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# ANALYTICAL SOLUTIONS OF MAGNETOHYDRODYNAMICAL PROBLEMS ON A FLOW OF CONDUCTING FLUID IN THE ENTRANCE REGION OF CHANNELS IN A STRONG MAGNETIC FIELD

**Ph.D. Thesis**

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## ABSTRACT

The PhD thesis is devoted to the theoretical study of new magnetohydrodynamic (MHD) problems on a flow of a conducting fluid in an initial part of a channel at the condition that fluid flows into the channel through the split of finite width or through a hole of finite radius on the channel's lateral side. Problems are solved for plane and circular channels of infinite length. The cases of longitudinal and transverse magnetic fields are studied in detail. The exact analytical solution of the problems is obtained in Stokes and inductionless approximation in the form of convergent improper integrals. On the basis of the obtained solutions the velocity field of the flow is analysed. In addition, the asymptotic solution at large Hartmann numbers is obtained.

The MHD problem on the influence of cross flow on a main flow in an infinitely long plane channel in a strong external magnetic field is also studied analytically in this work. This problem is solved in Oseen and inductionless approximation both for the longitudinal and transverse magnetic field. The field of perturbation velocity is analysed for different Hartmann and Reynolds numbers in detail. The dependence of the length of the initial part of the channel on the Hartman number is also studied.

All these problems are closely related to applications in metallurgy, where MHD phenomena are exploited to control and manipulate metals, and in design of fusion reactor where liquid metals are used both to produce tritium that fuels the reactor and for removing plasma's impurities and protecting solid surfaces from overheating and corrosion.

Solving problems on an inflow of a conducting fluid into a half-space through a split of finite width or studying a developing flow the initial part of channels, the uniform velocity profile or parabolic velocity profile usually is given at the channel's entrance region. One part of the PhD thesis is devoted to the analytical study of dependence of full pressure force at the entrance region on the profile of boundary velocity at this region. Using the example of two hydrodynamic and two MHD problems on an inflow of viscous fluid (conducting fluid for MHD problem) into a half-space through a split of finite width or through a hole of finite radius it is shown that if an uniform inlet velocity profile is given at the entrance then the full pressure force at the entrance of region is equal to infinity and, consequently, such flow is physically unrealistic. It is shown also, that if the velocity of the fluid at the entrance of region is given as a parabolic function then the full pressure force at the entrance will be finite. To the author's knowledge, this fact is not reported in the literature. The new asymptotic solutions for MHD problems are obtained at large Hartmann numbers for physically realistic cases.

Method of integral transforms is used in the thesis for problems solving. Numerical calculations were carried out by using the package "Mathematica".

# ANOTĀCIJA

Šis darbs ir teorētisks pētījums jaunu magnetohidrodinamisku (MHD) problēmu risināšanai par elektrovadoša šķidruma plūsmu kanālos stiprā magnētiskā laukā, kad šķidrums ietek kanālos caur galīga platuma spraugu vai caur galīga rādiusa atveri kanāla sānu malā. Tiek aplūkots bezgalīga garuma plakans un cilindrisks kanāls. Detalizēti ir izpētīti garenvirziena un šķērsvirziena magnētiskā lauka gadījumi. Sevišķa uzmanība pievērsta plūsmas izpētei kanālu sākotnējā apgabalā, kurā notiek plūsmas attīstība. Precīzi analītiskie atrisinājumi ir iegūti konverģentu neīsto integrāļu formā. Pamatojoties uz iegūtajiem atrisinājumiem, veikta plūsmas ātruma lauka skaitliskā analīze. Turklāt ir iegūti uzdevumu asimptotiskie atrisinājumi lieliem Hartmana skaitļiem.

Darbā analītiski tiek pētīta jauna MHD problēma par šķērsplūsmas ietekmi uz galveno plūsmu plakanā kanālā spēcīga magnētiskā lauka klātbūtnē. Problēma ir atrisināta šķērsvirziena un garenvirziena magnētiskajam laukam Ozeena un bezindukcijas tuvinājumā. Ātruma vektoru lauks ir izpētīts dažādiem Hartmana un Reinoldsa skaitļiem. Ir izpētīta sākotnējā apgabala garuma atkarība no Hartmana un Reinoldsa skaitļiem.

Visas šīs problēmas ir cieši saistītas ar lietojumiem metalurģijā un materiālu apstrādē, kur MHD parādības, ko rada elektrovadošu šķidrumu mijiedarbība ar magnētisko lauku, plaši izmanto, lai kontrolētu un manipulētu dažādus elektrovadošus materiālus, kā arī kodolsintēzes reaktora projektēšanā, kur šķidrais metāls tiek izmantots gan tritija ražošanai, kas kalpo kā degviela reaktoram, gan arī plazmas piemaisījumu noņemšanai un cieto virsmu aizsargāšanai no pārkāršanas un korozijas.

Risinot uzdevumus par šķidruma ietecēšanu pustelpā caur galīga platuma spraugu vai pētot attīstošo plūsmu kanālu sākotnējā apgabalā, parasti uz ieejas pustelpā vai uz kanāla ieejas uzdot vai nu vienmērīgu ātruma profilu vai arī parabolisko ātruma profilu. Promocijas darbā analītiski tiek pētīta pilna spiediena spēka atkarība no funkcijas veida, ar kuru ir uzdots robežnosacījums ātrumam šķidruma ieplūdes apgabalā. Šīs atkarības pētījums tiek veikts, risinot divas hidrodinamiskas problēmas un divas līdzīgas MHD problēmas par viskoza šķidruma (vadoša šķidruma MHD problēmām) ietecēšanu pustelpā caur plakanu spraugu vai caur apaļu atveri. Šķidruma ātrums ieplūdes apgabalā tika uzdots kā konstanta funkcija (funkcija ar galīgiem pārtraukumiem) vai kā paraboliska funkcija (kā nepārtraukta funkcija). Cik autoram ir zināms, šī atkarība netika pētīta agrāk. Jauni asimptotiski atrisinājumi pie lieliem Hartmana skaitļiem ir iegūti MHD uzdevumam gadījumā, kad uz ieejas apgabalā ir uzdots paraboliskais ātruma profils.

Uzdevumu risināšanai tika izmantota integrālo transformāciju metode. Skaitliskie aprēķini veikti ar programmu paketi „Mathematica”.

# CONTENTS

# INTRODUCTION

This work is devoted to a theoretical study of some new magnetohydrodynamic (MHD) problems on a flow of a conducting fluid in the initial part of a plane or circular channels in the presence of a strong external magnetic field. Namely, the case of inflow of conducting fluid through a finite width split or finite radius hole on the lateral side of the channel is considered.

Flows of a conducting fluid in external magnetic field produce a variety of new effects, which are not realizable in usual hydrodynamics. Magnetohydrodynamics (MHD) analyzes these phenomena. It also studies the arising of a flow of a conducting fluid due to the current passing through the fluid (so-called electrically induced vortex-type flows). MHD describes the frontier area combining classical fluid mechanics and electrodynamics. It is a relatively young discipline in natural science and engineering, starting with the pioneering work of Hartmann [138] on a liquid metal duct flow under the influence of a strong external magnetic field (1937). Today MHD has developed into a vast field of applied and fundamental research in engineering and physical science.

Nowadays electromagnetic methods of action on electrically conducting medium are widely used both in technical devices such as pumps, flow meters, generators and industrial processes in metallurgy and material processing, in chemical industry, industrial power engineering and nuclear engineering. MHD effects are exploited to control and manipulate various conducting materials.

In metallurgy and material processing different kind of magnetic fields (alternating, travelling, rotating, steady magnetic fields) are routinely used to heat, melt and cast conducting materials, to stir and levitate liquid metals, and to control the motion of melt ([31], [32], [39], [40], [48], [124]). For instance, in the continuous casting of large steel stabs, an intense static magnetic field (around 0.5 T) is commonly used to suppress motion within mould, i.e. suppress unwanted eddies vortices and the motion of submerged jets that feeds the mould from above. Electromagnetic stirring is used in casting operations to homogenize liquid zones of partially solidified ingots. The result is a fine structured, homogeneous ingot. Rotating magnetic fields are used in stirring. To melt and cast a metal in a single operation, induction furnaces, skull melting furnaces and cold crucible induction furnaces are used. Another common application of MHD in metallurgy is MHD separation that is used for electromagnetic removal of non-metallic inclusions from melts and metal extraction from oxides and slag. I.e. MHD is used for cleaning liquid metals of impurities as well as for the separation of multiphase systems into their components.

Nowadays electromagnetic pumps and their modifications are widely used in metallurgy and materials processing in order to transport and dose (exact batching) melting metal (see [39], [124]). The advantages of MHD pumps over mechanical pumps are: the absence of moving and rotating parts (this increases their reliability), noiseless operations (better vibration and noise characteristics), relative simplicity of control, being completely hermetically sealed. They can be utilized even with chemically aggressive, reactive and very hot fluids. Therefore, they are used also in chemical industry. [31], [32] contains an outline of the latest advances in metallurgical applications of MHD and a short history of electrometallurgy. See also [124].

Channels, in particular narrow channels, are common parts of many MHD devices. Therefore, investigation of MHD phenomena in plane layers and channels with conducting fluids is quite important both for understanding the basic mechanisms and for improving the existing industrial processes and for developing new MHD devices.

The study of MHD flows of liquid metal in ducts and channels in the presence of a strong external magnetic field is of a particular interest in the last four decades. It is related to the development of fusion reactors TOKAMAKS where plasma is confined by a strong magnetic field (see [108], [119], [53], [50]). It is planned to use liquid metals (LM) in different blocks of the reactor to produce tritium that fuels the reactor, to remove plasma's impurities and to protect solid surfaces from overheating and corrosion. One such block is the blanket of the reactor (see [68], [16], [53], [50]).

Let us briefly consider the main ideas of fusion reactor and liquid metal applications in it. The goal of this development is the realization of controlled thermonuclear fusion. It could lead to a significant contribution to future energy demands. In a fusion reaction, two light atomic nuclei fuse together to form heavier ones with a release of a large amount of energy. These reactions are the energy source of Sun and stars. In the fusion reactor deuterium and tritium will be used as the fuel. If a nucleus of deuterium fuses with a nucleus of tritium, a nucleus of Helium is produced and neutron is released. The energy released is 17.6 MeV per reaction. In macroscopic terms, just 1kg of this fuel would release $10^8$ kWh of energy and would provide the requirements of 1 GW (electrical) power station for a day. It should be noted that about 80% of the fusion energy is released in the form of high-energy neutrons. Reactions between the fuel components tritium and deuterium require temperatures above 100 million degrees centigrade, so that confinement using solid walls is not possible.

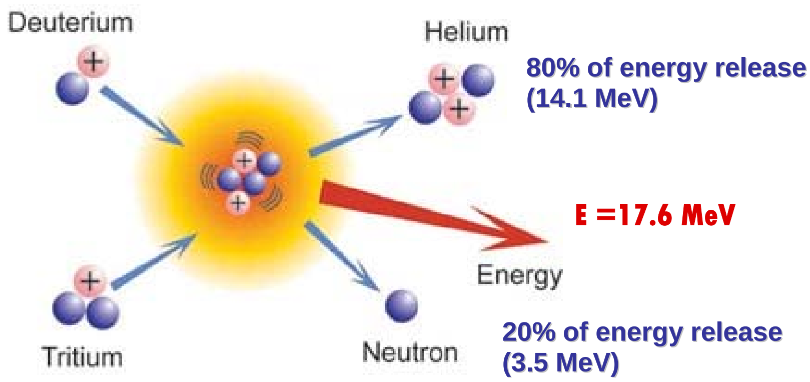



Figure 0.1. Two nuclei of deuterium and tritium fuse together to form helium, a neutron, and a large amount of energy.

At these temperatures the fuels are ionized and form electrically highly conducting plasma and can be confined by strong magnetic field to a defined volume. During the past decades different concepts of magnetic confinement have been investigated and studied, but the tokamak system with plasma confinement in toroidal vacuum vessel by a strong magnetic field (~10T), is considered as more attractive and suitable for the creation of fusion reactor ([106], [108]). Tokamaks have proven their capabilities for plasma confinement in a number of experiments in several countries (see, for example [106]). But in all experiments the time of plasma confinement was too short and energy realised in experiments was much smaller than the energy required for experiments. Nowadays Europe, USA, Russia and Japan are combining their efforts in developing the first thermonuclear reactor based on tokammak, which is known as ITER –International Thermonuclear Experimental Reactor. It will be the first fusion reactor to create more energy than it requires. The goal of ITER is to demonstrate the scientific and technological feasibility of fusion energy for peaceful purposes ([106], [1]). Details about ITER and about advantages of fusion energy can be found at the official Internet site of ITER [http://www.iter.org].

Liquid metal is used in different blocks of a reactor with the aim of removing plasma's impurities and protecting solid surfaces from overheating and corrosion. One such block is the blanket of the reactor ([108], [10]; [42], [116], [16], [106]).

Blanket is the solid component that, surrounding the plasma, covers the interior surfaces of the Vacuum Vessel and has three main functions. The neutrons are slowed down in the Blanket, where their kinetic energy is transformed into heat energy and collected by the coolants. In a fusion power plant, this energy will be used for electrical power production.

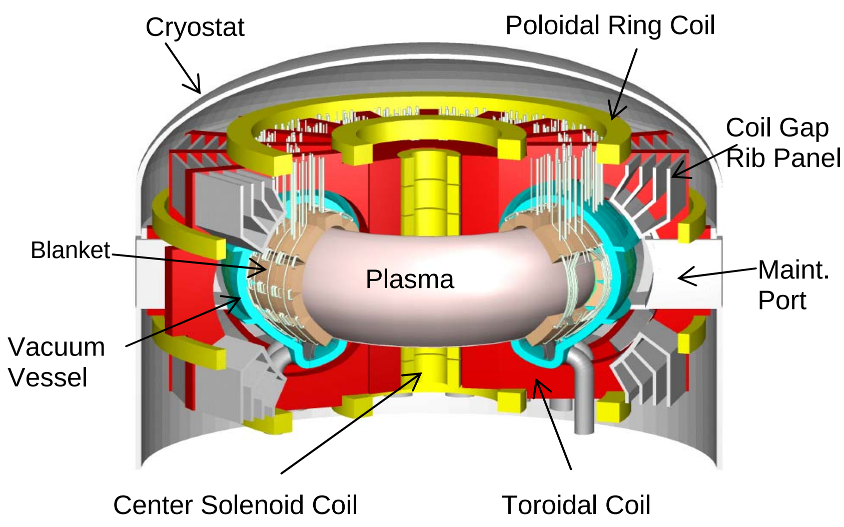



Figure 0.2. Scheme of reactor Tokamak

In absorbing neutrons the blanket provides shielding to solid surface of the Vessel and the superconducting Magnets from the heat and neutron fluxes of the fusion reaction, thus avoiding both radiation damage and heating of the magnetic field coils. The blanket is also responsible for the breeding of tritium, one of the fuel components that is not a naturally occurring isotope but can be obtained from liquid metal ([53], [16], [68]). At present there are several blankets under development in the international fusion community ([122], [53], [15], [16], [68]). In some design concepts for the blankets, liquid metals (Lithium or eutectic alloy lead-lithium) is used only as breeder material and helium or water is considered as possible coolants (so-called separately cooled blanket). In such a blanket the breeder is circulated at low speed only for tritium extraction. There are other concepts in which the liquid metal serves as breeder material and as coolant (self-cooled blanket) in which all released heat energy is removed only by a molten metal, flowing in the blanket at high enough speed ([122], [87]). A combination of both ideas results in the so-called dual-coolant blanket, where first wall and structure are cooled by helium and breeding zone is self-cooled ([106]). In Buhler [16] some liquid metal blankets are outlined that have been promoted during the last decades of fusion research and key issues concerning the liquid-metal flow in a fusion environment are briefly discussed.

Since the LM motion in the systems of Tokamak reactor, including the blanket, occurs in a strong magnetic field (10T) (see [108], [119]; [50]), the MHD effects become significant and it is important to take them into account already on the stage of projecting. The interaction between the circulating liquid metal with a strong magnetic field leads to a formation of large electromagnetic body forces. These forces determine the flow distribution (its structure) of liquid metal and produce large MHD pressure gradients. The resulting MHD pressure drop leads to the necessity of making the large pressure gradients for metal pumping, which, in turn, cause excessive pumping power loss and prohibitively large metal stresses. Besides, the flow distribution strongly affects the heat transfer characteristics of the blanket in general, and the first wall coolant channels in particular. Thus, the influence of the magnetic field on the LM flow in blanket's ducts and MHD effects must be taken into account on developing a multi-channel design of the blanket. In the recent past the issue of LM employment in blankets, including MHD effects, was the subject of many articles and reviews. There are several reviews of LM MHD flows in fusion-relevant conditions, focusing on the common types of LM blankets (see, for example, [47], [69], [68], [16], [61]), and a few papers considering specific issues associated with concrete blanket design ([87], [78], [79], [83], [84]). The main MHD issues to be discussed for the liquid metal coolant systems are the MHD pressure drop, flow distribution, electrical insulation, magnetic field non-uniformity effects, multi-channel (Madarame) effect, heat transfer, etc.

Design and construction of LM cooled fusion blankets requires detailed knowledge of MHD duct and channel flow. Special attention should be paid to the MHD phenomena occurring in the LM path transit sections, where a substantial change of the flow direction relative to the magnetic field direction takes place. Thus, it is important to study flows in bends, expansions, contractions, manifolds and other complex geometry flows. It is to be noted that for the self-cooled and dual-cooled blanket, the most important MHD task is the determination of pressure losses. Significant MHD pressure drop, occurring in above mentioned geometric complexities of channels, is needed to direct the LM flow through the flow system and blanket ([15]).

So, in order to ensure a successful and effective use of electromagnetic phenomena in industrial processes and technical systems, a very good understanding of the effects of the application of a magnetic field (also of strong magnetic fields) on the flow of electrically conducting fluid in channels and various geometric elements is required. Both experimental and theoretical studies of these problems are important.

The motion of a conducting fluid in an external magnetic field is described by the system of partial differential equations that consists of Navier-Stokes equation including the term of electromagnetic force, Maxwell's equations and Ohm's low (see [115], [111]). Due to the nonlinearity of these equations, the number of exact solutions in MHD is limited. (By the exact solution we mean an analytical solution, which is obtained when all the terms in governing equations are taken into account). Exact solutions are possible only in the case of the simplest flow geometry, simplest form of magnetic field, in the presence of symmetry and so on. For the MHD flows in channels the possibility of obtaining exact solutions was practically exhausted by 1970. Review of the exact solutions can be found, for instance, in [113]. Therefore most solutions for MHD problem are approximate, and have been obtained either by simplified flow models or by approximate methods, for example, by asymptotic expansions. With more complex flows and different forms of the magnetic fields the task of obtaining analytical solutions become more complicated. Therefore numerical methods are widely used for solving these MHD problems, especially 3D problems (see e.q. [75], [44]). Nowadays numerical codes are developed that are able to simulate MHD flows in arbitrary geometries and for any orientation of the applied field. During the last decade few attempts are made to develop 3D software also for analysis of MHD flows in strong magnetic fields (e.q. [3], [62], [73], [84]).

However, analytical solutions do not loose their importance (even if these solutions are obtained for simplified flows). First, analytical solutions give the qualitative picture of the flow and the main characteristic of the flow can be estimated. It is important for a general understanding of MHD flows. Second, analytical solutions are useful because they can be used as benchmark problems for testing numerical algorithms. Hence, it is always of interest to extend the class of problems for which an analytical solution can be found.

The present work is devoted to the theoretical study of the new MHD problems on a flow of conducting fluid in channels in the presence of a strong magnetic field for the case where fluid is flowing into channels through a split of finite width or through a hole of finite radius on the lateral side of the channel. Plane and circular channels of infinite length are considered. Particular attention is paid to the study of the flow in the initial part of the channel.

To solve problems on an inflow of conducting fluid into a half-space through a split of finite width or to study developing flow in the initial part of channels, a uniform velocity profile or parabolic velocity profile is usually given at the entrance of region (for example [139], [140], [141], [142], [143], [144]). In some works (for example [127], [137]) the case of an arbitrary velocity profile at the channel's inlet region has been also considered. In all these works the velocity distribution, the distribution of pressure or the temperature field have been studied, but to the author's knowledge, the dependence of full pressure force at the entrance on the entrance velocity profile has been not studied in literature. Therefore, one part of the

PhD thesis is devoted to the analytical study of this dependence by solving two hydrodynamic and two MHD problems on an inflow of viscous fluid (or conducting fluid for MHD problems) into a half-space through a split of finite width or through a hole of finite radius. Both the case of a uniform inlet velocity profile and the case of a parabolic inlet velocity profile are considered in the thesis.

In this work another new problem is studied analytically, i.e. the MHD problem on an influence of a cross flow on a main flow in an infinitely long plane channel in a strong magnetic field.

All problems considered in this work are described mathematically by the system of partial differential equations and are solved by means of integral transform, i.e. by using Fourier or Hankel transforms.

**The structure of the thesis:**

This work consists of an introduction, four chapters and conclusions.

1) The 1st chapter is dedicated to the analytical study of the new MHD problem on an inflow of a conducting fluid into a plane channel through a split of finite width on the lateral side of the channel. Cases of longitudinal and transverse magnetic fields are studied in detail. Stokes and inductionless approximations are used for problem solving. The exact analytical solution of the problem is obtained in the form of convergent improper integrals by using Fourier transform. On the basis of obtained results the velocity field is studied numerically. In the case of longitudinal magnetic field, numerical calculations show that the velocity has the M-shaped profiles in the entrance region of the channel at large Hartmann numbers. The physical explanation of these velocity profiles is given in the thesis. The asymptotic solution of the problem at Hartmann number $Ha \to \infty$ is also obtained. In this part of the thesis the important practical result has been obtained for the case of transverse magnetic field, namely, a pressure gradient which is proportional to the square of the Hartmann number is needed for the turning the flow by the angle 90 degrees at large Hartmann numbers, while for pumping the fluid we need a pressure gradient which is proportional to only the first power of the Hartmann number.

2) The 2nd chapter of the thesis is devoted to the theoretical study of problems on an inflow of conducting fluid into channels in the presence of the rotational symmetry in the geometry of the flow. Namely, two problems are solved analytically: the MHD problem on an inflow of a conducting fluid into a plane channel through a round hole of finite radius on a channel's

lateral side and the MHD problem on an inflow of conducting fluid into a circular channel through a split on channel's lateral side.

For the plane channel the case of transverse magnetic fields is studied in detail. For the round channel the problem is solved for longitudinal magnetic field. Both problems are solved in Stokes and inductionless approximations. Hankel transform is used for the solution. Analytical solution of the problem for cylindrical channel is obtained by transforming the $4^{th}$ order linear differential operator with varying coefficients into a product of two differential operators of second order. Similar M-shaped velocity's profiles in the entrance region of the round channel were obtained for longitudinal magnetic field at large Hartman numbers. Asymptotic solution of these problems at Hartmann number $Ha\to\infty$ is obtained.

3) In the $3^{rd}$ part of the thesis the dependence between the boundary velocity profile given at the entrance and the full force of the pressure at the entrance region is studied by means of analytical solutions of two hydrodynamic problems and two similar MHD problems. Namely, the analytical solutions of two hydrodynamic and two MHD problems on an inflow of a viscous fluid (conducting fluid for MHD problems) into a half-space through a plane slit of finite width or through a round hole are obtained. The problems are solved in the Stokes approximation (for MHD problems the inductionless approximation is additionally used). For each problem two cases are considered: the case of uniform inlet velocity profile at the entrance of region and the case of parabolic inlet velocity profile at the entrance. It is shown that in the first case the full pressure force at the entrance is equal to infinity and, consequently, such flow is physically unrealistic. In the second case the full pressure at the entrance of the region has finite value. In the case of the plane split the solution hydrodynamic problem is obtained in terms of elementary functions, but in the case of the round hole the solution is obtained in the form of integrals containing Bessel functions. For the MHD problem the solution is obtained in the form of improper convergent integrals. The new asymptotic solutions for MHD problems are obtained at large Hartmann numbers for the case of parabolic inlet velocity profile.

4) In the $4^{th}$ chapter the analytical solution is obtained for the problem on a MHD flow of conducting fluid in the initial part of a plane channel at the presence of a cross flow. Both the longitudinal magnetic field and transverse magnetic field is considered. The problem is solved in Oseen and inductionless approximation by using Fourier transform provided that only the velocity of fluid is prescribed at the entrance of the channel. Note, that in solving the problem of a flow of viscous fluid in the initial part of the channel by using the Oseen approximation,

it is usually assumed that the velocity and the pressure of the fluid are given at the entrance of the channel ([127], [113]). In our opinion, these boundary conditions overdetermine the problem. It is sufficient to prescribe only the velocity at the entrance of the channel, and that is done in the present work**.** The dependence of the length of the initial part $L_{init}$ on the Hartman number is analyzed. Additionally, the field of perturbation velocity is studied for different Hartmann and Reynolds numbers.

5) The results of all the work are described in the conclusion.

## 0.1. Governing equations.

In this section the governing equations of MHD are presented. In addition, the important dimensionless parameters and assumptions used in this work are discussed.

In general, the system of MHD equations describing the motion of conducting fluid in external magnetic field consists of (see [115], [111], [72]):

1) Navier-Stokes equation of motion including electromagnetic body force $\vec{F}_e = \vec{\tilde{j}} \times \vec{\tilde{B}}$ (so-called Lorentz force):

$$\frac{\partial \vec{\tilde{V}}}{\partial t} + \left( \vec{\tilde{V}} \cdot \nabla \right) \vec{\tilde{V}} = -\frac{1}{\rho} \nabla \tilde{P} + \gamma \nabla^2 \vec{\tilde{V}} + \frac{1}{\rho} \left[ \vec{\tilde{j}} \times \vec{\tilde{B}} \right]; \tag{0.1}$$

2) Equation of mass continuity for incompressible fluids, such as liquid metals:

$$div \vec{\tilde{V}} = 0; \tag{0.2}$$

3) The generalized Ohm's law for a slowly movable medium in a magnetic field (the velocity of medium is much smaller than speed of light):

$$\vec{\tilde{j}} = \sigma \left( \vec{\tilde{E}} + \vec{\tilde{V}} \times \vec{\tilde{B}} \right); \tag{0.3}$$

4) Maxwell's equations:

$$\frac{\partial \vec{\tilde{B}}}{\partial \tilde{t}} = -curl \vec{\tilde{E}}, \tag{0.4}$$

$$\vec{\tilde{j}} = \frac{1}{\mu} curl \vec{\tilde{B}} - \varepsilon \frac{\partial \vec{\tilde{E}}}{\partial \tilde{t}}, \tag{0.5}$$

$$div\tilde{\vec{B}} = 0, \qquad (0.6)$$

$$div\tilde{\vec{E}} = \frac{\rho_e}{\varepsilon}, \qquad (0.7)$$

where $\nabla = \frac{\partial}{\partial \tilde{x}}\vec{e}_x + \frac{\partial}{\partial \tilde{y}}\vec{e}_y$, $\nabla^2 = \frac{\partial^2}{\partial \tilde{x}^2} + \frac{\partial^2}{\partial \tilde{y}^2}$ is the Laplace operator;

$\tilde{\vec{V}}$ is the velocity of the fluid; $\tilde{P}$ is the pressure; $\sigma$ is the electrical conductivity, $\rho$ is the density, $\gamma$ is the kinematic viscosity of the fluid; $\tilde{\vec{B}}$ is the vector of magnetic induction, $\tilde{\vec{E}}$ is the electric field vector, $\tilde{\vec{j}}$ is the electric current density; $\varepsilon$ is the absolute electric permittivity of the fluid; $\mu$ is the absolute magnetic permeability of the fluid. If the fluid is not magnetic, then $\mu \approx \mu_0 = 4\pi \cdot 10^{-7}$ V · s/A · m, where $\mu_0$ is a magnetic permeability in free space.

Note: in this work dimensional quantities are denoted by symbols with tilde, while their dimensionless counterparts – by the same symbols without the tilde.

Let us briefly consider the physical significance of these equations.
Equation (0.1) describes the motion of a conducting fluid in magnetic field and takes into account the action of pressure force $grad\tilde{P}$, viscosity force $\gamma\nabla^2\tilde{\vec{V}}$, inertia force $\left(\tilde{\vec{V}} \cdot \nabla\right)\tilde{\vec{V}}$ and electromagnetic body Lorenz force $\tilde{\vec{j}} \times \tilde{\vec{B}}$ on the element of moving medium. The presence of the electromagnetic Lorenz force generated in the conducting fluid due to the interaction of current $\tilde{\vec{j}}$ with the magnetic field $\tilde{\vec{B}}$, represents the major difference between magnetohydrodynamics and classical hydrodynamics. The electromagnetic force not only exerts an influence on the flow of conducting fluid, but it itself is also changed under the influence of the flow ([89]; [72], [111]).

The Ohm's low (0.3) states that the motion of the conducting fluid in magnetic field induces an electrical current in the fluid. In general, electric field $\tilde{\vec{E}}$ in this law depends not only on the difference of the potentials acting from outside, but also on the particular conditions of the moving medium.

Eq.(0.4), or Faraday's law, states that an alternating magnetic field also creates a vortex of electrical field. In turn, the electrical current passing through the conducting fluid induces its own magnetic field according to Eq.(0.5), or Amper's law. The second term in Eq.(0.5), $-\varepsilon \frac{\partial \vec{\tilde{E}}}{\partial \tilde{t}}$, is called the displacement current. One must take this term into account for fast electromagnetic processes like the propagation of electromagnetic waves, but for slowly varying electromagnetic process in which velocity and related time scales are much smaller compared to the speed of light, one can neglect this term. Working with a conducting fluid one usually neglects this term, due to the fact that conducting current is much larger than the displacement current ([71], [32]). Therefore, the conducting fluid Amper's law (0.5) can be written in the form:

$$\vec{\tilde{j}} = \frac{1}{\mu} curl \vec{\tilde{B}} . \tag{0.8}$$

Taking the divergence of Eq. (0.8) one obtains the charge conservation law:

$$div \vec{\tilde{j}} = 0 . \tag{0.9}$$

It also follows from Eq.(0.4), that in a steady case ($\partial / \partial \tilde{t} = 0$) an electrical field is a potential field and it can be expressed in term of the gradient of a potential φ:

$$\vec{\tilde{E}} = -grad \tilde{\varphi} . \tag{0.10}$$

Last equation Eq.(0.7) of the system (0.1)-(0.7) states that the electric field is created by electric charge. For a good conducting fluid such as liquid metal (there isn't point charge in the fluid) this equation has the form

$$div \vec{\tilde{E}} = 0 . \tag{0.11}$$

Another frequently used equation is the Induction equation:

$$\frac{\partial \vec{\tilde{B}}}{\partial \tilde{t}} - curl(\vec{\tilde{V}} \times \vec{\tilde{B}}) = \frac{1}{\sigma\mu} \nabla^2 \vec{\tilde{B}} . \tag{0.12}$$

This equation is obtained by equating $\vec{j}$ of Eqs. (0.3.) and (0.8), applying the operator *curl* and using Eqs.(0.4) and (0.6).

Using formula $curl(\vec{a} \times \vec{b}) = (\vec{b} \cdot \nabla)\vec{a} - (\vec{a} \cdot \nabla)\vec{b} + \vec{a} \cdot div\vec{b} - \vec{b} \cdot div\vec{a}$ and Eqs.(0.2), (0.6), the Induction equation (0.12) can be also written in the form:

$$\frac{\partial \vec{\tilde{B}}}{\partial \tilde{t}} + \left( \vec{\tilde{V}} \cdot \nabla \right) \vec{\tilde{B}} - \left( \vec{\tilde{B}} \cdot \nabla \right) \vec{\tilde{V}} = \frac{1}{\mu\sigma} \Delta \vec{\tilde{B}}. \qquad (0.13)$$

This induction equation suggests (describes) that in the applied external magnetic field $\vec{B}^e$ the motion of a conducting fluid induces electrical currents, which induce their own magnetic field $\vec{B}^i$ and the total magnetic field is the sum of the external magnetic field and induced magnetic field $\vec{\tilde{B}} = \vec{\tilde{B}}^e + \vec{\tilde{B}}^i$. Thus, the flow of conducting fluid in the external magnetic field modifies the magnetic field inside the volume of the fluid according to Eq.(0.13).

The equations (0.12) and (0.13) are mostly used in problems that analyse the dependence of a full magnetic field on liquid's speed, and conversely, i.e. in formulations for the induced magnetic field instead of the current density. In this case, it is convenient to eliminate the current density and electrical field from the full system of equations (0.1)-(0.7). For these problems the electromagnetic force in the equation of fluid motion (0.1), is represented only by induction of the magnetic field $\vec{\tilde{F}}_e = \frac{1}{\mu} curl \vec{\tilde{B}} \times \vec{\tilde{B}}$. Therefore, the equation of fluid motion has the form

$$\frac{\partial \vec{\tilde{V}}}{\partial \tilde{t}} + \left( \vec{\tilde{V}} \cdot \nabla \right) \vec{\tilde{V}} = -\frac{1}{\rho} \nabla \tilde{P} + \gamma \nabla^2 \vec{\tilde{V}} + \frac{1}{\rho\mu} \left[ curl \vec{\tilde{B}} \times \vec{\tilde{B}} \right] \qquad (0.14)$$

and Eqs. (0.13) and (0.6) are used instead of the system (0.1)-(0.7).
Note that Eq.(0.14) can be also written in the form:

$$\frac{\partial \vec{\tilde{V}}}{\partial t} + \left( \vec{\tilde{V}} \cdot \nabla \right) \vec{\tilde{V}} = -\frac{1}{\rho} \nabla \tilde{P}_m + \gamma \nabla^2 \vec{\tilde{V}} + \frac{1}{\rho\mu} \left( \vec{\tilde{B}} \cdot \nabla \right) \vec{\tilde{B}}, \qquad (0.15)$$

where $\tilde{P}_m = \tilde{P} + \tilde{B}^2 / 2\mu$ is so-called total pressure.

We introduce dimensionless variables using $h$, $V_0$ , $B_0$, $h/V_0$ , $V_0B_0$ , $\rho\nu V_0/h$ as scales of the length, the velocity, magnetic field, time, electrical field and pressure, respectively. Scales $V_0$ and $h$ are chosen in the thesis depending on the problem. For problems on an inflow of conducting fluid in channels, we choose the half-width of the channel as a scale of length $h$, and the magnitude of average velocity $V_0$ at the entrance of the channel is chosen as a scale of the velocity. In all problems, considered in the thesis, $B_0$ is the magnitude of external magnetic field.

Therefore we obtain the following non-dimensional form of the system of MHD equations for the flow of electrically conducting fluid in an external magnetic field:

$$\mathrm{Re}\frac{\partial \vec{V}}{\partial t} + \mathrm{Re}\left(\vec{V}\cdot\nabla\right)\vec{V} = -\nabla P + \nabla^2\vec{V} + Ha^2\left[\vec{j}\times\vec{B}\right], \qquad (0.16)$$

$$div\vec{V} = 0\,, \qquad (0.17)$$

$$\vec{j} = \vec{E} + \vec{V}\times\vec{B}\,, \qquad (0.18)$$

$$\frac{\partial \vec{B}}{\partial t} = -curl\vec{E}\,, \qquad (0.19)$$

$$\mathrm{Re}_m\,\vec{j} = curl\vec{B}\,, \qquad (0.20)$$

$$div\vec{j} = 0\,, \qquad (0.21)$$

$$div\vec{E} = 0\,, \quad div\vec{B} = 0\,. \qquad (0.22)\,,(0.23)$$

The nondimensional Induction equation Eq.(0.13) has the form:

$$\frac{\partial \vec{B}}{\partial t} + \left(\vec{V}\cdot\nabla\right)\vec{B} - \left(\vec{B}\cdot\nabla\right)\vec{V} = \frac{1}{\mathrm{Re}_m}\nabla^2\vec{B}\,. \qquad (0.24)$$

In this system the following dimensionless parameters are used:

$\mathrm{Re} = \dfrac{V_0 h}{\gamma}$ is the Reynolds number. It represents the ratio of inertial to viscous forces in the flow.

$\mathrm{Re}_m = \mu\sigma h V_0$ is the magnetic Reynolds number and it can be interpreted as the ratio of the induced magnetic field to the applied external magnetic field.

$N = \dfrac{\sigma h B_0^2}{\rho V_0} = \dfrac{Ha^2}{\mathrm{Re}}$ is the Stuart number or interaction parameter, which characterizes the ratio of electromagnetic to inertial forces.

$Ha = \sqrt{N\cdot\mathrm{Re}} = B_0 h\sqrt{\dfrac{\sigma}{\rho\gamma}}$ , the Hartmann number, which characterizes the ratio of electromagnetic to viscous forces. The Hartmann number is sufficiently high for all liquid metals.

**The mathematical description of the problems in this work is based on the following assumptions:**

1) We consider the stationary ($\partial / \partial t = 0$) laminar flow of incompressible, viscous and electrically conducting fluid.

2) All physical properties of the fluid (magnetic permeability $\mu$, the electrical conductivity $\sigma$ and the kinematic viscosity $\nu$) are assumed to be constant.

3) We consider MHD flows in a uniform external magnetic field $\vec{B}^e$.

4) The magnetic Reynolds number is much less than one $\mathrm{Re}_m << 1$ (for Lithium blankets, for example, $\mathrm{Re}_m = 10^{-3} - 10^{-1}$, (see [42]). In this case we can use so-called inductionless approximation (see [111], [115],[130]), where we take into account the induced currents, but neglect the magnetic field created by these currents. In other words, we assume that the fluid flows in a strong magnetic field $\vec{B}^e$ and the magnetic field induced by currents in the fluid is negligible compared to the externally applied field $\left|\vec{B}^i\right| << \left|\vec{B}^e\right|$. The dimensional Navier-Stokes equation (0.1) in inductionless approximation has the form:

$$\frac{\partial \vec{\tilde{V}}}{\partial t} + \left(\vec{\tilde{V}} \cdot \nabla\right) \vec{\tilde{V}} = -\frac{1}{\rho} \nabla \vec{\tilde{P}} + \gamma \nabla^2 \vec{\tilde{V}} + \frac{\sigma}{\rho} \left[ \vec{\tilde{E}} + \vec{\tilde{V}} \times \vec{\tilde{B}}^e \right] \times \vec{\tilde{B}}^e . \qquad (0.25)$$

At these assumptions the system of non-dimensional MHD equations (0.16)-(0.23) has the form:

$$\mathrm{Re}\left(\vec{V} \cdot \nabla\right)\vec{V} = -\nabla P + \nabla^2 \vec{V} + Ha^2 \left[\vec{E} + \vec{V} \times \vec{B}^e\right] \times \vec{B}^e , \qquad (0.26)$$

$$div \vec{V} = 0 , \qquad (0.27)$$

$$div \vec{j} = 0 , \qquad (0.28)$$

$$div \vec{E} = 0 . \qquad (0.29)$$

As it was mentioned before, due to the nonlinearity of MHD equations, solution of both two-dimensional and three-dimensional MHD equations is associated with considerable difficulties. Exact solutions for these equations have been obtained only for very specific problems. (By the exact solution we mean an analytical solution, which is obtained when all the terms in governing equations are taken into account). However, for many applications these equations can be linearized by using two linearization schemes known as Oseen and

Stokes approximations (See, for example, [111], [135], [20]). These two approximations are also used in the PhD thesis in order to construct analytical solutions of MHD equations:

1) In the Stokes approximation, the nonlinear term $\left(\vec{\tilde{\tilde{V}}}\cdot\nabla\right)\vec{\tilde{\tilde{V}}}$ that corresponds to the inertia force, is neglected in Eq.(0.26);

2) In the Oseen approximation inertia force is partially taken into account by means of linearization of nonlinear term $\left(\vec{\tilde{\tilde{V}}}\cdot\nabla\right)\vec{\tilde{\tilde{V}}}$. For this purpose the perturbation of the velosity $\vec{\tilde{\tilde{U}}}=\vec{\tilde{\tilde{V}}}-V_0\vec{e}_u$ is introduced and only linear terms with respect to the perturbation are retained in equation. As a result, one uses linearized term $\left(\vec{\tilde{\tilde{V}}}_0\cdot\nabla\right)\vec{\tilde{\tilde{U}}}$ instead of the nonlinear term, where $V_0$=const (for example, $V_0$ is a magnitude of an averaged velocity of the fluid at infinity) ([109]).

The theorem of the existence and uniqueness of solutions both for the problem of Stokes flow and for the problem of Oseen flow is proved in [111]. Note that the Stokes paradox, which takes place in ordinary hydrodynamics, is absent in MHD problems, at least in the case of slowly decreasing at infinity magnetic field (see [111]). Therefore, in MHD there are more cases where one can use Stokes approximation and Oseen approximation, than in hydrodynamics and these approximations give good results (see [111], [135], [109], [108], [120], [121]). For example, these approximations are widely used for the problems on MHD flow in channels and for jet flows. Note that Stokes and Oseen approximations work well in strong magnetic field (N >> 1, M >> 1).

Different techniques have been applied to analyticaly solve linearized MHD equations for simple configurations, such as the method of separation of variables, matched asymptotic expansions, a singular perturbations method, integral equation techniques and so on. In this work we use integral transform technique, namely Fourier and Hankel transforms are used for the solution of linearized MHD equations.

Solving the problems on an inflow of conducting fluid into a plane channel through a split on the channel's lateral side, the following fact is taken into account in the thesis: the electric field is constant in all domain of fluid motion in the case of a plane-parallel flow and this electrical field does not affect the velocity distribution ([111]). If a transverse magnetic field is located in the plane of the flow and both the flow and magnetic field are stationary, then the intensity of electric field is constant in the whole domain of the flow and this intensity vector is perpendicular to the plane of the flow. $E = E_z = const$. Even if $E_x$ and $E_y \neq 0$, they don't affect the motion of the fluid and one can put $E_x = E_y = 0$ (see. [111])]

Solving new problems on MHD flow in channels, the results are also compared with well-known fully developed Poiseuille and Hartmann flows (see [113], [111]). The Poiseuille flow problem is the classic and simple problem of viscous and laminar flow in hydrodynamics (see for example [123]). Poiseuille flow occurs in viscous fluid moving between two plates whose length and width is much greater than the distance separating them, when the flow is driven by a pressure gradient in the direction of the flow in the absence of the magnetic field. Poiseuille flow has the parabolic distribution of velocity (Fig.0.3). The same velocity profile occurs in the similar MHD problem on conducting flow between two nonconducting parallel plates in the presence of longitudinal external magnetic field (i.e. when magnetic field is parallel the flow direction), since the Lorenz force doesn't affect the flow in this case ([111]).

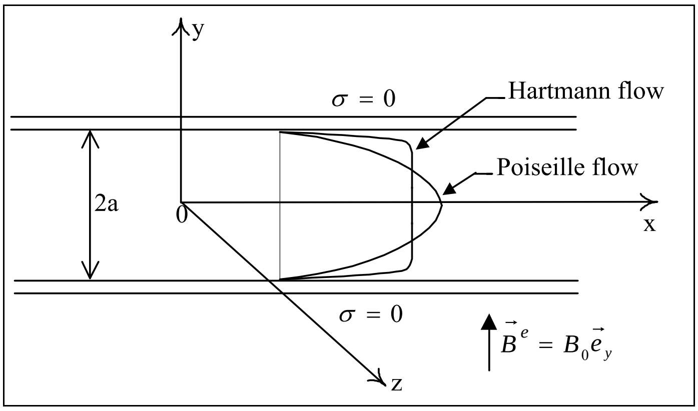



Figure 0.3. The geometry of the flow in a plane channel.

The MHD analogue of the plane Poiseuielle flow problem is the Hartmann problem on a fully developed flow of conducting fluid in the gap between two infinitely long parallel nonconducting plates under the influence of pressure gradient in external transverse magnetic field (see, for instance, [111], [72]). In this case, the velocity profile has a plane shape in the centre of the channel, and it sharply goes to zero near the channel's walls. This flow is known as Hartman flow (see Fig.0.3). With an increasing Hartmann number the velocity profile flattens in the channel core and exhibits thin boundary layers near the walls. The thickness of this boundary layer is $\delta \sim Ha^{-1}$. This phenomenon is called Hartman effect.

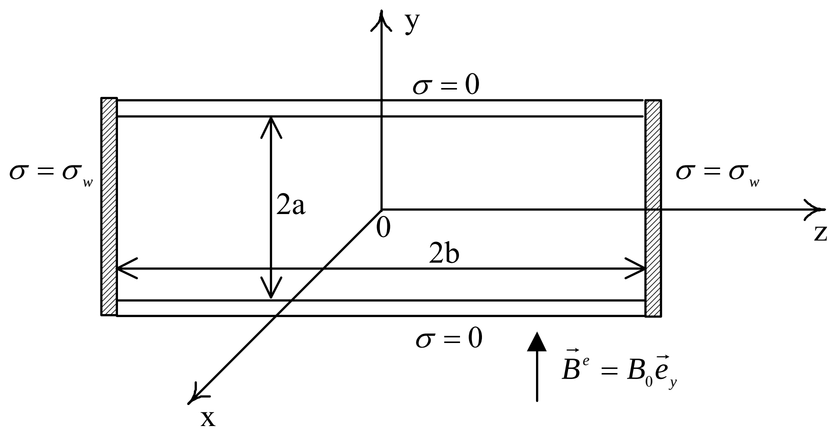



Figure 0.4. The geometry of the channel for the Hartmann flow.

Note, that the same Hartmann flow occurs in a rectangular duct with the large aspect ratio b/a>>1 in the presence of a transverse external magnetic field if the channel's walls $z = \pm b$ have the conductivity $\sigma_w$ which is much larger than conductivity of the fluid, but the walls $y = \pm a$ are non-conducting and external magnetic field is perpendicular to channel walls $y = \pm a$ (see Fig.0.4).

## 0.2. Review of literature

Beginning in the twenties, MHD flows of viscous electrically conducting fluids inside channels have been considered by many researchers for various situations. As was mentioned above, the number of exact solutions in MHD is limited due to nonlinearity of the Navier-Stokes equations. By the exact solution we mean an analytical solution, which is obtained when all the terms in governing equations are taken into account. For the MHD flows in channels the possibility of obtaining exact solutions was practically exhausted by 1970. Review of the analytical solutions of the problems on MHD flows in channels, including exact solutions, can be found, for example, in ([113], [72]). Therefore, only the most important theoretical papers, which are close to the theme of the thesis, are considered below. We also consider the most important theoretical papers devoted to jet flows in magnetic fields. In addition, experimental works dedicated to MHD flows in channels of different geometry, which are closely related to the PhD thesis are briefly summarized below.

We start the survey with the analysis of fundamental papers devoted to fully developed flows in pipes whose cross-section does not vary with respect to the longitudinal coordinate. In addition, only the case of a constant magnetic field is considered here. In this case, exact solutions of MHD equations exist for some simple shapes of the cross-section of the pipe. The flow is fully developed if the velocity profile is no longer changing in the main flow direction, i.e. a flow that has reached a stable steady state driven by a constant pressure gradient.

Hartmann (1937) first investigated experimentally and theoretically a steady fully developed MHD flow in the gap between two infinite parallel nonconducting plates driven by a constant pressure gradient (see [138]). The external magnetic field is uniform and perpendicular to the two channel walls. The lateral channel walls are at infinity. The velocity *V* and induced magnetic field B depend only on one coordinate so that it is a one dimensional problem. The dimensionless solution for velocity *V* was obtained in the form:

$$V(y) = \frac{Ha \cdot \cosh(Ha)}{Ha \cdot \cosh(Ha) - \sinh(Ha)} \left( 1 - \frac{\cosh(Ha \cdot y)}{\cosh(Ha)} \right)$$

For sufficiently high Hartmann numbers the flow is characterized by a core with uniform velocity and by thin boundary layers along the plates. These layers are called Hartmann layers and they are present along walls on which the magnetic field has a normal component. This investigation provided fundamental knowledge for the development of several MHD devices such as MHD pumps, generators, brakes and flow meters (see [72]). Later Chang and Lundgren in work [19] found an analytical solution for a similar problem for conducting walls. The case of two plates with equal conductivities $\sigma_1 = \sigma_2 \neq 0$ was considered. More general case where the conductivites are not equal ($\sigma_1 \neq \sigma_2$) was presented in [25].

There are two classes of analytical solutions for two-dimensional MHD channel flows. The first class, limited to certain channel geometries, comprises the exact solutions which may be represented by series expansion of eigenfunctions (Fourier series). The second class of solutions may be obtained as an asymptotic approximation for large Hartmann number (see, for example, [72]).

For simple rectangular channel geometries (where two-dimensional MHD flows occur) exact analytical solutions have been obtained only in the case of certain combinations of relative conductivities of the walls that are parallel and perpendicular to the magnetic field (see [72], [113]). Shercliff (see [89]) was the first who found an exact solution for 2D MHD flows in rectangular ducts with insulating walls in transverse magnetic field. He pointed out the existence of a second type of boundary layers near walls parallel to the magnetic field, known as side or parallel layers. Later a similar problem was solved for sloping magnetic field by Sloan in [81] and Eraslan in [22]. Chang and Lundgren in [19] and Ufland in [129] obtained exact solutions for MHD flow in channel with perfectly conducting walls. This configuration represents one of the several cases considered by Hunt in [45]. Hunt analyzed two cases: 1) the case of MHD flows in ducts with perfectly conducting Hartmans walls (which are perpendicular to the magnetic field) and thin conducting side walls (which are parallel to the magnetic field), and 2) the case with thin Hartmann walls of arbitrary conductivity and insulating side walls. The solution was found by using Fourier series expansion for the velocity and magnetic field. He examined also the solution for large Hartmann numbers and found out that a variation in the conductivity of the sidewalls has a strong effect on the velocity profile in the parallel boundary layers and on the volume flux carried by these layers. It is shown that in the first case for sufficiently large Ha the velocity in the core of the flow is almost constant. Thin boundary layers appear only near the walls: Hartmann layers of thickness $O(Ha^{-1})$ near the walls perpendicular to the field and side layers (or shear layers) of thickness $O(Ha^{-1/2})$ near the walls parallel to the field. In the second case the velocity in side boundary layers is of order $O(Ha^{1/2})$ while it is of order $O(Ha^{-1/2})$ in the core. This means that the velocity profile along the channel and perpendicular to magnetic field has a typical M-type profile. In the case of perfectly conducting lateral walls this effect disappears and the velocity profile is similar to the profile for the channel where all walls are nonconducting.

Cases with arbitrary conductivity of the walls have to be solved by numerical methods or analytically by asymptotic analysis that focuses on the main phenomena and valid for high Hartmann numbers and interaction parameters (see, for example, [104], [93], [92]). The same comment applies also to 3D flows that occur in bends, expansions, contractions, manifolds and other complex geometry flows. Three-dimensional MHD flows in channels with nonconducting walls are considered in [120] for the first time.

Note that the most useful approaches for asymptotical analysis of MHD flows in channels for blanket conditions are so called “core flow approximation” (for large Hartmann numbers) which neglects inertial and viscous terms ([13], [96], [95]) and methods of matched asymptotic expansions (see, for example, [12] and [72]). For obtaining the approximate solutions by using the second method, the flow domain is divided into the ‘core’ region and the boundary layer regions. The ‘core’ region is located far from the walls and the viscosity is not taken into account in this region. Boundary layer regions are located near channel walls where the viscosity plays an important role. For each region a simplified set of conservation equations holds, which may be solved analytically. The solutions are adjusted to the boundary conditions at the wall and they are matched to each other in order to get a smooth transition between subdomains.

Devices, which have a cylindrical pipe of large length, located in almost constant magnetic field often occur in applications as the major component (in particular, in flow meters). The action of the magnetic field on the fluid flow in such a pipe is physically similar to the flow in a square duct. However, in this case it is difficult to define precisely the domain of existence of Hartmann layers and layers of larger thickness. The exact analytical solution was obtained for the problem on MHD flow in insulating circular ducts in uniform magnetic field in [128], [26]. The solution was obtained in the form of a series of orthogonal functions (Bessel functions depending on the radius and cosine function depending on the angle), which, as many other series in magnetohydrodynamics, is slowly convergent for large Ha. Asymptotic solution of this problem for Ha>>1 was obtained in the paper [91] by Shercliff using the method of matched asymptotic expansions. The analysis of the solution indicated that side layers do not form for the flow in a circular pipe and there exists a boundary layer of another type which does not affect the flow in other regions. This boundary layer appears in the neigborhood of the points where the normal component of the magnetic field is equal to zero. The problem for this boundary layer was solved later in [80].

It was shown later in the papers [120], [121], [90] that abrupt change of the cross-section of the channel, conductivity of the walls, and induction of the magnetic field can lead to the formation of free boundary layers and M-shaped velocity profile. The mechanism of the formation of M-shaped velocity profile is related to the vortex character of the electromagnetic force and is discussed in detail in [115]. It was pointed out in the same work that M-shaped velocity profile is observed only in flows, which are symmetric with respect to the longitudinal axis of the flow direction and induction vector of the magnetic field. In general the action of constant magnetic field can lead to highly nonuniform and asymmetric structure of the flow with respect to the channel cross-section.

As was mentioned above, it is important to study MHD flows in channels of different cross-sections: bends, expansions, contractions, manifolds and other complex geometry flows for metallurgy applications and applications in fusion technology. In this case it is important to analyse MHD processes that occur when the flow makes a turn in a magnetic field, when transition from the flow in a circular channel occurs to the flow in a channel with high aspect ratio, when abrupt expansion of the channel occurs and so on.

It is important to study MHD flows in channels with high aspect ratio for metallurgical applications and the development of Tokamak reactor. For example, MHD flow in magnetohydrodynamic devices often can be considered as a plane parallel flow since the part of a MHD device that is located between the poles of a magnet usually has the form of a narrow rectangular channel. The conceptual idea of a liquid blanket for a Tokamak nuclear reactor was presented in [50], [122] where heat exchange is organized using a system of slotted channels (channels with high aspect ratio) and the slotted channel concept was considered as a method for MHD pressure drop reduction in a liquid metal cooled blanket design. Hence, analysis of MHD flows in such channels in a strong magnetic field is very important.

Many papers are devoted to the experimental analysis of flows in channels with high aspect ratio. A review of experimental results on MHD flow in slotted channels is presented for example in [23]. Analytical solution of the problem for a steady flow of a viscous electrically conducting fluid in a channel with high aspect ratio under external pressure gradient was obtained by [125]. Approximate method of Galerkin - Kantorovich was used in the analysis. The obtained solution takes into account finite conductivity of all walls (parallel and perpendicular to the field).

Later MHD processes in slotted channels that are similar for blanket problems were studied experimentally (see [38], [118]). Namely the MHD phenomena caused by a $90^0$ turn of the flow, flow transition from a circular pipe flow to a flow in a channel with high aspect ratio, and flow resulting in an abrupt expansion of the channel was studied. The dependence of the pressure drop in the model on the value of the mean flow rate velocity was studied. Distributions of the electric potential on the wide walls of the slotted channel and the velocity distribution of the developed flow in the channel were obtained in experiments. In addition, M-shaped velocity distributions of the velocity of such developed flow were obtained in the work. The results indicate that as the intensity of the magnetic field increases the influence of inertia forces on the flow of a liquid metal in a channel with high aspect ratio and M-shaped velocity profiles become more pronounced.

As was pointed out above flows in a plane channel can be considered as the first approximation to the flow in a channel with a high aspect ratio and in the present thesis we study analytically the flow of a conducting fluid in a plane channel for two cases: (a) for the case where fluid flows into a channel through a plane split at the channel’s lateral side and (b) for the case where the fluid inflows a channel through a round hole. Note that problems on inflow of a submerged jet into bounded space are fundamental MHD problems and are quite often considered in literature.

We consider in short the most important theoretical works devoted to the problems on inflow of a submerged jet into semi-space in a magnetic field. These problems have already been fairly fully studied and the generalization of the results obtained before the seventies can be found, for example, in [135]. The study of the MHD problems on the plane or round submerged jet flowing into half-space is usually restricted by the case where width of split or the radius of hole are negligibly small (see [135], [28]). The majority of such problems are analyzed using boundary layers approximations and inductionless approximations. A number of self-similar solutions are obtained. The existence of self-similar solutions for plane submerged jet are considered in [70].

The effect of a transverse magnetic field on the development of a two-dimensional submerged plane jet (the jet comes out of an infinitely long thin slit) of conducting fluid was examined by Moreau ([70], [21]) for the first time. He obtained a self-similar solution for the dimensionless velocity profile in a boundary-layer approximation representing the balance between inertia, magnetic and viscous forces. Quantitative characteristics are proposed in [94] and later in [117] which can be used to describe all the major parameters of the jet (flow rate etc.) and obtain the complete solution of the problem in a boundary layer approximation. In [94], the impulse in the initial cross-section of the jet was used as such characteristic for the case of a uniform magnetic field. In this work the problem was solved also for the case where applied transverse magnetic field is an arbitrary function of the x coordinate. In this work self-similar solution was obtained also for the case of laminar radial slit jet.

The same problem on the effect of an applied uniform transverse magnetic field on the development of a two-dimensional jet of incompressible fluid was examined also in [64]. In this paper qualitatively the same phenomenon – disappearance of the jet at the finite distance from the source – was obtained when all the viscous forces were neglected. It is shown that the viscous similarity solution obtained by Moreau is relevant when M=RN<<1, but for M=RN>>1 the inviscid treatment is appropriate and this solution is irrelevant and boundary layer theory breaks down. This article provides the alternative description of the inviscid flow for M>>1 and a general solution for this case is obtained. For inviscid jet two cases were considered: jet has top-hat profile at x=0 and the initial profile is $U(0, y) = \mathrm{sech}^2 y$. It was noted in this work that the boundary layer theory breaks down as the stopping plane is approached. Finally, the development of a jet of conducting fluid into in nonconducting environment is considered.

The problems on a free jet in uniform homogeneous magnetic field (for two-dimensional boundary layer equations) were also solved by other authors. They employed different methods to solve these problems, for example, it was solved by method of expansion of stream function in a series of small parameters in ( [76], [82], [126], [132]), by the integral method in ([88], [34], [35]) and also by methods involving the **asymptotic** boundary layer, for example in [94]. All the obtained solutions indicate that by applying external magnetic field one can change the shape of the expanded jet.

Several papers were also devoted to the analytical solution of problems for a submerged jet in longitudinal magnetic field (see, for example ([43], [35], [60], [49],[27]).

An axisymmetric jet flow of a viscous incompressible conducting fluid in a strong longitudinal magnetic field in a satellite flow is studied in [29] by using a linear approximation. Neglecting the induced magnetic field, within the framework of the theory of a MHD boundary layer the authors obtained a self-similar solution for a jet with given momentum. Authors investigated the effect of the satellite flow on the profile of the axial velocity component. The exact self-similar solution was obtained for a linearized system of equations of a MHD boundary layer. Authors studied jet flow at large distance from the source in a strong magnetic field. Jets within a concurrent flow were studied also in [117], [107].

Note that under the presence of co-flowing stream where in addition to some integral characteristic of the jet one needs to use also a characteristic of the coflowing stream it is not possible to construct self-similar solutions [135]. Traditional methods of analysis in these cases are based on linearization of the equations of motion which leads to either complete elimination of inertia terms (Stokes' approximation) or to partial retention of these terms (Oseen's approximation).

As it was mentioned above, the study of the MHD problems on the plane or round submerged jet flowing into a half-space is usually restricted to cases of negligibly small width of a split or negligibly small radius of a hole and only some papers are devoted to jet flows from a split of finite width. For example, two-dimensional mixing problem in a uniform co-flowing stream is solved in [131] where initial velocity $u=u_0$ and magnetic field perpendicular to the flow were specified. The solution is obtained under boundary layer approximation and inductionless approximation ([135]).

Explicit solution of the similar problem on jet flow from a nozzle with finite crossection of elliptic shape into a halfspace with co-flowing stream is obtained in ([86]). The problem was solved in inductionless approximation. Oseen's approximation is used to linearize the governing system of equations. The viscous boundary layer approximation is used for the solution.

Later a self-similar solution in a boundary layer approximation is obtained in [136] for a two-dimensional problem of a plane jet flowing into a halfspace from a split of a finite width. The impulse in the initial cross-section of the jet and half-width of the jet at x=0 are considered as quantitative characteristics for the complete solution of the problem. Initial flow rate can be used instead of the half-width of the jet.

Exact analytical solution for the similar problem on a plane submerged jet flowing into half space through split of finite width under the action of a sloping uniform magnetic field was obtained in Stokes and inductionless approximation in [5]. The problem was solved by means of the method of integral transform, namely, by using the Fourier transform. Two cases of a longitudinal and transverse magnetic field are examined in detail. The asymptotic solution of the problem was obtained for these cases for large Hartmann numbers. All boundary layers as well as the asymptotic of the pressure force at the fluid inflowing into half-space have been found for longitudinal and perpendicular strong magnetic fields.

In the thesis we consider the similar problem on a MHD flow, where the conducting fluid is flowing into a channel through a split of finite width or through a hole of finite radius on the lateral side of channel. Plane and round channels are considered in present work. These problems are solved analytically using integral transforms in Stokes' and inductionless approximation.

It was pointed out above that from a practical point of view the problem of a submerged jet in a finite region is the most important one. It cannot be solved theoretically using an exact formulation of the problem.

One problem that is related to it is the flow in a channel with abrupt enlargement. Expansions and contractions, as it was mentioned above, are important geometric elements in liquid metal devices and the study of the flow in such geometries is a key issue for applications in fusion reactor blankets where the flow is distributed from small pipes to large boxes. They are also basic geometric components of any liquid metal device.

Flow in channels with sudden enlargement of the cross-section is well investigated both experimentally (in 60s and 70s) and numerically (see, for example, [112], [110]; [24], [17], [11], [18], [77]).

A detailed survey of experimental results can be found in [112], [135]. One of the first experiments, which studied the influence of confinement of a jet in a rectangular duct with insulating walls, was carried out by Branover and Sherbinin (see [112]) . They analyzed a flow in a plane channel with sudden expansion in a perpendicular magnetic field. Nonmonotonic M-shaped velocity profiles are found for the first time. It is shown in experiments that the jet flowing from the plane slit into a wide rectangular part of the channel in a magnetic field that is perpendicular to the split is separated under the action of a magnetic field into two plane jets flowing near walls parallel to the field and rotated by 90 degrees relative to the output stream. This effect is clearly seen in strong magnetic fields ([101], [102], [103]). Experimental studies were related both to the study of the velocity distribution in the flow ([112] [110]) and potential distribution near sudden expansions and contractors ([24]); pressure distribution along the duct under sudden expansion ([17], [11]).

In [97] the numerical study is presented for the problem on a flow of a viscous incompressible fluid, arising when a flat laminar jet flows into a plane channel through the split of finite width in the front section of the channel under the influence of a magnetic field (for Rem<<1). The problem was reduced to the following problem: a jet of fluid flows into a plane channel through a slit whose width is one-tenth of the channel width. An external uniform magnetic field is in the direction transverse to the channel. The induced currents are assumed to be short circuited through electrodes (the side walls of the channel). The initial velocity profile is taken to be uniform (the value of the velocity at the entrance is equal to unity). In the calculations the position of the slit on the end wall of the channel is varied relative to its axis, and the values of the Reynold's R and Hartmann number Ha are also varied.

Now let us in short consider most important theoretical works devoted to these flows in channels with expansions.

Hunt and Leibovich in [46] studied the two-dimensional MHD flow in linear duct expansion under the action of a strong uniform transverse magnetic field under the assumption that N>>1, Ha>>1 and $Re_m$<<1. As an example, they considered MHD flows in a duct with single side linear expansion and in a duct formed by two different channels of uniform cross-section connected by a smooth linear expansion i.e. the expansions was given as $y = \pm 1$ for $x < 0$ and $y = \pm(1 + x\tan\alpha)$ for $x > 0$. They first tried to understand the structure of parallel layer in duct flow. These layers are formed along the magnetic field either at the solid walls parallel to the applied magnetic field, or inside the fluid at the discontinuities of the duct geometry, electrical conductivity of duct walls, etc. They examined the flow in three separate regions: in the 'core' region in which the pressure gradient is balanced by electro- magnetic forces; in Hartmann boundary layers where electromagnetic forces are balanced by viscous forces and in thin layers parallel to the magnetic field in which electromagnetic forces, inertial forces, and the pressure gradient balance each other. By expanding the solution as a series in descending powers of N they calculated the velocity distribution for the 'core' region, and in Hartmann boundary layer for finite values of N attainable in the laboratory.

Three-dimensional flow in a rectangular channel with smooth expansion in a strong perpendicular uniform magnetic field is considered in [98], [99] for the case of insulating walls. Authors analyzed the flow by using inertialess approximation. Method of asymptotic expansions is used for the solution. It is found that jets are formed in the neighbourhood of lateral walls in the case of nonconducting walls. The velocity in the jets is of the order of $O(Ha^{1/2})$ while in the core the velocity is of order $O(Ha^{-1/2})$, i.e., the velocity profile in the plane perpendicular to the magnetic field has a clearly seen M-shaped form. Similar problem for smooth expansions with thin conducting walls was considered later by Walker in [104] by using the same assumptions, namely high Hartmann number and interaction parameters. Similar but not so pronounced changes of the velocity profile were also obtained in ([101], [102], [103]) for the flow in a circular duct with expansion for the cases of conducting or nonconducting walls.

Later Molokov in [65] solved two-dimensional problem for the sudden duct expansion. The problem was solved by means of two methods: the method of matched asymptotic expansions for high values of Ha , N and Re and the numerical method to verify the asymptotic analysis.

Buhler in [14] considered MHD flow in a duct formed by two different rectangular channels of uniform cross-section connected by a smooth expansion. The problem was studied also by using asymptotic technique. For the analysis, inertia forces are neglected in comparison with Lorentz or pressure force for $N \rightarrow \infty$. He performed a study of the flow, reducing progressively the length of the expansion. This analysis shows that 3D effects disappear for very long expansion. With decreasing expansion length 3D effects become more important. For approaching the geometry of a sudden expansion, namely for an infinitesimally small expansion length, the pressure drop increases and in this limiting situation viscous internal layers become important and stronger 3D phenomena occur.

Note that all the previous analytical investigations are based on the assumptions of strong applied magnetic fields (Ha >>1) and negligible inertial effects in the core region (N>>1). The study of a wider range of parameters which allows, for example, the investigation of the effects of inertia forces, can be performed only by a fully numerical approach. A range of papers are devoted to numerical analysis of 3D MHD flows in duct with single-sided sudden expansion (a sudden expansion of one channel's wall) ([2], [74], [51]). Several works are devoted also to numerical analysis of 3D MHD phenomena in symmetric sudden expansions (see, for example, [63]).

In closing of review of literature about the jets in the bounded space, we note some more works where the geometry of the flow is similar to problem considered in these theses. In the last couple years several works were devoted to the numerical study of the problems on impinging jets in magnetic field. For example, in [52] authors numerically investigate the effects of magnetic field on the two-dimensional fluid flow and heat transfer of confined impinging slot jet in channel (investigate two-dimensional fluid and heat transfer in the confined impinging slot jet flow) in the presence of applied magnetic field. In this problem the uniform slot jet with velocity u=1 flows into the channel though the split of width D and impinge on the opposite wall. The magnetic field is perpendicular to the flow plane. They obtained the numerical solution for unsteady two dimensional Navier-Stokes and energy equations using (equations for liquid flow and heat transfer) by using the finite volume method.

# 1. MHD PROBLEM ON AN INFLOW OF A CONDUCTING FLUID INTO A PLANE CHANNEL THROUGH A SPLIT ON THE CHANNEL'S LATERAL SIDE

This part of thesis is devoted to the analytical study of the new MHD problem on an inflow of conducting fluid into a plane channel through a split of finite width on the channel's lateral side. Cases of longitudinal and transverse magnetic fields are studied in detail (see also author's papers [6], [7], [58]).

## 1.1. General formulation of the problem. The case of a sloping external magnetic field.

The mathematical and geometric formulation of the problem is given in this part of the thesis and main idea of solution is described for the case of a sloping magnetic field.

***Formulation of the problem:*** A plane channel with a conducting fluid is located in the region D=$\{-h<\tilde{y}<h,\ -\infty<\tilde{x}<\infty,\ -\infty<\tilde{z}<\infty\}$. On the channel's lateral side $\tilde{y}=-h$ there is a split in the region $\{-\tilde{L}<\tilde{x}<\tilde{L},\ \tilde{y}=-h,\ -\infty<\tilde{z}<\infty\}$. Conducting fluid flows into the channel through the split with constant velocity $V_0\vec{e}_y$. A strong uniform external magnetic field $\vec{B}^e$ is applied under the angle α to the split, i.e.

$$\vec{B}^e = B_0\cos\alpha\cdot\vec{e}_x + B_0\sin\alpha\cdot\vec{e}_y. \tag{1.1}$$

The geometry of the flow is shown in Fig.1.1.

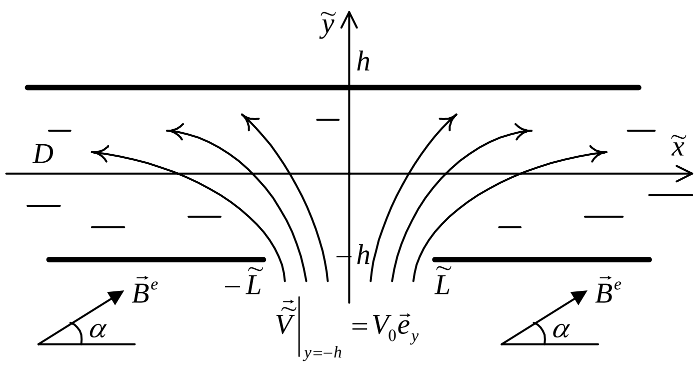



Figure 1.1. The geometry of the flow in the plane channel with a split on the channel's lateral side. The case of a sloping magnetic field.

We consider the case of nonconducting walls $\tilde{y} = \pm h$ and perfectly conducting lateral sidewalls $\tilde{z} = \pm\infty$. In this case the electrical field can be assumed to be zero. This is not an essential assumption for two-dimensional flows. It is shown in [111] that in the case of stationary external magnetic field $\vec{\mathrm{B}}^{\mathrm{e}}$ located in the plane of flow, the intensity of electrical field is of constant magnitude in all the domain of the flow and the vector of this intensity is perpendicular to the plane of flow. Thus, if the motion of fluid occurs in the *X Y* plane, then $E_x$ and $E_y$ don't affect the motion and we can assume that $E_x = E_y = 0$. In this case $E_z = const$ (see [111]).

One more assumption is used below. We suppose that induced streams do not flow through the split $\tilde{y} = -h$, $-\tilde{L} < \tilde{x} < \tilde{L}$ in the region $-\infty < \tilde{y} < -h$.

We introduce dimensionless variables using *h* (half-width of the channel) as the scale of length, $V_0$ (velocity of fluid in the split in the entrance region) as the scale of velocity and $B_0$, $V_0B_0$, $\rho\nu V_0/h$ as the scales of magnetic field, electrical field and pressure, respectively, where $\sigma$, $\rho$, $\nu$ are, respectively , the conductivity, the density and the viscosity of the fluid.

MHD equations (0.26)-(0.27) in Stokes and inductionless approximation have the form (see [115]):

$$-\nabla P + \Delta\vec{V} + Ha^2(\vec{E} + \vec{V} \times \vec{e}_B) \times \vec{e}_B = 0, \tag{1.2}$$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0, \tag{1.3}$$

where $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$,

$\vec{V} = V_x(x,y)\vec{e}_x + V_y(x,y)\vec{e}_y$ is the velocity of the fluid,

$P(x,y)$ is the pressure,

$\vec{e}_B = \cos\alpha \cdot \vec{e}_x + \cos\beta \cdot \vec{e}_y$ is the unit vector of external magnetic field,

$\vec{E} = E_z \cdot \vec{e}_z$ is the intensity of electrical field,

$Ha = B_0 h\sqrt{\dfrac{\sigma}{\rho\nu}}$ is the Hartmann number.

For the determination of the constant $E_z = const$, we use the fact that this flow is the Hartmann flow in a plane channel in external magnetic field $\vec{B}^e = B_0 \sin\alpha \cdot \vec{e}_y$ as x→∞. Therefore, in the case of nonconductive channel walls, we have

$$\vec{E} = -\frac{\sin\alpha}{2} \cdot sign(x)\vec{e}_z \,. \tag{1.4}$$

Projecting Eq.(1.2) onto the x and y axes, we obtain the problem in the form:

$$-\frac{\partial P}{\partial x} + \Delta V_x - Ha^2 \sin\alpha \cdot (E_z + V_x \sin\alpha - V_y \cos\alpha) = 0 \,, \tag{1.5}$$

$$-\frac{\partial P}{\partial y} + \Delta V_y + Ha^2 \cos\alpha \cdot (E_z + V_x \sin\alpha - V_y \cos\alpha) = 0 \,, \tag{1.6}$$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0 \,. \tag{1.7}$$

This problem can be written as:

$$-\frac{\partial P_m}{\partial x} + \Delta V_x - Ha^2 \sin\alpha \cdot (V_x \sin\alpha - V_y \cos\alpha) = 0 \,, \tag{1.8}$$

$$-\frac{\partial P_m}{\partial y} + \Delta V_y + Ha^2 \cos\alpha \cdot (V_x \sin\alpha - V_y \cos\alpha) = 0 \,, \tag{1.9}$$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0 \tag{1.10}$$

where $P_m = P - Ha^2 \cdot E_z \sin\alpha \cdot x + Ha^2 \cdot E_z \cos\alpha \cdot y$,

or $$P_m = P - \left( \frac{Ha^2 \cdot \sin^2\alpha}{2} \cdot x - \frac{Ha^2 \cdot \sin\alpha \cdot \cos\alpha}{2} \cdot y \right) \cdot \mathrm{sign}(x) \,. \tag{1.11}$$

The boundary conditions are:

$$y = -1: \quad V_x = 0, \quad V_y = \begin{cases} 0, & x \notin (-L, L) \\ 1, & x \in (-L, L) \end{cases}; \tag{1.12}$$

$$y = 1: \quad V_x = V_y = 0 \,, \quad \text{where} \quad L = \widetilde{L}/h \,. \tag{1.13}$$

$$x \to \pm\infty: \quad V_x \to V_\infty(y) \cdot \mathrm{sign}(x) \,, \quad \frac{\partial P}{\partial x} \to \frac{dP_\infty}{dx} \equiv A = const \,, \tag{1.14}$$

where $\vec{V}_\infty(y) = V_\infty(y) \cdot \vec{e}_x$ and $dP_\infty / dx \equiv A = const$ are the velocity of the flow and the pressure gradient in the channel sufficiently far away from the entrance region.

Functions $V_\infty(y)$ and $dP_\infty / dx$ depend on the external magnetic field and satisfy the following equations (see [111], [113]):

1) In the case of the longitudinal external magnetic field $\vec{B}^e = B_0 \cdot \vec{e}_x$ ($\alpha = 0$) the Poiseuille flow takes place at $x \to \pm\infty$ and the velocity $\vec{V}_\infty(y)$ with pressure gradient $dP_\infty / dx$ satisfy equation

$$\frac{dP_\infty}{dx} = \frac{d^2 V_\infty(y)}{dy^2} = const \equiv A. \tag{1.15}$$

2) In the case of the transverse magnetic field $\vec{B}^e = B_0 \cdot \vec{e}_y$ ($\alpha = \pi/2$) the Hartmann flow takes place at $x \to \pm\infty$ and $\vec{V}_\infty(y)$ with $dP_\infty / dx$ satisfy equation

$$\frac{dP_\infty}{dx} = \frac{d^2 V_\infty(y)}{dy^2} - Ha^2 V_\infty(y) = const \equiv A. \tag{1.16}$$

3) In the case of sloping magnetic field $\vec{B}^e = B_0 \cos\alpha \cdot \vec{e}_x + B_0 \sin\alpha \cdot \vec{e}_y$ the Hartman flow with Ha·sinα instead of Ha takes place at $x \to \infty$:

$$\frac{dP_\infty}{dx} = \frac{d^2 V_\infty(y)}{dy^2} - Ha^2 \sin^2\alpha \cdot V_\infty(y) = const \equiv A \tag{1.17}$$

The boundary conditions for these equations are: $y = \pm 1: \quad V_\infty(y) = 0$.

***Solution of the problem:*** To obtain the solution of the problem we use **the complex Fourier transform** with respect to x:

$$\hat{f}(\lambda, y) = F[f(x,y)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x,y) e^{-i\lambda x} dx. \tag{1.18}$$

Since $V_x$ and $\partial P / \partial x$ do not tend to zero at $x \to \pm\infty$, we introduce the following new functions for the velocity and pressure gradient before using the Fourier transform:

$$\vec{V}^{new} = \vec{V} - \frac{2}{\pi}\arctan(x) \cdot \vec{V}_\infty(y) \qquad \text{and} \qquad \frac{\partial P^{new}}{\partial x} = \frac{\partial P_m}{\partial x} - \frac{2}{\pi}\arctan(x) \cdot A \tag{1.19}$$

so that $V_x^{new} \to 0$ and $\dfrac{\partial P^{new}}{\partial x} \to 0$ as $x \to \pm\infty$.

As a result, the velocity of the fluid in the channel can be written as:

$$\vec{V} = \left( V_x^{new}(x,y) + \frac{2}{\pi}\arctan(x)\cdot V_\infty(y) \right)\cdot \vec{e}_x + V_y(x,y)\cdot \vec{e}_y \,. \qquad (1.20)$$

Eq.(1.8) becomes:

$$-\frac{\partial P_m^{new}}{\partial x} - \frac{2}{\pi}\arctan(x)\cdot A + \Delta V_x^{new} - \frac{4}{\pi}\frac{x}{(1+x^2)^2}V_\infty(y) - Ha^2\sin\alpha\cdot(V_x^{new}\sin\alpha - V_y\cos\alpha) +$$

$$+\frac{2}{\pi}\arctan(x)\cdot\left( \frac{\partial^2 V_\infty(y)}{\partial y^2} - Ha^2\sin^2\alpha\cdot V_\infty(y) \right) = 0 \,.$$

Since the velocity $V_\infty(y)$ in this case satisfies Eq.(1.17) for the Hartmann flow, the last equation can be written as:

$$-\frac{\partial P_m^{new}}{\partial x} + \Delta V_x^{new} - \frac{4}{\pi}\frac{x}{(1+x^2)^2}V_\infty(y) - Ha^2\sin\alpha\cdot(V_x^{new}\sin\alpha - V_y\cos\alpha) = 0 \,.$$

As a result, problem (1.8)-(1.14) for the new functions has the form:

$$-\frac{\partial P_m^{new}}{\partial x} + \Delta V_x^{new} - Ha^2\sin\alpha\cdot(V_x^{new}\sin\alpha - V_y\cos\alpha) - \frac{4}{\pi}\frac{x}{(1+x^2)^2}V_\infty(y) = 0 \,, \qquad (1.21)$$

$$-\frac{\partial P_m^{new}}{\partial y} + \Delta V_y + Ha^2\cos\alpha\cdot(V_x^{new}\sin\alpha - V_y\cos\alpha) + \frac{Ha^2\sin 2\alpha}{\pi}\arctan(x)\cdot V_\infty(y) = 0 \,, \qquad (1.22)$$

$$\frac{\partial V_x^{new}}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{2}{\pi}\frac{1}{1+x^2}V_\infty(y) = 0 \,. \qquad (1.23)$$

Boundary conditions are:

$$y=-1: \quad V_x^{new} = 0, \qquad V_y = \begin{cases} 0, & x \notin (-L,\, L) \\ 1, & x \in (-L,\, L) \end{cases}; \qquad (1.24)$$

$$y = 1: \quad V_x^{new} = V_y = 0 \,. \qquad (1.25)$$

$$x \to \pm\infty: \quad V_x^{new} \to 0 \,, \qquad \frac{\partial P^{new}}{\partial x} \to 0 \,. \qquad (1.26)$$

Applying the complex Fourier transform (1.18) to Eqs.(1.21)-(1.25) we get the system of ordinary differential equations for the Fourier transforms $\hat{V}_x(\lambda, y) = F[V_x^{new}(x,y)]$, $\hat{V}_y(\lambda, y) = F[V_y(x,y)]$, $\hat{P}(\lambda, y) = F[P_m^{new}(x,y)]$ with the corresponding boundary conditions:

$$-i\lambda\hat{P} + \mathbf{L}\hat{V}_x - Ha^2 \sin\alpha \cdot (\hat{V}_x \sin\alpha - \hat{V}_y \cos\alpha) + \hat{f}_3(\lambda)V_\infty(y) = 0, \tag{1.27}$$

$$-\frac{d\hat{P}}{dy} + \mathbf{L}\hat{V}_y + Ha^2 \cos\alpha \cdot (\hat{V}_x \sin\alpha - \hat{V}_y \cos\alpha) + Ha^2 \cos\alpha \sin\alpha \cdot \hat{f}_1(\lambda)V_\infty(y) = 0, \tag{1.28}$$

$$i\lambda\hat{V}_x + \frac{d\hat{V}_y}{dy} + \hat{f}_2(\lambda)V_\infty(y) = 0, \tag{1.29}$$

where operator $\mathbf{L}$ is $\mathbf{L}f = -\lambda^2 \cdot f + \dfrac{d^2 f}{dy^2}$, (1.30)

$$\hat{f}_1(\lambda) = F\left[\frac{2}{\pi}\cdot \arctan(x)\right] = -i\sqrt{\frac{2}{\pi}}\cdot\frac{e^{-|\lambda|}}{\lambda}, \tag{1.31}$$

$$\hat{f}_2(\lambda) = F\left[\frac{2}{\pi}\cdot\frac{1}{1+x^2}\right] = \sqrt{\frac{2}{\pi}}\cdot e^{-|\lambda|}, \tag{1.32}$$

$$\hat{f}_3(\lambda) = F\left[\frac{-4}{\pi}\cdot\frac{x}{(1+x^2)^2}\right] = i\cdot\sqrt{\frac{2}{\pi}}\cdot e^{-|\lambda|}\cdot\lambda. \tag{1.33}$$

The boundary conditions have the form:

$$y=-1: \quad \hat{V}_x = 0, \quad \hat{V}_y = \sqrt{\frac{2}{\pi}}\frac{\sin(\lambda L)}{\lambda}; \tag{1.34}$$

$$y=1: \quad \hat{V}_x = 0, \quad \hat{V}_y = 0. \tag{1.35}$$

Eliminating $\hat{V}_x$ and $\hat{P}$ from the system (1.27)-(1.29), we get the 4th order differential equation for $\hat{V}_y$:

$$\hat{V}_y^{(4)} - a_1\hat{V}_y'' - a_2\hat{V}_y' + a_3\hat{V}_y - Z(\lambda, y) = 0, \tag{1.36}$$

where

$$a_1 = 2\lambda^2 + Ha^2\sin^2\alpha, \quad a_2 = i\lambda Ha^2 \sin 2\alpha, \quad a_3 = \lambda^4 + \lambda^2 Ha^2 \cos^2\alpha, \tag{1.37}$$

$$Z(\lambda,y) = -V_\infty'''' \cdot \hat{f}_2(\lambda) + V_\infty'(y) Ha^2 \sin^2\alpha \cdot \hat{f}_2(\lambda) + V_\infty'(y)\left(\lambda^2 \hat{f}_2(\lambda) + i\lambda \cdot \hat{f}_3(\lambda)\right) +$$

$$+ V_\infty''''(y) Ha^2 \cos\lambda \sin\lambda \left(\lambda^2 \cdot \hat{f}_1(\lambda) + i\lambda \cdot \hat{f}_2(\lambda)\right).$$

Taking into account formulae (1.31)-(1.33) and the fact that $V_\infty(y)$ satisfies Eq.(1.17), we obtain

$$Z(\lambda,y) = 0.$$

Thus, the differential equation for $\hat{V}_y$ has the form:

$$\hat{V}_y^{(4)} - a_1 \hat{V}_y'' - a_2 \hat{V}_y' + a_3 \hat{V}_y = 0. \tag{1.38}$$

The characteristic equation for Eq.(1.38) is:

$$k^4 - a_1 k^2 - a_2 k + a_3 = 0. \tag{1.39}$$

The 4$^{\text{th}}$ order algebraic equation (1.39) with coefficients (1.37) can be transformed to the following form:

$$(k^2 - \lambda^2)^2 = (k \cdot Ha \cdot \sin\alpha + i\lambda \cdot Ha \cdot \cos\alpha)^2. \tag{1.40}$$

The roots of this equation are:

$$k_{1,2} = \mu\sin\alpha \pm \sqrt{D_1}, \quad k_{3,4} = -\mu\sin\alpha \pm \sqrt{D_2}, \tag{1.41}$$

where

$$D_{1,2} = \mu^2 \sin^2\alpha + \lambda(\lambda \pm 2i\mu\cos\alpha) \quad \text{and} \qquad Ha = 2\mu. \tag{1.42}$$

As a result, the general solution to the differential equation (1.38) has the form

$$\hat{V}_y(\lambda,y) = C_1 \sinh(k_1 y) + C_2 \sinh(k_2 y) + C_3 \cosh(k_3 y) + C_4 \cosh(k_4 y) \tag{1.43}$$

or

$$\hat{V}_y(\lambda,y) = C_1 e^{iBy} \sinh((+)y) + C_2 e^{-iBy} \sinh((-)y) + C_3 e^{iBy} \cosh((+)y) + C_4 e^{-iBy} \cosh((-)y),$$

where $C_1,...,C_4$ are arbitrary constants, $(+) = \mu\sin\alpha + A$, $(-) = \mu\sin\alpha - A$

$A = \operatorname{Re}\sqrt{D_1} = \lambda^2 + \mu^2 \sin^2\lambda$, $B = \operatorname{Im}\sqrt{D_1} = 2\lambda\mu\cos\lambda$.

In order to determine constants $C_1$-$C_4$, one has to use boundary conditions (1.34) and (1.35) and Eq. (1.29) which gives

$$d\hat{V}_y / dy = 0 \quad \text{at} \quad y = \pm 1. \tag{1.44}$$

One can get the solution of the problem by determining $\hat{V}_x$ from Eq.(1.29), $-i\lambda\hat{P}$ from Eq.(1.27), $d\hat{P}/dy$ from (1.28) and applying the inverse Fourier transform in form

$$f(x,y) = F^{-1}[\hat{f}(\lambda,y)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \hat{f}(\lambda,y)e^{i\lambda x}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \hat{f}(\lambda,y)(\cos\lambda x + i\sin\lambda x)dx. \tag{1.45}$$

In the next two paragraphs **two special cases** are considered in detail**:**

1) the external magnetic field $\vec{B}^e = B_0\vec{e}_x$ is parallel to the x-axis (so-called **longitudinal magnetic field** ($\alpha = 0$))
2) the external magnetic field $\vec{B}^e = B_0\vec{e}_y$ is parallel to the y-axis (so-called **transverse magnetic field** ($\alpha = \pi/2$))

In order to simplify the solution of these problems and reduce the number of constants *C₁,..., C₄* in Eq.(1.43) we divide each problems into two sub-problems: odd and even problems with respect to y, by considering a plane channel with two splits on its lateral sides $\tilde{y} = \pm h$ in the region $-\tilde{L} < \tilde{x} < \tilde{L}$.

1) The **odd problem** with respect to y (see Fig.1.2): the fluid with velocities $\mp(V_0\vec{e}_y)/2$, flows into the channel through both splits at $\tilde{y} = \pm h$.

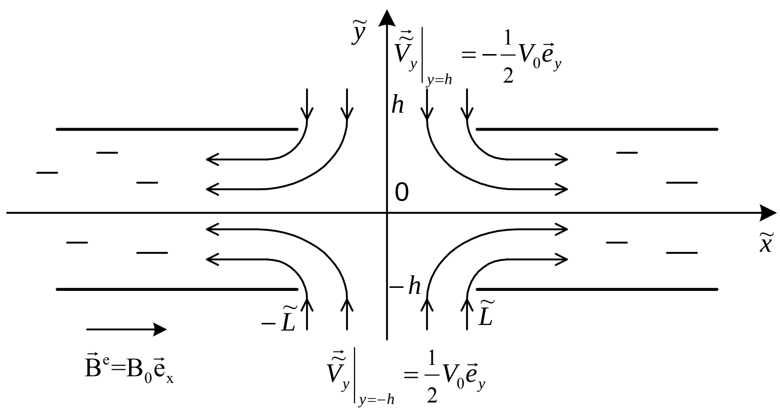



Figure 1.2. Plane channel with a split. The geometry of the flow for the odd problem.

The nondimensional boundary conditions are:

$$y = \pm 1 : V_x = 0,\ \ V_y = \begin{cases} 0, & x \notin (-L,\, L) \\ \mp 1/2, & x \in (-L,\, L) \end{cases} \tag{1.46}$$

$$\text{and } \ x \to \pm\infty : \quad V_x \to V_\infty(y)\cdot \mathrm{sign}(x), \qquad \frac{\partial P}{\partial x} \to \frac{\partial P_\infty}{\partial x} = A = const. \tag{1.47}$$

To solve this problem, new functions for the velocity and pressure gradient are to be introduced according to (1.19). Boundary conditions for these functions are:

$$y = \pm 1 : V_x^{new} = 0,\ \ V_y = \begin{cases} 0, & x \notin (-L,\, L) \\ \mp 1/2, & x \in (-L,\, L) \end{cases} \tag{1.48}$$

$$x \to \pm\infty :\ V_x^{new} \to 0, \qquad \partial P^{new} / \partial x \to 0. \tag{1.49}$$

After the applying to it the Fourier transform becomes:

$$y = \pm 1 :\ \ \hat{V}_y = \mp \frac{1}{2}\sqrt{\frac{2}{\pi}}\frac{\sin(\lambda L)}{\lambda},\ \ \hat{V}_x = 0. \tag{1.50}$$

In this problem $V_y$ is an **odd function with respect to y**, therefore the coefficients $C_3 = C_4 = 0$ in Eq.(1.43).

2) The **even problem** with respect to y (see Fig.1.3): the fluid with velocity $(V_0\vec{e}_y)/2$ flows into the channel through the split in plane $\tilde{y} = -h$ and flows out with the same velocity through the split in $\tilde{y} = h$.

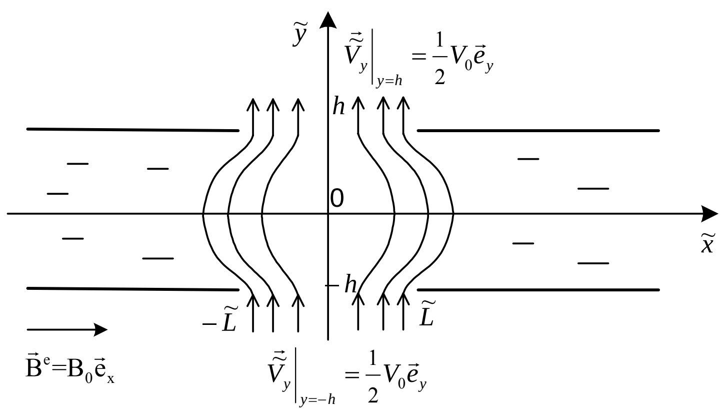



Figure 1.3. Plane channel with a split. The geometry of the flow for the even problem.

The nondimensional boundary conditions for this problem are:

$$y=\pm 1: V_x=0,\ V_y=\begin{cases}0, & x\notin(-L,\,L)\\ 1/2, & x\in(-L,\,L)\end{cases} \tag{1.51}$$

$$\text{and} \qquad x\to\pm\infty: \quad V_x\to 0, \qquad \partial P/\partial x\to 0. \tag{1.52}$$

There is no need to introduce new functions in this case, due to $V_x$ and $\partial P/\partial x$ tend to zero at $x\to\pm\infty$. Therefore, one has to solve the problem (1.8)-(1.11) with boundary conditions (1.51), (1.52) or the problem (1.21)-(1.23) with the same boundary conditions and with

$$V_\infty(y)=0 \qquad \text{and} \qquad \partial P_\infty/\partial x=A=0 \tag{1.53}$$

After applying the Fourier transform to the boundary conditions (1.51) we have:

$$y=\pm 1: \quad \hat{V}_y=\frac{1}{2}\sqrt{\frac{2}{\pi}}\frac{\sin(\lambda L)}{\lambda}, \qquad \hat{V}_x=0. \tag{1.54}$$

Solving the even problem, one must take into account that $V_y$ is an even function with respect to y in this problem and the coefficients $C_1=C_2=0$ in Eq.(1.43).

3) The solution of **the general problem** is equal to the sum of solutions for the odd and even problems with respect to y.

## 1.2. Solution of the problem for the longitudinal magnetic field

In this part of the thesis we solve the problem for the longitudinal magnetic field $\vec{B}^e=B_0\vec{e}_x$. The problem is described by the system (1.8)-(1.11) with $\alpha=0$ that has the form:

$$-\frac{\partial P}{\partial x}+\Delta V_x=0\ , \tag{1.55}$$

$$-\frac{\partial P}{\partial y}+\Delta V_y-Ha^2V_y=0, \tag{1.56}$$

$$\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}=0. \tag{1.57}$$

Boundary conditions for the problem are given by (1.12) - (1.14).

Note that in the case $\alpha=0$, $P_m=P$ since $E_z=0$ (see formula (1.4)). We solve the problem by dividing it into odd and even sub problems with respect to y.

### 1.2.1. Longitudinal magnetic field. Solution of the odd problem with respect to y

We solve the odd problem with respect to y, i.e. the system (1.55)-(1.57) with boundary conditions (1.46), (1.47). The geometry of the flow is shown in Fig.1.2
As it was mentioned above, we have to introduce new functions for the velocity and pressure gradient according to (1.19) to solve the odd problem:

$$\vec{V}^{new} = \vec{V} - \frac{2}{\pi}\arctan(x)\cdot \vec{V}_\infty(y) \quad \text{and} \quad \frac{\partial P^{new}}{\partial x} = \frac{\partial P}{\partial x} - \frac{2}{\pi}\arctan(x)\cdot A .$$

In the case of longitudinal magnetic field, the Poiseuille flow takes place at $x \to \pm\infty$ and functions $\vec{V}_\infty(y)$ and $\partial P_\infty / \partial x \equiv A$ satisfy Eq.(1.15).
The system of equations (1.21)-(1.23) for new functions in the case of longitudinal magnetic field has the form:

$$-\frac{\partial P^{new}}{\partial x} + \Delta V_x^{new} - \frac{4}{\pi}\frac{x}{(1+x^2)^2}V_\infty(y) = 0 , \qquad (1.58)$$

$$-\frac{\partial P^{new}}{\partial y} + \Delta V_y - Ha^2 V_y = 0 , \qquad (1.59)$$

$$\frac{\partial V_x^{new}}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{2}{\pi}\frac{1}{1+x^2}V_\infty(y) = 0 . \qquad (1.60)$$

Boundary conditions for this system are given by (1.48) and (1.49).

The corresponding system of ordinary differential equations for the Fourier transforms $\hat{V}_x(\lambda, y) = F[V_x^{new}(x,y)]$, $\hat{V}_y(\lambda, y) = F[V_y(x,y)]$, $\hat{P}(\lambda, y) = F[P^{new}(x,y)]$ consist of Eqs.(1.27)-(1.30) with $\alpha = 0$ and boundary conditions (1.50) and has the form:

$$-i\lambda\hat{P} + \mathbf{L}\hat{V}_x + \hat{f}_3(\lambda)V_\infty(y) = 0 , \qquad (1.61)$$

$$-\frac{d\hat{P}}{dy} + \mathbf{L}\hat{V}_y - Ha^2\hat{V}_y = 0 , \qquad (1.62)$$

$$i\lambda\hat{V}_x + \frac{d\hat{V}_y}{dy} + \hat{f}_2(\lambda)V_\infty(y) = 0 , \qquad (1.63)$$

$$y = \pm 1: \quad \hat{V}_y = \mp \frac{1}{2}\sqrt{\frac{2}{\pi}}\frac{\sin(\lambda L)}{\lambda}, \quad \hat{V}_x = 0. \tag{1.64}$$

where $\mathbf{L}f = -\lambda^2 f + \dfrac{d^2 f}{dy^2}$, functions $\hat{f}_2(\lambda)$, $\hat{f}_3(\lambda)$ are defined by (1.32) and (1.33).

Science the function $\hat{V}_y$ is the odd function with respect to y, we obtain the general solution of the system (1.61)-(1.63) for $\hat{V}_y$ using formula (1.43) and taking $\alpha = 0$ and $C_3 = C_4 = 0$. As a result we have:

$$\hat{V}_y(\lambda, y) = C_1 \sinh k_1 y + C_2 \sinh k_2 y, \tag{1.65}$$

where $\quad k_1 = \sqrt{\lambda^2 + i \cdot Ha\lambda} \quad , \quad k_2 = \sqrt{\lambda^2 - i \cdot Ha\lambda}$ . (1.66)

In order to determine constant $C_1$ and $C_2$ we use boundary condition (1.64) and following additional boundary condition derived from Eq.(1.63):

$$\frac{d\hat{V}_y}{dy} = 0 \quad \text{at} \quad y = 1. \tag{1.67}$$

As a result, we have

$$\hat{V}_y(\lambda, y) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\Delta_1}(k_1 \cosh k_1 \cdot \sinh k_2 y - k_2 \cosh k_2 \cdot \sinh k_1 y) \cdot \frac{\sin(\lambda L)}{\lambda}, \tag{1.68}$$

where $\Delta_1 = k_2 \cosh k_2 \cdot \sinh k_1 - k_1 \cosh k_1 \cdot \sinh k_2$.

Determining $\hat{V}_x$ from Eq.(1.63), we obtain:

$$\hat{V}_x(\lambda, y) = -\frac{i}{\sqrt{2\pi}} \cdot \frac{\sqrt{\lambda^4 + Ha^2\lambda^2}}{\Delta_1}(\cosh k_2 \cdot \cosh k_1 y - \cosh k_1 \cdot \cosh k_2 y) \cdot \frac{\sin(\lambda L)}{\lambda^2} + \frac{i}{\lambda}\hat{f}_2(\lambda) V_\infty(y), \tag{1.69}$$

Taking into account formulae (1.32) and (1.33), the last term of (1.69) can be written as

$$\frac{i}{\lambda}\hat{f}_2(\lambda) V_\infty(y) = \frac{i}{\lambda}\sqrt{\frac{2}{\pi}} \cdot e^{-|\lambda|} = -\hat{f}_1(\lambda).$$

Thus,

$$\hat{V}_x(\lambda, y) = -\frac{i}{\sqrt{2\pi}} \cdot \frac{\sqrt{\lambda^4 + Ha^2\lambda^2}}{\Delta_1}(\cosh k_2 \cdot \cosh k_1 y - \cosh k_1 \cdot \cosh k_2 y) \cdot \frac{\sin(\lambda L)}{\lambda^2} - \hat{f}_1(\lambda)V_\infty(y) \cdot \tag{1.70}$$

Determining $i\lambda\hat{P}$ from Eq. (1.61) and $\frac{d\hat{P}}{dy}$ from Eq.(1.62) we obtain:

$$\frac{d\hat{P}}{dy} = -\frac{Ha}{\sqrt{2\pi}}\frac{1}{\Delta_1}((i\lambda - Ha)k_2 \cosh k_2 \cdot \sinh k_1 y + (i\lambda + Ha)k_1 \cosh k_1 \cdot \sinh k_2 y) \cdot \frac{\sin \lambda L}{\lambda} \tag{1.71}$$

$$i\lambda\hat{P} = \frac{Ha}{\sqrt{2\pi}}\frac{\sqrt{\lambda^4 + Ha^2\lambda^2}}{\Delta_1}(\cosh k_1 \cdot \cosh k_2 y + \cosh k_2 \cdot \cosh k_1 y) \cdot \frac{\sin(\lambda L)}{\lambda} +$$

$$+ \left(\lambda^2 \hat{f}_1(\lambda) + \hat{f}_3(\lambda)\right) - \hat{f}_1(\lambda) \cdot V''_\infty(y) \tag{1.72}$$

Taking into account formulae (1.31) and (1.33), we have $\lambda^2 \hat{f}_1(\lambda) = -\hat{f}_3(\lambda)$.
Therefore, Eq.(1.72) can be written in the form

$$i\lambda\hat{P} = \frac{Ha}{\sqrt{2\pi}}\frac{\sqrt{\lambda^2 + Ha^2}}{\Delta_1}(\cosh k_1 \cdot \cosh k_2 y + \cosh k_2 \cdot \cosh k_1 y) \cdot \frac{\sin(\lambda L)}{\lambda} - \hat{f}_1(\lambda) \cdot V''_\infty(y). \tag{1.73}$$

In order to obtain the solution to the problem we apply the inverse complex Fourier transform (1.45) to functions $\hat{V}_x(\lambda, y)$, $\hat{V}_y(\lambda, y)$, $\partial\hat{P}(\lambda, y)/\partial y$ and $i\lambda\hat{P}$ in (1.68),(1.70),(1.71),(1.73) taking into account the fact that the functions $\hat{V}_y(\lambda, y)$ and $d\hat{P}/dy$ are even functions with respect to $\lambda$, and the functions $\hat{V}_x(\lambda, y)$ and $i\lambda\hat{P}$ are odd functions with respect to $\lambda$. We also use formula (1.31), i.e. $F^{-1}\left[\hat{f}_1(\lambda)\right] = \frac{2}{\pi} \cdot \arctan(x)$, formulae (1.19) and the fact that $F^{-1}\left[i\lambda\hat{P}\right] = \frac{\partial P}{\partial x}$.

As a result, we get the solution to the odd problem for the longitudinal magnetic field in the form of convergent improper integrals:

$$V_x(x, y) = \frac{1}{\pi}\int_0^\infty \frac{\sqrt{\lambda^2 + Ha^2}}{\Delta_1}(\cosh k_2 \cdot \cosh k_1 y - \cosh k_1 \cdot \cosh k_2 y) \cdot \frac{\sin \lambda L \sin \lambda x}{\lambda} d\lambda, \tag{1.74}$$

$$V_y(x,y)=\frac{1}{\pi}\int_0^\infty \frac{1}{\Delta_1}(k_1\cosh k_1\cdot\sinh k_2 y-k_2\cosh k_2\cdot\sinh k_1 y)\cdot\frac{\sin\lambda L\cos\lambda x}{\lambda}d\lambda\,, \qquad (1.75)$$

$$\frac{\partial P}{\partial x}=\frac{Ha}{\pi}i\int_0^\infty \frac{\sqrt{\lambda^2+Ha^2}}{\Delta_1}(\cosh k_1\cdot\cosh k_2 y+\cosh k_2\cdot\cosh k_1 y)\cdot\sin\lambda L\sin\lambda x d\lambda\,, \qquad (1.76)$$

$$\frac{\partial P}{\partial y}=-\frac{Ha}{\pi}\int_0^\infty \frac{1}{\Delta_1}((i\lambda-Ha)k_2\cosh k_2\cdot\sinh k_1 y+(i\lambda+Ha)k_1\cosh k_1\cdot\sinh k_2 y)\cdot\frac{\sin\lambda L\cos\lambda x}{\lambda}d\lambda \qquad (1.77)$$

where $\Delta_1=k_2\cosh k_2\cdot\sinh k_1-k_1\cosh k_1\cdot\sinh k_2$,

$$k_1=\sqrt{\lambda^2+i\cdot Ha\lambda},\quad k_2=\sqrt{\lambda^2-i\cdot Ha\lambda}\,.$$

We obtain the limits $\lim_{x\to\infty}V_x(x,y)$, $\lim_{x\to\infty}\partial P_m/\partial x$ for this problem to check solution. For this purpose we put $\lambda x=t$ in (1.74), (1.75) and (1.76) and pass to limit as x→∞. The investigation shows that $V_y\to 0$, $V_x\to V_p(y)$ at $x\to\infty$, where $V_p(y)$ Poiseille's flow velocity in the plane channel of width $2h$ with nonconducting walls, if the average velocity in the channel is equal to $V_0$.

### 1.2.2. Longitudinal magnetic field. Solution of the even problem with respect to y

In the present part of the thesis we solve the even problem with respect to y for the longitudinal magnetic field, i.e. we solve the system (1.55)-(1.57) with boundary conditions (1.51), (1.52). The geometry of the flow is shown in Fig.1.3. As it was mentioned above, there is no need to introduce new functions for velocity and pressure gradient in this case, due to $V_x$ and $\partial P/\partial x$ tends to zero at $x\to\pm\infty$. The system of ordinary differential equations for the Fourier transforms $\hat{V}_x(\lambda,y)=F[V_x(x,y)]$, $\hat{V}_y(\lambda,y)=F[V_y(x,y)]$, $\hat{P}(\lambda,y)=F[P(x,y)]$ for this problem has the form (1.61)-(1.63) with conditions (1.53), i.e.:

$$-i\lambda\hat{P}+\mathbf{L}\hat{V}_x=0\,, \qquad (1.78)$$

$$-\frac{d\hat{P}}{dy}+\mathbf{L}\hat{V}_y-Ha^2\hat{V}_y=0\,, \qquad (1.79)$$

$$i\lambda\hat{V}_x+\frac{d\hat{V}_y}{dy}=0\,, \qquad (1.80)$$

where operator $\mathbf{L}$ is given by formula (1.30).

Boundary conditions for this system are given by (1.54) and have the form

$$y = \pm 1: \quad \hat{V}_y = \frac{1}{2}\sqrt{\frac{2}{\pi}}\frac{\sin(\lambda L)}{\lambda}, \quad \hat{V}_x = 0. \tag{1.81}$$

Since $\hat{V}_y$ is the even function with respect to y in this problem and the constants $C_1 = C_2 = 0$ in Eq.(1.43), the general solution of the system (1.78)-(1.80) for $\hat{V}_y$ is:

$$\hat{V}_y(\lambda, y) = C_3 \cosh k_1 y + C_4 \cosh k_2 y. \tag{1.82}$$

In order to determine $C_3$ and $C_4$ we use boundary conditions (1.81) and (1.67).

Determining $\hat{V}_x$ from Eq.(1.80), $i\lambda\hat{P}$ from (1.78) and $d\hat{P}/dy$ from (1.79) and applying the inverse Fourier transform (1.45) to these functions we obtain the solution to the even problem (1.55)-(1.57) with boundary conditions (1.51), (1.52) in the form of convergent improper integrals:

$$V_x(x,y) = \frac{1}{\pi}\int_0^\infty \frac{\sqrt{\lambda^2 + Ha^2}}{\Delta_2}(\sinh k_1 \cdot \sinh k_2 y - \sinh k_2 \cdot \sinh k_1 y)\cdot\frac{\sin\lambda L \sin\lambda x}{\lambda}d\lambda, \tag{1.83}$$

$$V_y(x,y) = \frac{1}{\pi}\int_0^\infty \frac{1}{\Delta_2}(k_2 \sinh k_2 \cdot \cosh k_1 y - k_1 \sinh k_1 \cdot \cosh k_2 y)\cdot\frac{\sin\lambda L \cos\lambda x}{\lambda}d\lambda, \tag{1.84}$$

$$\frac{\partial P}{\partial x} = -\frac{Ha}{\pi} i\int_0^\infty \frac{\sqrt{\lambda^2 + Ha^2}}{\Delta_2}(\sinh k_2 \cdot \sinh k_1 y + \sinh k_1 \cdot \sinh k_2 y)\cdot \sin\lambda L \sin\lambda x d\lambda, \tag{1.85}$$

$$\frac{\partial P}{\partial y} = \frac{Ha}{\pi}\int_0^\infty \frac{1}{\Delta_2}((i\lambda - Ha)k_2 \sin k_2 \cdot \cosh k_1 y + (i\lambda + Ha)k_1 \sinh k_1 \cdot \cosh k_2 y)\cdot\frac{\sin\lambda L \cos\lambda x}{\lambda}d\lambda, \tag{1.86}$$

where $\Delta_2 = k_2 \sinh k_2 \cdot \cosh k_1 - k_1 \sinh k_1 \cdot \cosh k_2$,

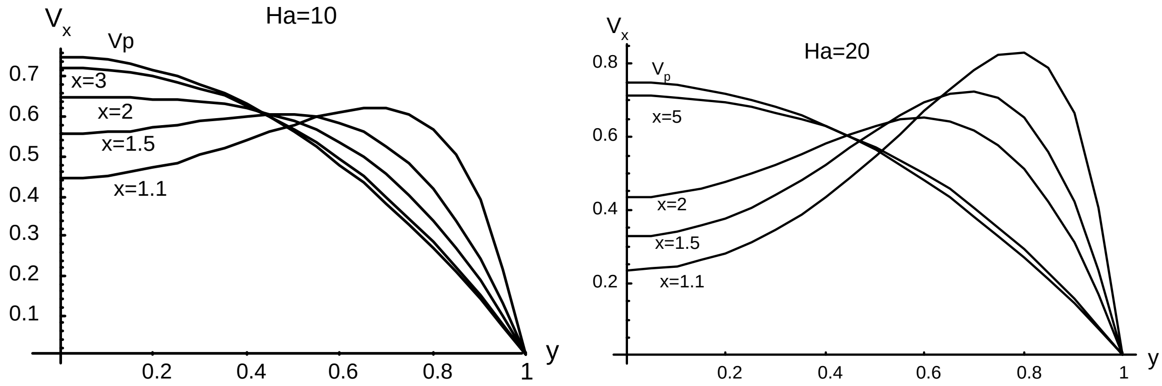
$$k_1 = \sqrt{\lambda^2 + i\cdot Ha\lambda}, \quad k_2 = \sqrt{\lambda^2 - i\cdot Ha\lambda}.$$

### 1.2.3. Longitudinal magnetic field. Numerical results and discussion.

On the basis of the obtained results in the form of improper integrals, the velocity field was studied numerically by using the package “Mathematica”.

**<u>1) Odd problem with respect to y:</u>**

Fig.1.4 plots the profiles of the velocity component $V_x$ calculated by means formula (1.74) for Ha=10, Ha=20, Ha=50. Note that component $V_x$ is an **even function** with respect to **y**.



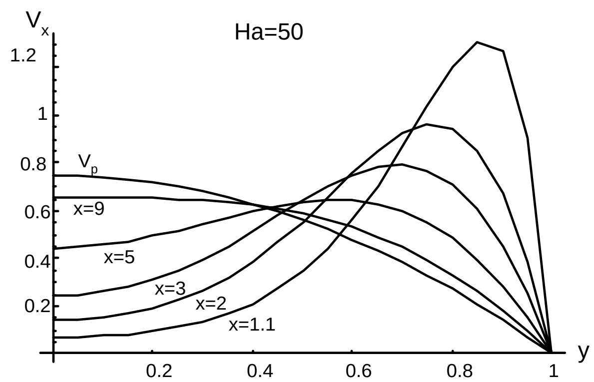




Figure 1.4. Profiles of the velocity component $V_x$ for the odd problem and $\vec{B}^e = B_0\vec{e}_x$

($V_p$ is the Poiseuille flow).

It can be seen from Fig.1.4 that at $Ha \geq 10$ the velocity component $V_x$ has the M-shaped profiles in the channel’s initial part. Note, that the M-shaped profiles become more pronounced as the Hartmann number increases.

For qualitative explanation of this phenomenon, we evaluate the electromagnetic force at the entrance of the channel. The vector of current density is equal to

$$\vec{j}\Big|_{y=-1} = \sigma\vec{V}\Big|_{y=-1} \times \vec{B}^e = \sigma \cdot \left\{0, \frac{1}{2}V_0, 0\right\} \times \left\{B^e, 0, 0\right\} = -\sigma\frac{V_0}{2}B^e\vec{e}_z .$$

Consequently, the electromagnetic force on the entrance of the channel is

$$\vec{F}\Big|_{y=-1} = \vec{j} \times \vec{B}^e = \left\{0,\, 0,\, -\frac{\sigma V_0}{2} B^e \right\} \times \left\{B^e,\, 0,\, 0\right\} = -\frac{\sigma V_0}{2}(B^e)^2 \vec{e}_y\,.$$

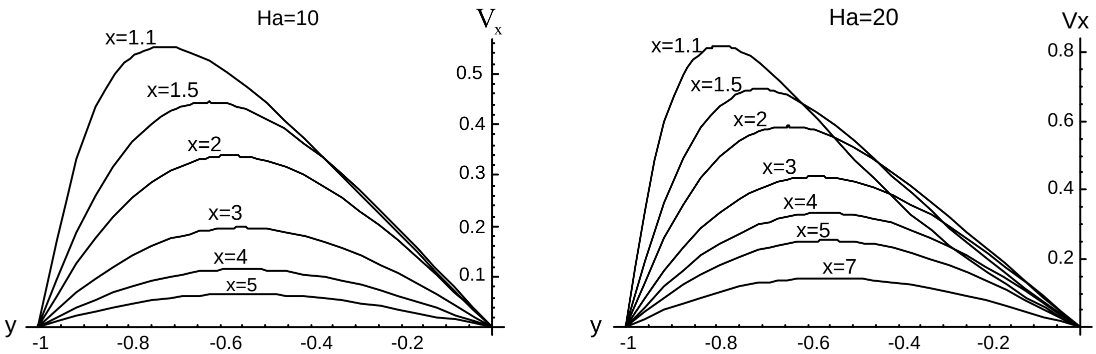
Similarly, at $y = 1$ we have: $\vec{F}\Big|_{y=1} = \vec{j} \times \vec{B}^e = \frac{\sigma V_0}{2}(B^e)^2 \vec{e}_y\,.$

It means that the electromagnetic force acts on the fluid in the direction of the channel's wall. In addition, we can see that the larger velocity's component $V_y$ and Hartmann number, the larger the electromagnetic force. This electromagnetic force creates the M-shaped profile of velocity's component $V_x$.

In addition, it can be seen from Fig. 1.4 that the stronger is the magnetic field (the Hartmann number is larger) the farther away from the entrance region the flow approaches the Poiseille's flow, i.e. the length of the initial part increases. In the present problem the initial part of the channel is defined as the part where the x-component of the velocity $\vec{V}(x, y)$ differs from the Poiseuille flow $V_p$ by less than 1%.

**2) Even problem with respect to y:**

Fig.1.5 presents the profiles of the velocity component $V_x$ calculated by means formula (1.83) for $-1 \le y \le 0$. In this case the function $V_x$ is an **odd function** with respect to **y**.



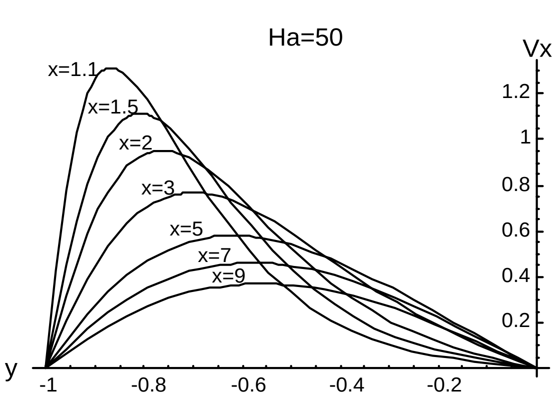
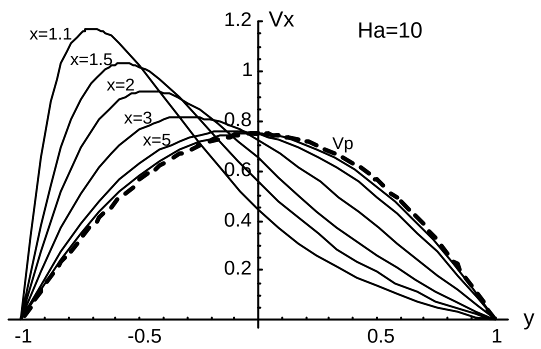



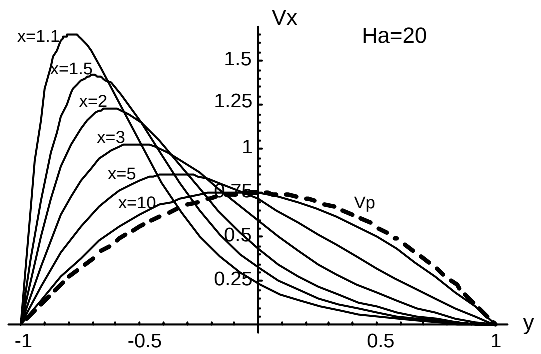

Figure 1.5. Profiles of the velocity component $V_x$ for the even problem and $\vec{B}^e = B_0\vec{e}_x$.

It can be seen from Fig. 1.5 that $V_x \to 0$ at $x \to \infty$ and the magnitude of the velocity component $V_x$ slower approach zero as the Hartmann number increases. In addition, in the channel's entrance region the component $V_x(x, y)$ of the velocity has the M-shaped profiles for small x ($1 < x < 5$) and large Hartmann numbers ($Ha = 50$).

**<u>3) The general problem for the case of longitudinal magnetic field</u>**:

The solutions of the general problem (1.8)-(1.10), (1.12)-(1.13) at $\alpha = 0$ are equal to the sum of solutions for the even and odd problems. The profiles of the velocity component $V_x$ are shown in Fig.1.6 for the Hartmann numbers Ha=10, Ha=20 and Ha=50.



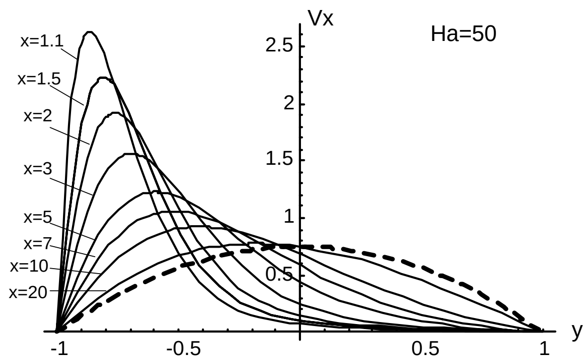




Figure 1.6. Profiles of the velocity component $V_x$ for the general problem and $\vec{B}^e = B_0\vec{e}_x$ ( – – – $V_p$ is the Poiseuille flow)

One can see that near the entrance split the flow mostly occurs along the wall with the split (at $y = -1$). As the Hartmann number increases, the layer of the flow is getting narrower and the velocity increases in this layer. The Poiseuille flow takes place at a distance from the entrance split. In addition, when the Hartmann number grows (i.e. with the increase of the intensity of the magnetic field), the flow in the channel slowly approaches Poiseuille flow. For instance, for $Ha = 10$ Poiseuille flow takes place already at $x = 6$, for $Ha = 20$ at $x = 12$

and for $Ha = 50$ only at $x > 20$. So $L_{init}$ increases as the Hartmann number grows. $L_{init}$ is the length of the initial part of the channel, where the x component of the velocity $\vec{V}(x, y)$ of the fluid differs from the velocity of the Poiseuille flow $V_P$ by less than 1%.

## 1.3. Solution of the problem for the transverse magnetic field

In this part of the thesis we solve the problem on an inflow of a conducting fluid into a plane channel through the split on the channel's lateral side for the transverse external field $\vec{B}^e = B_0 \vec{e}_y$. In this case $\alpha = \dfrac{\pi}{2}$ in Eqs. (1.8)-(1.10) and the system of dimensionless equations that describes this problem has the form:

$$-\frac{\partial P_m}{\partial x} + \Delta V_x - Ha^2 V_x = 0 \ , \tag{1.87}$$

$$-\frac{\partial P_m}{\partial y} + \Delta V_y = 0 \, , \tag{1.88}$$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0 \, . \tag{1.89}$$

$P_m = P - \dfrac{Ha^2}{2} x \cdot \operatorname{sign}(x)$ (see formula (1.11)).

Similarly as it was done for the longitudinal magnetic field, we solve the problem by splitting it into odd and even sub problems with respect to y.

### 1.3.1. Transverse magnetic field. Solution of the odd problem with respect to y

The geometry of the flow is shown in Fig.1.2, but the external magnetic field is perpendicular to the channel's walls. We solve the odd problem with respect to y, i.e. the system (1.87)-(1.89) with boundary conditions (1.46), (1.47).

In the case of transverse magnetic field $V_\infty(y)$ is the velocity of the Hartmann flow and satisfies Eq.(1.16).

To solve the problem we have to introduce new functions for velocity and pressure gradient according to (1.19), as it was done in the case of slopping magnetic field:

$$\vec{V}^{new} = \vec{V} - \frac{2}{\pi}\arctan(x)\cdot \vec{V}_\infty(y) \quad \text{and} \quad \frac{\partial P^{new}}{\partial x} = \frac{\partial P_m}{\partial x} - \frac{2}{\pi}\arctan(x)\cdot A.$$

Then the system of equations (1.21)-(1.23) for new functions $\vec{V}^{new}$ and $P^{new}$ in the case of transverse magnetic field has the form:

$$-\frac{\partial P^{new}}{\partial x} + \Delta V_x^{new} - \frac{4}{\pi}\frac{x}{(1+x^2)^2}V_\infty(y) - Ha^2 V_x = 0, \tag{1.90}$$

$$-\frac{\partial P^{new}}{\partial y} + \Delta V_y = 0, \tag{1.91}$$

$$\frac{\partial V_x^{new}}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{2}{\pi}\frac{1}{1+x^2}V_\infty(y) = 0. \tag{1.92}$$

Boundary conditions for the odd problem are given by (1.48) and (1.49).

The corresponding system (1.27)-(1.29) of ordinary differential equations for Fourier transforms $\hat{V}_x(\lambda, y) = F[V_x^{new}(x,y)]$, $\hat{V}_y(\lambda, y) = F[V_y(x,y)]$, $\hat{P}(\lambda, y) = F[P^{new}(x,y)]$ with the boundary conditions (1.50) has the form:

$$-i\lambda\hat{P} + \mathbf{L}\hat{V}_x - Ha^2\hat{V}_x + \hat{f}_3(\lambda)V_\infty(y) = 0, \tag{1.93}$$

$$-\frac{d\hat{P}}{dy} + \mathbf{L}\hat{V}_y = 0, \tag{1.94}$$

$$i\lambda\hat{V}_x + \frac{d\hat{V}_y}{dy} + \hat{f}_2(\lambda)V_\infty(y) = 0, \tag{1.95}$$

$$y = \pm 1: \quad \hat{V}_y = \mp\frac{1}{2}\sqrt{\frac{2}{\pi}}\frac{\sin(\lambda L)}{\lambda}, \quad \hat{V}_x = 0. \tag{1.96}$$

where $\mathbf{L}f = -\lambda^2 f + \dfrac{d^2 f}{dy^2}$ and $\hat{f}_2(\lambda)$, $\hat{f}_3(\lambda)$ are defined by formulae (1.32) and (1.33).

Since the function $\hat{V}_y$ is the odd function with respect to y in the odd problem, the general solution for the $\hat{V}_y$ can be obtained from (1.43) putting $C_3 = C_4 = 0$ and taking into account that $\alpha = \dfrac{\pi}{2}$.

As a result we have:

$$\hat{V}_y(\lambda, y) = C_1 \sinh k_1 y + C_2 \sinh k_2 y \tag{1.97}$$

where $C_1,...,C_2$ are arbitrary constants and

$$k_1 = \mu + \sqrt{\mu^2 + \lambda^2} \quad , \quad k_2 = \mu - \sqrt{\mu^2 + \lambda^2} \, . \tag{1.98}$$

In order to determine $C_1$ and $C_2$ we use boundary condition (1.96) and following additional boundary condition derived from Eq.(1.95):

$$\frac{d\hat{V}_y}{dy} = 0 \quad \text{at} \quad y = 1 \, . \tag{1.99}$$

As a result, we have

$$\hat{V}_y(\lambda, y) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\Delta_1} (k_1 \cosh k_1 \cdot \sinh k_2 y - k_2 \cosh k_2 \cdot \sinh k_1 y) \cdot \frac{\sin(\lambda L)}{\lambda} \, , \tag{1.100}$$

where $\Delta_1 = k_2 \cosh k_2 \cdot \sinh k_1 - k_1 \cosh k_1 \cdot \sinh k_2$.

Determining $\hat{V}_x$ from Eq.(1.95) we obtain:

$$\hat{V}_x(\lambda, y) = \frac{i}{\sqrt{2\pi}} \cdot \frac{1}{\Delta_1} (\cosh k_2 \cdot \cosh k_1 y - \cosh k_1 \cdot \cosh k_2 y) \cdot \sin(\lambda L) + \frac{i}{\lambda} \hat{f}_2(\lambda) V_\infty(y) \, , \tag{1.101}$$

Taking into account formulae (1.32) and (1.31), the last term of (1.101) can be simplified. Thus we have:

$$\hat{V}_x(\lambda, y) = \frac{i}{\sqrt{2\pi}} \cdot \frac{1}{\Delta_1} (\cosh k_2 \cdot \cosh k_1 y - \cosh k_1 \cdot \cosh k_2 y) \cdot \sin(\lambda L) - \hat{f}_1(\lambda) V_\infty(y) \, . \tag{1.102}$$

Determining $i\lambda\hat{P}$ from Eq. (1.93) and $d\hat{P}/dy$ from Eq.(1.94) we obtain:

$$\frac{\partial \hat{P}}{\partial y} = -\frac{Ha}{\sqrt{2\pi}} \frac{\lambda}{\Delta_1} (\cosh k_1 \cdot \sinh k_2 y - \cosh k_2 \cdot \sinh k_1 y) \cdot \sin(\lambda L) \tag{1.103}$$

$$i\lambda\hat{P} = -\frac{i \cdot Ha}{\sqrt{2\pi}} \frac{1}{\Delta_1} (k_1 \cosh k_1 \cdot \cosh k_2 y - k_2 \cosh k_2 \cdot \cosh k_1 y) \cdot \sin(\lambda L) + S \qquad (1.104)$$

where

$$S = \hat{f}_3(\lambda) \cdot V_\infty(y) - i\lambda \hat{f}_2(\lambda) V_\infty + i \frac{\hat{f}_2(\lambda)}{\lambda} \left(-Ha^2 V_\infty + V''_\infty\right) \qquad (1.105)$$

Taking into account formulae (1.32), (1.33) and the fact that function $V_\infty$ satisfies Eq.(1.16), we obtain

$$S = -\hat{f}_1(\lambda) \cdot A.$$

Therefore, Eq.(1.104) can be written in the form

$$i\lambda\hat{P} = -\frac{i \cdot Ha}{\sqrt{2\pi}} \frac{1}{\Delta_1} (k_1 \cosh k_1 \cdot \cosh k_2 y - k_2 \cosh k_2 \cdot \cosh k_1 y) \cdot \sin(\lambda L) - \hat{f}_1(\lambda) \cdot A. \qquad (1.106)$$

Applying the inverse complex Fourier transform (1.45) to the functions $\hat{V}_x$, $\hat{V}_y$, $d\hat{P}/dy$ and $i\lambda\hat{P}$ in (1.102), (1.100), (1.103), (1.106) , we take into account the fact that the functions $\hat{V}_y$ and $d\hat{P}/dy$ are even functions with respect to $\lambda$ and the functions $\hat{V}_x$ and $i\lambda\hat{P}$ are odd functions with respect to $\lambda$. We also use formula (1.31), i.e. $F^{-1}\left[\hat{f}_1(\lambda)\right] = \frac{2}{\pi} \cdot \arctan(x)$, formulae (1.19) and the fact that $F^{-1}\left[i\lambda\hat{P}\right] = \frac{\partial P}{\partial x}$.

As a result, we have the solution to problem odd with respect problem (1.87)-(1.89), (1.46)-(1.47) for the case of transverse magnetic field in the form of convergent improper integrals:

$$V_x(x, y) = -\frac{1}{\pi} \int_0^\infty \frac{1}{\Delta_1} (\cosh k_2 \cdot \cosh k_1 y - \cosh k_1 \cdot \cosh k_2 y) \cdot \sin\lambda L \sin\lambda x d\lambda, \qquad (1.107)$$

$$V_y(x, y) = \frac{1}{\pi} \int_0^\infty \frac{1}{\Delta_1} (k_1 \cosh k_1 \cdot \sinh k_2 y - k_2 \cosh k_2 \cdot \sinh k_1 y) \cdot \frac{\sin\lambda L \cos\lambda x}{\lambda} d\lambda, \qquad (1.108)$$

$$\frac{\partial P_m}{\partial x} = -\frac{Ha}{\pi} \int_0^\infty \frac{1}{\Delta_1} (k_1 \cosh k_1 \cdot \cosh k_2 y - k_2 \cosh k_2 \cdot \cosh k_1 y) \cdot \sin\lambda L \sin\lambda x d\lambda, \qquad (1.109)$$

$$\frac{\partial P_m}{\partial y} = -\frac{Ha}{\pi} \int_0^\infty \frac{\lambda}{\Delta_1} (\cosh k_1 \cdot \sinh k_2 y - \cosh k_2 \cdot \sinh k_1 y) \cdot \sin\lambda L \cos\lambda x d\lambda \qquad (1.110)$$

where $\Delta_1 = k_2 \cosh k_2 \cdot \sinh k_1 - k_1 \cosh k_1 \sinh k_2$,

$$k_1 = \mu + \sqrt{\mu^2 + \lambda^2} \quad , \quad k_2 = \mu - \sqrt{\mu^2 + \lambda^2}.$$

We obtain limits $\lim\limits_{x\to\infty} V_x(x,y)$, $\lim\limits_{x\to\infty} \partial P_m / \partial x$. For this purpose we put $\lambda x = t$ in (1.107), (1.109) and pass to limit as x→∞.: The analysis shows that

$$\lim_{x\to\infty} V_x = V_\infty(y) = \frac{L}{2}\frac{Ha}{(Ha - \tanh Ha)}\left(1 - \frac{\cosh(Ha \cdot y)}{\cosh Ha}\right) \quad \text{and} \quad V_y \to 0, \qquad (1.111)$$

$$\lim_{x\to\infty} \frac{\partial P_m}{\partial x} = -\frac{L}{2}\frac{Ha^3}{Ha - \tanh Ha} \qquad (1.112)$$

Formulae (1.111)-(1.112) coincide with the Hartmann flow in the plane channel of width 2 with nonconducting walls, if the average velocity in the channel is equal to ½L (see [6]). Consequently, formula (1.111) may be used for the calculation of the length of the initial part of the channel $L_{init}$ on which flow in the channel pass into the Hartmann flow (1.111).

**The asymptotic evaluation of the solution at Ha →∞ for the odd case:**

As it was mentioned in the introduction of the thesis, the greatest interest for applications in the reactor Tokamak is the determination of the following limits:

$$\lim_{\mu\to\infty} \left.\frac{\partial P_m}{\partial y}\right|_{y=-1} \quad \text{and} \quad \lim_{\mu\to\infty} \left.\frac{\partial P_m}{\partial x}\right|_{y=-1}.$$

We have $k_1 = \sqrt{\lambda^2 + \mu^2} + \mu \to 2\mu, \quad k_2 = \mu - \sqrt{\lambda^2 + \mu^2} \to -\frac{\lambda^2}{2\mu}$ at $\mu \to \infty$, where $\mu = Ha/2$. Then it follows from (1.109) and (1.110) that

$$\lim_{\mu\to\infty} \left.\frac{\partial P_m}{\partial y}\right|_{y=-1} = -\frac{Ha}{\pi} \cdot \int_0^\infty \frac{\sin \lambda L \cos \lambda x}{\lambda} d\lambda = \begin{cases} -\frac{Ha}{2}, & 0 < x < L \\ -\frac{Ha}{4}, & x = L \\ 0, & x > L \end{cases} \qquad (1.113)$$

$$\lim_{\mu\to\infty} \frac{\partial P_m}{\partial x} = -\frac{Ha^2}{\pi} \cdot \int_0^\infty \frac{\sin(\lambda L)\sin(\lambda x)}{\lambda^2} d\lambda = \begin{cases} -L \cdot \frac{Ha^2}{2} x, & 0 < x < L \\ -L \cdot \frac{Ha^2}{2}, & L < x < +\infty \end{cases} \qquad (1.114)$$

It follows from (1.113) and (1.114), that at large Hartmann numbers, the pressure gradient which is proportional to the square of the Hartmann number is needed for turning the flow on the angle 90 degrees, while for pumping of the fluid we need the pressure gradient which is proportional to only the first power of the Hartmann number.

### 1.3.2. Transverse magnetic field. Solution of the even problem with respect to y

The geometry of the flow is shown in Fig.1.3, but the magnetic field is perpendicular to the channel's wall. The problem is described by the system of equations is (1.87)-(1.89) with boundary conditions (1.51)-(1.52).

The system of ordinary differential equations for the Fourier transforms $\hat{V}_x(\lambda, y) = F[V_x(x,y)]$, $\hat{V}_y(\lambda, y) = F[V_y(x,y)]$, $\hat{P}(\lambda, y) = F[P_m(x,y)]$ has the form:

$$-i\lambda \hat{P}_m + \mathbf{L}\hat{V}_x - Ha^2\hat{V}_x = 0, \quad (1.115)$$

$$-\frac{d\hat{P}_m}{dy} + \mathbf{L}\hat{V}_y = 0, \quad (1.116)$$

$$i\lambda \hat{V}_x + \frac{d\hat{V}_y}{dy} = 0, \quad (1.117)$$

where $\mathbf{L}f = -\lambda^2 \cdot f + \dfrac{d^2 f}{dy^2}$.

Boundary conditions for this problem are described by (1.54) and have the form:

$$y = \pm 1: \ \hat{V}_y = \frac{1}{2}\sqrt{\frac{2}{\pi}}\frac{\sin(\lambda L)}{\lambda}, \ \hat{V}_x = 0. \quad (1.118)$$

Since the function $\hat{V}_y$ is the even function with respect to y in this problem, the general solution for the $\hat{V}_y$ can be obtained from (1.43) putting $C_1 = C_2 = 0$ and taking $\alpha = \dfrac{\pi}{2}$. As a result we have:

$$\hat{V}_y(\lambda, y) = C_3 \cosh k_1 y + C_4 \cosh k_2 y \quad (1.119)$$

where $C_3$,...,$C_4$ are arbitrary constants and $k_1, k_2$ are given by formula (1.98)

In order to determine $C_3$ and $C_4$ we use boundary condition (1.118) and Eq.(1.99).

Determining after that $\hat{V}_x$, $i\lambda\hat{P}$, $d\hat{P}_m / dy$ from the system and applying the inverse Fourier transform to them we obtain the solution to the even problem (1.87)-(1.89) with boundary conditions (1.51)-(1.52). in the form of convergent improper integrals:

$$V_x(x,y) = \frac{1}{\pi}\int_0^\infty \frac{1}{\Delta_2}(\sinh k_2 \cdot \sinh k_1 y - \sinh k_1 \cdot \sinh k_2 y)\cdot \sin\lambda L \sin\lambda x d\lambda\,, \qquad (1.120)$$

$$V_y(x,y) = \frac{1}{\pi}\int_0^\infty \frac{1}{\Delta_2}(k_2\sinh k_2 \cdot \cosh k_1 y - k_1\sinh k_1 \cdot \cosh k_2 y)\cdot \frac{\sin\lambda L \cos\lambda x}{\lambda} d\lambda\,, \qquad (1.121)$$

$$\frac{\partial P_m(x,y)}{\partial x} = -\frac{Ha}{\pi}\int_0^\infty \frac{1}{\Delta_2}(k_2\sinh k_2 \cdot \sinh k_1 y - k_1\sinh k_1 \cdot \sinh k_2 y)\cdot \sin\lambda L \sin\lambda x d\lambda\,, \qquad (1.122)$$

$$\frac{\partial P_m(x,y)}{\partial y} = -\frac{Ha}{\pi}\int_0^\infty \frac{\lambda}{\Delta_2}(\sinh k_2 \cdot \cosh k_1 y - \sinh k_1 \cdot \cosh k_2 y)\cdot \sin\lambda L \cos\lambda x d\lambda\,. \qquad (1.123)$$

$$\Delta_2 = k_2 \sinh k_2 \cdot \cosh k_1 - k_1 \cdot \sinh k_1 \cdot \cosh k_2 \qquad (1.124)$$

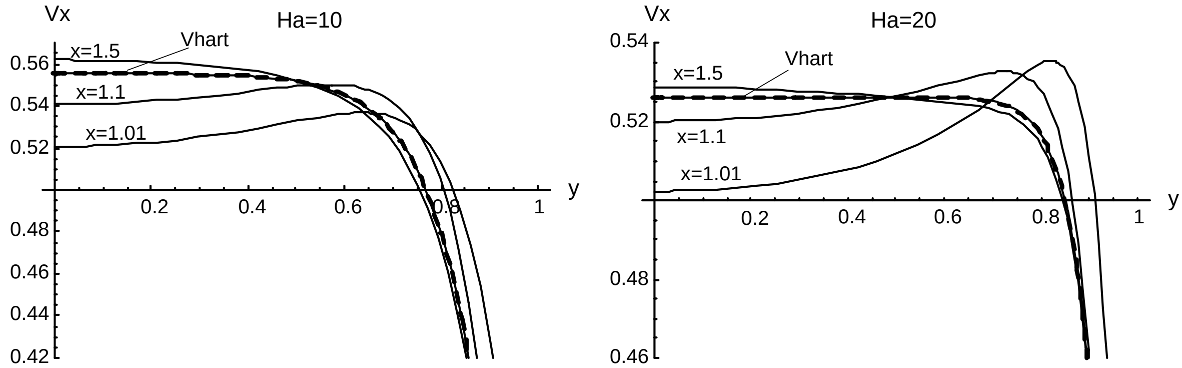
In this case $P_m=O(1)$ as $Ha\to\infty$ , so the principal contribution to the pressure gradient as $Ha\to\infty$ gives the odd with respect to y case.

### 1.3.3. Transverse magnetic field. Numerical results and discussion

**1) The odd problem with respect to y:**

Fig.1.7 plots the profiles of the velocity component $V_x$ calculated by means formula (1.107) for Ha=10, Ha=20 and Ha=50 for L=1 (Fig.1.7A) and for L=4 ((Fig.1.7B). Note that the component $V_x$ is an **even function** with respect to **y**.

It can be seen from Fig.1.7 that $V_x$ has the M-shaped profiles only near to the entrance hole ($1 \le r \le 1.1$ at Ha=10 and $1 \le r < 1.1$ Ha=50) for L=1. However, even at small distance from the entrance, the flow approaches the Hartmann flow in a plane channel in the transverse magnetic field. With the increase of L, the length of initial part of the channel increases. For L=4 the Hartmann flow takes place only at x=4. Note that in the present problem the initial part of the channel is defined to be the part where the x-component of the velocity $\vec{V}(x,y)$ differs from the Hartmann flow $V_{hart}$ by less than 1%. In addition, at L=4, velocity component $V_r$ don't have M-shaped profiles.



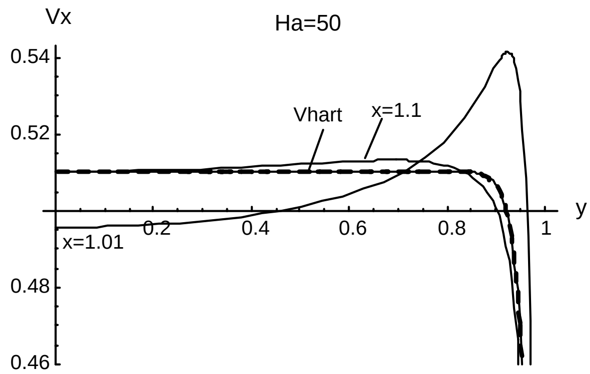
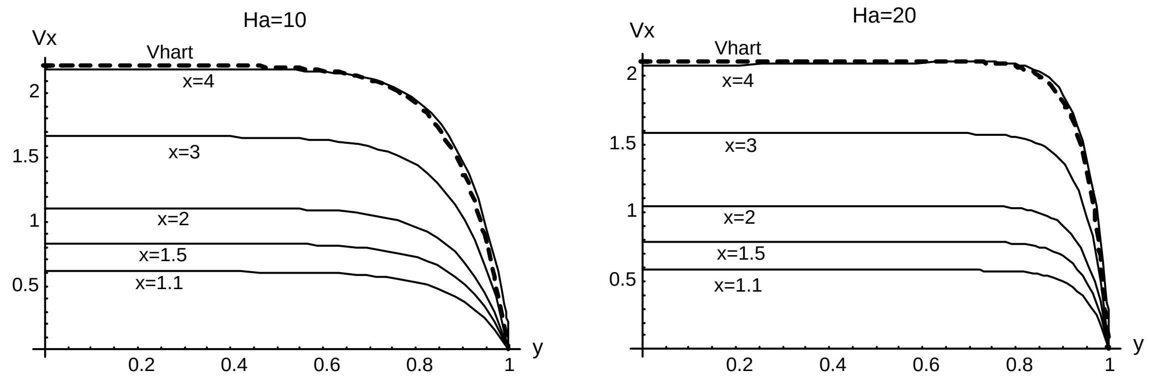




Figure 1.7A. Profiles for the x component $V_x$ of the velocity $\vec{V}$ for the odd problem and $\vec{B}^e = B_0\vec{e}_y$ at L=1 (--- $V_{hart}$ is the Hartmann flow).



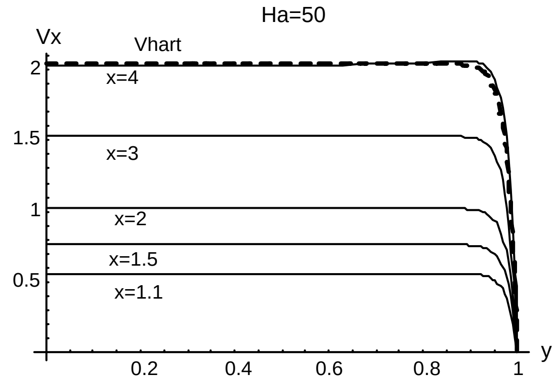




Figure 1.7B. Profiles for the x component $V_x$ of the velocity $\vec{V}$ for the odd problem and $\vec{B}^e = B_0\vec{e}_y$ at L=4 ( --- $V_{hart}$ is the Hartmann flow).

## 2) The even problem with respect to y:

Fig.1.8 plots the x profiles of the velocity component $V_x$ calculated by means formula (1.120) for the Hartmann numbers Ha=10, Ha=20 and Ha=50 for $-1 \le y \le 0$. In this case the function $V_x$ is an **odd function** with respect to **y**.

One can see from Fig.1.8 that $V_x$ differs from zero only near the entrance region ($r \le 2$ for $Ha = 10$ and $r < 1.5$ for $Ha = 50$). In addition, in Fig.1.8 for some values of x the component $V_x$ is negative at $-1 < y < 0$ and Ha=10. However, since the fluid inflows into the channel through the hole on $y = -1$, the x-component of the velocity must be positive for $-1 < y < 0$ at *Ha=0*. It means that there exists an opposite flow in the region in transverse magnetic field. It occurs due to a vortex generated in the channel (see Fig.1.9 ). Note that the velocity of the fluid in this vortex is very small. The vector field of velocity for Ha=10 is shown in Fig.1.9.

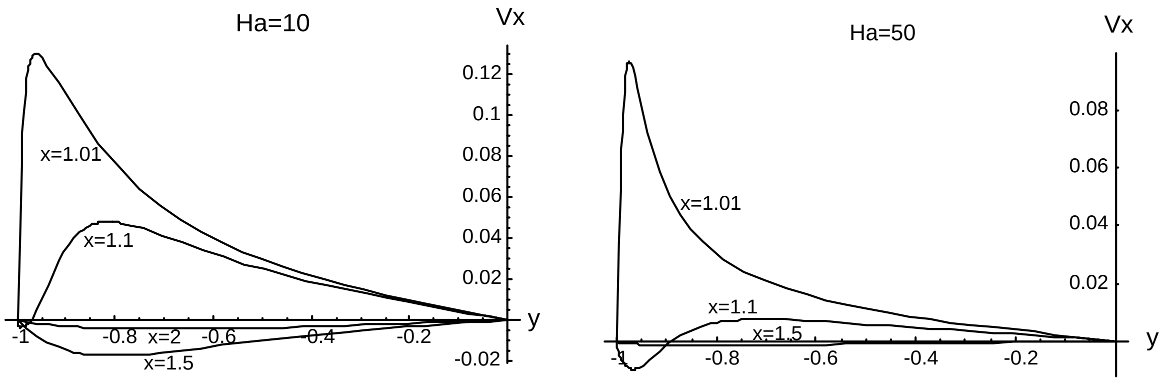



Figure 1.8. Profiles of the velocity x component $V_x$ for the even problem and $\vec{B}^e = B_0 \vec{e}_y$, L=1

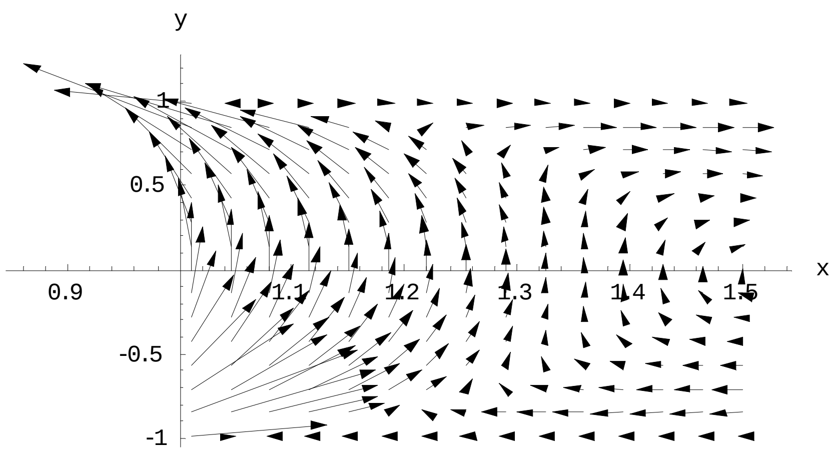



Figure 1.9. Velocity field for the even problem and $\vec{B}^e = B_0 \vec{e}_y$, L=1 at Ha=10.

**3) Numerical results for the general problem:**

The solution to general problem (1.5)-(1.7) at $\alpha=\pi/2$ with boundary conditions (1.12), (1.13) is equal to the sum of the solutions of odd and even problems with respect to y. Figure 1.10 plots the results of calculation of the x-component $V_x(x,y)$ of the velocity for the general problem for the Hartmann numbers Ha=10 and Ha=50. One can see that, similarly to the previous case, the profiles of the velocity component $V_x$ differ from the Hartmann flow profiles only near the entrance region. For Ha=10 the flow approaches the Hartmann flow at $x \geq 2$ and in the case Ha=50 the Hartmann flow takes the place at $x \geq 1.5$

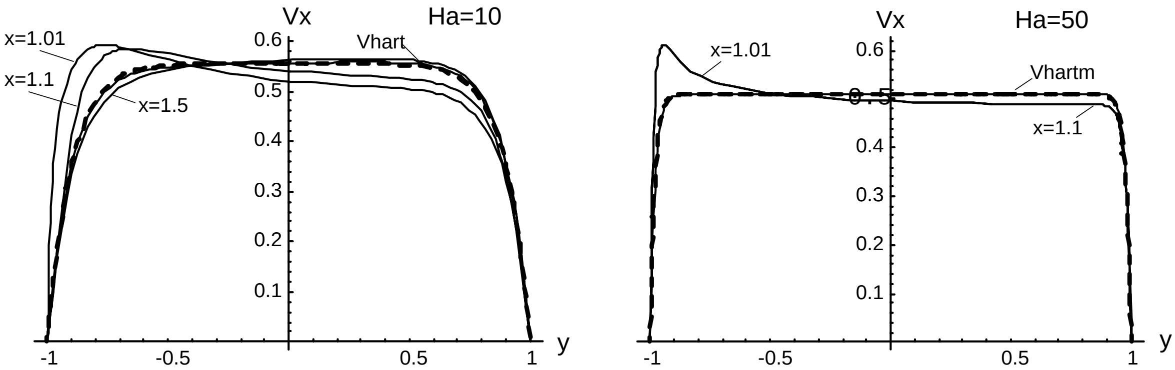



Figure 1.10. Profiles of the velocity component $V_x$ for the general problem and $\vec{\mathbf{B}}^e = B_0\vec{\mathbf{e}}_y$, L=1.

# 2. MHD PROBLEM ON AN INFLOW OF A CONDUCTING FLUID INTO A CHANNEL THROUGH THE CHANNEL'S LATERAL SIDE IN THE PRESENCE OF ROTATIONAL SYMMETRY

In this part of the PhD thesis we consider problems similar to the problems in the previous part, but with a presence of the rotational symmetry in the geometry of the flow. Two problems are solved in this chapter:

1) MHD problem on an inflow of conducting fluid into a plane channel through a round hole of finite radius in the channel's lateral side (see also [6], [57], [59]).

2) MHD problem on an inflow of conducting fluid into a round channel through a split in the channel's lateral side (see also [7]).

## 2.1. MHD problem on an inflow of a conducting fluid into a plane channel through a round hole of finite radius in the channel's lateral side

### 2.1.1. Formulation of the problem.

The plane channel with a conducting fluid is located in the region D:$\{ 0 \le \tilde{r} \le +\infty, 0 \le \tilde{\varphi} \le 2\pi, -h \le \tilde{z} \le h \}$ ($\tilde{r}, \tilde{\varphi}, \tilde{z}$ - cylindrical coordinates). The conducting fluid flows into the channel with the constant velocity $V_0 \vec{e}_z$ through a round hole of finite radius $\tilde{R}$, located in the channel's lateral side. It is supposed that the channel's walls $\tilde{z} = \pm h$ are non-conducting. It is also assumed that induced streams do not flow through the hole $\{ \tilde{z} = -h, \; 0 < \tilde{r} < \tilde{R} \}$ in the region $-\infty < \tilde{z} < -h$. The geometry of the flow is shown in Fig.2.1.

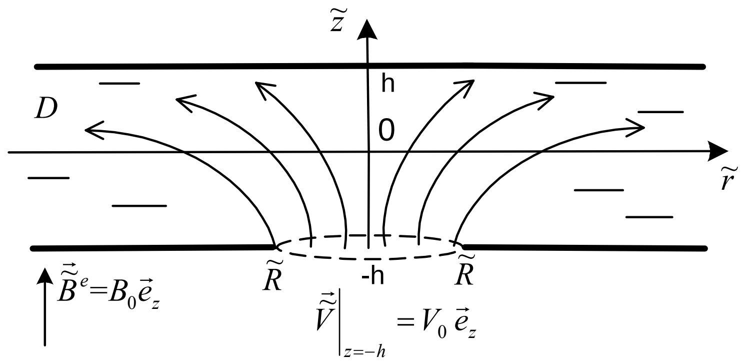



Figure 2.1. The geometry of the flow for the plane channel with the hole in the channel's lateral side.

We consider the case of the transverse magnetic field, i.e. the external magnetic field $\vec{B}^e = B_0 \vec{e}_z$ is parallel to the $\tilde{z}$ -axis.

We introduce dimensionless variables using the half-width of channel $h$ as the length scale; the magnitude of the velocity of fluid in the entrance region $V_0$ as the velocity scale, and $B_0$, $V_0 B_0$ , $\rho\nu V_0/h$ as the scales of magnetic field, electrical field and pressure, respectively, where $\sigma$ is the conductivity, $\rho$ is the density and $\nu$ is the viscosity of the fluid.

Dimensionless MHD equations in cylindrical coordinates in Stokes and inductionless approximation have the form (see [111]):

$$\Delta\vec{V} + Ha^2(\vec{E} + \vec{V} \times \vec{e}_B) \times \vec{e}_B = \nabla P, \tag{2.1}$$

$$div\vec{V} = 0, \tag{2.2}$$

where $\vec{V} = V_r(r,z)\vec{e}_r + V_z(r,z)\vec{e}_z$ is the velocity of the fluid, $P(r,z)$ is the pressure, $\vec{e}_B = \cos\alpha \cdot \vec{e}_r + \sin\alpha \cdot \vec{e}_z$ is the unit vector of external magnetic field,

$$\Delta\vec{V} = \vec{e}_r(L_0 V_r - \frac{V_r}{r^2}) + \vec{e}_z(L_0 V_z), \quad L_0 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

For this problem the intensity of electrical field $E = 0$ (see for example [111]).

In the case of transverse magnetic the system (2.1)- (2.2) in projections onto the r and z axes has the form:

$$-\frac{\partial P}{\partial r} + L_1 V_r - Ha^2 V_r = 0, \tag{2.3}$$

$$-\frac{\partial P}{\partial z} + L_0 V_z = 0, \tag{2.4}$$

$$\frac{\partial V_z}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r \cdot V_r) = 0. \tag{2.5}$$

The dimesionless boundary conditions are:

$$z = -1: \; V_r = 0, \quad V_z = \begin{cases} 1, & 0 \le r \le R \\ 0, & r > R \end{cases} \tag{2.6}$$

$$z = 1: \; V_r = 0, \quad V_z = 0, \tag{2.7}$$

$$r \to \pm\infty: \quad V_r \to 0, \quad \frac{\partial P}{\partial r} \to 0,$$

where $L_1$ is the operator $L_1 = L_0 - \frac{1}{r^2}$, $R = \tilde{R}/h$.

Due to the axial symmetry of the problem with respect to r, the Hankel transform (see [4]) is used for the solution of the problem. The Hankel transform of order 1 with respect to r is applied to the functions $V_r(r,z)$, $\partial P(r,z)/\partial r$ and the Hankel transform of order 0 is applied to the functions $V_z(r,z)$ and $\partial P(r,z)/\partial z$, i.e.

$$\hat{V}_r(\lambda,z) = \int_0^\infty V_r J_1(\lambda r) r dr, \qquad \hat{V}_z(\lambda,z) = \int_0^\infty V_z J_0(\lambda r) r dr, \qquad \hat{P}(\lambda,z) = \int_0^\infty P J_0(\lambda r) r dr, \quad (2.8)$$

where $J_\nu(\lambda r)$ is the Bessel function of order ν ($\nu$=0, 1).

As a result, we get the system of ordinary differential equations for the Hankel transforms $\hat{V}_r(\lambda,z)$, $\hat{V}_z(\lambda,z)$, $\hat{P}(\lambda,z)$:

$$\lambda\hat{P} - \lambda^2\hat{V}_r + \frac{d^2\hat{V}_r}{dz^2} - Ha^2\hat{V}_r = 0, \quad (2.9)$$

$$-\frac{d\hat{P}}{dz} - \lambda^2\hat{V}_z + \frac{d^2\hat{V}_z}{dz^2} = 0, \quad (2.10)$$

$$\frac{d\hat{V}_z}{dz} + \lambda\hat{V}_r = 0. \quad (2.11)$$

Eliminating $\hat{V}_r$ and $\hat{P}$ from this system, we obtain the differential equation for $\hat{V}_z$:

$$\hat{V}_z^{(4)} - \left(2\lambda^2 + Ha^2\right)\cdot\hat{V}_z'' + \lambda^4\cdot\hat{V}_z = 0, \quad (2.12)$$

The general solution to Eq. (2.12) has the form:

$$\hat{V}_z(\lambda,z) = C_1 \sinh k_1 z + C_2 \sinh k_2 z + C_3 \cosh k_1 z + C_4 \cosh k_2 z \quad (2.13)$$

where

$$k_1 = \mu + \sqrt{\mu^2 + \lambda^2}, \quad k_2 = \mu - \sqrt{\mu^2 + \lambda^2}, \quad (2.14)$$

$C_1$,...,$C_4$ are arbitrary constants and $\mu = Ha/2$.

In order to simplify the problem and reduce the number of constants $C_1$,..., $C_4$ in (2.13), we divide the problem into two sub-problems: odd problem with respect to z and the even problem, as it was done in Chapter 1. For this purpose, we consider a plane channel with two holes in it's lateral sides $z = \pm h$ in region $0 < \tilde{r} < \tilde{R}$.

**1) Odd problem** with respect to z (Fig.2.2): the fluid with velocities $\mp\left(V_0\vec{e}_z\right)/2$ flows into the channel through the both holes at $\tilde{z} = \pm h$.

**2) Even problem** with respect to z (Fig.2.3): the fluid with velocity $\left(V_0\vec{e}_z\right)/2$ flows into the channel through the hole at $\tilde{z} = -h$ and flows out with the same velocity through the hole at $\tilde{z} = h$.

**3)** The solution of the **general problem** is equal to the sum of solutions to odd end even problems.

### 2.1.2. Solution of the odd problem with respect to z

The **odd problem** with respect to z is the problem on an inflow of fluid into the channel through the holes at $\tilde{z} = \pm h$. The geometry of the flow is shown in Fig.2.2.

The dimensionless boundary conditions for the problem are

$$z = \pm 1: \; V_r = 0\,, \quad V_z = \begin{cases} \mp 1/2, & 0 \le r \le R \\ 0, & r > R \end{cases} \tag{2.15}$$

$$r \to \pm\infty: \quad V_x \to 0\,, \quad \partial P/\partial x \to 0\,. \tag{2.16}$$

Applying the Hankel transforms (2.8) to these boundary conditions, we obtain

$$z = \pm 1: \quad \hat{V}_r = 0\,, \quad \hat{V}_z = \mp \frac{1}{2}\frac{R \cdot J_1(\lambda R)}{\lambda} \tag{2.17}$$

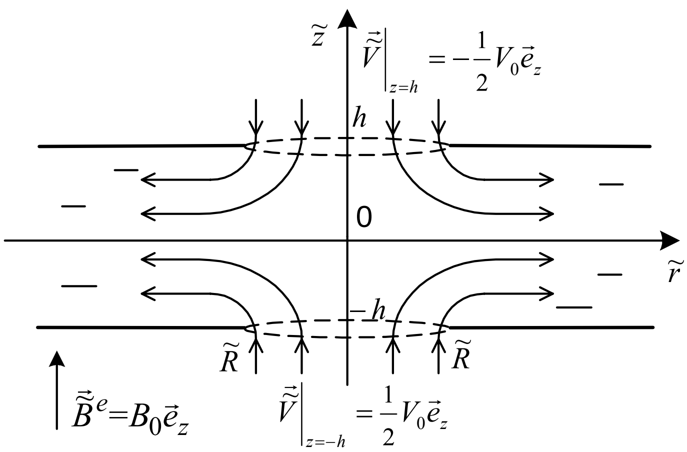



Figure 2.2. Plane channel with a round hole. The odd problem with respect to z.

For the odd problem, $\hat{V}_z$ is the odd function with respect to z and, therefore $C_3 = C_4 = 0$ in Eq.(2.13), i.e.

$$\hat{V}_z(\lambda, z) = C_1 \sinh k_1 z + C_2 \sinh k_2 z \tag{2.18}$$

In order to determine $C_1$ and $C_2$ we use boundary condition (2.17) and the additional boundary condition obtained from Eq.(2.11), i.e.

$$z = 1: \quad \hat{V}_z = -\frac{1}{2}\frac{R \cdot J_1(\lambda R)}{\lambda} \qquad \text{and} \qquad \frac{d\hat{V}_z}{dz} = 0\,. \tag{2.19}$$

As a result, we have

$$\hat{V}_z(\lambda, z) = \frac{R}{2} \cdot \frac{1}{\Delta_1}(k_1 \cosh k_1 \cdot \sinh k_2 z - k_2 \cosh k_2 \cdot \sinh k_1 z) \cdot \frac{J_1(\lambda R)}{\lambda}, \tag{2.20}$$

where $$\Delta_1 = k_2 \cosh k_2 \cdot \sinh k_1 - k_1 \cosh k_1 \cdot \sinh k_2, \tag{2.21}$$

and $k_1, k_2$ are given by formula (2.14).

Determining $\hat{V}_r$ from Eq.(2.11), we obtain:

$$\hat{V}_r = \frac{R}{2}\frac{1}{\Delta_1}(\cosh k_1 \cdot \cosh k_2 z - \cosh k_2 \cdot \cosh k_1 z) \cdot J_1(\lambda R) \tag{2.22}$$

We also determine $\lambda\hat{P}$ from Eq. (2.9) and $d\hat{P}/dz$ from Eq.(2.10):

$$\lambda\hat{P} = -\frac{Ha^2 \cdot R}{2} \cdot \frac{1}{\Delta_1}(k_1 \cosh k_1 \cdot \cosh k_2 z - k_2 \cosh k_2 \cdot \cosh k_1 z) \cdot J_1(\lambda R), \tag{2.23}$$

$$\frac{d\hat{P}}{dz} = -\frac{Ha \cdot R}{2} \cdot \frac{1}{\Delta_1}(\cosh k_1 \cdot \sinh k_2 z - \cosh k_2 \cdot \sinh k_1 z) \cdot \lambda \cdot J_1(\lambda R)\,. \tag{2.24}$$

In order to obtain the solution to the odd problem, we apply the inverse complex Hankel transform. As a result, the solution of problem (2.3)–(2.5) with boundary conditions (2.15), (2.16) takes the form of convergent improper integrals:

$$V_r(r, z) = \frac{R}{2}\int_0^\infty \frac{1}{\Delta_1}(\cosh k_1 \cdot \cosh k_2 z - \cosh k_2 \cdot \cosh k_1 z) \cdot \lambda \cdot J_1(\lambda R) \cdot J_1(\lambda r) d\lambda, \tag{2.25}$$

$$V_z(r, z) = \frac{R}{2}\int_0^\infty \frac{1}{\Delta_1}(k_1 \cosh k_1 \cdot \sinh k_2 z - k_2 \cosh k_2 \cdot \sinh k_1 z) \cdot J_1(\lambda R) \cdot J_0(\lambda r) d\lambda, \tag{2.26}$$

$$\frac{\partial P}{\partial r} = -\frac{Ha^2 \cdot R}{2}\int_0^\infty \frac{1}{\Delta_1}(k_1 \cosh k_1 \cdot \cosh k_2 z - k_2 \cosh k_2 \cdot \cosh k_1 z)\cdot \lambda \cdot J_1(\lambda R)\cdot J_1(\lambda r) d\lambda\,, \quad (2.27)$$

$$\frac{\partial P}{\partial z} = -\frac{Ha \cdot R}{2}\int_0^\infty \frac{1}{\Delta_1}(\cosh k_1 \cdot \sinh k_2 z - \cosh k_2 \cdot \sinh k_1 z)\cdot \lambda^2 \cdot J_1(\lambda R)\cdot J_0(\lambda r) d\lambda\,, \quad (2.28)$$

where $k_1 = \mu + \sqrt{\mu^2 + \lambda^2}$ , $k_2 = \mu - \sqrt{\mu^2 + \lambda^2}$ ,

$\Delta_1 = k_2 \cosh k_2 \cdot \sinh k_1 - k_1 \cosh k_1 \cdot \sinh k_2$.

**<u>Asymptotic evaluations at</u> $r \to \infty$ and $Ha \to \infty$ for the odd problem:**

1) In order to obtain $\lim\limits_{r\to\infty} V_r$ , $\lim\limits_{r\to\infty} \frac{\partial P}{\partial r}$, we substitute $\lambda r = t$ in (2.25) and (2.27) and pass to limit as $r \to \infty$. As a result, we obtain

$$\lim_{r\to\infty} V_r(r,z) = \frac{R}{2r} E \cdot \int_0^\infty J_1(t) dt = \frac{R}{2r} E\,, \quad (2.29)$$

where

$$E = \frac{Ha}{Ha - \tanh Ha}\left(1 - \frac{\cosh(Ha \cdot z)}{\cosh Ha}\right), \quad (2.30)$$

and

$$\lim_{r\to\infty} \frac{\partial P}{\partial r} = -\frac{Ha^2 R}{2r} G \cdot \int_0^\infty J_1(t) dt = -\frac{Ha^2 R}{2r} G \quad (2.31)$$

where $G = \dfrac{1}{Ha - \tanh(Ha)}$ . (2.32)

It follows from Eq.(2.29) and (2.31) that at sufficient distance from the entrance region, the profiles of the $V_r$ component are the same as for a Hartmann flow in a plane channel in transverse magnetic field (i.e. the Hartmann flow takes place at $r \to \infty$), but the magnitude of the velocity and pressure gradient are inversely proportional to the distance from the hole. It corresponds to the conservation law of flow rate.

2) The analysis of the pressure gradient at large Hartmann numbers has the greatest interest for applications in the reactor Tokamak. Thus, we determine the limits $\lim\limits_{\mu\to\infty} \frac{\partial P}{\partial r}$ and

$\lim\limits_{\mu\to\infty} \left.\frac{\partial P}{\partial z}\right|_{z=-h}$ ($\mu = \frac{1}{2} Ha$).k.

We have $k_1 = \sqrt{\lambda^2 + \mu^2} + \mu \to 2\mu, \quad k_2 = \mu - \sqrt{\lambda^2 + \mu^2} \to -\frac{\lambda^2}{2\mu}$ at $\mu \to \infty$, where $\mu = Ha/2$. Then it follows from (2.27) and (2.28) that

$$\lim_{\mu\to\infty} \frac{\partial P}{\partial r} = -\frac{Ha^2 R}{2} \int_0^\infty \frac{J_1(\lambda R) J_1(\lambda r)}{\lambda} d\lambda = \begin{cases} -\dfrac{Ha^2}{4} r, & 0 < r \le R \\ -\dfrac{Ha^2}{4} \dfrac{R^2}{r}, & R \le r < +\infty \end{cases} \tag{2.33}$$

(see [105], p.443),

$$\lim_{\mu\to\infty} \frac{\partial P}{\partial z} = -\frac{Ha \cdot R}{2} \int_0^\infty \frac{J_1(\lambda R) J_0(\lambda r)}{\lambda} d\lambda = -\frac{Ha \cdot R}{2} \widetilde{F}(r), \tag{2.34}$$

where
$$\widetilde{F}(r) = \begin{cases} F\left(\dfrac{1}{2}, -\dfrac{1}{2}, 1, \dfrac{r^2}{R^2}\right), & 0 < r < R \\ \dfrac{R}{2r} \cdot F\left(\dfrac{1}{2}, \dfrac{1}{2}, 2, \dfrac{R^2}{r^2}\right), & R < r < \infty \end{cases} \tag{2.35}$$

and F(α, β, γ, z) is the hypergeometric function ( see [30], formulae 6.574(1), 6.574(3)).

### 2.1.3. Solution of the even problem with respect to z

The geometry of the flow for the even problem with respect to z is shown in Fig.2.3. The dimensionless boundary conditions for this problem are:

$$z = \pm 1: \; V_r = 0, \quad V_z = \begin{cases} 1/2, & 0 \le r \le R \\ 0, & r > R \end{cases} \tag{2.36}$$

$$r \to \pm\infty: \quad V_x \to 0, \quad \partial P / \partial x \to 0. \tag{2.37}$$

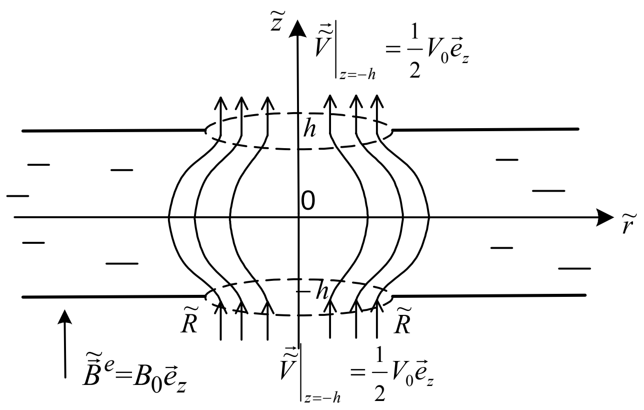



Figure 2.3. Plane channel with a round hole. The even problem with respect to z.

After applying the Hankel transform (2.8) to boundary conditions (2.36)-(2.37) we have:

$$z = \pm 1: \quad \hat{V}_r = 0, \quad \hat{V}_z = \frac{1}{2}\frac{R \cdot J_1(\lambda R)}{\lambda} \tag{2.38}$$

It must be taken into account that $\hat{V}_z$ is the even function with respect to z for the even problem and, therefore, $C_1 = C_2 = 0$ in (2.13), i.e.

$$\hat{V}_z(\lambda, z) = C_3 \cosh k_1 z + C_4 \cosh k_2 z \tag{2.39}$$

In order to determine $C_3$ and $C_4$ we use boundary condition (2.38) and Eq.(2.11).

As a result, we obtain:

$$\hat{V}_z(\lambda, z) = \frac{R}{2} \cdot \frac{1}{\Delta_2}(k_2 \sinh k_2 \cdot \cosh k_1 z - k_1 \sinh k_1 \cdot \cosh k_2 z) \cdot \frac{J_1(\lambda R)}{\lambda}, \tag{2.40}$$

where $\Delta_2 = k_2 \cosh k_1 \cdot \sinh k_2 - k_1 \cosh k_2 \cdot \sinh k_1$.

Determining $\hat{V}_r$ from Eq.(2.11), we get:

$$\hat{V}_r(\lambda, z) = -\frac{R}{2} \cdot \frac{1}{\Delta_2}(\sinh k_2 \cdot \sinh k_1 z - \sinh k_1 \cdot \sinh k_2 z) \cdot J_1(\lambda R) \tag{2.41}$$

Determining $\lambda \hat{P}$ and $d\hat{P}/dz$ from the system (2.9)-(2.10) we obtain:

$$\lambda \hat{P} = \frac{Ha \cdot R}{2} \cdot \frac{1}{\Delta_2}(k_2 \sinh k_2 \cdot \sinh k_1 z - k_1 \sinh k_1 \cdot \sinh k_2 z) \cdot J_1(\lambda R), \tag{2.42}$$

$$\frac{\partial \hat{P}}{\partial z} = -\frac{Ha \cdot R}{2} \cdot \frac{1}{\Delta_2}(\sinh k_2 \cdot \cosh k_1 z - \sinh k_1 \cdot \cosh k_2 z) \cdot \lambda \cdot J_1(\lambda R) \tag{2.43}$$

Applying the inverse complex Hankel transform to (2.40), (2.41), (2.42), (2.43) we obtain the solution to the problem (2.3)–(2.5) with boundary conditions (2.36), (2.37) in the form of convergent improper integrals:

$$V_r(r, z) = \frac{R}{2}\int_0^\infty \frac{1}{\Delta_2}(\sinh k_2 \cdot \sinh k_1 z - \sinh k_1 \cdot \sinh k_2 z) \cdot \lambda \cdot J_1(\lambda R) \cdot J_1(\lambda r) d\lambda, \tag{2.44}$$

$$V_z(r, z) = \frac{R}{2}\int_0^\infty \frac{1}{\Delta_2}(k_2 \sinh k_2 \cdot \cosh k_1 z - k_1 \sinh k_1 \cdot \cosh k_2 z) \cdot J_1(\lambda R) \cdot J_0(\lambda r) d\lambda, \tag{2.45}$$

$$\frac{\partial P}{\partial r} = -\frac{Ha \cdot R}{2} \int_0^{\infty} \frac{1}{\Delta_2} (k_2 \sinh k_2 \cdot \sinh k_1 z - k_1 \sinh k_1 \cdot \sinh k_2 z) \cdot \lambda \cdot J_1(\lambda R) \cdot J_1(\lambda r) d\lambda , \qquad (2.46)$$

$$\frac{\partial P}{\partial z} = -\frac{Ha \cdot R}{2} \int_0^{\infty} \frac{1}{\Delta_2} (\sinh k_2 \cdot \cosh k_1 z - \sinh k_1 \cdot \cosh k_2 z) \cdot \lambda^2 \cdot J_1(\lambda R) \cdot J_0(\lambda r) d\lambda , \qquad (2.47)$$

where $\quad k_1 = \mu + \sqrt{\mu^2 + \lambda^2} \quad , \quad k_2 = \mu - \sqrt{\mu^2 + \lambda^2} \; ,$

$$\Delta_2 = k_2 \cosh k_1 \cdot \sinh k_2 - k_1 \cosh k_2 \cdot \sinh k_1 .$$

### 2.1.4. Numerical results and discussion

**1) *The odd problem:***

The results of calculations of the velocity component $V_r$ for the odd problem at the Hartmann numbers Ha=10 and Ha=50 are shown in Fig.2.4. The component $V_r$ is an odd function with respect to z. One can see that $V_r$ has the M-shaped profiles only near the entrance hole ($1 \le r \le 1.1$ at Ha=10 and $1 \le r < 1.1$ Ha=50). Even at a small distance from the entrance, the profiles of the $V_r$ - component are the same as for a Hartmann flow in a plane channel in transverse magnetic field. The magnitude of the velocity is inversely proportional to the distance from the hole.

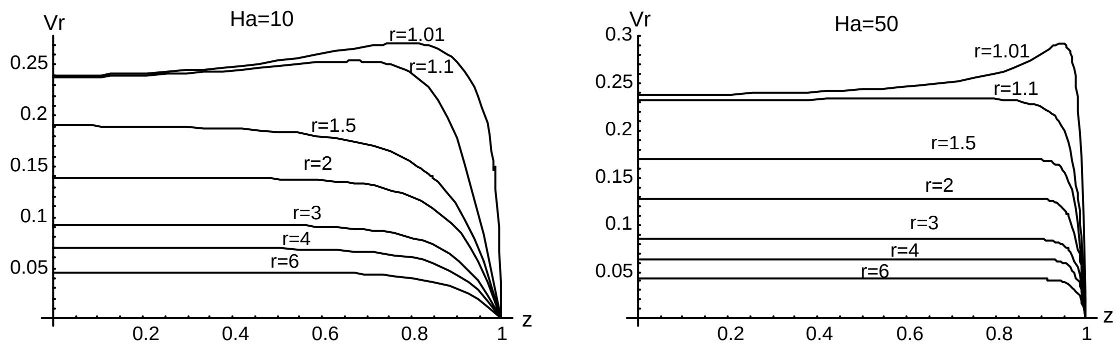



Figure 2.4. Profiles of the velocity radial component $V_r$ for the odd problem at R=1.

## 2) ***The even problem:***

Fig.2.5 shows the results of calculation of the r-component $V_r(r,z)$ of the velocity by means of formula (2.44). One can see that $V_r$ differs from zero only near the entrance region ($r \le 2$ for $Ha = 10$ and $r < 1.5$ for $Ha = 50$). Besides, in Fig.2.5 for some values of r the component $V_r$ is positive at $0 < r \le 1$ and Ha=10. However, since the fluid flows out through the hole at $z = 1$, the r-component of the velocity must be negative for $0 < r < 1$ at *Ha=0*. It means that in the transverse magnetic field there exists the opposite flow in the region $0<z<1$. It occurs due to vortices generated in the channel (see Fig.2.6 ). The velocity of fluid in this vortex is very small. The vector field of velocity for Ha=10 is shown in Fig.2.6

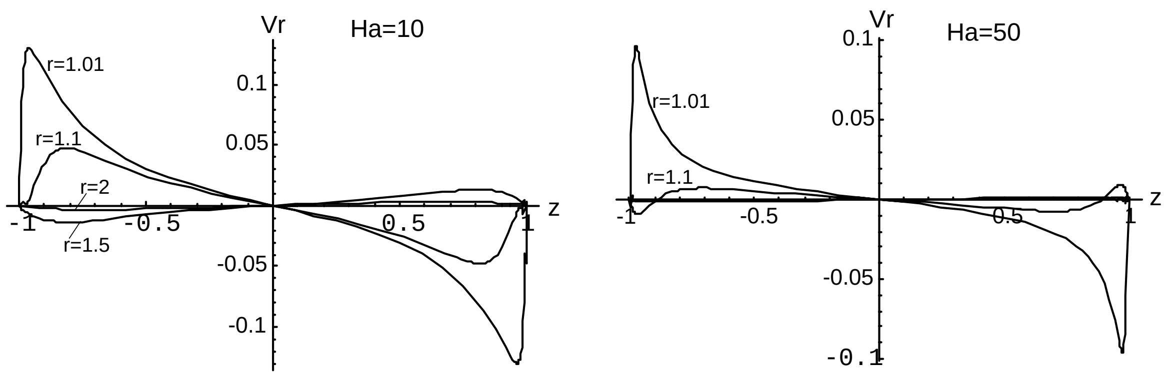



Figure 2.5. Profiles of the velocity radial component $V_r$ for the even problem at R=1.

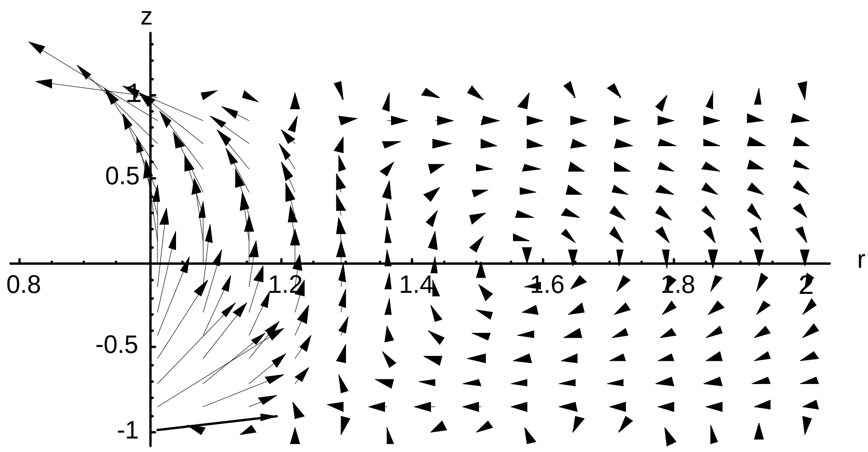



Figure 2.6. Velocity field for the even problem at R=1 and Ha=10.

**3) *<u>The general problem</u>*:**

The solution of the general problem is equal to the sum of solutions to the odd and even problems with respect to z. Fig.2.7 plots the results of calculation of the r-component $V_r(r,z)$ of the velocity for the general problem at the Hartmann numbers Ha=10 and Ha=50. One can see that similarly to the previous case, the profiles of the velocity component $V_r$ differ from the Hartmann flow profiles only near the entrance region. The magnitude of the velocity is inversely proportional to the distance from the hole.

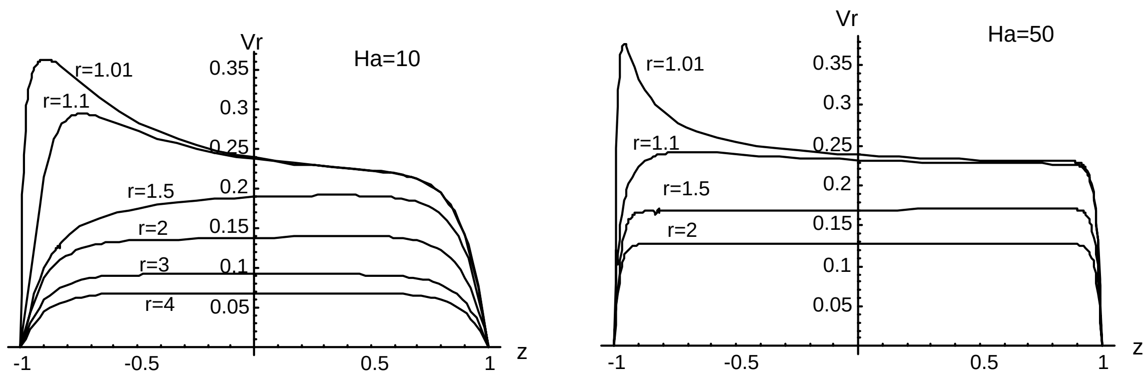



Figure 2.7. Profiles of the velocity radial component $V_r$ for the general problem at R=1.

## 2.2. Analytical solution for magnetohydrodynamical problem on a flow of conducting fluid in the initial part of a circular channel.

### 2.2.1. Formulation of the problem.

The circular channel is located in region $\widetilde{D}=\{0 \le \widetilde{r} < R,\ 0 \le \widetilde{\varphi} < 2\pi,\ -\infty < \widetilde{z} < +\infty\}$. There is a split in the channel's lateral surface in the region $\{\widetilde{r} = R, -\widetilde{d} \le \widetilde{z} \le \widetilde{d}\}$. The conducting fluid flows into the channel through this split with constant velocity $\vec{\widetilde{V}} = -V_0\vec{e}_r$. We consider the case of longitudinal magnetic field, i.e. the external magnetic field $\vec{B}^e = B_0\vec{e}_z$ is parallel to the z axis. The geometry of the flow is shown in Fig.2.8. Both the velocity field of the flow and the distribution of pressure in the channel will be studied in the problem.

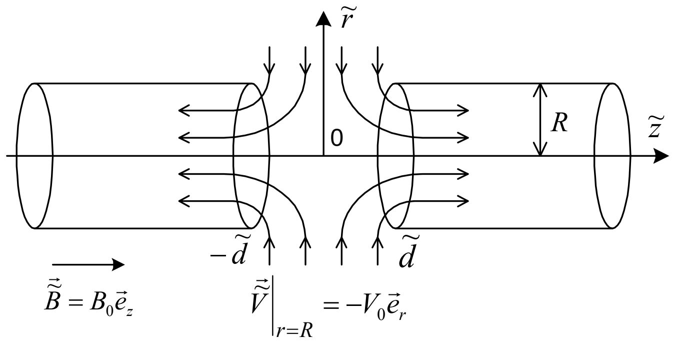



Figure 2.8. Circular channel. The geometry of the flow.

We consider the case of nonconducting walls $\tilde{r} = R$. We assume also, that induced streams do not flow through the split $\tilde{r} = R$, $-\tilde{d} < \tilde{z} < \tilde{d}$ in the region $R < \tilde{r} < +\infty$. In this problem $\vec{E} = 0$ (see [111]).

The dimensionless variables are introduced using the radius $R$ of the channel as the length scale; the magnitude of the velocity of the fluid in the entrance region $V_0$ as the velocity scale, and $B_0$, $V_0B_0$ , $\rho_0\nu V_0/R$ as the scales of magnetic field, electrical field and pressure, respectively, where $\sigma$ is the conductivity, $\rho$ is the density and $\nu$ is the viscosity of the fluid.

We use the dimensionless MHD equations in cylindrical coordinates in Stokes and inductionless approximation Eqs.(2.1)-(2.2). In projections on the r- and z- axes the problem has the form:

$$-\frac{\partial P}{\partial r} + (L_1 - Ha^2)V_r = 0, \tag{2.48}$$

$$-\frac{\partial P}{\partial z} + L_0 V_z = 0, \tag{2.49}$$

$$\frac{\partial V_z}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(rV_r) = 0, \tag{2.50}$$

where $L_0 = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{\partial^2}{\partial z^2}$, $L_1 = L_0 - 1/r^2$.

The boundary conditions are:

$$r = 1: \; V_z = 0, \quad V_r = \begin{cases} -1, & z \in (-d, d) \\ 0, & z \notin (-d, d) \end{cases} \tag{2.51}$$

$$z \to \pm\infty: \quad V_z \to V_\infty(r)\cdot \mathrm{sign}(z), \; \frac{\partial P}{\partial z} \to \frac{\partial P_\infty}{\partial z}\cdot sign(z) \equiv A \cdot sign(z), \tag{2.52}$$

where $d = \tilde{d}/R$, $A = const$.

Function $V_\infty(r)$ and $dP_\infty / dz$ are the velocity of the flow and the pressure gradient in the channel sufficiently far away from the entrance region and satisfy the following equation (see [111], [113])

$$-\frac{dP_\infty}{dz} + \frac{1}{r}\frac{d}{d\,r}(r \cdot \frac{dV_\infty}{dr}) = 0 \tag{2.53}$$

with the boundary condition: $r = 1: \quad V_\infty(r) = 0$.

### 2.2.2. Solution of the problem

In order to solve problem (2.48)-(2.52), we use the symmetry of the problem with respect to z, that is, we use the fact that the velocity component $V_r(r,z)$ and pressure $P(r,z)$ are even functions with respect to z, but the component $V_z(r,z)$ is the odd function with respect to z. This means that the functions $V_r(r,z)$ and $V_z(r,z)$ satisfy additional boundary conditions:

$$z = 0,\ 0 < r < 1 : \quad V_z = 0\,, \ \frac{\partial V_r}{\partial z} = 0. \tag{2.54}$$

Therefore, problem (2.48)-(2.52) can be solved by the Fourier cosine and Fourier sine transforms. Since $V_z$ and $\partial P / \partial z$ do not tends to zero at $z \to \pm\infty$, we introduce new functions for the velocity and pressure gradient before using these transforms:

$$\vec{V}^{new} = \vec{V} - \frac{2}{\pi}\arctan(z) \cdot V_\infty(r) \cdot \vec{e}_z \quad \text{and} \quad \frac{\partial P^{new}}{\partial z} = \frac{\partial P}{\partial z} - \frac{2}{\pi}\arctan(z) \cdot A \tag{2.55}$$

As a result, the problem has the form:

$$-\frac{\partial P^{new}}{\partial r} + (L_1 - Ha^2)V_r = 0\,, \tag{2.56}$$

$$-\frac{\partial P^{new}}{\partial z} + L_0 V_z^{new} - \frac{2}{\pi}V_\infty(r) \cdot \frac{2z}{\left(1+z^2\right)^2} = 0\,, \tag{2.57}$$

$$\frac{\partial V_z^{new}}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{2}{\pi}V_\infty(r) \cdot \frac{1}{1+z^2} = 0\,. \tag{2.58}$$

Boundary conditions are:

$$r=1: \; V_z^{new}=0, \qquad V_r=\begin{cases}-1, & z\in(-d,d)\\ 0, & z\notin(-d,d)\end{cases} \tag{2.59}$$

$$z\to\pm\infty: \quad V_z^{new}\to 0, \qquad \frac{\partial P^{new}}{\partial z}\to 0. \tag{2.60}$$

We apply the Fourier cosine transform with respect to z to equations (2.56), (2.58) and to $V_r$ in boundary conditions (2.59) and the Fourier sine transform to equation (2.57) and to $V_z$ in boundary conditions (2.59):

$$V_r^c(r,\lambda)=\sqrt{\frac{2}{\pi}}\int_0^\infty V_r(r,z)\cos\lambda z\,dz, \qquad V_z^s(r,\lambda)=\sqrt{\frac{2}{\pi}}\int_0^\infty V_z^{new}(r,z)\sin\lambda z\,dz,$$

$$P^c(r,\lambda)=\sqrt{\frac{2}{\pi}}\int_0^\infty P^{new}(r,z)\cos\lambda z\,dz.$$

We also use:

$$F^s\left[\frac{2z}{\left(1+z^2\right)^2}\right]=\lambda F^c\left[\frac{1}{1+z^2}\right]. \tag{2.61}$$

As a result, we obtain the following system of ordinary differential equations for the unknown functions $V_r^c(r,\lambda)=F^c\left[V_r(r,z)\right]$, $V_z^s(r,\lambda)=F^s\left[V_z^{new}(r,z)\right]$, $P^c(r,\lambda)=F^c\left[P^{new}(r,z)\right]$:

$$-\frac{dP^c}{dr}+(L_{1r}-Ha^2)V_r^c=0, \tag{2.62}$$

$$\lambda P^c+L_{0r}V_z^s-\frac{2}{\pi}V_\infty(r)\cdot\lambda\cdot F^c(\lambda)=0, \tag{2.63}$$

$$\lambda V_z^s+\frac{1}{r}\frac{d}{dr}\left(rV_r^c\right)+\frac{2}{\pi}V_\infty(r)\cdot F^c(\lambda)=0, \tag{2.64}$$

were $L_{0r}=\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\lambda^2$, $L_{1r}=L_{0r}-\frac{1}{r^2}$, un $F^c(\lambda)=F^c\left[\frac{1}{1+z^2}\right]$.

The boundary conditions are:

$$r=1: \; V_r^c(r,\lambda)=-\sqrt{\frac{2}{\pi}}\frac{\sin(\lambda d)}{\lambda}, \qquad V_z^s=0. \tag{2.65}$$

Eliminating the functions $V_z^s(r,\lambda)$, $P^c(r,\lambda)$ from the system (2.62)- (2.64), we obtain the equation for $V_r^c(r,\lambda)$:

$$\frac{d}{dr}(L_r\widetilde{L}_{0r})V_r^c - 2\lambda^2\widetilde{L}_{1r}V_r^c + \lambda^2(\lambda^2 + Ha^2)V_r^c = 0. \qquad (2.66)$$

where $\widetilde{L}_{0r} = \frac{1}{r} + \frac{d}{dr}$, $\quad L_r = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}$, $\quad \widetilde{L}_{1r} = L_r - \frac{1}{r^2}$. (2.67)

Operators $\widetilde{L}_{0r}$ and $\widetilde{L}_{1r}$ satisfy the relationship

$$\frac{d}{dr}[\widetilde{L}_{0r}a] = \widetilde{L}_{1r}a\,.$$

By using the previous formula, we simplify Eq. (2.66) in the following way:

$$\frac{d}{dr}(L_r\widetilde{L}_{0r})V_r^c = \frac{d}{dr}\left(\frac{d^2}{dr^2}\widetilde{L}_{0r}V_r^c + \frac{1}{r}\frac{d}{dr}\widetilde{L}_{0r}V_r^c\right) = \frac{d}{dr}\left(\frac{d}{dr}\left(\frac{d}{dr}\left[\widetilde{L}_{0r}V_r^c\right]\right) + \frac{1}{r}\left(\frac{d}{dr}\left[\widetilde{L}_{0r}V_r^c\right]\right)\right) =$$

$$= \frac{d}{dr}\left(\frac{d}{dr}\left[\widetilde{L}_{1r}V_r^c\right] + \frac{1}{r}\left[\widetilde{L}_{1r}V_r^c\right]\right) = \frac{d}{dr}\widetilde{L}_{0r}\left[\widetilde{L}_{1r}V_r^c\right] = \widetilde{L}_{1r}\left[\widetilde{L}_{1r}V_r^c\right].$$

Thus, we have

$$\frac{d}{dr}(L_r\widetilde{L}_{0r})V_r^c = \widetilde{L}_{1r}\left[\widetilde{L}_{1r}V_r^c\right]. \qquad (2.68)$$

Substituting Eq.(2.68) into Eq.(2.66), we obtain a linear differential operator of 4$^{th}$ order for the function $V_r^c(r, \lambda)$:

$$\widetilde{L}_{1r}^{\,2}V_r^c - 2\lambda^2\widetilde{L}_{1r}V_r^c + \lambda^2(\lambda^2 + Ha^2)V_r^c = 0. \qquad (2.69)$$

Operator in left-hand side of (2.69) can be transformed into the product of two linear differential operators of second order:

$$(\widetilde{L}_{1r} - k_1^2)(\widetilde{L}_{1r} - k_2^2)V_r^c = 0, \qquad (2.70)$$

where $\quad k_1 = \sqrt{\lambda^2 + iHa\lambda}, \quad k_2 = \sqrt{\lambda^2 - iHa\lambda}\,.$ (2.71)

The solution of Eq.(2.70), bounded at $r = 0$, has the form

$$V_r^c(r,\lambda) = C_1 I_1(k_1 r) + C_2 I_1(k_2 r) \tag{2.72}$$

where $C_1$, $C_2$ are arbitrary constants, $I_1(z)$ is the modified Bessel function of the first kind of order 1.

To determine constants $C_1$, $C_2$ we use boundary condition (2.65) and the following condition obtained from Eq.(2.64):

$$\left.\frac{d}{dr}(rV_r{}^c)\right|_{r=1} = 0\,. \tag{2.73}$$

As a result, we have:

$$C_1 = \frac{1}{\Delta} A(\lambda) k_2 I_0(k_2), \quad C_2 = -\frac{1}{\Delta} A(\lambda) k_1 I_0(k_1)\,, \tag{2.74}$$

where

$$A(\lambda) = \sqrt{\frac{2}{\pi}} \frac{\sin(\lambda d)}{\lambda}, \qquad \Delta = k_1 I_0(k_1) I_1(k_2) - k_2 I_0(k_2) I_1(k_1)\,. \tag{2.75}$$

We obtain function $\hat{V}_z^s$ from Eq.(2.64):

$$V_z^s(\mathrm{r},\lambda) = -\frac{1}{\lambda}\left(C_1 k_1 I_0(k_1 r) + C_2 k_2 I_0(k_2 r)\right) - \frac{2}{\pi} \cdot \frac{F^c(\lambda)}{\lambda} \cdot V_\infty(r) \tag{2.76}$$

We get functions $\lambda P^c$ and $\frac{dP^c}{dr}$ from Eqs. (2.62)-(2.63) taking into account the following formulae

$$\widetilde{L}_{1r} I_1(k\cdot r) = k^2 I_1(k\cdot r) \quad \text{un} \quad L_r I_0(k\cdot r) = k^2 I_0(k\cdot r).$$

Then

$$\frac{dP^c}{dr} = \widetilde{c}_1(\lambda) I_1(k_1 r) - \widetilde{c}_2(\lambda) I_1(k_2 r)\,, \tag{2.77}$$

$$-\lambda P^c = -iHa\cdot\left(c_1 k_1 I_0(k_1 r) - c_2 k_2 I_0(k_2 r)\right) - \frac{2}{\pi}\cdot\frac{F^c(\lambda)}{\lambda}\cdot A\,, \tag{2.78}$$

In order to obtain the solution to the problem, we apply the inverse Fourier sine and Fourier cosine transforms to functions $V_r^{\,c}(r,\lambda)$, $V_z^{\,s}(r,\lambda)$, $dP^c/dr$ and $-\lambda P^c$ in (2.76) (2.77), (2.78) taking into account the fact that $F^s\left[\frac{\partial P^{new}}{\partial z}\right]=-\lambda P^c$. We use also formulae (2.55) and te following formula:

$$\frac{F^c(\lambda)}{\lambda}=\frac{1}{\lambda}F^c\left[\frac{1}{1+z^2}\right]=F^s\left[\arctan(z)\right].$$

As a result, we get the solution to the problem (2.48)-(2.52) in the form of convergent improper integrals:

$$V_r=\sqrt{\frac{2}{\pi}}\int_0^\infty\left[c_1(\lambda)I_1(k_1r)+c_2(\lambda)I_1(k_2r)\right]\cos\lambda z d\lambda\,, \tag{2.79}$$

$$V_z(r,z)=-\sqrt{\frac{2}{\pi}}\int_0^\infty\left[c_1(\lambda)k_1I_0(k_1r)+c_2(\lambda)k_2I_0(k_2r)\right]\frac{\sin\lambda z}{\lambda}d\lambda\,, \tag{2.80}$$

$$\frac{\partial P}{\partial r}=\sqrt{\frac{2}{\pi}}\int_0^\infty\left[\tilde{c}_1(\lambda)I_1(k_1r)-\tilde{c}_2(\lambda)I_1(k_2r)\right]\cos\lambda z d\lambda\,, \tag{2.81}$$

$$\frac{\partial P}{\partial z}=-iHa\sqrt{\frac{2}{\pi}}\int_0^\infty\left[c_1(\lambda)k_1I_0(k_1r)-c_2(\lambda)k_2I_0(k_2r)\right]\sin\lambda z d\lambda\,, \tag{2.82}$$

where $c_1(\lambda)=\frac{1}{\Delta}A(\lambda)k_2I_0(k_2),\quad c_2(\lambda)=-\frac{1}{\Delta}A(\lambda)k_1I_0(k_1)$,

$\tilde{c}_1(\lambda)=Ha\cdot(i\lambda-Ha)\cdot c_1(\lambda),\quad \tilde{c}_2(\lambda)=Ha\cdot(i\lambda+Ha)\cdot c_2(\lambda)$,

$A(\lambda)=\sqrt{\frac{2}{\pi}}\frac{\sin(\lambda d)}{\lambda},\qquad \Delta=k_1I_0(k_1)I_1(k_2)-k_2I_0(k_2)I_1(k_1)$.

### 2.2.3. Numerical results for the circular channel.

The results of calculation of $V_z$ by means of formula (2.80) at Ha=10, Ha=20, Ha=50 and *d=1* are shown in Fig.2.9.

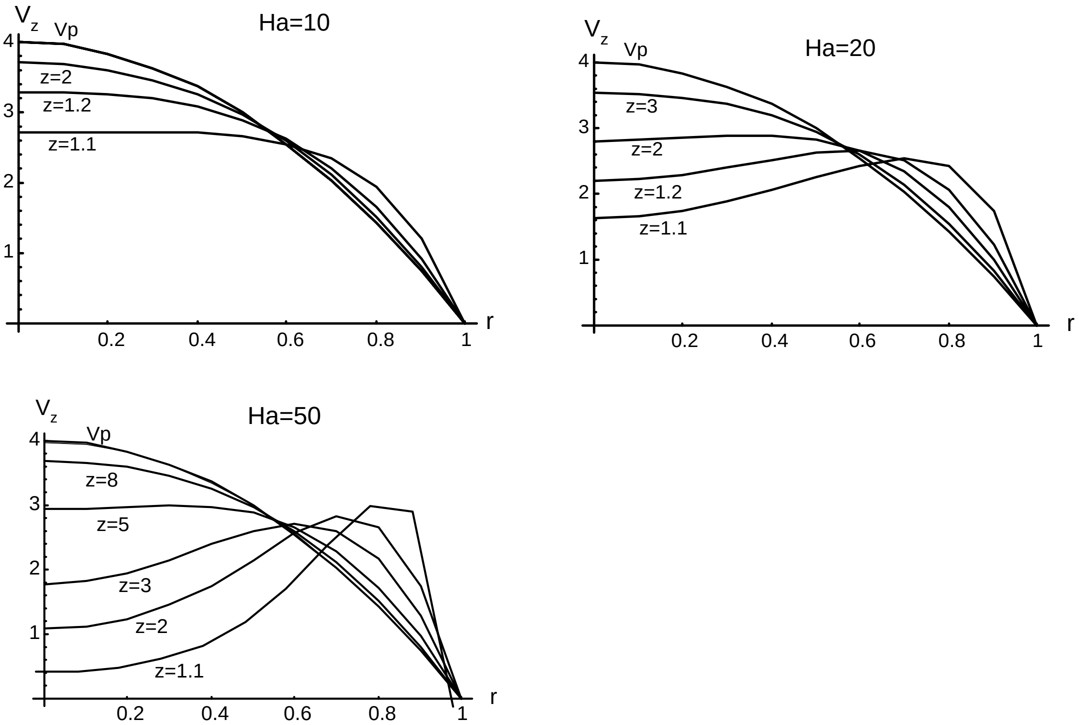



Figure 2.9. Circular channel. The velocity profiles at d=1 ($V_p$ is the Poiseuille flow).

It follows from calculations that at Ha ≥20 the velocity component $V_z(r,z)$ has the M-shaped profile in the initial part of the channel. It happens due to the electromagnetic force at the entrance of the channel is

$$\vec{F}\Big|_{\substack{r=1\\0<z<d}} = \vec{j}\times\vec{B}^e = (\sigma\vec{V}\Big|_{\substack{r=1\\0<z<d}}\times\vec{B}^e)\times\vec{B}^e = \sigma V_0 (B^e)^2\,\vec{e}_r .$$

This electromagnetic force acts on the fluid in the direction of the channel's wall. The greater are the values of $V_r$ and Ha, the larger is this force. This force creates the M-shaped profile of the velocity component $V_z$.

### 2.2.4. The asymptotic evaluation of the solution at z→∞ and Ha →∞

1) We calculate the limit $\lim_{z\to\infty} V_z(r,z)$. Substituting λz = t into Eq.(2.80) and passing to limit as z→∞, we obtain

$$\lim_{z\to\infty} V_z(r,z) = V(r,\infty) = 4L(1-r^2). \tag{2.83}$$

This formula gives the Poiseuille flow in a circular channel. Consequently, formula (2.83) may be used for calculation of length $L_\infty$ on which the solution (2.80) approaches the Poiseuille flow (2.83). The calculations give that at *d=1* and Ha=10; 20; 50 the values of $L_\infty$ are equal to 3.5; 7; 15, respectively (see Fig.2.9).

2) The greatest interest for applications is to determine the limit $\lim_{Ha\to\infty} \left(\frac{\partial P}{\partial r}\right)\Big|_{r=1}$.

We use the following asymptotic formula

$$I_n(z) = \frac{e^z}{\sqrt{2\pi z}}\left[1 + O\left(\frac{1}{z}\right)\right], \quad \text{n=0,1,} \quad |z| \to \infty, \tag{2.84}$$

in formula (2.81), which is valid in the following cases: a) $\lambda \to \infty$, Ha=$const$; b) $Ha \to \infty$, $\lambda = \lambda_0 = const > 0$ and c) $\lambda \to \infty$ and $Ha \to \infty$. Integral (2.81) converge uniformly in region $0 < r_0 \le r < 1$. The integrand in (2.81) is continuous in the region $0 < \lambda_0 \le \lambda < +\infty$. At $\lambda = 0$ the integrand has a removable singularity. We split the region of integration in integral (2.81) into two regions $0 \le \lambda \le \lambda_0$ and $\lambda_0 \le \lambda < +\infty$. We can choose $\lambda_0$ to be so small that the integral in region $0 \le \lambda \le \lambda_0$ becomes as small as we need for all Ha>0. Therefore, we use the asymptotic formula (2.84) below at once in the whole region $0 \le \lambda < +\infty$.

Using Eq.(2.84) we obtain that in region $0 < r_0 \le r < 1$ the integral (2.81) has the following form at $Ha \to \infty$:

$$\frac{\partial P}{\partial r} = \int_0^\infty \frac{\varphi(\lambda)}{\pi\sqrt{r}} \cdot \left(\sqrt{\lambda + iHa} \cdot e^{-(1-r)k_1} + \sqrt{\lambda - iHa} \cdot e^{-(1-r)k_2}\right) \frac{\sin\lambda L \cos\lambda z}{\lambda} d\lambda \tag{2.85}$$

where $\varphi(\lambda) = \sqrt{\lambda^2 + Ha^2}\left(\sqrt{\lambda + iHa} + \sqrt{\lambda - iHa}\right)$.

We will improve the convergence of integral (2.85) in region $0 < r_0 \le r < 1$ before passing to the limit as $r \to 1$ in the integrand. We take into account that the multiplier at the exponents in Eq.(2.85) is equivalent to $2\lambda^2$ at $\lambda \to \infty$. Therefore, in order to improve the convergence of integral (2.85), we rewrite it in the form

$$\frac{\partial P}{\partial r} = \frac{1}{\pi\sqrt{r}} \int_0^\infty \Big[ \left( \varphi(\lambda)\sqrt{\lambda + iHa} - \lambda^2 + \lambda^2 \right) e^{-k_1(1-r)} +$$

$$+ \left( \varphi(\lambda)\sqrt{\lambda - iHa} - \lambda^2 + \lambda^2 \right) \cdot e^{-k_2(1-r)} \Big] \frac{\sin\lambda L \cos\lambda z}{\lambda} d\lambda \qquad (2.86)$$

where $0 < r_0 \le r < 1$ and $Ha \to \infty$.

The calculation shows that

$$\lim_{r \to 1-0} \int_0^\infty \lambda \sin\lambda L \cdot e^{-k_i(1-r)} d\lambda = 0, \qquad i = 1,2 . \qquad (2.87)$$

Passing to limits $r \to 1$ and using (2.87) we obtain from (2.86) that

$$\left.\frac{\partial P}{\partial r}\right|_{r=1} = \lim_{Ha \to \infty} \frac{2}{\pi} \int_0^\infty \left[ \sqrt{\lambda^2 + Ha^2} \left( \lambda + \sqrt{\lambda^2 + Ha^2} \right) - 2\lambda^2 \right] \frac{\sin\lambda L \cos\lambda z}{\lambda} d\lambda = \frac{2Ha^2}{\pi} \int_0^\infty \frac{\sin\lambda L \cos\lambda z}{\lambda}$$

Consequently, at $Ha \to \infty$ we have

$$\left.\frac{\partial P}{\partial r}\right|_{r=1} = \begin{cases} Ha^2, & -L < z < L \\ 0, & z \notin (-L, L). \end{cases} \qquad (2.89)$$

Formula (2.89) gives the asymptotic of integral (2.81) at $Ha \to \infty$ and $r = 1$. It follows from (2.89) that at large Hartmann numbers, the pressure gradient which is proportional to the square of the Hartmann number is needed for turning the flow on the angle 90 degrees in the circular channel.

For calculations of $\partial P / \partial r$ at $r = 1$ and finite Hartmann number $Ha \ne 0$ by using Eq.(2.81) we also have to improve the convergence of integral (2.81) in region $0 < r \le r_0 < 1$ before passing to the limit as $r \to 1$ by similar way as it was done for integral in (2.85). As a result we obtain

$$\left.\frac{\partial P}{\partial r}\right|_{r=1} = \frac{2}{\pi}\int_0^\infty \left[ iHa\sqrt{\lambda^2 + Ha^2}\, \frac{\sqrt{\lambda + iHa}\, I_0(k_2) I_1(k_1) + \sqrt{\lambda - iHa}\, I_0(k_1) I_1(k_2)}{\sqrt{\lambda + iHa}\, I_0(k_2) I_1(k_1) - \sqrt{\lambda - iHa}\, I_0(k_1) I_1(k_2)} - \right.$$

$$\left. - 2\lambda^2 \right] \frac{\sin\lambda L \cos\lambda z}{\lambda}\, d\lambda\,. \qquad (2.90)$$

The integral (2.90) converges in ordinary sense and it is available for the calculation by using package “Mathematica”.

The results of calculation by means formula (2.90) at $z = 0$ and formula (2.89) are given in Table 2.1. Relative error between numerical results by formulae (2.90) and (2.89) tends to zero as $Ha \to \infty$.

| Ha | $\partial P / \partial r\|_{r=1}$, (2.102) | $\partial P / \partial r\|_{r=1} = Ha^2$, (2.101) | *Absolute error δ* | *Relative error δ/Ha² *100%* |
|---|---|---|---|---|
| 20 | 413.38 | 400 | 13.38 | 3.34 |
| 40 | 1628.20 | 1600 | 28.2 | 1.76 |
| 60 | 3644.30 | 3600 | 44.3 | 1.23 |
| 80 | 6461.70 | 6400 | 61.7 | 0.96 |
| 100 | 10080.00 | 10000 | 80 | 0.8 |
| 120 | 14500.00 | 14400 | 100 | 0.7 |
| 140 | 19722.00 | 19600 | 122 | 0.62 |
| 160 | 25744.00 | 25600 | 144 | 0.57 |
| 180 | 32568.00 | 32400 | 168 | 0.52 |
| 200 | 40194.00 | 40000 | 194 | 0.49 |

Table 2.1. Absolute and relative error between results by formulae (2.90) and (2.89).

# 3. THE DEPENDENCE BETWEEN THE BOUNDARY VELOCITY PROFILE OF THE FLUID AND THE FULL PRESSURE FORCE AT THE ENTRANCE OF REGION.

In this part of the PhD thesis we study the dependence between the profile of the boundary velocity of the fluid and the full pressure force at the entrance region by means of analytical solution of two hydrodynamic problems on an inflow of a viscous fluid into a half-space through a plane split or through a round hole and two similar MHD problems (see also [8], [54]). For each problem two cases are considered: the case of uniform inlet velocity profile at the entrance of region and the case of parabolic inlet velocity profile at the entrance. It is shown that the full pressure force at the entrance region is equal to infinity if the uniform velocity profile is given at the entry into a half-space, consequently, such problem is physically unrealistic. But if the boundary velocity profile is given as a continuous function, the full pressure force at the entrance region has finite value and the problem is physically realistic. The problems are solved in the Stokes approximation. The inductionless approximation was used for the MHD problems. Author assumes that this phenomenon also takes place in exact nonlinear formulation. The new asymptotic solutions for MHD problems are obtained at large Hartmann number for the case of parabolic boundary velocity profile.

## 3.1. The dependence between the boundary velocity profile of the fluid and the full pressure force at the entrance of region in hydrodynamic problems.

### 3.1.1. The problem on a plane jet flowing into a half-space through a split of finite width

We consider a half-space with viscous fluid located in the region D:$\{ 0 < \tilde{y} < +\infty$, $-\infty < \tilde{x},\ \tilde{z} < +\infty \}$. This half-space is bounded by the impermeable plate in the plane $\tilde{y} = 0$ with the split in the region $\{ -L \le \tilde{x} \le L,\ \tilde{y} = 0, -\infty < \tilde{z} < +\infty \}$. Viscous fluid flows into the half-space through this split with given velocity $\vec{\tilde{V}} = V_0 \cdot \psi(x) \cdot \vec{e}_y$. The geometry of the flow one can see on Fig.3.1.

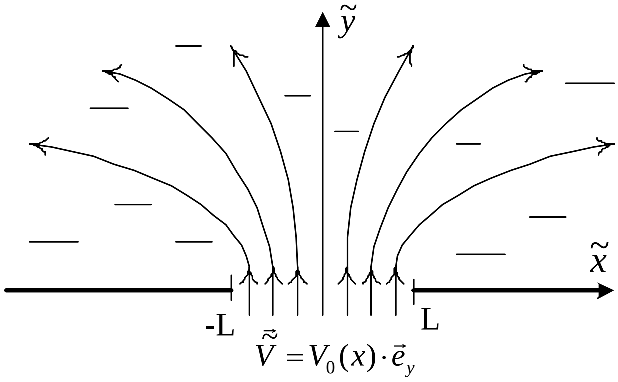



Fig.3.1. Plate with the plane split. The geometry of the flow.

We introduce the dimensionless variables in this problem using *L*-half width of the split as the scale of length, $V_0$ as the scale of the velocity $\rho \nu V_0 / L$ as the scale of pressure.

We use the nondimensional Navier-Stokes equations in Stokes approximation:

$$-\nabla P + \Delta \vec{V} = 0, \tag{3.1}$$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0, \tag{3.2}$$

where $\vec{V} = V_x(x,y)\vec{e}_x + V_y(x,y)\vec{e}_y$ is the velocity of the fluid, P is the pressure,

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \nabla = \frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y \quad .$$

Projecting Eq (3.1) on the x- and y- axis, we obtain the problem in the form

$$-\frac{\partial P}{\partial x} + \Delta V_x = 0, \tag{3.3}$$

$$-\frac{\partial P}{\partial y} + \Delta V_y = 0, \tag{3.4}$$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0. \tag{3.5}$$

The boundary conditions are:

$$y = 0: \ V_x = 0, \ V_y = \begin{cases} \psi(x), & \mathrm{x} \in (-1;1) \\ 0, & \mathrm{x} \notin (-1;1) \end{cases}, \tag{3.6}$$

$$\sqrt{x^2 + y^2} \to \infty: \quad V_x, \ V_y \to 0. \tag{3.7}$$

To study the dependence between the profile of the boundary velocity and the full pressure force on this boundary, two cases for the velocity of the fluid at the entrance of the region are considered:

1) The **uniform inlet velocity** profile is given at the entrance**:** $\vec{V} = 1 \cdot \vec{e}_y$. In this case the function $\psi(x) = 1$;

2) The **parabolic inlet velocity** profile is given at the entrance: $\vec{V} = (1 - x^2)\vec{e}_y$. Then $\psi(x) = (1 - x^2)$.

In order to solve problem (3.3)-(3.7) we use the symmetry of the problem with respect to x, that is, we use the fact that the velocity component $V_y(x, y)$ and pressure $P(x, y)$ are even functions with respect to x, but the component $V_x(x, y)$ is odd function with respect to x. Therefore, the problem can be solved by means of Fourier cosine and Fourier sine transforms, namely, we apply the Fourier sine transform with respect to x to equations (3.3), (3.5) and to $V_x$ in boundary conditions (3.6) and we apply Fourier cosine transform to equation (3.4) and to $V_y$ to boundary conditions (3.6):

$$V_y^c(\lambda, y) = \sqrt{\frac{2}{\pi}} \int_0^\infty V_y(x, y) \cos \lambda x \, dx, \tag{3.8}$$

$$V_x^s(\lambda, y) = \sqrt{\frac{2}{\pi}} \int_0^\infty V_x(x, y) \sin \lambda x \, dx, \tag{3.9}$$

$$P^c(\lambda, y) = \sqrt{\frac{2}{\pi}} \int_0^\infty P(x, y) \cos \lambda x \, dx. \tag{3.10}$$

As a result, we obtain the following system of ordinary differential equations for unknown functions $V_x^s(\lambda, y)$, $V_y^c(\lambda, y)$, $P^c(\lambda, y)$:

$$\lambda P^c - \lambda^2 V_x^s + \frac{d^2}{dy^2} V_x^s = 0, \tag{3.11}$$

$$-\frac{dP^c}{dy} - \lambda^2 V_y^c + \frac{d^2}{dy^2} V_y^c = 0 \tag{3.12}$$

$$\lambda V_x^s + \frac{dV_y^c}{dy} = 0. \tag{3.13}$$

We also apply the transforms (3.8)-(3.9) to boundary conditions (3.6):

$$\text{y=0:} \qquad V_y^c(\lambda,0) = \hat{\psi}(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^1 \psi(x)\cos\lambda x\, dx, \qquad V_x^s(\lambda,0) = 0. \tag{3.14}$$

Eliminating the functions $V_x{}^s(\lambda, y)$ and $P^c(\lambda, y)$, from system (3.11)-(3.13) we obtain the equation for $V_y{}^c(\lambda, y)$ :

$$\frac{d^4 V_y{}^c}{dy^4} - 2\lambda^2 \frac{d^2 V_y{}^c}{dy^2} + \lambda^4 V_y{}^c = 0. \tag{3.15}$$

The solution of Eq.(3.15), that tends to 0 at $x \to \infty, y \to \infty$ has the form

$$V_y{}^c = C_1 e^{-\lambda y} + C_2 y e^{-\lambda y}, \tag{3.16}$$

where $C_1$, $C_2$ are constants.

To determine constants $C_1$, $C_2$ we use boundary conditions (3.14) and the following additional boundary obtained from (3.13) and (3.14):

$$\left.\frac{d}{dy} V_y{}^c\right|_{y=o} = 0. \tag{3.17}$$

As a result, we have

$$C_1 = \hat{\psi}(\lambda), \quad C_2 = \lambda C_1 = \lambda \hat{\psi}(\lambda) \tag{3.18}$$

Therefore,

$$V_y{}^c = \hat{\psi}(\lambda) e^{-\lambda y}(1 + \lambda y). \tag{3.19}$$

Determining $V_x{}^s(\lambda, y)$ and $\mathrm{P}^c(\lambda, y)$ from the system (3.11)-(3.13) and using the inverse Fourier sine and Fourier cosine transforms, we obtain the solution to the problem (3.3)-(3.7) with boundary conditions (3.6)-(3.7) in the form:

$$V_y = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\lambda y}(1+\lambda y)\hat{\psi}(\lambda)\cos\lambda x d\lambda, \tag{3.20}$$

$$V_x = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\lambda y}\lambda y \hat{\psi}(\lambda)\sin\lambda x d\lambda, \tag{3.21}$$

$$\frac{\partial P}{\partial y} = -2\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\lambda y}\lambda^2\hat{\psi}(\lambda)\cos\lambda x d\lambda\,, \tag{3.22}$$

$$\frac{\partial P}{\partial x} = -2\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\lambda y}\lambda^2\hat{\psi}(\lambda)\sin\lambda x d\lambda\,, \tag{3.23}$$

$$P = 2\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\lambda y}\lambda\hat{\psi}(\lambda)\cos\lambda x d\lambda\,, \tag{3.24}$$

function $\hat{\psi}(\lambda)$ is given by (3.14).

The full pressure force at the entrance into the half-space is equal to

$$F = 2\int_0^1 P\big|_{y=o} dx = 4\sqrt{\frac{2}{\pi}}\int_0^1 \left\{\int_0^\infty e^{-\lambda y}\lambda\hat{\psi}(\lambda)\cos\lambda x d\lambda\right\}\Bigg|_{y=0} dx = 4\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{\psi}(\lambda)\sin\lambda x d\lambda \tag{3.25}$$

**<u>1) The viscous fluid flows into the half-space through the split with velocity $\vec{\mathbf{V}} = \mathbf{1}\cdot\vec{\mathbf{e}}_{\mathbf{y}}$ :</u>**

In this case $\psi(x) = 1$ in (3.6), i.e. the boundary conditions (3.6) have the form:

$$y = 0:\;\; V_x = 0\,,\;\; V_y = \begin{cases} 1, & \mathrm{x} \in (-1;1) \\ 0, & \mathrm{x} \notin (-1;1) \end{cases} \;. \tag{3.26}$$

$$\sqrt{x^2 + y^2} \to \infty : V_x, V_y \to 0$$

In this case function $\hat{\psi}(\lambda)$ in (3.14) has the form:

$$\hat{\psi}(\lambda) = V_y^c(\lambda,0) = \sqrt{\frac{2}{\pi}}\frac{\sin\lambda}{\lambda}\,. \tag{3.27}$$

Substituting $\hat{\psi}(\lambda)$ into Eqs.(3.20)-(3.24) we obtain the solution of problem (3.3)-(3.5) with boundary conditions (3.26) in the form of convergent improper integrals:

$$V_y = \frac{2}{\pi}\int_0^\infty e^{-\lambda y}(1+\lambda y)\,\frac{\sin\lambda\cos\lambda x}{\lambda}d\lambda\,, \tag{3.28}$$

$$V_x = \frac{2}{\pi}\int_0^\infty e^{-\lambda y} y\sin\lambda\sin\lambda x d\lambda\,, \tag{3.29}$$

$$\frac{\partial P}{\partial y} = -\frac{4}{\pi}\int_0^\infty e^{-\lambda y}\lambda \sin\lambda \cos\lambda x d\lambda\,, \tag{3.30}$$

$$\frac{\partial P}{\partial x} = -\frac{4}{\pi}\int_0^\infty e^{-\lambda y}\lambda \sin\lambda \sin\lambda x d\lambda\,, \tag{3.31}$$

$$P = \frac{4}{\pi}\int_0^\infty e^{-\lambda y} \sin\lambda \cos\lambda x d\lambda\,. \tag{3.32}$$

Integrals (2.28)-(3.32) were analytically evaluated by using Laplace transform $L[f(y)] = \int_0^\infty F(\lambda)e^{-\lambda y}d\lambda$ and its properties (see [4]). As a result, we obtained the solution of the problem in the form of elementary functions:

$$V_y = \frac{1}{\pi}\left[\pi - \arctan\frac{y}{1-x} - \arctan\frac{y}{1+x} + \frac{y(1-x)}{(1-x)^2+y^2} + \frac{y(1+x)}{(1+x)^2+y^2}\right], \qquad \text{if } 0<x<1$$

$$V_y = \frac{1}{\pi}\left[-\arctan\frac{y}{1-x} - \arctan\frac{y}{1+x} + \frac{y(1-x)}{(1-x)^2+y^2} + \frac{y(1+x)}{(1+x)^2+y^2}\right], \qquad \text{if } x>1$$

$$V_x = \frac{y^2}{\pi}\left[\frac{1}{(1-x)^2+y^2} - \frac{1}{(1+x)^2+y^2}\right],$$

$$\frac{\partial P}{\partial y} = -\frac{4y}{\pi}\left[\frac{1-x}{\left(y^2+(1-x)^2\right)^2} + \frac{1+x}{\left(y^2+(1+x)^2\right)^2}\right],$$

$$\frac{\partial P}{\partial x} = -\frac{2}{\pi}\left[\frac{y^2-(1-x)^2}{\left(y^2+(1-x)^2\right)^2} - \frac{y^2-(1+x)^2}{\left(y^2+(1+x)^2\right)^2}\right],$$

$$P = \frac{2}{\pi}\left[\frac{1-x}{(1-x)^2+y^2} - \frac{1+x}{(1+x)^2+y^2}\right].$$

Substituting $\hat{\psi}(\lambda)$ from formula (3.27) into (3.25) we obtain the full pressure force at the entrance into the half-space in the form:

$$F = \frac{8}{\pi}\int_0^\infty \frac{\sin^2\lambda}{\lambda}d\lambda \tag{3.33}$$

The integral (3.33) diverges and, consequently, the full pressure in this case is $F = \infty$.

**<u>2) The fluid flows into a half-space through the split with velocity</u>** $\vec{\mathbf{V}} = (\mathbf{1} - \mathbf{x}^2)\vec{\mathbf{e}}_y$

In this case $\psi(x) = 1 - x^2$ in (3.6), i.e. the boundary conditions (3.6) have the form:

$$y = 0: \ V_x = 0, \ V_y = \begin{cases} 1 - x^2, & x \in (-1;1) \\ 0, & x \notin (-1;1) \end{cases} . \tag{3.34}$$

$$\sqrt{x^2 + y^2} \to \infty : V_x, V_y \to 0 . \tag{3.35}$$

Function $\hat{\psi}(\lambda)$ from (3.14) in the case of parabolic profile of the inlet velocity has the form:

$$\hat{\psi}(\lambda) = V_y^c(\lambda, 0) = 2\sqrt{\frac{2}{\pi}} \frac{\sin\lambda - \lambda\cos\lambda}{\lambda^3} . \tag{3.36}$$

Substituting $\hat{\psi}(\lambda)$ from (3.36) into Eqs.(3.20)-(3.24), we obtain the solution of problem (3.3)-(3.5) with boundary conditions (3.34) in the form of convergent improper integrals:

$$V_y = \frac{4}{\pi}\int_0^\infty e^{-\lambda y}(1 + \lambda y) \ \frac{\sin\lambda - \lambda\cos\lambda}{\lambda^3} \cos\lambda x d\lambda , \tag{3.37}$$

$$V_x = \frac{4}{\pi}\int_0^\infty e^{-\lambda y} y \frac{\sin\lambda - \lambda\cos\lambda}{\lambda^2} \sin\lambda x d\lambda , \tag{3.38}$$

$$\frac{\partial P}{\partial y} = -\frac{8}{\pi}\int_0^\infty e^{-\lambda y} \frac{\sin\lambda - \lambda\cos\lambda}{\lambda} \cos\lambda x d\lambda , \tag{3.39}$$

$$\frac{\partial P}{\partial x} = -\frac{8}{\pi}\int_0^\infty e^{-\lambda y} \frac{\sin\lambda - \lambda\cos\lambda}{\lambda} \sin\lambda x d\lambda , \tag{3.40}$$

$$P = \frac{8}{\pi}\int_0^\infty e^{-\lambda y} \frac{\sin\lambda - \lambda\cos\lambda}{\lambda^2} \cos\lambda x d\lambda . \tag{3.41}$$

Integrals (3.37)-(3.41) were evaluated analytically by using Laplace transform and its properties. As a result, the solution of the problem was obtained in the form of elementary functions:

$$P = \frac{4}{\pi}\left[2 - \pi y + y\,arctg\frac{y}{1 - x} + y\,arctg\frac{y}{1 + x} - \frac{x}{2}Ln\left|\frac{(1 + x)^2 + y^2}{(1 - x)^2 + y^2}\right|\right], \qquad \text{if } 0<x<1;$$

$$P = \frac{4}{\pi}\left[2 + y\,arctg\frac{y}{1 - x} + y\,arctg\frac{y}{1 + x} - \frac{x}{2}Ln\left|\frac{(1 + x)^2 + y^2}{(1 - x)^2 + y^2}\right|\right], \qquad \text{if } x>1;$$

$$\frac{\partial P}{\partial x} = -\frac{4}{\pi}\left[\frac{1-x}{(1-x)^2+y^2} - \frac{1+x}{(1+x)^2+y^2} + \frac{1}{2}Ln\left|\frac{(1+x)^2+y^2}{(1-x)^2+y^2}\right|\right], \qquad \text{if } x>0;$$

$$\frac{\partial P}{\partial y} = -\frac{4}{\pi}\left[\pi - arctg\frac{y}{1-x} - arctg\frac{y}{1+x} - \frac{y}{y^2+(1-x)^2} - \frac{y}{y^2+(1+x)^2}\right], \qquad \text{if } 0<x<1;$$

$$\frac{\partial P}{\partial y} = -\frac{4}{\pi}\left[-arctg\frac{y}{1-x} - arctg\frac{y}{1+x} - \frac{y}{y^2+(1-x)^2} - \frac{y}{y^2+(1+x)^2}\right], \qquad \text{if } x>1;$$

$$V_y = -y^2 + \frac{2y}{\pi} - \frac{1}{\pi}(1-x^2-y^2)\left(arctg\frac{y}{1-x} + arctg\frac{y}{1+x}\right) + 1 - x^2, \qquad \text{if } 0<x<1;$$

$$V_y = \frac{2y}{\pi} - \frac{1}{\pi}(1-x^2-y^2)\left(arctg\frac{y}{1-x} + arctg\frac{y}{1+x}\right), \qquad \text{if } x>1;$$

$$V_x = \frac{2y}{\pi}\left(\pi x - x\,arctg\frac{y}{1-x} - x\,arctg\frac{y}{1+x} + \frac{y}{2}Ln\left|\frac{(1-x)^2+y^2}{(1+x)^2+y^2}\right|\right), \qquad \text{if } 0<x<1;$$

$$V_x = \frac{2y}{\pi}\left(-x\,arctg\frac{y}{1-x} - x\,arctg\frac{y}{1+x} + \frac{y}{2}Ln\left|\frac{(1-x)^2+y^2}{(1+x)^2+y^2}\right|\right), \qquad \text{if } x>1.$$

The full pressure at the entrance of the half-space in this case has the form:

$$F = \frac{16}{\pi}\int_0^\infty \frac{\sin\lambda - \lambda\cos\lambda}{\lambda^3}\sin\lambda d\lambda \qquad (3.42)$$

Integral in (3.42) converges and it is equal to F=1/2. So, if the profile of the boundary velocity at the entrance into the half-space is given as a parabolic function, then the full pressure at the entrance of the half-space, in contrast to the previous case, has finite value.

### 3.1.2. The problem on a round jet flowing into a half-space through a round hole of finite radius.

In this part of the thesis the problem similar to the one considered in the previous section of the thesis, is solved for the case where fluid flows into the half-space $\tilde{z} \geq 0$ through a **round hole** of finite radius. The geometry of the flow is shown in Fig.3.2.

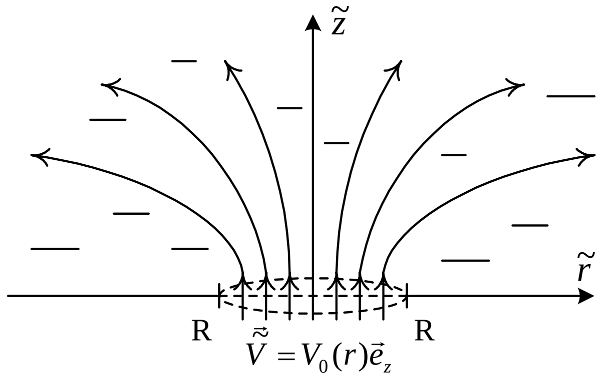



Figure 3.2. Plate with a round hole. The geometry of the fow.

The half-space with a fluid is located in the region D:{ $0 < \tilde{z} < +\infty$, $0 < \tilde{r} < +\infty$ } ($\tilde{r}$, $\tilde{\varphi}$, $\tilde{z}$ are the cylindrical coordinates). There is an impermeable plate in the plane $\tilde{z} = 0$ with a round hole in the region { $0 \le \tilde{r} \le R$, $\tilde{z} = 0$ }. The viscous fluid flows into the half-space through this hole with given velocity $\tilde{\vec{V}} = V_0 \cdot \varphi(r) \cdot \vec{e}_z$.

We introduce the dimensionless quantities using the values R, $V_0$, $\rho\nu V_0 / R$ as scales of length, velocity and pressure, respectively.

We use the nondimensional Navier-Stokes equations in polar coordinate system and Stokes approximation. In projections on the r- and z- axis these equations have the form:

$$-\frac{\partial P}{\partial r} + \left( \Delta V_r - \frac{V_r}{r^2} \right) = 0, \tag{3.43}$$

$$-\frac{\partial P}{\partial z} + \Delta V_z = 0, \tag{3. 44}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left( rV_r \right) + \frac{\partial V_z}{\partial z} = 0. \tag{3.45}$$

where $\vec{V} = V_r(r,z)\vec{e}_r + V_z(r,z)\vec{e}_z$ is the velocity of the fluid, P is the pressure,

$\Delta = \frac{\partial^2}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left( r\frac{\partial}{\partial r} \right).$

The dimensionless boundary conditions are:

$$z = 0: \quad V_z = \begin{cases} \varphi(r), & r < 1 \\ 0, & r > 1 \end{cases}, \quad V_r = 0. \tag{3.46}$$

$$r^2 + z^2 \to \infty: \quad V_r, V_z \to 0 . \tag{3.47}$$

In order to solve problem (3.45)-(3.49) we use the Hankel transform (2.8) with respect to r:

$$\hat{V}_r(z,\lambda) = \int_0^\infty r V_r(r,z)\, J_1(\lambda r) dr \,, \tag{3.48}$$

$$\hat{V}_z(z,\lambda) = \int_0^\infty r V_z(r,z)\, J_0(\lambda r) dr \,, \tag{3.49}$$

$$\hat{P}(z,\lambda) = \int_0^\infty r P(r,z)\, J_0(\lambda r) dr \,. \tag{3.50}$$

where $J_\nu(\lambda r)$ are the Bessel functions of order $\nu$ $(\nu = 0{,}1)$.

As a result we obtain the following system of ordinary differential equations for unknown functions $\hat{P}, \hat{V}_r, \hat{V}_z$:

$$\lambda \hat{P} - \lambda^2 \hat{V}_r + \frac{d^2}{dz^2}\hat{V}_r = 0 \,, \tag{3.51}$$

$$-\frac{d\hat{P}}{dz} - \lambda^2 \hat{V}_z + \frac{d^2}{dz^2}\hat{V}_z = 0 \tag{3.52}$$

$$\lambda \hat{V}_r + \frac{d\hat{V}_z}{dz} = 0 \,. \tag{3.53}$$

We also apply Hankel transforms to boundary conditions (3.46):

$$\text{z=0:} \quad \hat{V}_z = \hat{\varphi}(\lambda) = \int_0^1 \varphi(r) r J_0(\lambda r) dr \,, \quad \hat{V}_r = 0 \,. \tag{3.54}$$

Eliminating the functions $\hat{V}_r(\lambda, r)$, $\hat{P}(\lambda, r)$ from system (3.51)-(3.53) we obtain the differential equation for $\hat{V}_z(\lambda, r)$:

$$\frac{d^4\hat{V}_z}{dz^4} - 2\lambda^2 \frac{d^2\hat{V}_z}{dz^2} + \lambda^4 \hat{V}_y = 0 \,. \tag{3.55}$$

The general solution to this equation that tends to 0 at $z \to \infty$ has the form:

$$\hat{V}_z = C_1 e^{-\lambda z} + C_2 z e^{-\lambda z} \,, \tag{3.56}$$

where $C_1$, $C_2$ are constants.

To determine constants $C_1$ and $C_2$ we use boundary condition (3.54) and the following additional boundary condition, that one can obtain from (3.53) and (3.54):

$$\frac{d}{dz}\hat{V}_z\bigg|_{z=o} = 0\,. \tag{3.57}$$

As a result, we get the solution for $\hat{V}_z(\lambda, z)$ in the form:

$$\hat{V}_z(\lambda, z) = \hat{\varphi}(\lambda)\cdot e^{-\lambda z}\cdot(1+\lambda z) \tag{3.58}$$

Determining $\hat{V}_r$ and $\hat{P}$ from the system (3.51)-(3.53) and using the inverse Hankel transforms, we obtain the solution of the problem:

$$V_z = \int_0^\infty \hat{\varphi}(\lambda)e^{-\lambda z}(1+\lambda z)\lambda J_0(\lambda r)\,d\lambda\,, \tag{3.59}$$

$$V_r = \int_0^\infty \lambda^2 z\hat{\varphi}(\lambda)e^{-\lambda z}J_1(\lambda r)\,d\lambda \tag{3.60}$$

$$P = \int_0^\infty 2\lambda^2\hat{\varphi}(\lambda)e^{-\lambda z}J_0(\lambda r)\,d\lambda\,, \tag{3.61}$$

$$\frac{\partial P}{\partial z} = \int_0^\infty -2\lambda^3\hat{\varphi}(\lambda)e^{-\lambda z}J_0(\lambda r)\,d\lambda\,, \tag{3.62}$$

$$\frac{\partial P}{\partial r} = \int_0^\infty -2\lambda^3\hat{\varphi}(\lambda)e^{-\lambda z}J_1(\lambda r)\,d\lambda \tag{3.63}$$

The full pressure force at the entrance into the half-space is:

$$F = 2\pi\int_0^1 P\big|_{z=0}\,rdr = 4\pi\int_0^1\left\{\int_0^\infty \lambda^2\hat{\varphi}(\lambda)J_0(\lambda r)d\lambda\right\}rdr \tag{3.64}$$

Changing the order of integration in (3.64) and evaluating the integral with respect to r we obtain:

$$F = 4\pi\int_0^\infty \lambda\hat{\varphi}(\lambda)J_1(\lambda)d\lambda \tag{3.65}$$

Similarly to the previous paragraph, we **consider two cases:**

1) the viscous fluid flows into the half-space through the hole with velocity $\vec{V} = 1\cdot\vec{e}_z$

2) the viscous fluid flows into the half-space through the hole with velocity $\vec{V} = (1-r^2)\vec{e}_z$.

**1) The fluid flows into the half-space through the round hole with velocity** $\vec{\mathbf{V}} = \mathbf{1} \cdot \vec{\mathbf{e}}_{\mathbf{z}}$

In this case $\varphi(r) = 1$ in boundary condition (3.46) and the function $\hat{\varphi}(\lambda)$ in (3.54) has the form:

$$\hat{\varphi}(\lambda) = \hat{V}_z(\lambda, 0) = \int_0^1 1 \cdot r J_0(\lambda r) dr = \frac{1}{\lambda} J_1(\lambda) \,. \tag{3.66}$$

Substituting $\hat{\varphi}(\lambda)$ from (3.66) into Eqs (3.59)-(3.63) we obtain the solution of problem in the form of convergent improper integrals:

$$V_z = \int_0^\infty e^{-\lambda z}(1 + \lambda z) J_1(\lambda) J_0(\lambda r)\, d\lambda \,, \tag{3.67}$$

$$V_r = \int_0^\infty \lambda z e^{-\lambda z} J_1(r) J_1(\lambda r)\, d\lambda \,, \tag{3.68}$$

$$P = \int_0^\infty 2\lambda e^{-\lambda z} J_1(r) J_0(\lambda r)\, d\lambda \,, \tag{3.69}$$

$$\frac{\partial P}{\partial z} = \int_0^\infty -2\lambda^2 e^{-\lambda z} J_1(r) J_0(\lambda r)\, d\lambda \,, \tag{3.70}$$

$$\frac{\partial P}{\partial r} = \int_0^\infty -2\lambda^2 e^{-\lambda z} J_1(r) J_1(\lambda r)\, d\lambda \,. \tag{3.71}$$

Substituting function $\hat{\varphi}(\lambda)$ from (3.66) into (3.65) we obtain the full pressure force at the entrance region in the form

$$F = 4\pi \int_0^\infty {J_1}^2(\lambda) d\lambda \,. \tag{3.72}$$

To evaluate this integral we use the following asymptotic formula (see [114]):

$$J_n(\lambda) = \sqrt{\frac{2}{\pi\lambda}} \left[ \cos\left( \lambda - \frac{\pi n}{2} - \frac{\pi}{4} \right) + O\left( \frac{1}{\lambda} \right) \right].$$

For $n = 1$ we have

$$J_1(\lambda) = \sqrt{\frac{2}{\pi\lambda}} \left[ \cos\left( \lambda - \frac{3\pi}{4} \right) + O\left( \frac{1}{\lambda} \right) \right]. \tag{3.73}$$

We consider integral $\int_N^\infty J_1^{\,2}(\lambda)d\lambda$. For sufficiently large $N$ one can substitute (3.73) into this integral instead $J_1(\lambda)$.

As a result, one obtains integral $\int_N^\infty \frac{\cos^2\lambda}{\lambda}d\lambda$. This integral diverges, consequently the integral (3.72) also diverges and full pressure at the entrance is $F=\infty$.

**2) The fluid flows into the half-space through the round hole with velocity** $\vec{\mathbf{V}}=(\mathbf{1}-\mathbf{r}^2)\vec{\mathbf{e}}_z$

In this case $\varphi(r)=1-r^2$ in boundary conditions (3.46) and the function $\hat{\varphi}(\lambda)$ in (3.54) has the form:

$$\hat{\varphi}(\lambda)=\int_0^1(1-r^2)\cdot rJ_0(\lambda r)dr=\frac{2}{\lambda^2}J_2(\lambda)\,. \tag{3.74}$$

Substituting function $\hat{\varphi}(\lambda)$ from (3.74) into the formulae (3.59)-(3.63) we obtain the solution of the problem for the case of the continuous velocity on the entrance region in the form:

$$V_z=2\int_0^\infty e^{-\lambda z}(1+\lambda z)\frac{J_2(\lambda)J_0(\lambda r)}{\lambda}\,d\lambda\,, \tag{3.75}$$

$$V_r=2\int_0^\infty ze^{-\lambda z}J_2(r)J_1(\lambda r)\,d\lambda \tag{3.76}$$

$$P=4\int_0^\infty e^{-\lambda z}J_2(r)J_0(\lambda r)\,d\lambda\,, \tag{3.77}$$

$$\frac{\partial P}{\partial z}=-4\int_0^\infty \lambda e^{-\lambda z}J_2(r)J_0(\lambda r)\,d\lambda\,, \tag{3.78}$$

$$\frac{\partial P}{\partial r}=-4\int_0^\infty \lambda e^{-\lambda z}J_2(r)J_1(\lambda r)\,d\lambda \tag{3.79}$$

Substituting $\hat{\varphi}(\lambda)$ also into (3.65) and using formula from [114] we obtain the full pressure force at the entrance into half-space:

$$F = 8\pi \int_0^{\infty} \frac{J_1(\lambda) J_2(\lambda)}{\lambda} d\lambda = 8\pi \frac{\Gamma(1)\Gamma\left(\frac{3}{2}\right)}{2\Gamma\left(\frac{3}{2}\right)\Gamma\left(1+\frac{3}{2}\right)\Gamma(1)} = \frac{16}{3}\sqrt{\pi} \ . \qquad (3.80)$$

So, integral (3.80) converges, consequently, the full pressure F at the entrance into a half-space has a finite value and, therefore, the problem is physically realistic in the case of parabolic profile of the inlet velocity.

## 3.2. The dependence between the boundary velocity profile of the fluid and the full pressure force at the entrance of region for MHD problems

In this part we consider the MHD analogue of problems considered in the first part of the Chapter 3.

### 3.2.1. The MHD problem on a plane jet flowing fluid into a half-space through a plane split of finite width

We solve the problem considered in §3.1.1 on an inflow of conducting fluid into a half-space $\tilde{y} > 0$ with a given velocity $V_0 \psi(x)$ through a split of finite width located in the region $\{ \tilde{y} = 0 ,\ -L \le \tilde{x} \le L ,\ -\infty < \tilde{z} < +\infty \}$, but with the presence of external magnetic field that is parallel to the y-axis, i.e. $\vec{\tilde{B}}^e = B_0 \vec{e}_y$. We introduce the dimensionless quantities similarly to §3.1.1, but as a scale of magnetic field the magnitude $\mathrm{B}_0$ is used.

We use dimensionless MHD equations (1.5)-(1.7) in the Stokes and inductionless approximation, i.e.:

$$-\frac{\partial P}{\partial x} + \Delta V_x - Ha^2 V_x = 0 , \qquad (3.81)$$

$$-\frac{\partial P}{\partial y} + \Delta V_y = 0 , \qquad (3.82)$$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0 , \qquad (3.83)$$

$\vec{V} = V_x(x, y)\vec{e}_x + V_y(x, y)\vec{e}_y$ is the velocity of fluid, $P = P(x, y)$ is the pressure of the fluid.

Boundary conditions are described by Eqs.(3.6), (3.7), i.e.

$$y = 0: \; V_x = 0\,, \;\; V_y = \begin{cases} \psi(x), & x \in (-1,1) \\ 0, & x \notin (-1,1) \end{cases}, \tag{3.84}$$

$$x^2 + y^2 \to \infty : V_x, V_y \to 0\,. \tag{3.85}$$

For the solution of the problem we use the Fourier cosine and Fourier sine transforms (similarly to the problem considered in paragraph 3.1.1). As the result, the solution to the problem has the form:

$$V_x(x,y) = \frac{4}{\pi Ha}\sinh(\mu y)\int_0^\infty e^{-y\cdot f(\lambda)}\hat{\psi}(\lambda)\,\lambda\sin(\lambda x)\,d\lambda\,, \tag{3.86}$$

$$V_y(x,y) = \frac{4}{\pi Ha}\int_0^\infty e^{-yf(\lambda)}[\,f(\lambda)\sinh(\mu y) + \mu\cosh(\mu y)]\hat{\psi}(\lambda)\cos(\lambda x)\,d\lambda\,, \tag{3.87}$$

$$P(x,y) = \frac{4}{\pi}\int_0^\infty e^{-yf(\lambda)}[\,f(\lambda)\cosh(\mu y) + \mu\sinh(\mu y)]\hat{\psi}(\lambda)\cos(\lambda x)\,d\lambda\,, \tag{3.88}$$

where

$$f(\lambda) = \sqrt{\lambda^2 + \mu^2}\,,\;\; \mu = 0.5Ha\,, \tag{3.89}$$

$$\hat{\psi}(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^1 \psi(x)\cos\lambda x\,dx\,. \tag{3.90}$$

The full pressure force on the entrance of the half-space is equal to:

$$F = 2\int_0^1 P\big|_{y=0}\,dx = 8\pi^{-1}\int_0^\infty \sqrt{\lambda^2 + \mu^2}\;\hat{\psi}(\lambda)\frac{\sin\lambda}{\lambda}d\lambda\,. \tag{3.91}$$

Similarly to the paragraph 3.1.1., two cases are considered for the problem:

1) the conducting fluid flows into the half-space through the split with velocity $\vec{V} = 1\cdot\vec{e}_y$;

2) the conducting fluid flows into the half-space with the parabolic velocity $\vec{V} = (1 - x^2)\cdot\vec{e}_y$.

**1) the conducting fluid flows into the half-space through the split with velocity** $\vec{\mathbf{V}} = \mathbf{1} \cdot \vec{\mathbf{e}}_\mathbf{y}$

It means that function $\psi(x) = 1$ in (3.84) and the function $\hat{\psi}(\lambda)$ has the form (3.27). The full pressure at the entrance into half-space for this case is:

$$F = \frac{8}{\pi}\int_0^\infty \sqrt{\lambda^2 + \mu^2}\ \frac{\sin^2(\lambda)}{\lambda^2} d\lambda\ . \qquad (3.92)$$

This integral diverge, $\mathrm{F} = \infty$, and, consequently, the flow with boundary velocity $\vec{V} = 1 \cdot \vec{e}_y$ is physically unrealistic.

**2) the conducting fluid flows into the half-space through the split with velocity** $\vec{\mathbf{V}} = (\mathbf{1} - \mathbf{x}^2) \cdot \vec{\mathbf{e}}_\mathbf{y}$

In this case the function $\hat{\psi}(\lambda)$ has the form (3.36) and the full pressure at the entrance into half-space is

$$F = 16\sqrt{\frac{2}{\pi}} \cdot \frac{1}{\pi}\int_0^\infty \sqrt{\lambda^2 + \mu^2} \left(\frac{\sin\lambda - \lambda\cos\lambda}{\lambda^2}\right) \sin\lambda d\lambda\ . \qquad (3.93)$$

This integral is converged and, consequently, the flow with parabolic boundary velocity is physically realistic.

The **asymptotic solution at** $\mathrm{Ha} \to \infty$ is also obtained for this case:

$$\lim_{Ha\to\infty} F = 8\sqrt{\frac{2}{\pi}}\ \frac{Ha}{\pi}\int_0^\infty \left(\frac{\sin\lambda - \lambda\cos\lambda}{\lambda^4}\right)\sin\lambda d\lambda = \frac{4}{3}\sqrt{\frac{2}{\pi}}\ Ha\ , \qquad (3.94)$$

$$\lim_{Ha\to\infty} V_x = 0\,, \qquad \lim_{Ha\to\infty} V_y = (1 - x^2)\eta(1 - |x|)\,, \qquad (3.95)$$

$$\lim_{Ha\to\infty} P = 4\sqrt{\frac{2}{\pi}}\ \frac{Ha}{\pi}\int_0^\infty \left(\frac{\sin\lambda - \lambda\cos\lambda}{\lambda^3}\right)\cos\lambda x d\lambda = \frac{1}{2}\sqrt{\frac{2}{\pi}}\ Ha \cdot (1 - x^2)\eta(1 - |x|)\,. \qquad (3.96)$$

where $\eta(x)$ is the Heaviside step function.

### 3.2.2. The MHD problem on a round jet flowing into a half-space through a round hole of finite radius.

In this paragraph we study the dependence between the profile of the boundary velocity and the full pressure force at the entrance region for the problem considered in §3.1.2 on an inflow of conducting fluid into the half-space $\tilde{z} > 0$ through the round hole of finite radius R, but with the presence of the external magnetic field $\tilde{\vec{B}}^e = B_0 \vec{e}_z$ that is perpendicular to the plane. The fluid flows into the half-space with the given velocity $V_0\varphi(r)$. The dimensionless quantities are the same as in §3.2.1, only as the scale of length in this problem we take radius of the hole $R$.

Dimensionless MHD equations in Stokes and inductionless approximation in cylindrical coordinate system have the form of Eqs.(2.3)-(2.5), i.e.:

$$\frac{\partial P}{\partial r} + (L_1 - Ha^2)V_r = 0, \tag{3.97}$$

$$\frac{\partial P}{\partial z} + L_0 V_z = 0, \tag{3.98}$$

$$\frac{\partial V_z}{\partial z} + \frac{1}{r}\frac{\partial (rV_r)}{\partial r} = 0, \tag{3.99}$$

where $L_0 = \frac{\partial^2}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right), \quad L_1 = L_0 - \frac{1}{r^2}$,

$\vec{V} = V_r\vec{e}_r + V_z\vec{e}_z$ is the velocity of the fluid, *P(r,z)* is the pressure.

The boundary conditions are given by Eqs.(3.46)-(3.47) and have the form:

$$z = 0: \ V_r = 0, \ \ V_z = \begin{cases} \varphi(r), & r \in (-1,1) \\ 0, & r \notin (-1,1) \end{cases}, \tag{3.100}$$

$$r^2 + z^2 \to \infty: \ V_r \to 0, \ V_z \to 0. \tag{3.101}$$

For the solution of the problem we use the Hankel transform of order 1 with respect to r for function $V_r(r,z)$ and the Hankel transform of order 0 with respect to r for functions $V_z(r,z)$ and *P(r, z)* (similarly, as it was done in § 3.1.2.). As a result we obtain the solution of the problem (3.97)-(3.101) in the form:

$$V_r(r,z)=\frac{1}{Ha}\int_0^\infty (e^{k_2 z}-e^{k_1 z})\lambda^2\hat{\varphi}(\lambda)J_1(\lambda)J_1(\lambda r)\,d\lambda\,, \tag{3.102}$$

$$V_z(r,z)=\frac{2}{Ha}\int_0^\infty (k_2 e^{k_1 z}-k_1 e^{k_2 z})\hat{\varphi}(\lambda)\lambda J_0(\lambda r)\,d\lambda\,, \tag{3.103}$$

$$P(r,z)=-2\int_0^\infty (k_2 e^{k_1 z}+k_1 e^{k_2 z})\hat{\varphi}(\lambda)\lambda J_0(\lambda r)\,d\lambda\,, \tag{3.104}$$

where

$$k_1=-(\sqrt{\lambda^2+\mu^2}+\mu)\,,\ k_2=-(\sqrt{\lambda^2+\mu^2}-\mu)\,, \tag{3.105}$$

$$\hat{\varphi}(\lambda)=\int_0^1 \varphi(r)rJ_0(\lambda r)\,dr\,, \tag{3.106}$$

$Ha=2\mu$, $J_\nu(\lambda r)$ $(\nu=0,1)$ are the Bessel functions of order $\nu$.

The full pressure force at the entrance of the half-space is equal to:

$$F=2\pi\int_0^1 P\big|_{z=0}rdr=4\pi\int_0^\infty \hat{\varphi}(\lambda)\sqrt{\lambda^2+\mu^2}\;J_1(\lambda)d\lambda\,. \tag{3.107}$$

**1) the conducting fluid flows into a half-space through the hole with velocity** $\vec{\mathbf{V}}=\mathbf{1}\cdot\vec{\mathbf{e}}_{\mathbf{z}}$

Then $\varphi(r)=1$ and the function $\hat{\varphi}(\lambda)$ in Eq.(3.106) has the form (3.66), i.e.

$$\hat{\varphi}(\lambda)=\int_0^1 rJ_0(\lambda r)\,dr=\frac{J_1(\lambda)}{\lambda}\,. \tag{3.108}$$

Formulae (3.102)-(3.106) with (3.108) describe the solution of the problem for the case of the constant velocity $\vec{V}=1\cdot\vec{e}_z$ in the entrance region. The full pressure force at the entrance region for the problem is:

$$F=4\pi\int_0^\infty \sqrt{\lambda^2+\mu^2}\;\frac{J_1{}^2(\lambda)}{\lambda}d\lambda\,. \tag{3.109}$$

With the aim to study the convergence of the integral (3.109) we consider the following integral:

$$\int_N^\infty \sqrt{\lambda^2+\mu^2}\;\frac{J_1{}^2(\lambda)}{\lambda}d\lambda\,. \tag{3.110}$$

For sufficiently large N we can use the asymptotic formula (3.73) for $J_1(\lambda)$, as a result we obtain the integral in the form:

$$\int_N^{\infty} \sqrt{\lambda^2 + \mu^2} \, \frac{\cos^2(\lambda - 3\pi/4)}{\lambda^2} d\lambda \,. \qquad (3.111)$$

This integral diverges and, consequently, the integral (3.109) also diverges, $F = \infty$, then the flow with inlet velocity $\vec{V} = 1 \cdot \vec{e}_z$ is physically unrealistic.

**<u>2) the conducting fluid flows into a half-space with the parabolic velocity</u> $\vec{\mathbf{V}} = (\mathbf{1} - \mathbf{r}^2)\vec{\mathbf{e}}_z$ .**

In this case $\varphi(r) = (1 - r^2)$ and the function in Eq.(3.106) has the form of (3.74), i.e.,

$$\hat{\varphi}(\lambda) = \int_0^1 (1 - r^2) r J_0(\lambda r)\, dr = 2 \frac{J_2(\lambda)}{\lambda^2} \,. \qquad (3.112)$$

Formulae (3.102)-(3.106) with (3.112) describe the solution of the problem in the case of parabolic boundary velocity and the full pressure force at the entrance region is:

$$F = 8\pi \int_0^{\infty} \frac{\sqrt{\lambda^2 + \mu^2}}{\lambda^2} J_1(\lambda) J_2(\lambda) d\lambda \,, \qquad (3.113)$$

Integral (3.113) converges and, consequently, the flow with parabolic boundary velocity is physically realistic.

**The asymptotic solution at** $\text{Ha} \to \infty$ is also obtained for the problem:

$$\lim_{Ha \to \infty} F = 4\pi Ha \int_0^{\infty} \frac{J_1(\lambda) J_2(\lambda)}{\lambda^2} d\lambda = \text{(see [30], formula 6.574(2))} = \frac{1}{2} \pi \cdot Ha \,. \qquad (3.114)$$

$$\lim_{Ha \to \infty} V_r = \frac{r}{Ha} \eta(1 - r), \quad \lim_{Ha \to \infty} V_z = (1 - r^2) \cdot \eta(1 - r),$$

$$\lim_{Ha \to \infty} P = Ha(1 - r^2) \cdot \eta(1 - r) \,. \qquad (3.115)$$

Note, that formulae (3.115) were obtained by using formula 6.575(1) in [30].

Thus, in this part of the thesis, on the considering two hydrodynamic and two similar MHD problems on an inflow of viscous fluid (conducting fluid for MHD problems) into a half-space through a plane split and through a round hole, it is shown that the full pressure force at the entrance region is equal to the infinity if the boundary velocity has a uniform profile and, consequently, such problem is physically unrealistic. But if the boundary velocity is given as a parabolic (continuous) function, the full pressure force at the entrance region has finite value and the problem is physically realistic. Moreover, the new asymptotic solutions for MHD problems are obtained at large Hartmann numbers for the case of parabolic boundary velocity profile.

# 4. ANALYTICAL SOLUTION TO THE MHD PROBLEM ON THE INFLUENCE OF CROSS FLOW ON THE MAIN FLOW IN THE INITIAL PART OF A PLANE CHANNEL

In this chapter the analytical solution is obtained for the problem on a MHD flow of a conducting fluid in a plane channel in the presence of a cross flow. Namely, we solve the problem on a MHD flow in the initial part of a plane channel if the conducting fluid flows into the channel through a split in one channel's lateral side and flows out through the split in its other lateral side in the presence of a main flow in the channel (see also author's papers [9], [55], [56]). The influence of the cross flow on the main flow in the channel is studied. The problem is solved analytically in Oseen and inductionless approximations by using the Fourier transform.

As it was mentioned in the Introduction, on solving the problem on a flow of viscous fluid in the initial part of the channel by using the Oseen approximation it is usually assumed, that the velocity and the pressure of the fluid are given at the entrance of the channel [127]. In our opinion, these boundary conditions overdetermine the problem and it is sufficient to prescribe only the velocity at the entrance of the channel. Thus, in the thesis the problem is solved by using the Oseen approximation with the assumption that only the velocity of the fluid is given at the entrance of the channel, i.e. without the pressure assignment at the entrance. Note, that the results of previous chapter will be taken into account on solving the problem, namely, the velocity at the entrance of the channel is given as a continuous function, i.e. as a parabolic function. The problem is solved for a transverse and longitudinal magnetic field. The dependence of the length of the initial part $L_{init}$ on the Hartman and Reynolds numbers is analyzed. Besides, the field of perturbation velocity is studied for different Hartmann and Reynolds numbers.

## 4.1. Formulation of the problem

The plane channel with a flowing conducting fluid is located in the region $\tilde{D}:\{-h \le \tilde{y} < h; -\infty < \tilde{x}, \tilde{z} < +\infty\}$. There are two splits in the region $\{-\tilde{L} \le \tilde{x} \le \tilde{L}\}$ on the channel's lateral sides $\tilde{y} = \pm h$. The conducting fluid with the constant velocity $\tilde{\vec{V}} = \tilde{V}_{ent}\left(\tilde{L}^2 - \tilde{x}^2\right)\vec{e}_y$ flows into the channel through a split in the channel's wall at $\tilde{y} = -h$ and flows out with the same velocity through the split at $\tilde{y} = h$.

Two cases are considered:

1) the uniform external magnetic field $\vec{B}^e$ is parallel to the y-axis (transverse magnetic field)

2) the uniform external magnetic field $\vec{B}^e$ is parallel to the x-axis (longitudinal magnetic field)

The velocity of the main flow in the channel at the sufficient distance from the entrance region $\vec{\tilde{V}}_\infty(\tilde{y})$ depends on the external magnetic field. In the transverse magnetic field the Hartman flow exists at a sufficient distance from the entrance region of the channel, i.e. $\lim_{\tilde{x}\to\pm\infty}\vec{\tilde{V}} = \vec{\tilde{V}}_\infty(\tilde{y}) = \vec{\tilde{V}}_{Hartm}(\tilde{y})$. The geometry of the flow for the transverse magnetic field is shown in Fig.4.1. For the longitudinal magnetic field, the Poiseuille flow takes place at a sufficient distance from the entrance region of the channel $\lim_{\tilde{x}\to\pm\infty}\vec{\tilde{V}} = \vec{\tilde{V}}_\infty(\tilde{y}) = \vec{\tilde{V}}_p(\tilde{y})$.

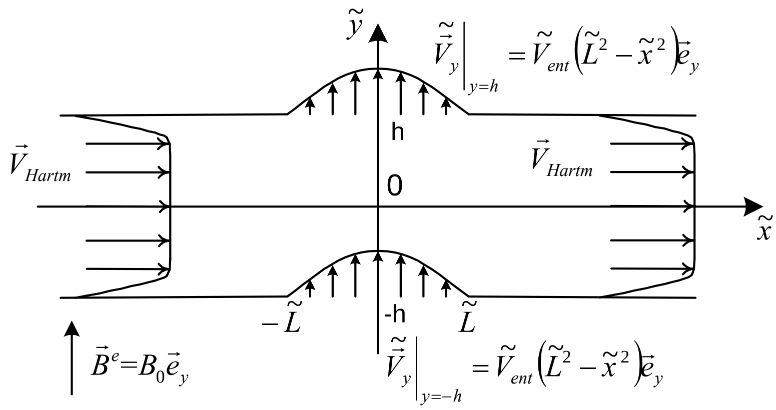



Figure 4.1. Plane channel with a cross flow. The geometry of the flow for transverse magnetic field.

The velocity $\vec{\tilde{V}}(\tilde{x},\tilde{y})$ of the fluid in the channel can be written as:

$$\vec{\tilde{V}} = \left(\tilde{V}_x(\tilde{x},\tilde{y}) + \tilde{V}_\infty\right)\vec{e}_x + \tilde{V}_y(\tilde{x},\tilde{y})\vec{e}_y\,. \tag{4.1}$$

To solve the problem, we define a new function, so-called the perturbation velocity

$$\vec{\tilde{V}}_{new} = \vec{\tilde{V}} - \vec{\tilde{V}}_\infty = \tilde{V}_x(\tilde{x},\tilde{y})\vec{e}_x + \tilde{V}_y(\tilde{x},\tilde{y})\vec{e}_y\,. \tag{4.2}$$

The MHD equations in inductionless and Oseen approximation for the new velocity have the form:

$$\left(\vec{V}_0 \cdot \nabla\right) \tilde{\vec{V}}_{new} = -\frac{1}{\rho}\nabla P + \gamma \Delta \tilde{\vec{V}}_{new} + \frac{\sigma}{\rho}\left[\vec{E} + \tilde{\vec{V}}_{new} \times \vec{B}^e\right] \times \vec{B}^e , \tag{4.3}$$

$$\nabla \tilde{\vec{V}}_{new} = 0 , \tag{4.4}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ , $\nabla = \frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y$ ;

$\sigma, \rho, \nu$ are the conductivity, density and viscosity of the fluid, respectively;

$\vec{V}_0 = V_0 \cdot \vec{e}_x$ , $V_0$ is the average velocity of the fluid at the cross section of the channel.

We introduce dimensionless variables using the half-width of channel, *h*, as the length scale, the average velocity of the fluid at the cross section of the channel, $V_0$, as the velocity scale and $B_0$, $V_0B_0$ , $\rho\nu V_0/h$ as scales of magnetic field, electrical field and pressure, respectively.

Then the dimensionless MHD equations (4.3)-(4.4) in inductionless approximation are:

$$\mathrm{Re}\left(\vec{e}_x \nabla\right) \vec{V}_{new} = -\nabla P + \Delta \vec{V}_{new} + Ha^2 (\vec{V}_{new} \times \vec{e}_B) \times \vec{e}_B , \tag{4.5}$$

$$\nabla V_{new} = 0 , \tag{4.6}$$

where $\vec{e}_B$ is the unit vector of the external magnetic field,

$\mathrm{Re} = V_0 L / \nu$ is the Reynolds number,

$Ha = B_0 h \sqrt{\sigma / (\rho \nu)}$ is the Hartmann number.

The boundary conditions are

$$y = \pm 1 : V_x = 0, V_y = \begin{cases} 0, & x \notin (-L, L) \\ V_{ent}\left(L^2 - x^2\right), & x \in (-L, L) \end{cases} \tag{4.7}$$

$$x \to \pm\infty : \quad V_x \to 0, \quad V_y \to 0 , \tag{4.8}$$

where $L = \tilde{L} / h$ , $V_{ent} = \tilde{V}_{ent} / V_0$ , $2\tilde{L}$ is the width of the split in the channel's wall.

## 4.2. The solution of the problem for the transverse magnetic field

In the case of transverse external magnetic field $\vec{B}^e = B_0 \vec{e}_y$ there exists the Hartman flow at the sufficient distance from the entrance region of the channel, i.e. $\vec{V}_\infty(y) = \vec{V}_{Hartm}(y)$. In the inductionless approximation the velocity of the Hatrmann flow has the form (see [111]):

$$V_{Hartm} = \frac{Ha}{Ha - \tanh(Ha)}\left(1 - \frac{\cosh(Ha \cdot y)}{\cosh(Ha)}\right) \tag{4.9}$$

Projecting equations (4.5), (4.6) onto the x- and y- axis and taking into account that $\vec{e}_B = \vec{e}_y$, the problem takes the form:

$$\mathrm{Re}\frac{\partial V_x}{\partial x} = -\frac{\partial P}{\partial x} + \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} - Ha^2 V_x, \tag{4.10}$$

$$\mathrm{Re}\frac{\partial V_y}{\partial x} = -\frac{\partial P}{\partial y} + \frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2}, \tag{4.11}$$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0 \tag{4.12}$$

We use the complex Fourier transform with respect to x for the solution of this system (see formula (1.18)). As a result, the following system of ordinary differential equations is obtained for the Fourier transforms $\hat{V}_x(\lambda, y)$, $\hat{V}_y(\lambda, y)$, $\hat{P}(\lambda, y)$:

$$-i\,\mathrm{Re}\lambda \hat{V}_x = i\lambda \hat{P} + \lambda^2 \hat{V}_x - \frac{d^2 \hat{V}_x}{dy^2} - Ha^2 V_x, \tag{4.13}$$

$$-i\,\mathrm{Re}\lambda \hat{V}_y = \frac{d\hat{P}}{dy} + \lambda^2 \hat{V}_y - \frac{d^2 \hat{V}_y}{dy^2}, \tag{4.14}$$

$$i\lambda \hat{V}_x + \frac{d\hat{V}_y}{dy} = 0. \tag{4.15}$$

The boundary conditions (4.7)-(4.8) become

$$y = \pm 1: \quad \hat{V}_x = 0, \quad \hat{V}_y = \frac{4V_{ent}}{\lambda^2\sqrt{\pi}}\left(\frac{\sin \lambda L}{\lambda} - L\cos \lambda L\right) \equiv \hat{V}_{ent}(\lambda), \tag{4.16}$$

Eliminating $\hat{V}_x$ and $\hat{P}$ from Eqs.(4.13)-(4.15), one gets the 4[th] order differential equation for $\hat{V}_y$:

$$\hat{V}_y^{(4)} - (i\lambda \operatorname{Re} + 2\lambda^2 + Ha^2)\hat{V}_y'' + \lambda^2 (i\lambda \operatorname{Re} + \lambda^2)\hat{V}_y = 0 . \tag{4.17}$$

In order to solve this equation, the following additional boundary condition is used, which can be obtained from Eq.(4.15) taking into account boundary condition (4.16) for $\hat{V}_x$:

$$y = \pm 1: \quad \frac{d\hat{V}_y}{dy} = 0 . \tag{4.18}$$

The characteristic equation for differential equation (4.17) can be written in the form:

$$\left(k^2 - \lambda^2\right)^2 - i\lambda \operatorname{Re}\left(k^2 - \lambda^2\right) + Ha^2 k^2 = 0 . \tag{4.19}$$

The roots of this equation are

$$k_1 = \sqrt{\lambda^2 + D_1}\ , \quad k_2 = \sqrt{\lambda^2 + D_2}\ , \quad k_3 = -k_1 , \quad k_4 = -k_2 , \tag{4.20}$$

where

$$D_1 = \frac{1}{2}\left( i \operatorname{Re}\cdot \lambda + Ha^2 + \sqrt{\left(i \operatorname{Re}\cdot \lambda + Ha^2\right)^2 + 4Ha^2} \right),$$

$$D_2 = \frac{1}{2}\left( i \operatorname{Re}\cdot \lambda + Ha^2 - \sqrt{\left(i \operatorname{Re}\cdot \lambda + Ha^2\right)^2 + 4Ha^2} \right).$$

Due to the velocity component $V_y(x, y)$ is an odd function with respect to y, the solution of differential equation (4.17) with boundary conditions (4.16), (4.18) takes the form:

$$\hat{V}_y = \frac{k_2 \sinh k_2 \cdot \cosh k_1 y - k_1 \sinh k_1 \cdot \cosh k_2 y}{\Delta} \cdot \hat{V}_{ent} , \tag{4.21}$$

where $$\Delta = k_2 \sinh k_2 \cdot \cosh k_1 - k_1 \sinh k_1 \cdot \cosh k_2 . \tag{4.22}$$

Functions $\hat{V}_x(\lambda, y)$ and $\hat{P}(\lambda, y)$ are determined by substituting Eq.(4.21) into Eq.(4.14) and Eq.(4.15). Applying to them the inverse Fourier transform, we obtain the solution of problem (4.10)-(4.12) with boundary conditions (4.7)-(4.8) in the form of convergent improper integrals:

$$V_x(x,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{i\cdot k_1k_2}{\Delta\cdot\lambda}(k_2\sinh k_2\cdot\sinh k_1y-k_1\sinh k_1\cdot\sinh k_2y)\hat{V}_{ent}\cdot e^{i\lambda x}d\lambda\,, \quad (4.23)$$

$$V_y(x,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{k_2\sinh k_2\cdot\cosh k_1y-k_1\sinh k_1\cdot\cosh k_2y}{\Delta}\hat{V}_{ent}\cdot e^{i\lambda x}d\lambda\,, \quad (4.24)$$

$$\frac{\partial P}{\partial x}=\frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{ik_1k_2}{\Delta\cdot\lambda}\left(D_2\sinh k_2\cdot\sinh k_1y-D_1\sinh k_1\cdot\sinh k_2y\right)\hat{V}_{ent}\cdot e^{i\lambda x}d\lambda\,, \quad (4.25)$$

$$\frac{\partial P}{\partial y}=\frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{k_1k_2}{\Delta\cdot\lambda}(D_2k_1\sinh k_2\cdot\cosh k_1y-D_1k_2\sinh k_1\cdot\cosh k_2y)\hat{V}_{ent}\cdot e^{i\lambda x}d\lambda\,, \quad (4.26)$$

where $$\hat{V}_{ent}(\lambda)=\frac{4V_{ent}}{\lambda^2\sqrt{\pi}}\left(\frac{\sin\lambda L}{\lambda}-L\cos\lambda L\right). \quad (4.27)$$

***Numerical results for the transverse magnetic field*:**

On the basis of obtained solution, the length of the initial part $L_{init}$, where the x component of the velocity $\vec{V}(x,y)$ of fluid in channel differs from the velocity of the Hartman flow $V_{Hartm}$ by less than 1%, is analyzed numerically. $L_{init}$ is usually calculated in the center of the channel, i.e. at y=0. In our case, due to the x component of the velocity is equal to $V_{Hartm}$ at y=0 for all x, $L_{init}$ is calculated at y=-0.5. The dependence of the length of the initial part $L_{init}$ on the Reynolds number for different Hartmann numbers is shown in Fig.4.2. $L_{init}$ was calculated provided that L=1 and $V_{ent}=1$.

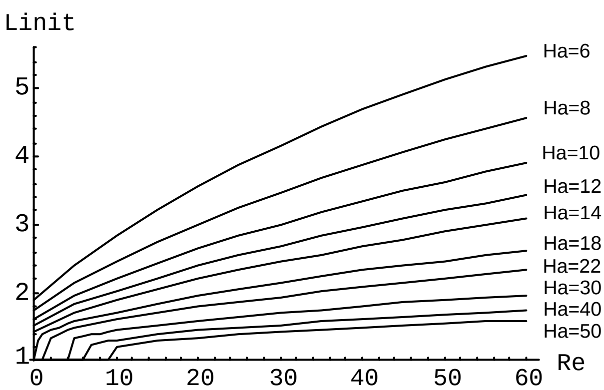



Figure 4.2. The dependence of length of the initial part $L_{init}$ on Re at y=-0.5 for $\vec{B}^e=B_0\vec{e}_y$.

As one can see from Fig.4.2, at fixed Hartmann number the increase in Reynolds number (i.e. in effect of inertial force) leads to the increase in the length of the initial part $L_{init}$. Besides, $L_{init}$ decreases as the Hartmann number grows. It means that with the increase in the intensity of magnetic field, the flow of conducting fluid in channel faster approaches Hartmann flow.

Fig. 4.3 presents the results of calculation of the x-component $V_x$ of the perturbation velocity $\vec{V}_{new}$ by means of formula (4.23) at Ha=8, Ha=20 and Re=5 and Re=35.

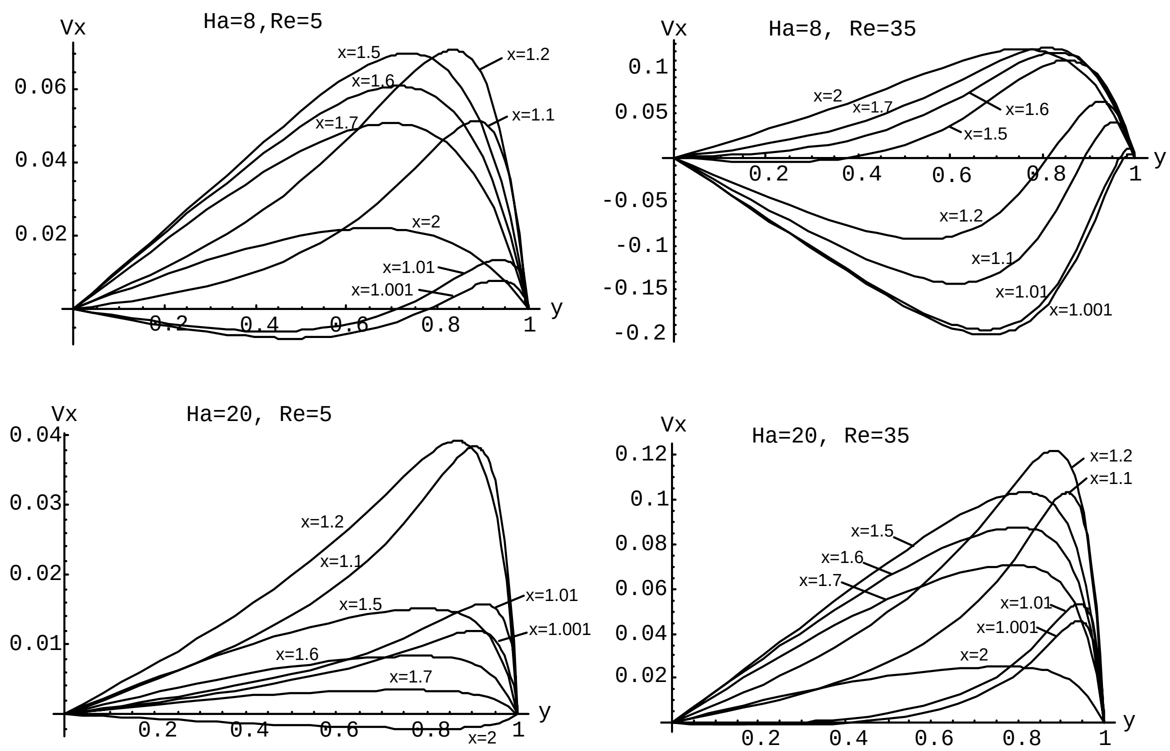



Figure 4.3. Profiles for the x component $V_x(x, y)$ of the perturbation velocity $\vec{V}_{new}$ for $\vec{B}^e = B_0 \vec{e}_y$.

One can see that in the channel's entrance region the component $V_x(x, y)$ of velocity $\vec{V}_{new}$ has M-shaped profile for small x ($1 < x < 1.5$). Besides, for some values of x the component $V_x(x, y)$ is positive. But due to the fluid flowing out through the split on $y = 1$, the x-component of the velocity $\vec{V}_{new}$, must be negative for $0 < y < 1$ at *Ha=0*. It means that in the region $0 < y < 1$ there exists the opposite flow in transverse magnetic field. It happens because the cross-flow generates vortexes in the channel ( Fig.4.5 ).

The vector field of perturbation velocity $\vec{V}_{new}$ are shown in Fig.4.4 and Fig.4.5.

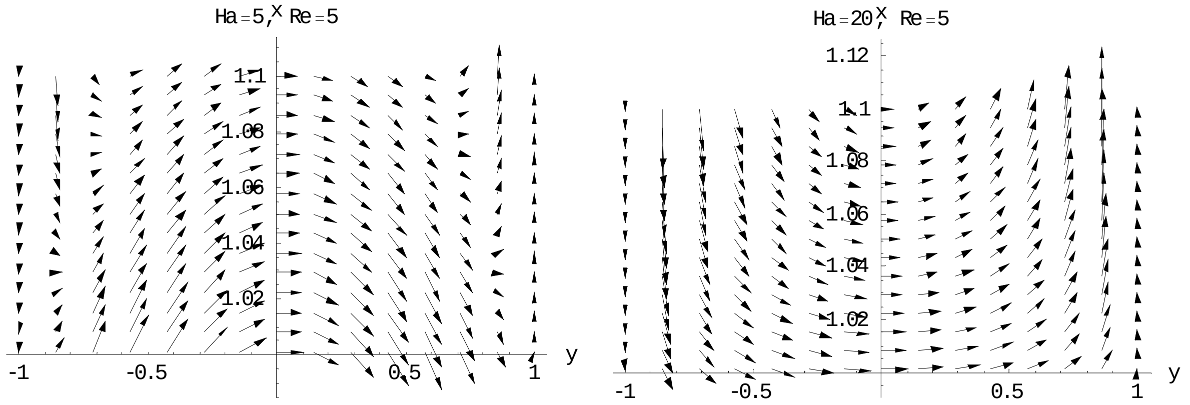



Figure 4.4. The vector field of perturbation velocity $\vec{V}_{new}$ for $0 < x \leq 1.1$ and $\vec{B}^e = B_0 \vec{e}_y$.

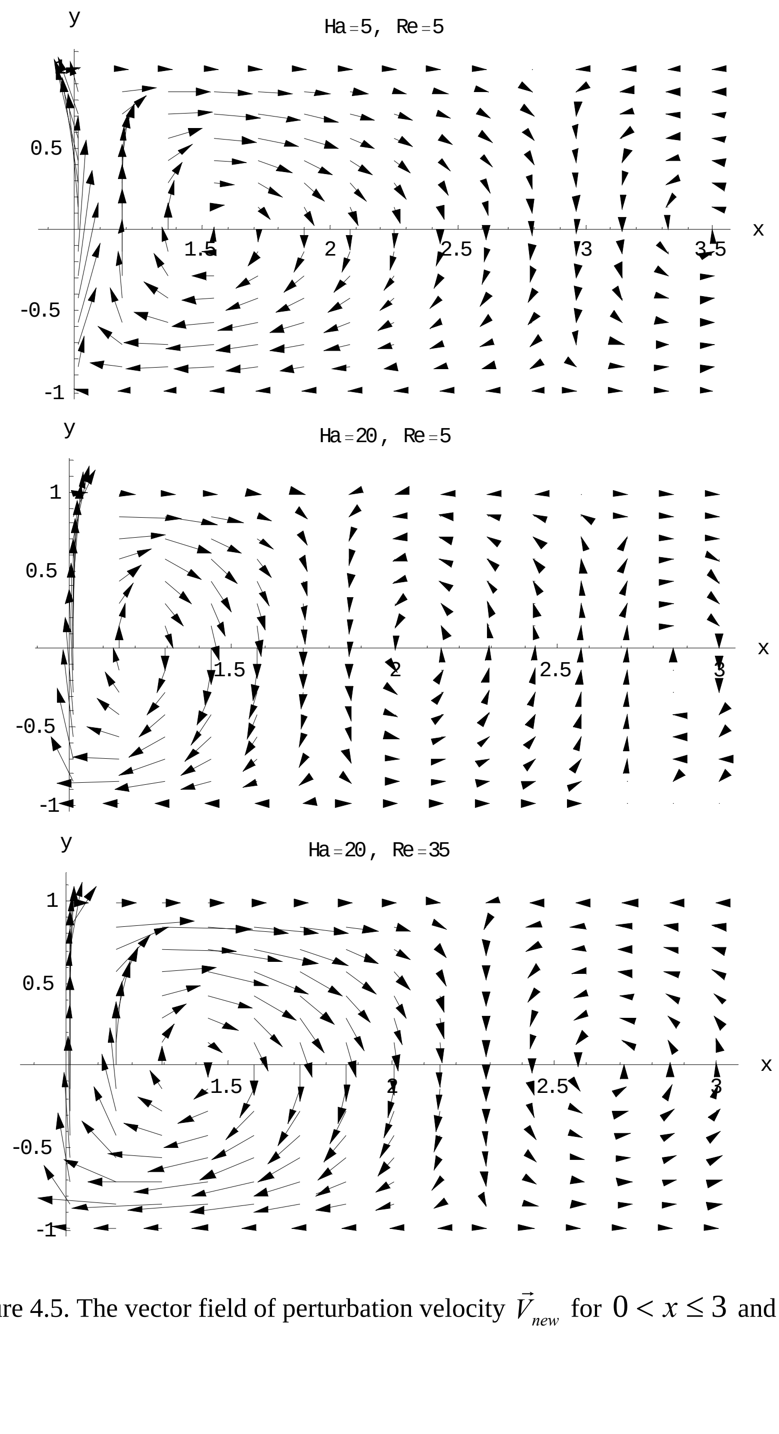



Figure 4.5. The vector field of perturbation velocity $\vec{V}_{new}$ for $0 < x \leq 3$ and $\vec{B}^e = B_0 \vec{e}_y$.

### 4.3. The solution of the problem for the longitudinal magnetic field

In the case of the longitudinal magnetic field $\vec{B}^e = B_0\vec{e}_x$ the Poiseuille flow takes the place at a sufficient distance from the entrance region of the channel, i.e.

$$\tilde{\vec{V}}_\infty(\tilde{y}) = \tilde{\vec{V}}_p(\tilde{y}) = \frac{3}{4}V_0\left(1 - \frac{\tilde{y}^2}{h^2}\right)\vec{e}_x, \tag{4.28}$$

where $V_0$ is a magnitude of a average velocity of the fluid at the cross section of the channel.

Projecting Eqs. (4.5)-(4.6) onto the x- and y- axis and taking into account that $\vec{e}_B = \vec{e}_x$, the problem has the form:

$$\mathrm{Re}\frac{\partial V_x}{\partial x} = -\frac{\partial P}{\partial x} + \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2}, \tag{4.29}$$

$$\mathrm{Re}\frac{\partial V_y}{\partial x} = -\frac{\partial P}{\partial y} + \frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} - Ha^2 V_y, \tag{4.30}$$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0. \tag{4.31}$$

The boundary conditions for the problem are described by (4.7) and (4.8).

Applying the complex Fourier transform (1.18) with respect to x to the problem (4.29)-(4.31), one gets the system of ordinary differentials equations for the Fourier transforms $\hat{V}_x(\lambda, y), \hat{V}_y(\lambda, y), \hat{P}(\lambda, y)$ in the following form:

$$-i\,\mathrm{Re}\,\lambda\hat{V}_x = i\lambda\hat{P} + \lambda^2\hat{V}_x - \frac{d^2\hat{V}_x}{dy^2}, \tag{4.32}$$

$$-i\,\mathrm{Re}\,\lambda\hat{V}_y = \frac{d\hat{P}}{dy} + \lambda^2\hat{V}_y - \frac{d^2\hat{V}_y}{dy^2} - Ha^2\hat{V}_y, \tag{4.33}$$

$$i\lambda\hat{V}_x + \frac{d\hat{V}_y}{dy} = 0. \tag{4.34}$$

Boundary conditions for this problem have the form of (4.16) and (4.18), i.e.

$$y = \pm 1: \quad \hat{V}_x = 0, \quad \hat{V}_y = \frac{4V_{ent}}{\lambda^2\sqrt{\pi}}\left(\frac{\sin\lambda L}{\lambda} - L\cos\lambda L\right) \equiv \hat{V}_{ent}(\lambda), \tag{4.35}$$

$$y = \pm 1: \quad d\hat{V}_y / dy = 0. \tag{4.36}$$

Eliminating $\hat{V}_x$ and $\hat{P}$ from the system (4.32)-(4.34), we obtain the 4th order differential equation for $\hat{V}_y$:

$$\hat{V}_y^{(4)} - (i\lambda \,\mathrm{Re} + 2\lambda^2)\hat{V}_y'' + \lambda^2 (i\lambda \,\mathrm{Re} + \lambda^2 + Ha^2)\hat{V}_y = 0 . \qquad (4.37)$$

The characteristic equation for this differential equation can be transform to the form:

$$\left(k^2 - \lambda^2\right)^2 - i\lambda \,\mathrm{Re}\left(k^2 - \lambda^2\right) + Ha^2\lambda^2 = 0 . \qquad (4.38)$$

The roots of (4.38) are:

$$k_1 = \sqrt{\lambda^2 + i\lambda D_1^*}\ ,\ k_2 = \sqrt{\lambda^2 + i\lambda D_2^*}\ ,\ k_3 = -k_1,\ k_3 = -k_2 , \qquad (4.39)$$

where $\quad D_1^* = \frac{1}{2}\left(\mathrm{Re} + \sqrt{\mathrm{Re}^2 + 4Ha^2}\right),\ D_2^* = \frac{1}{2}\left(\mathrm{Re} - \sqrt{\mathrm{Re}^2 + 4Ha^2}\right).$

As a result, the solution of equation (4.37) with the boundary conditions (4.35), (4.36) has the form:

$$\hat{V}_y(\lambda, y) = \frac{1}{\Delta}\left(k_2 \sinh k_2 \cdot \cosh k_1 y - k_1 \sinh k_1 \cdot \cosh k_2 y\right)\hat{V}_{ent}\ , \qquad (4.40)$$

where $\quad \Delta = k_2 \sinh k_2 \cdot \cosh k_1 - k_1 \sinh k_1 \cdot \cosh k_2 . \qquad (4.41)$

We determine the functions $\hat{V}_x(\lambda, y)$ and $\hat{P}(\lambda, y)$, substituting (4.40) into (4.34) and (4.32). Using after that the inverse Fourier transform, we obtain the solution of problem for the case of longitudinal magnetic field in the form of improper convergent integrals:

$$V_x(x,y) = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{k_1 k_2}{\Delta} (\sinh k_2 \cdot \sinh k_1 y - \sinh k_1 \cdot \sinh k_2 y) \cdot \frac{\hat{V}_{ent}(\lambda)}{\lambda} \cdot e^{i\lambda x} d\lambda\ , \qquad (4.42)$$

$$V_y(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\Delta} (k_2 \sinh k_2 \cdot \cosh k_1 y - k_1 \sinh k_1 \cdot \cosh k_2 y) \cdot \hat{V}_{ent}(\lambda) \cdot e^{i\lambda x} d\lambda\ , \qquad (4.43)$$

$$\frac{\partial P}{\partial x} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{k_1 k_2}{\Delta} (D_2^* \sinh k_2 \cdot \sinh k_1 y - D_1^* \sinh k_1 \cdot \sinh k_2 y) \cdot \hat{V}_{ent}(\lambda) \cdot e^{i\lambda x} d\lambda\ , \qquad (4.44)$$

$$\frac{\partial P}{\partial y} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{i k_1 k_2}{\Delta} (D_2^* k_1 \sinh k_2 \cdot \cosh k_1 y - D_1^* k_2 \sinh k_1 \cdot \cosh k_2 y) \cdot \frac{\hat{V}_{ent}(\lambda)}{\lambda} \cdot e^{i\lambda x} d\lambda\ , \qquad (4.45)$$

where

$$k_1 = \sqrt{\lambda^2 + i\lambda D_1^{\ *}}\ , \qquad\qquad k_2 = \sqrt{\lambda^2 + i\lambda D_2^{\ *}}\ ,$$

$$D_1^{\ *} = \frac{1}{2}\left(\mathrm{Re} + \sqrt{\mathrm{Re}^2 + 4Ha^2}\right), \quad D_2^{\ *} = \frac{1}{2}\left(\mathrm{Re} - \sqrt{\mathrm{Re}^2 + 4Ha^2}\right),$$

$$\Delta = k_2 \sinh k_2 \cdot \cosh k_1 - k_1 \sinh k_1 \cdot \cosh k_2 .$$

***Numerical results for the longitudinal magnetic field***

On the basis of obtained results, we study the dependence of the length of the initial part $L_{init}$ on the Reynolds and Hartmann numbers. In the present problem the initial part of the channel is the part, where the x-component of the velocity $\vec{V}(x,y)$ differs from the Poiseuille flow $V_p$ by less than 1%.

Fig.4.6 presents the results of calculation of the length of the initial part $L_{init}$ at y=0.5 and at Ha=5; 6; 7; 8; 9; 10; 15; 20 on the condition that L=1 and $V_{ent} = 1$. $L_{init}$ is calculated with precision $\pm 0.001$.

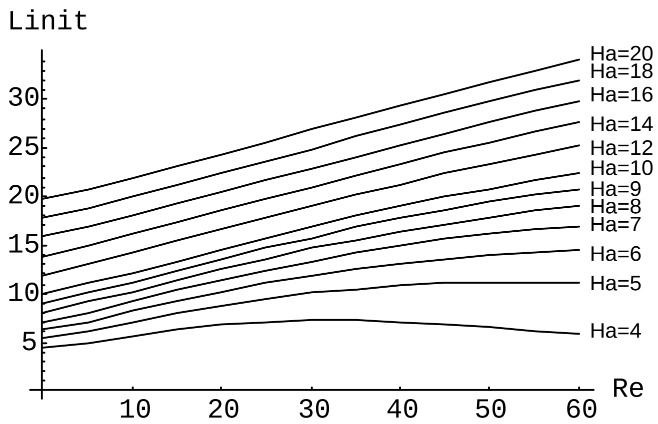



Figure 4.6. The dependence of the length of the initial part $L_{init}$ on Re at y=0.5 for $\vec{B}^e = B_0 \vec{e}_x$ .

One can see from Fig.4.6., that $L_{init}$ increases at growing Hartmann number. Besides, the increase in the Reynolds number leads to the increase in the length of the initial part at fixed Hartmann numbers. The dependence of the value $L_{init}$ on Reynolds number becomes linear at large Hartmann numbers.

Fig.4.7 presents the profiles of perturbation velocity $\vec{V}_{new} = \vec{V} - \vec{V}_p$ and Fig.4.8 shows the vector field of perturbation velocity for different Ha and Re numbers.

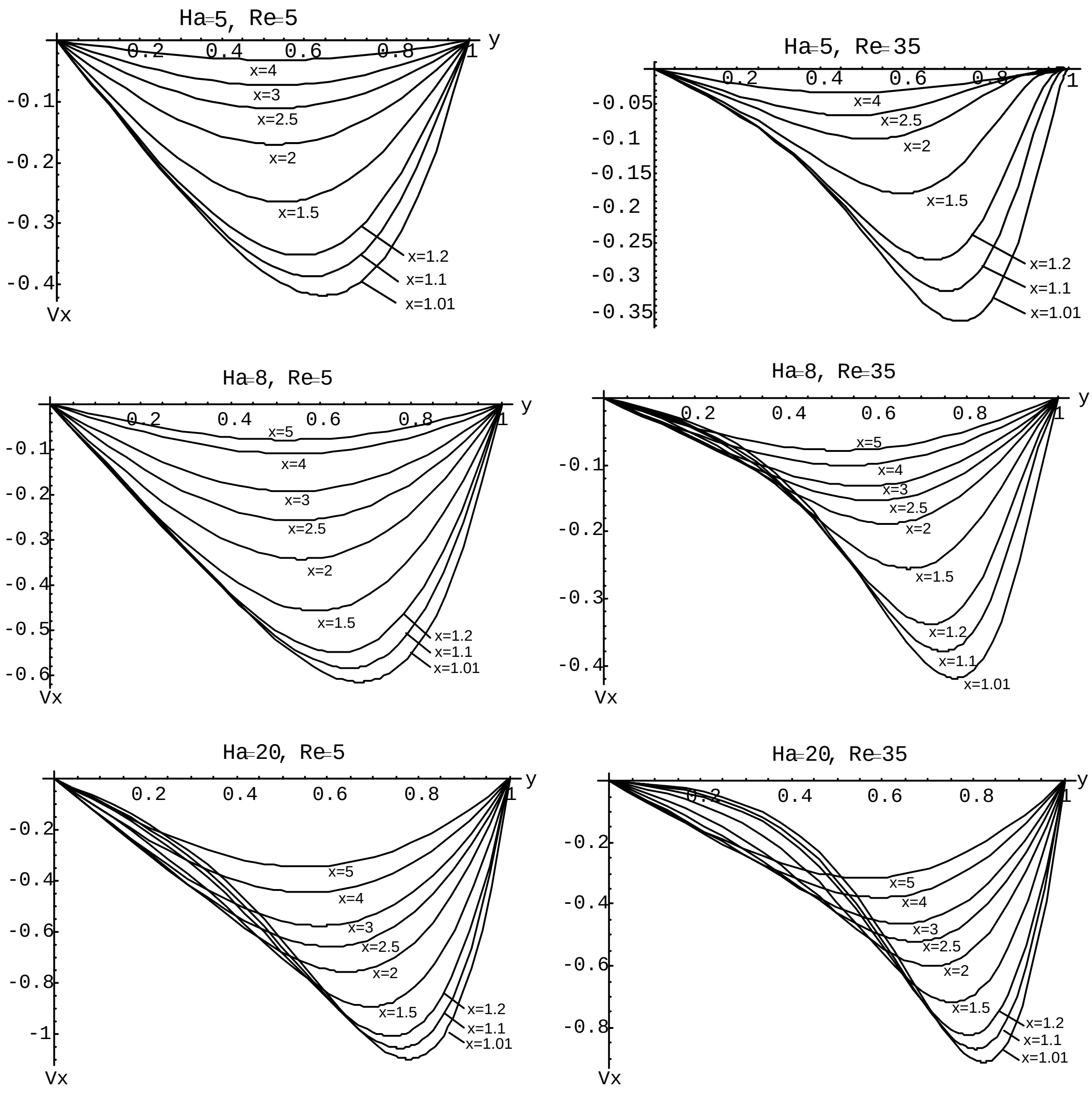



Figure 4.7. Profiles for x component $V_x(x, y)$ of the perturbation velocity $\vec{V}_{new}$ for $\vec{B}^e = B_0\vec{e}_x$.

As it can be seen from the Fig.4.7, the opposite flow doesn't appear in the channel in the case of the longitudinal field, because the cross flow doesn't generate vortexes in the channel (see Fig.4.8). In the channel's entrance region the component $V_x(x, y)$ of velocity $\vec{V}_{new}$ has more pronounced M-shaped profiles for small x than in the case of transverse magnetic field.

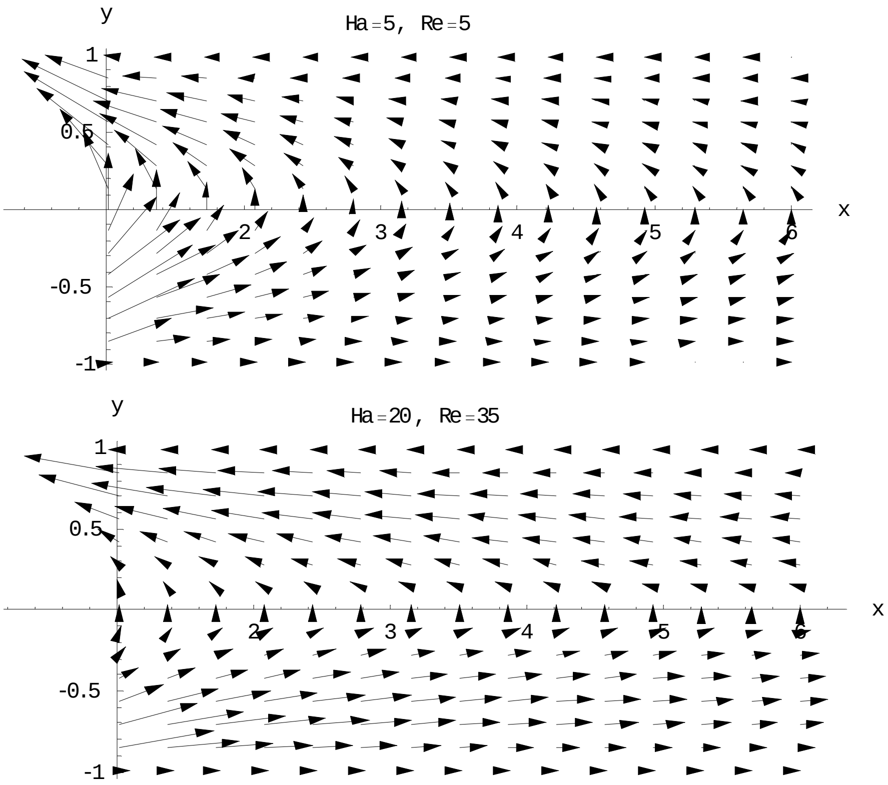



Figure 4.8. The vector field of perturbation velocity $\vec{V}_{new}$ for $\vec{B}^e = B_0\vec{e}_x$ ..

The value $L_{init}$ is the function of 4 parameters: Reynolds number Re, Hartmann number Ha, velocity of the fluid at the entrance region of the channel $V_{ent}$ and the half-width L of the splits in the channel's walls. The count of the parameters can be reduced to three, if we suppose, that the half-width L of the splits tends to zero ($L \to 0$) and $V_{ent} \to \infty$, so that product $V_{ent}L = Q_{ent} = const$. It means that the fluid inflows into the channel through the infinitesimal narrow split $x = 0$ in the plate $y = -1$ with given consumption $Q_{ent}$ and flows out from the channel with the same consumption through the narrow split in the plate $y = 1$. Besides, $\lim_{L\to 0}\hat{V}_{ent}(\lambda) = 4Q_{ent}/(\lambda^2\sqrt{\pi})$ and $L_{init} = L_{init}(\mathrm{Re}, Ha, Q_{ent})$ in this case.

# CONCLUSIONS

The present thesis is devoted to the theoretical study of new magnetohydrodynamical (MHD) problems on a flow of a conducting fluid in the initial part of a channel under the condition that a conducting fluid flows into a channel through a split of finite width or through a hole of finite radius on the channel's lateral side in the presence of external magnetic field. Problems were solved for plane and circular channels of infinite length. On the basis of the obtained solutions the vector field of the velocity was analysed and the dependence of the length of the initial part of the channel on Hartmann number was studied. Asymptotic solutions of the problems at Hartmann large numbers were also obtained.

In this thesis one more problem was studied, namely, the dependence of the full pressure force at the entrance region on the profile of the boundary velocity at entrance was studied analytically by means of analytical solution of two hydrodynamic problems and two similar MHD problems on an inflow of the viscous fluid (or conducting fluid for MHD problems) into a half-space through a split of finite width or through a hole of finite radius.

For the solution of the problems integral transforms were used, namely, Fourier transform and Hankel transform. Numerical calculations were performed using the package "Mathematica 5.0".

The problems considered in this work are closely related to applications in blanket of a fusion reactor.

Main results obtained in this PhD thesis are:

1) **In the 1st part** of PhD thesis the new MHD problem on an inflow of a conducting fluid into a plane channel through a split of finite width on the channel's lateral side was solved analytically in Stokes and inductionless approximation by using the Fourier transform. The problem was solved by dividing it into two subproblems: even and odd problems with respect to the axis perpendicular to the channel's walls. The cases of longitudinal and transverse magnetic fields were studied in detail. The solutions of the problems were obtained in the form of convergent improper integrals. On the basis of obtained analytical solutions the velocity fields were studied numerically. It was shown that the effect of the longitudinal magnetic field is more pronounced than the effect of the transverse magnetic field.

*Results* ***for longitudinal magnetic field****:*

- For the odd problem with respect to y, numerical calculations show that the velocity component $V_x$ has M-shaped profiles in the entrance region of the channel at large Hartmann numbers. The M-shaped profiles become more pronounced as the Hartmann number increases. The physical explanation of these velocity profiles was given in the thesis.
- For the even problem with respect to y these M-shaped profiles aren't so much pronounced.
- For the general problem, which solution is equal to the sum of even and odd problems, the numerical results show, that in strong longitudinal magnetic field the flows mostly occur along the wall with the split. As the Hartmann number increases, the layer of the flow is getting narrower and the velocity increases in this layer. The Poiseuille flow takes place at a distance from the entrance split. In addition, when the Hartmann number grows (i.e. the intensity of the magnetic field increases) the length of the initial part of the channel $L_{init}$ increases and the flow in the channel slowly approaches Poiseuille flow as the Hartmann number grows.

*Results for* ***transverse magnetic field:***

- For the transverse magnetic field, the flow in the channel differs from the Hartmann flow only near the entrance region and Hartman flow is approached very quickly also for small Hartmann numbers.
- In the even problem with respect to y, the profiles of the velocity component $V_x$ also differ from zero only near the entrance region and there exists an opposite flow in the channel in the transverse magnetic field. It happens due to a vortex generated in the channel.
- In addition, asymptotic solutions of the problems at large Hartmann numbers are obtained for the transverse magnetic field and important practical result has been obtained for this problem. It says that at large Hartmann numbers, a pressure gradient which is proportional to the square of the Hartmann number is needed for turning the flow through an angle $90^0$, while for pumping the fluid we need a pressure gradient which is proportional to only the first power of the Hartmann number.

2) **In the 2$^{nd}$ part** of the thesis the analytical solutions of new MHD problems on an inflow of a conducting fluid into a channel through the channel's lateral side in the presence of rotational symmetry were obtained and the following two problems were solved:

a) the MHD problem on an inflow of a conducting fluid into a plane channel through a round hole of finite radius on channel's lateral side;

b) the MHD problem on an inflow of a conducting fluid into a round channel through a split of finite width on the channel's lateral side.

Both problems were solved in Stokes and inductionless approximation by using Hankel transform.

a) **The first problem** on an inflow of a conducting jet into a **plane channel** through a **round hole** of finite radius on the channel's lateral side was solved for the case of transverse magnetic field. The analytical solution of the problem was obtained by dividing the problem into the odd and even problems with respect to axis perpendicular to the walls of channel. Velocity profiles of the velocity radial component $V_r$ are obtained numerically at Ha=10 and Ha=50.

- The results of calculation of velocity component $V_r$ shows that the velocity profiles differ from the Hartmann flow only near the entrance region.
- In this problem the magnitude of the velocity is inversely proportional to the distance from the hole.
- Similarly to the problem solved in the 1$^{st}$ part, for the even problem, there exists an opposite flow in the transverse magnetic field. It occurs due to a vortex generated in the channel. At large Hartmann numbers ($Ha \geq 50$) the opposite flow is absent.

b) **The second problem** solved in this part of the theses, i.e. the MHD problem on an inflow of a conducting fluid into a **cylindrical channel** through a **split** of finite width on the channel's lateral side, was solved only for longitudinal magnetic field. The analytical solution of the problem was also obtained in the form of convergent improper integrals. The velocity field was studied numerically on the basis of the obtained analytical solutions. As a result, M-shaped velocity profiles were obtained in the entrance region of the channel at large Hartman numbers for the velocity component $V_z$. The M-shaped profiles become more pronounced as the Hartmann number increases. The physical explanation of these velocity profiles is given in the thesis. It was also shown that when the Hartmann number grows, the flow in the channel slowly approaches Poiseuille flow. Asymptotic solution of the problem at large Hartmann numbers was also obtained.

3) **In the 3rd part** of the PhD thesis the dependence of the full pressure force in the entrance region on the profile of the inlet velocity in this region was studied analytically by means of analytical solution of two hydrodynamic problems and two MHD problems on an inflow of a viscous fluid (or conducting fluid for MHD problems) into a half-space through a plane split of finite width or through a round hole of finite radius. Both the case of a uniform inlet velocity profile and the case of a parabolic velocity profile were considered in the thesis.

It was shown in this part of the thesis that the full pressure force at the entrance region is equal to infinity if the profile of the boundary velocity is uniform and, consequently, such problem is physically unrealistic. But if the profile of the boundary velocity is a parabolic function, the full pressure force at the entrance region has a finite value and the problem is physically realistic. The problems have been solved in the Stokes approximation (and inductionless approximation for MHD problems). In the case of the plane split the solution of the hydrodynamic problem was obtained in terms of elementary functions, but in the case of round hole the solution was obtained in the form of convergent integrals containing Bessel functions. For the MHD problems the solutions were obtained in the form of improver convergent integrals. The new asymptotic solutions for MHD problems were obtained at large Hartmann number ( $Ha \to \infty$ ) for the case of parabolic profile of the inlet velocity.

4) **In 4th chapter** of the PhD thesis the analytical solution was obtained for the new MHD problem on the influence of a cross flow on the main flow in an infinitely long plane channel in a presence of a strong external magnetic field. Both the longitudinal and transverse magnetic fields were considered. The problem was solved in Oseen and inductionless approximation by using the Fourier transform, provided that only the velocity of fluid is prescribed at the entrance of the channel. The solution of the problem was obtained in the form of convergent improper integrals. The field of perturbation velocity was analysed numerically for different Hartmann and Reynolds numbers. Numerical calculations give the M-shaped velocity profile at the initial part of channel at large Hartmann numbers. The vector fields of perturbation velocity were also presented for different Hartmann and Reynolds numbers. Besides, the dependence of the length of the initial part of the channel $L_{init}$ on the Hartman and Reynolds numbers was studied.

*Results for* ***transverse magnetic field:***

- It is shown that the increase in the Reynolds number leads to the increase in the length of the initial part of the channel $L_{init}$;
- With the increase in the intensity of magnetic field, the flow of conducting fluid in channel faster approaches Hartmann flow;
- The cross-flow in the transverse magnetic field generates vortices in the channel.

*Results* ***for longitudinal magnetic field****:*

- In the longitudinal magnetic field the increase in the Reynolds number also leads to the increase in the length of the initial part $L_{init}$;
- The initial part of the channel $L_{init}$ increase, when Hartmann number grows (the flow in the channel slower approaches Poiseuille flow);
- The dependence of the value $L_{init}$ on Reynolds number becomes linear in a strong magnetic field.

**Results of this work were also published in** [6]- [9], [54]-[59].

In addition, the results were reported at the following conferences:

1) 3$^{th}$ Latvian Mathematical Conference, April 14-15, 2000, Jelgava, Latvia;
2) 5$^{th}$ International Conference Mathematical Modeling and Applications, June 8-9, 2000, Jurmala, Latvia;
3) 3$^{rd}$ European Congress of Mathematics, July 10-15, 2000, Barcelona, Spain;
3) 6$^{th}$ International Conference Mathematical Modeling and Analysis, May 31-June 2, 2001, Vilnius, Lithuania;
4) 42$^{nd}$ International scientific conference of Riga Technical University, October 11-13, 2001, Riga, Latvia;
4) 5$^{th}$ International Conference on Fundamental and applied MHD, September 2002, France;
6) 4$^{th}$ European Congress of Mathematics, June 27- July 2, 2004, Stockholm, Sweden;
7) 45$^{th}$ International Scientific Conference of Riga Technical University, October 14 - 16, 2004, Riga, Latvia;
10) 6$^{th}$ Latvian Mathematical Conference, April 7 - 8, 2006, Liepaja, Latvia;
11) Join CTS-HYCON Workshop on Nonlinear and Hybrid Control, 10-12 July 2006, Paris, France;
12) 47$^{th}$ International scientific conference of Riga Technical University, October 12-14, 2006, Riga, Latvia;

12) 7th Pamir International Conference on Fundamental and Applied MHD, September 8-12, 2008, Presqu'ile de Giens (France);

13) 8th International PAMIR Conference on Fundamental and Applied MHD", September 5-9, 2011, Borgo, Corsica, France;

14) 52th International scientific conference of Riga Technical University, Subsection «Boundary Field Problems and Computer Simulation». October 12-14, 2011, Riga, Latvia;

15) 14th WSEAS International Conference on Mathematical and Computational Methods in Science and Engineering (MACMESE'12), September 7-9, 2012, Sliema, Malta.

16) 53th International scientific conference of Riga Technical University, Subsection «Boundary Field Problems and Computer Simulation». October 14-16, 2013, Riga, Latvia.

**Appendix 1**

# NOMENKLATURE

## List of Latin symbols

$\vec{B}$ the vector of magnetic induction

$\vec{B}^e$ the vector external magnetic field

$B_0$ the magnitude of applied external magnetic field

$\vec{E}$ the electric field vector

$h$ the half-width of the channel

$\vec{j}$ the electric current density

$J_\nu(\lambda r)$ the Bessel functions of order ν

$L$ the half-width of the split

**L** the operator $\mathbf{L}f = -\lambda^2 \cdot f + \frac{d^2 f}{dy^2}$

$P$ the pressure;

$t$ time

$\vec{V}$ the velocity of the fluid

$V_0$ the characteristic velocity

$Ha$ the Hartmann number

$N$ the Stuart number

Re the Reynolds number

$\mathrm{Re}_m$ the magnetic Reynolds number

## List of Greek symbols

$\Gamma(x)$ is Euler gamma function

$\nabla$ operator nabla $\nabla = \frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y$,

$$\nabla = \frac{\partial}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial}{\partial \varphi}\vec{e}_\varphi + \frac{\partial}{\partial z}\vec{e}_z,$$

$\Delta = \nabla^2$ Laplacian $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\Delta\vec{V} = \Delta V_x \vec{e}_x + \Delta V_y \vec{e}_y$$

$$\Delta\vec{V} = \vec{e}_r(\Delta V_r - \frac{V_r}{r^2} - \frac{2}{r^2}\frac{\partial V_\varphi}{\partial\varphi}) + \vec{e}_\varphi(\Delta V_\varphi + \frac{2}{r^2}\frac{\partial V_r}{\partial\varphi} - \frac{V_\varphi}{r^2}) + \vec{e}_z \Delta V_z$$

$\mu$ is the absolute magnetic permeability of the fluid

$\mu_0$ is a magnetic permeability in free space

$\sigma$ is the electrical conductivity of the fluid

$\rho$ is the density of the fluid

$\gamma$ is the kinematic viscosity of the fluid

$\varepsilon$ is the absolute electric permittivity of the fluid

$\eta(x)$ is the Heaviside step function

**Coordinate systems**

(x,y,z) Cartesian coordinates, $x,y,z \in R$

$(r,\varphi,z)$ cylindrical coordinates, $r \geq 0,\ 0 \leq \varphi \leq 2\pi,\ \in R$

## Appendix 2

# LIST OF FIGURES

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