dimensionless temperatures $\hat{V}^0(\hat{x}, \hat{y}, \hat{z}, \hat{t})$ and $\hat{V}(\hat{x}, \hat{y}, \hat{z}, \hat{t})$ in terms of the new variables (1.6):

$$\hat{V}^0(\hat{x}, \hat{y}, \hat{z}, \hat{t}) = \frac{V^0(x, y, z, t) - V_a}{V_b - V_a}, \ \hat{V}(\hat{x}, \hat{y}, \hat{z}, \hat{t}) = \frac{V(x, y, z, t) - V_a}{V_b - V_a}, \tag{1.7}$$

where V_a, V_b, are some characteristic environmental temperatures. Similarly, the new temperatures of the surrounding medium are

$$\hat{\Theta}(\hat{x}, \hat{y}, \hat{z}, \hat{t}) = \frac{\Theta(x, y, z, t) - V_a}{V_b - V_a}. \tag{1.8}$$

$\Theta^0(y, z, t)$, $\Theta(x, y, z, t)$ are the values of the temperature of the fluid at the left side of the wall, and at the other sides of the sample.

And the new parameters are

$$\hat{\beta}_0^0 = \frac{h_0(B+R)}{k_0}, \quad \hat{\beta}^0 = \frac{h(B+R)}{k_0}, \quad \hat{\beta} = \frac{h(B+R)}{k}, \quad \hat{\tau} = \frac{\tau}{T^*}, \quad \hat{a}^2 = \frac{a^2 T^*}{(B+R)^2}.$$

The quantities a^2, k, τ have the same meaning as in the preceding section, but $\beta = h/k$, h the heat transfer coefficient.

1.3 Using Hyperbolic Heat Conduction Equation to Describe IQ Process

In this section we are going to consider equation (1.5) for describing IQ process, and solve IBVP using Green's function method (see, e.g., [8], [9], [12], [16], [19], [22]). To modify the method to obtain a closed-form Green's function for so called regular non-canonical domain, we are going to represent the original domain as a finite union of canonical sub-domains with appropriate boundary conditions along the planes (or lines in 2D) connecting two neighbour domains.

1.3.1 General Statement of 3D Problem for L-shape Sample

We may therefore suppose that the given L-shaped sample is made up from two rectangular parallelepipeds (rectangles in 2D) G_0 and G, the *base* (sometimes called the *wall*) and the *foot*:

$$G_0 = \{(x, y, z) \mid x \in [0, \delta], y \in [0,1], z \in [0, \omega]\},$$
$$G = \{(x, y, z) \mid x \in [\delta, \delta + l], y \in [0, b], z \in [0, \omega]\},$$

joined along the surface $x = \delta$. By means of that we'll be able to define IQ process for each part separately. Thus, in terms of the dimensionless variables (1.6), (1.7) and dropping the hats, the hyperbolic equations of heat conduction have the following form

$$\tau_{r,0} \frac{\partial^2 V_0}{\partial t^2} + \frac{\partial V_0}{\partial t} = a^2 \left(\frac{\partial^2 V_0}{\partial x^2} + \frac{\partial^2 V_0}{\partial y^2} + \frac{\partial^2 V_0}{\partial z^2} \right) + S_0(x, y, z, t) \tag{1.9}$$

and

$$\tau_r \frac{\partial^2 V}{\partial t^2} + \frac{\partial V}{\partial t} = a^2 \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \right) + S(x, y, z, t), \tag{1.10}$$

where

$$S_0 = \frac{1}{\tilde{c}\rho} \left(f(x, y, z, t) + \tau_{r,0} \frac{\partial f}{\partial t}(x, y, z, t) \right),$$

$$S = \frac{1}{\tilde{c}\rho} \left(f(x, y, z, t) + \tau_r \frac{\partial f}{\partial t}(x, y, z, t) \right).$$

If there are no sources of heat, then S, $S_0 = 0$.

To state the IBVP for determining the temperature of the sample, we need to formulate both boundary and initial conditions. Along the planes $y = 0$, $y = 1$ symmetry conditions must be applied:

$$\left.\frac{\partial V_0}{\partial y}\right|_{y=0} = 0, \left.\frac{\partial V_0}{\partial y}\right|_{y=1} = 0, \tag{1.11}$$

$$\left.\frac{\partial V}{\partial y}\right|_{y=0} = 0. \tag{1.12}$$

But at the other sides of the sample a heat exchange takes place with the surrounding medium, whose temperature is given by (1.8). Thus

$$\left(\frac{\partial V_0}{\partial x} - \beta_0^0 (V_0 - \Theta)\right)\Big|_{x=0} = 0, \tag{1.13}$$

$$\left(\frac{\partial V_0}{\partial x} + \beta^0 (V_0 - \Theta)\right)\Big|_{x=\delta} = 0, \ y \in (b,1), \tag{1.14}$$

$$\left(\frac{\partial V_0}{\partial z} - \beta^0 (V_0 - \Theta)\right)\Big|_{z=0} = 0, \tag{1.15}$$

$$\left(\frac{\partial V_0}{\partial z} + \beta^0 (V_0 - \Theta)\right)\Big|_{z=\omega} = 0. \tag{1.16}$$

For the foot we have mixed boundary conditions as well:

$$\left(\frac{\partial V}{\partial x} + \beta (V - \Theta)\right)\Big|_{x=\delta+l} = 0, \tag{1.17}$$

$$\left(\frac{\partial V}{\partial y} + \beta (V - \Theta)\right)\Big|_{y=b} = 0, \tag{1.18}$$

$$\left(\frac{\partial V}{\partial z} - \beta (V - \Theta)\right)\Big|_{z=0} = 0, \tag{1.19}$$

$$\left(\frac{\partial V}{\partial z} + \beta (V - \Theta)\right)\Big|_{z=\omega} = 0. \tag{1.20}$$

For the future we will set the dimensionless environmental temperature $\Theta(x, y, z, t)$ to be zero for all x, y, z, with the exception at $x = 0$ where $\Theta(0, y, z, t) = 1$.

It is also possible to consider non-linear boundary conditions, see [4]. In that case solutions of hyperbolic and parabolic equations differ essentially.

As the base is in ideal thermal contact with the foot, continuity of temperature and heat flux are imposed at the interface $x = \delta$ between the adjacent parts of the sample:

$$V_0\big|_{x=\delta-0} = V\big|_{x=\delta+0}, \tag{1.21}$$

$$\beta\frac{\partial V_0}{\partial x}\Big|_{x=\delta-0} = \beta^0\frac{\partial V}{\partial x}\Big|_{x=\delta+0}. \tag{1.22}$$

Let us also establish the initial conditions:

$$V_0\big|_{t=0} = V_0^0(x, y, z), \ V\big|_{t=0} = V^0(x, y, z), \tag{1.23}$$

$$\frac{\partial V_0}{\partial t}\Big|_{t=0} = W_0^0(x, y, z), \ \frac{\partial V}{\partial t}\Big|_{t=0} = W^0(x, y, z). \tag{1.24}$$

In IQ conditions (1.24) are unrealistic: the initial time-rate of the temperature change can't be measured experimentally. It should be calculated to compare it with critical cooling rate to predict heat transfer modes, as the initial cooling rate can be in different ranges (see [39]). Therefore, we can assume that the temperature distribution and the speed of temperature change are given at the end of the process:

$$V_0\big|_{t=T} = V_0^T(x, y, z),\ V\big|_{t=T} = V^T(x, y, z),$$

(1.25)

$$\frac{\partial V_0}{\partial t}\bigg|_{t=T} = W_0^T(x, y, z),\ \frac{\partial V}{\partial t}\bigg|_{t=T} = W^T(x, y, z).$$

(1.26)

Putting together the partial differential equations, the boundary, and the initial conditions, we have a direct problem, or an inverse problem when using final conditions. Since finding solution of 3D problem is quite similar as in 2D case, we will consider only the latter.

1.3.2 Formulation of Direct Problem in 2D

Let's suppose the domains G_0, G are thin in the z–direction (for 2D model), so $\omega \ll 1$, $\omega \ll b$, $\omega \ll \delta$. Assuming that the temperature is almost constant in this direction, it allows us to use the simplest form of the conservative averaging method – approximation by a constant. Hence, to reduce the 3D problems (1.9), (1.10) considered before to 2D ones, we will introduce average values of all the functions used before over the interval $[0, \omega]$. For example,

$$v_0(x, y, t) = \frac{1}{\omega}\int_0^\omega V_0(x, y, z, t)dz,$$

(1.27)

$$v(x, y, t) = \frac{1}{\omega}\int_0^\omega V(x, y, z, t)dz.$$

(1.28)

We then use these approximations and apply the boundary conditions (1.15), (1.16), and (1.19), (1.20) to obtain two 2D hyperbolic heat-conduction equations

$$\tau_{r,0}\frac{\partial^2 v_0}{\partial t^2} + \frac{\partial v_0}{\partial t} = a^2\left(\frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 v_0}{\partial y^2}\right) - a^2\frac{2\beta^0}{\omega}v_0 + s_0(x, y, t),$$

(1.29)

$$\tau_r\frac{\partial^2 v}{\partial t^2} + \frac{\partial v}{\partial t} = a^2\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - a^2\frac{2\beta}{\omega}v + s(x, y, t).$$

(1.30)

Additionally, we will use the same boundary and initial conditions as formulated in the statement of the original problem in the previous subsection.

Using the continuity conditions (1.21), (1.22), the boundary condition for the right-hand side border of the base can be written in the form

$$\left(\frac{\partial v_0}{\partial x} + \beta^0 v_0\right)\bigg|_{x=\delta-0} = \beta^0 H_0(y, t),$$

$$H_0(y, t) = \begin{cases} 0 & y \in (b, 1] \\ \left(\dfrac{1}{\beta}\dfrac{\partial v}{\partial x} + v\right)\bigg|_{x=\delta+0} & y \in [0, b] \end{cases}.$$

(1.31)

At the left-hand side border of the foot we get 3rd type boundary condition as well:

$$\left(\frac{\partial v}{\partial x} - \beta v\right)\bigg|_{x=\delta+0} = \beta H(y, t),$$

$$H(y, t) = \left(\frac{1}{\beta^0}\frac{\partial v_0}{\partial x} - v_0\right)\bigg|_{x=\delta-0},. y \in [0, b]$$

(1.32)

But the initial conditions are

$$v_0\big|_{t=0} = v_0^0(x, y)\qquad \frac{\partial v_0}{\partial t}\bigg|_{t=0} = w_0^0(x, y),$$

$$v\big|_{t=0} = v^0(x,y) \qquad \frac{\partial v}{\partial t}\bigg|_{t=0} = w^0(x,y).$$

1.3.2.1 Solution for the Base

The problem for (1.29) can be solved by transforming the problem with the non-homogeneous boundary condition (1.31) into another one with zero boundary conditions. To do that, we assume a solution of the form (see [46])

$$v_0(x,y,t) = v_{0,1}(x,y,t) + v_{0,2}(x,y,t),$$

where

$$v_{0,1}(x,y,t) = -\frac{\beta^0 x - 1 - \beta^0 \delta}{\beta^0 + \beta_0^0 + \beta^0 \beta_0^0 \delta}\beta_0^0 + \frac{1 + \beta_0^0 x}{\beta^0 + \beta_0^0 + \beta^0 \beta_0^0 \delta}\beta^0 H_0(y,t)$$

$$\equiv A_0(x) + B_0(x)H_0(y,t) \tag{1.33}$$

satisfies the non-homogeneous boundary condition. Substituting (1.33) into the original problem, we get the transformed problem in $v_{0,2}(x,y,t)$:

$$\tau_{r,0}\frac{\partial^2 v_{0,2}}{\partial t^2} + \frac{\partial v_{0,2}}{\partial t} = a^2\left(\frac{\partial^2 v_{0,2}}{\partial x^2} + \frac{\partial^2 v_{0,2}}{\partial y^2}\right) - a^2\frac{2\beta^0}{\omega}v_{0,2} + \phi_0(x,y,t), \tag{1.34}$$

$$\phi_0(x,y,t) = s_0(x,y,t) - \tau_{r,0}\frac{\partial^2 v_{0,1}}{\partial t^2} - \frac{\partial v_{0,1}}{\partial t} + a^2\frac{\partial^2 v_{0,1}}{\partial y^2} - a^2\frac{2\beta^0}{\omega}v_{0,1},$$

with these boundary conditions:

$$\left(\frac{\partial v_{0,2}}{\partial x} - \beta_0^0 v_{0,2}\right)\bigg|_{x=0} = 0 \qquad \left(\frac{\partial v_{0,2}}{\partial x} + \beta^0 v_{0,2}\right)\bigg|_{x=\delta-0} = 0, \tag{1.35}$$

$$\frac{\partial v_{0,2}}{\partial y}\bigg|_{y=0} = 0 \qquad \frac{\partial v_{0,2}}{\partial y}\bigg|_{y=1} = 0, \tag{1.36}$$

and the initial conditions:

$$v_{0,2}\big|_{t=0} = \varphi_0(x,y) = v_0^0(x,y) - v_{0,1}^0(x,y), \tag{1.37}$$

$$\frac{\partial v_{0,2}}{\partial t}\bigg|_{t=0} = \psi_0(x,y) = w_0^0(x,y,z) - \frac{\partial v_{0,1}}{\partial t}\bigg|_{t=0}. \tag{1.38}$$

The solution of the new problem (1.34) − (1.38) can be written as

$$v_{0,2}(x,y,t) = \bar{v}_{2,1}(x,y,t) + \bar{v}_{2,2}(x,y,t) + \bar{v}_{2,3}(x,y,t) =$$

$$= \left(\frac{1}{\tau_{r,0}} + \frac{\partial}{\partial t}\right)Z_{\varphi_0}(x,y,t) + Z_{\psi_0}(x,y,t) + \int_0^t Z_{\phi_{0,t}}(x,y,t-\iota)d\iota. \tag{1.39}$$

Here $\bar{v}_{2,2}(x,y,t)$ is the solution of (1.34) at $\varphi_0 = \phi_0 = 0$; $\bar{v}_{2,1}(x,y,t)$ is the solution of (1.34) at $\psi_0 = \phi_0 = 0$; $\bar{v}_{2,3}(x,y,t)$ is the solution of (1.34) at $\varphi_0 = \psi_0 = 0$. The function $\phi_{0,t}$ is defined by $\phi_{0,t} = \frac{1}{\tau_{r,0}}\phi_0(x,y,\iota)$.

First, let's develop the function $Z_{\psi_0}(x,y,t)$. Based on the given boundary conditions we expand the solution as

$$\bar{v}_{2,2}(x,y,t) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}T_{ij}(t)X_i(x)Y_j(y),$$

where the eigenfunctions have the following expressions

$$X_i(x) = \sin\left(\frac{\lambda_i}{\delta}x + \vartheta_i\right), \quad \tan(\vartheta_i) = \frac{\lambda_i}{\delta\beta_0^0}; \qquad Y_j(y) = \cos(j\pi y)$$

with normal squares

$$M_i = \int_0^\delta X_i^2(\zeta)d\zeta = \frac{\delta}{2}\left[1 - \frac{\sin\lambda_i}{\lambda_i}\cos(\lambda_i + 2\vartheta_i)\right]; \qquad N_j = \int_0^1 Y_j^2(\upsilon)d\upsilon = \begin{cases} \dfrac{1}{2} & j = 1,2,\dots \\ 1 & j = 0 \end{cases}.$$

But the eigenvalues are $\left(\dfrac{\lambda_i}{\delta}\right)^2$, $(j\pi)^2$, where λ_i are positive zero-points of

$$F(x) = \cot x - \frac{1}{(\beta_0^0 + \beta^0)\delta}\left(x - \frac{\beta_0^0\beta^0\delta^2}{x}\right).$$

The functions $T_{ij}(t)$ can be found in the form

$$T_{ij}(t) = \exp\left(-\frac{t}{2\tau_{r,0}}\right)\left(a_{ij}\underline{\cos}(\bar{\gamma}_{ij}t) + b_{ij}\underline{\sin}(\bar{\gamma}_{ij}t)\right)$$

with

$$\underline{\sin}(\bar{\gamma}_{ij}t) = \begin{cases} \sinh(\bar{\gamma}_{ij}t) & \bar{\gamma}_{ij} = \dfrac{1}{2\tau_{r,0}}\sqrt{1 - 4\tau_{r,0}\gamma_{ij}} \\ t & \bar{\gamma}_{ij} = 0 \\ \sin(\bar{\gamma}_{ij}t) & \bar{\gamma}_{ij} = \dfrac{1}{2\tau_{r,0}}\sqrt{4\tau_{r,0}\gamma_{ij} - 1} \end{cases},$$

$$\underline{\cos}(\bar{\gamma}_{ij}t) = \begin{cases} \cosh(\bar{\gamma}_{ij}t) & \bar{\gamma}_{ij} = \dfrac{1}{2\tau_{r,0}}\sqrt{1 - 4\tau_{r,0}\gamma_{ij}} \\ 1 & \bar{\gamma}_{ij} = 0 \\ \cos(\bar{\gamma}_{ij}t) & \bar{\gamma}_{ij} = \dfrac{1}{2\tau_{r,0}}\sqrt{4\tau_{r,0}\gamma_{ij} - 1} \end{cases}$$

and

$$\gamma_{ij} = a^2\left(\left(\frac{\lambda_i}{\delta}\right)^2 + (j\pi)^2 + \frac{2\beta^0}{\omega}\right).$$

Here a_{ij} and b_{ij} are undetermined constants. Applying the initial condition $\bar{v}_{2,2}(x,y,0) = 0$ leads to $a_{ij} = 0$. But $\dfrac{\partial}{\partial t}\bar{v}_{2,2}(x,y,0) = \psi_0(x,y)$ yields

$$\sum_{i=1}^\infty \sum_{j=0}^\infty b_{ij}\underline{\bar{\gamma}}_{ij}X_i(x)Y_j(y) = \psi_0(x,y), \qquad \underline{\bar{\gamma}}_{ij} = \begin{cases} \bar{\gamma}_{ij} & \bar{\gamma}_{ij} \neq 0 \\ 1 & \bar{\gamma}_{ij} = 0 \end{cases}.$$

Solving for b_{ij} gives

$$b_{ij} = \frac{1}{\underline{\bar{\gamma}}_{ij}}\psi_{ij}^0, \qquad \psi_{ij}^0 = \frac{1}{M_i N_j}\int_0^\delta d\zeta \int_0^1 \psi_0(\zeta,\upsilon)X_i(\zeta)Y_j(\upsilon)d\upsilon.$$

Finally, we have

$$\bar{v}_{2,2}(x,y,t) = Z_{\psi_0}(x,y,t) = \int_0^\delta d\zeta \int_0^1 G_0(x,\zeta,y,\upsilon,t)\psi_0(\zeta,\upsilon)d\upsilon \qquad (1.40)$$

with the Green function given by

$$G_0(x,\zeta,y,\upsilon,t) = \exp\left(-\frac{t}{2\tau_{r,0}}\right)\sum_{i=1}^\infty \sum_{j=0}^\infty \frac{1}{\underline{\bar{\gamma}}_{ij}} \frac{1}{M_i N_j} X_i(x)X_i(\zeta)Y_j(y)Y_j(\upsilon)\underline{\sin}(\bar{\gamma}_{ij}t).$$

From the given definition (1.39) and (1.40) we see that

$$\bar{v}_{2,1}(x,y,t) = \int_0^\delta d\zeta \int_0^1 \left(\frac{1}{\tau_{r,0}} + \frac{\partial}{\partial t}\right)G_0(x,\zeta,y,\upsilon,t)\varphi_0(\zeta,\upsilon)d\upsilon, \qquad (1.41)$$

$$\bar{v}_{2,3}(x,y,t) = \int_0^t d\iota \int_0^\delta d\zeta \int_0^1 \frac{1}{\tau_{r,0}} G_0(x,\zeta,y,\upsilon,t-\iota)\phi_0(\zeta,\upsilon,\iota)d\upsilon. \qquad (1.42)$$

Therefore the solution of (1.34) - (1.38) is

$$v_0(x,y,t) =$$

$$= \int_0^\delta d\zeta \int_0^1 \left(\frac{1}{\tau_{r,0}} + \frac{\partial}{\partial t}\right)G_0(x,\zeta,y,\upsilon,t)\left[v_0^0(\zeta,\upsilon) - A_0(\zeta)\right]d\upsilon$$

$$+ \int_0^\delta d\zeta \int_0^1 G_0(x,\zeta,y,\upsilon,t)w_0^0(\zeta,\upsilon)d\upsilon$$

$$+ \int_0^t d\iota \int_0^\delta d\zeta \int_0^1 \frac{1}{\tau_{r,0}} G_0(x,\zeta,y,\upsilon,t-\iota)s_0(\zeta,\upsilon,\iota)d\upsilon$$

$$- a^2 \frac{2\beta^0}{\omega} \int_0^t d\iota \int_0^\delta d\zeta \int_0^1 \frac{1}{\tau_{r,0}} G_0(x,\zeta,y,\upsilon,t-\iota)A_0(\zeta)d\upsilon$$

$$+ \int_0^\delta d\zeta \int_0^1 \frac{\partial}{\partial t} G_0(x,\zeta,y,\upsilon,0)A_0(\zeta)d\upsilon$$

$$- \int_0^\delta d\zeta \int_0^1 \left(\frac{1}{\tau_{r,0}} + \frac{\partial}{\partial t}\right)G_0(x,\zeta,y,\upsilon,t)B_0(\zeta)H_0(\upsilon,0)d\upsilon$$

$$- \int_0^\delta d\zeta \int_0^1 G_0(x,\zeta,y,\upsilon,t)\left[B_0(\zeta)\frac{\partial}{\partial t}H_0(\upsilon,0)\right]d\upsilon$$

$$- \int_0^t d\iota \int_0^\delta d\zeta \int_0^1 \frac{1}{\tau_{r,0}} G_0(x,\zeta,y,\upsilon,t-\iota)B_0(\zeta)\left[\tau_{r,0}\frac{\partial^2}{\partial \iota^2}H_0(\upsilon,\iota) + \frac{\partial}{\partial \iota}H_0(\upsilon,\iota)\right]d\upsilon$$

$$- \int_0^t d\iota \int_0^\delta d\zeta \int_0^1 \frac{1}{\tau_{r,0}} G_0(x,\zeta,y,\upsilon,t-\iota)B_0(\zeta)\left[-a^2\frac{\partial^2}{\partial \upsilon^2}H_0(\upsilon,\iota) + a^2\frac{2\beta^0}{\omega}H_0(\upsilon,\iota)\right]d\upsilon$$

$$+ \int_0^\delta d\zeta \int_0^1 \frac{\partial}{\partial t} G_0(x,\zeta,y,\upsilon,0)B_0(\zeta)H_0(\upsilon,t)d\upsilon. \qquad (1.43)$$

1.3.2.2 Solution for the Foot

The solution to the second problem (1.30) is quite similar:

$$v(x,y,t) =$$

$$= \int_\delta^{\delta+l} d\xi \int_0^b \left(\frac{1}{\tau_r} + \frac{\partial}{\partial t}\right)G(x,\xi,y,\eta,t)v^0(\xi,\eta)d\eta$$

$$+ \int_{\delta}^{\delta+l} d\xi \int_{0}^{b} G(x,\xi,y,\eta,t)w^{0}(\xi,\eta)d\eta$$

$$+ \int_{0}^{t} d\tau \int_{\delta}^{\delta+l} d\xi \int_{0}^{b} \frac{1}{\tau_{r}} G(x,\xi,y,\eta,t-\tau)s(\xi,\eta,\tau)d\eta$$

$$- \int_{\delta}^{\delta+l} d\xi \int_{0}^{b} \left(\frac{1}{\tau_{r}} + \frac{\partial}{\partial t} \right) G(x,\xi,y,\eta,t)A(\xi)H(\eta,0)d\eta$$

$$- \int_{\delta}^{\delta+l} d\xi \int_{0}^{b} G(x,\xi,y,\eta,t)A(\xi)\frac{\partial H}{\partial t}(\eta,0)d\eta$$

$$- \int_{0}^{t} d\tau \int_{\delta}^{\delta+l} d\xi \int_{0}^{b} \frac{1}{\tau_{r}} G(x,\xi,y,\eta,t-\tau)A(\xi)\left[\tau_{r} \frac{\partial^{2}H}{\partial \tau^{2}} + \frac{\partial H}{\partial \tau} - a^{2} \frac{\partial^{2}H}{\partial \eta^{2}} + a^{2}\frac{2\beta}{\omega}H(\xi,\tau) \right]d\eta$$

$$+ \int_{\delta}^{\delta+l} d\xi \int_{0}^{b} \frac{\partial}{\partial t} G(x,\xi,y,\eta,0)A(\xi)H(\eta,t)d\eta,\tag{1.44}$$

where

$$A(x) = \frac{\beta(x-\delta)-1-\beta l}{2\beta + \beta^{2}l}\beta.$$

Thus the Green function is

$$G(x,\xi,y,\eta,t) = \exp\left(-\frac{t}{2\tau_{r}} \right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{\underline{\bar{\gamma}}_{mn}} \frac{1}{M_{m}N_{n}} X_{m}(x)X_{m}(\xi)Y_{n}(y)Y_{n}(\eta)\underline{\sin}(\bar{\gamma}_{mn}t).$$

Functions $\underline{\sin}(\bar{\gamma}_{mn}t)$, $\bar{\gamma}_{mn}$ have the same meaning as in the previous subsection, only

$$\gamma_{mn} = a^{2}\left(\left(\frac{\mu_{m}}{l} \right)^{2} + \left(\frac{\nu_{n}}{b} \right)^{2} + \frac{2\beta}{\omega} \right)$$

and

$$X_{m}(x) = \sin\left(\frac{\mu_{m}}{l}(x-\delta) + \vartheta_{m} \right), \quad \tan(\vartheta_{m}) = \frac{\mu_{m}}{l\beta}; \quad Y_{n}(y) = \cos\left(\frac{\nu_{n}}{b}y \right),$$

$$M_{m} = \frac{l}{2}\left[1 - \frac{\sin\mu_{m}}{\mu_{m}}\cos(\mu_{m}+2\vartheta_{m}) \right]; \quad N_{n} = \frac{b}{2}\left(1 + \frac{\sin 2\nu_{n}}{2\nu_{n}} \right).$$

Here μ_{m} are positive zero-points of

$$f(x) = \cot x - \frac{1}{2\beta l}\left(x - \frac{\beta^{2}l^{2}}{x} \right),$$

but ν_{n} are positive zero-points of

$$g(x) = \cot x - \frac{x}{\beta b}.$$

1.3.2.3   Conjugation of Both Solutions

Upon coupling equation (1.44) with the formula (1.31), we have a representation for the combination of the solution for the base and its derivative at the border between both parts - $H_{0}(y,t)$. In order to determine a similar representation for $H(y,t)$, we substitute the expression (1.43) into (1.32). As the function $H_{0}(y,t)$ is dependent of $H(y,t)$, we plug the

latter into the former, and obtain non-homogeneous Volterra-Fredholm integral equation of the 2nd kind at the interface between both parts of the L-shape sample:

$$H_0(y,t) = \Phi_0(y,t) - \int_0^t d\iota \int_0^b L(H_0, y, \upsilon, t, \iota) K_0(y, \upsilon, t, \iota) d\upsilon .$$

After finding solution to this integral equation, we can calculate the temperature distribution in the sample and find the rate of the temperature change.

1.3.3   Formulation of Inverse Problem in 2D

As it was mentioned before, the initial rate of temperature change is not known but sometimes must be determined. So we now address the inverse problem of determining that rate given the final data: the temperature distribution and the rate of change of the temperature at $t = T$. This corresponds to solving the given equations backwards. We can transform this problem into a direct problem by introducing a new time variable

$$\tilde{t} = T - t ,\tag{1.45}$$

so that

$$\tilde{u}^0(x, y, \tilde{t}) = u^0(x, y, T - \tilde{t}) ,\tag{1.46}$$

$$\tilde{u}(x, y, \tilde{t}) = u^0(x, y, T - \tilde{t}) .\tag{1.47}$$

In terms of (1.45) – (1.47), the wave equations (1.29), (1.30) transform to

$$\tau_{r,0} \frac{\partial^2 \tilde{v}_0}{\partial \tilde{t}^2} - \frac{\partial \tilde{v}_0}{\partial \tilde{t}} = a^2 \left(\frac{\partial^2 \tilde{v}_0}{\partial x^2} + \frac{\partial^2 \tilde{v}_0}{\partial y^2} \right) - a^2 \frac{2\beta^0}{\omega} \tilde{v}_0 + \tilde{s}_0(x, y, \tilde{t}) ,\tag{1.48}$$

$$\tau_r \frac{\partial^2 \tilde{v}}{\partial \tilde{t}^2} - \frac{\partial \tilde{v}}{\partial \tilde{t}} = a^2 \left(\frac{\partial^2 \tilde{v}}{\partial x^2} + \frac{\partial^2 \tilde{v}}{\partial y^2} \right) - a^2 \frac{2\beta}{\omega} \tilde{v} + \tilde{s}(x, y, \tilde{t}) .\tag{1.49}$$

The boundary conditions do not change, but the initial conditions become

$$\tilde{v}_0\big|_{\tilde{t}=0} = v_0^T(x, y) , \ \tilde{v}\big|_{\tilde{t}=0} = v^T(x, y) ,\tag{1.50}$$

$$\frac{\partial \tilde{v}_0}{\partial \tilde{t}}\bigg|_{\tilde{t}=0} = -\frac{\partial v_0}{\partial t}\bigg|_{t=T} = -w_0^T(x, y) ,\tag{1.51}$$

$$\frac{\partial \tilde{v}}{\partial \tilde{t}}\bigg|_{\tilde{t}=0} = -\frac{\partial v}{\partial t}\bigg|_{t=T} = -w^T(x, y) .\tag{1.52}$$

These direct problems can be solved as before.

1.4 Using Parabolic Heat Conduction Equation to Describe IQ Process

1.4.1 General Statement of 2D Problem

In this section we'll outline how to find the rate of change of the temperature when using a parabolic heat conduction equation to describe the temperature in the sample (see [17], [28] for more). Parabolic equations of heat conduction in 3D are given by

$$\frac{\partial V_0}{\partial t} = a^2 \left(\frac{\partial^2 V_0}{\partial x^2} + \frac{\partial^2 V_0}{\partial y^2} + \frac{\partial^2 V_0}{\partial z^2} \right) + S_0(x, y, z, t) ,\tag{1.53}$$

$$S_0 = \frac{1}{\tilde{c}\rho} f(x, y, z, t)$$

and

$$\frac{\partial V}{\partial t} = a^2 \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \right) + S(x, y, z, t),$$ (1.54)

$$S = \frac{1}{\tilde{c}\rho} f(x, y, z, t).$$

By using average value of the functions over the interval $[0, \omega]$ and applying appropriate boundary conditions, (1.15), (1.16); (1.19), (1.20), equations (1.53) and (1.54) are transformed into

$$\frac{\partial v_0}{\partial t} = a^2 \left(\frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 v_0}{\partial y^2} \right) - a^2 \frac{2\beta^0}{\omega} v_0 + s_0(x, y, t),$$ (1.55)

$$\frac{\partial v}{\partial t} = a^2 \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - a^2 \frac{2\beta}{\omega} v + s(x, y, t)$$ (1.56)

with two spatial variables. All the additional conditions are the same as in the preceding section.

1.4.1.1 Exact Solution of 2D Problem

Now we apply the technique used in Section 1.3 (see subsection 1.3.2) to derive the solutions to IBVPs (1.55) and (1.56). Thus the solution for the base has a form

$$v_0(x, y, t) =$$

$$= \int_0^\delta d\zeta \int_0^1 G_0(x, \zeta, y, \upsilon, t) \left[v_0^0(\zeta, \upsilon) - A_0(\zeta) \right] d\upsilon$$

$$+ \int_0^t d\iota \int_0^\delta d\zeta \int_0^1 G_0(x, \zeta, y, \upsilon, t - \iota) s_0(\zeta, \upsilon, \iota) d\upsilon$$

$$- a^2 \frac{2\beta^0}{\omega} \int_0^t d\iota \int_0^\delta d\zeta \int_0^1 G_0(x, \zeta, y, \upsilon, t - \iota) A_0(\zeta) d\upsilon$$

$$+ \int_0^\delta d\zeta \int_0^1 G_0(x, \zeta, y, \upsilon, 0) A_0(\zeta) d\upsilon$$

$$- \int_0^\delta d\zeta \int_0^1 G_0(x, \zeta, y, \upsilon, t) B_0(\zeta) H_0(\upsilon, 0) d\upsilon$$

$$- \int_0^t d\iota \int_0^\delta d\zeta \int_0^1 G_0(x, \zeta, y, \upsilon, t - \iota) B_0(\zeta) \left[\frac{\partial}{\partial \iota} H_0(\upsilon, \iota) - a^2 \frac{\partial^2}{\partial \upsilon^2} H_0(\upsilon, \iota) + a^2 \frac{2\beta^0}{\omega} H_0(\upsilon, \iota) \right] d\upsilon$$

$$+ \int_0^\delta d\zeta \int_0^1 G_0(x, \zeta, y, \upsilon, 0) B_0(\zeta) H_0(\upsilon, t) d\upsilon,$$ (1.57)

where

$$A_0(x) = -\frac{\beta^0 x - 1 - \beta^0 \delta}{\beta^0 + \beta_0^0 + \beta^0 \beta_0^0 \delta} \beta^0, \quad B_0(x) = \frac{1 + \beta_0^0 x}{\beta^0 + \beta_0^0 + \beta^0 \beta_0^0 \delta} \beta^0.$$

We use the eigenfunctions and eigenvalues from Section 1.3 to expand the Green function:

$$G_0(x, \zeta, y, \upsilon, t) = \sum_{i=1}^\infty \sum_{j=0}^\infty \frac{1}{M_i N_j} X_i(x) X_i(\zeta) Y_j(y) Y_j(\upsilon) \exp(-\gamma_{ij} t).$$

But the function satisfying the stated problem for (1.56) assumes the following form:

$$v(x, y, t) =$$

$$= \int_{\delta}^{\delta+l} d\xi \int_0^b G(x,\xi,y,\eta,t)v^0(\xi,\eta)d\eta$$

$$+ \int_0^t d\tau \int_{\delta}^{\delta+l} d\xi \int_0^b G(x,\xi,y,\eta,t-\tau)s(\xi,\eta,\tau)d\eta$$

$$- \int_{\delta}^{\delta+l} d\xi \int_0^b G(x,\xi,y,\eta,t)A(\xi)H(\eta,0)d\eta$$

$$- \int_0^t d\tau \int_{\delta}^{\delta+l} d\xi \int_0^b G(x,\xi,y,\eta,t-\tau)A(\xi)\left[\frac{\partial H}{\partial \tau} - a^2 \frac{\partial^2 H}{\partial \eta^2} + a^2 \frac{2\beta}{\omega}H(\xi,\tau)\right]d\eta$$

$$+ \int_{\delta}^{\delta+l} d\xi \int_0^b G(x,\xi,y,\eta,0)A(\xi)H(\eta,t)d\eta \tag{1.58}$$

with

$$A(x) = \frac{\beta(x-\delta)-1-\beta l}{2\beta + \beta^2 l}\beta .$$

Here the Green function is defined by

$$G(x,\xi,y,\eta,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{M_m N_n}X_m(x)X_m(\xi)Y_n(y)Y_n(\eta)\exp(-\gamma_{mn}t).$$

Using the same method as in the preceding section we obtain Volterra-Fredholm integral equation and then get the solution in the sample and the initial time-rate of the temperature change.

1.5 Comparing the Rate of the Temperature

When v_p^0, v_h^0, the solutions of parabolic and hyperbolic heat conduction equations in the wall are found, we can differentiate these expressions with respect to t and compare the rates of change of the temperatures in a small neighbourhood of the initial time $t = 0$ by setting

$$\left|\frac{\partial v_h^0}{\partial t} - \frac{\partial v_p^0}{\partial t}\right|_{t=\varepsilon},$$

where

$$\left.\frac{\partial v_h^0}{\partial t}\right|_{t=\varepsilon} = -\left.\frac{\partial \tilde{v}_h^0}{\partial \tilde{t}}\right|_{\tilde{t}=T-\varepsilon}.$$

The same expressions can be derived for the functions v_h, v_p.

2   Stationary Heat Transfer in System with Double Wall and Double Fins

There have been numerous studies on systems with extended surfaces where the entire structure is made of the same material. In many research areas, e.g., on modern computers, more complex elements have to be employed. Usually these are structures of micrometre scales and smaller. Various techniques are presented for manipulating these microscale elements to control the surface roughness. In one of those (see [29], [35], [36]) a plain surface is roughened by adding densely distributed vertical silicon nanowires, and then covered by some kind of coating, e.g., fluorine carbon C_4F_8 (see Fig. 2.1). Such micro/nano structures are often developed to control and enhance the performance of boiling heat transfer.
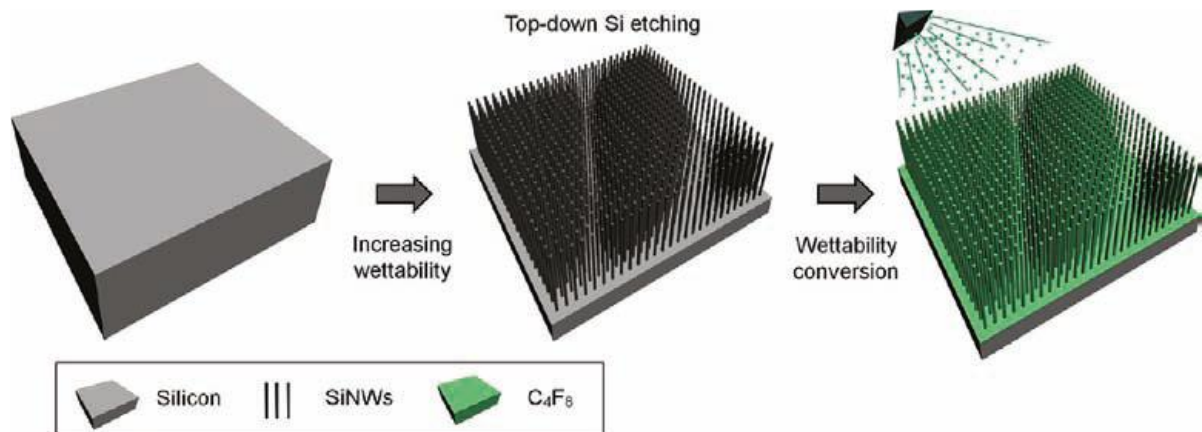
Fig. 2.1 Schematic of the fabrication processes for increasing hydrophilicity and converting hydrophilic/superhydrophilic surfaces into hydrophobic/superhydrophobic surfaces, [35]

In this chapter and one that follows we consider heat conduction in these kind of assemblies (we'll call those *systems with a double wall and double fins*). Let's assume that the assembly is 2D with straight fins of rectangular profile, Fig. 2.2.

Fig. 2.2 2D system with fins

In the first part of this chapter we focus on the simplest case when the process is stationary, linear and homogeneous and the assembly has constant properties (these results are published in [6], [7] and [10]). The linear case analysis is followed by an analysis of stationary heat conduction in the given system, when there is boiling occurring at some sides of the fin ([6], [7]). Just like in the publications [6], [7], [10], [13] - [15], [20], [21], [24], [25] conservative averaging method for L-type domains is exploited to reduce the dimensions for the given problem. For the linear case an approximate analytical solution is constructed. In non-linear case we construct approximate solution using finite difference method and its modification for boundary conditions.

Our mathematical models are new and quite a bit different than those where relatively simple fin assemblies are considered, e.g., [1], [13], [14], [25], [34], [41], [44], [52], [53], [54].

2.1 Geometry and Mathematical Statement of the Original Problem

Because of the geometrical and thermal symmetry of the model, we can divide it into several symmetrical parts. It is sufficient to describe and analyse the problem for only one of those L-shaped parts (see Fig. 2.3).

Fig. 2.3 L-type domain

Such a domain can be represented as a finite union of canonical non-overlapping subdomains with appropriate conjugation conditions along the lines connecting two neighbour domains. We may therefore suppose that this L-shaped sample is made up from five rectangles (see Fig. 2.4)
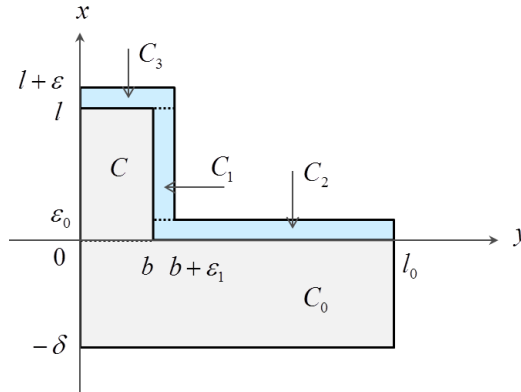
Fig. 2.4 Definition of geometrical parameters for the sample

Let us denote the temperatures of the domains C_i by the symbols $V_i(x, y)$. The basic properties, such as thermal conductivity, heat transfer coefficient, are constant and denoted by k_i, h_i, respectively. For simplicity reasons we are going to assume that $k = k_0$ and $k_2 = k_3 = k_1$.

As the process is stationary, the temperature fields are described by Laplace's equations:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 , \ x, y \in C , \tag{2.1}$$

$$\frac{\partial^2 V_0}{\partial x^2} + \frac{\partial^2 V_0}{\partial y^2} = 0 , \ x, y \in C_0 , \tag{2.2}$$

$$\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} = 0 , \ x, y \in C_1 , \tag{2.3}$$

$$\frac{\partial^2 V_2}{\partial x^2} + \frac{\partial^2 V_2}{\partial y^2} = 0 , \ x, y \in C_2 , \tag{2.4}$$

$$\frac{\partial^2 V_3}{\partial x^2} + \frac{\partial^2 V_3}{\partial y^2} = 0 , \ x, y \in C_3 . \tag{2.5}$$

Besides the equations, the following boundary conditions are imposed. We have a heat flux at $x = -\delta$:

$$\left. \frac{\partial V_0}{\partial x} \right|_{x=-\delta} = -Q_0(y). \tag{2.6}$$

As the geometry of interest has mirror symmetry along the lines $y=0$ and $y=l_0$, symmetry boundary conditions are applied:

$$\left. \frac{\partial V_0}{\partial y} \right|_{y=0} = 0, \tag{2.7}$$

$$\left. \frac{\partial V}{\partial y} \right|_{y=0} = 0, \tag{2.8}$$

$$\left. \frac{\partial V_3}{\partial y} \right|_{y=0} = 0, \tag{2.9}$$

$$\left. \frac{\partial V_0}{\partial y} \right|_{y=l_0} = 0, \tag{2.10}$$

$$\left. \frac{\partial V_2}{\partial y} \right|_{y=l_0} = 0. \tag{2.11}$$

The particular choice of boundary conditions to be used at the other sides of the system depends on the situation that is being modelled.

Along the lines connecting two neighbour domains the continuity of temperature and heat flux are ensured by conjugation conditions:

$$\left. V_0 \right|_{x=-0} = \left. V \right|_{x=+0}, \tag{2.12}$$

$$\left. \frac{\partial V_0}{\partial x} \right|_{x=-0} = \left. \frac{\partial V}{\partial x} \right|_{x=+0}, \tag{2.13}$$

$$\left. V_0 \right|_{x=-0} = \left. V_2 \right|_{x=+0}, \tag{2.14}$$

$$\left. \frac{\partial V_0}{\partial x} \right|_{x=-0} = \frac{k_1}{k_0} \left. \frac{\partial V_2}{\partial x} \right|_{x=+0}, \tag{2.15}$$

$$\left. V_2 \right|_{x=\varepsilon_0-0} = \left. V_1 \right|_{x=\varepsilon_0+0}, \tag{2.16}$$

$$\left. \frac{\partial V_2}{\partial x} \right|_{x=\varepsilon_0-0} = \left. \frac{\partial V_1}{\partial x} \right|_{x=\varepsilon_0+0}, \tag{2.17}$$

$$\left. V \right|_{x=l-0} = \left. V_3 \right|_{x=l+0}, \tag{2.18}$$

$$\left. \frac{\partial V}{\partial x} \right|_{x=l-0} = \frac{k_1}{k_0} \left. \frac{\partial V_3}{\partial x} \right|_{x=l+0}, \tag{2.19}$$

$$\left. V_1 \right|_{x=l-0} = \left. V_3 \right|_{x=l+0}, \tag{2.20}$$

$$\left. \frac{\partial V_1}{\partial x} \right|_{x=l-0} = \left. \frac{\partial V_3}{\partial x} \right|_{x=l+0}, \tag{2.21}$$

$$\left. V \right|_{y=b-0} = \left. V_2 \right|_{y=b+0}, \tag{2.22}$$

$$\left. \frac{\partial V}{\partial y} \right|_{y=b-0} = \frac{k_1}{k_0} \left. \frac{\partial V_2}{\partial y} \right|_{y=b+0}, \tag{2.23}$$

$$\left. V \right|_{y=b-0} = \left. V_1 \right|_{y=b+0}, \tag{2.24}$$

$$\left.\frac{\partial V}{\partial y}\right|_{y=b-0} = \frac{k_1}{k_0}\left.\frac{\partial V_1}{\partial y}\right|_{y=b+0}. \tag{2.25}$$

As the upper layer is quite thinner than the substrate, we may assume that the temperature variations across the layer thickness are as small as to be negligible. In this way the temperature can be taken constant here. Hence from the continuity conditions (2.14), (2.24), (2.18) we get these approximate analytic expressions for calculating the temperatures in the upper layer:

$$V_2(x, y) = v_2(y) = V_0(0, y), \tag{2.26}$$

$$V_1(x, y) = v_1(x) = V(x, b), \tag{2.27}$$

$$V_3(x, y) = v_3(y) = V(l, y). \tag{2.28}$$

Therefore only those steady-state conduction problems that are defined for the basic layer, need to be solved. So, we have Laplace's equations (2.1), (2.2). They require boundary conditions on all sides of the new domain $\{(x, y) \mid x \in (0, l), y \in (0, l_0)\}$ in which the solution is to be obtained. We'll see to it in the next section.

2.2 Approximate Solution to Heat Transfer with Heat Exchange

Assuming that at the other sides there is heat exchange between the sample and its surroundings, third type boundary conditions have to be specified here:

$$\left.\left(\frac{\partial V_3}{\partial x} + \beta_1^1 V_3\right)\right|_{x=l+\varepsilon} = 0, \tag{2.29}$$

$$\left.\left(\frac{\partial V_3}{\partial y} + \beta_1^1 V_3\right)\right|_{y=b+\varepsilon_1} = 0, \tag{2.30}$$

$$\left.\left(\frac{\partial V_1}{\partial y} + \beta_1^1 V_1\right)\right|_{y=b+\varepsilon_1} = 0, \tag{2.31}$$

$$\left.\left(\frac{\partial V_2}{\partial x} + \beta_1^1 V_2\right)\right|_{x=\varepsilon_0} = 0, \tag{2.32}$$

where $\beta_1^1 = \dfrac{h_1}{k_1}$.

To derive expressions for the boundary conditions in the new domain at $x = l$, $y = b$ and $x = 0$ let's use the boundary conditions (2.29), (2.31), (2.32), appropriate conjugation conditions (2.18) and (2.19), (2.24) and (2.25), (2.14) and (2.15), and formulae (2.26) – (2.28). For example, at $x = l$ we would have

$$0 = \left.\left(\frac{\partial V_3}{\partial x} + \beta_1^1 V_3\right)\right|_{x=l+\varepsilon} \approx \frac{1}{k_1}\left.\left(k_0\frac{\partial V}{\partial x} + h_1 V\right)\right|_{x=l},$$

or equivalently,

$$\left.\left(\frac{\partial V}{\partial x} + \beta_0^1 V\right)\right|_{x=l} = 0, \quad y \in (0, b), \tag{2.33}$$

but at $x = l$, $y = b$:

$$\left.\left(\frac{\partial V}{\partial y} + \beta_0^1 V\right)\right|_{y=b} = 0, \quad x \in (0, l), \tag{2.34}$$

$$\left(\frac{\partial V_0}{\partial x} + \beta_0^1 V_0\right)\bigg|_{x=0} = 0, \quad y \in (b, l_0). \tag{2.35}$$

In addition, there are the conjugation conditions (2.12), (2.13) that state that the fin and the wall are in ideal thermal contact at the interface $x = 0$. And, of course, the boundary conditions (2.6) – (2.8), (2.10).

Using conservative averaging method we are going to transform the given mathematical model into a more usable form and construct an approximate solution for this problem. The main idea of the method is to replace the unknown function by a certain combination of functions depending on in which direction the behaviour of the solution is predictable.

2.2.1   Solution for the Fin

So, let's use an exponential approximation in the y-direction for the 2D temperature field $V(x, y)$ in the fin. Then the general form of the function is given by

$$V(x, y) = f_0(x) + \left(e^{\rho y} - 1\right)f_1(x) + \left(1 - e^{-\rho y}\right)f_2(x) \tag{2.36}$$

with $\rho = b^{-1}$ and three unknown functions $f_i(x)$, $i = 0,1,2$. To calculate these functions we'll use all the conditions that are defined for $V(x, y)$.

When imposing symmetry condition (2.8) on (2.36) we find that

$$f_2(x) = -f_1(x).$$

So, (2.36) now becomes:

$$V(x, y) = f_0(x) + 2(\cosh(\rho y) - 1)f_1(x). \tag{2.37}$$

Let's introduce a new function $v(x)$ defined as integral average value of $V(x, y)$ over the interval $[0, b]$:

$$v(x) = \rho \int_0^b V(x, y)dy. \tag{2.38}$$

Now, integrating the expression (2.37) with respect to y, we can find the function $f_1(x)$:

$$f_1(x) = \frac{v(x) - f_0(x)}{2(\sinh(1) - 1)}. \tag{2.39}$$

Let's substitute (2.39) in (2.37):

$$V(x, y) = \frac{\cosh(\rho y) - 1}{\sinh(1) - 1}v(x) + \frac{\sinh(1) - \cosh(\rho y)}{\sinh(1) - 1}f_0(x). \tag{2.40}$$

Applying the boundary condition (2.34) on (2.40) leads to

$$v(x)\left(\rho \sinh(1) + \beta_0^1(\cosh(1) - 1)\right) + f_0(x)\left(-\rho \sinh(1) + \beta_0^1(\sinh(1) - \cosh(1))\right) = 0.$$

Hence our calculation yields the formula

$$f_0(x) = \psi v(x) \tag{2.41}$$

with

$$\psi = \frac{\sinh(1) + \beta_0^1 b(\cosh(1) - 1)}{\sinh(1) + \beta_0^1 b(\cosh(1) - \sinh(1))}. \tag{2.42}$$

Consequently,

$$V(x, y) = v(x)\Phi(y), \tag{2.43}$$

where

$$\Phi(y) = \frac{\sinh(1) + \beta_0^1 b\big(\cosh(1) - \cosh(\rho y)\big)}{\sinh(1) + \beta_0^1 b\big(\cosh(1) - \sinh(1)\big)}. \tag{2.44}$$

Thus the function $V(x, y)$ is reduced to the form (2.43) containing only one unknown $v(x)$. We are going to determine this function by requiring that it satisfies a certain differential equation. When integrating the partial differential equation (2.1) from 0 to b, we get

$$\frac{d^2 v}{dx^2} + \rho \left. \frac{\partial V}{\partial y} \right|_{y=0}^{y=b} = 0. \tag{2.45}$$

The difference of the derivatives may be found via the boundary conditions (2.34) and (2.8), and the expression (2.43). Thereupon the differential equation becomes

$$\frac{d^2 v}{dx^2} - \lambda^2 v(x) = 0, \tag{2.46}$$

where

$$\lambda^2 = \rho \beta_0^1 \Phi(b). \tag{2.47}$$

We apply the operator (2.38) on (2.33) to get a boundary condition for (2.46):

$$v'(l) + \beta_0^1 v(l) = 0. \tag{2.48}$$

The solution to the problem (2.46), (2.48) is hence found to be

$$v(x) = c_1 \big(e^{\lambda x} + \mu e^{-\lambda x}\big), \tag{2.49}$$

where

$$\mu = \frac{\lambda + \beta_0^1}{\lambda - \beta_0^1} e^{2\lambda l} \tag{2.50}$$

and c_1 is an unknown constant. Therefore

$$V(x, y) = c_1 \big(e^{\lambda x} + \mu e^{-\lambda x}\big) \Phi(y). \tag{2.51}$$

2.2.2 Solution for the Base

We act almost equally for the wall, using the same method of conservative averaging to describe the 2D temperature field $V_0(x, y)$ here. Let's introduce exponential temperature distribution in the x–direction by

$$V_0(x, y) = g_0(y) + \big(e^{-dx} - 1\big) g_1(y) + \big(1 - e^{dx}\big) g_2(y) \tag{2.52}$$

with $d = \delta^{-1}$. Once again we use the properties of the function to solve for the unknown functions $g_i(y)$, $i = 0,1,2$.

Before we proceed, let's obtain average value function by the integral:

$$v_0(y) = d \int_{-\delta}^{0} V_0(x, y) dx. \tag{2.53}$$

After integrating (2.52) over the segment $[-\delta, 0]$, it gives

$$v_0(y) = g_0(y) + (e - 2) g_1(y) + e^{-1} g_2(y). \tag{2.54}$$

As the function $V_0(x, y)$ satisfies the boundary condition (2.6), we apply it to (2.52) to find $g_2(y)$:

$$g_2(y) = e\delta Q_0(y) - e^2 g_1(y). \tag{2.55}$$

Then by combining (2.54) and (2.55) together, we have

$$g_1(y) = \frac{1}{2}\left(g_0(y) - v_0(y) + \delta Q_0(y)\right).$$

(2.56)

Putting the last two expressions, (2.55) and (2.56), in formula (2.52) it becomes

$$V_0(x, y) = g_0(y)\left(1 + \frac{1}{2}\left(e^{-dx} - 1\right) - e^2\frac{1}{2}\left(1 - e^{dx}\right)\right)$$

$$+ v_0(y)\left(-\frac{1}{2}\left(e^{-dx} - 1\right) + e^2\frac{1}{2}\left(1 - e^{dx}\right)\right)$$

$$+ Q_0(y)\left(\frac{1}{2}\delta\left(e^{-dx} - 1\right) + \left(\delta e - e^2\frac{1}{2}\delta\right)\left(1 - e^{dx}\right)\right).$$

(2.57)

Equation (2.57) still contains two unknown functions $g_0(y)$ and $v_0(y)$. Before these are determined, it is convenient to divide the base into two parts, the right part of which occupies the domain $x \in [-\delta, 0]$, $y \in [b, l_0]$, but the left one is for $x \in [-\delta, 0]$, $y \in [0, b]$.

The integration has to be performed on (2.2) over the interval $[-\delta, 0]$ to get 1D equation from which we would find $v_0(y)$:

$$d\frac{\partial V_0}{\partial x}\Big|_{x=-\delta}^{x=0} + \frac{d^2 v_0}{dy^2} = 0.$$

(2.58)

2.2.2.1 Solution for the Right Part of the Base

The function $g_0(y)$ for the right part of the base is obtained by imposing the boundary condition (2.35) on (2.57):

$$g_0(y)\left(-\frac{1}{2}d + \beta_0^1 + \frac{1}{2}de^2\right) + v_0(y)\left(\frac{1}{2}d - \frac{1}{2}de^2\right) + Q_0(y)\left(-\frac{1}{2} - e + \frac{1}{2}e^2\right) = 0,$$

which yields

$$g_0(y) = v_0(y)a_0 + Q_0(y)b_0$$

(2.59)

with

$$a_0 = \frac{d - de^2}{d - 2\beta_0^1 - de^2}, \quad b_0 = -\frac{1 + 2e - e^2}{d - 2\beta_0^1 - de^2}.$$

Substituting (2.59) into the representation (2.57) yields

$$V_0(x, y) = v_0(y)\left(a_0 + \frac{1}{2}(a_0 - 1)\left(e^{-dx} - 1\right) + e^2\frac{1}{2}\left(1 - a_0\right)\left(1 - e^{dx}\right)\right)$$

$$+ Q_0(y)\left(b_0 + \frac{1}{2}(b_0 + \delta)\left(e^{-dx} - 1\right) + \left(\delta e - e^2\frac{1}{2}\left(\delta + b_0\right)\right)\left(1 - e^{dx}\right)\right).$$

(2.60)

This shows that the function now depends only on one unknown – the function $v_0(y)$. We'll find it by solving (2.58). Owing to the boundary condition (2.6) and (2.60), the equation for the right part of the wall results in an ordinary differential equation

$$\frac{d^2 v_0}{dy^2} - \kappa^2 v_0(y) = \gamma Q_0(y),$$

(2.61)

where

$$\kappa^2 = d\beta_0^1 a_0,$$
$$\gamma = d\left(\beta_0^1 b_0 - 1\right).$$

For simplicity reasons we henceforth assume the function $Q_0(y)$ to be constant, that is, $Q_0(y) = Q_0$.

Integrating (2.10), we obtain a boundary condition:

$$v_0'(l_0) = 0. \tag{2.62}$$

The solution of (2.61), (2.62) is

$$v_0(y) = c_2\left(e^{\kappa y} + \mu_0 e^{-\kappa y}\right) - \frac{\gamma Q_0}{\kappa^2} \tag{2.63}$$

with

$$\mu_0 = e^{2\kappa l_0}$$

and c_2 as an unknown constant.

2.2.2.2 Solution for the Left Part of the Base

We are still left with two unknown functions, $g_0(y)$ and $v_0(y)$, to be determined for the temperature in the left part of the base. To get those we are going to use the conjugation conditions (2.12), (2.13). From the first of these conditions and the expression (2.51) for $V(x,y)$ we get that

$$g_0(y) = c_1(1 + \mu)\Phi(y). \tag{2.64}$$

But $v_0(y)$ is found by solving 1D equation (2.58) at first modifying it, so that it is valid for the left part of the base. As the functions $V_0(x,y)$, $V(x,y)$ satisfy (2.13) at $x = 0$, from (2.13) and (2.51) it follows that

$$d\frac{\partial V_0}{\partial x}\bigg|_{x=-0} = d\frac{\partial V}{\partial x}\bigg|_{x=+0}$$

$$= dc_1\lambda(1-\mu)\Phi(y). \tag{2.65}$$

So, now we can remove the term $d\dfrac{\partial V_0}{\partial x}\bigg|_{x=-\delta}^{x=0}$ in (2.58) by using (2.65) and the boundary condition

(2.6). And the equation (2.58) becomes

$$\frac{d^2 v_0}{dy^2} = -dc_1\lambda(1-\mu)\Phi(y) - dQ_0, \tag{2.66}$$

which can be rewritten as

$$\frac{d^2 v_0}{dy^2} = -c_1 B_1 - dQ_0 + c_1 B_2 \cosh(\rho y), \tag{2.67}$$

where

$$B_1 = d\lambda(1-\mu)\Phi_0,$$
$$B_2 = \beta_0^1 bd\lambda(1-\mu)\Phi_1,$$
$$\Phi_1 = \left(\sinh(1) + \beta_0^1 b(\cosh(1) - \sinh(1))\right)^{-1},$$
$$\Phi_0 = \left(\sinh(1) + \beta_0^1 b\cosh(1)\right)\Phi_1.$$

In addition we get 1D boundary condition when integrating (2.7):

$$v_0'(0) = 0. \tag{2.68}$$

The solution of the problem (2.67), (2.68) is then

$$v_0(y) = c_1 B_2 \cosh(\rho y) b^2 + \frac{1}{2}\left(-c_1 B_1 - dQ_0\right)y^2 + c_3, \tag{2.69}$$

with c_3 as integration constant.

2.2.3 Conjugation of Solutions

We have just found solution to the given problem. But we are still left with finding the unknown constants in the formulas (2.49), (2.63) and (2.69). To determine those, we need several requirements to be fulfilled. First, the temperatures $V(x, y)$, $V_0(x, y)$ must coincide at the contact point $x = 0, y = b$ between the fin and the right part of the wall. So,

$$c_1(1+\mu)\Phi(b) = c_2 a_0 \left(e^{\kappa b} + \mu_0 e^{-\kappa b}\right) - a_0 \frac{\gamma Q_0}{\kappa^2} + Q_0 b_0. \tag{2.70}$$

Second, the mean temperature values in the wall have continuity at $y = b$:

$$c_2\left(e^{\kappa b} + \mu_0 e^{-\kappa b}\right) - \frac{\gamma Q_0}{\kappa^2} = c_1\left(B_2 \cosh(1)b^2 - \frac{1}{2}B_1 b^2\right) - \frac{1}{2}dQ_0 b^2 + c_3. \tag{2.71}$$

Third, we claim that the mean fluxes also coincide at $y = b$:

$$c_2\kappa\left(e^{\kappa b} - \mu_0 e^{-\kappa b}\right) = c_1\left(B_2 \sinh(1)b - B_1 b\right) - dQ_0 b. \tag{2.72}$$

All the constants can be found from the system (2.70), (2.71), and (2.72). Hence the approximate analytical solution to the 2D problem is uniquely determined by (2.51); (2.60) and (2.63); and (2.57) together with (2.64) and (2.69).

2.2.4 Numerical Results

To get some kind of notion if this model could describe the actual situation in computer cooling systems for L-shaped micro elements, we used the following geometrical parameters:

$$\delta = 5\,\mu m,$$
$$l = 1\,\mu m,$$
$$b = 5 \cdot 10^{-2}\,\mu m,$$
$$l_0 = 1 \cdot 10^{-1}\,\mu m.$$

But for the termophysical properties we chose:

$$h_1 = 4.48 \cdot 10^{-7}\,W\mu m^{-2}K^{-1},$$
$$k_0 = 1.412 \cdot 10^{-4}\,W\mu m^{-1}K^{-1} \text{ (for silicon)},$$
$$Q_0 = 10\,K\mu m^{-1}.$$

x\y	0	0.01	0.02	0.03	0.04	0.05
1	9.81	9.80	9.80	9.80	9.79	9.78
0.9	9.99	9.98	9.98	9.98	9.97	9.96
0.8	10.34	10.34	10.34	10.34	10.33	10.32
0.7	10.88	10.88	10.88	10.88	10.87	10.86
0.6	11.60	11.60	11.60	11.60	11.59	11.58
0.5	12.50	12.50	12.50	12.50	12.49	12.48
0.4	13.58	13.58	13.58	13.58	13.57	13.56
0.3	14.85	14.85	14.84	14.84	14.83	14.83
0.2	16.29	16.29	16.29	16.29	16.28	16.27
0.1	17.92	17.92	17.92	17.92	17.91	17.90
0	19.74	19.74	19.73	19.73	19.72	19.72

Table 2.1 Temperature distribution in the fin, in °C

x\y	0	0.01	0.02	0.03	0.04	0.05
0	19.74	19.74	19.73	19.73	19.72	19.72
-0.5	20.38	20.38	20.38	20.37	20.37	20.36
-1	21.39	21.39	21.39	21.39	21.39	21.38
-1.5	22.79	22.79	22.79	22.79	22.78	22.78
-2	24.58	24.58	24.58	24.58	24.58	24.58
-2.5	26.78	26.78	26.78	26.79	26.79	26.79
-3	29.43	29.43	29.43	29.43	29.43	29.43
-3.5	32.53	32.53	32.53	32.53	32.54	32.54
-4	36.13	36.13	36.13	36.13	36.14	36.14
-4.5	40.26	40.26	40.26	40.26	40.26	40.27
-5	44.96	44.96	44.96	44.96	44.96	44.97

Table 2.2 Temperature distribution in the left part of the base, in °C

x\y	0.05	0.06	0.07	0.08	0.09	0.1
0	19.72	19.72	19.72	19.72	19.72	19.72
-0.5	20.36	20.36	20.37	20.37	20.37	20.37
-1	21.38	21.38	21.38	21.38	21.39	21.39
-1.5	22.78	22.78	22.78	22.79	22.79	22.79
-2	24.58	24.58	24.58	24.58	24.58	24.58
-2.5	26.79	26.79	26.79	26.79	26.79	26.79
-3	29.43	29.43	29.43	29.44	29.44	29.44
-3.5	32.54	32.54	32.54	32.54	32.54	32.54
-4	36.14	36.14	36.14	36.14	36.14	36.14
-4.5	40.27	40.27	40.27	40.27	40.27	40.27
-5	44.97	44.97	44.97	44.97	44.97	44.97

Table 2.3 Temperature distribution in the right part of the base, in °C

We can conclude that the model describes the real-life situation quite precisely.

2.3 Approximate Solution to Heat Transfer with Boiling

In some applications the boiling is localized at certain sides of the given structure. In this section, this problem is illustrated by the case of non-linear boundary conditions.

Like in the linear case, the temperature fields satisfy Laplace's equations (2.1) – (2.5) under the same boundary conditions as imposed in Section 2.1, with the exception of non-linear conditions defined at the boundaries where there is boiling occurring. So, we have (2.6) that defines a heat flux at $x = -\delta$, symmetry conditions (2.7) – (2.11) at $y = 0$ and $y = l_0$, and, of course, conjugation conditions (2.12) – (2.25).

For the problem with partial boiling we take the boundary conditions (2.29) and (2.32). Also, we add the following boundary conditions defined at $y = b + \varepsilon_1$:

$$\left(\frac{\partial V_1}{\partial y} + \beta_1^1 V_1^m \right)\Bigg|_{y=b+\varepsilon_1} = 0, \tag{2.73}$$

$$\left(\frac{\partial V_3}{\partial y} + \beta_1^1 V_3^m \right)\Bigg|_{y=b+\varepsilon_1} = 0. \tag{2.74}$$

But for full boiling we would have (2.73), (2.74) and

$$\left.\left(\frac{\partial V_3}{\partial x} + \beta_1^1 V_3^m\right)\right|_{x=l+\varepsilon} = 0, \tag{2.75}$$

$$\left.\left(\frac{\partial V_2}{\partial y} + \beta_1^1 V_2^m\right)\right|_{x=\varepsilon_0} = 0. \tag{2.76}$$

As it was mentioned in the publication [15], for the case of boiling the value of the index m should belong to the interval $\left[3; 3\frac{1}{3}\right]$. It is usually taken to be equal to 3 or $3\frac{1}{3}$.

The model is non-linear because of the boiling condition. Therefore, unlike it was done in the previous section, we are going to take a slightly different approach to solve this problem.

In view of the previous assumption, that the temperature is uniform across the substrate thickness (see formulae (2.26) - (2.28)), we simplify the original problem. So, now in this new model the boundary conditions (2.73) - (2.76) transform to

$$\left.\left(\frac{\partial V}{\partial y} + \beta_0^1 V^m\right)\right|_{y=b} = 0, . \tag{2.77}$$

$$\left.\left(\frac{\partial V}{\partial x} + \beta_0^1 V^m\right)\right|_{x=l} = 0, \tag{2.78}$$

$$\left.\left(\frac{\partial V_0}{\partial x} + \beta_0^1 V_0^m\right)\right|_{x=0} = 0. \tag{2.79}$$

2.3.1 Partial Boiling

Let's start with the case of partial boiling. First, conservative averaging method will be used for the given 2D problem to be transformed into 1D form. Second, an iterative method for approximately solving the non-linear equation for the fin will be presented.

2.3.1.1 Solution for the Fin

For the temperature in the fin we use the simplest approximation assuming that it is constant with respect to the argument y. We take

$$V(x, y) \approx v(x), \tag{2.80}$$

were $v(x)$ is the average value of $V(x, y)$, namely, (2.38).

The function $v(x)$ is found from the 1D equation (2.45) for the fin. Using the boundary conditions (2.77) and (2.8), this equation becomes

$$\frac{d^2 v}{dx^2} - \lambda^2 v^m(x) = 0, \tag{2.81}$$

where

$$\lambda^2 = \rho \beta_0^1.$$

The boundary condition for (2.81) follows from integration of the condition (2.33) and is given by (2.48).

Before continuing on our analysis of the model, let us rewrite the ordinary differential equation (2.81) as follows:

$$\frac{d^2 v}{dx^2} - \lambda^2 v v^{m-1} = 0. \tag{2.82}$$

As the equation is non-linear, we are going to solve it using a finite difference scheme and iterations. So, let's set up a regular grid

$$x_i = ih_x, \ i = 0,\ldots,N, \ h_x = \frac{l}{N}$$

and replace the continuous problem by its discrete approximation:

$$\frac{v_{i+1}^k - 2v_i^k + v_{i-1}^k}{h_x^2} - \lambda^2 \left(v_i^{k-1}\right)^{m-1} v_i^k = 0, \ i = 1,\ldots,N-1. \tag{2.83}$$

Here v_i^k is a difference approximation to the value of $v(x)$ at the mesh point x_i, but the superscript k denotes the k-th iteration.

In this case, we can rearrange the above equation (2.83) as (see [48]):

$$Av_{i-1}^k - C_i^{k-1} v_i^n + Bv_{i+1}^k = -F \tag{2.84}$$

with coefficients

$$F = 0, \ A = B = \frac{1}{h_x^2}, \ C_i^{k-1} = A + B + \lambda^2 \left(v_i^{k-1}\right)^{m-1}. \tag{2.85}$$

To obtain a second order approximation for the boundary condition (2.48) at first the function $v(x)$ is developed as a Taylor series to give

$$\frac{dv}{dx}\bigg|_{x=l} = \frac{v(l) - v(x)}{l - x} + \frac{l - x}{2} v''(l) + O\left((l-x)^2\right). \tag{2.86}$$

Making use of the differential equation (2.82), the second order derivative in (2.86) is replaced by $\lambda^2 v v^{m-1}$. Then, with appropriate use of this and (2.83), the approximation of the condition (2.48) leads to the expression

$$\frac{v_N^k - v_{N-1}^k}{h_x} + \frac{h_x}{2} \lambda^2 \left(v_N^{k-1}\right)^{m-1} v_N^k + \beta_0^1 v_N^k = 0 \tag{2.87}$$

or

$$v_N^k = \chi_1^{k-1} v_{N-1}^k, \tag{2.88}$$

with the coefficient χ_1^{k-1} to be

$$\chi_1^{k-1} = \left[1 + h_x \left(\beta_0^1 + \frac{h_x}{2} \lambda^2 \left(v_N^{k-1}\right)^{m-1}\right)\right]^{-1}. \tag{2.89}$$

Taking into account (2.88), for any given value of k (2.84) constitutes a set of N linear equations:

$$v_{i+1}^k = \alpha_{i+1}^{k-1} v_i^k + \beta_{i+1}^{k-1}, \ i = 0,\ldots,N-1 \tag{2.90}$$

with

$$\alpha_i^{k-1} = \frac{A}{C_i^{k-1} - B\alpha_{i+1}^{k-1}}, \ \beta_i^{k-1} = \frac{B\beta_{i+1}^{k-1} + F}{C_i^{k-1} - B\alpha_{i+1}^{k-1}}, \ i = N-1,\ldots,1 \tag{2.91}$$

and

$$\alpha_N^{k-1} = \chi_1^{k-1}, \ \beta_N^{k-1} = 0. \tag{2.92}$$

From (2.85) and (2.92) it can be seen that $\beta_i^{k-1} = 0$ for all $i = 1,\ldots,N$.

2.3.1.2 Solution for the Base

As all the additional conditions for the base are the same as in the linear case, the solution for the wall is obtained in the same manner as in subsection 2.2.2. Thus, the solution of the 1D problem for the right part of the base is (2.63):

$$v_0(y) = c_2\left(e^{\kappa y} + \mu_0 e^{-\kappa y}\right) - \frac{\gamma Q_0}{\kappa^2}.$$

When deriving 1D equation for the left part of the wall, it is important to remember that for $x = 0$ the functions $V_0(x, y)$, $V(x, y)$ must fulfil the conjugation conditions (2.12), (2.13). So, from (2.12) we get that

$$g_0(y) = v(0), \quad y \in (0, b).$$

That means the function $g_0(y)$, when considering the left part of the wall, is constant and its value is equal to

$$g_0(y) = v_0^k. \tag{2.93}$$

Whereas from (2.13) we get

$$d\frac{\partial V_0}{\partial x}\bigg|_{x=-0} = d\frac{dv}{dx}\bigg|_{x=+0}$$
$$= d\left[\frac{v_1^k - v_0^k}{h_x} - \frac{h_x}{2}\lambda^2\left(v_0^{k-1}\right)^{m-1}v_0^k\right].$$

Inserting (2.90) into this expression, after taking $i = 0$, we can write

$$d\frac{\partial V_0}{\partial x}\bigg|_{x=-0} = d\left[\frac{\alpha_1^{k-1} - 1}{h_x} - \frac{h_x}{2}\lambda^2\left(v_0^{k-1}\right)^{m-1}\right]v_0^k. \tag{2.94}$$

From this expression we see that we should use iteration process to solve the equation for the left part of the wall here as well. Before we find this equation, let's rewrite the last expression as

$$d\frac{\partial V_0}{\partial x}\bigg|_{x=-0} = G_0^{k-1}v_0^k. \tag{2.95}$$

Here

$$G_0^{k-1} = d\left(\frac{\alpha_1^{k-1} - 1}{h_x} - \frac{h_x}{2}\lambda^2\left(v_0^{k-1}\right)^{m-1}\right).$$

Now, if we use conditions (2.95) and (2.6), the integrated equation (2.58) becomes

$$\frac{d^2 v_0}{dy^2} = H_0^{k-1}, \tag{2.96}$$

where H_0^{k-1} is defined by

$$H_0^{k-1} = -G_0^{k-1}v_0^k - dQ_0. \tag{2.97}$$

So, the solution of the problem (2.96), (2.68) is:

$$v_0(y) = \frac{1}{2}H_0^{k-1}y^2 + c_1. \tag{2.98}$$

In this case we get an expression that is dependent of the previous iteration.

2.3.1.3   Conjugation of Solutions

In order to ensure the continuity of the solution of the given problem in all the domain, it is necessary for the temperatures $V(x, y)$, $V_0(x, y)$ to coincide at the contact point $x = 0$, $y = b$ between the fin and the right part of the wall. So, from the assumption (2.80) and expressions (2.57), (2.59) and (2.63) we get:

$$v_0^k = V_0(0, b)$$

$$= a_0 v_0(b) + b_0 Q_0$$
$$= a_0\left(c_2\left(e^{\kappa b} + \mu_0 e^{-\kappa b}\right) - \frac{\gamma Q_0}{\kappa^2}\right) + b_0 Q_0. \tag{2.99}$$

Although the approximate solution to the problem has been found, there are still two constants left to be calculated, c_1 and c_2. It is required that

$$v_0(b-0) = v_0(b+0), \tag{2.100}$$

$$\frac{d}{dy}v_0(b-0) = \frac{d}{dy}v_0(b+0). \tag{2.101}$$

In accordance with (2.98), (2.97) and (2.99) , and (2.63), this system becomes

$$\frac{1}{2}b^2\left(-G_0^{k-1}\left(a_0\left(c_2\left(e^{\kappa b} + \mu_0 e^{-\kappa b}\right) - \frac{\gamma Q_0}{\kappa^2}\right) + b_0 Q_0\right) - dQ_0\right) + c_1 = c_2\left(e^{\kappa b} + \mu_0 e^{-\kappa b}\right) - \frac{\gamma Q_0}{\kappa^2},$$

$$-G_0^{k-1}b\left(a_0\left(c_2\left(e^{\kappa b} + \mu_0 e^{-\kappa b}\right) - \frac{\gamma Q_0}{\kappa^2}\right) + b_0 Q_0\right) - bdQ_0 = c_2\kappa\left(e^{\kappa b} - \mu_0 e^{-\kappa b}\right).$$

When solving this we find the constants and get that these are iteration-dependent, c_1^{k-1}, c_2^{k-1}:

$$c_1^{k-1} = c_2^{k-1}\left[\left(e^{\kappa b} + \mu_0 e^{-\kappa b}\right)\frac{1}{2}b^2 G_0^{k-1}a_0 + \left(e^{\kappa b} + \mu_0 e^{-\kappa b}\right)\right]$$
$$- \frac{\gamma Q_0}{\kappa^2} - \frac{1}{2}b^2 G_0^{k-1}Q_0\left(a_0\frac{\gamma}{\kappa^2} - b_0\right) + \frac{1}{2}b^2 dQ_0;$$

$$c_2^{k-1} = \left(-b_0 Q_0 G_0^{k-1}b - bdQ_0 + G_0^{k-1}ba_0\frac{\gamma Q_0}{\kappa^2}\right)\left[G_0^{k-1}ba_0\left(e^{\kappa b} + \mu_0 e^{-\kappa b}\right) + \kappa\left(e^{\kappa b} - \mu_0 e^{-\kappa b}\right)\right]^{-1}.$$

The iteration algorithm can be organised in this way: at first we must choose some "initial" value for $v_0(y)$, that is v_0^0, and find α_i^0 and constants c_1^0, c_2^0. After that we perform the first iteration in the fin by calculating v_0^1 from (2.99). Knowing this we can use (2.90) to calculate v_i^1. Afterwards we can proceed with the iteration process.

To get solution of the 1D problem for the base we use formulae (2.98) and (2.63). In the end we get the 2D temperature in the system, which can be calculated directly from (2.80) for the fin and from (2.57) for the base. Here the function $g_0(y)$ for the left part of the base is found from the formulae (2.93) and (2.99), but for the right part of the wall we use (2.59).

2.3.2   Full Boiling

When finding solution to heat transfer with full boiling, the principle remains the same as in the previous subsection 2.3.1.

2.3.2.1   Numerical Solution for the Fin

Here the problem is formulated in the same way as in subsection 2.3.1.1, with the exception of boundary condition at $x = l$. Rewriting the boundary condition (2.78) as follows

$$\left(\frac{\partial V}{\partial x} + \beta_0^1 V^{m-1}V\right)\bigg|_{x=l} = 0$$

and integrating the last term (see [40], Ch. 4.7):

$$\int_0^b V^{m-1}(x,y)V(x,y)\big|_{x=l}\,dy = \int_0^b V^{m-1}(l,y)V(l,y)dy = \left(v(l)\right)^{m-1}bv(l)$$

leads to a non-linear 1D boundary condition

$$v'(l) + \overline{\beta}_0^1 \big(v(l)\big)^{m-1} v(l) = 0, \quad \overline{\beta}_0^1 = \beta_0^1 b.$$
(2.102)

In the end we get the same difference scheme as in (2.90) - (2.92). But here the coefficient χ_1^{k-1} (2.89) changes to

$$\chi_1^{k-1} = \left(1 + h_x\left(\frac{h_x}{2}\lambda^2 + \overline{\beta}_0^1\right)\big(v_N^{k-1}\big)^{m-1}\right)^{-1}.$$

2.3.2.2 Numerical Solution for the Base

As 1D equation for the base is likely to be non-linear, we'll consider a discrete realization of the iterative process for the base as well and use iterative difference scheme. So, let's define the following grid points:

$$y_j = \begin{cases} jh_{y,1} & j = 0,\dots,M_0 \\ b + (j - M_0)h_{y,2} & j = M_0 + 1,\dots,M \end{cases}, \quad h_{y,1} = \frac{b}{M_0}, \quad h_{y,2} = \frac{l_0 - b}{M - M_0}.$$

The Right Part of the Base

Upon applying the boundary condition (2.79) to (2.57) we get

$$g_0(y)\left(-\frac{1}{2}d + \frac{1}{2}de^2 + \beta_0^1\big(g_0(y)\big)^{m-1}\right) + v_0(y)\left(\frac{1}{2}d - \frac{1}{2}de^2\right)$$
$$+ Q_0(y)\left(-\frac{1}{2} - e + e^2\frac{1}{2}\right) = 0.$$
(2.103)

We can rewrite this expression as

$$g_0(y) = v_0(y)a_0^{k-1}(y) + Q_0 b_0^{k-1}(y),$$
(2.104)

where

$$a_0^{k-1}(y) = \frac{d - de^2}{d - 2\beta_0^1\big(g_0^{k-1}(y)\big)^{m-1} - de^2},$$

$$b_0^{k-1}(y) = \frac{-1 - 2e + e^2}{d - 2\beta_0^1\big(g_0^{k-1}(y)\big)^{m-1} - de^2}.$$

So the 1D equation (2.61) for the right part of the base becomes

$$\frac{d^2 v_0}{dy^2} - \kappa_0^{k-1}(y)v_0(y) = \gamma_0^{k-1}(y)Q_0$$
(2.105)

with

$$\kappa_0^{k-1}(y) = -d\big(1 - a_0^{k-1}(y)\big)\left(\frac{1}{2}d - \frac{1}{2}de^2\right),$$

$$\gamma_0^{k-1}(y) = -d\left(b_0^{k-1}(y)\left(-\frac{1}{2}d + \frac{1}{2}de^2\right) + \left(-\frac{1}{2} - e + e^2\frac{1}{2}\right) + 1\right).$$

And the boundary condition for the equation is (2.62).

We discretize the equation (2.105) as

$$\frac{v_{0,j+1}^k - 2v_{0,j}^k + v_{0,j-1}^k}{h_{y,2}^2} - \kappa_{0,j}^{k-1} v_{0,j}^k = \gamma_{0,j}^{k-1} Q_0, \quad j = M_0 + 1,\dots,M - 1.$$

Using these notations

$$A_2 = \frac{1}{h_{y,2}^2} = B_2, \quad C_{2,j}^{k-1} = \frac{2}{h_{y,2}^2} + \kappa_{0,j}^{k-1}, \quad F_{2,j}^{k-1} = -\gamma_{0,j}^{k-1} Q_0$$

the difference scheme is equivalent to

$$A_2 v_{0,j-1}^k - C_{2,j}^{k-1} v_{0,j}^k + B_2 v_{0,j+1}^k = -F_{2,j}^{k-1}, \quad j = M_0 + 1, \ldots, M - 1. \tag{2.106}$$

When approximating the boundary condition (2.62) we use the same approach as in subsection 2.3.1 to get

$$v_{0,M}^k = \varpi_{2,1}^{k-1} v_{0,M-1}^k + \varpi_{2,2}^{k-1} \tag{2.107}$$

with

$$\varpi_{2,1}^{k-1} = \frac{1}{h_{y,2}} \left(\frac{1}{h_{y,2}} + \frac{h_{y,2}}{2} \kappa_{0,M}^{k-1} \right)^{-1},$$

$$\varpi_{2,2}^{k-1} = -\frac{h_{y,2}}{2} \left(\frac{1}{h_{y,2}} + \frac{h_{y,2}}{2} \kappa_{0,M}^{k-1} \right)^{-1} \gamma_{0,M}^{k-1} Q_0.$$

Combining (2.106) and (2.107) gives

$$v_{0,j+1}^k = \varsigma_{2,j+1}^{k-1} v_{0,j}^k + \eta_{2,j+1}^{k-1}, \quad j = M_0, \ldots, M - 1, \tag{2.108}$$

where

$$\varsigma_{2,M}^{k-1} = \varpi_{2,1}^{k-1}, \quad \eta_{2,M}^{k-1} = \varpi_{2,2}^{k-1},$$

$$\varsigma_{2,j}^{k-1} = \frac{A_2}{C_{2,j}^{k-1} - B_2 \varsigma_{2,j+1}^{k-1}}, \quad \eta_{2,j}^{k-1} = \frac{B_2 \eta_{2,j+1}^{k-1} + F_{2,j}^{k-1}}{C_{2,j}^{k-1} - B_2 \varsigma_{2,j+1}^{k-1}}, \quad j = M - 1, \ldots, M_0 + 1.$$

The Left Part of the Base

At first we find 1D equation using continuity condition (2.13) and the definition (2.80):

$$d \left. \frac{\partial V_0}{\partial x} \right|_{x=-0} = d \left. \frac{\partial V}{\partial x} \right|_{x=+0}$$

$$= d \left. \frac{dv}{dx} \right|_{x=+0}. \tag{2.109}$$

So, (2.58) becomes

$$\frac{d^2 v_0}{dy^2} + d \left. \frac{dv}{dx} \right|_{x=+0} + dQ_0 = 0. \tag{2.110}$$

For finding numerical approximation to the solution of the problem (2.110), (2.68) for the left side of the base, equation (2.110) is replaced by

$$\frac{v_{0,j+1}^k - 2v_{0,j}^k + v_{0,j-1}^k}{h_{y,1}^2} + d \left(\frac{v_1^k - v_0^k}{h_x} - \frac{h_x}{2} \lambda^2 \left(v_0^{k-1} \right)^{m-1} v_0^k \right) + dQ_0 = 0$$

or

$$A_1 v_{0,j-1}^k - C_1 v_{0,j}^k + B_1 v_{0,j+1}^k + D_1^{k-1} v_0^k = -F_1, \quad j = 1, \ldots, M_0 - 1, \tag{2.111}$$

$$v_{0,0}^k = v_{0,1}^k - \varpi_{1,3}^{k-1} v_0^k + \mu_1, \tag{2.112}$$

where the coefficients are expressed by

$$A_1 = \frac{1}{h_{y,1}^2} = B_1, \quad C_1 = \frac{2}{h_{y,1}^2}, \quad D_{1,}^{k-1} = d \left(\frac{\alpha_1^{k-1} - 1}{h_x} - \frac{h_x}{2} \lambda^2 \left(v_0^{k-1} \right)^{m-1} \right), \quad F_1 = dQ_0,$$

$$\varpi_{1,3}^{k-1} = dh_{y,1} \frac{h_{y,1}}{2}\left(\frac{1-\alpha_1^{k-1}}{h_x} + \frac{h_x}{2}\lambda^2\left(v_0^{k-1}\right)^{m-1}\right), \quad \mu_1 = dh_{y,1}\frac{h_{y,1}}{2}Q_0.$$

It is easy to show that the expression (2.111) together with (2.112) gives this scheme for the left part of the base:

$$v_{0,j}^k = \xi_{j+1} v_{0,j+1}^k - \psi_{j+1}^{k-1} v_0^k + \chi_1, \quad j = M_0 - 1, \dots, 0. \tag{2.113}$$

The expression uses the abbreviations

$$\xi_1 = 1, \quad \psi_1^{k-1} = \varpi_{1,3}^{k-1}, \quad \chi_1 = \mu_1,$$

$$\xi_{j+1} = \frac{B_1}{C_1 - A_1\xi_j}, \quad \psi_{j+1}^{k-1} = \frac{A_1\psi_j^{k-1} - D_1^{k-1}}{C_1 - A_1\xi_j}, \quad \chi_{j+1} = \frac{A_1\chi_j + F_1}{C_1 - A_1\xi_j}, \quad j = 1, \dots, M_0 - 1.$$

Conjugation of Solutions

By examining the schemes (2.90), (2.108), (2.113) closely, we can see that they depend on the values of v_{0,M_0}^k, v_0^k. These two should be calculated for the average temperatures to fit the continuity conditions (2.100), (2.101). (2.100) yields the following equality

$$v_0^k = v_{0,M_0}^k a_{0,M_0}^{k-1} + Q_0 b_{0,M_0}^{k-1}. \tag{2.114}$$

But (2.101) is estimated using the approximation

$$\frac{v_{0,M_0}^k - v_{0,M_0-1}^k}{h_{y,1}} + \frac{h_{y,1}}{2}\left[-d\left(\frac{\alpha_1^{k-1}-1}{h_x} - \frac{h_x}{2}\lambda^2\left(v_0^{k-1}\right)^{m-1}\right)v_0^k - dQ_0\right] =$$
$$= \frac{v_{0,M_0+1}^k - v_{0,M_0}^k}{h_{y,2}} - \frac{h_{y,2}}{2}\left(\kappa_{0,M_0}^{k-1} v_{0,M_0}^k + \gamma_{0,M_0}^{k-1} Q_0\right). \tag{2.115}$$

In order to evaluate the terms v_{0,M_0-1}^k, v_{0,M_0+1}^k we use the equations (2.113), (2.108) when $j = M_0 - 1$ and $j = M_0$, respectively. Substituting those and (2.114) in (2.115) we get

$$\left(\begin{array}{c}\dfrac{1}{h_{y,1}} - \dfrac{\xi_{M_0}}{h_{y,1}} + \dfrac{\psi_{M_0}^{k-1} a_{0,M_0}^{k-1}}{h_{y,1}} - d\dfrac{h_{y,1}}{2}a_{0,M_0}^{k-1}\left(\dfrac{\alpha_1^{k-1}-1}{h_x} - \dfrac{h_x}{2}\lambda^2\left(v_0^{k-1}\right)^{m-1}\right) \\[3mm] -\left(\dfrac{\varsigma_{2,M_0+1}^{k-1}-1}{h_{y,2}} - \dfrac{h_{y,2}}{2}\kappa_{0,M_0}^{k-1}\right)\end{array}\right) v_{0,M_0}^k =$$

$$= \frac{\eta_{2,M_0+1}^{k-1}}{h_{y,2}} - \frac{h_{y,2}}{2}\gamma_{0,M_0}^{k-1} Q_0 - \frac{\psi_{M_0}^{k-1} Q_0 b_{0,M_0}^{k-1} - \chi_1}{h_{y,1}}$$

$$+ d\frac{h_{y,1}}{2}Q_0\left[\left(\frac{\alpha_1^{k-1}-1}{h_x} - \frac{h_x}{2}\lambda^2\left(v_0^{k-1}\right)^{m-1}\right)b_{0,M_0}^{k-1} + 1\right].$$

from which we find v_{0,M_0}^k, giving the possibility to calculate v_0^k from (2.114). Eventually, the numerical solution of the given problem can be obtained using the explicit schemes (2.90), (2.108), (2.113).

36

3 Transient Heat Transfer in System with Double Wall and Double Fins

This chapter follows on directly from Chapter 2. It deals with the mathematical aspects of transient heat conduction for double wall with double fins in 2D geometry. The non-stationary heat conduction problem is examined, when assigning third type linear boundary conditions and partial boiling conditions on the fin's surface. Conservative averaging and finite difference methods are applied to the given problem to construct numerical solution.

3.1 Mathematical Statement of the Problem

Using the same notations and assumptions as in Chapter 2, we are going to describe these non-stationary temperature fields by the following partial differential equations:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{1}{\tilde{a}_0^2}\frac{\partial V}{\partial t}, \ x, y \in C, \tag{3.1}$$

$$\frac{\partial^2 V_0}{\partial x^2} + \frac{\partial^2 V_0}{\partial y^2} = \frac{1}{\tilde{a}_0^2}\frac{\partial V_0}{\partial t}, \ x, y \in C_0, \tag{3.2}$$

$$\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} = \frac{1}{\tilde{a}_1^2}\frac{\partial V_1}{\partial t}, \ x, y \in C_1 \tag{3.3}$$

$$\frac{\partial^2 V_2}{\partial x^2} + \frac{\partial^2 V_2}{\partial y^2} = \frac{1}{\tilde{a}_1^2}\frac{\partial V_2}{\partial t}, \ x, y \in C_2 \tag{3.4}$$

$$\frac{\partial^2 V_3}{\partial x^2} + \frac{\partial^2 V_3}{\partial y^2} = \frac{1}{\tilde{a}_1^2}\frac{\partial V_3}{\partial t}, \ x, y \in C_3. \tag{3.5}$$

where $\tilde{a}_0^2 = \dfrac{k_0}{c_0 \tilde{\rho}_0}$, $\tilde{a}_1^2 = \dfrac{k_1}{c_1 \tilde{\rho}_1}$. c_0, c_1 denote specific heat, but $\tilde{\rho}_0$, $\tilde{\rho}_1$ stand for the density.

Here we use the same boundary conditions (2.6) – (2.11), (2.29) – (2.32) and ideal thermal contact conditions (2.12) – (2.25), as stated in Chapter 2. Beside these, initial conditions should be provided as well:

$$V(x, y, 0) = V^0(x, y), \tag{3.6}$$

$$V_0(x, y, 0) = V_0^0(x, y), \tag{3.7}$$

$$V_1(x, y, 0) = V_1^0(x, y), \tag{3.8}$$

$$V_2(x, y, 0) = V_2^0(x, y), \tag{3.9}$$

$$V_3(x, y, 0) = V_3^0(x, y). \tag{3.10}$$

3.2 Approximate Solution of the Problem

Just like we did in the previous chapter, we make the simplest approximation for the upper layer taking the temperature to be constant in the appropriate directions. This leads to the problem for the basic layer only which is represented by equations (3.1), (3.2), the initial conditions (3.6), (3.7), the boundary conditions

$$\left.\frac{\partial V_0}{\partial x}\right|_{x=-\delta} = -Q_0(y, t), \tag{3.11}$$

$$\left.\frac{\partial V_0}{\partial y}\right|_{y=0} = 0, \tag{3.12}$$

$$\left.\frac{\partial V}{\partial y}\right|_{y=0} = 0, \tag{3.13}$$

$$\left.\frac{\partial V_0}{\partial y}\right|_{y=l_0} = 0, \tag{3.14}$$

$$\left.\left(\frac{\partial V}{\partial x} + \beta_0^1 V\right)\right|_{x=l} = 0, \tag{3.15}$$

$$\left.\left(\frac{\partial V}{\partial y} + \beta_0^1 V\right)\right|_{y=b} = 0, \tag{3.16}$$

$$\left.\left(\frac{\partial V_0}{\partial x} + \beta_0^1 V_0\right)\right|_{x=0} = 0, \tag{3.17}$$

and the conjugation conditions

$$V_0\big|_{x=-0} = V\big|_{x=+0}, \tag{3.18}$$

$$\left.\frac{\partial V_0}{\partial x}\right|_{x=-0} = \left.\frac{\partial V}{\partial x}\right|_{x=+0}. \tag{3.19}$$

Using conservative averaging method for rectangular fins, [6], [7], [10], [13], [14], [25], [26], we are going to reduce the given 2D equations to 1D equations. To make a numerical approximation to the solution of the problem the finite difference method can be used.

3.2.1 Reduction of the 2D Problem for the Fin

First, we approximate the temperature in the fin by

$$V(x, y, t) = f_0(x, t) + \left(e^{\rho y} - 1\right) f_1(x, t) + \left(1 - e^{-\rho y}\right) f_2(x, t), \quad \rho = b^{-1} \tag{3.20}$$

and define the integral average value over the interval $[0, b]$:

$$v(x, t) = \rho \int_0^b V(x, y, t) dy. \tag{3.21}$$

The unknown functions $f_i(x, t)$, $i = 0, 1, 2$ are found in the same way as in Chapter 2 by applying appropriate conditions, namely, (3.13), (3.16), (3.21). This yields

$$f_2(x, t) = -f_1(x, t), \tag{3.22}$$

$$f_1(x, t) = \frac{v(x, t) - f_0(x, t)}{2(\sinh(1) - 1)}, \tag{3.23}$$

$$f_0(x, t) = \psi v(x, t), \tag{3.24}$$

with

$$\psi = \frac{\sinh(1) + \beta_0^1 b(\cosh(1) - 1)}{\sinh(1) + \beta_0^1 b(\cosh(1) - \sinh(1))}.$$

Substitution of (3.22) − (3.24) into (3.20) gives

$$V(x, y, t) = v(x, t)\Phi(y) \tag{3.25}$$

with

$$\Phi(y) = \frac{\sinh(1) + \beta_0^1 b(\cosh(1) - \cosh(\rho y))}{\sinh(1) + \beta_0^1 b(\cosh(1) - \sinh(1))},$$

while the function $v(x,t)$ is yet unknown. In order to find it, at first we use the definition (3.21) and integrate the equation (3.1):

$$\frac{\partial^2 v}{\partial x^2} + \rho \frac{\partial V}{\partial y}\bigg|_{y=0}^{y=b} = \frac{1}{\tilde{a}_0^2} \frac{\partial v}{\partial t}.$$

According to the boundary conditions (3.16), (3.13) and the expression (3.25), this can be written as

$$\frac{\partial^2 v}{\partial x^2} - \lambda^2 v(x,t) = \frac{1}{\tilde{a}_0^2} \frac{\partial v}{\partial t}, \tag{3.26}$$

with

$$\lambda^2 = \rho \beta_0^1 \Phi(b).$$

The boundary and the initial condition for (3.26) can also be obtained by simply applying the operator (3.21) to (3.15) and (3.6):

$$\left(\frac{\partial v}{\partial x} + \beta_0^1 v\right)\bigg|_{x=l} = 0, \tag{3.27}$$

$$v(x,0) = u^0(x). \tag{3.28}$$

3.2.2   Reduction of the 2D Problem for the Base

For temperature approximation in the wall we use the following representation:

$$V_0(x,y,t) = g_0(y,t) + \left(e^{-dx} - 1\right)g_1(y,t) + \left(1 - e^{dx}\right)g_2(y,t), \; d = \delta^{-1} \tag{3.29}$$

and the average value of $V_0(x,y,t)$,

$$v_0(y,t) = d \int_{-\delta}^{0} V_0(x,y,t)dx. \tag{3.30}$$

From the definition (3.30) and the boundary condition (3.11) it is possible to calculate these functions:

$$g_2(y,t) = e\delta Q_0(y,t) - e^2 g_1(y,t), \tag{3.31}$$

$$g_1(y,t) = \frac{1}{2}\left(g_0(y,t) - v_0(y,t) + \delta Q_0(y,t)\right). \tag{3.32}$$

Then the formula (3.29) reduces to

$$V_0(x,y,t) = g_0(y,t)\left(1 + \frac{1}{2}\left(e^{-dx} - 1\right) - e^2 \frac{1}{2}\left(1 - e^{dx}\right)\right)$$

$$+ v_0(y,t)\left(-\frac{1}{2}\left(e^{-dx} - 1\right) + e^2 \frac{1}{2}\left(1 - e^{dx}\right)\right)$$

$$+ Q_0(y,t)\left(\frac{1}{2}\delta\left(e^{-dx} - 1\right) + \left(\delta e - e^2 \frac{1}{2}\delta\right)\left(1 - e^{dx}\right)\right). \tag{3.33}$$

Before continuing, we divide the wall into two parts along the line $y = b$. And integrate the main equation (3.2) with respect to x:

$$d\frac{\partial V_0}{\partial x}\bigg|_{x=-\delta}^{x=0} + \frac{\partial^2 v_0}{\partial y^2} = \frac{1}{\tilde{a}_0^2} \frac{\partial v_0}{\partial t}. \tag{3.34}$$

3.2.2.1 The Right Part of the Base

Here we examine the part of the base that occupies the domain $x \in (-\delta, 0)$, $y \in (b, l_0)$. In order to compute $g_0(y,t)$ let us apply the boundary condition (3.17) to (3.33) to compute:

$$g_0(y,t) = v_0(y,t)a_0 + Q_0(y,t)b_0,$$ (3.35)

where

$$a_0 = \frac{d - de^2}{d - 2\beta_0^1 - de^2}, \quad b_0 = -\frac{1 + 2e - e^2}{d - 2\beta_0^1 - de^2}.$$

The expression (3.35) is plugged into the representation (3.33), so that we obtain

$$V_0(x, y, t) = v_0(y,t)\left(a_0 + \frac{1}{2}(a_0 - 1)(e^{-dx} - 1) + e^2 \frac{1}{2}(1 - a_0)(1 - e^{dx}) \right)$$
$$+ Q_0(y,t)\left(b_0 + \frac{1}{2}(b_0 + \delta)(e^{-dx} - 1) + e\left(\delta - e\frac{1}{2}(\delta + b_0) \right)(1 - e^{dx}) \right).$$ (3.36)

Then with appropriate use of the boundary conditions (3.17), (3.11) and the formula (3.36), the equation (3.34) for the right part of the wall takes the form

$$\frac{\partial^2 v_0}{\partial y^2} - \kappa^2 v_0(y,t) = \frac{1}{\tilde{a}_0^2} \frac{\partial v_0}{\partial t} + \gamma Q_0(y,t),$$ (3.37)

with

$$\kappa^2 = d\beta_0^1 a_0, \quad \gamma = d(\beta_0^1 b_0 - 1).$$

When integrating (3.14) over $[-\delta, 0]$, the boundary condition for (3.37) becomes

$$\left. \frac{\partial v_0}{\partial y} \right|_{y=l_0} = 0.$$ (3.38)

But the 1D initial condition

$$v_0(y, 0) = u_0^0(y)$$ (3.39)

is obtained by integration of (3.7).

3.2.2.2 The Left Part of the Base

To find the functions $g_0(y,t)$ and $v_0(y,t)$ for the temperature in the left part of the base we use the conjugation conditions (3.18), (3.19). From (3.18) and (3.25) we derive

$$g_0(y,t) = v(0,t)\Phi(y).$$ (3.40)

From (3.19) and (3.25) we derive

$$d\left. \frac{\partial V_0}{\partial x} \right|_{x=-0} = d\left. \frac{\partial V}{\partial x} \right|_{x=+0}$$
$$= d\left. \frac{\partial v}{\partial x} \right|_{x=0} \Phi(y),$$ (3.41)

Using (3.41), the boundary condition (3.11), and proceeding in the same way we did in the derivation of (2.66) the equation (3.34) becomes

$$\frac{\partial^2 v_0}{\partial y^2} + d\left. \frac{\partial v}{\partial x} \right|_{x=0} \Phi(y) + dQ_0(y,t) = \frac{1}{\tilde{a}_0^2} \frac{\partial v_0}{\partial t},$$ (3.42)

with the additional conditions to be satisfied:

$$\left.\frac{\partial v_0}{\partial y}\right|_{y=0} = 0 \qquad (3.43)$$

and (3.39).

In what follows, the function $Q_0(y,t)$ is taken to be constant.

3.2.3 Construction of the Difference Scheme

We are going to solve these 1D problems in a numerical way. For approximating the solution we'll use finite-difference method discretizing the differential equations with three-point scheme with non-negative weights. The second order approximations of boundary conditions are obtained by using differential equations and their approximations (see [6], [7], [25], [26], and [48]).

Let's begin by introducing the grid points by

$$x_i = ih_x, \ i = 0,\dots,N, \ h_x = \frac{l}{N}, \qquad (3.44)$$

$$y_j = \begin{cases} jh_{y,1} & j = 0..M_0 \\ b + (j - M_0)h_{y,2} & j = M_0 + 1..M \end{cases}, \ h_{y,1} = \frac{b}{M_0}, \ h_{y,2} = \frac{l_0 - b}{M - M_0}, \qquad (3.45)$$

to discretize the domain in space. In addition, we consider $\tau > 0$ to discretize the domain in time $[0,T]$, defining a grid

$$t_n = n\tau, \ n = 0,1,\dots,\frac{T}{\tau}. \qquad (3.46)$$

But v_i^n and $v_{0,j}^n$ denote the values of numerical solutions at points (x_i, t_n) and (y_j, t_n), respectively.

3.2.3.1 Difference Scheme for the Fin

Hence for (3.26) we have the following discrete equation:

$$\frac{v_i^{n+1} - v_i^n}{\tilde{a}_0^2 \tau} = \sigma \frac{v_{i+1}^{n+1} - 2v_i^{n+1} + v_{i-1}^{n+1}}{h_x^2} + (1-\sigma)\frac{v_{i+1}^n - 2v_i^n + v_{i-1}^n}{h_x^2}$$
$$- \sigma\lambda^2 v_i^{n+1} - (1-\sigma)\lambda^2 v_i^n, \qquad (3.47)$$

where $i = 1,\dots,N-1$ and the weight $0 \le \sigma \le 1$. We can rewrite the scheme (3.47) as

$$Av_{i-1}^{n+1} - Cv_i^{n+1} + Bv_{i+1}^{n+1} = -F_i^n, \qquad (3.48)$$

where

$$A = \frac{\sigma}{h_x^2} = B, \ C = \frac{2\sigma}{h_x^2} + \lambda^2\sigma + \frac{1}{\tilde{a}_0^2 \tau},$$

$$F_i^n = (1-\sigma)\frac{v_{i+1}^n - 2v_i^n + v_{i-1}^n}{h_x^2} - \lambda^2(1-\sigma)v_i^n + \frac{v_i^n}{\tilde{a}_0^2 \tau}. \qquad (3.49)$$

To get the second order approximation to the first derivative in the boundary condition (3.27), let's use the differential equation (3.26) and its approximation (3.47) (see [6], [7], [25], [26], and [48]):

$$\sigma\frac{v_N^{n+1} - v_{N-1}^{n+1}}{h_x} + (1-\sigma)\frac{v_N^n - v_{N-1}^n}{h_x} + \frac{h_x}{2}\left[\frac{1}{\tilde{a}_0^2}\frac{v_N^{n+1} - v_N^n}{\tau} + \sigma\lambda^2 v_N^{n+1} + (1-\sigma)\lambda^2 v_N^n\right]$$
$$+ \sigma\beta_0^1 v_N^{n+1} + (1-\sigma)\beta_0^1 v_N^n = 0.$$

We can rewrite this as

41

$$v_N^{n+1} = \varpi_1 v_{N-1}^{n+1} + \varpi_2^n \tag{3.50}$$

with coefficients

$$\varpi_1 = \frac{\sigma}{h_x}\left(\frac{\sigma}{h_x} + \frac{h_x}{2}\left(\sigma\lambda^2 + \frac{1}{\tilde{a}_0^2\tau}\right) + \sigma\beta_0^1\right)^{-1},$$

$$\varpi_2^n = \left(\frac{\sigma}{h_x} + \frac{h_x}{2}\left(\sigma\lambda^2 + \frac{1}{\tilde{a}_0^2\tau}\right) + \sigma\beta_0^1\right)^{-1}\left(\frac{h_x}{2}\frac{v_N^n}{\tilde{a}_0^2\tau} - (1-\sigma)\left(\frac{v_N^n - v_{N-1}^n}{h_x} + \left(\frac{\lambda^2 h_x}{2} + \beta_0^1\right)v_N^n\right)\right).$$

Using new notations

$$\varsigma_N = \varpi_1, \ \eta_N^n = \varpi_2^n, \tag{3.51}$$

(3.50) becomes

$$v_N^{n+1} = \varsigma_N v_{N-1}^{n+1} + \eta_N^n. \tag{3.52}$$

Let's solve (3.48) for v_i^{n+1}:

$$v_i^{n+1} = \frac{A}{C}v_{i-1}^{n+1} + \frac{B}{C}v_{i+1}^{n+1} + \frac{F_i^n}{C} \tag{3.53}$$

and take $i = N-1$:

$$v_{N-1}^{n+1} = \frac{A}{C}v_{N-2}^{n+1} + \frac{B}{C}v_N^{n+1} + \frac{F_{N-1}^n}{C}.$$

Substituting (3.52) in this and rearranging terms yields

$$v_{N-1}^{n+1} = \varsigma_{N-1}v_{N-2}^{n+1} + \eta_{N-1}^n \tag{3.54}$$

with

$$\varsigma_{N-1} = \frac{A}{C - B\varsigma_N}, \ \eta_{N-1}^n = \frac{B\eta_N^n + F_{N-1}^n}{C - B\varsigma_N}.$$

In the next step we take the expression (3.53) for $i = N-2$ and replace v_{N-1}^{n+1} by (3.54). So, in the end we get an expression for v_{N-2}^{n+1}.

Doing the same procedure for the indices $i = N-3,\ldots,1$, it follows that

$$v_{i+1}^{n+1} = \varsigma_{i+1}v_i^{n+1} + \eta_{i+1}^n, \quad i = 0,\ldots,N-1, \tag{3.55}$$

with coefficients (3.51) and

$$\varsigma_i = \frac{A}{C - B\varsigma_{i+1}}, \ \eta_i^n = \frac{B\eta_{i+1}^n + F_i^n}{C - B\varsigma_{i+1}}, \ i = N-1,\ldots,1.$$

Having used the recurrence relations to find the coefficients, the values of v_i^{n+1} are easily obtained from (3.55). The only problem here is that we don't know v_0^{n+1} yet. We'll find it in due course.

3.2.3.2 Difference Scheme for the Right Part of the Base

The differential equation for the right part of the base (3.37) is approximated by the scheme

$$\frac{v_{0,j}^{n+1} - v_{0,j}^n}{\tilde{a}_0^2\tau} = \sigma_2\frac{v_{0,j+1}^{n+1} - 2v_{0,j}^{n+1} + v_{0,j-1}^{n+1}}{h_{y,2}^2} + (1-\sigma_2)\frac{v_{0,j+1}^n - 2v_{0,j}^n + v_{0,j-1}^n}{h_{y,2}^2}$$

$$- \sigma_2\kappa^2 v_{0,j}^{n+1} - (1-\sigma_2)\kappa^2 v_{0,j}^n - \gamma Q_0,$$

where $i = M_0 + 1,\ldots,M-1$ and $0 \le \sigma_2 \le 1$, which we rewrite as

$$A_2 v_{0,j-1}^{n+1} - C_2 v_{0,j}^{n+1} + B_2 v_{0,j+1}^{n+1} = -F_{2,j}^n,$$

using following notations:

$$A_2 = \frac{\sigma_2}{h_{y,2}^2} = B_2, \quad C_2 = \frac{2\sigma_2}{h_{y,2}^2} + \kappa^2 \sigma_2 + \frac{1}{\tilde{a}_0^2 \tau},$$

$$F_{2,j}^n = (1-\sigma_2) \frac{v_{0,j+1}^n - 2v_{0,j}^n + v_{0,j-1}^n}{h_{y,2}^2} - \kappa^2(1-\sigma_2)v_{0,j}^n + \frac{v_{0,j}^n}{\tilde{a}_0^2 \tau} - \gamma Q_0. \tag{3.56}$$

When approximating the boundary condition (3.38), we have

$$\sigma_2 \frac{v_{0,M}^{n+1} - v_{0,M-1}^{n+1}}{h_{y,2}} + (1-\sigma_2) \frac{v_{0,M}^n - v_{0,M-1}^n}{h_{y,2}}$$

$$+ \frac{h_{y,2}}{2} \left(\frac{v_{0,M}^{n+1} - v_{0,M}^n}{\tilde{a}_0^2 \tau} + \sigma_2 \kappa^2 v_{0,M}^{n+1} + (1-\sigma_2)\kappa^2 v_{0,M}^n + \gamma Q_0 \right) = 0$$

or

$$v_{0,M}^{n+1} = \varpi_{2,1} v_{0,M-1}^{n+1} + \varpi_{2,2}^n$$

with

$$\varpi_{2,1} = \frac{\sigma_2}{h_{y,2}} \left(\frac{\sigma_2}{h_{y,2}} + \frac{h_{y,2}}{2} \left(\sigma_2 \kappa^2 + \frac{1}{\tilde{a}_0^2 \tau} \right) \right)^{-1},$$

$$\varpi_{2,2}^n = -\left(\frac{\sigma_2}{h_{y,2}} + \frac{h_{y,2}}{2} \left(\sigma_2 \kappa^2 + \frac{1}{\tilde{a}_0^2 \tau} \right) \right)^{-1}$$

$$\left((1-\sigma_2) \frac{v_{0,M}^n - v_{0,M-1}^n}{h_{y,2}} + \frac{h_{y,2}}{2}(1-\sigma_2)\kappa^2 v_{0,M}^n - \frac{h_{y,2}}{2} \left(\frac{v_{0,M}^n}{\tilde{a}_0^2 \tau} - \gamma Q_0 \right) \right).$$

On carrying through the similar analysis as for the fin, the difference scheme for the right part of the base can be expressed as

$$v_{0,j+1}^{n+1} = \varsigma_{2,j+1} v_{0,j}^{n+1} + \eta_{2,j+1}^n, \quad j = M_0, \ldots, M-1, \tag{3.57}$$

with

$$\varsigma_{2,M} = \varpi_{2,1}, \quad \eta_{2,M}^n = \varpi_{2,2}^n,$$

$$\varsigma_{2,j} = \frac{A_2}{C_2 - B_2 \varsigma_{2,j+1}}, \quad \eta_{2,j}^n = \frac{B_2 \eta_{2,j+1}^n + F_{2,j}^n}{C_2 - B_2 \varsigma_{2,j+1}}$$

for $j = M-1, \ldots, M_0+1$.

3.2.3.3 Difference Scheme for the Left Part of the Base

The scheme for (3.42) is extended in this way to give

$$\frac{v_{0,j}^{n+1} - v_{0,j}^n}{\tilde{a}_0^2 \tau} = \sigma_1 \frac{v_{0,j+1}^{n+1} - 2v_{0,j}^{n+1} + v_{0,j-1}^{n+1}}{h_{y,1}^2}$$

$$+ (1-\sigma_1) \frac{v_{0,j+1}^n - 2v_{0,j}^n + v_{0,j-1}^n}{h_{y,1}^2} + dQ_0$$

$$+ d \left(\sigma \frac{v_1^{n+1} - v_0^{n+1}}{h_x} + (1-\sigma) \frac{v_1^n - v_0^n}{h_x} \right) \Phi_j$$

43

$$-\frac{h_x}{2}d\left(\frac{v_0^{n+1}-v_0^n}{\tilde{a}_0^2\tau}+\sigma\lambda^2 v_0^{n+1}+(1-\sigma)\lambda^2 v_0^n\right)\Phi_j.\tag{3.58}$$

It depends on the values of v_0^{n+1} and v_1^{n+1}. The latter can be computed from (3.55):

$$v_1^{n+1}=\varsigma_1 v_0^{n+1}+\eta_1^n.\tag{3.59}$$

Substituting (3.59) into (3.58), the difference scheme results in

$$A_1 v_{0,j-1}^{n+1}-C_1 v_{0,j}^{n+1}+B_1 v_{0,j+1}^{n+1}+D_{1,j}v_0^{n+1}=-F_{1,j}^n,\tag{3.60}$$

where

$$A_1=\frac{\sigma_1}{h_{y,1}^2}=B_1,\ C_1=\frac{2\sigma_1}{h_{y,1}^2}+\frac{1}{\tilde{a}_0^2\tau},$$

$$D_{1,j}=-d\left(\frac{\sigma}{h_x}(1-\varsigma_1)+\frac{h_x}{2}\left(\frac{1}{\tilde{a}_0^2\tau}+\sigma\lambda^2\right)\right)\Phi_j,$$

$$F_{1,j}^n=\frac{v_{0,j}^n}{\tilde{a}_0^2\tau}+(1-\sigma_1)\frac{v_{0,j+1}^n-2v_{0,j}^n+v_{0,j-1}^n}{h_{y,1}^2}$$

$$+d\left(\frac{\sigma}{h_x}\eta_1^n+(1-\sigma)\frac{v_1^n-v_0^n}{h_x}-\frac{h_x}{2}\left(-\frac{v_0^n}{\tilde{a}_0^2\tau}+(1-\sigma)\lambda^2 v_0^n\right)\right)\Phi_j+dQ_0.$$

The boundary condition (3.43) has the following finite difference representation:

$$\sigma_1\frac{v_{0,1}^{n+1}-v_{0,0}^{n+1}}{h_{y,1}}+(1-\sigma_1)\frac{v_{0,1}^n-v_{0,0}^n}{h_{y,1}}$$

$$+d\frac{h_{y,1}}{2}\left(\sigma\frac{v_1^{n+1}-v_0^{n+1}}{h_x}+(1-\sigma)\frac{v_1^n-v_0^n}{h_x}-\frac{h_x}{2}\left(\frac{v_0^{n+1}-v_0^n}{\tilde{a}_0^2\tau}+\sigma\lambda^2 v_0^{n+1}+(1-\sigma)\lambda^2 v_0^n\right)\right)\Phi_0$$

$$-\frac{h_{y,1}}{2}\frac{v_{0,0}^{n+1}-v_{0,0}^n}{\tilde{a}_0^2\tau}+\frac{h_{y,1}}{2}dQ_0=0.$$

This can be rewritten as

$$v_{0,0}^{n+1}=\varpi_{1,1}v_{0,1}^{n+1}-\varpi_{1,3}v_0^{n+1}+\mu_1^n$$

using

$$\varpi_{1,1}=\frac{\sigma_1}{h_{y,1}}\left(\frac{\sigma_1}{h_{y,1}}+\frac{h_{y,1}}{2a_0^2\tau}\right)^{-1},$$

$$\varpi_{1,3}=\left(\frac{\sigma_1}{h_{y,1}}+\frac{h_{y,1}}{2\tilde{a}_0^2\tau}\right)^{-1}d\frac{h_{y,1}}{2}\left(\frac{\sigma}{h_x}(1-\varsigma_1)+\frac{h_x}{2}\left(\frac{1}{\tilde{a}_0^2\tau}+\sigma\lambda^2\right)\right)\Phi_0,$$

$$\mu_1^n=\left(\frac{\sigma_1}{h_{y,1}}+\frac{h_{y,1}}{2\tilde{a}_0^2\tau}\right)^{-1}\times$$

$$\times\left(\begin{array}{l}(1-\sigma_1)\dfrac{v_{0,1}^n-v_{0,0}^n}{h_{y,1}}+\dfrac{h_{y,1}}{2}dQ_0+\dfrac{h_{y,1}}{2}\dfrac{v_{0,0}^n}{\tilde{a}_0^2\tau}\\[2mm]+d\dfrac{h_{y,1}}{2}\left(\dfrac{\sigma}{h_x}\eta_1^n+(1-\sigma)\dfrac{v_1^n-v_0^n}{h_x}-\dfrac{h_x}{2}\left(-\dfrac{v_0^n}{\tilde{a}_0^2\tau}+(1-\sigma)\lambda^2 v_0^n\right)\right)\Phi_0\end{array}\right).$$

After some substitutions and algebraic manipulations the difference scheme (3.60) for equation (3.42) becomes

$$v_{0,j}^{n+1} = \xi_{j+1} v_{0,j+1}^{n+1} - \psi_{j+1} v_0^{n+1} + \chi_{j+1}^n, \quad j = M_0 - 1, \ldots, 0, \tag{3.61}$$

with coefficients

$$\xi_1 = \varpi_{1,1}, \ \psi_1 = \varpi_{1,3}, \ \chi_1^n = \mu_1^n,$$

and

$$\xi_{j+1} = \frac{B_1}{C_1 - A_1 \xi_j}, \ \psi_{j+1} = \frac{A_1 \psi_j - D_{1,j}}{C_1 - A_1 \xi_j}, \ \chi_{j+1}^n = \frac{A_1 \chi_j^n + F_{1,j}^n}{C_1 - A_1 \xi_j}$$

for $j = 1, \ldots, M_0 - 1$.

3.2.3.4   Conjugation of Solutions

We see that the schemes (3.55), (3.57) and (3.61) can only be used once the values of v_0^{n+1} and v_{0,M_0}^{n+1} are found. They could be obtained from these requirements:

$$V(0, b - 0, t) = V_0(0, b + 0, t), \tag{3.62}$$

$$\left. \frac{\partial v_0}{\partial y} \right|_{y=b-0} = \left. \frac{\partial v_0}{\partial y} \right|_{y=b+0}, \tag{3.63}$$

that state that the temperatures $V_0(x, y, t)$, $V(x, y, t)$ and the average heat fluxes must coincide at the contact point $x = 0$, $y = b$ between the fin and the right part of the wall. So, from (3.25) and (3.36) it follows that

$$V(0, b, t) = v(0, t) \Phi(b),$$
$$V_0^r(0, b, t) = a_0 v_0^r(b, t) + b_0 Q_0.$$

If Φ_{M_0} represents the approximation to $\Phi(y)$ at the point y_{M_0}, then (3.62) becomes

$$v_0^{n+1} \Phi_{M_0} = a_0 v_{0,M_0}^{n+1} + b_0 Q_0.$$

In other words,

$$v_0^{n+1} = \frac{a_0}{\Phi_{M_0}} v_{0,M_0}^{n+1} + \frac{b_0 Q_0}{\Phi_{M_0}}. \tag{3.64}$$

But for the condition (3.63) we have

$$\sigma_1 \frac{v_{0,M_0}^{n+1} - v_{0,M_0-1}^{n+1}}{h_{y,1}} + (1 - \sigma_1) \frac{v_{0,M_0}^n - v_{0,M_0-1}^n}{h_{y,1}} + \frac{h_{y,1}}{2} \left(\frac{v_{0,M_0}^{n+1} - v_{0,M_0}^n}{\tilde{a}_0^2 \tau} - dQ_0 \right)$$

$$-d \frac{h_{y,1}}{2} \left(\sigma \frac{v_1^{n+1} - v_0^{n+1}}{h_x} + (1 - \sigma) \frac{v_1^n - v_0^n}{h_x} - \frac{h_x}{2} \left(\begin{array}{c} \frac{v_0^{n+1} - v_0^n}{\tilde{a}_0^2 \tau} + \sigma \lambda^2 v_0^{n+1} \\ + (1 - \sigma) \lambda^2 v_0^n \end{array} \right) \right) \Phi_{M_0} =$$

$$= \sigma_2 \frac{v_{0,M_0+1}^{n+1} - v_{0,M_0}^{n+1}}{h_{y,2}} + (1 - \sigma_2) \frac{v_{0,M_0+1}^n - v_{0,M_0}^n}{h_{y,2}}$$

$$- \frac{h_{y,2}}{2} \left(\frac{v_{0,M_0}^{n+1} - v_{0,M_0}^n}{\tilde{a}_0^2 \tau} + \sigma_2 \kappa^2 v_{0,M_0}^{n+1} + (1 - \sigma_2) \kappa^2 v_{0,M_0}^n + \gamma Q_0 \right). \tag{3.65}$$

According to formulas (3.55), (3.57) and (3.61) when $i = 0$, $j = M_0$ and $j = M_0 - 1$, respectively, this is true:

$$v_1^{n+1} = \varsigma_1 v_0^{n+1} + \eta_1^n,$$
$$v_{0,M_0+1}^{n+1} = \varsigma_{2,M_0+1} v_{0,M_0}^{n+1} + \eta_{2,M_0+1}^n,$$

$$v_{0,M_0-1}^{n+1} = \xi_{M_0} v_{0,M_0}^{n+1} - \psi_{M_0} v_0^{n+1} + \chi_{M_0}^n.$$

Let us insert these into (3.65) and thereafter compute

$$H_1 v_{0,M_0}^{n+1} + J_1 v_0^{n+1} = -G_1^n,$$ (3.66)

where

$$H_1 = \frac{\sigma_1}{h_{y,1}}\left(1 - \xi_{M_0}\right) + \frac{h_{y,1}}{2\tilde{a}_0^2 \tau} + \frac{\sigma_2}{h_{y,2}}\left(1 - \varsigma_{2,M_0+1}\right) + \frac{h_{y,2}}{2}\left(\frac{1}{\tilde{a}_0^2 \tau} + \sigma_2 \kappa^2\right),$$

$$J_1 = \frac{\sigma_1}{h_{y,1}} \psi_{M_0} + d\,\frac{h_{y,1}}{2}\left(\frac{\sigma}{h_x}\left(1 - \varsigma_1\right) + \frac{h_x}{2}\left(\frac{1}{\tilde{a}_0^2 \tau} + \sigma\lambda^2\right)\right)\Phi_{M_0},$$

$$G_1^n = \left(1 - \sigma_1\right)\frac{v_{0,M_0}^n - v_{0,M_0-1}^n}{h_{y,1}} - \frac{\sigma_1}{h_{y,1}} \chi_{M_0}^n$$

$$+ \frac{h_{y,1}}{2}\left(\begin{array}{c} -\dfrac{v_{0,M_0}^n}{\tilde{a}_0^2 \tau} - dQ_0 \\[2mm] -d\left(\sigma\dfrac{1}{h_x}\eta_1^n + (1-\sigma)\dfrac{v_1^n - v_0^n}{h_x} - \dfrac{h_x}{2}\left(-\dfrac{v_0^n}{\tilde{a}_0^2 \tau} + (1-\sigma)\lambda^2 v_0^n\right)\right)\Phi_{M_0} \end{array}\right)$$

$$- \frac{\sigma_2}{h_{y,2}}\eta_{2,M_0+1}^n - (1-\sigma_2)\frac{v_{0,M_0+1}^n - v_{0,M_0}^n}{h_{y,2}} + \frac{h_{y,2}}{2}\left(-\frac{v_{0,M_0}^n}{\tilde{a}_0^2 \tau} + (1-\sigma_2)\kappa^2 v_{0,M_0}^n + \gamma Q_0\right).$$

It is easy to eliminate v_0^{n+1} from (3.66) with (3.64):

$$H_1 v_{0,M_0}^{n+1} + J_1 \frac{a_0}{\Phi_{M_0}} v_{0,M_0}^{n+1} = -J_1 \frac{b_0 Q_0}{\Phi_{M_0}} - G_1^n.$$ (3.67)

Solving this equation for v_{0,M_0}^{n+1}, we get

$$v_{0,M_0}^{n+1} = -\frac{J_1 b_0 Q_0 + \Phi_{M_0} G_1^n}{H_1 \Phi_{M_0} + J_1 a_0}.$$ (3.68)

Substitution of this into the expression (3.64) results in

$$v_0^{n+1} = -\frac{a_0}{\Phi_{M_0}} \cdot \frac{J_1 b_0 Q_0 + \Phi_{M_0} G_1^n}{H_1 \Phi_{M_0} + J_1 a_0} + \frac{b_0 Q_0}{\Phi_{M_0}}.$$ (3.69)

Then given the resulting values for v_0^{n+1} and v_{0,M_0}^{n+1}, equations (3.55), (3.57), (3.61) together with approximations of the initial conditions (3.28), (3.39) that give the values for all nodes at the first time level,

$$v_i^0 = u_i^0,\ i = 0,\ldots,N,$$
$$v_{0,j}^0 = u_{0,j}^0,\ j = 0,\ldots,M,$$

constitute the approximate solution of the given system in 1D.

3.3 Numerical Results

Here we used the same parameters as in Chapter 2. The geometrical parameters:

$$\delta = 5\,\mu m,$$
$$l = 1\,\mu m,$$
$$b = 5 \cdot 10^{-2}\,\mu m,$$

$l_0 = 1 \cdot 10^{-1} \, \mu m$.

And the termophysical properties:

$h_1 = 4.48 \cdot 10^{-7} \, W \mu m^{-2} K^{-1}$,

$k_0 = 1.412 \cdot 10^{-4} \, W \mu m^{-1} K^{-1}$ (for silicon),

$c_0 = 700 \, Jkg^{-1} K^{-1}$ (for silicon),

$\tilde{\rho}_0 = 2.33 \cdot 10^{-15} \, kg \mu m^{-3}$ (for silicon).

Additional conditions:

$m = 3$,

$Q_0 = 10 K \mu m^{-1}$,

$u^0(x) = 25°C$, $u_0^0(y) = 35°C$,

$T = 8s$.

x\y	0	0.01	0.02	0.03	0.04	0.05
1	9.80	9.80	9.80	9.79	9.79	9.78
0.9	9.98	9.98	9.98	9.97	9.97	9.96
0.8	10.34	10.34	10.34	10.33	10.33	10.32
0.7	10.88	10.88	10.88	10.87	10.87	10.86
0.6	11.60	11.60	11.60	11.59	11.59	11.58
0.5	12.50	12.50	12.50	12.49	12.49	12.48
0.4	13.58	13.58	13.58	13.57	13.57	13.56
0.3	14.84	14.84	14.84	14.84	14.83	14.82
0.2	16.29	16.29	16.29	16.28	16.28	16.27
0.1	17.92	17.92	17.92	17.91	17.91	17.90
0	19.73	19.73	19.73	19.73	19.72	19.71

Table 3.1 Temperature distribution in the fin when $t=7.5$ s, in °C

x\y	0	0.01	0.02	0.03	0.04	0.05
0	19.73	19.73	19.73	19.73	19.72	19.71
-0.5	20.37	20.37	20.37	20.37	20.37	20.36
-1	21.39	21.39	21.39	21.38	21.38	21.38
-1.5	22.78	22.78	22.78	22.78	22.78	22.78
-2	24.57	24.57	24.57	24.57	24.57	24.57
-2.5	26.78	26.78	26.78	26.78	26.78	26.78
-3	29.42	29.42	29.42	29.42	29.43	29.43
-3.5	32.53	32.53	32.53	32.53	32.53	32.54
-4	36.12	36.12	36.13	36.13	36.13	36.14
-4.5	40.25	40.25	40.25	40.26	40.26	40.27
-5	44.95	44.95	44.95	44.96	44.96	44.96

Table 3.2 Temperature distribution in the left part of the base when $t=7.5$ s, in °C

x\y	0.05	0.06	0.07	0.08	0.09	0.1
0	19.71	19.71	19.71	19.71	19.71	19.71
-0.5	20.36	20.36	20.36	20.36	20.36	20.36
-1	21.38	21.38	21.38	21.38	21.38	21.38
-1.5	22.78	22.78	22.78	22.78	22.78	22.78
-2	24.57	24.58	24.58	24.58	24.58	24.58
-2.5	26.78	26.79	26.79	26.79	26.79	26.79

x\y	0.05	0.06	0.07	0.08	0.09	0.1
-3	29.43	29.43	29.43	29.43	29.43	29.43
-3.5	32.54	32.54	32.54	32.54	32.54	32.54
-4	36.14	36.14	36.14	36.14	36.14	36.14
-4.5	40.27	40.27	40.27	40.27	40.27	40.27
-5	44.96	44.97	44.97	44.97	44.97	44.97

Table 3.3 Temperature distribution in the right part of the base when $t=7.5$ s, in °C

From the tables (see, Table 2.1 − Table 2.3, Table 3.1 − Table 3.3) we can see that at the end of the process there isn't much difference between the transient case and the stationary one. Actually, from the numerical results we got in both cases we can conclude that the transient temperature tends to the stationary one.

3.4 Approximate Solution of the Problem with Partial Boiling

Just like in the stationary case, for partial boiling we have this boundary condition along the line $y = b + \varepsilon_1$:

$$\left(\frac{\partial V_1}{\partial y} + \beta_1^1 V_1^m\right)\Bigg|_{y=b+\varepsilon_1} = 0, \tag{3.70}$$

But the rest of the formulation of the problem remains unchanged.

3.4.1 Numerical Solution for the Fin

Once again using conservative averaging method and approximating the function $V(x,y,t)$ by its mean value over the interval $[0,b]$, namely,

$$V(x, y, t) \approx v(x, t),$$

we get the following 1D problem:

$$\frac{\partial^2 v}{\partial x^2} - \lambda^2 v^m(x,t) = \frac{1}{\tilde{a}_0^2}\frac{\partial v}{\partial t}, \tag{3.71}$$

$$\left(\frac{\partial v}{\partial x} + \beta_0^1 v\right)\Bigg|_{x=l} = 0, \tag{3.72}$$

$$v(x,0) = u^0(x). \tag{3.73}$$

As the problem is non-linear, for finding approximate solution to this problem we are going to use iterations. We use the same discrete grid as defined in subsection 3.2.3 by (3.44) - (3.46).

So, the equation (3.71) is approximated by

$$\frac{v_i^{n+1,k} - v_i^n}{\tilde{a}_0^2 \tau} = \sigma\frac{v_{i+1}^{n+1,k} - 2v_i^{n+1,k} + v_{i-1}^{n+1,k}}{h_x^2} + (1-\sigma)\frac{v_{i+1}^n - 2v_i^n + v_{i-1}^n}{h_x^2}$$

$$- \sigma\lambda^2\left(v_i^{n+1,k-1}\right)^{m-1}v_i^{n+1,k} - (1-\sigma)\lambda^2\left(v_i^n\right)^m, \quad i = 1,\ldots,N-1. \tag{3.74}$$

When the process at the k-th time level has become stable, we don't use the iteration number any more. Obviously, (3.74) can be rewritten as

$$Av_{i-1}^{n+1,k} - C_i^{n+1,k-1}v_i^{n+1,k} + Bv_{i+1}^{n+1,k} = -F_i^n$$

with

$$A = \frac{\sigma}{h_x^2} = B, \quad C_i^{n+1,k-1} = \frac{2\sigma}{h_x^2} + \lambda^2\sigma\left(v_i^{n+1,k-1}\right)^{m-1} + \frac{1}{\tilde{a}_0^2\tau},$$

$$F_i^n = (1-\sigma)\frac{v_{i+1}^n - 2v_i^n + v_{i-1}^n}{h_x^2} - \lambda^2(1-\sigma)(v_i^n)^m + \frac{v_i^n}{\tilde{a}_0^2 \tau}.$$

The boundary condition (3.72) is handled much as was done in the linear case. That is,

$$\sigma\frac{v_N^{n+1,k} - v_{N-1}^{n+1,k}}{h_x} + (1-\sigma)\frac{v_N^n - v_{N-1}^n}{h_x} + \frac{h_x}{2}\left(\frac{v_N^{n+1,k} - v_N^n}{\tilde{a}_0^2 \tau}\right)$$

$$+ \frac{h_x}{2}\left(\sigma\lambda^2(v_N^{n+1,k-1})^{m-1} v_N^{n+1,k} + (1-\sigma)\lambda^2(v_N^n)^m\right) + \sigma\beta_0^1 v_N^{n+1,k} + (1-\sigma)\beta_0^1 v_N^n = 0. \tag{3.75}$$

Taking this all into account, an approximate description of the problem for the fin is given by the system of equations

$$v_{i+1}^{n+1,k} = \varsigma_{i+1}^{n+1,k-1} v_i^{n+1,k} + \eta_{i+1}^{n,k-1}, \quad i = 0,\ldots,N-1. \tag{3.76}$$

with

$$\varsigma_i^{n+1,k-1} = \frac{A}{C_i^{n+1,k-1} - B\varsigma_{i+1}^{n+1,k-1}}, \quad \eta_i^{n,k-1} = \frac{B\eta_{i+1}^{n,k-1} + F_i^n}{C_i^{n+1,k-1} - B\varsigma_{i+1}^{n+1,k-1}}, \quad i = N-1,\ldots,1$$

and

$$\varsigma_N^{n+1,k-1} = \varpi_1^{n+1,k-1}, \quad \eta_N^{n,k-1} = \varpi_2^{n,k-1},$$

where,

$$\varpi_1^{n+1,k-1} = \frac{\sigma}{h_x}\left(\frac{\sigma}{h_x} + \sigma\beta_0^1 + \frac{h_x}{2}\left(\frac{1}{\tilde{a}_0^2 \tau} + \sigma\lambda^2(v_N^{n+1,k-1})^{m-1}\right)\right)^{-1},$$

$$\varpi_2^{n,k-1} = \left(\frac{\sigma}{h_x} + \sigma\beta_0^1 + \frac{h_x}{2}\left(\frac{1}{\tilde{a}_0^2 \tau} + \sigma\lambda^2(v_N^{n+1,k-1})^{m-1}\right)\right)^{-1} \times$$

$$\left(\frac{h_x}{2}\frac{v_N^n}{\tilde{a}_0^2 \tau} - (1-\sigma)\left(\frac{v_N^n - v_{N-1}^n}{h_x} + \frac{h_x}{2}\lambda^2(v_N^n)^m + \beta_0^1 v_N^n\right)\right).$$

And, of course, the initial values

$$v_i^0 = u_i^0, \quad i = 0,\ldots,N.$$

Once the values v_0^{n+1} have been found (see subsection 3.4.2.3) the approximations v_i^{n+1} can be calculated by recursively applying formula (3.76).

3.4.2   Numerical Solution for the Base

The problem for the base in partial boiling is the same as in subsections 3.1, 3.2, when all the boundary conditions were linear. So, the formulae (3.29) − (3.43) do not change. We just make a slight modification by taking $\Phi(y) = 1$.

3.4.2.1.   Difference Scheme for the Right Part of the Base

Let's construct a discrete approximation of the differential equation for the right part of the wall:

$$\frac{v_{0,j}^{n+1,k} - v_{0,j}^n}{\tilde{a}_0^2 \tau} = \sigma_2\frac{v_{0,j+1}^{n+1,k} - 2v_{0,j}^{n+1,k} + v_{0,j-1}^{n+1,k}}{h_{y,2}^2} + (1-\sigma_2)\frac{v_{0,j+1}^n - 2v_{0,j}^n + v_{0,j-1}^n}{h_{y,2}^2}$$

$$- \sigma_2\kappa^2 v_{0,j}^{n+1,k} - (1-\sigma_2)\kappa^2 v_{0,j}^n - \gamma Q_0, \quad i = M_0+1,\ldots,M-1 \tag{3.77}$$

and for the boundary condition (3.38):

$$\sigma_2 \frac{v_{0,M}^{n+1,k} - v_{0,M-1}^{n+1,k}}{h_{y,2}} + \left(1 - \sigma_2\right) \frac{v_{0,M}^{n} - v_{0,M-1}^{n}}{h_{y,2}}$$

$$+ \frac{h_{y,2}}{2} \left(\frac{v_{0,M}^{n+1,k} - v_{0,M}^{n}}{\tilde{a}_0^2 \tau} + \sigma_2 \kappa^2 v_{0,M}^{n+1,k} + (1 - \sigma_2) \kappa^2 v_{0,M}^{n} + \gamma Q_0 \right) = 0. \qquad (3.78)$$

In summary, the difference approximation to the problem for the right part of the base takes the form

$$v_{0,j+1}^{n+1,k} = \varsigma_{2,j+1} v_{0,j}^{n+1,k} + \eta_{2,j+1}^{n}, \quad j = M_0, \ldots, M-1, \qquad (3.79)$$

where

$$\varsigma_{2,M} = \varpi_{2,1}, \quad \eta_{2,M}^{n} = \varpi_{2,2}^{n}, \quad \varsigma_{2,j} = \frac{A_2}{C_2 - B_2 \varsigma_{2,j+1}}, \quad \eta_{2,j}^{n} = \frac{B_2 \eta_{2,j+1}^{n} + F_{2,j}^{n}}{C_2 - B_2 \varsigma_{2,j+1}}$$

for $j = M-1, \ldots, M_0 + 1$. And

$$A_2 = \frac{\sigma_2}{h_{y,2}^2} = B_2, \quad C_2 = \frac{2\sigma_2}{h_{y,2}^2} + \kappa^2 \sigma_2 + \frac{1}{\tilde{a}_0^2 \tau},$$

$$F_{2,j}^{n} = (1 - \sigma_2) \frac{v_{0,j+1}^{n} - 2v_{0,j}^{n} + v_{0,j-1}^{n}}{h_{y,2}^2} - \kappa^2 (1 - \sigma_2) v_{0,j}^{n} + \frac{v_{0,j}^{n}}{\tilde{a}_0^2 \tau} - \gamma Q_0,$$

$$\varpi_{2,1} = \frac{\sigma_2}{h_{y,2}} \left(\frac{\sigma_2}{h_{y,2}} + \frac{h_{y,2}}{2} \left(\sigma_2 \kappa^2 + \frac{1}{\tilde{a}_0^2 \tau} \right) \right)^{-1},$$

$$\varpi_{2,2}^{n} = -\left(\frac{\sigma_2}{h_{y,2}} + \frac{h_{y,2}}{2} \left(\sigma_2 \kappa^2 + \frac{1}{\tilde{a}_0^2 \tau} \right) \right)^{-1} \times$$

$$\left((1 - \sigma_2) \frac{v_{0,M}^{n} - v_{0,M-1}^{n}}{h_{y,2}} + \frac{h_{y,2}}{2} \left((1 - \sigma_2) \kappa^2 v_{0,M}^{n} - \frac{v_{0,M}^{n}}{\tilde{a}_0^2 \tau} + \gamma Q_0 \right) \right).$$

But for (3.39):

$$v_{0,j}^{0} = u_{0,j}^{0}, \quad j = 0, \ldots, M.$$

3.4.2.2. Difference Scheme for the Left Part of the Base

The equation for the left part of the wall reduces to the discrete equation

$$\frac{v_{0,j}^{n+1,k} - v_{0,j}^{n}}{\tilde{a}_0^2 \tau} = \sigma_1 \frac{v_{0,j+1}^{n+1,k} - 2v_{0,j}^{n+1,k} + v_{0,j-1}^{n+1,k}}{h_{y,1}^2} + (1 - \sigma_1) \frac{v_{0,j+1}^{n} - 2v_{0,j}^{n} + v_{0,j-1}^{n}}{h_{y,1}^2} + dQ_0$$

$$+ d\left(\sigma \frac{v_1^{n+1,k} - v_0^{n+1,k}}{h_x} + (1 - \sigma) \frac{v_1^{n} - v_0^{n}}{h_x} \right)$$

$$- \frac{h_x}{2} d\left(\frac{v_0^{n+1,k} - v_0^{n}}{\tilde{a}_0^2 \tau} + \sigma \lambda^2 \left(v_0^{n+1,k-1} \right)^{m-1} v_0^{n+1,k} + (1 - \sigma) \lambda^2 \left(v_0^{n} \right)^{m} \right) \qquad (3.80)$$

for $i = M_0 + 1, \ldots, M-1$.

Whereas the discrete boundary condition is

$$\sigma_1 \frac{v_{0,1}^{n+1,k} - v_{0,0}^{n+1,k}}{h_{y,1}} + (1 - \sigma_1) \frac{v_{0,1}^{n} - v_{0,0}^{n}}{h_{y,1}}$$

$$-\frac{h_{y,1}}{2}\frac{v_{0,0}^{n+1,k}-v_{0,0}^{n}}{\tilde{a}_0^2\tau}+\frac{h_{y,1}}{2}dQ_0+d\frac{h_{y,1}}{2}\left(\sigma\frac{v_1^{n+1,k}-v_0^{n+1,k}}{h_x}+(1-\sigma)\frac{v_1^{n}-v_0^{n}}{h_x}\right)$$

$$+d\frac{h_{y,1}}{2}\left(-\frac{h_x}{2}\left(\frac{v_0^{n+1,k}-v_0^{n}}{\tilde{a}_0^2\tau}+\sigma\lambda^2\left(v_0^{n+1,k-1}\right)^{m-1}v_0^{n+1,k}+(1-\sigma)\lambda^2\left(v_0^{n}\right)^{m}\right)\right)=0. \quad (3.81)$$

The above discretization of the problem for the left part of the base produces

$$v_{0,j}^{n+1,k}=\xi_{j+1}v_{0,j+1}^{n+1,k}-\psi_{j+1}^{n+1,k-1}v_0^{n+1,k}+\chi_{j+1}^{n,k-1}, \quad j=M_0-1,\ldots,0 \quad (3.82)$$

with

$$\xi_1=\varpi_{1,1}, \quad \psi_1^{n+1,k-1}=\varpi_{1,3}^{n+1,k-1}, \quad \chi_1^{n,k-1}=\mu_1^{n,k-1}$$

and

$$\xi_{j+1}=\frac{B_1}{C_1-A_1\xi_j}, \quad \psi_{j+1}^{n+1,k-1}=\frac{A_1\psi_j^{n+1,k-1}-D_{1,j}^{n+1,k-1}}{C_1-A_1\xi_j}, \quad \chi_{j+1}^{n,k-1}=\frac{A_1\chi_j^{n,k-1}+F_{1,j}^{n,k-1}}{C_1-A_1\xi_j},$$

for $j=1..M_0-1$. Here we use the notation

$$A_1=\frac{\sigma_1}{h_{y,1}^2}=B_1, \quad C_1=\frac{2\sigma_1}{h_{y,1}^2}+\frac{1}{\tilde{a}_0^2\tau},$$

$$D_1^{n+1,k-1}=-d\left(\frac{\sigma}{h_x}+\frac{h_x}{2}\left(\frac{1}{\tilde{a}_0^2\tau}+\sigma\lambda^2\left(v_0^{n+1,k-1}\right)^{m-1}\right)-\frac{\sigma}{h_x}\varsigma_1^{n+1,k-1}\right),$$

$$F_{1,j}^{n,k-1}=\frac{v_{0,j}^{n}}{\tilde{a}_0^2\tau}+(1-\sigma_1)\frac{v_{0,j+1}^{n}-2v_{0,j}^{n}+v_{0,j-1}^{n}}{h_{y,1}^2}+dQ_0,$$

$$+d\left(\sigma\frac{\eta_1^{n+1,k-1}}{h_x}+(1-\sigma)\frac{v_1^{n}-v_0^{n}}{h_x}-\frac{h_x}{2}\left(-\frac{v_0^{n}}{\tilde{a}_0^2\tau}+(1-\sigma)\lambda^2\left(v_0^{n}\right)^{m}\right)\right),$$

$$\varpi_{1,1}=\frac{\sigma_1}{h_{y,1}}\left(\frac{\sigma_1}{h_{y,1}}+\frac{h_{y,1}}{2\tilde{a}_0^2\tau}\right)^{-1},$$

$$\varpi_{1,3}^{n+1,k-1}=\left(\frac{\sigma_1}{h_{y,1}}+\frac{h_{y,1}}{2\tilde{a}_0^2\tau}\right)^{-1}d\frac{h_{y,1}}{2}\left(\frac{\sigma}{h_x}\left(1-\varsigma_1^{n+1,k-1}\right)+\frac{h_x}{2}\left(\frac{1}{\tilde{a}_0^2\tau}+\sigma\lambda^2\left(v_0^{n+1,k-1}\right)^{m-1}\right)\right),$$

$$\mu_1^{n,k-1}=\left(\frac{\sigma_1}{h_{y,1}}+\frac{h_{y,1}}{2\tilde{a}_0^2\tau}\right)^{-1}\times$$

$$\left(\begin{array}{l}(1-\sigma_1)\dfrac{v_{0,1}^{n}-v_{0,0}^{n}}{h_{y,1}}+\dfrac{h_{y,1}}{2}dQ_0+\dfrac{h_{y,1}}{2}\dfrac{v_{0,0}^{n}}{\tilde{a}_0^2\tau}\\[2em]+d\dfrac{h_{y,1}}{2}\left(\dfrac{\sigma}{h_x}\eta_1^{n,k-1}+(1-\sigma)\dfrac{v_1^{n}-v_0^{n}}{h_x}+\dfrac{h_x}{2}\left(\dfrac{1}{\tilde{a}_0^2\tau}-(1-\sigma)\lambda^2\left(v_0^{n,k-1}\right)^{m-1}\right)v_0^{n}\right)\end{array}\right).$$

3.4.2.3. Conjugation of Solutions

Since both $v_0^{n+1,k}$ and $v_{0,M_0}^{n+1,k}$ are not known all we need to do know is to apply the conditions (3.62), (3.63). These result in

$$v_{0,M_0}^{n+1,k}=-\frac{G_1^{n,k-1}+b_0Q_0J_1^{n+1,k-1}}{H_1+a_0J_1^{n+1,k-1}},$$

$$v_0^{n+1,k} = a_0 v_{0,M_0}^{n+1,k} + b_0 Q_0,$$

where the following notations are used:

$$H_1 = \frac{\sigma_1}{h_{y,1}}\left(1 - \xi_{M_0}\right) + \frac{h_{y,1}}{2\tilde{a}_0^2 \tau} + \frac{\sigma_2}{h_{y,2}}\left(1 - \varsigma_{2,M_0+1}\right) + \frac{h_{y,2}}{2}\left(\frac{1}{\tilde{a}_0^2 \tau} + \sigma_2 \kappa^2\right),$$

$$J_1^{n+1,k-1} = d\,\frac{h_{y,1}}{2}\left(\frac{h_x}{2}\left(\frac{1}{\tilde{a}_0^2 \tau} + \sigma\lambda^2\left(v_0^{n+1,k-1}\right)^{m-1}\right) + \frac{\sigma}{h_x}\left(1 - \varsigma_1^{n+1,k-1}\right)\right) + \sigma_1\,\frac{\psi_{M_0}^{n+1,k-1}}{h_{y,1}},$$

$$G_1^{n,k-1} = -\frac{\sigma_1}{h_{y,1}}\chi_{M_0}^{n,k-1} + \left(1 - \sigma_1\right)\frac{v_{0,M_0}^n - v_{0,M_0-1}^n}{h_{y,1}} - \frac{h_{y,1}}{2}\left(\frac{v_{0,M_0}^n}{\tilde{a}_0^2 \tau} + dQ_0\right)$$

$$- d\,\frac{h_{y,1}}{2}\left(\frac{\sigma}{h_x}\eta_1^{n,k-1} + (1-\sigma)\frac{v_1^n - v_0^n}{h_x} + \frac{h_x}{2}\left(\frac{v_0^n}{\tilde{a}_0^2 \tau} - (1-\sigma)\lambda^2\left(v_0^n\right)^m\right)\right)$$

$$- \frac{\sigma_2}{h_{y,2}}\eta_{2,M_0+1}^n - \left(1-\sigma_2\right)\frac{v_{0,M_0+1}^n - v_{0,M_0}^n}{h_{y,2}} + \frac{h_{y,2}}{2}\left(-\frac{v_{0,M_0}^n}{\tilde{a}_0^2 \tau} + (1-\sigma_2)\kappa^2 v_{0,M_0}^n + \gamma Q_0\right).$$

Thus, the approximation algorithm we have just described in this section produces a discrete solution to the given 1D continuous problem.

4   Mathematical Model for a Willow Flute

The willow flute (Norwegian: *seljefløyte*, Finnish: *pitkähuilu* or *pajupilli,* Swedish: *sälgflöjt* or *sälgpipa*) is a Scandinavian folk instrument of the recorder family existing in two forms: an end-blown flute, often called a whistling flute, and a side-blown flute. This thesis focuses on the side-blown flute (see Fig. 4.1) which is between 40 and 80 centimetres in length. It consists of a tube with a transverse fipple mouthpiece that is constructed by putting a wood plug with a groove in one end of the tube, and cutting a sound hole (edged opening) a short distance away from the plug (see Fig. 4.2). The air flow is directed through a passageway across the edge creating a sound.

Fig. 4.1 Two forms of willow flute: side-blown flute (left), [55], whistling flute (right), [57]

"Similar, however not the same instruments were made by peasants in Poland, usually using a different method described in sources as "*kręcenie*" (that nowadays means literally "rolling", at that time possibly also "drilling-gouging"), "*ukręcanie*", "*ulinianie*" (nowadays literally meaning: "making moulted"). Such instruments are mentioned in folk poems or songs. There is also a Karelian variant of the willow flute that is made in Finnish Karelia and the Russian Republic of Karelia. It is made the same way as the willow flute, however instead of using willow bark; it is made out of birch bark.

Modern willow flutes are usually made of plastic (PVC tubing is often used), but the original willow flutes were made from sections of bark cut from green willow branches. Willow flutes could only be made this way during the spring, and became unplayable when the bark dried out."[1]

In this chapter we propose a mathematical model for a Norwegian flute, [11]. We find the possible frequencies of the sound produced by the flute, analyse how the pitches can be altered when changing parameters that are used in the problem, and determine the energy distribution in the sound.
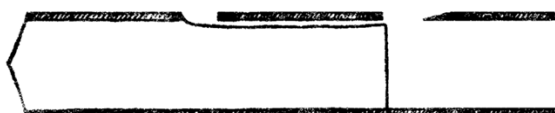

Fig. 4.2 Longitudinal section of the flute, [56]

As the flute has no finger holes, different pitches are produced by overblowing and by using a finger to cover, half-cover or uncover the hole at the far end of the tube. The *seljefløyte* plays tones in the harmonic series called the *natural scale*. When the end of the tube is left

---

[1] Citation from [57].

open, the flute produces one fundamental and its overtones, playing it with the end closed produces another harmonic scale.

4.1 Problem Formulation

In general the process is quite complex, as there are many things that should be taken into account. Although a non-linear equation could be considered when dealing with mass-transfer, we are going to use a linear one. Secondly, there is a question about hydrodynamic behaviour of the jet of air. Both experimental and theoretical results suggest (see, e.g., [50]) that the airstream tends to form vortices when impinging the edged opening. In the case of one dimensional plane wave, we are going to use a source function to describe this process. Thirdly, the mass flow in the passageway (see Fig. 4.3) can be approximated by a 3rd type boundary condition.

The willow flute can be considered as a first approximation as a cylindrical tube open, closed or partially open at one end. To derive the mathematical model we are going to make the following assumptions:
  a) the diameter of the flute is small enough in comparison to its length and the wavelength of sound;
  b) the diameter of the cylinder is large enough so that the effects of viscosity can be neglected;
  c) the walls of the pipe are rigid, smooth and thermally insulating, so the enclosure has no effect on wave propagation.

Assuming that the sound wave in the flute is a plane wave of sound, the vibration can be represented by

$$\frac{\partial^2 p}{\partial t^2} = c^2 \frac{\partial^2 p}{\partial x^2} + F(x), \ c^2 = \frac{B}{\rho},$$ (4.1)

where $p(x,t)$ denotes the acoustic pressure, function $F(x)$ is some external source, ρ density of air ($\approx 1.2041 \ kg/m^3$). Let's suppose that there is no appreciable conduction of heat, therefore the behaviour of sound waves is adiabatic. So, B is the adiabatic bulk modulus of air, which is approximately given by $B = \gamma p_a$. Here γ (≈ 1.4) refers to the ratio of the specific heats of air at constant pressure and at constant volume, and p_a ($\approx 101,325 \ Pa$) is the ambient pressure.
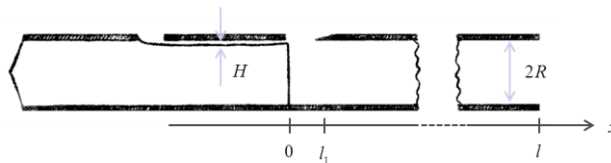

Fig. 4.3 Definition of geometrical parameters for the flute

Let's establish boundary and initial conditions. At $x = 0$ we have the 3rd type boundary condition

$$p_x(0,t) - \kappa_1 p(0,t) = 0.$$ (4.2)

The boundary condition at the far end of the tube depends upon whether the end is open, closed or partially open. For an open end of a tube, the total pressure at the end must be approximately equal to atmospheric pressure. In other words, the acoustic pressure p is zero:

$$p(l,t) = 0.$$ (4.3)

At a closed end there is an antinode of pressure, as most of the sound is reflected:

$$p_x(l,t) = 0.$$ (4.4)

In many research papers and works where wind instruments have been considered (see, e.g., [30], [33], [37], [47]), only these two types of boundary conditions are used. But one should bear in mind that the player can also close the end of the tube only halfway. In that case the Robin condition

$$p_x(l,t) + \kappa_2 p(l,t) = 0 \tag{4.5}$$

must be applied (see [49]). The value of κ_2 depends on how much of the end is closed. Conditions (4.3) – (4.5) can be rewritten in the following form

$$(1-h)p_x(l,t) + hp(l,t) = 0, \ 0 \le h \le 1. \tag{4.6}$$

But the initial conditions are

$$p(x,0) = \phi(x), \tag{4.7}$$
$$p_t(x,0) = \psi(x). \tag{4.8}$$

Nonhomogeneous equation (4.1) can be solved by using the method of separation of variables, namely, we look for a solution expressed in the following non-closed form:

$$p(x,t) = \sum_n X_n(x)T_n(t),$$

where $X_n(x)$, $n = 1,2,3,\ldots$ are the eigenfunctions of the associated homogeneous problem. Our boundary conditions (4.5), (4.6) imply that the eigenfunctions of the problem are

$$X_n(x) = \sin(\mu_n x + \varphi_n),$$

with

$$\tan(\varphi_n) = \frac{\mu_n}{\kappa_1}, \ \|X_n\| = \frac{l}{2}\left(1 - \frac{\sin(\mu_n l)}{\mu_n l}\cos(\mu_n l + 2\varphi_n)\right).$$

But the corresponding eigenvalues μ_n are roots of these transcendental equations:

- $\tan(\mu_n l) = -\dfrac{\mu_n}{\kappa_1}$ (end open);

- $\cot(\mu_n l) = \dfrac{\mu_n}{\kappa_1}$ (end closed);

- $\cot(\mu_n l) = \dfrac{1}{\kappa_1 + \kappa_2}\left(\mu_n - \dfrac{\kappa_1\kappa_2}{\mu_n}\right)$, $\kappa_2 = \dfrac{h}{1-h}$ (end partially open).

To determine the functions $T_n(t)$, we expand the source function and the initial conditions (4.7), (4.8) in a Fourier series. So $T_n(t)$ will satisfy the initial value problem

$$T_n''(t) + c^2\mu_n^2 T_n(t) = F_n,$$
$$T_n(0) = \phi_n,$$
$$T_n'(0) = \psi_n.$$

Hence the solution of problem (4.1), (4.2), (4.6) – (4.8) is

$$p(x,t) =$$
$$-\int_0^l \phi(\xi)\frac{\partial^2}{\partial t^2}G(x,\xi,t)d\xi - \int_0^l F(\xi)G(x,\xi,t)d\xi$$
$$-\int_0^l \psi(\xi)\frac{\partial}{\partial t}G(x,\xi,t)d\xi + \int_0^l F(\xi)G(x,\xi,0)d\xi \tag{4.9}$$

with the Green's function

$$G(x, \xi, t) = \sum_{n=1}^{\infty} \frac{\cos(c\mu_n t)}{c^2 \mu_n^2} \frac{X_n(x) X_n(\xi)}{\|X_n\|^2}.$$

4.2 Pitch and Frequency

One of the fundamental properties of sound is its pitch. Pitch is a subjective attribute of sound related to the frequency of a sound wave. Increasing the frequency causes a rise in pitch. But when decreasing the frequency, the pitch of the note diminishes.

The tone of the lowest frequency is known as the *fundamental* or *first harmonic*. All other possible pitches whose frequencies are integer multiples of the fundamental frequency are called the *upper harmonics* or *overtones*.

Using relation (4.9), we obtain the frequencies of the vibration mode of the willow pipe:

$$f_n = \frac{c\mu_n}{2\pi}, \quad n = 1,2,3,\dots$$

If the parameter κ_1 is sufficiently large, the possible frequencies for the flute are close to harmonics:

- $f_n \approx \dfrac{cn}{2l}, \quad n = 1,2,3,\dots$ (end open),

- $f_n \approx \dfrac{cn}{4l}, \quad n = 1,3,5,\dots$ (end closed).

As μ_n is dependent on l, we can change the pitch by changing the length of the tube. If the pipe has finger holes, one can also change the effective length of the instrument by opening different holes. By altering the pressure of the air blown into the instrument, we change the value of n, that is, we jump between solutions p_n, resulting in discrete differences in pitch. The pitch can also be changed by modifying h in the boundary condition at $x = l$.

4.3 Energy of Tones

As the sound of the willow flute playing is relatively harmonic, the energy of the sound is concentrated at certain frequencies of vibration. The more dominant the certain pitch, the greater the concentration of energy at that frequency. A person playing this particular instrument can get different pitches through manipulation of the supplied air, which is, changing initial conditions.

Formula for calculating the energy distribution is

$$E(t) = \frac{1}{2} \int_0^l \rho p_t^2(x,t)\,dx - \int_0^t d\iota \int_0^l B p_{xx}(x,\iota) p_t(x,\iota)\,dx. \tag{4.10}$$

If the initial conditions are defined as in Section 4.4, you can modify the distribution of energy between the fundamental and its overtones by changing ε.

4.4 Numerical Results

All the results in the next section were obtained for the following parameter values:
  a) geometrical parameters (m)

$$l \approx 0.656, \quad l_1 \approx 0.01, \quad H = \frac{R}{5},$$

  b) additional parameters

$$\kappa_1 = p_1 \frac{H^2 \pi}{R^2 \pi}, \quad p_1 = 101{,}324$$

  c) initial conditions

$$\phi(x) = 0 \,,$$

$$\psi(x) = \begin{cases} 0 & 0 < x < l_1 \\ 1 - \dfrac{x - l_1}{\varepsilon} & l_1 \le x \le l_1 + \varepsilon \,, \\ 0 & l_1 + \varepsilon < x < l \end{cases} \quad \varepsilon = \frac{l - l_1}{d} \,,$$

where d is a positive constant:

d) source function

$$F(x) = \frac{\kappa_1}{100} \cdot \frac{l - x}{l} \,.$$

4.4.1 Open-end scale

For the given parameters the fundamental for the open flute is Middle C or C4. The second harmonic is then an octave above the fundamental; the third harmonic is G5, and so on.
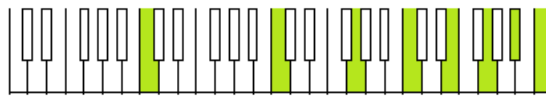

Fig. 4.4 Notes of open-end C scale

One can notice (see Table 4.1) that some of these frequencies differ from the frequencies of the notes on a piano in twelve-tone equal temperament (12-TET) with the 49th key, the fifth A (called A4), defined as 440 Hz. The frequency of the mth key in 12-TET can be found from the equation

$$f_m = 440 \cdot 2^{\frac{m - 49}{12}} \,.$$

Many people should be capable of detecting the difference if it is as little as 2 Hz.

Note	Piano	Willow flute
C4	261.63	261.63
C5	523.25	523.27
G5	783.99	784.90
C6	1046.50	1046.53
E6	1318.51	1308.16
G6	1567.98	1569.80
B♭6	1864.66	1831.43
C7	2093.00	2093.06

Table 4.1 Frequencies (Hz) of willow flute's open-end C scale compared with frequencies of 12-TET scale

4.4.2 Closed-end scale

Closing the end drops the fundamental an octave below the pitch of the pipe that is open at the end. The next possible note, G4, has approximately three times the frequency of the fundamental C3; the next one, E5, five times, and so on. This means that only the odd harmonics are present.
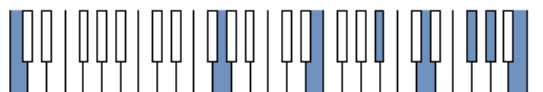

Fig. 4.5 Notes of closed-end C scale

Table 4.2 shows that the willow flute's pitches do not quite conform to those produced by the piano.

Note	Piano	Willow flute
C3	130.81	130.82
G4	392.00	392.45
E5	659.26	654.08
B♭5	932.33	915.72
D6	1174.66	1177.35
F♯6	1479.98	1438.98
G♯6	1661.22	1700.61
B6	1975.53	1962.25

Table 4.2 Frequencies (Hz) of willow flute's closed-end C scale compared with frequencies of 12-TET scale

4.4.3 Intermediate scale

We can get further effects, when covering the end hole only halfway. This implies that we should change the parameter h in the boundary condition (4.6). In this case the flute produces an intermediate set of pitches that fall in between those produced with the end closed and those when the end is open.
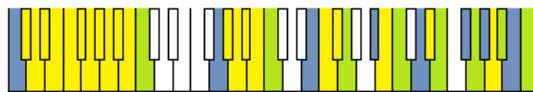

Fig. 4.6 Approximate location of the willow flute's pitches on a piano. C scale

■ Pitches produced with end closed
■ Pitches produced with end open
■ Pitches produced with end partially open

In Fig. 4.8 and Table 4.3 you can see how the parameter h affects location of pitches played by the flute when leaving the end partially open.
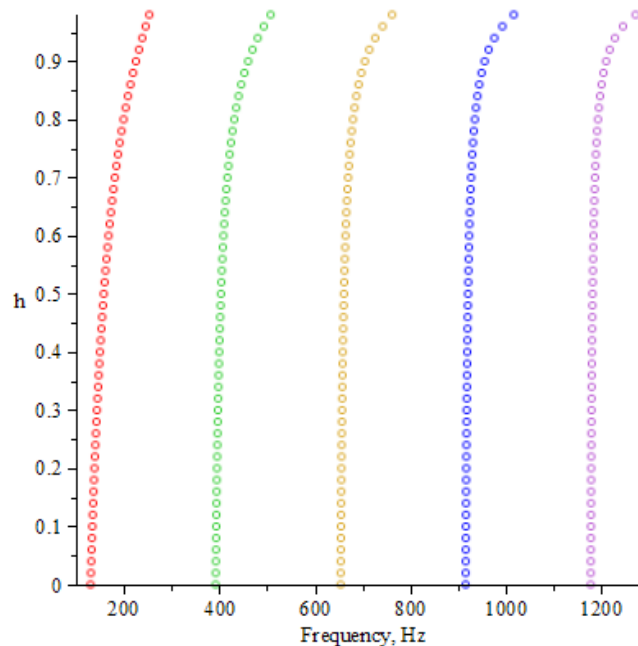

Fig. 4.7 Effect of the parameter h on the frequencies (Hz) of tones. C scale

58

h=0	130.82	392.45	654.08	915.72	1177.35	1438.98	1700.61	1962.25
h=1/10	134.57	393.73	654.85	916.27	1177.78	1439.33	1700.91	1962.50
h=2/10	138.97	395.33	655.82	916.96	1178.31	1439.77	1701.28	1962.83
h=3/10	144.22	397.35	657.05	917.84	1179.00	1440.33	1701.76	1963.24
h=4/10	150.58	400.01	658.68	919.01	1179.92	1441.09	1702.40	1963.79
h=5/10	158.46	403.65	660.95	920.65	1181.20	1442.13	1703.28	1964.56
h=6/10	168.51	408.92	664.30	923.09	1183.11	1443.70	1704.61	1965.72
h=7/10	181.75	417.15	669.74	927.09	1186.26	1446.30	1706.82	1967.63
h=8/10	199.92	431.49	679.98	934.84	1192.44	1451.43	1711.19	1971.44
h=9/10	225.78	460.57	704.81	955.30	1209.50	1465.93	1723.75	1982.49
h=1	261.63	523.27	784.90	1046.53	1308.16	1569.80	1831.43	2093.06

Table 4.3 Effect of the parameter h on the frequencies (Hz) of tones. C scale
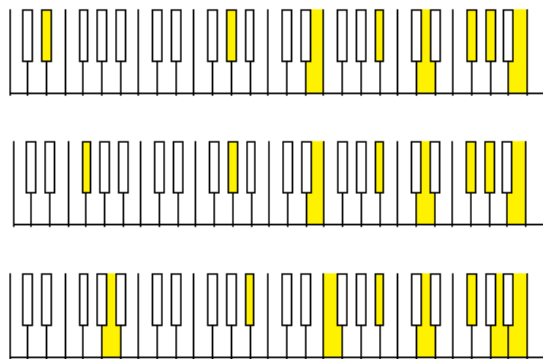

Fig. 4.8 Approximate location of pitches when $h = 0.5, 0.7, 0.9$

4.4.4 Energy of Tones

Using formula (4.10) we can find the energies of different harmonics and discover which harmonic is dominant when changing the width of the interval over which the blow is delivered. The examples given below (Fig. 4.9, Fig. 4.10) show that it has a very considerable effect on the energy of the overtones. You must blow as softly as possible to produce the fundamental, as it is hard to get ($\varepsilon \approx l$). Blow a little harder and you get the first overtone and so on. The harder you blow the higher harmonics you get to be dominant. This means that the presence of high overtones increases when decreasing ε.
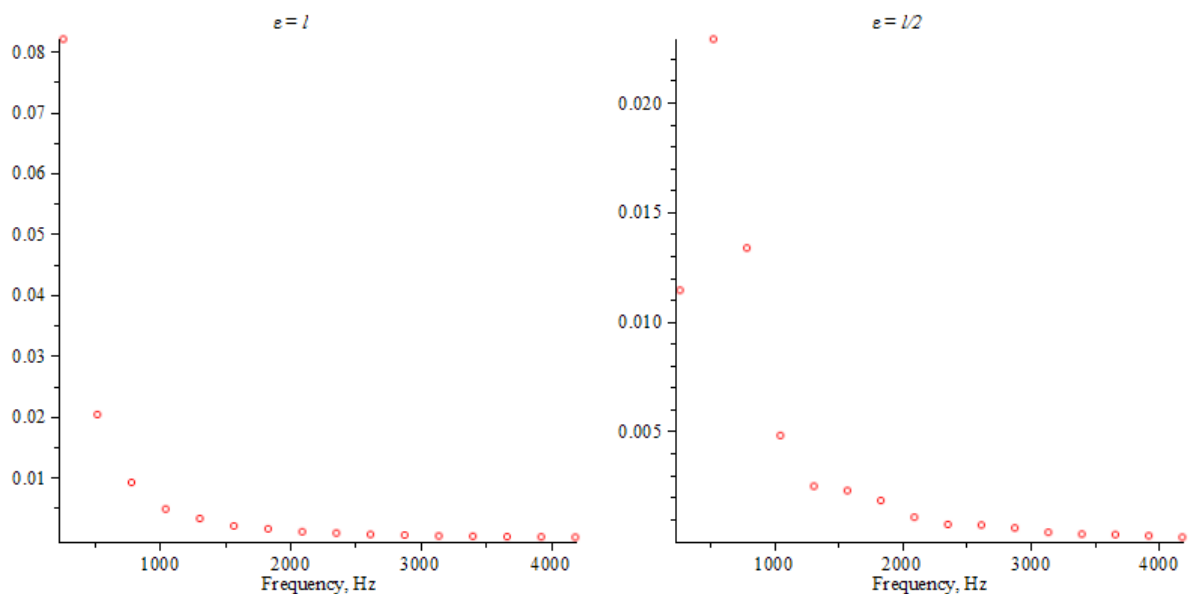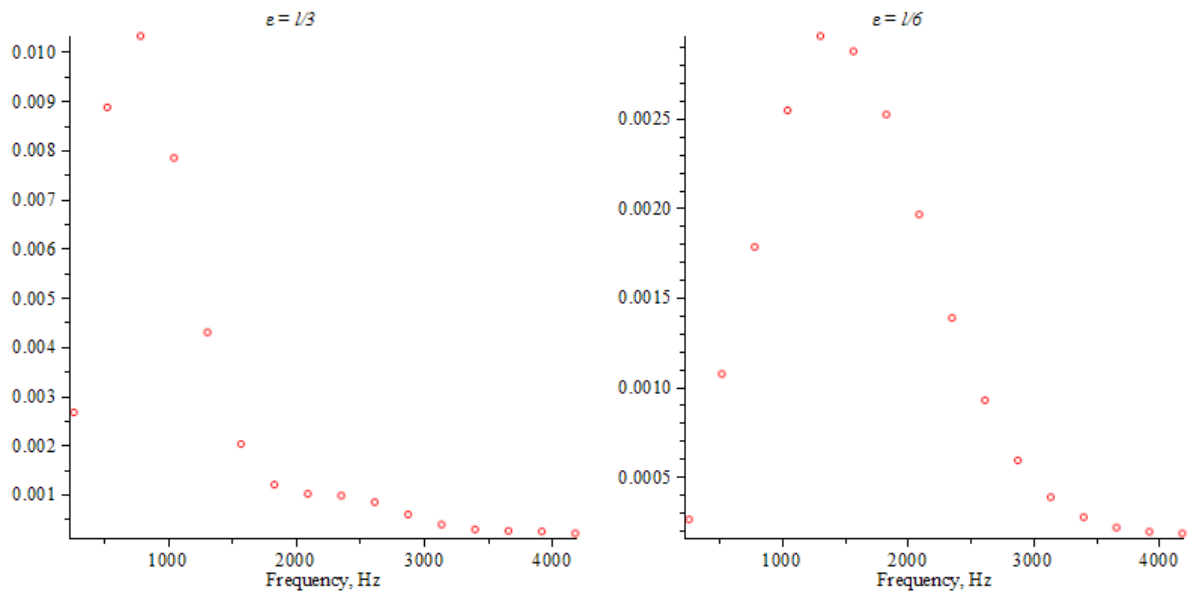
Fig. 4.9 Energy distribution (J) across the frequencies when changing ε. Open end
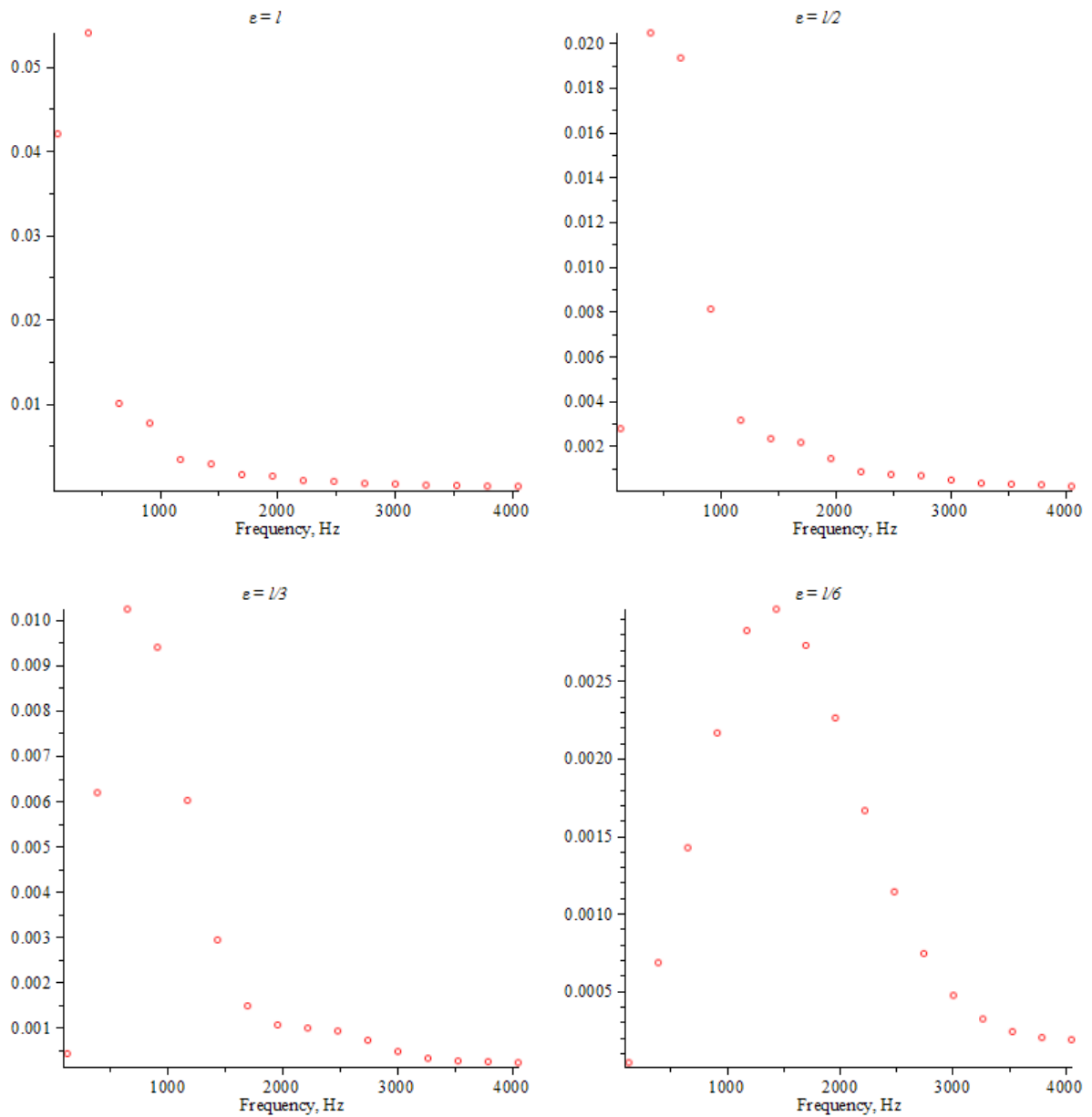

Fig. 4.10 Energy distribution (J) across the frequencies when changing ε. Closed end

Bibliography

[1]   A.H.M. Al-Essa, One Dimensional Finite Element Solution of the Rectangular Fin with Rectangular Perforations, *WSEAS Transactions on Heat and Mass Transfer*, Vol.1, No.10, 2006, pp. 762-768.

[2]   M.A. Aronov, N.I. Kobasko, J.A. Powell, I*ndustrial-Scale Intensive Quenching Process for Tool Products Commercialization*. EMTEC CT-76-II, Final Report, August 2005.

[3]   M.A. Aronov, N. Kobasko, J.A. Powell, Intensive Quenching of Carburized Steel Parts *IASME Transactions,* Issue 9, Vol.2, November 2005, pp. 1841-1845.

[4]   S. Blomkalna, M. Buike, A. Buikis, Several Intensive Steel Quenching Models for Rectangular and Spherical Samples, *Proceedings of the 9th IASME/WSEAS International Conference on HEAT TRANSFER, THERMAL ENGINEERING and ENVIRONMENT (HTE'11), Florence, Italy, 2011*, WSEAS Press, 2011, pp. 390-395.

[5]   T. Bobinska, M. Buike, A. Buikis. Comparing Solutions of Hyperbolic and Parabolic Heat Conduction Equations for L-shape Samples, *Proceedings of the 9th IASME/WSEAS International Conference on HEAT TRANSFER, THERMAL ENGINEERING and ENVIRONMENT (HTE'11), Florence, Italy, 2011*, WSEAS Press, 2011, pp. 384-389.

[6]   T. Bobinska, M. Buike, A. Buikis, Heat Transfer with Full Boiling in System with Double Wall and Double Fins, *Proceedings of the 11th WSEAS International Conference on Heat Transfer, Thermal Engineering and Environment, Vouliagmeni, Greece*, WSEAS Press, 2013, pp. 148-153.

[7]   T. Bobinska, M. Buike, A. Buikis, Heat Transfer with Partial Boiling in System with Double Wall and Double Fins, *Proceedings of the 11th WSEAS International Conference on Heat Transfer, Thermal Engineering and Environment, Vouliagmeni, Greece*, WSEAS Press, 2013, pp. 132-137.

[8]   T. Bobinska, M. Buike, A. Buikis, Hyperbolic Heat Equation as Mathematic Model for Steel Quenching of L-shape and T-shape Samples, Direct and Inverse Problems, *WSEAS TRANSACTIONS on HEAT and MASS TRANSFER*, Vol.5, No.3, 2010, pp. 63-72.

[9]   T. Bobinska, M. Buike, A. Buikis, Hyperbolic Heat Equation as Mathematical Model for Steel Quenching of L-shape Samples, Part 2 (Inverse Problem), *Proceedings of the 5th IASME/WSEAS International Conference on CONTINUUM MECHANICS (CM'10), Cambridge, UK, 2010*, WSEAS Press, 2010, pp. 21-26.

[10]   T. Bobinska, M. Buike, A. Buikis, H.H. Cho, Stationary Heat Transfer in System with Double Wall and Double Fins, *Proceedings of the 11th WSEAS International Conference on Data Networks, Communications, Computers (DNCOCO'12) and the 5th WSEAS Int. Conf. on Materials Science (MATERIALS'12), Sliema, Malta*, WSEAS Press, 2012, pp. 260-265.

[11]   T. Bobinska, A. Buikis, A Mathematical Model for a Willow Flute, *Proceedings of the 9th IASME/WSEAS International Conference on HEAT TRANSFER, THERMAL ENGINEERING and ENVIRONMENT (HTE'11), Florence, Italy, 2011*, WSEAS Press, 2011, pp. 188-192.

[12]   M. Buike, Exact Solution of Full 3-D Steady-State Problem for One Element of the System with Rectangular Fin, *New Aspects of Heat Transfer, Thermal Engineering and Environment, Part II*, WSEAS Press, 2008, pp. 402-406.

[13] M. Buike, Simulation of Steady-State Heat Process for the Rectangular Fin-Containing System, *Mathematical Modelling and Analysis*, Vol.4, Technika, 1999, pp. 33-43.

[14] M. Buike, A. Buikis, Approximate Analytical Two-Dimensional Solution for a Longitudinal Fin of Rectangular Profile, *Acta Universitatis Latviensis*, Vol.612, 1998, pp. 33-44.

[15] M. Buike, A. Buikis, Approximate Solutions of Heat Conduction Problems in Multi-Dimensional Cylinder Type Domain by Conservative Averaging Method, Part 1&2, *Proceedings of the 5th IASME/WSEAS International Conference on Heat Transfer, Thermal Engineering and Environment,* Vouliagmeni, Athens, August 25 -27, 2007, pp. 15–20, pp. 21-26.

[16] A. Buikis, M. Buike, Exact 3-D Solution for System with a Rectangular Fin, Part 1, *Proceedings of the 12th WSEAS International Conference on APPLIED MATHEMATICS (MATH'07-II)*, *Cairo, Egypt*, WSEAS Press, 2007, pp. 64-69.

[17] M. Buike, A. Buikis, Exact Transient Solution for System with Rectangular Fin, *Proc. of the 5th WSEAS International Conference on HEAT AND MASS TRANSFER (HMT'08), Acapulco, Mexico*, WSEAS Press, 2008, pp. 25-30.

[18] M. Buike, A. Buikis, Hyperbolic Heat Equation as Mathematical Model for Steel Quenching of L-shape Samples, Part 1 (Direct Problem), *Proceedings of the 13th WSEAS International Conference on APPLIED MATHEMATICS (MATH'08), Puerto De La Cruz, Canary Islands, Spain*, WSEAS Press, 2008, pp. 198-203.

[19] A. Buikis, Conservative averaging as an approximate method for solution of some direct and inverse heat transfer problems, *Advanced Computational Methods in Heat Transfer,* IX, WIT Press, 2006, pp. 311-320.

[20] A. Buikis, Two-Dimensional Solution for Heat Transfer in a Regular Fin Assembly, *Latvian Journal of Physics and Technical Sciences*, No.5, 1995, pp. 51-58.

[21] A. Buikis, M. Buike, Closed Two-Dimensional Solution for Heat Transfer in a Periodical System with a Fin, *Proceedings of the Latvian Academy of Sciences*, Section B, Vol.52, No.5, 1998, pp. 218-222.

[22] A. Buikis, M. Buike, Some analytical 3-D steady-state solutions for systems with rectangular fin, *IASME Transactions*, Issue 7, Vol.2, September 2005, pp. 1112-1119.

[23] A. Buikis, M. Buike, T. Bobinska, Solution of hyperbolic heat equation reduction to integral equation for a right corner, *Proceedings of the International Conference Differential Equations and Their Applications (DETA'09) Panevezys, Lithuania, 2009*, KTU, 2009, pp. 14-16.

[24] A. Buikis, M. Buike, S. Guseinov, Analytical Two-Dimensional Solutions for Heat Transfer in a System with Rectangular Fin, *Advanced Computational Methods in Heat Transfer*, *VIII*, WIT Press, 2004, pp. 35-44.

[25] A. Buikis, M. Buike, N. Ulanova, Analytically-Numerical Solution for Transient Process in the System with Rectangular Fins, *Proc. of the 5th WSEAS International Conference on HEAT and MASS TRANSFER, Acapulco, Mexico*, WSEAS Press, 2008, pp. 31-36.

[26] A. Buikis, M. Buike, N. Ulanova, Transient solution for heat transfer in periodical system with fins (А.Буйкис, М.Буйке, Н.Уланова. Нестационарные решения для теплопереноса в периодических системах с ребрами), Proceedings of the Ukrainian Congress of Mathematics, Section 8, 2002, pp.15-23 (In Russian).

[27] A. Buikis, Sh. Guseinov, Inverse Initial Heat Flux and Relaxation Time Finding Problems for Hyperbolic Heat Equation, *Proceedings of the 10th WSEAS International*

Conference on APPLIED MATHEMATICS, Dallas, Texas, USA, WSEAS Press, 2006, pp.10-14.

[28]   A. Buikis, Sh. Guseinov, Solution on Reverse Hyperbolic Heat Equation for Intensive Carburized Steel Quenching, *Proceedings of the International Conference on Computational & Experimental Engineering and Sciences (ICCES'05)*, India, 2005, pp. 741-746.

[29]   H. H. Cho, Plenary lecture 8. Boiling Heat Transfer Enhancement Using Nano and Micro Technology. *Recent Advances in Fluid Mechanics and Heat & Mass Transfer*, Florence, Italy, August 23-25, 2011, p. 23.

[30]   N. H. Fletcher, T. D. Rossing, *The Physics of Musical Instruments*, *Second Edition*, Springer Science+Business Media, Inc., 2010.

[31]   D.D. Joseph, L. Preziosi, Heat waves, *Reviews of Modern Physics*, Vol.61, No.1, 1989, pp. 41-73.

[32]   D.D. Joseph, L. Preziosi, Addendum to the paper: Heat waves, *Reviews of Modern Physics*, Vol.62, No.2, 1990, pp. 375-391.

[33]   R. W. Hall and K. Josić, The Mathematics of Musical Instruments, *American Mathematical Monthly*, Vol.108, No.4, 2001, pp. 347-357. Available from Internet: http://www.sju.edu/~rhall/newton/article.pdf (last checked 28/10/2013).

[34]   D.Q. Kern, A.D. Kraus, *Extended Surface Heat Transfer*, McGraw-Hill Book Company, New York, 1972.

[35]   B.S. Kim, S. Shin, S.J. Shin, K.M. Kim, H.H. Cho, Control of Superhydrophilicity/Superhydrophobicity using Silicon Nanowires via Electroless Etching Method and Fluorine Carbon Coatings, *Langmuir*, Vol.27, No.16, 2011, pp. 10148-10156.

[36]   B.S. Kim, S. Shin, S.J. Shin, K.M. Kim, H.H. Cho, Micro-nano hybrid structures with manipulated wettability using a two-step silicon etching on a large area, *Nanoscale Research Letters, a SpringerOpen Journal*, Vol.6 (ID:333), 2011.

[37]   L. E. Kinsler, A. R. Frey, A. B. Coppens, J. V. Sanders, *Fundamentals of Acoustics, Fourth Edition*, John Wiley& Sons, Inc., 2000.

[38]   N.I. Kobasko, *Intensive Steel Quenching Methods. Theory and Technology of Quenching*, Springer-Verlag, New-York, N.Y., 1992.

[39]   N. Kobasko, P. Krukovskyi, D. Yurchenko, Initial and Critical Heat Flux Densities Evaluated on the Basis of CFD Modeling and Experiments during Intensive Quenching, *Proceedings of the 5th IASME/WSEAS International Conference on HEAT TRANSFER, THERMAL ENGINEERING AND ENVIRONMENT*, *Athens, Greece*, WSEAS Press, 2007, pp. 295-298.

[40]   G. A. Korn, T. M. Korn, Mathematical Handbook for Scientists and Engineers. Definitions, Theorems and Formulas for Reference and Review. McGRAW-HILL BOOK COMPANY, INC. 1961.

[41]   A.D. Kraus, *Analysis and Evaluation of Extended Surface Thermal Systems*, Hemisphere Publishing Corporation, 1982.

[42]   A. Lehtinen, R. Karvinen, Analytical Three-Dimensional Solution for Heat Sink Temperature, *Proceedings of IMECE2004-61062*, 2004, 7 p.

[43]   M.Y. Malik, A.S. Wood, A. Buikis, An Approximate Analytical Solution to a Familiar Conjugate Heat Transfer Problem, *International Journal of Pure and Applied Mathematics,* Vol.10, No.1, 2004, pp. 89-105.

[44] M. Manzoor, *Lecture Notes in Engineering. Heat Flow through Extended Surfaces Heat Exchangers*. Edited by C.A. Brebbia and S.A. Orszag. Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1984.

[45] Y.S. Muzychka, M.M. Yovanovich, J.R. Culham, Thermal Spreading Resistance in Compound and Orthotropic Systems, *Journal of Thermophysics and Heat Transfer*, Vol.18, No.1, 2004, pp. 45-51.

[46] A.D. Polyanin, *Handbook of Linear Partial Differential Equations for Engineers and Scientists*, Chapman&Hall/CRC Press, Boca Raton, 2002.

[47] T. D. Rossing (Ed.), *Springer Handbook of Acoustics*, Springer Science+Business Media, LLC New York, 2007.

[48] A.A. Samarskii, *The Theory of Difference Schemes (Pure and Applied Mathematics Vol.240)*, New York, Marcel Dekker, 2001.

[49] L. Simanoviča, *Boundary Conditions Effect on Musical Instruments Timbre*, Faculty of Physics and Mathematics, University of Latvia, Bachelor's Thesis, 2010 (in Latvian).

[50] P. A. Skordos, *Modeling flue pipes: subsonic flow, lattice Boltzmann, and parallel distributed computers*, Department of Electrical Engineering and Computer Science, MIT, PhD. Dissertation, January 1995. Available from Internet: http://hdl.handle.net/1721.1/36534 (last checked 28/10/2013).

[51] L. Wang, X. Zhou, X. Wei, *Heat Conduction*, Springer-Verlag Berlin Heidelberg, 2008.

[52] A.S. Wood, G.E. Tupholme, M.I.H. Bhatti, P.J. Heggs, Performance Indicators for Steady-State Heat Transfer Through Fin Assemblies, *ASME*. *Journal of Heat Transfer*, Vol.118, No.2, 1996, pp. 310-316.

[53] A.S. Wood, G.E. Tupholme, M.I.H. Bhatti, P.J. Heggs, Steady-State Heat Transfer through Extended Plane Surfaces, *International Communication in Heat and Mass Transfer*, Vol.22, No.1, 1995, pp. 99-109.

[54] Z.M. Zhang, *Nano/Microscale Heat Transfer*, McGraw-Hill, 2007.

[55] http://fenja-og-menja.blogspot.com/p/vikingmusikken-til-fenja-og-menja.html (last checked 28/10/2013)

[56] http://www.hf.ntnu.no/rff/organisasjonar/skrifter/vol1/html/olakailedang.html

[57] http://en.wikipedia.org/wiki/Willow_flute (last checked 28/10/2013)