LATVIJAS UNIVERSITĀTES
ZINĀTNISKIE RAKSTI

ACTA UNIVERSITATIS LATVIENSIS

606

MATEMĀTIKA

MATHEMATICS

Matemātika: Zinātniskie raksti / Zin. red. A. Šostaks. Rīga: LU, 1997. - 76 lpp.

Krājumā publicēti zinātniskā darba rezultāti, kas iegūti LU matemātikas katedru un LU Matemātikas un informātikas institūta docētāju, darbinieku un doktorandu darbos. Krājumā ir iekļauti arī dažu citu augstskolu speciālistu darbi, kas iegūti ciešā sadarbībā ar LU matemātiķiem. Krājumā publicētie rezultāti lielākoties ir iegūti 1995. - 1996. gadā.

The "Scientific Papers" contain the results of the investigations carried out mainly in the years 1995-1996 by lecturers, researchers and post-graduate students of the Department of Mathematics and the Institute of Mathematics and Computer Sciences of the University of Latvia. Some of the papers included reveal also the results of the researches carried out by specialists from some other universities collaborating with the mathematicians of the University of Latvia.

В сборник включены результаты научных исследований, проведенных сотрудниками и докторантами математических кафедр Латвийского университета, Института математики и информатики при Латвийском университете, а также отдельные работы сотрудников других вузов, сотрудничающих с математиками Латвийского университета. Большинство из опубликованных в сборнике результатов получены в период 1995-1996 гг.

ISBN 9984-516-04-0

M A T E M Ā T I K A

Zinātniskie raksti

606. sējums

Zinātniskais redaktors A. Šostaks

Literārās redaktores L. Paegle, R. Pavlova

Reģ. apl. Nr. 2-0266. Pasūt. Nr. _____

Metiens 100 eks.

Iespiests tipogrāfijā

L A T V I J A S   U N I V E R S I T Ā T E
Matemātiskās analīzes katedra

M A T E M Ā T I K A

Zinātniskie raksti

606. sējums

Rīga 1997

UNIVERSITY OF LATVIA
Department of Mathematical Analysis

MATHEMATICS

Scientific papers

Volume 606

Riga 1997

CONTENTS

SATURA RĀDĪTĀJS

СОДЕРЖАНИЕ

Latvijas Universitātes Zinātniskie Raksti, 606 (1997). Matemātika

EXTREMAL PROBLEMS OF APPROXIMATION THEORY OF FUZZY SETS

Svetlana V. Asmuss, Alexander P. Šostak

Abstract. The problem of approximation of a fuzzy subset of a normed space is considered in the paper. We study the error of approximation, which in this case is characterized by a fuzzy number. In order to do this we define the supremum of a fuzzy set of real numbers as well as the supremum and the infimum of crisp sets of fuzzy numbers. The introduced concepts allow us to investigate the best approximation and the optimal linear approximation. The fuzzy counterparts of duality theorems are proved.
AMS SC 04A72, 41A65

1 Introduction

Extremal problems of Approximation Theory is a field of mathematics whose setting allows one to solve such important problems as precise estimation of the error of fixed approximation methods and determination of the best method of approximation.

The central one in this theory is the notion of the best approximation introduced by P.L.Chebyshev [11]. While at the beginning researche's attention was attracted to the investigation of the best approximation to a single element x

$$E(x,\mathcal{U}) = \inf_{u\in\mathcal{U}} \|x-u\|_X \tag{1}$$

(here X is a normed space, $x \in X$ and $\mathcal{U} \subset X$), starting with thirties the emphasis moved to the problem of approximation to the whole class $\mathcal{V} \subset X$

$$E(\mathcal{V},\mathcal{U}) = \sup_{x\in\mathcal{V}} E(x,\mathcal{U}). \tag{2}$$

The statement of the problem (2) is caused by the fact that estimation of the value $E(\mathcal{V},\mathcal{U})$ is being searched in terms of some characteristics of the element x, determining not the given element but a certain set $\mathcal{V}$ containing this element. The problem (2) can be interpreted also as follows: we have only a certain incomplete information about the element v to be approximated; this information determines not a single element but a whole set $\mathcal{V}$, and our task is to get the best possible estimation of the value $E(v,\mathcal{U})$ based only on this information. It is clear that the value $E(\mathcal{V},\mathcal{U})$ obtain as the result of such reasoning will be true for any element x of $\mathcal{V}$, although, generally, it will not be exact for each element separately. Therefore the set $\mathcal{V}$ must be sufficiently narrow in order that the characteristics which determine it could reflect the basic properties of the element v more fully.

Under such interpretation it seems natural to consider the set $\mathcal{V}$ as a fuzzy, rather than as a crisp one, i.e. to realize it as a function $\mathcal{V}: X \to [0,1]$, where the value $\mathcal{V}(x)$ describes the "belongness degree" of the element x to the set $\mathcal{V}$.

Example 1.1 The problem (2) can be interpreted also as the problem of approximation to a fixed element on the basis of non-precise, or fuzzy information. For example, one can consider approximation to a function f by values r_i in points $t_i \in [a,b], i = 1,2,\ldots,n$, which are known up to a certain error: $r_i = f(t_i) + \xi_i$. If one knows the principle of distribution F of errors ξ_i, than as $\mathcal{V}: C[a,b] \to [0,1]$ one can take

$$\mathcal{V}(x) = 1 - (F(\|r(x) - r\|) - F(-\|r(x) - r\|)),$$

where $r(x) = (x(t_1),\ldots,x(t_n)), r = (r_1,\ldots,r_n)$ and $\|\ldots\|$ is a norm in $\mathbb{R}^n$.

Example 1.2 When approximating to a function in order to estimate the value $E(v,\mathcal{U})$ one usually uses characteristics of its smoothness. In many practically important cases the problem (2) is solved for the class $\mathcal{V} = L_p^r[a,b]$ of $(r-1)$-times absolutely continuously differentiable functions whose r-th derivative is p-summable. Along with this one can often assume that the function to be approximated is infinitely many times differentable. Here as $\mathcal{V}$ one can take

$$\mathcal{V}(x) = \alpha(m) \text{ for } x \in L_p^m[a,b] \setminus L_p^{m+1}[a,b],$$

where α is a certain weight function of the natural variable such that

- $\alpha(0) = 0$
- $\lim_{m\to\infty} \alpha(m) = 1$
- α is strictly increasing.

2 Fuzzy sets. Upper and lower bounds in fuzzy setting

In this section we recollect some concepts and results concerning fuzzy sets and related notions which are needed in the sequel. Most of them are well-known to those working with fuzzy sets, some are probably new.

2.1 Fuzzy sets

The concept of a fuzzy set was introduced by L.A. Zadeh [8]. Following Zadeh, by *a fuzzy set*, or more precisely, by *a fuzzy subset of a set X* we realize a mapping $\mathcal{V}: X \to I := [0,1]$. The value $\mathcal{V}(x)$ can be interpreted as the "belongness degree" of an element x to a fuzzy set $\mathcal{V}$. A fuzzy set $\mathcal{V}: X \to I$ will be called *normed* if $\sup_x \mathcal{V}(x) = 1$ (cf e.g. [1]).
In case X is a metric space, a fuzzy set $\mathcal{V}$ is called *bounded* if for every $\varepsilon > 0$ the set $\mathcal{V}^{-1}[\varepsilon, 1]$ is bounded. Further, let X be a vector space over $\mathbf{R}$ and let $\lambda \in \mathbf{R}$. Then the product $\lambda\mathcal{V}: X \to \mathbf{R}$ is defined by the equality $\lambda\,\mathcal{V}(x) = \mathcal{V}(\frac{x}{\lambda})$. (In case $\mathcal{V}$ is a crisp set this definition obviously reduces to the classical one).The image of a given fuzzy set $\mathcal{V}: X \to I$ under a mapping $\varphi: X \to Y$ is defined as a fuzzy set

$$\varphi(\mathcal{V})(y) = \begin{cases} \sup_{\varphi(x)=y} \mathcal{V}(y) & \text{if} \quad \varphi^{-1}(y) \neq \emptyset \\ 0 & \quad \textit{otherwise.} \end{cases}$$

2.2 Fuzzy real numbers. Fuzzy real line

Definition 2.2.1 *(see [3], [2], cf also [6],[7]).* A fuzzy real number *is a function* $z: \mathbf{R} \to I$ *such that*

- *z is non-increasing;*
- $\sup_x z(x) = 1, \quad \inf_x z(x) = 0;$
- *z is upper semicontinuous, i.e.* $\lim_{x \to a^-} z(x) = z(a)$.

The set of all fuzzy real numbers is called a fuzzy real line *and it is denoted by* $\mathbf{R}(I)$.

Remark 2.2.2 (1) Those working in "fuzzy mathematics" usually consider these concepts in a more general setting, namely the so called L−fuzzy real numbers and the L−fuzzy real line $\mathbf{R}(L)$ where L is a bounded lattice satisfying certain conditions. However in our paper we deal only with "classic" fuzzy sets, i.e. with L−fuzzy sets for $L = I$, and therefore, naturally we shall use just the (I-)fuzzy real line $\mathbf{R}(I)$.

(2) In the original papers on this subject ([3], [2] et.al.) fuzzy real numbers were defined not as functions themselves, but as certain classes of equivalence of functions. In this paper we accept the definition first suggested by S.E. Rodabaugh [7] which is essentially equivalent to the original one.

(3) The ordinary real line $\mathbf{R}$ can be identified with a subspace of $\mathbf{R}(I)$ by assigning to a real number $a \in \mathbf{R}$ the fuzzy real number z_a defined by

$$z_a(x) = \begin{cases} 1, & \text{if} \quad x \leq a \\ 0, & \text{if} \quad x > a. \end{cases}$$

2.2.3 Fuzzy topology on R(I) (cf [2] etc).
Given $a, b \in \mathbb{R}$ let $\lambda_b, \rho_a : \mathbb{R}(I) \to I$ be defined by

$$\lambda_b(z) = 1 - z(b), \text{ and}$$

$$\rho_a(z) = z(a^+), \quad \text{where} \quad z(a^+) = \sup_{x<a} z(x).$$

Then the family $\{\lambda_b, \rho_a : a, b \in \mathbb{R}\}$ generates a fuzzy topology τ on $\mathbb{R}(I)$; on the real line $\mathbb{R}$ viewed as a subspace of $\mathbb{R}(I)$ this fuzzy topology induces the usual (order) topology .

2.2.4 Addition of fuzzy numbers *(see [6], [7]).* Given $z_1, z_2 \in \mathbb{R}(I)$ let

$$(z_1 \oplus z_2)(x) = \sup_t z_1(t) \wedge z_2(x - t).$$

The operation of fuzzy addition $\oplus$ is a jointly continuous extension of addition from the real line $\mathbb{R}$ to the fuzzy real line $\mathbb{R}(I)$.

2.2.5 Product of fuzzy numbers was defined and thoroughly studied by S.E. Rodabaugh (see e.g. [7]). The definition of product $\odot$ of fuzzy numbers is much more complicated if compared with the definion of their sum. Fortunately, for our purposes we need only multiplication of a fuzzy number by a *positive real* number $t \in \mathbb{R}$. In this case the general definition is equivalent to the one given by the following simple formula:

$$(t \odot z)(x) = z\left(\frac{x}{t}\right).$$

The operation of product is in accordance with addition and is jointly continuous.

2.3 Supremum of a fuzzy set of real numbers

Definition 2.3.1 *By the supremum of a normed bounded fuzzy set M $\mathbb{R} \to I$ we call the fuzzy real number $Sup(M) : \mathbb{R} \to I$ such that*

(1s) $M \leq Sup(M)$ *and*

(2s) $\forall x \in \mathbb{R}, \forall \varepsilon > 0, \exists y \geq x - \varepsilon$ such that $M(y) \geq Sup(M)(x) - \varepsilon$.

Notice that in case M is crisp, this definition reduces to the usual definition of supremum.

In case $M = \varphi(\mathcal{V})$ where $\mathcal{V} : X \to I$ and $\varphi : X \to \mathbb{R}$ (see 2.1) we shall usually write $Sup_{x \in \mathcal{V}} \varphi(x)$ or $Sup\{\varphi(x) : x \tilde{\in} \mathcal{V}\}$ instead of $Sup(\varphi(\mathcal{V}))$.

Theorem 2.3.2 *Let M $\mathbb{R} \to I$ be a normed bounded fuzzy set and let S_M $\mathbb{R} \to I$ be a mapping. Then the following are equivalent:*

(1) $S_M = Sup(M)$, *i.e.* S_M *is the supremum of* M;

(2) $S_M(x) = \sup\{\alpha : \sup M^{-1}[\alpha, 1] \geq x\} \quad \forall x \in \mathbb{R}$;

(3) $S_M(x) = \sup\{\alpha : \sup M^{-1}(\alpha, 1] \geq x\} \quad \forall x \in \mathbb{R}$;

(4) $S_M = \bigwedge\{s \mid s : \mathbb{R} \to I, s \geq M, \ s \text{ is non-increasing}, \ s \text{ is u.s.c.}\}$

Since the fuzzy set S_M defined in (4) is obviously a fuzzy number, this theorem, among other, implies the existence of the supremum for every bounded normed fuzzy set of real numbers.

Remark 2.3.3 If we omit the condition that M is normed and bounded, then one can also define a mapping $Sup(M) : \mathbb{R} \to I$ by the properties $(1s)$ and $(2s)$ in 2.3.1. However, in this case $Sup(M)$ may fail to be a fuzzy number. Namely, $Sup(M)$ is non-increasing and u.s.c. but generally does not satisfy the second condition in the definition of a fuzzy real number. A statement analogous to Theorem 2.3.2 remains valid in this case, too.

Proposition 2.3.4 *If* $M : \mathbb{R} \to I$ *is a normed bounded fuzzy set and* $t > 0$, *then* $Sup(M \odot t) = t \odot Sup(M)$

Remark 2.3.5 Patterned after the situation in classic analysis one can define the infimum of a normed bounded fuzzy set $M : \mathbb{R} \to I$ as the fuzzy number $Inf(M) = -Sup(-M)$. However, we shall not deepen into consideration of this concept here, because it is not needed for the purposes of our work.

2.4 Infimum and supremum of a crisp set of fuzzy numbers

Let $F \subset \mathbb{R}(I)$. Then the infimum of this set is defined by the equality

$$\inf F = \bigwedge\{s \mid s \in F\}.$$

Obviously $\inf F$ is a fuzzy real number. It is also easy to see that $\inf F$ can be characterised as the largest one $(\geq)$ in the family of fuzzy numbers which are less than or equal to any one of fuzzy numbers $s \in F$. The supremum of the set F is defined by the equality

$$\sup F = \inf\{s \mid s \text{ is a fuzzy number and } s \geq s' \quad \forall s' \in F\}$$

3 Extremal problems of approximation theory

In this section we generalise the well-known notions of "classic" or "crisp" approximation theory (see, e.g. [10], [4], [9]) to the fuzzy case. We consider the problem of approximation of a fuzzy subset V in a normed space X.

3.1 On the best approximation of a fuzzy set

Let $\mathcal{U}$ be a fixed non-empty subset of a space X. Speaking about the best approximation of a fuzzy set $\mathcal{V} : X \to I$ by the set $\mathcal{U}$, the exact upper bound in (2) must be realized in the fuzzy sense (see 2.3.1).

Definition 3.1.1 *The best approximation of a fuzzy set $\mathcal{V}$ by a set $\mathcal{U}$ is defined as the fuzzy number*

$$E(\mathcal{V},\mathcal{U}) = Sup_{x \in \mathcal{V}} E(x,\mathcal{U})$$

The fuzzy value $E(\mathcal{V},\mathcal{U})$ can be interpreted as the deviation of the fuzzy set $\mathcal{V}$ from the crisp set $\mathcal{U}$.

Proposition 3.1.2 *For each $\alpha \in I$ the value* $\sup E(\mathcal{V},\mathcal{U})^{-1}([\alpha,1])$ *gives the best approximation of the set $\mathcal{V}^{-1}[\alpha,1]$ in the crisp sense.*

3.2 On the error of the method of approximation of a fuzzy set

Having the precise estimation of approximation in the form $E(\mathcal{V},\mathcal{U})$ we usually cannot construct for a given element $v \in \mathcal{V}$ an element in the set $\mathcal{U}$ realizing such an error. Instead of the best approximation operator $P : X \to \mathcal{U}$ defined as $\|x - Px\|_X = E(x,\mathcal{U})$ we should prefer a constructively realizable method of approximation. Any one of such methods is defined by a certain operator $A : X \to \mathcal{U}$.

Definition 3.2.1 *The error of approximation of a fuzzy set $\mathcal{V}$ by a method $A : X \to \mathcal{U}$ is given by the value*

$$e(A,\mathcal{V},\mathcal{U}) = Sup_{x \in \mathcal{V}} \|x - Ax\|_X. \tag{3}$$

Proposition 3.2.2

$$E(\mathcal{V},\mathcal{U}) = \inf_A e(A,\mathcal{V},\mathcal{U}).$$

Definition 3.2.3 *The operator (method) A_0 is called the* optimal approximation *operator (method) if*

$$e(A_0,\mathcal{V},\mathcal{U}) = E(\mathcal{V},\mathcal{U}).$$

The best approximation operator P, whenever it exists, is the optimal one, but it is not necessarily the unique among possible ones. Taking into account that in fuzzy case the value $E(\mathcal{V},\mathcal{U})$ is a fuzzy real number, i.e. a mapping of R into I, one could hardly be able to construct the optimal approximation method.

Definition 3.2.4 *The operator (method) A_ε is called the* ε-optimal approximation *operator (method) for non-negative $\varepsilon \in \mathbb{R}(I)$ if*

$$e(A_\varepsilon, \mathcal{V}, \mathcal{U}) \leq E(\mathcal{V}, \mathcal{U}) \oplus \varepsilon.$$

If we are interested only in linear methods, then for fixed $\mathcal{V}$ and $\mathcal{U}$ (in this case we suppose that $\mathcal{U}$ is a subspace of X), it is natural to search for those linear operators $L \in \mathcal{L}(X, \mathcal{U})$, whose upper bound (3) takes the minimal value.

Definition 3.2.5 *The best linear approximation of a fuzzy set $\mathcal{V}$ by a subspace $\mathcal{U}$ is defined as a fuzzy number*

$$\mathcal{E}(\mathcal{V}, \mathcal{U}) = \inf_{L \in \mathcal{L}(X, \mathcal{U})} e(L, \mathcal{V}, \mathcal{U}).$$

Definition 3.2.6 *The linear operator (method) $L_0 : X \to \mathcal{U}$ is called the* optimal linear approximation *operator (method) if*

$$e(L_0, \mathcal{V}, \mathcal{U}) = \mathcal{E}(\mathcal{V}, \mathcal{U}).$$

Definition 3.2.7 *The linear operator (method) $L_\varepsilon : X \to \mathcal{U}$ is called the* ε -optimal linear approximation *operator (method) for non-negative $\varepsilon \in \mathbb{R}(I)$ if*

$$e(L_\varepsilon, \mathcal{V}, \mathcal{U}) \leq \mathcal{E}(\mathcal{V}, \mathcal{U}) \oplus \varepsilon.$$

The inequalities

- $E(\mathcal{V}, \mathcal{U}) \leq \mathcal{E}(\mathcal{V}, \mathcal{U}) \leq e(L, \mathcal{V}, \mathcal{U})$ for each linear method L,
- $E(\mathcal{V}, \mathcal{U}) \leq e(A, \mathcal{V}, \mathcal{U})$ for each operator A

explain the practical importance to know the value $E(\mathcal{V}, \mathcal{U})$. This value provides an orient which allows to judge about the dignity and shortage of a given concrete method.

3.3 On the widths of a fuzzy set

The introduced concepts allow us to consider in fuzzy case the notion of the width, connected with the search of the optimal apparatus of approximation.

Definition 3.3.1 *The fuzzy value*

$$d_N(\mathcal{V}, X) = \inf_{\mathcal{U} \subset X, \dim \mathcal{U} \leq N} E(\mathcal{V}, \mathcal{U})$$

is called the Kolmogoroff N-width *of a fuzzy set $\mathcal{V}$.*

Definition 3.3.2 *The fuzzy value*

$$\lambda_N(\mathcal{V}, X) = \inf_{\mathcal{U} \subset X, \dim \mathcal{U} \leq N} E(\mathcal{V}, \mathcal{U})$$

is called the linear N-width *of a fuzzy set* $\mathcal{V}$.

By analogy the counterparts of other widths (see e.g. [5]) can be considered in the context of fuzzy sets.

4 Approximation in L_q -metric

The most important normed functional space is the space $L_q(I)$. This space consists of all integrable functions defined on I for which the following norm is finite

$$\|f\|_q = \left(\int_I |f(\tau)|^q d\tau \right)^{1/q} \qquad 1 \leq q < \infty. \tag{4}$$

When $q = \infty$, the right side of (4) is replaced by the essential supremum of f.

We consider the best approximation to elements of a fuzzy subset $\mathcal{V}$ of the space $L_p^r(I) = \{f : f^{(r)} \in L_p(I)\}$ by a finite dimensional subspace $\mathcal{U} \subset L_p^r(I)$ in L_q -metric

$$E(\mathcal{V}, \mathcal{U})_q = \sup_{x \in \mathcal{V}} E(x, \mathcal{U})_q, \qquad E(x, \mathcal{U})_q = \inf_{u \in \mathcal{U}} \|x - u\|_q.$$

4.1 On the best approximation of functions

For the best approximation to a function $f \in L_p^r(I)$ in a subspace $\mathcal{U}$, including the class P_m of all polynomials of degree m it holds

Theorem 4.1 *([9]). If $f \in L_p^r(I) \setminus \mathcal{U}$, where $P_m \subset \mathcal{U} \subset L_p^r(I)$,* $\dim \mathcal{U} < \infty$, $1 \leq p \leq \infty$, $r = 0, 1, \cdots$, *then*

$$E(f, \mathcal{U})_q = \sup \left\{ \int_I f^{(r)}(\tau) g(\tau)\, d\tau : \quad g \in W_{q'}^r(\mathcal{U})_m \right\} \qquad \textit{for} \;\; m = r - 1,$$

$$E(f, \mathcal{U})_q = \sup \left\{ \int_I f^{(r)}(\tau) g(\tau)\, d\tau : \quad g \in W_{q'}^r(\mathcal{U})_{r-1}, g \perp P_{m-r} \right\} \qquad \textit{for} \;\; m \geq r$$

where
$W_{q'}^r(\mathcal{U})_m = \left\{ g \in L_{q'}^r(I) : \|g^{(r)}\|_{q'} \leq 1, g^{(r)} \perp \mathcal{U}, g^{(k)}(0) = g^{(k)}(1) = 0, 0 \leq k \leq m \right\}$,
$1/q + 1/q' = 1$ *and* $g^{(r)} \perp \mathcal{U}$ *means that* $\int_I g^{(r)}(\tau) u(\tau) d\tau = 0 \;\; \forall u \in \mathcal{U}$.

One can see that the problem of estimation of $E(\mathcal{V},\mathcal{U})_q$ will be correct, if derivatives $f^{(r)}$ of all approximated functions $f\tilde{\in}\mathcal{V}$ will be bounded in norm. Therefore we consider $f\tilde{\in}\mathcal{V}\wedge W_p^r$ for $W_p^r = \{f \in L_p^r(I) : \|f^{(r)}\|_p \leq 1\}$.

4.2 On the best approximation of a crisp set

In "crisp" approximation theory the most powerful methods of the solution of extremal problems are based on the duality correlations. The obtained results are their fuzzy analogies. They allow to reduce the problem of estimation of the value $E(\mathcal{V},\mathcal{U})_q$ to a more visible extremal problem in the conjugate space.

Theorem 4.2 *Let* $\mathcal{V} : L_p^r(I) \to I, P_m \subset \mathcal{U} \subset L_p^r(I), \dim\mathcal{U} < \infty$, $1 \leq p \leq \infty, r = 0, 1, \cdots,$ *Then*

$$E(\mathcal{V},\mathcal{U})_q \geq Sup\left\{\|g\|_{p'} : g\tilde{\in}\mathcal{W} \wedge W_{q'}^r(\mathcal{U})_m\right\} \qquad for\ m = r-1,$$

$$E(\mathcal{V},\mathcal{U})_q \geq Sup\left\{\|g\|_{p'} : g\tilde{\in}\mathcal{W} \wedge W_{q'}^r(\mathcal{U})_{r-1}, g \perp P_{m-r}\right\} \qquad for\ m \geq r.$$

where $1/p + 1/p' = 1, 1/q + 1/q' = 1, \mathcal{W} : L_{q'}^r(I) \to I,$

$$\mathcal{W}g = \sup_{f \in W_p^r(I)} \left\{\mathcal{V}f : \int_I f^{(r)}(\tau)g(\tau)d\tau = \|g\|_{p'}\right\}$$

The proof of this theorem follows from Theorem 4.1 by means of

Lemma 4.2 *Let* $\mathcal{V} : L_p(I) \to I, \mathcal{U} \subset L_{p'}(I), 1/p + 1/p' = 1, 1 \leq p, p' \leq \infty$. *Then*

$$\sup_{g\in\mathcal{U}} Sup\left\{\int_I f(\tau)g(\tau)d\tau : f\tilde{\in}\mathcal{V}, \|f\|_p \leq 1\right\} \geq Sup\left\{\|g\|_{p'} : g\tilde{\in}\mathcal{U} \wedge \mathcal{W}\right\},$$

where

$$\mathcal{W}g = \sup_{f \in L_p(I)} \left\{\mathcal{V}f : \int_I f(\tau)g(\tau)d\tau = \|g\|_{p'}, \|f\|_p \leq 1\right\}.$$

References

[1] P.Eklund, W.Gähler, Basic notions for fuzzy topology, Part II, *Fuzzy Sets and Syst.* 27 (1988), 171-195.

[2] T.E. Gantner, R.C. Steinlage, R.H. Warren, Compactness in fuzzy topological spaces, *J. Math. Anal. Appl.*, 62 (1978) 547-562.

[3] B. Hutton, Normality in fuzzy topological spaces, J. Math. Anal. Appl., 50 (1975), 74-79

[4] H.Keisewetter. Vorlesungen über lineare Approximation. - Berlin, VEB; 1973.
[5] A.Pinkus. n-widths in approximation theory. - Berlin: Springer Verlag, 1985.
[6] S.E.Rodabaugh. Fuzzy addition and the fuzzy real lines. Fuzzy Sets and Syst., 8 (1982), 39-52.
[7] S.E.Rodabaugh. Complete fuzzy topological hyperfields and fuuzy multiplication in the fuzzy real lines. Fuzzy Sets and Syst., 15 (1988), 285-310.
[8] L.A.Zadeh. Fuzzy sets. Inform. and Control, 8 (1965), 338-365.
[9] Н.П.Корнейчук. Точные константы в теории приближения. - Москва, 1987.
[10] В.М.Тихомиров. Некоторые вопросы теории приближений. - Москва, 1976
[11] П.Л.Чебышев. Избранные труды. - Москва, АН СССР, 1955, с. 462 - 578.

<u>С. Асмусс, А. Шостак. Экстремальные задачи аппроксимации нечетких множеств.</u>

Аннотация. Рассматривается задача аппроксимации нечеткого множества в нормированном пространстве. Исследуется погрешность приближения, которая в этом случае характеризуется нечетким числом. С этой целью определяется понятие супремума нечеткого числового множества, а также инфимум и супремум множества нечетких чисел. Введенные понятия позволяют исследовать наилучшее приближение и погрешность оптимального линейного приближения. Доказан нечеткий аналог теорем двойственности.

<u>S. Asmuss, A. Šostaks. Fazi kopu aproksimācijas ekstremālie uzdevumi.</u>

Anotācija. Tiek apskatīta normētas telpas fazi-apakškopas aproksimācijas problēma. Mēs pētam aproksimācijas kļūdu, kas šajā gadījumā ir raksturota ar fazi-skaitli. Ar šo nolūku mēs definējam reālo skaitļu fazi-kopas suprēmu un fazi-skaitļu kopas suprēmu un infimu. Ievieatās koncepcijas ļauj pētīt labāko tuvinājumu un optimāla lineāra tuvinājuma kļūdu. Ir pierādītas dualitāšu teorēmu fazi-versijas.

Department of Mathematics
University of Latvia
Rainis blvd. 19
Riga LV-1586
LATVIA

Latvijas Universitātes Zinātniskie Raksti, 606 (1997). Matemātika

AN APPROXIMATION OF NOISY DATA BY SMOOTHING SPLINES

Natalia Budkina

Abstract. In this paper an approximation of elements of a Hilbert space on the basis of inexact information is considered as the problem of conditional minimisation of a smoothing functional. For this problem we introduce an auxiliary problem of unconditional minimization. The connection equation of the parameters of the initial problem and of the auxiliary problem is investigated.
AMS SC 41A15, 65D10

1 Introduction

We consider the problem of approximation of elements of a Hilbert space X by the information given by linear functionals $k_i : X \to R, i = 1, \ldots, n$. We denote $A = (k_1, \ldots, k_n)$ and assume that the data Ag about an approximated element $g \in X$ is noisy, i.e. the known information $r \in R^n$ is inexact:

$$|k_i g - r_i| \leq \varepsilon_i, \; i = 1, \ldots, n, \quad \varepsilon_i \geq 0.$$

As approximation of g we are looking for the solution of the conditional minimisation problem of the functional

$$J(x) = \| Tx \|_Y \longrightarrow \min_{x \in X} \tag{1}$$

$$R_i(x) = |k_i x - r_i| \leq \varepsilon_i, \quad i = 1, \ldots, n, \tag{2}$$

defined by a linear operator $T : X \to Y$ in Hilbert spaces.

In the special case of approximation of a function f on the interval $[a, b]$ by the measurments

$$k_i f = f(t_i) \tag{3}$$

at the knots $t_1, \ldots, t_n$ $\quad a \leq t_1 < t_2 < \ldots < t_n \leq b$ $\quad$ and when the operator T is the operator of the q times differentiation

$$Tx = x^{(q)} \tag{4}$$

the problem (1)-(2) gets the form

$$\int_a^b (f^{(q)}(t))^2\,dt \longrightarrow \min_{f\in \mathbf{W}_2^q[a,b]} \tag{1'}$$

$$|f(t_i) - r_i| \le \varepsilon_i^2,\ i = 1,\dots,n, \tag{2'}$$

($\mathbf{W}_2^q[a,b]$ is the Sobolev space). It is known that the solution of the problem (1')-(2') is a natural spline.

We denote by $\tilde{S}_{2q-1,1}$ the space of natural splines. A natural spline s of degree $(2q-1)$ over the grid $t_1,\dots,t_n$ is a function which satisfies conditions :

1. s is a polynomial of degree $(2q-1)$ on each $[t_i, t_{i+1}]$, $i = 1,\dots,n-1$;
2. $s \in C^{2q-2}[a,b]$.
3. $s^{(q)}(t) \equiv 0$, if $t \in [a,t_1]\bigcup[t_n,b]$.

If $n > q$ and no algebriac polynomial of degree $(q-1)$ satisfies the conditions (2'), then a natural spline of degree $(2q-1)$ gives the unique solution of the problem (1')-(2') (see e.g.[1]). If $n \le q$, then the solution of the problem (1')-(2') is any polynomial P of degree $(q-1)$, which satisfies the conditions $\mathrm{P}(t_i) = z_i, i = 1,\dots,n$.

In [3] for the conditional problem (1')-(2') we introduce the auxiliary problem of unconditional minimization

$$F(f) = \int_a^b (f^{(q)}(t))^2\,dt + \sum_{j=1}^{n} \frac{1}{\alpha_j}(f(t_i) - r_i)^2 \longrightarrow \min_{f\in \mathbf{W}_2^q[a,b]}, \tag{5'}$$

with the smoothing parameters $\alpha_j > 0, j = 1,\dots,n$. The solution of the problem (5') will be obtained as the solution of a system of linear algebraic equations.

The main result of the paper [3] connects the solution of the problem (1') - (2') with the solution of the problem (5'). There is proved that if the parameters α and ε are connected with the connection equation

$$\varphi(\alpha) = \varepsilon, \tag{6}$$

then the spline s_α, i.e. the solution of the problem (5'), gives the unique solution of the problem (1')-(2').

In the present paper the main problem, the auxiliary problem and connection equation are considered in general case.

We assume that $kerT + kerA$ is closed and $kerT \bigcap kerA = \{0\}$ ($kerA$ is the kernel of the operator A). It is known that in this case the problem (1)-(2) has the unique solution and this solution is the spline from the space

$$S(T,A) = \{s \in X : \forall x \in kerA \quad < Ts, Tx >= 0\}$$

corresponding to the given operators $T : X \to Y, A : X \to R^n$ (see [1] p.185, [2] p.9). In special case (3)-(4) the set $S(T,A)$ is the space $\tilde{S}_{2q-1,1}$ of natural splines.

A spline $s \in S(T,A)$ is called an interpolating spline for a vector $z = (z_1, \ldots, z_n)$ if $As = z$. Under stated assumptions for every vector z there is a unique interpolating spline (see [1] p.186). So $dim S(T,A) = n$.

2 The auxiliary problem and the connection equation

We formulate the auxiliary problem as follows

$$F(f) = \| Tx \|_Y + \| Ax - r \|_a \longrightarrow \min_{x \in X} \tag{5}$$

where $\| z \|_a = \sum_{j=1}^{n} \frac{1}{a_j} z_i^2$ is a norm in R^n defined by the coefficient $a \in R^n$.

It is known that the unique solution of the problem (5) exists and this solution is the spline from the space $S(T,A)$. The solution $s \in S(T,A)$ of the problem (5) may be uniquely restored by a vector $\lambda \in R^n$

$$T^*Ts = A^*\lambda$$

where A^* and T^* are conjugate operators (see [2] p.9). It is known (see [2] p.13), that the components of the vector λ of the spline-solution of (5) satisfy the conditions

$$\lambda_i = \frac{1}{\alpha_i}(r_i - k_i s), i = 1, \ldots, n. \tag{7}$$

It is important for us that the smoothing parameters α and ϵ of the main and auxiliary problems are connected with the equality

$$\varphi(\alpha) = \epsilon \tag{6}$$

where

$$\alpha = (\alpha_i : i = 1, \ldots, n); \quad \epsilon = (\epsilon_i^2 : i = 1, \ldots, n);$$

$$\varphi_i(\alpha) = R_i(s_\alpha) = (s_\alpha(t_i) - r_i)^2, i = 1, \ldots, n; \quad \varphi(\alpha) = (\varphi_i(\alpha) : i = 1, \ldots, n),$$

(we denote by s_α the solution of the problem (5)).

Theorem 2.1 *If the parameters α and ϵ are connected with the equation (4), then the spline s_α as the solution of the problem (5) gives the unique solution of the problem (1)-(2).*

Proof. Let f be the solution of (1)-(2). Note that the spline s_α satisfies the equation (6), i.e. $R_i(s_\alpha) = \epsilon_i^2, i = 1, \ldots, n$. So s_α satisfies also the conditions (2). Let us compare the values $J(f)$ and $J(s_\alpha)$.

Suppose $J(f) \leq J(s_\alpha)$. Then

$$F(f) = J(f) + \sum_{j=1}^{n} \frac{1}{\alpha_j} R_j(f) \leq J(s_\alpha) + \sum_{j=1}^{n} \frac{1}{\alpha_j} \epsilon_j{}^2 =$$

$$= J(s_\alpha) + \sum_{j=1}^{n} \frac{1}{\alpha_j}(s_\alpha(t_j) - r_j)^2) = F(s_\alpha),$$

i.e. $F(f) \le F(s_{\alpha_j})$. We know that s_α is the unique solution of the problem (5), therefore $f \equiv s_\alpha$. Thus for $f \not\equiv s_\alpha$ it holds $J(f) > J(s_{\alpha_j})$.

Theorem is proved.

3 Investigation of the connection equation

We investigate the connection equation (6) . Let us define the operation L by the pair of operators T and A as $Lx = (Tx, Ax)$ and operate to the space $E = Y \times R^n$ with the scalar product

$$< (y^1, z^1), (y^2, z^2) >_E = < y^1, y^2 >_Y + \sum_{i=1}^{n} \frac{1}{\alpha_i} < z_i^1, z_i^2 >_R$$

The main result of the present section is

Theorem 3.1 *For the derivatives $\frac{\partial \varphi_i}{\partial \alpha_j}$ we have*

$$\frac{\partial \varphi_i}{\partial \alpha_j}(\alpha) = 2\alpha_i^2 < L\frac{\partial s}{\partial \alpha_i}, L\frac{\partial s}{\partial \alpha_j} > .$$

So the Jacobi matrix $\frac{\partial \varphi}{\partial \alpha} = \frac{\partial(\varphi_1, \ldots, \varphi_n)}{\partial(\alpha_1, \ldots, \alpha_n)}$ of the vector-function $\varphi(\alpha)$ can be written as

$$\begin{pmatrix} 2\alpha_1^2 < L\frac{\partial s}{\partial \alpha_1}, L\frac{\partial s}{\partial \alpha_1} > & 2\alpha_1^2 < L\frac{\partial s}{\partial \alpha_1}, L\frac{\partial s}{\partial \alpha_n} > \\ \vdots & \vdots \\ 2\alpha_n^2 < L\frac{\partial s}{\partial \alpha_n}, L\frac{\partial s}{\partial \alpha_1} > & 2\alpha_n^2 < L\frac{\partial s}{\partial \alpha_n}, L\frac{\partial s}{\partial \alpha_n} > \end{pmatrix}$$

It can be easily proved that the operator L is linear and continuous. The following properties of L and the conjugate operator L^* will be used in the proof of Theorem 3.1.

Lemma 3.1 *1.The conjugate operator A^* can be written as*

$$A^* z = \sum_{i=1}^{n} k_i^* z_i, \quad z = (z_1, \ldots, z_n) \in R^n.$$

2.The conjugate operator L^* can be written as

$$L^* e = T^* y + \sum_{i=1}^{n} \frac{1}{\alpha_i} k_i^* z_i, \quad e = (y, z), y \in Y \ z \in R^n.$$

3. The operator L^*L can be written as

$$L^*Lx = T^*Tx + \sum_{i=1}^{n} \frac{1}{\alpha_i} k_i^* k_i x, \quad x \in X.$$

Proof. 1. By definition of the conjugate operator

$$< Ax, s >_{R^n} = < x, A^* s >_X$$

we have

$$< Ax, s >_{R^n} = \sum_{i=1}^{n} < k_i x, s_i >_R = \sum_{i=1}^{n} < x, k_i^* s_i >_X = < x, \sum_{i=1}^{n} k_i^* s_i >_X .$$

Therefore $A^* s = \sum_{i=1}^{n} k_i^* s_i$.

2. Taking into account that

$$< Lx, e >_B = < (Tx, Ax), (y, s) > = < Tx, y >_Y + \sum_{i=1}^{n} \frac{1}{\alpha_i} < k_i x, s_i >_R =$$

$$= < x, T^* y >_X + \sum_{i=1}^{n} \frac{1}{\alpha_i} < x, k_i^* s_i >_X = < x, T^* y + \sum_{i=1}^{n} \frac{1}{\alpha_i} k_i^* s_i >_X$$

by definition $< Lx, e >_B = < x, L^* e >_X$ we get

$$L^* e = T^* y + \sum_{i=1}^{n} \frac{1}{\alpha_i} k_i^* s_i$$

3. To prove the equality we transform the scalar product $< L^*Lx^1, x^2 >_X$

$$< L^*Lx^1, x^2 >_X = < Lx^1, Lx^2 >_B = < (Tx^1, Ax^1), (Tx^2, Ax^2) >_B = < Tx^1, Tx^2 >_Y +$$

$$+ \sum_{i=1}^{n} \frac{1}{\alpha_i} < k_i x^1, k_i x^2 >_R = < T^*Tx^1, x^2 >_X + \sum_{i=1}^{n} \frac{1}{\alpha_i} < k_i^* k_i x^1, x^2 >_X =$$

$$= < T^*Tx^1 + \sum_{i=1}^{n} \frac{1}{\alpha_i} k_i^* k_i x^1, x^2 >_X .$$

Therefore $L^*Lx = T^*Tx + \sum_{i=1}^{n} \frac{1}{\alpha_i} k_i^* k_i x$.

Lemma 8.2 *The spline $s_\alpha(t)$ is continuously differentiable with respect to α.*

Proof. Let us take as basis in the space $S(T,A)$ the system $s_1,\ldots,s_n$, where s_i is the interpolating spline for $z_i = (0,\ldots,0,1,0,\ldots,0)$(where 1 is on the i-th place). The spline s_α can be written as $s_\alpha = \sum_{i=1}^n c_i s_i$, where $c_i = k_i s_\alpha$. Note that s_i don't depend on α, so it is sufficient to establish that the coefficients c_i of the spline are continuously differentiable. By $T^*Ts = A^*\lambda$ we obtain

$$T^*Ts_\alpha = \sum_{i=1}^n c_i T^*Ts_i = A^*\lambda.$$

We take scalar product by s_j

$$< \sum_{i=1}^n c_i T^*Ts_i, s_j > = < A^*\lambda, s_j >$$

So $\sum_{i=1}^n c_i < Ts_i, Ts_j > = < \lambda, As_j > = < \lambda, z_j > = \lambda_j$ and by (7) we obtain

$$\sum_{i=1}^n c_i < Ts_i, Ts_j > = \frac{1}{\alpha_j}(z_j - c_j).$$

The splines s_i are fixed and so $< Ts_i, Ts_j >$ are fixed. Denoting $b_{ij} = < Ts_i, Ts_j >$ and obtain

$$\alpha_j \sum_{i=1}^n c_i b_{ij} + c_j = z_j \tag{8}$$

So the coefficients $c_i, i = 1,\ldots,n$ of the spline s_α can be obtained as the solution of a system of linear equations with non-zero determinant (since the problem (6) has the unique solution). This system defines c_i as implicit functions depending on $(\alpha_1,\ldots,\alpha_n)$ By the theorem of implicit function, coefficients $c_i, i = 1,\ldots,n$ will be continuously differentiable if the Jacobian (i.e. the determinant of the system (8)) is distinct from zero. So the coefficients of the spline $s_\alpha(t)$ are continuously differentiable with respect to α.

Lemma 3.2 is proved.

Lemma 3.3 *The operators T, T^*T and k_i are commuting with the operator $\frac{\partial}{\partial \alpha_j}$ in the following sense*

$$a).\frac{\partial}{\partial \alpha_j} k_i s = k_i \frac{\partial s}{\partial \alpha_j},$$

$$b).\frac{\partial}{\partial \alpha_j} Ts = T \frac{\partial s}{\partial \alpha_j},$$

$$c).\frac{\partial}{\partial \alpha_j} Ls = L \frac{\partial s}{\partial \alpha_j},$$

$$d).\frac{\partial}{\partial \alpha_j} T^*Ts = T^*T \frac{\partial s}{\partial \alpha_j}$$

where $s \in S(T,A)$.

Proof. The spline $s_\alpha = s(\alpha_1, \dots, \alpha_n)$ can be written as

$$s(\alpha_1, \dots, \alpha_n) = \sum_{i=1}^{n} c_i(\alpha) s_i$$

where $c_i(\alpha) \in R$, and $s_1, \dots, s_n$ is a basis of $S(T, A)$. Therefore the order of operations with respect to α and elements of X is not important. So

$$\frac{\partial}{\partial \alpha_j} k_i s = k_i \frac{\partial s}{\partial \alpha_j} \quad \text{and} \quad \frac{\partial}{\partial \alpha_j} Ts = T \frac{\partial s}{\partial \alpha_j}.$$

From a) and b) it follows that $\frac{\partial}{\partial \alpha_j} Ls = L \frac{\partial s}{\partial \alpha_j}$.

To prove d) we differentiate the equation $< Ts, Tx > = < T^* Ts, x >$ with respect to α_j. We get

$$< \frac{\partial}{\partial \alpha_j} Ts, Tx > + < Ts, \frac{\partial}{\partial \alpha_j} Tx > = < \frac{\partial}{\partial \alpha_j} T^* Ts, x > + < T^* Ts, \frac{\partial x}{\partial \alpha_j} > .$$

Since

$$< \frac{\partial}{\partial \alpha_j} Ts, Tx > + < Ts, \frac{\partial}{\partial \alpha_j} Tx > = < T \frac{\partial s}{\partial \alpha_j}, Tx > + < Ts, T \frac{\partial x}{\partial \alpha_j} > =$$

$$= < T^* T \frac{\partial s}{\partial \alpha_j}, x > + < T^* Ts, \frac{\partial x}{\partial \alpha_j} >,$$

we have $\frac{\partial}{\partial \alpha_j} T^* Ts = T^* T \frac{\partial s}{\partial \alpha_j}$.

Proof of theorem 3.1.

Differentiating $\varphi_i(\alpha) = < k_i s - z_i, k_i s - z_i > = (k_i s - z_i)^2$ with respect to α_j we obtain

$$\frac{\partial \varphi_i(\alpha)}{\partial \alpha_j} = 2(k_i s - z_i) k_i \frac{\partial s(\alpha)}{\partial \alpha_j} = 2 < k_i s - z_i, k_i \frac{\partial s(\alpha)}{\partial \alpha_j} >_R =$$

$$= 2 < k_i^* (k_i s - z_i), \frac{\partial s(\alpha)}{\partial \alpha_j} >_X .$$

By $T^* Ts = A^* \lambda$ and (7) we get

$$T^* Ts = A^* \lambda = \sum_{i=1}^{n} k_i^* \lambda_i = \sum_{i=1}^{n} k_i^* \frac{1}{\alpha_i} (z_i - k_i s).$$

So

$$T^* Ts + \sum_{j=1}^{n} \frac{1}{\alpha_j} k_j^* (k_j s - z_j) = 0. \tag{9}$$

According to Lemma 3.1

$$L^*Ls - \sum_{j=1}^{n} \frac{1}{\alpha_j} k_j^* z_j = 0,$$

or

$$L^*Ls = \sum_{j=1}^{n} \frac{1}{\alpha_j} k_j^* z_j \tag{10}$$

Differentiating the equality (9) with respect to α_i we obtain

$$T^*T\frac{\partial s(\alpha)}{\partial \alpha_i} + \sum_{j=1}^{n} \frac{1}{\alpha_j} k_j^* k_j \frac{\partial s(\alpha)}{\partial \alpha_i} - \frac{1}{\alpha_i^2} k_i^*(k_i s - z_i) = 0 \tag{11}$$

From (11) it follows

$$L^*L\frac{\partial s(\alpha)}{\partial \alpha_i} = \frac{1}{\alpha_i^2} k_i^*(k_i s - z_i).$$

We get the final result by substituting this equality into (9)

$$\frac{\partial \varphi_i(\alpha)}{\partial \alpha_j} = 2\alpha_i^2 < L^*L\frac{\partial s(\alpha)}{\partial \alpha_i}, \frac{\partial s(\alpha)}{\partial \alpha_j} > = 2\alpha_i^2 < L\frac{\partial s(\alpha)}{\partial \alpha_i}, L\frac{\partial s(\alpha)}{\partial \alpha_j} > .$$

Theorem is proved.

It is better to consider the function $\psi(\beta)$ instead of the function $\varphi(\alpha)$, where $\psi(\beta) = \varepsilon - \varphi(\frac{1}{\alpha})$. The connection equation in this case is

$$\psi(\beta) = 0 \tag{12}$$

The Jacobi matrix $\frac{\partial \psi}{\partial \beta} = \frac{\partial(\psi_1,\ldots,\psi_n)}{\partial(\beta_1,\ldots,\beta_n)}$ of the vector-function $\psi(\beta)$ can be written as

$$\begin{pmatrix} 2\frac{1}{\beta_1^4} < L\frac{\partial s}{\partial \beta_1}, L\frac{\partial s}{\partial \beta_1} > & 2\frac{1}{\beta_1^2\beta_n^2} < L\frac{\partial s}{\partial \beta_1}, L\frac{\partial s}{\partial \beta_n} > \\ \vdots & \vdots \\ 2\frac{1}{\beta_1^2\beta_n^2} < L\frac{\partial s}{\partial \beta_n}, L\frac{\partial s}{\partial \beta_1} > & 2\frac{1}{\beta_n^4} < L\frac{\partial s}{\partial \beta_n}, L\frac{\partial s}{\partial \beta_n} > \end{pmatrix}$$

The Jacobi matrix for ψ is symmetric and so the operator ψ is potential for a function F, i.e. $\psi = gradF$.

The Jacobi matrix for ψ is a Gram matrix and non-singular, since $\frac{1}{\beta_i^2} L\frac{\partial s}{\partial \beta_i}, i = 1, \ldots, n$, are linearly independent.

We will prove this fact. The vectors $\frac{1}{\beta_i^2} L\frac{\partial s}{\partial \beta_i}, i = 1, \ldots, n$, are linearly independent if

$$\sum_{i=1}^{n} p_i \frac{1}{\beta_i^2} L\frac{\partial s}{\partial \beta_i} = 0 \text{ if and only if } p_i = 0, i = 1, \ldots, n.$$

The spline s can be written as $s = \sum_{k=1}^{n} c_k s_k$, where $s_k, k = 1,\ldots,n$ are the base of the space $S(T,A)$. So

$$\sum_{i=1}^{n} p_i \frac{1}{\beta_i^2} \sum_{k=1}^{n} \frac{\partial c_k}{\partial \beta_i} L s_k = 0$$

or

$$\sum_{k=1}^{n} L s_k \sum_{i=1}^{n} \frac{p_i}{\beta_i^2} \frac{\partial c_k}{\partial \beta_i} = 0.$$

The elements Ls_k are linearly independent, so

$$\sum_{i=1}^{n} \frac{p_i}{\beta_i^2} \frac{\partial c_k}{\partial \beta_i} = 0, \quad k = 1,\ldots,n.$$

The spline s is the solution of the problem (5), therefore it satisfies (8)

$$\sum_{i=1}^{n} c_i b_{ij} + \beta_j c_j = \beta_j z_j, \; j = 1,\ldots,n,$$

i.e. the spline s is the smoothing spline for the vector $z = (z_j : j = 1,\ldots,n)$. We differentiate this equality

$$\sum_{i=1}^{n} \frac{\partial c_i}{\partial \beta_m} b_{ij} + \beta_j \frac{\partial c_j}{\partial \beta_m} = 0, \; j = 1,\ldots,n, \; j \neq m,$$

$$\sum_{i=1}^{n} \frac{\partial c_i}{\partial \beta_m} b_{im} + \beta_m \frac{\partial c_m}{\partial \beta_m} = z_m - c_m.$$

So the spline $s^j = \sum_{k=1}^{n} \frac{\partial c_k}{\partial \beta_j} s_k$ is the smoothing spline for the vector $(0,0,\ldots,0,\frac{z_j - c_j}{\beta_j},0,\ldots,0)$.

The spline $\tilde{s} = \sum_{i=1}^{n} \frac{p_i}{\beta_i^2} \sum_{k=1}^{n} \frac{\partial c_k}{\partial \beta_i} s_k$ is smoothing for the vector $\tilde{z} = (\frac{p_1(z_1 - c_1)}{\beta_1^3},\ldots,\frac{p_n(z_n - c_n)}{\beta_n^3})$.

It is easy to see that $\tilde{s}(t_k) = 0, k = 1,\ldots,n$, so $\tilde{s} \equiv 0$. Therefore $\tilde{z} \equiv 0$, so $p_i = 0, i = 1,\ldots,n$, and the Jacobi matrix for ψ is non-singular .

References

[1] P.J. Laurent. Approximation et optimization. - Hermann, Paris, 1972.

[2] В.В. Вершинин, Ю.С. Завьялов, Н.Н. Павлов. Экстремальные свойства сплайнов и задача сглаживания. - Новосибирск, 1988.

[3] N. Budkina. On a method of solving a smoothing problem. - Proc. Estonian Acad. Sci. Phys. Math., p.183-191, 1996.

N. Budkina. Nogludinošo splainu lietošana neprecizo datu aproksimācijā.
Anotācija. Rakstā ir aplūkots Hilberta telpas elementu aproksimācijas uzdevums pēc informācijas, kas ir uzdota ar zināmo neprecizitāti. Pēc būtības tas ir nogludinoša funkcionāļa nosacītas minimizācijas uzdevums. Šā uzdevuma risināšana ir reducēta uz beznosacījumu minimizācijas palīguzdevuma risināšanu. Ir pētīts vienādojums, kurš sasaista pamatuzdevuma un palīguzdevuma parametrus.

Н. Будкина. Аппроксимация зашумлённых данных сглаживающими сплайнами.
Аннотация. В статье рассматривается задача аппроксимации элементов гильбертого пространства по информации, известной с некоторой погрешностью. По своей постановке она является задачей условной минимизации сглаживающего функционала. Вводится вспомогательная задача на безусловный минимум. Исследуется уравнение, связывающее параметры основной и вспомогательной задач.
УДК 517.5

Department of Mathematics
University of Latvia
Rainis blvd. 19
LV 1586 Riga
LATVIA

EXTREMUM PROBLEMS FOR AN AFFINE FUNCTION DEFINED ON A SET OF PERMUTATIONS

G. Engelis

Summary. The composition of n given affine functions is $At+B$, where B depends on the permutation of these functions. Some necessary properties of the permutation corresponding to the minimal value of B are formulated and in some special cases a precise characteristic of the permutation in question is given.

AMS subject classification 05D99

Let us consider n fixed affine functions $R \to R$

$$f_i(t) = a_i t + b_i$$

(where $a_i \in R \setminus \{0\}, b_i \in R, i \in \{1,2,\ldots,n\}$) and the $n!$ compositions of these functions

$$f_{i_1,\ldots,i_n} := f_{i_n} \circ f_{i_{n-1}} \circ \ldots \circ f_{i_2} \circ f_{i_1}(t) = At + B.$$

It is clear that $A = \prod_{i=1}^{n} a_i$, but B depends on the permutation $(i_1, i_2, \ldots, i_n)$ and can have $n!$ different values. The extremal values $min\{B\}, max\{B\}$ can be found by computing all these $n!$ numbers, but it seems quite naturally to search other ways of solving this problem, too. In the literature we have found no mentioning of this situation. In the general case (without further restrictions for a_i, b_i) the problem seems to be a complicated one. The aim of this paper is to develop some methods effective in certain special cases.

In section 1 we give some inequalities necessary for the permutation corresponding to $min\{B\}$ (in the sequel they will be called optimal). If we are interested in $max\{B\}$, we have only to replace "$\leq$" with "$\geq$". In sections 1 and 2 these inequalities are used in some special cases, where they are especially effective.

1.

Further we often will use numbers

$$c_i = \frac{b_i}{a_i - 1}, \quad d_i = \frac{a_i - 1}{b_i}.$$

Both numbers are not defined only when $a_i = 1, b_i = 0$, but in this case the function $f_i(t) = t$ commutes with each $f_j (f_i \circ f_j = f_j \circ f_i)$ and the place of f_i in the permutation does not change the corresponding B. Therefore we assume that none of $f_i(t)$ equals to t. Then it is easy to conclude that f_j and f_k commute iff $c_j = c_k$ (or $d_j = d_k$). If c_{jk} denotes the corresponding number for $f_j \circ f_k$, then $c_{jk} = c_j = c_k$.

To simplify the notation we assume that

$$min\{B(i_1, \ldots, i_n)\} = B_0 = B(1, 2, \ldots, n)$$

(what can be obtained by reenumeration of the $\{f_i\}$). It is easy to compute that

$$B_0 = \sum_{k=2}^{n} (\prod_{i=k}^{n} a_i) b_{k-1} + b_n. \tag{1}$$

We will compare the B_0 with some B, corresponding to some "close" permutation. For this purpose we will use $n-1$ permutations $(1, 2, \ldots, k-2, k, k-1, k+1, \ldots, n)$ where $k \in \{2, 3, \ldots, n\}$. (But in 2.4 it will be necessary to use some other "close" permutations too).
If we write $B_k = B(1, \ldots, k-2, k, k-1, k+1, \ldots, n)$ as a sum like (1) and put both sums into the inequality

$$B_0 \leq B_k$$

we see that in both parts the summands not containing factors b_k and b_{k-1} are equal and therefore

$$\prod_{i=k}^{n} (a_i) b_{k-1} + (\prod_{i=k+1}^{n} a_i) b_k \leq (\prod_{i=k+1}^{n} a_i) a_{k-1} b_k + (\prod_{i=k+1}^{n} a_i) b_{k-1}. \tag{2}$$

Let $N(k)$ be the number of negatives in the set $\{a_{k+1}, a_{k+2}, \ldots, a_n\}$. Dividing both parts of (2) by $\prod_{k+1}^{n} a_i$ we obtain

$$(-1)^{N(k)} a_k b_{k-1} + b_k \leq (-1)^{N(k)} a_{k-1} b_k + b_{k-1}$$

or

$$(-1)^{N(k)} (a_k - 1) b_{k-1} \leq (-1)^{N(k)} (a_{k-1} - 1) b_k. \tag{3}$$

If $(a_{k-1} - 1)(a_k - 1) \neq 0$ and

$$sgn(a_{k-1} - 1)(a_k - 1) = (-1)^{M(k)}, \quad M(k) \in \{0, 1\}$$

then from (3) it follows

$$(-1)^{N(k)+M(k)}c_{k-1} \leq (-1)^{N(k)+M(k)}c_k. \tag{4}$$

If $b_{k-1}b_k \neq 0$ and

$$sgn b_{k-1}b_k = (-1)^{O(k)}, \quad O(k) \in \{0,1\}$$

then from (3) it follows

$$(-1)^{N(k)+O(k)}d_k \leq (-1)^{N(k)+O(k)}d_{k-1}. \tag{5}$$

In case when $a_k = 1, b_{k-1} = 0$ or $a_{k-1} = 1, b_k = 0$ we get

$$0 \leq (-1)^{N(k)}(a_{k-1} - 1)b_k \tag{6}$$

or

$$(-1)^{N(k)}(a_k - 1)b_{k-1} \leq 0. \tag{7}$$

We have proved the statement: if $B_0 = B(1,2,\ldots,n) = min\{B\}$, then for every $k \in \{2,3,\ldots,n\}$ at least one of the inequalities (4), (5), (6), (7) is valid. This statement allows us to exclude a large set of $n!$ permutations as being non optimal. However, in general the set of remaining permutations is still a large one, too (especially if there are many f_i with $a_i < 0$ and many f_j with $a_j > 0$). Further we investigate some cases when the inequalities (4)-(7) are effective.

2.

In this section we investigate the problem of the optimal permutation in cases, when the set of $\{f_i\}$ is a "homogeneous" one. It means that all numbers a_i belong to one of the six sets: $[1,+\infty[$, $\{1\}$, $]0,1[$, $]-1,0[$, $\{-1\}$, $]-\infty,-1[$.

2.1. If all $a_i > 1$, then $N(k) = M(k) = 0$ for all k and from (4) it follows that in the optimal permutation it holds:

$$c_{k-1} \leq c_k. \tag{8}$$

To find such a permutation we have to compute all c_i and to order the given functions in such a way, that the sequence (c_i) is non decreasing.

2.2. If all $a_i = 1$, then from (1) it follows

$$B_0 = \sum_{k=1}^{n} b_k \tag{9}$$

and every permutation is optimal.

2.3. If for all i $0 < a_i < 1$, then $N(k) = M(k) = 0$ and we get (8) in this case, too. Nevertheless we can not unite (2.1) and (2.3) ($a > 0, a \neq 1$) since then in some cases it will be $M(k) = 1$, but this essentially changes the situation.

2.4. If for all $i \quad -1 < a_i < 0$, then $N(k) = n - k$, $M(k) = 0$ and from (4) it follows

$$c_n \geq c_{n-1} \leq c_{n-2} \geq c_{n-3} \leq \dots \tag{10}$$

But for a given set $\{c_i\}$ the set of sequences satisfying property (10) is a *large* one (if $n = 2m$, then the number of sequences is greater than $(m!)^2$). Therefore we have to search further criteria for the optimal permutation. We consider here like in section 1 some permutations "close" to the optimal one:

$$B_k^l = B(1, 2, \dots, k-2, k+1, k, k-1, k+2, \dots, n), \quad k \in \{2, 3, \dots, n-1\}.$$

Here $N(k+1) = n - k - 1$ and from $B_0 \leq B_k^l$ it follows (like in section 1) that

$$(-1)^{n-k-1}(a_{k+1}a_k b_{k-1} + a_{k+1}b_k + b_{k+1}) \leq (-1)^{n-k-1}(a_{k-1}a_k b_{k+1} + a_{k-1}b_k + b_{k-1}). \tag{11}$$

It turns out that (11) gives some inequalities for the numbers c_{ij}, since if $f_{ij} = f_j \circ f_i$, then $a_{ij} = a_i a_j, \quad b_{ij} = a_j b_i + b_j$ and

$$c_{ij} = \frac{a_j b_i + b_j}{a_i a_j - 1}. \tag{12}$$

Multiplying by $a_k < 0$ and adding to both parts $-b_k$ we obtain from (11):

$$(-1)^{n-k}(a_k a_{k+1} - 1)(a_k b_{k-1} + b_k) \leq (-1)^{n-k}(a_k a_{k-1} - 1)(a_k b_{k+1} + b_k)$$

and recalling that $(a_k a_{k+1} - 1)(a_k a_{k-1} - 1) > 0$ we get

$$(-1)^{n-k} c_{k-1,k} \leq (-1)^{n-k} c_{k+1,k} \tag{13}$$

or

$$c_{n,n-1} \leq c_{n-2,n-1}, \quad c_{n-3,n-2} \leq c_{n-1,n-2}, \quad c_{n-2,n-3} \leq c_{n-4,n-3}, \dots \tag{14}$$

But there exists another possibility to use (11). Adding to both parts $a_{k-1}a_k a_{k+1} b_k$ we get

$$(-1)^{n-k-1}(a_k a_{k-1} - 1)(a_{k-1}b_k + b_{k-1}) \leq (-1)^{n-k-1}(a_{k+1}a_k - 1)(a_{k+1}b_k + b_{k+1})$$

or

$$(-1)^{n-k-1} c_{k,k-1} \leq (-1)^{n-k-1} c_{k,k+1}. \tag{15}$$

Taking here $k = n-1, k = n-2, \dots$ we get

$$c_{n-1,n-2} \leq c_{n-1,n}, \quad c_{n-2,n-1} \leq c_{n-2,n-3}, \quad c_{n-3,n-4} \leq c_{n-3,n-4}, \dots \tag{16}$$

From (14) and (16) it follows

$$c_{n-1,n} \geq c_{n-1,n-2} \geq c_{n-3,n-2} \geq c_{n-3,n-4}, \dots \tag{17}$$

and

$$c_{n,n-1} \leq c_{n-2,n-1} \leq c_{n-2,n-3} \leq c_{n-4,n-3}, \dots \tag{18}$$

But from (12) we obtain

$$c_{ij} - c_{ji} = \frac{(a_j-1)(a_i-1)(c_i-c_j)}{a_i a_j - 1}$$

and

$$sgn(c_{ij} - c_{ji}) = -sgn(c_i - c_j). \tag{19}$$

Combining this with (10) we get

$$c_{n-1,n} \geq c_{n,n-1}, c_{n-1,n-2} \geq c_{n-2,n-1}, \dots$$

and this means that the inequalities (16) contain the greatest and (18) the smallest of the numbers $c_{i,i+1}, c_{i+1,i}$. This gives an algorithm for the search of the optimal permutation.

For this purpose we compute all numbers c_j, and renumerate functions f_j in such a way that the sequence (c_j) is monotone. Now we compute the greater c_{ij}, the whole number of which will be $\frac{n(n-1)}{2}$. After this we search for a sequence containing $n-1$ numbers c_{uv} such that

$$c_{ij} \geq c_{ki} \geq c_{lk} \geq c_{sl} \geq \dots \geq c_{pq}. \tag{20}$$

Such sequence always exists, but it must not be unique. If the sequence (20) is found then we suspect that the permutation

$$(\dots, s, l, k, i, j)$$

starting with p or q, depending on the parity of n, can be the optimal one. If there is more than one sequence (20), we have to compute all corresponding B_h' and take the least one. Or, we can compute the set of $min(c_{ij}, c_{ji})$, form the sequence corresponding to (20) and combine the results of the both sequences.

It is not essential that we consider sequences "from the end to the beginning". Only by construction of the optimal permutation in the natural order we must distinguish the cases when n is even or odd.

2.5. If all $a_i = -1$, then

$$B = \sum_{i=1}^{n} k_i(-1)^{n-i} = \sum_{h=0}^{[\frac{n}{2}]} k_{n-2h} - \sum_{h=0}^{[\frac{n}{2}]} k_{n-1-2h}.$$

and a permutation is optimal iff the numbers $n-1, n-3, n-5, \dots$ are given to the functions with the largest k_i.

2.6. If all $a_i < -1$, we can use all reasonings of 2.4. up to (18). But (19) in this case changes to

$$sgn(c_{ij} - c_{ji}) = sgn(c_i - c_j)$$

from which it follows

$$c_{n,n-1} \geq c_{n-1,n}, \quad c_{n-2,n-1} \geq c_{n-1,n-2}, \dots$$

It means that in this situation (16) contains the smallest and (18) the largest of the $(c_{i,i+1}, c_{i+1,i})$. Further we can proceed on as in (2.4).

3

Here we will show the use of the results of section 1 in some unhomogeneous cases.

3.1. It is possible to combine 2.1 and 2.2 and assume that for all i

$$1 \leq a_i < +\infty.$$

From (4), (6), (7) we obtain that the optimal permutation starts with f_i that have $a_i = 1$ and $b_i < 0$ (here we can define $c_i = -\infty$), further come f_i with $a_i > 1$ and $c_{k-1} \leq c_k$ and the permutation ends with f_i who have $a_i = 1$ and $b_i = 0$ (here $c_i = +\infty$). The order of f_i with $a_i = 1$ at the beginning and the end of the permutation is not essential.

3.2. In an analogue manner we can combine 2.2 and 2.3 and assume that

$$0 < a_i \leq 1.$$

Then the optimal permutation begins with $f_i(t) = t + b_i, b_i > 0$ and ends with $f_i(t) = t + b_i, b_i < 0$.

3.3. It is possible to combine 2.1, 2.2 and 2.3 if for all i, j it holds:

$$b_i b_j > 0.$$

Then $N(k) = 0, O(k) = 0$ and from (8) it follows that the optimal permutation is characterized by

$$d_1 \geq d_2 \geq d_3 \geq \dots \geq d_n.$$

3.4. The case when for all i it holds $a_i = a$ is investigated in section 2, but it can be solved in quite a different way. From (1) we get

$$B_n = \sum_{k=2}^{n} a^{n-k+1} b_{k-1} + b_n$$

and here B_n is the scalar product of the vectors $(a^{n-1}, a^{n-2}, \dots, a, 1)$ and $(b_1, b_2, \dots, b_n)$. A permutation of f_i can change only the second vector, and it is well known that the scalar product is minimal if the coordinates of one vector form a nonincreasing and the coordinates of the other vector form a nondecreasing sequence. For example, if

$$-1 < a < 0, \quad n = 2m,$$

then $a < a^3 < a^5 < \ldots < a^{2m-1} < a^{2m} < a^{2m-2} < \ldots < a^2 < 1$ and the permutation is optimal, if

$$b_1 \geq b_3 \geq b_5 \geq \ldots \geq b_{2m-1} \geq b_{2m} \geq \ldots \geq b_2 \geq b_{2n}.$$

3.5 In an analogical manner we investigate the case when for all i it holds:

$$a_i > 0, b_i = b.$$

Then from (1) it follows

$$B_0 = b[\sum_{k=2}^{n}(\prod_{i=k}^{n} a_i) + 1].$$

If $b > 0$ then B_0 has the least value when

$$a_n \leq a_{n-1} \leq a_{n-2} \leq \ldots \leq a_2 \leq a_1.$$

If $b < 0$, then " $\leq$ " is replaced with " $\geq$ ". Analogically when $a_i < 0$ for all i.

3.6. Let $n-1$ functions f_i have $c_i = c$, but for an index j the number $c_j = \gamma \neq c$. We know that functions with equal c_i commute and therefore an optimal permutation begins with some commuting functions with the composition

$$F_1(t) = A_1 t + (A_1 - 1)c,$$

then follows a function $\phi(t) = \alpha t + (\alpha - 1)\gamma$, and finally the permutation ends with some commuting functions with the composition

$$F_2(t) = A_2 t + (A_2 - 1)c.$$

(But we must remember, that it can be $F_1(t) = 1$ or $F_2(t) = 1$.) It is easy to compute $A := A_1 A_2$ too and we have

$$F_2 \circ \phi \circ F_1 = A_1 A_2 \alpha t + B,$$

where

$$B = A\alpha c - c + A_2(\alpha - 1)(\gamma - c).$$

Here only A_2 depends on the permutation. The sign of $\alpha - 1$ and $\gamma - c$ are fixed and we can only take A_2 as the maximal or the minimal one of all possible products of the numbers a_i (for commuting functions), including the number 1 (if $A_1 = A$).

3.7. We can get some further results by applying our methods in cases when $n-1$ functions belong to one of the sets investigated in 2.1-2.3, but one function belongs to a set from 2.4-2.6. Only in these cases the formulation of the results becomes too complicated to be of any interest for concrete problems.

G.Engelis. Ekstrema problēmas afinai funkcijai, kas definēta permutāciju kopā.

Anotācija. n dotu afinu funkciju kompozīcija ir funkcija $At+B$, kur B ir atkarīgs no šo funkciju permutācijas. Darbā ir formulētas dažas nepieciešamas īpašības permutācijai, kas dod minimālo B vērtību, un parādīts, ka zināmos speciālgadījumos tas atļauj viennozīmīgi noteikt šo permutāciju.

Г.Энгелис. Экстремальные задачи для аффинной функции, определенной на множестве перестановок.

Аннотация. Композиция n данных аффинных функций является аффинной функцией $At+B$, где B зависит от перестановки функций. В работе указаны некоторые необходимые свойства перестановки, дающей минимальное значение B и показано, что в некоторых специальных случаях эти свойства однозначно определяют эту перестановку.

Department of Mathematics
University of Latvia
Rainis blvd. 19
LV-1586 Riga
LATVIA

Fixed points for non-invariant mappings

Daiga Grundmane

Abstract

In this paper we prove generalizations of Banach and Edelstein fixed point theorems for non-invariant mappings (i.e. mappings with, possibly, different domain and range), by introducing and using two different generalized notions of contractive mappings and a notion of *reflector*, introduced in [5].

AMS subject classification 54H25.

Introduction

In this paper we further explore the notion of *reflector*, introduced in [5]. The largest part of known fixed point theorems works for functions $f : X \to X$, i.e. for functions with equal domain and range. The notion of reflector have turned to be useful to prove the existence of fixed points for mappings f in case, when domain and range of f do not necessarily coincide.

In [5] we succeeded to prove the following result

Theorem 1 (Grundmane, Liepiņš) *Let Y be a complete metric space with distance d, $\emptyset \neq X \subseteq Y$, let X be a closed space, and let $f : X \to Y$ be contractive mapping, such that there exists a reflector $O \subseteq X$. Then there exists a unique fixed point of f.* $\diamond$

A similar generalized theorem for nonexpansive mappings was obtained in the M.Sc. thesis of the author.

Theorem 2 (Grundmane) *Let Y be a metric space with distance d, $\emptyset \neq X \subseteq Y$, let X be a compact space, and let $f : X \to Y$ be strongly nonexpansive mapping, such that there exists a reflector $O \subseteq X$. Then there exists a unique fixed point of f.*$\Diamond$

While it seems that the notion of reflector is a quite adequate tool for extending these two classical results to the case of non-invariant mappings, a question may arise, whether the requirements that mapping f must be contractive or strongly nonexpansive are not too strong. Here we propose two different ways how these requirements could be relaxed.

First, we further explore the idea of A. Liepiņš that the requirements of contractiveness or strong nonexpansiveness can be substituted by the requirement of existence of mappings $A : Y \to PY$ and $\varphi : PY \to \mathbf{R}$ which satisfy some natural properties. It turns out that such generalized notions really are sufficient to obtain generalized versions of Theorems 1 and 2, if we place some additional natural requirements on reflectors.

At the same time, it appears that the proposed existence of mappings $A : Y \to PY$ and $\varphi : PY \to \mathbf{R}$ more likely can be considered not as a direct generalization of contractiveness (or strong nonexpansiveness), but as a generalization of requirements in the form $d(x, f(x)) \leq qd(f(x), f(f(x)))$ (or $d(x, f(x)) < d(f(x), f(f(x)))$). For this reason it also turns out to be insufficient to guarantee the uniqueness of fixed points.

We propose a similar approach that more likely can be considered as a generalization of requirements that mapping must be contractive (strongly nonexpansive). Namely, we consider mappings $A : Y^2 \to PY$ and $\varphi : PY \to \mathbf{R}$ and place some natural restrictions on function φ. Such approach also allows to obtain generalizations of Theorems 1 and 2, besides it allows somewhat relaxed requirements on reflectors (we actually substitute the original reflector property with somewhat similar, but formulated in terms of mapping φ). While such approach also does not guarantee the uniqueness of fixed points, it remains an open problem, whether uniqueness can be achieved by placing some stronger conditions on reflectors.

Main definitions and notation

In general we follow the standard definitions and notation used in mathematical analysis, it can be found, for example, in [3]. Here we only include the definition of *reflector*, originally given in [5].

Definition 1 *Let Y be a metric space with distance d, $\emptyset \neq X \subseteq Y$, and let $f \quad X \to Y$ We call subset O of a metric space X a* reflector *if the following two conditions are satisfied:*

1. $f(O) \subseteq X$, *and*
2. $\forall x \in X$, *if* $f(x) \notin X$, *then there exists* $y \in O$ *such that* $d(x, f(x)) = d(x,y) + d(y, f(x))$.

Main results

Our first two theorems are generalizations of Theorems 1 and 2, obtained by using a generalized requirement of contractiveness (strong nonexpansiveness) proposed by A. Liepiņš.

Theorem 3 *Let Y be a complete metric space with distance d, $\emptyset \neq X \subseteq Y$, let X be a closed space, and let $f \quad X \to Y$ be a mapping, such that there exists a reflector $O \subseteq X$ and mappings $A : Y \to PY$, $\varphi : PY \to \mathbf{R}$ which for some $q \in]0,1[$ for all $x \in X$ satisfy the following properties:*

1. $\varphi(A(f(x))) \leq q\varphi(A(x))$,
2. $d(x, f(x)) \leq \varphi(A(x))$,
3. *if* $f(x) \notin X$, *then exists* $y \in O$ *with* $d(x, f(x)) = d(x,y) + d(y, f(x))$ *and* $\varphi(A(y)) \leq \varphi(A(f(x)))$.

Then there exists $x^* \in X$, such that $f(x^*) = x^*$.$\diamond$

Proof. Let $x \in X$. We recursively construct a sequence $x_0, x_1, x_2, \ldots, x_k, \ldots$ as follows. We take $x_0 = x$. Then, consecutively for all $i = 0, 1, 2, \ldots$, we take $x_{i+1} = f(x_i)$, if $f(x_i) \in X$, and we take $x_{i+1} = y_i$, if $f(x_i) \notin X$. Here $y_i \in O$ and is such that $d(x_i, f(x_i)) = d(x_i, y_i) + d(y_i, f(x_i))$ and $\varphi(A(y_i)) \leq \varphi(A(f(x_i)))$ (such y_i exists due to the theorem conditions, if there are several different choices for y_i, we arbitrarily choose one of them).

Now, we will prove that for all $i \in \mathbf{N}$, we have inequality

$$d(x_{i+1}, x_{i+2}) \leq q\varphi(A(x_i)).$$

We have to consider four different cases, depending on whether $f(x_i) \in X$ and whether $f(x_{i+1}) \in X$.

Case 1. $f(x_i) \in X$ and $f(x_{i+1}) \in X$.
In this case we have

$$d(x_{i+1}, x_{i+2}) = d(x_{i+1}, f(x_{i+1})) \leq \varphi(A(x_{i+1})) = \varphi(A(f(x_i))) \leq q\varphi(A(x_i)).$$

Hence, $d(x_{i+1}, x_{i+2}) \leq q\varphi(A(x_i))$.

Case 2. $f(x_i) \in X$ and $f(x_{i+1}) \notin X$.
In this case $d(x_{i+1}, f(x_{i+1})) = d(x_{i+1}, x_{i+2}) + d(x_{i+2}, f(x_{i+1}))$. Thus, $d(x_{i+1}, x_{i+2}) \leq d(x_{i+1}, f(x_{i+1}))$, and, similarly as in Case 1 we have

$$d(x_{i+1}, f(x_{i+1})) \leq \varphi(A(x_{i+1})) = \varphi(A(f(x_i))) \leq q\varphi(A(x_i)),$$

therefore $d(x_{i+1}, x_{i+2}) \leq q\varphi(A(x_i))$.

Case 3. $f(x_i) \notin X$ and $f(x_{i+1}) \notin X$.
If $f(x_i) \notin X$, then, by definition, $x_{i+1} \in O$, thus $f(x_{i+1}) \in X$. Therefore, Case 3 is not possible - there can not be two consecutive x_i, x_{i+1} with $f(x_i) \notin X$ and $f(x_{i+1}) \notin X$.

Case 4. $f(x_i) \notin X$ and $f(x_{i+1}) \in X$.
In this case we have

$$d(x_{i+1}, x_{i+2}) \leq \varphi(A(x_{i+1})) \leq \varphi(A(f(x_i))) \leq q\varphi(A(x_i)).$$

Hence, $d(x_{i+1}, x_{i+2}) \leq q\varphi(A(x_i))$.

Thus, in all cases we have $d(x_{i+1}, x_{i+2}) \leq q\varphi(A(x_i))$. By induction we can show that for all $i \in \mathbf{N}$ we have $\varphi(A(x_i)) \leq q^i\varphi(A(x_0))$, i.e. that $\{x_i\}_{i\in\mathbf{N}}$ is a Cauchy sequence. Since Y is complete and X is closed, the sequence $\{x_i\}_{i\in\mathbf{N}}$ converges to $x^* \in X$. From here it easily follows that $f(x^*) = x^*$.◇

Theorem 4 *Let Y be a metric space with distance d, $\emptyset \neq X \subseteq Y$, let X be a compact space, and let $f : X \to Y$ be a mapping, such that there exists a reflector $O \subseteq X$ and mappings $A : Y \to PY$, $\varphi : PY \to \mathbf{R}$ which for all $x \in X$ satisfy the following properties:*

1. $\varphi(A(f(x))) < \varphi(A(x))$,
2. $d(x, f(x)) \leq \varphi(A(x))$,
3. *if $f(x) \notin X$, then exists $y \in O$ with $d(x, f(x)) = d(x, y) + d(y, f(x))$ and $\varphi(A(y)) \leq \varphi(A(f(x)))$,*
4. $\forall \varepsilon \in \mathbf{R}_+ \exists \delta \in \mathbf{R}_+ \quad \forall y \in X \; d(x,y) < \delta \Longrightarrow | \varphi(A(x)) - \varphi(A(y)) | < \varepsilon$

Then there exists $x^* \in X$, such that $f(x^*) = x^*$.$\diamond$

Proof. Since mapping φ is continuous (requirement 4) and X is compact there exists $x^* \in X$, such that for all $y \in X$ we have $\varphi(A(x^*)) \leq \varphi(A(y))$. We will show that necessarily $f(x^*) = x^*$

Let assume that $f(x^*) = z \neq x^*$ If $z \in X$, then, due to the requirement 1, $\varphi(A(z)) < \varphi(A(x^*))$, which contradicts the choice of x^*. If $z \notin X$, then there exists $y \in O$ with $d(x^*, z) = d(x^*, y) + d(y, z)$ and $\varphi(A(y)) \leq \varphi(A(z))$. Hence, $\varphi(A(y)) \leq \varphi(A(z)) < \varphi(A(x^*))$, which again contradicts the choice of x^*. Therefore $f(x^*) = x^*$.$\diamond$

There are examples that show that the requirements 3 in both theorems are essential simply the existence of reflector $O \subseteq X$ is not sufficient to guarantee the existence of fixed point, we must place some conditions that relates the values of φ for $f(x) \notin X$ with the values of φ for corresponding points from reflector. At the same time, probably it is possible to substitute our requirement 3 with some modified version of it.

The requirement 4 in Theorem 4 guarantees that mapping φ is continuous, it is also clear that some form of such requirement is necessary for the result to hold.

The next two results are similar generalizations of Theorems 1 and 2. However, here we use another generalized version of requirement of contractiveness (strong nonexpansiveness), which, by our opinion, more precisely describes the situation. In this case we can also relax the requirements on reflector (at least for Theorem 5), and more conveniently to formulate them in terms of mapping φ.

Theorem 5 *Let Y be a complete metric space with distance d, $\emptyset \neq X \subseteq Y$, let X be a closed space, and let $f : X \to Y$ be a mapping, such that there exists a set $O \subseteq X$, with $f(O) \subseteq X$ and mappings $A : Y^2 \to PY$, $\varphi : PY \to \mathbf{R}$ which for some $q \in]0,1[$ for all $x, y \in X$ satisfy the following properties:*

1. $\varphi(A(f(x), f(y))) \leq q\varphi(A(x,y))$,
2. $d(x,y) \leq \varphi(A(x,y))$,
3. *if* $f(x) \notin X$, *then exists* $z \in O$ *with* $d(z, f(x)) + \varphi(A(x,z)) \leq \varphi(A(x, f(x)))$.

Then there exists $x^* \in X$, such that $f(x^*) = x^*$. $\diamond$

<u>**Proof.**</u> Let $x \in X$. Similarly as in proof of Theorem 3 we construct a sequence $\{x_i\}_{i \in \mathbf{N}}$ and prove that for all $i \in \mathbf{N}$ we have inequality

$$d(x_{i+1}, x_{i+2}) \leq q\varphi(A(x_i, x_{i+1})).$$

We have to consider four different cases, depending on whether $f(x_i) \in X$ and whether $f(x_{i+1}) \in X$.

<u>*Case 1.*</u> $f(x_i) \in X$ and $f(x_{i+1}) \in X$.
In this case we have

$$d(x_{i+1}, x_{i+2}) \leq \varphi(A(x_{i+1}, x_{i+2})) = \varphi(A(f(x_i), f(x_{i+1}))) \leq q\varphi(A(x_i, x_{i+1})).$$

Hence, $d(x_{i+1}, x_{i+2}) \leq q\varphi(A(x_i, x_{i+1}))$.

<u>*Case 2.*</u> $f(x_i) \in X$ and $f(x_{i+1}) \notin X$.
In this case $\varphi(A(x_{i+1}, x_{i+2})) + d(x_{i+2}, f(x_{i+1})) \leq \varphi(A(x_{i+1}, f(x_{i+1})))$.
Hence, $d(x_{i+1}, x_{i+2}) \leq \varphi(A(x_{i+1}, x_{i+2})) \leq \varphi(A(x_{i+1}, f(x_{i+1}))) \leq q\varphi(A(x_i, x_{i+1}))$.
<u>*Case 3.*</u> $f(x_i) \notin X$ and $f(x_{i+1}) \notin X$.
Similarly, as in proof of Theorem 3 we conclude that such case is not possible.

<u>*Case 4.*</u> $f(x_i) \notin X$ and $f(x_{i+1}) \in X$.
From the existence of reflector we have that

$d(x_{i+1}, x_{i+2}) \leq d(x_{i+1}, f(x_i)) + d(f(x_i), x_{i+2}) \leq d(x_{i+1}, f(x_i)) + \varphi(A(f(x_i), f(x_{i+1}))) \leq d(x_{i+1}, f(x_i)) + q\varphi(A(x_i, x_{i+1})) \leq \varphi(A(x_i, x_{i+1}))$.

Hence, $d(x_{i+1}, x_{i+2}) \leq q\varphi(A(x_i, x_{i+1}))$.

Thus, in all cases we have $d(x_{i+1}, x_{i+2}) \leq q\varphi(A(x_i))$. Again, by induction we can show that for all $i \in \mathbf{N}$ we have $\varphi(A(x_i)) \leq q^i\varphi(A(x_0))$, i.e. that $\{x_i\}_{i\in\mathbf{N}}$ is a Cauchy sequence. Since Y is complete and X is closed, the sequence $\{x_i\}_{i\in\mathbf{N}}$ converges to $x^* \in X$, from where $f(x^*) = x^*$.◇

Theorem 6 *Let Y be a metric space with distance d, $\emptyset \neq X \subseteq Y$, let X be a compact space, and let $f : X \to Y$ be a mapping, such that there exists a set $O \subseteq X$ with $f(O) \subseteq X$ and mappings $A \quad Y^2 \to PY$, $\varphi \quad PY \to \mathbf{R}$ which for all $x, y \in X$ satisfy the following properties:*

1. $\varphi(A(f(x), f(y))) < \varphi(A(x, y))$,

2. $d(x, y) \leq \varphi(A(x, y))$,

3. *if $f(x) \notin X$, then exists $z \in O$ with $d(z, f(x)) + \varphi(A(x, z)) \leq \varphi(A(x, f(x)))$ and $\varphi(A(z, f(z))) < \varphi(A(x, z))$,*

4. $\forall \varepsilon \in \mathbf{R}_+ \exists \delta \in \mathbf{R}_+ \quad \forall v, w \in X \; d(x, v) \;\&\; d(y, w) < \delta \Longrightarrow | \varphi(A(x, y)) - \varphi(A(v, w)) | < \varepsilon$

Then there exists $x^* \in X$, such that $f(x^*) = x^*$.◇

__Proof.__ Let us define a function $g \quad X \to X$ by equalities $g(x) = f(x)$, if $f(x) \in X$ and $g(x) = z$, if $f(x) \notin X$ (where $z \in O$ is a point, which exists due to the requirement 3).

Similarly, as in proof of Theorem 4, we conclude that there exists $x^*, \in X$, such that for all $y \in X$ we have $\varphi(A(x^*, g(x^*))) \leq \varphi(A(y, g(y)))$. We will show that necessarily $f(x^*) = x^*$.

Let assume that $f(x^*) = z \neq x^*$. If $z \in X$, then $\varphi(A(z, g(z))) = \varphi(A(z, f(z)) < \varphi(A(x^*, f(x^*))) = \varphi(A(x^*, g(x^*)))$, which contradicts the choice of x^*. If $z \notin X$, then $g(x^*) = z' \in X$, such that $\varphi(A(z', g(z'))) = \varphi(A(z', f(z'))) < \varphi(A(x^*, z')) = \varphi(A(x^*, g(x^*)))$, which again contradicts the choice of x. Therefore $f(x^*) = x^*$.◇

References

[1] W.A. Kirk. An abstract fixed point theorem for nonexpansive mappings. *Proceedings of American Mathematical Society*, 82:640 642, 1981.

[2] W.A Kirk. Fixed point theory for nonexpansive mappings. *Lecture Notes in Mathematics*, 886:484–505, 1981.

[3] A. Kolmogorov and S. Fomin. *Elements of the theory of functions and functional analysis.* Nauka, 1989.

[4] A. Liepiņš. A cradle-song for a little tiger on fixed points. *Topological spaces and their mappings*, pages 61–69, 1983.

[5] D. Zabarovska and A. Liepiņš. Mirrors and existence of fixed points for non-invariant mappings. *Topological spaces and their mappings*, pages 88-92, 1989.

Daiga Grundmane. Neinvariantu attēlojumu nekustīgie punkti

Anotācija

Dotajā darbā tiek definēti divi jauni saspiedējattēlojumu vispārinājumi. Šādiem vispārinātiem saspiedējattēlojumiem, izmantojot darbā [5] ieviesto spoguļa jēdzienu, tiek pierādīti Ēdelšteina un Banaha teorēmu vispārinājumi neinvariantiem attēlojumiem (t.i., attēlojumiem, kuriem definīcijas un vērtību kopas var būt dažādas).

Faculty of Physics and Mathematics
The University of Latvia
Rainis boulevard 19
Rīga, LV 1586
Latvia
email: `gdaiga@lanet.lv`

Latvijas Universitātes Zinātniskie Raksti, 606 (1997). Matemātika

CONSTRUCTION OF ALL EQUIVARIANT REALCOMPACT EXTENSIONS BY MEANS OF NETS

V.G. Jevstigneyev

Abstract *All equivariant realcompact extensions of a Tychonoff G-space are constructed by means of nets.*
AMS Subject Classification: 54D60

In [1] we developed a method allowing to construct all compact extensions of a Tychonoff space by means of nets. In [2] a similar method was applied to construct all realcompact extensions. Further, in [3] nets were used for construction of equivariant compactifications. Therefore, it seems natural to consider the possibility to use nets for construction of equivariant realcompact extensions of a Tychonoff space X on which a group G continuously acts.

Consider a class of functions $\emptyset \neq S^* \subset C^*$ where C^* is the family of all real-valued functions on a Tychonoff space X. A net $\{x_\alpha\} \subset X$ will be called an S^*-net iff for every $f \in S^*$ there exists $\lim f(x_\alpha)$. In [2] it is proved that a Tychonoff space is realcompact iff every C^*-net in it converges. It follows from here that a space is realcompact iff every S^*-net in it contains a convergent subnet.

Given a topological group G, a space X is called a G-space if G is continuously acting on X, i.e. if there exists a continuous mapping $\varphi : G \times X \to X$ such that $\varphi(e,x) = x$, $\varphi(h, \varphi(g,x)) = \varphi(h \cdot g, x)$, where $x \in X$, $h, g \in G$ and e is the unity of the group G. We shall say that Y is an equivariant realcompact extension, (or an equivariant realcompactification) of a space X, if Y is a realcompact space containing X as a dense subset and the given mapping $\varphi : G \times X \to X$ can be continuously extended to a mapping $\tilde{\varphi} : G \times Y \to Y$ with the above mentioned properties.

Theorem 1 *Every equivariant realcompact extension Y of a G-space X can be constructed by means of nets.*

Proof Consider a space Y_{S^*}, whose points are S^*-nets in the space X where $S^* = \{f \mid X \; f \in C^*(Y)\}$. Two S^*N-nets $\{x_\alpha\}$ and $\{x_\beta\}$ will be identified, if for every $f \in S^*$ it holds $\lim f(x_\alpha) = \lim f(x_\beta)$. Let Y_{S^*} be endowed with a topology such that a net $\{y_\alpha\} \subset Y_{S^*}$ converges to a point $y \in Y_{S^*}$ if and only if there exists $\lim_\alpha \lim_\beta f(x_{\alpha,\beta})$ and if moreover $\lim_\alpha \lim_\beta f(x_{\alpha,\beta}) = \lim_\gamma f(x_\gamma)$ where $y_\alpha = \{x_{\alpha,\beta}\}$, $y = \{x_\gamma\}$ are S^*-nets. We shall show that the equality $Y = Y_{S^*}$ holds.

If $y = \{x_\alpha\}$ is an S^*-net, then for every $f \in C^*(Y)$ there exists $\lim f(x_\alpha)$ and, according to the characteristic property of real compact spaces, $\{x_\alpha\}$ converges in Y to a point y. The mapping $\{x_\alpha\} \to y$ is a homeomorphism from Y_{S^*} onto Y, which is identical on points of X.

We define an action of a group G on a space Y_{S^*} by setting $g \cdot y = \{g(x_\alpha)\}$ for a point $y = \{x_\alpha\} \in Y_{S^*}$. Since Y is equivariant, it follows that a mapping $y \to gy$, $y \in Y$, is continuous for any fixed $g \in G$. Hence, gx_α converges to $g \cdot y$ in Y and for any $f \in S^*$ there exists $\lim_\alpha f \cdot g(x_\alpha) = f \cdot gy$. Therefore, $\{gx_\alpha\}$ is an S^*-net. If $\{x_\alpha\}$ and $\{x_\beta\}$ are equivalent S^*-nets, then for every $f \in S^*$ the composition $f \circ g$ belongs to S^*. Hence, $\lim_\alpha f(g(x_\alpha)) = \lim_\beta f(g(x_\beta))$ and the nets $\{gx_\alpha\}$ and $\{gx_\beta\}$ are also S^*- equivalent. This completes the proof of the theorem.

References

1. В.Г. Евстигнеев, Бикомпактность и меры, *Функциональный Анализ*, 4 , N 3 (1970) 51-60.
2. А.П. Шостак, Е-компактные расширения топологических пространств, *Функциональный Анализ*, 8 , N 1 (1974) 62-68.
3. В.Г. Евстигнеев, квивариантные бикомпактные расширения и функциональные направленности, *Топологические Пространства и Отображения*, изд. ЛГУ, Рига, 1989, 62-65.

V. Jevstignejevs: Visu ekvivariantu reālkompaktu paplašinājumu konstruešana ar vispārinātām virknēm.

Anotācija. Tiek izstrādāta metode, kas ļauj konstruēt dotajai G-telpai X visus ekvivariantus paplašinājumus ar vispārinātām virknēm.

В.Г. Евстигнеев: Построение всех эквивариантных вещественно-компактных расширений методом направленностей

Аннотация.. Предлагается метод построения всех вещественно компактных расширений данного пространства с помощью направленностей.

Chair of Mathematics
State Academy of Management (S. Ordzhenikidze GAU)
Ryazanski Prospekt 99
Moscow 109542
RUSSIA

ISOGEOMETRIC INTERPOLATION BY RATIONAL CUBIC SPLINES

Sergey Krivosheyev

Summary. The problem of interpolation by cubic rational splines with preserving of such geometrical properties as monotonicity and convexity is considered in this paper. We obtain sufficient conditions on parametrs of spline to provide the isogeometric interpolation. The received results are tested on examples.

AMS Subject Classification: 65D07

0. Introduction.

In many practical situations it is necessary to approximate a function f with given values

$$f_i = f(x_i) \tag{1}$$

at interpolating points x_i of a grid $\Delta: a = x_0 < x_1 < \ldots < x_n = b$ by splines with a similar geometric structure (see e. g. [2], [4], [5], [8], [9], [11], [12], [13], [14]). We denote by

$$\Delta_i = \frac{f_{i+1} - f_i}{h_i}, \quad i = \overline{0, n-1}$$

$$\delta_i = \frac{\Delta_i - \Delta_{i-1}}{h_{i-1} + h_i} \qquad i = \overline{1, n-1}$$

the first and second divided differences (here $h_i = x_{i+1} - x_i$, $i = \overline{0, n-1}$). The initial data are called increasing, if $\Delta_i > 0$, and decreasing, if $\Delta_i > 0$ for

$i=\overline{0,n-1}$. Increasing or decreasing data are called monotone. The initial data (1) are called convex, if $\delta_i < 0$, and concave, if $\delta_i > 0$, for $i=\overline{1,n-1}$.

The problem of isogeometric interpolation consists of construction of an interpolational spline, which preserves monotonicity and convexity of initial data. It is known (see [6], [7], [10]), that rational splines offer good possibilities for the solution of this problem. The present work is devoted to the study of geometrical properties of rational cubic splines, defined in [1]. The authors of [1] have introduced into a structure of a cubic spline parameters to operate the qualitative behaviour of the received curve. The purpose of the present paper is to obtain sufficient conditions on parameters of rational cubic splines from [1] to provide the isogeometric interpolation.

1. About rational cubic splines

By a rational cubic spline is called a function $s \in C^2[a,b]$, which on each interval $[x_i, x_{i+1}]$ has a representation

$$s(t) = A_i t + B_i(1-t) + \frac{C_i t^3}{1+p_i(1-t)} + \frac{D_i(1-t)^3}{1+p_i t} \tag{2}$$

where $t = \dfrac{x-x_i}{h_i}$ and $-1 < p_i$, $i=\overline{1,n}$, are given numbers.

The rational cubic spline is an interpolation for a function $f:[a,b]\to R$, if

$$s(x_i) = f_i\ ,\ i=\overline{0,n},\ \text{where}\ f_i = f(x_i). \tag{3}$$

To define an interpolation rational cubic spline uniquely we consider the following boundary condition:

$$s'(x_k) = f_k' \qquad k = 0,n. \tag{4}$$

From the interpolation conditions (3) it follows, that $B_i + D_i = f_i$, $A_i + C_i = f_{i+1}$. Then

$$s(t) = f_i(1-t) + f_{i+1}t + C_i\left(\frac{t^3}{1+p_i(1-t)} - t\right) + D_i\left(\frac{(1-t)^3}{1+p_i t} - (1-t)\right).$$

In [1] the construction of a rational cubic spline is reduced to calculation of the values of the first order derivative at points x_i. Let us design $m_i = s'(x_i)$, then

$$C_i = \frac{-(3+p_i)(f_{i+1}-f_i) + h_i m_i + (2+p_i)h_i m_{i+1}}{(2+p_i)^2-1},$$
$$D_i = \frac{(3+p_i)(f_{i+1}-f_i) - h_i m_{i+1} - (2+p_i)h_i m_i}{(2+p_i)^2-1} \tag{5}$$

For the calculation of m_i in [1] the system:

$$\begin{cases} m_0 = f_0' \\ \lambda_i P_{i-1} m_{i-1} + (\lambda_i P_{i-1}(2+p_{i-1}) + \mu_i P_i(2+p_i))m_i + \mu_i P_i m_{i+1} = \\ = \lambda_i P_{i-1}(3+p_{i-1})\Delta_{i-1} + \mu_i P_i(3+p_i)\Delta_i \\ m_N = f_N' \end{cases} \quad i = \overline{1, n-1}, \tag{6}$$

is given where $\lambda_i = h_i(h_{i-1}+h_i)^{-1}$ $\quad \mu_i = 1 - \lambda_i$ $\quad P_i = \dfrac{3+3p_i+p_i^2}{(2+p_i)^2-1}$

By analogy, using $M_i = s''(x_i)$, we find

$$D_i = \frac{M_i h_i^2}{2(p_i^2+3p_i+3)}, \quad C_i = \frac{M_{i-1} h_i^2}{2(p_i^2+3p_i+3)} \tag{7}$$

For the calculation of M_i we get the system

$$\begin{cases} (2+p_0)Q_0 M_0 + Q_0 M_1 = 2\dfrac{\Delta_0 - f_0'}{h_0} \\ \mu_i Q_{i-1} M_{i-1} + (\lambda_i Q_i(2+p_i) + \mu_i Q_{i-1}(2+p_{i-1}))M_i + \lambda_i Q_i M_{i+1} = 2\delta_i, \quad i = \overline{1, n-1}, \\ Q_{N-1} M_{N-1} + (2+p_{N-1})Q_{N-1} M_N = 2\dfrac{f_N' - \Delta_{N-1}}{h_{N-1}} \end{cases} \tag{8}$$

where $Q_i = \dfrac{1}{p_i^2+3p_i+3}$.

Existence and uniqueness of the solutions of the systems (6) and (8), are provided by the dominant main diagonal of the matrixes of those systems.

2. About the values of the derivatives of spline at the points of interpolation.

The study of the signs of the first and second order derivatives of an interpolation spline is reduced to the analysis of the sings of the derivatives at the interpolating points. This analysis is based on the following result [10].

Lemma 1. *Let the coefficients of a system of linear algebraic equations*

$$\begin{cases} a_0 z_0 + b_0 z_1 = d_0 \\ c_i z_{i-1} + a_i z_i + b_i z_{i+1} = d_i, \quad i = \overline{1, n-1} \\ c_n z_{n-1} + a_n z_n = d_n \end{cases} \tag{9}$$

satisfy the conditions

$a_i > 0$, $i = \overline{0, n}$; $c_i \geq 0$, $b_i \geq 0$, $a_i > b_i + c_i$, $i = \overline{1, n-1}$,

$b_0 < \dfrac{a_0 a_1}{c_1 + b_1}$, $c_n < \dfrac{a_{n-1} a_n}{c_{n-1} + b_{n-1}}$

If $\quad d_i - \dfrac{b_i d_{i+1}}{a_{i+1}} - \dfrac{c_i d_{i-1}}{a_{i-1}} \geq 0, \quad i = \overline{0, n}$, $\quad$ (10)

(here $c_0 = b_n = d_{-1} = d_{n+1} = 0$; $a_{-1} = a_{n+1} = 1$), then the system (9) is solvable and $z_i \ge 0$, $i = \overline{0,n}$.

Similarly, if $d_i - \frac{b_i d_{i+1}}{a_{i+1}} - \frac{c_i d_{i-1}}{a_{i-1}} \le 0$, $i = \overline{0,n}$ *then the system (9) is solvable and* $z_i \le 0$, *for all* $i = \overline{0,n}$

The following theorems contain the conditions on parameter p_i sufficient to preserve a sign of the derivatives of an interpolation spline at points.

Theorem 1. *Let a rational cubic spline* s *(2), interpolates growing (decreasing) data and satisfy the boundary conditions (4) with* $f_0', f_n' > 0$ $(f_0', f_n' < 0)$. *If*

$$p_i \ge \max\left\{\frac{\Delta_{i+1}}{\Delta_i} - 1, \frac{\Delta_{i-1}}{\Delta_i} - 1\right\},\ i = \overline{1,n-2},\quad p_0 \ge \frac{f_0'}{\Delta_0} - 3,\quad p_{n-1} \ge \frac{f_n'}{\Delta_{n-1}} - 3 \tag{11}$$

then $m_i \ge 0, (m_i \le 0), i = \overline{1,n-1}$.

Proof. For the proof we consider the case of growing data. By Lemma 1 the solution m_i, $i = \overline{0,n}$, of the system (6) is nonnegative, if for coefficients of this system the inequality (10) is true, that is

$$\lambda_i P_{i-1}(3+p_{i-1})\Delta_{i-1} + \mu_i P_i(3+p_i)\Delta_i - \frac{\mu_i P_i(\lambda_{i+1} P_i(3+p_i)\Delta_i + \mu_{i+1}P_{i+1}(3+p_{i+1})\Delta_{i+1})}{\lambda_{i+1}P_i(2+p_i) + \mu_{i+1}P_{i+1}(2+p_{i+1})} -$$
$$- \frac{\lambda_i P_{i-1}(\lambda_{i-1}P_{i-2}(3+p_{i-2})\Delta_{i-2} + \mu_{i-1}P_{i-1}(3+p_{i-1})\Delta_{i-1})}{\lambda_{i-1}P_{i-2}(2+p_{i-2}) + \mu_{i-1}P_{i-1}(2+p_{i-1})} \ge 0$$

We transform the last inequality as follows:

$$\mu_i P_i((3+p_i)\Delta_i - \frac{\lambda_{i+1}P_i(3+p_i)\Delta_i}{\lambda_{i+1}P_i(2+p_i) + \mu_{i+1}P_{i+1}(2+p_{i+1})} - \frac{\mu_{i+1}P_{i+1}(3+p_{i+1})\Delta_{i+1}}{\lambda_{i+1}P_i(2+p_i) + \mu_{i+1}P_{i+1}(2+p_{i+1})}) +$$
$$+\lambda_i P_{i-1}((3+p_{i-1})\Delta_{i-1} - \frac{\lambda_{i-1}P_{i-2}(3+p_{i-2})\Delta_{i-2}}{\lambda_{i-1}P_{i-2}(2+p_{i-2}) + \mu_{i-1}P_{i-1}(2+p_{i-1})} - \frac{\mu_{i-1}P_{i-1}(3+p_{i-1})\Delta_{i-1})}{\lambda_{i-1}P_{i-2}(2+p_{i-2}) + \mu_{i-1}P_{i-1}(2+p_{i-1})}) \ge 0$$

Because of

$P_i > 0$ and $\frac{\lambda_{i+1}P_i(3+p_i)\Delta_i}{\lambda_{i+1}P_i(2+p_i) + \mu_{i+1}P_{i-1}(2+p_{i+1})} \le \frac{3+p_i}{2+p_i}\Delta_i$,

it is enough to require

$$\begin{cases} (3+p_i)\Delta_i \ge \frac{3+p_i}{2+p_i}\Delta_i + \frac{3+p_{i+1}}{2+p_{i+1}}\Delta_{i+1} \\ (3+p_{i-1})\Delta_{i-1} \ge \frac{3+p_{i-2}}{2+p_{i-2}}\Delta_{i-2} + \frac{3+p_{i-1}}{2+p_{i-1}}\Delta_{i-1} \end{cases}$$

Taking into account, that $\frac{3+p_i}{2+p_i} \le 2$ for $p_i > -1$, these inequalities will be valid if

$$\begin{cases} (3+p_i)\Delta_i \geq 2(\Delta_i + \Delta_{i+1}) \\ (3+p_{i-1})\Delta_{i-1} \geq 2(\Delta_{i-2} + \Delta_{i-1}) \end{cases}$$

or (in equivalent form)

$$\begin{cases} p_i \geq \dfrac{\Delta_{i+1}}{\Delta_i} - 1 \\ p_i \geq \dfrac{\Delta_{i-2}}{\Delta_{i-1}} - 1 \end{cases}$$

Thus, by the conditions

$$p_i \geq \max\left\{\frac{\Delta_{i+1}}{\Delta_i} - 1, \frac{\Delta_{i-1}}{\Delta_i} - 1\right\}, \ i = \overline{1, n-2}, \quad p_0 \geq \frac{f_0'}{\Delta_0} - 3, \ p_{n-1} \geq \frac{f_n'}{\Delta_{n-1}} - 3$$

we guarantee that $m_i \geq 0$, for all $i = \overline{0, n}$. □

<u>**Theorem 2**</u>. *Let a rational cubic spline s (2) interpolates convex downwards (upwards) data (3) and satisfys the boundary conditions (4). If*

$$p_i \geq \max\left\{\frac{\delta_{i+1}}{\mu_{i+1}\delta_i} - 2, \frac{\delta_i}{\lambda_i\delta_{i+1}} - 2\right\}, \quad i = \overline{1, n-2}$$

$$p_0 \geq \frac{4(\Delta_0 - f_0')}{h_0\delta_1} - 2, \ p_{n-1} \geq \frac{4(f_n' - \Delta_{n-1})}{h_{n-1}\delta_{n-1}} - 2, \tag{12}$$

then $M_i \geq 0$ ($M_i \leq 0$), for all $i = \overline{0, n}$

<u>**Proof.**</u> In the proof we consider convex downwards data. By Lemma 1 the solution M_i, $i = \overline{0, n}$ of the system (8) is nonnegative, if the coefficients of this system satisfy the condition (10), that is

$$2\delta_i - \frac{2\lambda_i Q_i \delta_{i+1}}{\lambda_{i+1}(2+p_{i+1})Q_{i+1} + \mu_{i+1}(2+p_i)Q_i} - \frac{2\mu_i Q_{i-1}\delta_{i-1}}{\lambda_{i-1}(2+p_{i-1})Q_{i-1} + \mu_{i-1}(2+p_{i-2})Q_{i-2}} \geq 0.$$

Taking into account, that $\lambda_i + \mu_i = 1$, we transform the last inequality

$$\lambda_i(\delta_i - \frac{Q_i\delta_{i+1}}{\lambda_{i+1}(2+p_{i+1})Q_{i+1} + \mu_{i+1}(2+p_i)Q_i}) + \mu_i(\delta_i - \frac{Q_{i-1}\delta_{i-1}}{\lambda_{i-1}(2+p_{i-1})Q_{i-1} + \mu_{i-1}(2+p_{i-2})Q_{i-2}}) \geq 0$$

Because of

$$Q_i > 0 \quad \text{and} \quad \frac{Q_i\delta_{i+1}}{\lambda_{i+1}(2+p_{i+1})Q_{i+1} + \mu_{i+1}(2+p_i)Q_i} \leq \frac{2\delta_{i+1}}{2+p_i},$$

it is enough to require

$$\begin{cases} \delta_i \geq \dfrac{\delta_{i+1}}{\mu_{i+1}(2+p_i)} \\ \delta_i \geq \dfrac{\delta_{i-1}}{\lambda_{i-1}(2+p_{i-1})} \end{cases}$$

Those inequalities will be executed, if

$$p_i \geq \max\left\{\frac{\delta_{i+1}}{\mu_{i+1}\delta_i} - 2, \frac{\delta_i}{\lambda_i\delta_{i+1}} - 2\right\}, \quad i = \overline{1, n-2}$$

$$p_0 \geq \frac{4(\Delta_0 - f_0')}{h_0\delta_1} - 2, \ p_{n-1} \geq \frac{4(f_n' - \Delta_{n-1})}{h_{n-1}\delta_{n-1}} - 2 \quad □$$

3. Monotony and convexity of interpolational rational splines.

Theorem 3. *For a rational spline s (2)-(3)-(4) we have:*

1) if $m_i \geq 0$ and $M_i \geq 0$, the spline s is increasing and concave;

2) if $m_i \leq 0$ and $M_i \geq 0$, the spline s is decreasing and concave;

3) if $m_i \geq 0$ and $M_i \leq 0$, the spline s is increasing and convex;

4) if $m_i \leq 0$ and $M_i \leq 0$, the spline s is decreasing and convex ;

Proof. We prove only the first statement. If $M_i \geq 0$, then $C_i \geq 0$ and $D_i \geq 0$ by virtue of (7). Taking into account, that for $t \in [0,1]$

$2p_i^2 t^3 - 6p_i(1+p_i)t + 6(1+p_i)^2 \geq 0$,

$2p_i^2(1-t)^3 - 6p_i(1+p_i)(1-t) + 6(1+p_i)^2 \geq 0$

we receive $s'' \geq 0$. It means, that the derivative s' increases. Taking into account also that $m_i \geq 0$ we conclude that the spline s is increasing and concave.

The following final result (Theorem 4) is a corollary of the proven Thoorems 1-3.

Theorem 4: *Let a rational spline s (2)-(3)-(4) interpolates the increasing (decreasing) and convex (concave) data. If the parameters p_i satisfy the conditions (11) and (12), then the spline s is increasing (decreasing) and convex (concave), too.*

4. Examples.

We consider two test examples:

1) $f_1(x) = 1 - \dfrac{e^{50x} - e^{-50x}}{e^{50} - e^{-50}}$,

2) $f_2(x) = 1 - \sqrt{x(2-x)}$

The interpolation cubic spline δ_k and the cubic rational spline s_k for the function f_k, $k = 1,2$ are constructed. The graphs of them you can see at the figures 1 and 2 correspondingly:

a) the graph of the function f_k;

b) the graph of the cubic spline δ_k;

c) the graph of the rational cubic spline s_k.

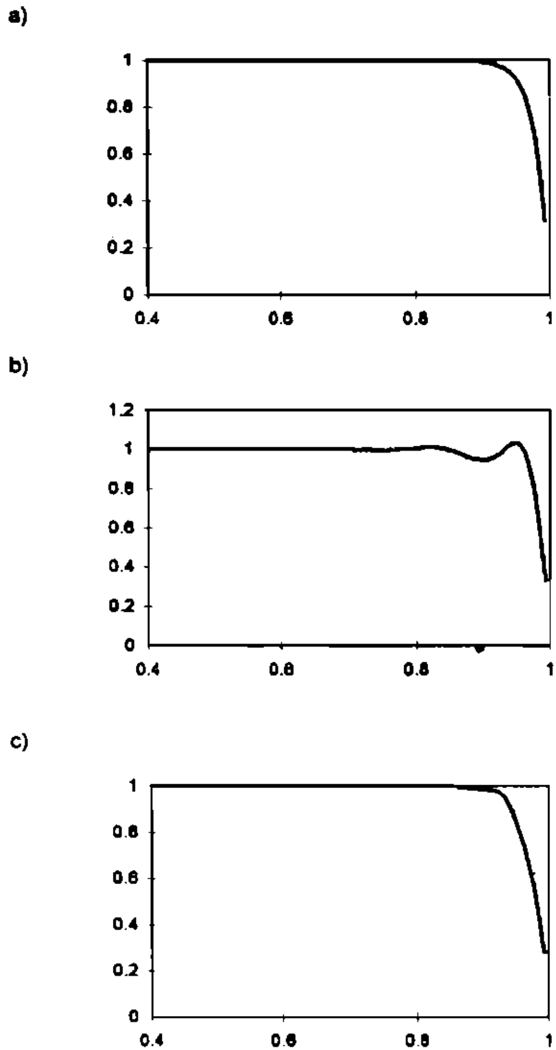
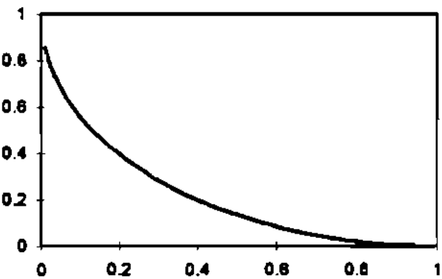
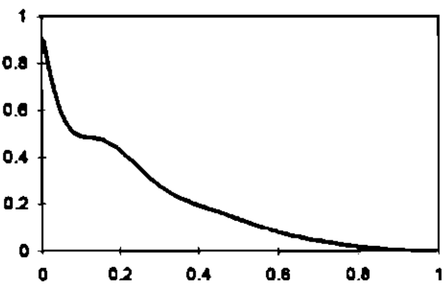


Fig.1 $f_1(x) = 1 - \dfrac{e^{50x} - e^{-50x}}{e^{50} - e^{-50}}$ [a, b]=[0.4; 1], n=8.

a)



b)



c)

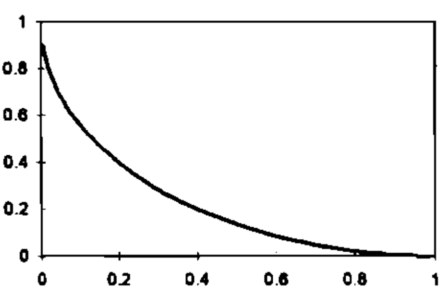


Fig. 2. $f_2(x) = 1 - \sqrt{x(2-x)}$, [a, b]=[0.005; 1], n=7.

Acknowledgement. The author would like to thank his advisor Dr. Math. S. Asmuss for her interest and guidance.

References.

1) Завьялов Ю. С.,Квасов Б. И., Мирошниченко В. Л. Методы сплайн-функции-М., 1980.

2)Мирошниченко В. Л. Интерполяция функций с большими градиентами //Методы аппроксимации и интерполяции. —Новосибирск, 1980. —С.98-107.

3) Beatson R. K. Monotone and convex approximation by splines, error estimates and a curve fitting algorithm // SIAM J. Numer. Anal.—1982. —V.19, #6. —P.1278-1285.

4) Pagallo G., Pereya V. Smooth monotone spline interpolation // Numer. Anal.—1982.—V.909.—P 142-147

5) Schumaker L. On shape preserving quadratic spline interpolation // SIAM J. Numer. Anal.—1983.— V.20—P 854-864.

6) Квасов Б.И. Интерполяция рациональными параболическими сплайнами// Численные методы механики сплошной среды.— Новосибирск, 1984. — Т. 15, №4. — С.60-70.

7) Квасов Б. И., Яценко С. А. Изогеометрическая интерполяция рациональными сплайнами// Аппроксимация сплайнами—1987. — С. 11-35.

8) Schmidt J., Hess W. Positive interpolation with rational quadratic splines // Computing.—1987.—V.38—P.261-267.

9) Schmidt J., Hess W. Positivity of cubic polynomials on intervals and positive spline interpolation // BIT.—1988.—V.28—P.340-352.

10) Назаренко Н. А. Об изогеометрической интерполяции обобщенными сплайнами// Некоторые проблемы анализа и дифференциальной топологии. — Киев, 1988. — С. 64-69.

11) Morandi R., Costantini P. Piecewise monotone quadratic histosplines // SIAM J. Sci. Statist. Comput. —1989. —V10, #2. —P.397-406.

12) Верлан И. И. Синтез положительных кривых обобщенными кубическими сплайнами// Мат. исслед. — 1990. — №114. — С. 27-32.

13) Lahtinen A. Spape preserving interpolation by quadratic splines // J. Comput. and Appl. Math. —1990. —V.29, #1. —C.15-24.

14) Lahtinen A. Positive Hermite interpolation by quadratic splines // SIAM J. Math.Anal.—1993,V.24—P.223-233.

С. Кривошеев. Изогеометрическая интерполяция рациональными кубическими сплайнами.

Аннотация. В работе рассматривается задача интерполяции рациональными кубическими сплайнами с сохранением геометрических свойств исходных данных. Обоснован выбор параметров, обеспечивающий решение рассматриваемой задачи изогеометрической интерполяции. Предложенный алгоритм реализован на тестовых примерах.

УДК 517.

S. Krivošejevs. Izoģeometriskā interpolācija ar racionāliem kubiskiem splainiem.

Anotācija. Darbā ir apskatīts uzdevums par izejas datu interpolēšanu ar monotonitātes un izliekuma īpašību saglabāšanu. Lai to atrisinātu, tika izmantoti racionālie kubiskie splaini. Darbā ir pamatota splaina parametru izvēle, kas nodrošina izoģeometrisko interpolāciju. Piedāvātais algoritms ir realizēts un izmēģināts testa uzdevumos.

Department of Mathematics
University of Latvia
Rainis blvd., 19
Riga LV-1586
LATVIA

Latvijas Universitātes Zinātniskie Raksti, 606 (1997). Matemātika

On existence of a solution to the boundary value problem for functional-differential equation

V.Ponomarev

Summary. Condition for the existence of a solution to a boundary value problem for functional-differential equation are given.
MSC 34K10

Consider boundary value problems

$$x' = Fx + F_0x, \tag{1}$$

$$Lx = r, \tag{2}$$

$$x' = Fx, \tag{3}$$

$$Lx = 0, \tag{4}$$

where $F, F_0 : AC(I, R^n) \to L(I, R^n)$, $L : AC(I, R^n) \to R^n$, $r \in R^n$, $n \in \{1, 2, \ldots\}$, $I = [a, b]$, $-\infty < a < b < \infty$, $AC(I, R^n)$ the space of absolutely continuous functions $x : I \to R^n$ with a norm

$$\|x\| = | x(a) | + \int_a^b | x'(s) | ds,$$

$L(I, R)$ the space of Lebesque-integrable functions $y : I \to R^n$ with a norm

$$\|y\| = \int_a^b | y(s) | ds,$$

where $| x | = max\{| x_i | : i \in \{1, 2, \ldots, n\}\}$ the norm in R^n.

I. We suppose in the sequel that the BVP has a unique solution, the trivial one.

Solutions of the problem (1), (2); (3), (4) are identical with solutions of the equations

$$x(t) = \int_a^t (Fx)(s)ds + \int_a^t (F_0x)(s)ds + Lx + x(a) - r,$$

$$x(t) = \int_a^t (Fx)(s)ds + Lx + x(a),$$

respectively.

Define the operators $B, B_0 \quad AC(I, R^n) \to AC(I, R^n)$, $A \quad [0, 1] \times AC(I, R^n) \to AC(I, R^n)$ as follows:

$$(Bx)(t) = \int_a^t (Fx)(s)ds + Lx + x(a),$$

$$(B_0x)(t) = \int_a^t (Fx)(s)ds - r,$$

$$A(\lambda, x) = Bx + \lambda B_0 x.$$

The problem (3), (4) can be written as

$$x = Bx,$$

and the problem (1,2) as

$$x = Bx + B_0 x.$$

First we show that there exists $\mu \in (1, \infty)$ such that for any solution v of the equation

$$x = A(\lambda, x), \quad 0 \le \lambda \le 1 \tag{5}$$

an estimate

$$\|v\|_{AC} \le \mu\lambda \|B_0 v\|_{AC} \tag{6}$$

is valid.

Suppose the contrary is true. Then one can find a sequence v_n, $n = 1, 2, \ldots$ of nontrivial solution of the equation (5), and λ_n, $n = 1, 2, \ldots$ such that

$$\|v_n\|_{AC} > n\lambda_n \|B_0 v_n\|_{AC}$$

or

$$\lim_{n\to\infty} \frac{\lambda_n \|B_0 v_n\|_{AC}}{\|v_n\|_{AC}} \le \lim_{n\to\infty} \frac{1}{n} = 0. \tag{7}$$

II. Suppose that the operators B and B_0 are completely continuous, and B is also homogeneous.

The equation (5) implies

$$\frac{v_n}{\|v_n\|_{AC}} = \frac{Bv_n}{\|v_n\|_{AC}} + \frac{\lambda_n B_0 v_n}{\|v_n\|_{AC}},$$

and, letting $v_n^* = \frac{v_n}{\|v_n\|_{AC}}$, $n = 1, 2, \dots$, one obtains

$$v_n^* = Bv_n^* + \frac{\lambda_n B_0 v_n}{\|v_n\|_{AC}}. \tag{8}$$

Since B is completely continuous, it maps the bounded set

$$\{v_n^* : v_n^* = \frac{v_n}{\|v_n\|_{AC}}, n = \{1, 2, \dots\}\}$$

into compact. Hence one can choose a subsequence $v_{n_k}^*$, $k = 1, 2, \dots$, from the sequence v_n^*, $n = 1, 2, \dots$, which converges to $v_0 \in AC(I, R^n)$, and $\|v_0\|_{AC} = 1$.

Passing to the limit in (8), we have $v_0 = Bv_0$, in view of (7), which contradicts the uniqueness of a solution to the problem (3), (4).

We prove now an a priori estimate.

III. Suppose that the inequality

$$\|B_0 x\|_{AC} \le f_0 + k\|x\|_{AC},$$

is true, where $f_0, k \in [0, \infty)$.

Then we get from (6) that

$$\|v\|_{AC} \le \mu\lambda\|B_0 v\| \le \mu\lambda f_0 + \mu\lambda k\|v\|_{AC} \le \mu\lambda f_0 + \mu\lambda(k+1)\|v\|_{AC}.$$

Hence

$$\|v\|_{AC} \le \frac{\mu\lambda f_0}{(1-\mu\lambda)(k+1)}.$$

IV. Let for some $k_0 \in (0, \infty)$ such that

$$k_0 > \frac{\mu\lambda f_0}{(1-\mu\lambda)(k+1)}$$

the condition

$$\|Bx\|_{AC} \le k_0, \quad \|x\|_{AC} = k_0$$

be fulfilled.

We get now the existence of a solution to the problem, by application of the method of Leray-Schauder (cfr. [1], v.5.37.6, p.298).

Thus, a theorem is proved.

Theorem. 1 *Let conditions I-IV hold. Then there exists a solution to the problem (1), (2).*

Remark. This note sharpens the results in [2] and generalizes the results in [3].

Список литературы

[1] Красносельский М.А. Забрейко П.П. Геометрические методы нелинейного анализа. М.: Наука, 1975.

[2] Пономарев В.Д. Существование решений краевой задачи для функционально-дифференциального уравнения первого порядка // Latv.Univ.Zinātn.Raksti. Matemātika. Diferenciālvienādojumi. - 1994. - V.593. 50.-53.lpp.

[3] Krakowiak A. On the existence of solution of homogeneous boundary value problem for ordinary differential equation // Zeza.nauk.U.J.Acta math. 1987. N 26. P.179-181.

Institute of Mathematics
and Computer Science,
Latvian University
Rainis blvd. 29
LV-1459, Riga

Пономарев В.Д. Существование решения краевой задачи для функционально-дифференциального уравнения.

Даются условия существования решения для краевой задачи для функционально-дифференциального уравнения.

УДК 517.985

V.Ponomarevs. Funkcionāla diferenciālvienādojuma robežproblēmas atrisinājuma eksistence.

Anotācija. Tiek doti atrisinājuma eksistences nosacījumi funkcionāla diferenciālvienādojuma robežproblēmai.

Latvijas Universitātes Zinātniskie Raksti, 606 (1997). Matemātika

ON κB-COMPACT SPACES

A. Sondore

SUMMARY. By a κB-compact space we call a topological space each cover of which by open sets with boundaries, whose cardinalities are less than or equal to some fixed cardinal number κ, contains a finite subcover. In this paper are studied basic properties of κB-compact spaces and some relations of these spaces to other classes of topological spaces.

KEY WORDS: compactness, κB-compactness, FB-compactness, clp-compactness, Hausdorffness, κB-Hausdorffness, FB-Hausdorffness.
1991 MSC 54D30, 54D20, 54A25

This work is one in a series of papers where we study compactness type topological properties which are defined by special open covers. Namely, here we are interested in those spaces each cover of which, consisting of open sets with boundaries whose cardinalities do not exceed a given cardinal κ, has a finite subcover. These spaces will be referred to as κB-compact. In particular, in case $\kappa = 0$ we come to the notion of clp-compact space (see the article [4]). We study also spaces each open cover of which consisting of sets with finite boundaries has a finite subcover. In the sequel such spaces will be called finite-boundary compact or FB-compact for short.

We consider also the concept of the so called κB-Hausdorffness which in the context of κB-compact spaces plays a role similar to the role of T_2 spaces in the classic theory of compact spaces. Besides we mention here the related properties of κB-T_1, κB-regularity, κB-normality and locally κB-compactness.

1 . BASIC PROPERTIES OF κB-COMPACT SPACES

Given a set A in a topological space X, let $\gamma_X(A)$ or just $\gamma(A)$ denote the corresponding boundary.

(1.1) Definition. A topological space X is called κB-compact if every its cover $\{U_i : i \in I\}$, all elements of which are open sets such that $|\gamma_X(U_i)| \leq \kappa$, contains a finite subcover. A topological space X is called FB-compact if every its open cover consisting of sets with finite boundaries contains a finite subcover.

In a natural way concepts of κB-countably compact, FB-countably compact, κB-Lindelöf, FB-Lindelöf, etc. can be introduced.

The following assertion shows some obvious relations of the introduced notions to other classes of topological spaces.

(1.2) Assertion. Every compact space is FB-compact and κB-compact for any cardinal κ. If $\kappa_1 < \kappa_2$ then a κ_2B-compact space is also κ_1B-compact. Every κB-compact space for any cardinal κ is FB-compact and every FB-compact space is clp-compact.

No opposite implication holds:

(1.3) Example. If $\kappa < c$, then the plane R^2 is a non-compact FB-compact space but fails to be $\aleph_0$B-compact. On the other hand the real line R is not even FB-compact. Furthermore, given two cardinals κ_1 and κ_2 such that $\kappa_1 < \kappa_2$ one can construct a κ_1B-compact space which fails to be κ_2B-compact.

From the definition one can imagine that there exists a certain analogy in the behaviour of compactness, FB-compactness and κB-compactness in different situations. We shall say more about this in the sequel. We start with one condition when κB-compactness and compactness become equivalent.

(1.4) Assertion. If every point x of a space X has a base $\eta=\{U_i : i\in I\}$, all elements of which are open sets such that $|\gamma_\lambda(U_i)|\le\kappa$ ($|\gamma_\lambda(U_i)|<\aleph_0$), then the space X is κB-compact (resp. FB-compact) iff it is compact.

(1.5) Example. The metric hedgehog J_σ, $\sigma<\aleph_0$ is a compact space. If $\sigma\ge\aleph_0$ then the hedgehog J_σ is not compact and is not κB-compact for $\kappa\ge\aleph_0$, but it remains an FB-compact space. Observe that for any σ for every point of hedgehog J_σ there exists a base of open subsets with boundaries whose cardinalities do not exceed σ.

(1.6) Example. For every σ the quotient hedgehog J_σ is a FB-compact space, but if $\sigma\ge\aleph_0$, this space is not κB-compact for $\kappa\ge\aleph_0$.

Generalising this example we come to a more general construction allowing to get new κB-compact and FB-compact spaces from old ones.

(1.7) Construction. Let X_α, $\alpha\in A$ (where A is a finite set) be κB-compact spaces and assume that in each space X_α there exists a point $x_\alpha^o\in X_\alpha$ having a neighbourhood U_α such that $|\gamma_{X_\alpha}(U_\alpha)|\le\kappa$. Let $Z=\oplus X_\alpha/$ denote the quotient space under equivalence relation $x_\alpha^o\sim x_\beta^o$ for each $\alpha,\beta\in A$. The space Z is also κB-compact. In case of FB-compact spaces the same construction leads to an FB-compact space for any index set A.

(1.8) Proposition. A topological space X is countably compact iff it is countably $\aleph_0$B-compact.

Proof. Assume, on the contrary, that X is countably $\aleph_0$B-compact but fails to be countably compact. Then there exists a countable closed discrete set $A=\{a_i : i=1,2,\dots\}$. Let us construct new sets $U_i = X\setminus\{a_{ij}\ \ j=1\ge i\}$ for $i=1,2,\dots$. It is clear, that these new sets are open with countable boundaries and make a countable cover of X which has no finite subcovers.

On the other hand a countably compact space is countably $\aleph_0$B-compact.

It is natural to call a subset M of a space X κB-compact if each cover $\{U_i : i \in I\}$ of M, **all elements of which are** open sets in the space X with $|\gamma_X(U_i)| \leq \kappa$, has a finite subcover. In a similar way one can define FB-compact subsets.

(1.9) Proposition. For any cardinal κ if a topological space X is κB-compact (FB-compact) and M is its closed subset such that $|\gamma_X(M)| \leq \kappa$ (resp. $|\gamma_X(M)| < \aleph_0$), then M is κB-compact (resp. FB-compact) subset of X.

<u>Proof</u> can be easily done patterned after its classical analogue.

(1.10) Proposition. If M is a κB-compact (FB-compact) subspace of X then M is also a κB-compact (resp. FB-compact) subset of X.

<u>Proof</u> follows from the next easily verifiable lemma.

(1.11) Lemma. If Y is the subspace of X and $A \subset Y$ then $\gamma_X(A) \supset \gamma_Y(A \cap Y)$

(1.12) Proposition. A finite union of κB-compact (FB-compact) subsets of a given space X is a κB-compact (resp. FB-compact) subset.

Recall that a system $\mathcal{A} = \{U_i : i \in I\}$ is said to have a finite intersection property if every finite intersection of sets from $\mathcal{A}$ is non-empty, i. e. $A_{i_1} \cap \ldots \cap A_{i_n} \neq \emptyset$ for every finite family $\{A_{i_1}, \ldots, A_{i_n}\}$ of $\mathcal{A}$.

(1.13) Proposition. A topological space is κB-compact (resp. FB-compact) if and only if every system $\mathfrak{u} = \{U_i : i \in I\}$, all elements of which are closed sets such that $|\gamma(U_i)| \leq \kappa$ (resp. $|\gamma(U_i)| < \aleph_0$), which has the finite intersection property has the non-empty intersection.

<u>Proof.</u> Since the boundary of a set U in the space X coincides with the boundary of its complement $X \setminus U$ this proposition can be proved similarly as the classical characterisation of compactness by systems of closed sets with finite intersection property.

Since the boundary of the intersection of two sets is contained in the union, of boundaries of these sets one can easily get the following modification of the previous proposition:

(1.14) Proposition. A topological space is κB-compact (resp. FB-compact) iff every system $\{U_i : i \in I\}$ of its non-empty closed subsets having $|\gamma(U_i)| \leq \kappa$ (resp. $|\gamma(U_i)| < \aleph_0$) which is invariant under finite intersections, has non-empty intersection.

(1.15) Proposition. Let Y be a continuous image of a κB-compact (FB-compact) space X under a mapping f such that $|f^{-1}(y)| \leq \kappa$ (resp. $|f^{-1}(y)| < \aleph_0$) for each point $y \in Y$, then Y is κB-compact (resp. FB-compact), too.

<u>Proof.</u>Notice firstly that $\overline{f^{-1}(V)} \setminus f^{-1}(V) \subset f^{-1}(\overline{V} \setminus V)$ for every subset V of Y. Therefore, under assumption of the proposition the preimage of any subset V of Y such that $|\gamma_Y(V)| \leq \kappa$, has also a boundary in the space X whose cardinality do not exceed κ.

Now the proof of the proposition can be easily done patterned after the proof of its classical prototype.

(1.16) Proposition. If a topological space Y is κB-compact (FB-compact) and f is a mapping from a space X into Y with the following properties:

(1) f is closed and open;

(2) for every point $y \in Y$ the preimage $f^{-1}(y)$ is a kB-compact (resp. FB-compact) subset of X;

then the space X is kB-compact (resp. FB-compact), too.

<u>Proof.</u> Notice firstly that since f is a closed and open mapping it holds

$$\overline{f(U)} \setminus Int\, f(U) = f(\bar{U}) \setminus Int\, f(U) \subset f(\bar{U} \setminus Int\, U)$$

for every subset U of X. Let $\mathcal{U} = \{U_i \quad i \in I\}$ be a system of nonempty closed subsets of X such that $|\gamma_X(U_i)| \leq k \neq \emptyset$.

Let $f(\mathfrak{U})=\{f(U_i):U_i\in\mathfrak{U}\}$. Then obviously $f(\mathfrak{U})$ is a system of closed subsets of Y such that also $|\gamma_Y f(U_i)|\leq\kappa$, and $f(\mathfrak{U})$ has the finite intersection property. Therefore $\cap f(\mathfrak{U})\neq\emptyset$.

Then there exists a point $y_o\in\cap f(\mathfrak{U})$. Hence for each $i\in I$: $f^{-1}(y_o)\cap U_i\neq\emptyset$. Taking in account that $\mathfrak{U}$ is invariant under finite intersections and $f^{-1}(y_o)$ is a κB-compact subset of X we conclude that there exists also a point $x\in\cap\{f^{-1}(y_o)\cap U_i:i\in I\}\subset\cap\mathfrak{U}$.

The case of FB-compact situation can be proved in a similar way.

As we can see in the paper [4] the property of clp-compactness is not multiplicative. We obtained some positive results about products of κB-compact spaces in special situations (see Propositions (1.18) and (1.19)). We start with the following lemma:

(1.17) Lemma. If X is locally connected, $A\subset X\times Y$ is a closed subset with compact boundary and p is the projection $p:X\times Y\to X$, then $p(\gamma_{X\times Y}(A))\supset\gamma_X(p(A))$.

Proof. Let $a\in\gamma_X(p(A))$, $a\notin p(\gamma_{X\times Y}(A))$ and take a connected neighbourhood U_a of a in X. Further, let $U_a'=U_a\times Y$ and $Y_a=\{a\}\times Y$ Since $a\notin p(\gamma_{X\times Y}(A))$ it follows that $Y_a\cap A=\emptyset$ Since $a\in\gamma_X(p(A))$ it follows that $U_a'\cap A\neq\emptyset$. Choose a point $(x_o,y_o)\in U_a'\cap A$. Moreover, since $\gamma_{X\times Y}(A)$ is compact this point can be chosen in such a way that $y_o\notin p_Y(\gamma_{X\times Y}(A))$

Now let $Z=(X\times\{y_o\})\cap U_a'$ and $W=A\cap Z$. From the construction it is clear, that Z is connected, W is a clopen set in Z (since $\gamma_Z(A\cap Z)=\emptyset$), $(a;y_o)\notin W$ and $(x_o;y_o)\in W$ The obtained contradiction completes the proof.

(1.18) Proposition. If X is a locally connected FB-compact T_1-space, Y is a FB-compact T_1-space then the product $X\times Y$ is FB-compact.

Proof. Let $\mathfrak{U}=\{U_i:i\in I\}$ be a system of non-empty closed subsets of $X\times Y$ with finite boundaries which is invariant under finite intersections.

Let $\mathcal{B}=\{\overline{p(U_i)}: U_i \in \mathcal{U}\}$ where p is the projection $p: X\times Y\to X$. From Lemma 1.17 follows that $\overline{p(\gamma_{X\times Y}(U_i))}\supset\gamma_X(p(U_i))$ and taking into account that obviously $\gamma_X(\overline{A})\subset\gamma_X(A)$ for every set A and that $\gamma_{X\times Y}(p(U_i))$ is finite, it follows that $\gamma_X(\overline{p(U_i)})\subset\overline{p(\gamma_{X\times Y}(U_i))}=p(\gamma_{X\times Y}(U_i))$. Thus $\mathcal{B}$ is a system of closed sets with finite boundaries in X satisfying the finite intersection property. Since X is FB-compact $\cap\overline{p(U_i)}\neq\varnothing$, i. e. there exists a point $x_o\in\cap\overline{p(U_i)}$; let $Y_o=\{x_o\}\times Y$

Further, let $\mathcal{V}=\{U_i\cap Y_o: U_i\in\mathcal{U}\}$. It is easy to note that $\mathcal{V}$ is a system of non-empty closed sets in Y_o with finite boundaries and which is invariant under finite intersections. Since Y_o is homeomorphic to Y, it is FB-compact and hence $\cap\mathcal{V}\neq\varnothing$. Let $y_o\in\cap\{V_i: V_i\in\mathcal{V}\}$. It is clear that the point $(x_o;y_o)\in\cap\{U_i:U_i\in\mathcal{U}\}\neq\varnothing$ and hence by Proposition 1.14. $X\times Y$ is FB-compact.

The proof of this proposition can not be extended to the case of κB-compactness for $\kappa\geq\aleph_o$ because we can not maintain that sets of the system $\mathcal{B}$ have finite boundaries.

(1.19) Proposition. The product of compact and a κB-compact (FB-compact) space is κB-compact for every κ (resp. FB-compact).

Proof. Assume that X is a compact space and Y is a κB-compact space for some cardinal κ. Then the projection $p_Y: X\times Y\to Y$ being a projection along a compact space is a closed mapping. Besides, all preimages of points under it are homeomorphic to the compact space X. Now the conclusion follows directly from Proposition 1.16.

2. κB-SEPARATION PROPERTIES

In this section we examine the modified separation properties: κB-T_1, κB-T_2 , κB-regular and κB-normal spaces.

(2.1) Definition. A topological space is called κB-T_1 if for any two different points x and y there exist open neighbourhoods A_x ($y\notin A_x$) or B_y ($x\notin B_y$) of these points with conditions that $|\gamma_X(A_x)|\leq\kappa$ and $|\gamma_X(B_y)|\leq\kappa$.

One can easily see that κB-T_1 and T_1 are equal concepts. Nevertheless we introduced this concept for the sake of completeness of our scheme. In particular in Proposition 1.18 we could use κB-T_1 instead of T_1

(2.2) Definition. A topological space X is called κB-Hausdorff (FB-Hausdorff) if for any two different points x and y there exist disjoint open neighbourhoods A_x and B_y of these points with conditions that $|\gamma_X(A_x)| \le \kappa$ and $|\gamma_X(B_y)| \le \kappa$ (resp. $|\gamma_X(A_x)| \le \aleph_0$ and $|\gamma_X(B_y)| \le \aleph_0$).

It is clear that every κB-Hausdorff space is Hausdorff. A Hausdorff space is not always κB-Hausdorff (for example, a plane is not κB-Hausdorff if $\kappa < c$). Obviously, κB-Hausdorffness is hereditary, **but fails to be** multiplicative (a plane is a product of a real line which is an $\aleph_0 B$-Hausdorff space, and even 1B-Hausdorff).

(2.3) Example. We can remark that both hedgehogs J_σ (see (1.5) and (1.6)) are FB-Hausdorff and hence κB-Hausdorff for any cardinal κ.

(2.4) Proposition. A κB-compact (resp. FB-compact) subset of a κB-Hausdorff (resp. FB-Hausdorff) space is closed.

<u>Proof.</u> We consider the case of κB-situation. Let A be a κB-compact subset of a κB-Hausdorff space X and let $x \notin A$. By κB-Hausdorffness of X in this space for each point $y \in A$ there exists an open neighbourhood U_y such that $|\gamma_X(U_y)| \le \kappa$ and an open neighbourhood V_x^y for a point x such that $|\gamma_X(V_x^y)| \le \kappa$ with $U_y \cap V_x^y = \emptyset$

Thus the system $\{U_y : y \in A\}$ is an open cover of the κB-compact subset A and all elements of this cover have boundaries of restricted cardinality in X. Therefore one can choose a finite subcover $\{U_{y_i} : i = 1, \dots, n\}$

Let V be the intersection of corresponding system $\{V_x^{y_i} : i = 1, \dots, n\}$ Clear, that V_x is an open neighbourhood of x and this set does not intersect with the union of the system $\{U_{y_i} : i = 1, \dots, n\}$ and moreover with the set A. As a result for each $x \notin A$ there exists an open set V which contains x but does not intersect A, i. e. A is a closed set of X.

(2.5) Definition. A topological space is called κB-regular if for every point x and a closed set M such that $|\gamma(M)|\leq\kappa$ and $x\notin M$ there exist disjoint open neighbourhoods U_x and U_M such that $|\gamma(U_x)|\leq\kappa$ and $|\gamma(U_M)|\leq\kappa$.

As for clp-situation (see [5]) we can extend the notion of a normality to κB-situation, too.

(2.6) Definition. A topological space is called κB-normal if every pair of disjoint closed sets A and B such that $|\gamma(A)|\leq\kappa$ and $|\gamma(B)|\leq\kappa$ can be separated by disjoint open neighbourhoods U_A and U_B such that $|\gamma(U_A)|\leq\kappa$ and $|\gamma(U_B)|\leq\kappa$.

Just in the same way FB-regularity and FB-normality are defined.

It is clear, every a κB-normal κB-T_1 space is κB-regular. Now, some propositions similar to the classical case.

(2.7) Proposition. Let F be an open subset of a κB-regular (resp. FB-regular) space X such that $\gamma_X(F)\leq\kappa$ (resp. [illegible] $(F)<\aleph_0$) and let $x\in F$. Then there exists an open neighbourhood U [illegible] in X such that [illegible] $)\leq\kappa$ (resp. [illegible] $<\aleph_0$) and $\overline{U}$ [illegible] $\subset F$. If for a point [illegible] and each open set $F\subset X$ that $x\in F$ there e[illegible] an open neighbourhood U [illegible] of x in X such th[illegible] $)\leq\kappa$ (resp. [illegible] $\aleph_0$ and $\overline{U_x}\subset F$ then X is a κB-regular (resp. FB-regular) space.

Proof. Let $x\in F$ and let $F\subset X$ be an arbitrary open subset with [illegible] $_X(F)\leq\kappa$. Then $X\setminus F$ is [illegible] closed set and in a κB-regular space we can separate [illegible] and $X\setminus F$ with open neighbourhoods U [illegible] and [illegible] with boundaries of restricted [illegible]ardinalities. It is clear, that $\overline{U}$ [illegible] F.

The converse proposition also is right, because the boundary of the closure of a set is contained in the boundary of the set.

The proof for FB-situation is similar.

(2.8) Proposition. A κB-Hausdorff κB-compact space is κB-normal. A FB-Hausdorff FB-compact space is FB-normal.

Proof. Let M and N be disjoint closed sets in a κB-compact space X such that $|\gamma_X(M)|\leq\kappa$ and $|\gamma_X(N)|\leq\kappa$. From Proposition 1.9 it follows that M and N are κB-compact subsets. Then from the proof of Proposition 2.4 we can conclude that for each point $y\in N$ there exists an open set U_y containing the set M with $|\gamma_X(U_y)|\leq\kappa$ and an open neighbourhood V_y of the point y such that $|\gamma_X(V_y)|\leq\kappa$, that $U_y\cap V_y=\varnothing$.

Obviously the system $\{V_y : y\in N\}$ is an open cover of the set N whose elements are open sets such that $|\gamma_X(V_y)|\leq\kappa$ in the space X. Therefore we can choose the finite subcover $\{V_{y_i} : i=1,\dots,n\}$ of N.

Now, let $V=\cup\{V_{y_i} : i=1,\dots,n\}$ and $U=\cap\{U_{y_i} : i=1,\dots,n\}$. The sets V and U are open disjoint neighbourhoods of the given sets M and N with boundaries of restricted cardinality and therefore X is a κB-normal space.

The case of FB-compact concept can be proved in a similar way.

As a κB-Hausdorff space is also a κB-T_1 space then from the previous proposition follows immediately the next:

(2.9) Proposition. A κB-Hausdorff κB-compact space is κB-regular.

And as usually, a similar statement holds for FB-situation.

3. LOCALLY κB-COMPACTNESS

(3.1) Definition. A topological space X is called locally κB-compact if for each $x\in X$ there exists an open neighbourhood U_x such that $|\gamma_X(U_x)|\leq \kappa$ and $\overline{U_x}$ is a κB-compact subset of X.

This concept is a known analogue of local compactness. In the next theorem a kind of Alexandroff's κB-compactification is constructed for locally κB-compact space.

(3.2) Theorem. If a κB-Hausdorff space X is locally κB-compact but is not κB-compact then there exists a κB-Hausdorff κB-compact space $X^{\blacktriangledown} = X \cup \{\psi\}$ containing X as a dense subspace.

Proof. Let the open sets in the space $X^{\blacktriangledown}$ will be those which are open in the space X and also sets $A \cup \{\psi\}$ where $X \setminus A$ is a closed κB-compact subset with $|\gamma_X(X \setminus A)| \leq \kappa$ in X.

Firstly we show κB-compactness of $X^{\blacktriangledown}$ Let $\mathfrak{U}=\{U_i; i \in I\}$ be a cover of $X^{\blacktriangledown}$ with open sets in the space $X^{\blacktriangledown}$ such that $|\gamma_{X^{\blacktriangledown}}(U_i)| \leq \kappa$. Then there exists $i_o \in I$ that $\psi \in U_{i_o}$ As U_{i_o} is an open set of $X^{\blacktriangledown}$, then $U_{i_o} = A \cup \{\psi\}$, where $X \setminus A$ is a κB-compact subset in X such that $|\gamma_X(X \setminus A)| \leq \kappa$. Besides, $\mathfrak{U}$ is also cover of the set $X \setminus A$ with open sets such that $|\gamma_X(U_i)| \leq \kappa$ in the space X (see Lemma 1.11). Hence there exists a finite subcover $\{U_{i_1},\dots,U_{i_n}\}$ of the κB-compact subset $X \setminus A$. Then the system $\{U_{i_o}, U_{i_1},\dots,U_{i_n}\}$ is a finite subcover of $\mathfrak{U}$ of the space $X^{\blacktriangledown}$ Thus $X^{\blacktriangledown} = X \cup \{\psi\}$ is a κB-compact space.

If we assume that X is a closed subset of $X^{\blacktriangledown}$, then $X^{\blacktriangledown} \setminus X = \{\psi\}$ is an open subset of $X^{\blacktriangledown}$, but this contradicts the given assumption that X is not a κB-compact subset. Therefore the closure of X is $X^{\blacktriangledown}$.

Finally, we show κB-Hausdorffness of $X^{\blacktriangledown}$ If $x, y \in X$ then these points can be separated in X by neighbourhoods with desired properties. It is easy to note that with the same sets we can separate these points in $X^{\blacktriangledown}$ We need to show the situation when $x \in X$ but $y=\{\psi\}$. From the local κB-compactness of X there exists an open neighbourhood U_x of the point x such that $\overline{U_x}$ (closure in X) is a κB-compact subset of X and $|\gamma_X(U_x)| \leq \kappa$. It is clear, that $|\gamma_{X^{\blacktriangledown}}(\overline{U_x})| \leq \kappa$. Then the set $V=(X \setminus \overline{U_x}) \cup \{\psi\}$ is an open subset of $X^{\blacktriangledown}$ and $|\gamma_{X^{\blacktriangledown}}(V)| \leq \kappa$. Obviously $U_x \cap V = \varnothing$. Hence $X^{\blacktriangledown}$ is a κB-Hausdorff space. ———

The opposite to the statement of Theorem 3.2 also holds:

(3.3) Remark. If X is a κB-Hausdorff κB-compact space then for each $x \in X$ the space $X_o = X \setminus \{x\}$ is locally κB-compact.

This follows immediately from the next proposition.

(3.4) Proposition. If X is a κB-Hausdorff κB-compact space and Y is its open subset such that $|\gamma_x(Y)| \le \kappa$ then Y is a locally κB-compact subset.

Proof. Let x be an arbitrary point of Y. From Proposition 2.11 X is a κB-regular space, i. e. there exists an open neighbourhood U_x of the point x such that $|\gamma_x(U_x)| \le \kappa$ and $\overline{U_x} \subset Y$. At last, $\overline{U_x}$ is a κB-compact subset (see Proposition 1.9).

Results similar to the ones included in this section can be proved also for FB-case.

REFERENCES

1. К.Куратовский, Топология, Том I, Москва, 1969.
2. К.Куратовский, Топология, Том II, Москва, 1969.
3. R. Engelking, General topology, Warszawa, PWN, 1977.
4. A. Sondore, A. Šostak, On clp-compact and countably clp-compact spaces, Acta Univ. Latv., 1994, vol. 595, p. 123-143.
5. A. Sondore, On clp-Lindelof and clp-paracompact spaces, Acta Univ. Latv., 1994, vol. 595, p. 143-156.

A. Сондоре. О кВ-компактных пространствах.

Аннотация. Топологическое пространство назовем кВ-компактным пространством, если каждое его покрытие открытыми множествами с границей, мощность которой не превосходит данный кардинал к, имеет конечное подпокрытие. В данной заметке развиваются основы теории кВ-компактных пространств и рассматриваются связи таких пространств с другими классами топологических пространств.

A. Sondore. Par kB-kompaktām telpām

Anotācija. Par kB-kompaktu telpu tiek saukta tāda topoloģiska telpa, kuras jebkurš pārklājums ar vaļējām kopām, kuru robežu apjoms nepārsniedz doto kardinālu k, satur galīgu apakšpārklājumu. Šis raksts ir veltīts kB-kompaktu telpu izpētei, kā arī apskata šo telpu saistību ar citām topoloģisku telpu klasēm.

Department of Mathematics
Daugavpils Pedagogical University
Daugavpils LV-5401 LATVIA

Impressions of the Council Meeting of the European Mathematical Society.

Alexander Šostak

On July 22-27, 1996 the second European Congress of Mathematics was held in Budapest. (The previous, first European Congress of Mathematics took place in Paris, in 1992.) Just before the Congress, on July 20-21 in the Institute of Mathematics of the Hungarian Academy of Sciences, the meeting of the Council of the European Mathematical Society was held. This meeting was of especial significance for the Mathematical community of Latvia, because at this meeting the Latvian Mathematical Society (LMS) was accepted as a corporate member of the European Mathematical Society (EMS). I had an honour to represent our society at this meeting by a request of the chairman of LMS, Professor Uldis Raitums. Therefore, I should like to share here my impressions and recollections of this important event.

The total number of delegates at the meeting was 62, some of whom represented national societies or institutional societies as corporate members, while others were individual members. Although the official opening of the meeting was on Saturday, July 20, the Council actually started its activities on Friday evening, when all present delegates were invited to a get-together party in a small restaurant located in the basement of a house just opposite the building of the Institute of Mathematics. There, in a cozy atmosphere, refreshing with wine and fortifying by light snack, the delegates had good opportunity to make them acquaintance and to discuss different subjects - mathematics, mathematical education, politics, standards of life, salaries, flowers, birds, food, etc., etc...
As I already mentioned, the official opening of the meeting was on Saturday morning. At the beginning the president of the EMS, professor Jean-Pierre Bourguignon from France, welcomed the delegates and addressed on them with not too long, but very saturated speech. In this address the president, in particular, noted the main achievements of the activity of the EMS during the last 2 years (after the previous meeting of the Council in August 1994 in Zurich) as well as touched most urgent, in his opinion, problems the Society will be having in the near future and the perspectives of its development.

The second item of the agenda was the election of new corporate, full (national or regional) and institutional members. In addition to Latvian there were three more societies-candidates: The Ural Mathematical Society, the Institute of Mathematics of the Academy of Sciences of Moldova and the Institute Non-Lineare de Nice (the last two pretended to be accepted as institutional members). The elections were organized in the following way: after a short presentation of the representative of a society-candidate, questions were asked him by the delegates. (These questions mainly concerned such subjects as the number of members in the society, how fully does it represent the mathematical community of the country, basic trends of mathematics being developed in the country or a region, publications etc.). Then the voting was held: its results were favourable for all 4 candidates: all societies-candidates were accepted into EMS. In particular, the Latvian Mathematical Society was accepted anonymously. When results of the voting were announced, the participants of the meeting warmly congratulated the new accepted societies and their representatives by loud applause and invited them to participate in the further work of the meeting.

The next point of agenda was the election to the vacancies of the Executive Committee for the period 1997-2000 (In accordance to the EMS statutes, there was one vacancy for a vice-president and 4 seats for ordinary members). This procedure turned out to be much more stormy if compared with the previous ones, since not infrequently the opinions of different members essentially diverged. Therefore the president suggested to postpone the election till Sunday, and in the meanwhile to ask to study this question by the Nominating Committee with Prof. F. Hirzebruch as the chair. The Council agreed with this suggestion. In the result, the procedure of elections continued on the next day when the delegates, taking into account the well-grounded proposal of the Nominating Committee, by voting elected A. Pelczar (Poland) as a vice-president and B. Branner (Denmark), M. Sanz-Sole (Spain), R. Jeltsch (Germany) and A. Vershik (Russia, St. Petersburg) as members.

Among items discussed by the Council were the accounts and auditor's reports, budget, membership fees for the next two years and some other financial type questions. These questions, although of extreme significance for the EMS, and its Council in particular, likely will not be of a large interest for our readers. Therefore, I shall better linger here on the next quite vacuous item of the agenda: namely the reports of the Committees. There are more than a dozen Committees at the Council, having special areas of activities. Here I shall tell briefly about the activities of some of these committees.

- *Education of Mathematics*. The Council noted a very large diversity in the level and quality of education in different countries, regions and schools, and as a consequence of this, a tremendous diversity in mathematical knowledge and in experience of children and students. To remedy, to some extent, this drawback, the Council suggested to the Committee of Education to define the minimal knowledge in mathematics for children and young people at different age. Another suggestion of the Council was to work on some approaches allowing to present mathematics at school in a more attractive and lively way.
- *Publicity of the EMS and Contacts with European Institutions*. Certain work was done towards establishing relations of the EMS with various European Institutions (European Parliament, Ministries of European Affairs, etc.) It was pointed out that contacts have to be established at the political level in all European countries.
- *Support of East-European Mathematicians*. In the last two years the activities of the committee were mainly directed to the following two goals: 1) to find finance which would allow to cover travel expenses to the European Congress of Mathematics and to its Satellite Conferences for mathematicians from East-European countries (local expenses in most cases were covered by the organizers) and 2) to support satellite conferences to European Congress of Mathematics. As the result 16 satellite conferences were supported with a budget of ECU 11000. A new programme accepted at the meeting, in addition to sponsoring travel expenses for organizers of conferences

includes also the so called "new library scheme" which foresees to find help for libraries in East-European universities for subscribing journals, to get them at the price of production costs.

- *European Mathematical Information Service*. The EMS server (EMIS) contains the so called Electronic Library (ELibEMS), general information on the EMS and information on mathematical activities and institutions, lists of conferences, etc. The ELibESM provides free access to a collection of electronic journals and electronic versions of printed journals. Before being included in ELibEMS, the quality of these periodicals and collections has to be approved by the Electronic Publishing Committee of the EMS. Server EMIS can be reached by

  http: //www.emis.de , or by anonymous ftp: ftp.emis.de/directory/pub/EMIS.
- *European Database*. Zentralblatt für Mathematik, from German is becoming a European enterprise. Now the reviewing process is being organized in different countries what allows to make it essentially faster and more qualitative. A new idea stated at the meeting is to organize "current awareness programme" on the basis of the Zentralblatt. In particular, the publishers of mathematical journals will be asked to send the contents of the journal and the abstracts of the papers electronically to the office of the Zentralblatt where this material will be made ready for being included in both the date base MATH and the current awareness service of EMIS.
- *Diderot Mathematical Forum*. A cycle of conferences, called "Diderot Mathematical Forum" consists of two conferences a year taking place simultaneously in three European cities. In the process of the conference the participants in different cities exchange information by telecommunication. The subject considered at conferences covers three different aspects: fundamental mathematics, applications of mathematics and their relation to society (e.g. ethical and epistemological dimensions).
- *Publications*. The Society continues publications of the so called *Newsletters* - a certain analogue of the much better known Notices published by the American Mathematical Society. Newsletters contain information about the Society, announcements of conferences, book reviews, a problem corner and articles of general interest. A new programme of the Council is to found a *new mathematical journal* of a general nature and a very high scientific quality. The Council hopes to make it, in a short time, a leading journal covering all aspects of mathematics. Prof. J. Jost was appointed as the Editor-in Chief of the journal.
- *Summer schools*. In order to promote the interaction of young mathematicians, two series of summer schools, one each year in mathematics and one in applications of mathematics will be organized. They will bring together about a hundred graduate students to attend advanced courses and to exchange their research experiences.

One of principal aims of the European Mathematical Society is to unify European mathematicians - both on the level of Mathematical Societies and on the individual level. But at the same time the EMS strives for establishing relations and developing fruitful collaboration with other, non-European, professional societies of mathematicians. As a certain evidence of this was that representatives of the three very influential mathematical societies arrived at the meeting in order to welcome the delegates, to share information about their societies and to discuss perspectives of collaboration. These high guests were:

Professor K.C. Chang, the President of the Chinese Mathematical Society. (It is worth mentioning here that China has made a bid for the organization of the International Congress of Mathematicians in 2002 in Beijing.)

2. Professor A. Kerkour (Morocco), the President of the African Mathematical Union.
3. Professor J. Ewing, The Executive Director of the American Mathematical Society.

Shortly before the closing of the Council meeting on Sunday's evening, another spectacular event took place: namely, the selection of the site for the European Congress of Mathematics (ECM) in 2000. (It is worth reminding here , that the Congress in 2000 is of special significance since year 2000 is announced as the International Year of Mathematics!)

To start from the beginning, I must say that at the previous Council meeting in 1994 four cities were accepted as candidates for this honourable role. They were: Copenhagen (proposed by the Danish Math. Society), Torino (proposed by the Italian Math. Union), Barcelona (proposed by the Catalonian Math. Society) and Brighton (proposed by the London Math. Society). A special Committee of the Council, after preliminary investigation of the situation, has acknowledged that all four candidates were able to provide adequate facilities for the organization of the Congress. However, short before the meeting, Torino decided to withdraw its application in favour of Barcelona. Between the rest three candidates a serious competition took place at the meeting. The representatives of the corresponding societies informed how the opening and closing ceremonies will be organized, the preliminary programmes of sessions and round tables, the perspectives for publishing Proceedings, availability of good libraries in the area, technical equipment, etc. They informed also about the financial situation, in particular, sketched the approximate budget of the Congress, described perspectives of finding sponsors, the perspectives of support from the Governments and local authorities.. They also showed films and slides about their cities and universities, and described their advantages. Then representatives were asked different specifying questions. Further, the voting procedure was held. The results of voting where extremely favourable for the capital of Catalonia and the capital of 99th Olympic games: namely, 36 voices were given for Barcelona, while only 13 for Brighton and 7 for Copenhagen. After the results were announced, stormy applause started, and the delegates warmly congratulated professor Sebastia Xambo, the president of the Catalonian Mathematical Society, and professor Manuel Castellet, the director of the Mathematical Center in Barcelona.

The election of the site for the 3rd ECM in year 2000 was the last item on the agenda of the Council meeting. After this the meeting was officially closed. However the delegates did not hurry to leave Budapest: most of them remained in the city and joined the ranks of a much more numerous and impressive assemblage consisting of participants and guests of the 2nd Mathematical Congress which was opened on Monday, July 22. However, this is another story. Here I shall mention only, that at the Congress, besides the author of these notes, there were two other Latvian participants: Andrejs Reinfelds and Felikss Sadyrbajevs.