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MATHEMATICAL MODELLING APPLIED PROBLEMS OF MATHEMATICAL PHYSICS

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LATVIJAS UNIVERSITĀTE

Latvijas ZA un LU Matemātikas institūts
Diferenciālvienādojumu un tuvināto metožu analīzes katedra

MATEMĀTISKĀ MODELĒŠANA

MATEMĀTISKĀS FIZIKAS LIETIŠĶĀS PROBLĒMAS

Zinātniskie raksti

612. sējums

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The "Scientific Papers" are devoted to the development and applying of numerical methods for certain problems of the mathematical physics.

Krājums veltīts skaitlisko metožu izstrādei, pamatošanai un pielietošanai konkrētu matemātiskās fizikas problēmu risināšanā.

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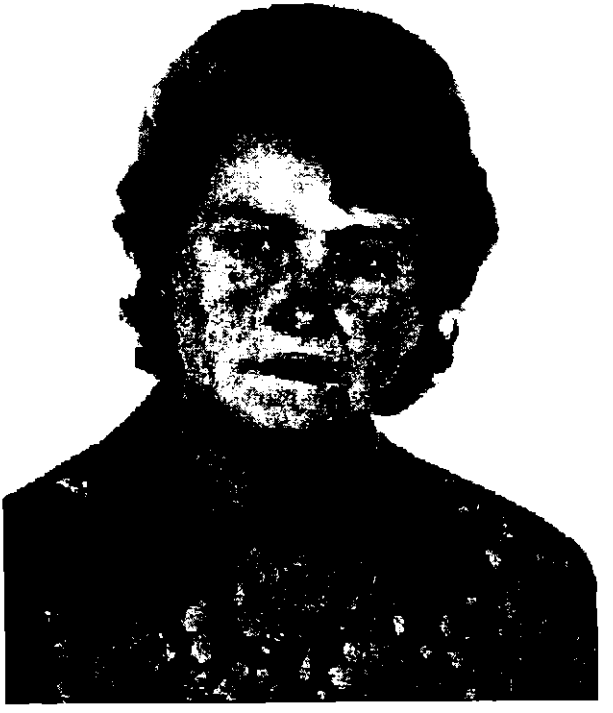
SATURA RĀDĪTAJS

ILGA PAGODKINA



1997. gada 26. jūnijā pēc smagas slimības pāragri, savu radošo spēju pilnbriedā, no mums šķīrās matemātikas zinātņu doktore, docente Ilga Pagodkina.

Ilga Pagodkina (dzim. Zaļkalne) dzimusi 1942. gada 6. janvārī Talsu rajonā. Agrā bērnībā, 1949. gada martā Ilga kopā ar saviem vecākiem no dzimtajām mājām tika aizvesta uz Omskas apgabalu, no kurienes Latvijā atgriezās tikai pēc 9 gadiem. Tālajā un savai latviskajai mentalitātei svešajā Sibīrijā viņa pabeidza pamatskolu, pēc atgriešanās Latvijā turpināja mācības Ventspilī, bet vēlāk – Sabilē. Pēc Sabiles vidusskolas absolvēšanas 1961. gadā viņa iestājās Latvijas Universitātes Fizikas un matemātikas fakultātē, kuru beidza 1966. gadā un ieguva matemātiķa kvalifikāciju.

Jau studiju laikā 1964. gadā I. Pagodkina, apvienojot mācības ar zinātnisko darbu, sāka strādāt pusslodzē LU Astronomijas observatorijā par laboranti. 1967. gadā viņa iestājās LU Astronomijas observatorijas aspirantūrā, kuru sekmīgi beidza ar profesora Kārļa Šteina vadībā uzrakstītu disertāciju "Neregulārie spēki komētu kustībā" Tā tika sekmīgi aizstāvēta Pulkovas observatorijā Ļeņingradā (tagad Sankt-Pēterburga) 1971. gadā. 1972. gadā Ilgai Pagodkinai tika piešķirts fizikas un matemātikas zinātņu kandidāta grāds, kas pēc Latvijas neatkarības atgūšanas 1992. gadā tika pielīdzināts (nostrificēts) matemātikas doktora grādam.

1971. gadā Ilga Pagodkina sāka strādāt LU Fizikas un matemātikas fakultātes Matemātiskās analīzes katedrā, bet gadu vēlāk iekļāvās jaunizveidotajā Lietišķās matemātikas katedrā. 1973. gadā viņa tika ievēlēta docenta amatā LU Fizikas un matemātikas fakultātes Diferenciālvienādojumu un tuvināto metožu katedrā, bet 1978. gadā ieguva minētās katedras docenta zinātnisko nosaukumu. Šajā katedrā Ilga Pagodkina aktīvi strādāja līdz pat sava mūža pēdējām dienām.

Ilga Pagodkina studentiem ir lasījusi vispārīgās astronomijas, elementārās matemātikas, programmēšanas kursus, bet īpaši nozīmīgs LU tradīciju izveidei ir viņas devums skaitlisko metožu pamatkursā un daudzos aktuālos speckursos. Docentes nolasītās lekcijas un vadītie laboratorijas darbi izcēlās ar labu sagatavotību un kontaktu ar auditoriju. Sarakstītie 16 mācību un metodiskie līdzekļi un viņas sastādītās 5 akadēmisko kursu programmas ievērojami aktivizēja mācību darbu fakultātē. Lielu vērību Ilga Pagodkina veltīja savas kvalifikācijas celšanai. Tā, piemēram, 1987. gadā viņa beidza skaitliskās matemātikas un kibernētikas kvalifikācijas celšanas kursus M. Lomonosova Maskavas Valsts universitātē. Ilga Pagodkina aktīvi piedalījās fakultātes un katedras sabiedriskajā dzīvē: bija ilggadēja labākā studentu grupas kuratore, bija atbildīgā par studentu ražošanas praksi, darbojās fakultātes metodiskajā komisijā, piedalījās zonālajos skolēnu profesionālās orientācijas semināros.

Ilgas Pagodkinas pētnieciskо darbību noteica dažādiem matemātiskās modelēšanas uzdevumiem veltītu tēmu izpilde gan LU Fizikas un matemātikas fakultātē, gan arī akadēmiskajās zinātniskās pētniecības iestādēs – LU Skaitļošanas centrā, kuru 1990. gadā pārveidoja par LU Matemātikas un informātikas institūtu, bet 1991. gadā viņa iesaistījās jaunizveidotā LZA un LU Matemātikas institūta darbā. Ilgas Pagodkinas veiktā zinātniskā darba tematika sākumā bija saistīta ar zvaigžņu un mazo planētu pētīšanu, vēlāk ar naftas un alumīnija ieguves matemātisko modelēšanu, bet pēdējā laikā ar siltuma procesu pētīšanu plānās kārtainās un slāņainās vidēs. Par Ilgas Pagodkinas zinātnē paveikto liecina 20 zinātniskās publikācijas, kurām viņa ir autore vai līdzautore. Veiktos pētījumus Ilga Pagodkina cieši sasaistīja ar mācību procesu, tie regulāri atspoguļojās viņas vadītajos kursa un diplomdarbos.

Ilga Pagodkina bija LZA un LU Matemātikas institūta zinātniskā sekretāre no šī institūta dibināšanas brīža. No 1994. gada janvāra, kad tika izveidota LU Matemātikas zinātņu nozares habilitācijas un promocijas padome, Ilga Pagodkina pildīja arī šīs padomes sekretāres pienākumus. Viņas vadībā

tika ieviestas matemātikas doktora eksāmenu programmas trijās matemātikas apakšnozarēs: 1) matemātiskajā analīzē un diferenciālvienādojumos, 2) diskrētajā matemātikā, 3) matemātiskās fizikas vienādojumos un metodēs, skaitliskajā analīzē.

Latvijas matemātiķu saime cietusi sāpīgu zaudējumu, atvadoties no prasmīgas pedagoģes, zinātnieces un organizatores, atsaucīgas un sirsnīgas kolēģes. Ilga Pagodkina paliks vienmēr gaišā piemiņā ne tikai viņas darba biedriem, bet arī visiem tiem, kurus viņa mācīja, kuri viņu kaut nedaudz pazina.

A. Buiķis
J. Cepītis
H. Kalis
A. Lobanoviča

ILGAS PAGODKINAS PUBLICĒTO DARBU SARAKSTS

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FOURIER–ASYMPTOTIC APPROXIMATION

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ABSTRACT

A new Fourier-asymptotic (F–A) method of functions approximation is presented. The method is based on the Fourier series and on the methods of asymptotic expansions of integrals. The main merit of the method is a multiple precision of approximation by relatively small number of points of interpolation. It is especially important for the solution of unstable problems. Numerical examples are presented that illustrate the effectiveness and capability of the application of F–A approximation in the solution of differential equations.

In this work a new Fourier-asymptotic (F–A) method of functions approximation is presented. The method is based on the Fourier series and on the methods of asymptotic expansions of integrals. The main merit of the method is a multiple precision of approximation by relatively small number of points of interpolation. It is especially important for the solution of unstable problems. Here are presented the numerical examples, that illustrates the effectiveness and capability of the application of F–A approximation in the solution of differential equations.

[1]This work was partly supported by Latvian Council of Science under Grant 96. 0691

1. BASIC STATEMENT

Let the function $v:[a,\ b] \to \mathbb{R}$ is sufficiently smooth and his cosine Fourier-series expansion is

$$v(x) = \sum_{k=0}^{+\infty} c_k u_k(x), \quad x \in [a,\ b], \quad u_k(x) = \cos\frac{k\pi}{l}(x-a), \quad l = b-a. \tag{1.1}$$

If $k > n \geq 0$, we replace the Fourier coefficients $c_k = \frac{2}{l}\int_a^b v(x)u_k(x)\,dx$ with their asymptotic approximations. We obtain these with the $2m+1$ - multiple integration by parts [1]. Then

$$v(x) = \sum_{k=0}^{n} c_k u_k(x) + A_0 + R_0, \tag{1.2}$$

$$A_0 = \frac{2}{l}\sum_{j=1}^{m} \varepsilon^{2j}\left(v^{(2j-1)}(b)\beta_{2j}\left(\frac{x-a+l}{2l}\right) - v^{(2j-1)}(a)\beta_{2j}\left(\frac{x-a}{2l}\right)\right), \tag{1.3}$$

$$\beta_{2\mu}(x) = (-1)^{\mu+1}\sum_{k=n+1}^{+\infty}\left(\frac{n+1}{k}\right)^{2\mu}\cos 2\pi kx, \quad \mu = 1,2,\ldots \tag{1.4}$$

$$\beta_{2\mu+1}(x) = (-1)^{\mu+1}\sum_{k=n+1}^{+\infty}\left(\frac{n+1}{k}\right)^{2\mu+1}\sin 2\pi kx, \quad \mu = 0,1,\ldots \tag{1.5}$$

where $\varepsilon \quad \frac{l}{\pi(n+1)}$ is a small parameter. The representation (1.2) will be called the Fourier-asymptotic (F-A) approximation or more exactly the F-A approximation of cosine Fourier-series. We have the following integral representation for the approximations error R_0

$$R_0 = \frac{\varepsilon^{2m+1}}{l}\int_a^b v^{(2m+1)}(\tau)\left[\beta_{2m+1}\left(\frac{x+\tau-2a}{2l}\right) + \beta_{2m+1}\left(\frac{\tau-x}{2l}\right)\right]d\tau. \tag{1.6}$$

For the derivatives $v^{(2\nu)}(x)$, $\nu \in \mathbb{N}$ we construct the F-A approximation in the form:

$$v^{(2\nu)}(x) = (-1)^{\nu}\varepsilon^{-2\nu}\sum_{k=1}^{n}\left(\frac{k}{n+1}\right)^{2\nu} c_k u_k(x) + A_{2\nu} + R_{2\nu}, \tag{1.7}$$

$$A_{2\nu} = \frac{2}{l}\sum_{j=-\nu+1}^{m}\varepsilon^{2j}\left(v^{(2\nu+2j-1)}(b)\beta_{2j}\left(\frac{x-a+l}{2l}\right) - v^{(2\nu+2j-1)}(a)\beta_{2j}\left(\frac{x-a}{2l}\right)\right), \tag{1.8}$$

$$R_{2\nu}=\frac{\varepsilon^{2m+1}}{l}\int_a^b v^{(2\nu+2m+1)}(\tau)\left[\beta_{2m+1}\left(\frac{x+\tau-2a}{2l}\right)+\beta_{2m+1}\left(\frac{\tau-x}{2l}\right)\right]d\tau, \tag{1.9}$$

where $\beta_0(x)=\frac{1}{2}+\sum_{k=1}^{n}\cos 2\pi kx$, but

$$\beta_{-2\mu}(x)=(-1)^{\mu}\sum_{k=1}^{n}\left(\frac{k}{n+1}\right)^{2\mu}\cos 2\pi kx, \quad \mu=1,2,\dots \tag{1.10}$$

$$\beta_{-2\mu-1}(x)=(-1)^{\mu+1}\sum_{k=1}^{n}\left(\frac{k}{n+1}\right)^{2\mu+1}\sin 2\pi kx, \quad \mu=0,1,\dots \tag{1.11}$$

We obtain the F–A approximation of derivatives $v^{(2\nu-j)}(x), j=1,2,\dots,2\nu$ from (1.7) by integration over x. For $j=2\nu$ we obtain the F–A approximation of function $v=v(x)$ from (1.2) with the substitution of $m+\nu$ for m.

The F–A approximation of functions system $v^{(k)}(x)$, $k.=0,1,\dots,2\nu$ is formed with $n+1$ leading Fourier coefficients of function $v=v(x)$ $c_0,c_1,\dots,c_n$ and with values of derivatives $v^{(2s-1)}(x)$ by $x=a$, $x=b$ and $s=1,2,\dots,m+\nu$. The F–A approximation has an asymptotic character with respect to a small parameter $\varepsilon=\frac{l}{\pi(n+1)}$.

If we put the values $x=x_j=a+\frac{l}{n}j$, $j=0,1,\dots,n$ in the representation (1.2) and from the obtained system express the Fourier coefficients $c_0,c_1,\dots,c_n$ by means of functions values $v_j=v(x_j)$, $j=0,1,\dots,n$ in the points of uniform net, we get another variant of F–A approximation by replacing the now obtained coefficients c_j in (1.2).

If we restrict the F–A approximation of functions system $v^{(k)}(x)$, $k=0,1,\dots,2\nu$ onto the points of net $\{x_j\}_0^n$ we get for $k=1,2,\dots,2\nu$ the approximations formulae of the derivatives. In matrix form they are :

$$V^{(k)}=V_p^{(k)}+R_k \qquad V^{(k)}=\left(v^{(k)}(x_0),\dots,v^{(k)}(x_n)\right)^T \tag{1.12}$$

$$V_p^{(k)}=\varepsilon^{-k}\left(T^{(k)}V+\frac{2}{l}\sum_{j=1}^{m+\nu}\varepsilon^{2j}P_j^{(k)}G_j\right), \qquad V=(v_0,\dots,v_n)^T, \tag{1.13}$$

$$R_k=\frac{\varepsilon^{2(m+\nu)-k+1}}{l}\int_a^b v^{(2m+2\nu+1)}(\tau)Q_k(\tau)\,d\tau. \tag{1.14}$$

where $G_j=\left(v^{(2j-1)}(b),-v^{(2j-1)}(a)\right)^T$ $T^{(k)},P_j^{(k)}$ are known matrices and $Q_k(\tau)$ is a known matrix function. These matrices depend on n only, i. e. they are independent of ν and of the approximations interval $[a,b]$. It is

obvious, that the obtained approximations have an asymptotic character. If the function $v = v(x)$ is a polynomial of degree less than $2m + 2\nu + 1$, the error R_k annuls and the equality $V^{(k)} = V_p^{(k)}$ holds.

2. NUMERICAL EXAMPLES

We consider some examples of approximation based on formula (1.12). Let $\nu = 1$, $K_1 = \|V^{(1)} - V_p^{(1)}\|$, $K_2 = \|V^{(2)} - V_p^{(2)}\|$ (norm is considered in $\mathbb{R}^{n+1}$, i. e. $\|V\|^2 = v_0^2 + v_1^2 + \cdots + v_n^2$), $I = \frac{\varepsilon^{2m+1}}{l} \max\limits_{x\in[a,b]} |v^{(2m+3)}(x)|$.

In the table 1 we give the results for the function $v(x) = \exp(-x^2)$, $x \in [0; 2]$ by $n = 4$. In the table 2 are given the results for $v(x) = J_5(x)$, $x \in [0; 10]$ by $n = 8$. In the tables 3, 4 and 5 are given the results for $v(x) = \frac{1}{1+x^2}$, $x \in [0; 2]$ by $n = 4$, $n = 8$ and $n = 16$ respectively. In the table 6 are given the results for $v(x) = 10\exp(-0.25x)((x-1)^2 + 9)^{-\frac{3}{2}}$, $x \in [0, 2]$ by $n = 8$. In the tables 7 and 8 are given the results for $v(x) = \frac{9}{9+x^3}$, $x \in [0, 2]$ by $n = 4$ and $n = 8$ respectively.

Table 1.

m	K_1	K_2	I
-1	$8 \cdot 10^{-2}$	0.5	
0	$3\ 10^{-3}$	$6\ 10^{-2}$	0.25
2	$8 \cdot 10^{-5}$	$4\ 10^{-4}$	$7\ 10^{-2}$
4	$9 \cdot 10^{-7}$	$3 \cdot 10^{-6}$	$6 \cdot 10^{-4}$

Table 2.

m	K_1	K_2	I
-1	0.1	0.2	–
0	$3 \cdot 10^{-3}$	$1.5 \cdot 10^{-2}$	$6 \cdot 10^{-2}$
2	$3 \cdot 10^{-5}$	$5 \cdot 10^{-5}$	$6 \cdot 10^{-5}$
4	$4 \cdot 10^{-7}$	$6 \cdot 10^{-7}$	$8 \cdot 10^{-7}$

Table 3.

m	K_1	K_2	I
-1	0.2	0.9	–
0	10^{-2}	$6\ 10^{-2}$	0.3
2	10^{-2}	$4\ 10^{-2}$	0.09
4	10^{-3}	$4 \cdot 10^{-2}$	0.15

Table 4.

m	K_1	K_2	I
−1	0.2	1.8	–
0	$3 \cdot 10^{-4}$	$9 \cdot 10^{-3}$	0.2
2	$6 \cdot 10^{-5}$	$2 \cdot 10^{-4}$	$4 \cdot 10^{-2}$
4	$6 \cdot 10^{-5}$	$2 \cdot 10^{-4}$	$8 \cdot 10^{-4}$

Table 5.

m	K_1	K_2	I
−1	0.2	3.6	–
0	10^{-4}	$5 \cdot 10^{-3}$	0.1
2	$3 \cdot 10^{-9}$	$3 \cdot 10^{-8}$	$2 \cdot 10^{-4}$
4	$5 \cdot 10^{-10}$	$2 \cdot 10^{-9}$	$3 \cdot 10^{-6}$

Table 6.

m	K_1	K_2	I
−1	0.1	1.2	–
0	10^{-4}	$4 \cdot 10^{-3}$	$4 \cdot 10^{-3}$
2	$4 \cdot 10^{-8}$	$2 \cdot 10^{-7}$	$1.5 \cdot 10^{-6}$
4	$7 \cdot 10^{-11}$	$3 \cdot 10^{-10}$	$5 \cdot 10^{-9}$

Table 7.

m	K_1	K_2	I
−1	0.4	2.14	–
0	$3 \cdot 10^{-2}$	$6 \cdot 10^{-2}$	$4 \cdot 10^{-2}$
2	10^{-5}	$8 \cdot 10^{-5}$	10^{-3}
4	10^{-5}	$6 \cdot 10^{-5}$	10^{-3}

These numerical results confirm the asymptotic behaviour of approximation (1.12). The exactness of results increase with m considerably by constant n. The two former functions are integer. The last three functions have singular points, whose influences are not taken into account in the asymptotic approximation of Fourier coefficients. It is necessary to increase n.

Further we consider the solution of the boundary problem:

$$\begin{cases} v^{(2)}(x) + \omega(x)v(x) = f(x), \qquad x \in [a, b] \\ \alpha_1 v(a) + \beta_1 v^{(1)}(a) + \gamma_1 v(b) + s_1 v^{(1)}(b) = \mu_1 \\ \alpha_2 v(a) + \beta_2 v^{(1)}(a) + \gamma_2 v(b) + s_2 v^{(1)}(b) = \mu_2, \end{cases} \tag{2.1}$$

Table 8.

m	K_1	K_2	I
−1	0.4	4.2	–
0	10^{-3}	$3\cdot10^{-2}$	$2\cdot10^{-2}$
2	$8\cdot10^{-7}$	$5\cdot10^{-6}$	$5\cdot10^{-5}$
4	$2\cdot10^{-9}$	10^{-8}	$7\ 10^{-7}$

where the functions $\omega = \omega(x)$ and $f = f(x)$ are sufficiently smooth. We approximate the derivative $v^{(2)}(x)$ using the formula (1.12) for $\nu = 1$ and $k = 2$. From the equation of problem (2.1) we find the values of derivatives, which are necessary for the calculation of $V_p^{(2)}$:

$$v^{(k)}(x) = q_k(x)v(x) + p_k(x)v'(x) + s_k(x), \qquad k = 0, 1, \ldots, \tag{2.2}$$

where $q_0 = 1$, $p_0 = 0$, $s_0 = 0$ and $q_{k+1}(x) = q_k'(x) - \omega(x)p_k(x)$, $p_{k+1}(x) = p_k'(x) + q_k(x)$, $s_{k+1}(x) = s_k'(x) + p_k(x)f(x)$, $k = 0, 1, \ldots$

As a result we get the matrix equation $ZV^* = F + R$, where $V^* = (v'(x_0), v(x_0), \ldots, v(x_n), v'(x_n))^T$ Z and F are known matrices and the methods error is $R = \frac{\varepsilon^{2m+1}}{l}\int_a^b v^{(2m+3)}(\tau)Q_2^*(\tau)\,d\tau$. Here $Q_2^*(\tau) = (0, Q_2(\tau), 0)^T$, $v^{(2m+3)}(\tau) = q_{2m+3}(\tau)v(\tau) + p_{2m+3}(\tau)v'(\tau) + s_{2m+3}(\tau)$. From the solution of equation $ZV_p^* = F$ we get the approximation V_p^* for V^* We give in the table 9 the numerical results for $\omega(x) = -\frac{2}{x^2}$, $f(x) = \frac{10}{x}$, $x \in [1;2]$. The boundary conditions are $v(1) = 0$; $v(2) + 2v'(2) = 0$ and $m = 4$. Moreover $K = \|V - V_p\|$, $V = (v(x_0), \ldots, v(x_n))^T$, V_p is a corresponding matrix of approximate values, $s = \max\limits_{x\in[1,2]} |\tilde{s}(x)|$ and $\tilde{s}(x) = \frac{2}{l}\varepsilon^{10}s_{11}(x)$. The parameters p and q are defined similarly. We note, that $\max\limits_{x\in[1,2]} |v(x)| = 1.75$.

For the functions $\omega(x) = -\frac{600}{x^2}$, $f(x) = \frac{100}{x}$ and the same boundary conditions we give the results by $m = 4$ in the table 10. In this case $\max\limits_{x\in[1,2]} |v(x)| = 0.3$.

Table 9.

n	s	p	q	K
2	2.5	$5\cdot10^{-2}$	$9\cdot10^{-2}$	10^{-2}
4	$2.5\cdot10^{-3}$	$4.5\cdot10^{-4}$	$9\cdot10^{-4}$	$3\cdot10^{-4}$
8	$1.2\cdot10^{-5}$	$2.5\cdot10^{-6}$	$4\ 10^{-6}$	$1.5\cdot10^{-5}$
16	$3.5\ 10^{-8}$	$8\cdot10^{-9}$	$1.4\cdot10^{-8}$	$8.4\cdot10^{-7}$

It is obvious that the high exactness of numerical results is achievable by comparatively small n. In the last case (table 10) the exactness of numerical

Table 10.

n	s	p	q	K
8	150	40	10^3	$1.7\ 10^{-1}$
16	0.5	0.14	3	$5 \cdot 10^{-4}$
32	10^{-3}	$4 \cdot 10^{-4}$	$8 \cdot 10^{-3}$	$2 \cdot 10^{-5}$

results are comparatively lower. It is caused by small parameter at the second degree derivative in the equation of problem (2.1).

We consider one more problem:

$$\begin{cases} u_t^{(1)} + (uu_x^{(3)})_x^{(1)} = 0, \quad x \in (0,1), \quad u|_{t=0} = f(x), \\ u_x^{(1)}|_{x=0} = u_x^{(1)}|_{x=1} = u_x^{(3)}|_{x=0} = u_x^{(3)}|_{x=1} = 0. \end{cases} \tag{2.3}$$

Here the unknown function $u = u(x,t)$ is dependent on two arguments. From (2.3) we get the equalities $u_x^{(2s-1)}|_{x=0,1} = 0$ for $\forall s = 1, 2, \ldots$. We solved the problem (2.3) using the method of lines [2]. We approximate the partial derivatives $u_x^{(k)}$, $k = 1, 2, 3$ and 4 in the points of net $x_j = \frac{j}{n}$, $j = 0, 1, \ldots, n$ by means of the formula (1.12). Then $U^{(k)} \equiv \left(u_x^{(k)}(x_0,t), \ldots, u_x^{(k)}(x_n,t)\right)^T = \varepsilon^{-k} T^{(k)} U + R_k$, $U = (u(x_0,t), \ldots, u(x_n,t))^T$ The error R_k satisfies the estimates: $R_k = o(\varepsilon^N)$, $\varepsilon \to +0$, $\forall N \in \mathbf{N}$. Thus, we approximate the problem (2.3) by means of Cauchy problem for system of $n+1$ equations:

$$\frac{d}{dt} U = F(U), \qquad U|_{t=0} = (f(x_0), \ldots, f(x_n))^T \tag{2.4}$$

where $F = F(U)$ is a known function. We solve the problem (2.4) with the help of the method of degenerate matrices [3].

The specific character of the nonlinear problem (2.3) is expressed in the following. If the function $f(x)$ is nonnegative, the solution of problem (2.3) is stable with increasing t and unstable with decreasing t (like in the inverse heat conduction problem). The numerical examples show, that by small n the F–A approximation work both by increasing t and by decreasing t. We carry out the calculations for $n = 5$ and solve the Cauchy problem (2.4) to within up 32 significant digits. If $U|_{t=0} = (0,\ 0.003,\ 0,\ 0,\ 0.003,\ 0)^T$, we get $U|_{t=0.1} = (10^{-33},\ 0.010374843\ldots\ 0.019625156\ldots,\ 0.019625156\ldots,\ 0.010374843\ldots,\ 10^{-33})^T$ By $t = 0.1$ the solution become stationary and we get 8 stable significant digits. If we solve the inverse problem with the initial value $U|_{t=0.1}$, we get $U|_{t=0} = (10^{-33}, 0.02\underbrace{9\ldots9}_{22}\ldots, 1.009 \cdot 10^{-25}, 1.083 \cdot 10^{-25}, 0.02\underbrace{9\ldots9}_{22}\ldots, 10^{-33})^T$ Similarly, if $U|_{t=0} = (0.05,\ 0,\ 0,\ 0,\ 0,\ 0.05)^T$, we get $U|_{t=0.145} = (0.010947\ldots, 0.010299\ldots, 0.009226\ldots, 0.009226\ldots, 0.010299\ldots,$

$0.010947\ldots)^T$ and for inverse solution we get $U|_{t=0} = (0.04\underbrace{9\ldots9}_{11}\ldots, 1.04 \cdot 10^{-14}, 7.85 \cdot 10^{-15}, 7.85 \cdot 10^{-15}, 1.04 \cdot 10^{-14}, 0.04\underbrace{9\ldots9}_{11}\ldots)^T$.

3. CONCLUSION

The high exactness of results, by relatively small number of the points of interpolation, is the basic merit of the F–A approximation. By using the integral representation of error, we can analyze a priori the exactness of results. These effects are achieved by using in the F–A approximation an additional analytic information about the investigated function, the structure of its asymptotic expansion. It is possibly to construct the algorithms of F–A approximation on the basis of other types of Fourier series.

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FURJĒ–ASIMPTOTISKĀ APROKSIMĀCIJA

MIHAILS BELOVS un JĀNIS SMOTROVS

Darbā ir aplūkota jauna Furjē-asimptotiskā (F–A) metode funkciju aproksimācijai. Metodē tiek izmantoti izvirzījumi Furjē rindās un integrāļu asimptotiskie attīstījumi. Šo metodi raksturo augsta aproksimācijas precizitāte pie relatīvi maza interpolācijas punktu skaita. Sevišķi svarīga šī īpašība ir nestabilu problēmu risināšanā. Ir doti skaitliskie piemēri, kas ilustrē F–A aproksimācijas efektivitāti un iespējas diferenciālvienādojumu risināšanā.

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THE NUMERICAL SOLUTION OF THE HEAT CONDUCTION PROBLEM FOR A WINDOW

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ABSTRACT

The task is to determine the temperature distribution in a window of living room, consisting of two glasses and air layer between them.

1. FORMULATION OF THE PROBLEM

We shall consider problem of mathematical physics for the steady-state heat conduction equation without convection. In general, the way a problem is posed in a 2-D domain D with the boundary $S = \partial D$ has the form:

$$\begin{cases} \operatorname{div}(k \operatorname{grad} u) = -f & \text{in domain } D \\ -k\dfrac{\partial u}{\partial n} \qquad\quad \alpha(u - u_A) & \text{on boundary } \partial S, \end{cases} \tag{1}$$

where

$k > 0$, $\alpha \geq 0$ are the coefficients of heat conductivity and heat transfer,

$\vec{n}$ - the vector of the external normal to boundary S,

$\dfrac{\partial u}{\partial n} = \dfrac{\partial u}{\partial x}\cos(\vec{n}, o\vec{x}) + \dfrac{\partial u}{\partial y}\cos(\vec{n}, o\vec{y})$ -the normal derivative,

u_A - the temperature of external medium,

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f – the density function of the heat source,

$u = u(x,y)$ – the distribution of temperature field in domain D.

If a boundary of domain D is isolated, then $\alpha = 0$ and $\dfrac{\partial u}{\partial n} = 0$ (Neumann boundary conditions), but in case such a boundary has a given temperature, $\alpha = \infty$ and $u = u_A$ (Diriclet boundary conditions). There are no heat sources in the domain – $f = 0$.

If a domain D consists of an N-layer medium (the layers being rectangular) with different physical parameters $k_i = const$, $i = \overline{1,N}$, $a_0 = 0$, that is,

$$D_i = \{ (x,y) \mid 0 < x < L,\ a_{i-1} < y < a_i \},$$

and its edges are parallel to coordinate axes, on the surfaces S_i where the media have continuity (namely, on the edges of rectangles D_i),

$$S_i = \{ (x,y_i) \mid 0 < x < L,\ y_i = a_i \}$$

we have the following continuity conditions for temperature u_i and heat flux $k_i \dfrac{\partial u_i}{\partial y}$:

$$\begin{aligned} u_i|_{y=a_i} &= u_{i+1}|_{y=a_i} \\ k_i \left.\frac{\partial u_i}{\partial y}\right|_{y=a_i} &= k_{i+1} \left.\frac{\partial u_i}{\partial y}\right|_{y=a_i} \end{aligned} \tag{2}$$

Here $u_i = u_i(x,y)$ is a temperature function in rectangle $D_i (i = \overline{1, N-1})$, but Eqs. (1) have the $k_i \Delta u_i = f_i$, (Δ – Laplace operator when $k_i = const$).

In this concrete case to compute the temperature in a window of a living house, we take x,y as directions of window edges (namely, the length and the width). A distinguishing feature of such solution of the problem – the window length is much greater than its width. Therefore we consider an $N = 3$ layer medium with the temperature distribution in a restricted area $D = \{ 0 \le y \le a_3, 0 \le x \le L \}$. So we have a medium of three layers:

1) two of them – window glasses (D_1 and D_3),

2) one layer of the air between the two (D_2).

The internal space presents a half of the plane $y < 0$, while external – $y > a_3$.

We assume the coefficients of heat conductivity in each layer to be $k_1 = k_3 = 0.74\ W/(m \cdot deg)$ in D_1 and D_3 and coefficient of air heat conductivity $k_2 = 0.0257\ W/(m \cdot deg)$ in D_2.

We can write the 2–D case of Poisson equation for each layer ($i = \overline{1,3}$) as follows:

$$k_i \left(\frac{\partial^2 u_i}{\partial x^2} + \frac{\partial^2 u_i}{\partial y^2} \right) = -f_i. \tag{3}$$

Functions $u_1(x,y)$, $u_2(x,y)$, $u_3(x,y)$ describe the temperatures in internal glass, air layer and external glass, respectively. Numbers a_1, a_2, a_3 are coordinates of the boundaries glasses–air–glasses, L – length of a part of window is shown on this figure:

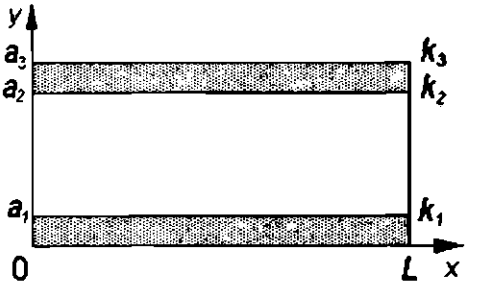

Figure 1.

On the external edges the following processes take place:

1) heat exchange with internal space ($y = 0$):

$$k_1 \frac{\partial u_1}{\partial y} = \alpha_{ist}(u_1 - u_{ist}), \tag{4}$$

2) heat exchange with external space ($y = a_3$):

$$k_1 \frac{\partial u_3}{\partial y} = \alpha_A(u_A - u_3), \tag{5}$$

where α_{ist}, α_A are the coefficients of heat transfer with indoors and outdoors, u_{ist}, u_A - temperature of room and outside air,

3) heat exchange with wooden parts of the window:

$$\begin{aligned} (x = 0) \quad & k_i \frac{\partial u_i}{\partial y} = \alpha_k(u_i - u_{0i}), \\ (x = L) \quad & k_i \frac{\partial u_i}{\partial y} = \tilde{\alpha}_k(\tilde{u}_{0i} - u_i), \end{aligned} \tag{6}$$

where $\tilde{\alpha}_k, \alpha_k$ are the coefficients of heat transfer, $\tilde{u}_{0i}, u_{0i}$ is the temperature of the wooden edge.

We have conditions (2) on the continuous surfaces of the layers:

$$S_1(y = a_1): \quad u_1 = u_2, \quad k_1 \frac{\partial u_1}{\partial y} = k_2 \frac{\partial u_2}{\partial y}, \tag{7}$$

$$S_2(y = a_2): \quad u_2 = u_3, \quad k_2 \frac{\partial u_2}{\partial y} = k_3 \frac{\partial u_3}{\partial y}. \tag{8}$$

2. SOLUTION OF THE 1–D HOMOGENEOUS PROBLEM

We assume that $f = 0$ and the heat spreads only in the direction of y–axis. Therefore Eqs. (3) are written as follows ($N = 3$):

$$k_i \frac{\partial^2 u_i}{\partial y^2} = 0 \quad i = \overline{1, N}.$$

(On the edges $x = 0$, $x = L$ we have Neumann boundary conditions). The temperature in every rectangle is described by a linear function dependent only on y:

$$u_i(y) = c_i y + d_i, \tag{9}$$

where c_i, d_i ($i = \overline{1, N}$) are unknown constants. To find them from conditions (4), (5), (7), (8) we should obtain the exact solution of this problem. As a result, we have a system of $2N$ equations enabling us to find $2N$ unknown constants:

$$\begin{cases} c_1(k_1 - \alpha_0 a_0) - \alpha_0 d_1 & = & -\alpha_0 u_0 & \\ k_i c_i - k_{i+1} c_{i+1} & = & 0 & i = \overline{1, N-1} \\ d_i + c_{i+1} a_i \left(\dfrac{k_{i+1}}{k_i} - 1 \right) - d_{i+1} & & 0 & i = \overline{1, N-1} \\ c_N(k_n + \alpha_A a_N) + \alpha_N d_N & & \alpha_A u_A. & \end{cases}$$

From these conditions we derive the system of algebraic equations

$$A\vec{p} = \vec{b} \tag{10}$$

where vectors are written as

$$\vec{p} = \begin{pmatrix} c_1 \\ d_1 \\ \vdots \\ c_N \\ d_N \end{pmatrix} \} 2N - 4, \quad \vec{b} = \begin{pmatrix} -\alpha_0 u_0 \\ 0 \\ \\ 0 \\ \alpha_A u_A \end{pmatrix} \} 2N - 2$$

and A is the tridiagonal matrix. System (10) we can solve by the factorisation method [4].

3. AN EXACT DIFFERENCE SCHEME

Consider an 1–D case of the given problem (the number of layers is $N = 3$). We write Eqs. (3) in the form:

$$(k_i u_i')' = -f_i. \tag{11}$$

Then choose nonuniform grid in every layer (D_1, D_2, D_3):

$$\{ y_0 = 0, y_1, \ldots, y_{L1} = a_1, y_{L1+1}, \ldots, y_{L2} = a_2, y_{L2+1}, \ldots, y_{L3} = a_3 \},$$

where $L1$, $L2$, $L3$ are the natural numbers.

By using the finite volumes method [3] we derive difference equations on surfaces S_1, S_2 ($y = a_1$, $y = a_2$). The approximation in others points of the grid is in its standard form, i. e., central-difference expressions of the 2nd order. We shall consider approximation in the grid point $y_L \equiv y_{L1} = a_1$ (the same holds for the grid point $y_{L2} = a_2$) and choose such points $\{ y_{L-1}, y_L, y_{L+1} \}$ with steps $g_L^+ = y_{L+1} - y_L$, $g_L^- = y_L - y_{L-1}$.

We define now the heat flux $W = (k_i u_i)'$, which is continuous in the interval $[y_-, y_+]$, where $y_- = (y_L + y_{L-1})/2$, $y_+ = (y_{L+1} + y_L)/2$, integrate Eqs. (11) from y_- to y_+ and get:

$$W_+ - W_- = -\int_{y_-}^{y_+} f\, dy = -\int_{y_-}^{y_L} f_-\, dy - \int_{y_L}^{y_+} f_+\, dy, \qquad (12)$$

where

$$W_\pm = W(y_\pm), \quad f = \begin{cases} f_-, & y \in (y_-, y_L) \\ f_+, & y \in (y_L, y_+). \end{cases}$$

This is an integral form of the conservation law for $[y_-, y_+]$. To make the exact difference scheme we integrate Eq. (11) from y_- to y where $y \in [y_{L-1}, y_L]$. Then

$$W(y) - W_- = -\int_{y_-}^{y} f_-\, dt, \quad W(y) = k_1 u'(y).$$

After dividing by k_1 and integrating from y_{L-1} to y_L we obtain:

$$u_L^- - u_{L-1}^- = \frac{g_L^-}{k_1} W_- - \frac{1}{k_1} \int_{y_{L-1}}^{y_L} \int_{y_-}^{y} f_-\, dt\, dy,$$

where $u^-(y) = u_1(y)$, $u_L^- = u^-(y_L)$, $u_{L-1}^- = u^-(y_{L-1})$, $k_- = k_1$.

Hence

$$W_- = \frac{k_-}{g_L^-}(u_L^- - u_{L-1}^-) - B_-,$$

with

$$B_- = -\frac{1}{g_L^-} \int_{y_{L-1}}^{y_L} \int_{y_-}^{y} f_-\, dt\, dy.$$

Similarly, by determining the heat flux W_+ of (12) in interval (y_L, y_{L+1}) one obtains:

$$W_+ = \frac{k_+}{g_L^+}(u_{L+1}^+ - u_L^+) - B_+,$$

where

$$B_+ = -\frac{1}{g_L^+}\int_{y_L}^{y_{L+1}}\int_{y_+}^{y} f_+\,dt\,dy,$$

$$u^+(y) \equiv u_2(y),\ u_L^+ = u^+(y_L) = u^-(y_L),\ u_{L+1}^+ = u^+(y_{L+1}),\ k_+ = k_2.$$

To derive a 3–point difference equation associated with the central grid point y_L we should apply an equation of the form:

$$\frac{k_+}{g_L^+}(u_{L+1}^+ - u_L^+) - \frac{k_-}{g_L^-}(u_L^- - u_{L-1}^-) = B_+ - B_- - \int_{y_-}^{y_L} f_-\,dy - \int_{y_L}^{y_+} f_+\,dy = -F_L, \tag{13}$$

where

$$F_L = \frac{1}{g_L^-}\int_{y_{L-1}}^{y_L}(y - y_{L-1})f_-\,dy + \frac{1}{g_L^+}\int_{y_L}^{y_{L+1}}(y_{L+1} - y)f_+\,dy,$$

integrals B_+, B_- being modified by partial integral formula. From (13) we get the exact difference scheme in case there are discontinuous coefficients $k_+ = k_2$, $k_- = k_1$ and right sides f_+, f_- of Eq. (11).

The exact difference equations we can obtain in grid points $y_0 = 0, y_{L3} = a_3$ where we have boundary conditions (4), (5). First, we apply the integral form of the conservation law to the interval $[y_0, y_+]$, with $y_+ = y_1/2 = g_0^+/2$. We get:

$$W_+ - W_0 = -\int_{y_0}^{y_+} f_+\,dy,$$

where $f_+ = f_1$, $W_+ = W(y_+)$, $W_0 = k_1 u_1'(0) = \alpha_{ist}(u_1(0) - u_{ist})$.

In the same manner as before we integrate Eq. (11) from y_+ to y where $y \in [y_0, y_1]$, and from y_0 to y_1 we derive the following 2–point difference equations:

$$\frac{k_1}{g_0^+}(u_1(y_1) - u_1(0)) - \alpha_{ist}(u_1(0) - u_{ist}) = -F_0, \tag{14}$$

where

$$F_0 = \frac{1}{g_0^+}\int_{y_0}^{y_1}(y_1 - y)f_+\,dy.$$

If $y_{L3} = a_3$, we integrate Eq. (11) from y_- to y_{L3}, where

$$y_- = (y_{L3} - y_{L3-1})/2 = g_{L3}^-/2:$$

$$W_{L3} - W_- = -\int_{y_-}^{y_{L3}} f_-\,dy,$$

with $f_- = f_3$, $W_L = k_3 u_3'(a_3) = \alpha_A(u_A - u_3(y_L))$.

By integrating (11) from y_- to $y \in [y_-, y_{L3}]$ and from y_{L3} to y_{L3-1} we determine W_- and obtain 2-point difference equation in the following form:

$$\alpha_A(u_A - u_3(y_{L3})) - \frac{k_3}{g_{L3}}(u_3(y_{L3}) - u_3(y_{L3-1})) = -B_- - \int_{y_-}^{y_{L3}} f_-\, dy = -F_{L3}, \tag{15}$$

where

$$F_{L3} = \frac{1}{g_{L3}} \int_{y_{L3-1}}^{y_{L3}} (y - y_{L3-1}) f_-\, dy.$$

We can see now that difference equations (13), (14), (15) are exact approximations for solving 1-D problem (3-5), (7) depending only on y. We consider these equations in a nonuniform grid with 4 points:

$$\{ (y_j) \mid j = 0, \ldots, 3;\ y_0 = 0;\ y_1 = a_1;\ y_2 = a_2;\ y_3 = a_3 \},$$

with steps $g_1 = a_1$, $g_2 = a_2 - a_1$, $g_3 = a_3 - a_2$. By denoting $v_j = u(y_j)$ one obtains the system of 4 equations:

$$\begin{cases} \dfrac{k_1}{g_1}(v_1 - v_0) - \alpha_{ist}(v_1 - u_{ist}) = -F_0 & (j = 0) \\ \dfrac{k_2}{g_2}(v_2 - v_1) - \dfrac{k_1}{g_1}(v_1 - v_0) = -F_1 & (j = 1) \\ \dfrac{k_3}{g_3}(v_3 - v_2) - \dfrac{k_2}{g_2}(v_2 - v_1) = -F_2 & (j = 2) \\ \alpha_A(u_A - v_3) - \dfrac{k_3}{g_3}(v_3 - v_2) = -F_3 & (j = 3) \end{cases} \tag{16}$$

where

$$F_0 = \int_{y_0}^{y_1} \left(1 - \frac{y - y_0}{g_1}\right) f_1(y)\, dy,$$

$$F_1 = \int_{y_0}^{y_1} \left(1 - \frac{y_1 - y}{g_1}\right) f_1(y)\, dy + \int_{y_1}^{y_2} \left(1 - \frac{y - y_1}{g_2}\right) f_2(y)\, dy,$$

$$F_2 = \int_{y_1}^{y_2} \left(1 - \frac{y_2 - y}{g_2}\right) f_2(y)\, dy + \int_{y_2}^{y_3} \left(1 - \frac{y - y_2}{g_3}\right) f_3(y)\, dy,$$

$$F_3 = \int_{y_2}^{y_3} \left(1 - \frac{y_3 - y}{g_3}\right) f_3(y)\, dy.$$

4. SOLVING OF EXACT DIFFERENCE SCHEME

System (16) we can solve by the factorisation method or more simply. For this purpose, from the first equation we conclude that

$$-\tilde{\alpha}_1(v_1 - u_{ist}) + \frac{k_1}{h_1}(v_1 - v_0) = -\frac{\tilde{\alpha}_1}{\alpha_{ist}} F_0, \tag{17}$$

where

$$(\tilde{\alpha}_1)^{-1} = \frac{1}{\alpha_{ist}} + \frac{1}{k_1/h_1}.$$

We put (17) into the second equation of (16) and get

$$\frac{k_2}{h_2}(v_2 - v_1) - \tilde{\alpha}_1(v_1 - u_{ist}) = -\tilde{\alpha}_1\tilde{F}_1,$$

where $\tilde{F}_1 = \dfrac{F_1}{\tilde{\alpha}_1} + \dfrac{F_0}{\alpha_{ist}}$.

Hence

$$\frac{k_3}{h_3}(v_3 - v_2)^{\cdot} - \tilde{\alpha}_3(v_2 - u_{ist}) = -\tilde{\alpha}_3\tilde{F}_3, \tag{18}$$

where $\tilde{F}_3 = \dfrac{F_0}{\alpha_{ist}} + \dfrac{F_1}{\tilde{\alpha}_1} + \dfrac{F_2}{\tilde{\alpha}_2}$, $(\tilde{\alpha}_3)^{-1} = \dfrac{1}{\alpha_{ist}} + \dfrac{1}{k_1/h_1} + \dfrac{1}{k_2/h_2}$ is the inverse value of the interaction coefficient of 3 layers in the same direction.

From the last equation (16) ($j = 3$) we conclude that

$$\alpha_A(u_A - v_2) - \left(\frac{k_3}{h_3} + \alpha_A\right)(v_3 - v_2) = -F_3. \tag{19}$$

from which it follows

$$\alpha_2^*(u_A - v_2) - \frac{k_3}{h_3}(v_3 - v_2) = -\frac{\alpha_2^*}{\alpha_A}F_3,$$

where $(\alpha_2^*)^{-1} = \dfrac{1}{\alpha_A} + \dfrac{1}{k_3/h_3}$ is the inverse value of the interaction coefficient of two layers in opposite direction.

The last equation we put into (18) and get v_2 as:

$$v_2 = \frac{\tilde{\alpha}_2 u_{ist} + \alpha_2^* u_A + \tilde{F}_2^*}{\tilde{\alpha}_2 + \alpha_2^*}, \tag{20}$$

where $\tilde{F}_2^* = \alpha_2\tilde{F}_2 + \dfrac{\alpha_2^*}{\alpha_A}F_3$.

From (19) we derive v_3, where v_2 is expression (20):

$$v_3 = \left(\frac{k_3}{h_3} + \alpha_A\right)^{-1}\left(\alpha_A u_A + \frac{k_3}{h_3}v_2 + F_3\right) \tag{21}$$

We can also consider the opposite direction. At $j = 3$ and $j = 2$ from (16) it follows:

$$\alpha_2^*(u_A - v_2) - \frac{k_2}{h_2}(v_2 - v_1) = -F_2^*\alpha_2^*,$$

where $F_2^* = \dfrac{F_2}{\alpha_2^*} + \dfrac{F_3}{\alpha_A}$, $(\alpha_2^*)^{-1} = \dfrac{1}{\alpha_A} + \dfrac{1}{k_3/h_3}$.

Therefore

$$\alpha_1^*(u_A - v_1) - \frac{k_1}{h_1}(v_1 - v_0) = -F_1^*\alpha_1^* \tag{22}$$

where $F_1^* = \dfrac{F_1}{\alpha_1^*} + \dfrac{F_2}{\alpha_2^*} + \dfrac{F_3}{\alpha_A}$,

$$(\alpha_1^*)^{-1} = \frac{1}{\alpha_A} + \frac{1}{k_3/h_3} + \frac{1}{k_2/h_2}.$$

We put Eq. (17) into (22) and get

$$\alpha_1^*(u_A - v_1) + \tilde{\alpha}_1(u_{ist} - v_1) = -F_1^*\alpha_1^* - \frac{\tilde{\alpha}_1}{\alpha_{ist}}F_0.$$

It follows that

$$v_1 = \frac{\alpha_1^* u_A + \tilde{\alpha}_1 u_{ist} + \tilde{F}_1^*}{\alpha_1^* + \tilde{\alpha}_1} \tag{23}$$

where $\tilde{F}_1^* = F_1^*\alpha_1^* + \dfrac{\tilde{\alpha}_1}{\alpha_{ist}}F_0$.

From the first differential equation (16) we can write an expression where v_1 is Eq. (23):

$$v_0 = \left(\frac{k_1}{h_1} + \alpha_{ist}\right)^{-1}\left(\frac{k_1}{h_1}v_1 + \alpha_{ist}u_{ist} + F_0\right) \tag{24}$$

By expressions (20), (21), (23), (24) we obtain the solutions of the given problem for surfaces a_i, $i = \overline{0,3}$. Assuming that the heat source function $f = 0$, we can write the temperature inside the window for every $y_0 \in [0, a_3]$:

$$u(y_0) = \frac{\alpha^* u_A + \tilde{\alpha} u_{ist}}{\alpha^* + \tilde{\alpha}},$$

where

1) if $y_0 \in [0, a_1]$: $(\tilde{\alpha})^{-1} = \dfrac{1}{\alpha_{ist}} + \dfrac{y_0}{k_1}$, $(\alpha^*)^{-1} = \dfrac{1}{\alpha_A} + \dfrac{h_3}{k_3} + \dfrac{h_2}{k_2} + \dfrac{a_1 - y_0}{k_1}$,

2) if $y_0 \in [a_1, a_2]$: $(\tilde{\alpha})^{-1} = \dfrac{1}{\alpha_{ist}} + \dfrac{h_1}{k_1} + \dfrac{y_0 - a_1}{k_2}$, $(\alpha^*)^{-1} = \dfrac{1}{\alpha_A} + \dfrac{h_3}{k_3} + \dfrac{a_2 - y_0}{k_2}$.

3) if $y_0 \in [a_2, a_3]$: $(\tilde{\alpha})^{-1} = \dfrac{1}{\alpha_{ist}} + \dfrac{h_1}{k_1} + \dfrac{h_2}{k_2} + \dfrac{y_0 - a_2}{k_3}$, $(\alpha^*)^{-1} = \dfrac{1}{\alpha_A} + \dfrac{a_3 - y_0}{k_3}$.

5. DISCRETE APPROXIMATION OF THE PROBLEM IN THE 2-D CASE

We consider a case with boundary condition (6) on the edge $x = L$ and Neumann boundary condition on the edge $x = 0$ in domain D. Having chosen a measured grid, we replace Eqs. (3) and all boundary conditions by difference expressions of the second order and then solve this difference scheme by iteration methods ([1] – methods of iterations).

6. THE FINITE VOLUMES METHOD IN 2-D CASE

We consider 2–D problem approximation applying the finite volumes method to assume that in (11) $f_i = k_i \dfrac{\partial^2 u_i}{\partial x^2}$. We must approximate the right sides of expressions (13), (14), (15) to get a difference scheme of the 2nd order. This having done, we use the Tailor series for functions $f_\pm = f_\pm(y)$ in point $y = y_L$:

$$f_\pm(y) = f_\pm(y_L) + f'_\pm(y)(y - y_L) + (1/2)f''_\pm(y_L)(y - y_L)^2 + o(g^3).$$

We know that function $f(y)$ and derivatives $\dfrac{\partial^2 u}{\partial x^2}$ and $\dfrac{\partial^3 u}{\partial x^2 \partial y} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial^2 u}{\partial x^2}\right)$ are continuous in grid point $y = y_L$. As a result, we get

$$\begin{aligned} F_0 &= \left(\frac{1}{2}g + \frac{\alpha_{ist}}{6k_1}g^2\right) f'_+(0) + o(g^3); \\ F_{L3} &= \left(\frac{1}{2}g + \frac{\alpha_A}{6k_3}g^2\right) f'_-(a_3) + o(g^3); \\ F_L &= -\frac{k_1 + k_2}{2} g \left.\frac{\partial^2 u}{\partial x^2}\right|_{y=y_L} + o(g^3). \end{aligned} \tag{25}$$

By dividing expressions (13), (14), (15), (25) by g we obtain difference scheme:

$$\begin{cases} \dfrac{k_1}{g^2}(v_{ij+1} - v_{ij}) - \dfrac{\alpha_{ist}}{g}(v_{ij} - u_{ist}) \\ + \left(\dfrac{k_1}{2} + \dfrac{\alpha_{ist}g}{6}\right) \dfrac{v_{i+1j} - 2v_{ij} + v_{i-1j}}{h^2} = 0 \\ \dfrac{k_2}{g^2}(v_{ij+1} - v_{ij}) - \dfrac{k_1}{g^2}(v_{ij} - v_{ij-1}) + \dfrac{k_1 + k_2}{2h^2}(v_{i+1j} - 2v_{ij} + v_{i-1j}) = 0 \\ \dfrac{k_3}{g^2}(v_{ij+1} - v_{ij}) - \dfrac{k_2}{g^2}(v_{ij} - v_{ij-1}) + \dfrac{k_3 + k_2}{2h^2}(v_{i+1j} - 2v_{ij} + v_{i-1j}) = 0 \\ \dfrac{\alpha_A}{g}(u_A - v_{ij}) - \dfrac{k_3}{g^2}(v_{ij} - v_{ij-1}) \\ + \left(\dfrac{k_3}{2} + \dfrac{\alpha_A g}{6}\right) \dfrac{v_{i+1j} - 2v_{ij} + v_{i-1j}}{h^2} = 0. \end{cases} \tag{26}$$

For equations (26):

1) $j = 0$, $y = 0$;

2) $j = 1$, $y = a_1$;

3) $j = 2$, $y = a_2$;

3) $j = 3$, $y = a_3$.

The coefficient matrix for system (26) is positive definite and we can solve it by iteration methods.

7. AVERAGING PROBLEM

We consider 2–D case as in Chapt. 6. The way the problem is posed we replace by that of averaging the temperature $\bar{u}_1(x)$ on a width $0 < y < a_1$ and the temperature $\bar{u}_3(x)$ on a width $a_2 < y < a_3$:

$$\bar{u}_1(x) = \frac{1}{a_1} \int_0^{a_1} u_1(x, y)\, dy \tag{27}$$

$$\bar{u}_3(x) = \frac{1}{a_3 - a_2} \int_{a_2}^{a_3} u_3(x, y)\, dy. \tag{28}$$

Then temperature can be found for an contracted domain ($a_1 \le y \le a_2$) only. First,we do modifications in layer D_1 ($0 \le y \le a_1$). By integrating Eq. (3) ($i = 1$) from 0 to a_1 we get:

$$\frac{k_1}{a_1} \left\{ \left. \frac{\partial u_1}{\partial y} \right|_{y=a_1} - \left. \frac{\partial u_1}{\partial y} \right|_{y=0} \right\} + k_1 \frac{\partial^2 \bar{u}_1}{\partial x^2} = -\bar{f}_1(x),$$

where

$$\bar{f}_1(x) = \frac{1}{a_1} \int_0^{a_1} f_1(x, y)\, dy.$$

To use condition (7) we can write

$$k_2 \left. \frac{\partial u}{\partial y} \right|_{y=a_1} - k_1 \left. \frac{\partial u_1}{\partial y} \right|_{y=0} + a_1 k_1 \frac{\partial^2 \bar{u}_1}{\partial x^2} = -a_1 \bar{f}_1(x), \tag{29}$$

where $u = u_2(x, y)$.

We consider approximation of function $u_1(x, y)$ in the direction of y–axis, assuming that $u_1(x, y)$ is linear function:

$$u_1(x, y) = \frac{a_1 - y}{y} u(x, 0) + \frac{y}{a_1} u(x, a_1). \tag{30}$$

Similarly as in [2] we use condition (4), expressions (27), (30), (29) and obtain the boundary condition

$$k_2 \left.\frac{\partial u}{\partial y}\right|_{y=a_1} = \frac{k_1\alpha_{ist}}{k_1+\alpha_{ist}a_1}(u-u_{ist}) - a_1k_1\frac{\partial^2}{\partial x^2}\left(\frac{2k_1+\alpha_{ist}a_1}{2(k_1+\alpha_{ist}a_1)}u\right)$$
$$-a_1\bar{f}_1(x) - a_1k_1\frac{\partial^2}{\partial x^2}\left(\frac{a_1\alpha_{ist}}{2(k_1+\alpha_{ist}a_1)}u_{ist}\right) \tag{31}$$

We find the boundary condition when $y = a_2$ in case $y = a_1$ by assuming $u_3(x,y)$ is linear function depending on y:

$$u_3(x,y) = \frac{y-a_2}{a_3-a_2}u(x,a_3) + \frac{a_3-y}{a_3-a_2}u(x,a_2).$$

Further, we obtain

$$k_2 \left.\frac{\partial u}{\partial y}\right|_{y=a_2} = \frac{k_1\alpha_A}{k_1+\alpha_A(a_3-a_2)}(u_A-u)$$
$$+k_1(a_3-a_2)\frac{\partial^2}{\partial x^2}\left(\frac{2k_1+\alpha_A(a_3-a_2)}{2(k_1+\alpha_A(a_3-a_2))}u\right) \tag{32}$$
$$+k_1(a_3-a_2)\frac{\partial^2}{\partial x^2}\left(\frac{\alpha_A(a_3-a_2)}{2(k_1+\alpha_A(a_3-a_2))}u_A\right) + (a_3-a_2)\bar{f}_3(x)$$

where

$$\bar{f}_3(x) = \frac{1}{a_3-a_2}\int_{a_2}^{a_3} f_3(x,y)\,dy.$$

In this case we approximate Eq. (3) ($i = 2$) and conditions by difference expressions of the 2nd order. Similarly, we can consider averaging when $u_1(x,y)$, $u_2(x,y)$ are quadratic functions [2].

8. ANALYSIS OF RESULTS

In the 1–D case ($f = 0$) the same solutions have been obtained with two various methods. We consider case there is the heat source in layer D_1 (in interior glass). We use the exact difference scheme to obtain results at $f_1 = 10$ ($u_A = 0^0\text{C}, u_i = 20^0\text{C}, \alpha_{ist} = \alpha_A = 2, k_1 = 0.74, k_2 = 0.0257$):

$$u(0) = 19.7578; u(a_1) = 19.7082; u(a_2) = 0.5586; u(a_3) = 0.4921.$$

We conclude that the heat source has an influence not only in interior glass but in all domain D.

Dependence of heat transfer coefficient $\alpha = \alpha_{ist} = \alpha_A$. Values of function $u(y)$ can be seen in table 1:

Table 1.

	$\alpha = 0,01$	$\alpha = 0,1$	$\alpha = 1$	$\alpha = 2$	$\alpha = 5$	$\alpha = 10$
$u(0,6)$	10.892	5.0517	0.9263	0.4856	0.2	0.101
$u(0,55)$	10.8859	5.0858	0.9889	0.5512	0.2677	0.1693
$u(0,5)$	9.114	14.9141	19.011	19.4487	19.7323	19.83
$u(0)$	9.1078	14.9482	19.0736	19.5143	19.7999	19.898

The numerical experiments show: the smaller is coefficient α, the greater is an isolation on boundaries $y = 0$, $y = a_3$, and when $\alpha \to \infty$, temperature $u \to u_i$ on boundary $y = 0$ and $u \to u_A$ on boundary $y = a_3$.

Dependence of coefficients k_1, k_2. Values of function $u(y)$ are in table 2 ($u_i = 20^0\text{C}, u_A = 0^0\text{C}, \alpha = 2$):

Table 2.

	$k_1 = 7,4$ $k_2 = 0,0257$	$k_1 = 0,74$ $k_2 = 0,0257$	$k_1 = 0,0074$ $k_2 = 0,0257$	$k_1 = 0,74$ $k_2 = 2,57$	$k_1 = 0,74$ $k_2 = 0,00257$
$u(0,6)$	0.48855	0.4856	0.2943	7.5205	0.0511
$u(0,55)$	0.49515	0.551	4.272	8.5368	0.058
$u(0,5)$	19.5048	19.448	15.727	11.463	19.9441
$u(0)$	19.5114	19.514	19.705	12.479	19.9489

The greater are the coefficients k_1, k_2, the smaller changes of temperature are in the glass and air layers.

Change of the thickness of air layer $h = a_2 - a_1$. Values of function $u(y)$ are shown in table 3 ($u_A = 0^0\text{C}, u_i = 20^0\text{C}, k_1 = 0.74, k_2 = 0.0257$):

Table 3.

	$h = 0,5$	$h = 1,0$	$h = 1,2$
$u(0,6)$	0.4856	0.249	0.209
$u(0,55)$	0.551	0.283	0.237
$u(0,5)$	19.448	19.716	19.762
$u(0)$	19.514	19.7502	19.7909

So the air layer is wider: 1) the temperature on boundaries $y = 0$, $y = a_3$ is closer to indoor and outdoor temperatures; 2) influence of the outdoor temperature on the indoor temperature decreases.

The results of the 2–D case have been obtained by 3 methods. Conclusion the best of them are using the difference scheme (26). The corresponding results ($u_A = 0^0\text{C}, u_i = 20^0\text{C}, u_k = 20^0\text{C}, \alpha_{ist} = \alpha_A = 2, k_1 = 0.74, k_2 = 0.0257, L = 1, a_1 = 0.05, a_2 = 0.55, a_3 = 0.6$, steps $g = 0.05, h = 0.1$) are as follows:

0.49	0.49	0.49	0.49	0.49	0.49	0.50	0.51	0.54	0.58	0.64
0.55	0.55	0.55	0.55	0.56	0.56	0.57	0.58	0.61	0.66	0.78
2.44	2.44	2.44	2.44	2.45	2.45	2.46	2.48	2.52	2.58	2.71
4.33	4.33	4.33	4.34	4.34	4.35	4.36	4.38	4.42	4.49	4.61
6.22	6.22	6.22	6.23	6.23	6.24	6.25	6.28	6.32	6.39	6.49
8.11	8.11	8.11	8.12	8.12	8.13	8.14	8.17	8.21	8.27	8.37
10.00	10.00	10.00	10.01	10.01	10.02	10.03	10.05	10.09	10.15	10.23
11.89	11.89	11.89	11.90	11.90	11.91	11.92	11.94	11.97	12.02	12.09
13.78	13.78	13.78	13.78	13.79	13.79	13.80	13.82	13.84	13.88	13.93
15.67	15.67	15.67	15.67	15.68	15.68	15.69	15.70	15.72	15.74	15.78
17.56	17.56	17.56	17.56	17.56	17.57	17.57	17.58	17.59	17.60	17.62
19.45	19.45	19.45	19.45	19.45	19.45	19.45	19.45	19.46	19.46	19.46
19.51	19.51	19.51	19.51	19.52	19.52	19.52	19.52	19.52	19.52	19.53

Dependence of coefficient α_k. The greater is coefficient α_k, the more influence temperature u_k has over all domain.

The results for the averaging problem are the same as those obtained by two other methods in the domain $y \in [a_1, a_2]$.

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TEMPERATŪRAS SADALĪJUMA APRĒĶINĀŠANA ĒKAS LOGĀ

ANDA BROKA, HARIJS KALIS un ILGA PAGODKINA

Darbā aplūko stacionāro siltuma vadīšanas vienādojuma skaitlisko risināšanu, nosakot temperatūras sadalījumu dzīvojamās ēkas logā, kas sastāv no diviem stikliem un gaisa slāņa starp tiem.

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A. Buiķis, H. Kalis and A. Reinfelds (Eds)

APPROXIMATE ANALYTICAL TWO–DIMENSIONAL SOLUTION FOR A LONGITUDINAL FIN OF RECTANGULAR PROFILE

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Dedicated to Dr. Ilga Pagodkina's memory

ABSTRACT

In this paper it is constructed an approximate analytical solution for steady state heat exchange in a regular assembly with rectangular fins.

In this paper, a way is shown of finding an approximate analytical solution for heat exchange in a regular assembly with longitudinal fins. The paper is a continuation of the investigations initiated in [1, 2]. In [1] we started with attempts to generalize the conservative averaging method for the L-type domains. This method was primarily developed by A. Buiķis in [3] for domains of a rectangular form. Here we are solving the problem in a way that differs from [1, 2]. This way allows the conjugation conditions to be fulfilled on the contact line between the wall and the fin.

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1. PROBLEM STATEMENT

Consider a periodical system with a rectangular fin whose elements, in dimensionless arguments, are a wall $\{x \in [-\delta, 0], y \in [0, 1]\}$ and a fin $\{x \in [0, l], y \in [0, b]\}$. The symmetry axis (or periodicity conditions) goes through the median of the fin, $y = 0$, and the line $y = 1$.

The physical and mathematical content of the problem is practically identical to the formulations given in [1, 2]. The difference is that here a normalized temperature is employed; that is, the temperature of surroundings to the left of the wall (at $x < -\delta$) is equal to the unity, and to the right of the wall (at $x > 0$) it is equal to zero.

The stationary heat process in such a system is described as follows: in the wall – by the differential equation

$$\frac{\partial^2 U_0}{\partial x^2} + \frac{\partial^2 U_0}{\partial y^2} = 0, \quad -\delta < x < 0, \; 0 < y < 1 \tag{1}$$

and in the fin –

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0, \quad 0 < x < l, \; 0 < y < b \tag{2}$$

with the boundary conditions: on the left border of the wall

$$\frac{\partial U_0}{\partial x} + \beta_0^0 (1 - U_0) = 0, \quad x = -\delta, \; 0 < y < 1, \tag{3}$$

and on the right border –

$$\frac{\partial U_0}{\partial x} + \beta_0 U_0 = 0, \quad x = 0, \; b < y < 1, \tag{4}$$

while for the side of the fin

$$\frac{\partial U}{\partial y} + \beta U = 0, \quad 0 < x < l, \; y = b, \tag{5}$$

and for its end

$$\frac{\partial U}{\partial x} + \beta U = 0, \quad x = l, \; 0 < y < b; \tag{6}$$

the periodicity (symmetry) conditions are as follows:

$$\left.\frac{\partial U_0}{\partial y}\right|_{y=0} = \left.\frac{\partial U_0}{\partial y}\right|_{y=1} = 0, \quad -\delta < x < 0, \tag{7}$$

$$\left.\frac{\partial U}{\partial y}\right|_{y=0} = 0, \quad 0 < x < l. \tag{8}$$

To complete the problem formulation, the conjugation conditions should be mentioned. They can be written in the form:

$$U_0|_{x=-0} = U|_{x=+0}, \quad \beta \left.\frac{\partial U_0}{\partial x}\right|_{x=-0} = \beta_0 \left.\frac{\partial U}{\partial x}\right|_{x=+0}, \quad 0 < y < b. \tag{9}$$

Here we have employed the dimensionless arguments:

$$x = \frac{\bar{x}}{B+R}, \; y = \frac{\bar{y}}{B+R},$$

and the dimensionless parameters:

$$\delta = \frac{\Delta}{B+R}, \; l = \frac{L}{B+R}, \; b = \frac{B}{B+R},$$

where $2B$ is the fin thickness, $2R$ is the distance between two adjacent fins; L stands for the fin length, and Δ denotes the wall thickness.

The parameters of boundary conditions (the Biot criteria) are as follows:

$$\beta_0^0 = \frac{h_0(B+R)}{k_0}, \; \beta_0 = \frac{h(B+R)}{k_0}, \; \beta = \frac{h(B+R)}{k},$$

where $k_0(k)$ is the heat conduction coefficient for the wall (relative to the fin); $h_0(h)$ is Newton's coefficient of heat exchange with the surroundings on the left (right) side.

2. SOLUTION FOR THE FIN

We will seek a solution to the problem for the fin in the form [1]:

$$U(x,y) = f_0(x) + (e^{\rho y} - 1)f_1(x) + (1 - e^{-\rho y})f_2(x), \; \rho = b^{-1}, \tag{10}$$

where $f_i(x)$, $i = 0, 1, 2$ are unknown functions. To found them, we will require that (10) fulfill the boundary conditions (5), (6) and (8) and basic equations (2).

Fulfillment of boundary condition (8) leads to the relationship $f_2(x) = -f_1(x)$, from which it follows that (10) can be presented in the form:

$$U(x,y) = f_0(x) + 2(\cosh(\rho y) - 1)f_1(x). \tag{11}$$

Now we will define the mean integral value of the function $U(x,y)$ in the y–direction:

$$u(x) = \rho \int_0^b U(x,y)\,dy. \tag{12}$$

Having integrated (11), we obtain

$$f_1(x) = \frac{u(x) - f_0(x)}{2(\sinh(1) - 1)},$$

which gives for (11), by excluding $f_1(x)$ from there, the following:

$$U(x,y) = \frac{\cosh(\rho y) - 1}{\sinh(1) - 1}u(x) + \frac{\sinh(1) - \cosh(\rho y)}{\sinh(1) - 1}f_0(x). \tag{13}$$

Having required that boundary condition (5) be fulfilled, we derive:

$$\rho\sinh(1)u(x) - \rho\sinh(1)f_0(x)$$

$$+\beta(\cosh(1) - 1)u(x) + \beta(\sinh(1) - \cosh(1))f_0(x) = 0,$$

which in a shorter form reads as

$$f_0(x) = \psi u(x), \tag{14}$$

where

$$\psi = \frac{\sinh(1) + \beta b(\cosh(1) - 1)}{\sinh(1) + \beta b(\cosh(1) - \sinh(1))}. \tag{15}$$

This means that representation (13) of the solution $U(x,y)$ can be reduced to the form:

$$U(x,y) = u(x)\Phi(y). \tag{16}$$

Here

$$\Phi(y) = \frac{(1 - \psi)\cosh(\rho y) + \psi\sinh(1) - 1}{\sinh(1) - 1} \tag{17}$$

or

$$\Phi(y) = 1 + (1 - \psi)\frac{\cosh(\rho y) - \sinh(\rho y)}{\sinh(1) - 1}.$$

Using (15) we can convert this function into the ψ–free form:

$$\Phi(y) = \frac{\sinh(1) + \beta b(\cosh(1) - \cosh(\rho y))}{\sinh(1) + \beta b(\cosh(1) - \sinh(1))}. \tag{18}$$

from which one can see that $\Phi(y) > 0$ for all $y \in [0, b]$ and that it is a descendent function.

The expression for solution (16) shows that there has remained only one unknown function - $u(x)$. It can be deduced if we have required fulfillment of the integrated basic equation (2):

$$\frac{d^2u(x)}{dx^2} + \frac{1}{b}\left(\left.\frac{\partial U}{\partial y}\right|_{y=b} - \left.\frac{\partial U}{\partial y}\right|_{y=0}\right) = 0, \quad 0 < x < l.$$

The derivatives of function $U(x, y)$ at the points $y = 0$ and $y = b$ can be expressed via boundary conditions (5) and (8) as

$$\frac{d^2u(x)}{dx^2} - \mu^2 u(x) = 0, \quad 0 < x < l, \tag{19}$$

where

$$\mu^2 = \frac{\beta}{b}\Phi(b). \tag{20}$$

The latter expression can be rewritten, with the help of (18), as

$$\mu^2 = \frac{1}{b^2(\coth(1) - 1 + (\beta b)^{-1})}. \tag{21}$$

Integrating boundary condition (6) in the y–direction, we get:

$$u'(l) + \beta u(l) = 0.$$

This allows the solution of equation (19) to be written as

$$u(x) = C_1(\mu_1 e^{\mu x} + e^{-\mu x}), \tag{22}$$

where

$$\mu_1 = \frac{\mu - \beta}{\mu + \beta} e^{-2\mu l}$$

Then from (16) it follows:

$$U(x, y) = C_1(\mu_1 e^{\mu x} + e^{-\mu x})\Phi(y). \tag{23}$$

Later, we shall define the free constant C_1, when setting up a correspondence between the solution for the fin and that for the wall.

3. THE MAIN RELATIONSHIPS FOR THE WALL

We will express the exponential approximation for the wall in the x–direction:

$$U_0(x,y) = g_0(y) + (e^{-dx} - 1)g_1(y) + (1 - e^{dx})g_2(y),\ d = \delta^{-1}, \tag{24}$$

where the unknown function $g_i(y)$, $i = 0,1,2$ will be sought in a similar way: by demanding that the boundary conditions and the basic equation be fulfilled.

First, we will define the integral mean in the x–direction:

$$u_0(x) = d\int_{-\delta}^{0} U_0(x,y)\,dx. \tag{25}$$

Integrating representation (24) of function $U_0(x,y)$ we obtain the relationship:

$$u_0(y) = g_0(y) + (e-2)g_1(y) + e^{-1}g_2(y). \tag{26}$$

We should require fulfillment of boundary condition (3). We obtain then, that

$$(e + \beta_0^0\delta(e-1))g_1(y) + (e^{-1} + \beta_0^0\delta(1-e^{-1}))g_2(y) = \beta_0^0\delta(1 - g_0(y)). \tag{27}$$

Excluding unknown function $g_2(y)$ from equations (26) and (27) we get:

$$K_1 g_1(y) = A_1 g_0(y) - B_1 u_0(y) + D_1, \tag{28}$$

where

$$K_1 = e^{-1}(2 + \beta_0^0\delta(e-1)(3-e)), \tag{29_0}$$

$$A_1 = e^{-1}(1 + \beta_0^0\delta(e-2)), \tag{29_1}$$

$$B_1 = e^{-1}(1 + \beta_0^0\delta(e-1)), \tag{29_2}$$

$$D_1 = e^{-1}\beta_0^0\delta. \tag{29_3}$$

In its turn, elimination of function $g_1(y)$ from equations (26) and (27) gives the relationship:

$$K_1 g_2(y) = -A_1 g_0(y) + B_2 u_0(y) - D_2. \tag{30}$$

where

$$A_2 = e + \beta_0^0\delta, \tag{31_1}$$

$$B_2 = e + \beta_0^0\delta(e-1), \tag{31_2}$$

$$D_2 = \beta_0^0 \delta(e - 2). \tag{31$_3$}$$

Expressions (28) and (30) are more convenient to write in the form:

$$g_1(y) = a_1 g_0(y) - b_1 u_0(y) + d_1, \tag{32$_1$}$$

$$g_2(y) = -a_2 g_0(y) + b_2 u_0(y) - d_2. \tag{32$_2$}$$

Here

$$a_i = \frac{A_i}{K_1}, \quad b_i = \frac{B_i}{K_1}, \quad d_i = \frac{D_i}{K_1}, \quad i = 1, 2.$$

In the earlier publications [1] we made an attempt to write an equation for the temperature on the right border of the wall - $g_0(y)$. Here we will obtain an equation for the mean integral temperature $u_0(y)$.

First, we exclude $g_1(y)$ and $g_2(y)$ from expression (24) with (32$_i$), $i = 1, 2$. We then get:

$$U_0(x, y) = (1 + (e^{-dx} - 1)a_1 - (1 - e^{dx})a_2)g_0(y)$$
$$+((1 - e^{dx})b_2 - (e^{-dx} - 1)b_1)u_0(y) + (e^{-dx} - 1)d_1 - (1 - e^{dx})d_2. \tag{33}$$

Further transformations are associated with conditions on the border $x = 0$. These conditions are different for the interval $y \in [0, b]$, where the wall conjugates with the fin, and for the interval $y \in [b, 1]$, where the former contacts with surroundings. Therefore further calculations for the wall will be different in these intervals.

4. SOLUTION FOR THE UPPER PART OF THE WALL

We will make use of boundary condition (4). Let us demand that the expression for $U_0(x, y)$ meet this condition. This gives:

$$(a_2 - a_1)g_0(y) + (b_1 - b_2)u_0(y) + d_2 - d_1 + \beta_0 \delta g_0(y) = 0.$$

This relationship can be rewritten in the form similar to (32$_i$):

$$g_0(y) = b_0 u_0(y) - d_0, \tag{34}$$

where

$$b_0 = \frac{B_0}{K_0}, \quad d_0 = \frac{D_0}{K_0}, \tag{35$_0$}$$

while

$$K_0 = A_2 - A_1 + \beta_0 \delta K_1 = e - e^{-1} + 2\beta_0^0 \delta e^{-1} + \beta_0 \delta K_1, \tag{35$_1$}$$

$$B_0 = B_2 - B_1 = e^{-1}(e^2 - 1 + \beta_0^0 \delta (e - 1)^2), \tag{35$_2$}$$

$$D_0 = D_2 - D_1 = \beta_0^0 \delta (e - e^{-1} - 2). \tag{35_3}$$

The expression (34) allows representation (33) of $U_0(x, y)$ to be rewritten in a form depending on one unknown function only–on the integral mean $u_0(y)$:

$$U_0(x,y) = (b_0 + (a_1 b_0 - b_1)(e^{-dx} - 1) + (b_2 - a_2 b_0)(1 - e^{dx}))u_0(y) - d_0 + (d_1 - a_1 d_0)(e^{-dx} - 1) + (a_2 d_0 - d_2)(1 - e^{dx}). \tag{36}$$

Next, we integrate the differential equation (1) in the x-direction:

$$\frac{d^2 u_0(y)}{dy^2} + \frac{1}{\delta} \frac{\partial U_0}{\partial x}\Big|_{x=-\delta}^{x=0} = 0, \tag{37}$$

find the derivative of function $U_0(x, y)$ and substitute it into this integrated equation. The result reads:

$$\frac{d^2 u_0}{dy^2} - \kappa^2 u_0 = -\Theta_2, \quad b < y < 1, \tag{37_1}$$

where:

$$\kappa^2 = \delta^{-2}((b_1 - a_1 b_0)(e - 1) + (b_2 - a_2 b_0)(1 - e^{-1})), \tag{38_1}$$

$$\Theta_2 = \delta^{-2}((d_1 - a_1 d_0)(e - 1) + (d_2 - a_2 d_0)(1 - e^{-1})). \tag{38_2}$$

By means of (29_i), (31_i), and (32_i), the expression for κ^2 can be rewritten as

$$\kappa^2 = 2\delta^{-2} K_0^{-1} \left((\beta_0 + \beta_0^0)\delta \sinh(1) + 2\beta_0 \beta_0^0 \delta^2 (\cosh(1) - 1)\right), \tag{39}$$

which clearly shows that $\kappa^2 > 0$.

Integrating boundary condition (7) in the x-direction at $y = 1$, we obtain:

$$u_0'(1) = 0.$$

With the help of this boundary condition, the general solution of equation (37_1) is found in the form:

$$u_0(y) = C_2 \cosh(\kappa(1 - y)) + \theta_2, \tag{40}$$

where

$$\theta_2 = \frac{\Theta_2}{\kappa^2}$$

that is,

$$\theta_2 = \frac{(D_1 K_0 - A_1 D_0)(e - 1) + (D_2 K_0 - A_2 D_0)(1 - e^{-1})}{(B_1 K_0 - A_1 B_0)(e - 1) + (B_2 K_0 - A_2 B_0)(1 - e^{-1})}. \tag{40_1}$$

Similarly to the fin case, the remaining free constant C_2 will be defined later.

5. SOLUTION FOR THE LOWER PART OF THE WALL

Consider, first, representation (33) for the wall part $y \in [0, b]$.

Let us require that there be fulfilled the conjugation condition (9_1). Using (23), we get

$$g_0(y) = C_1(1 + \mu_1)\Phi(y). \tag{41}$$

Consider, next, the derivatives of function $U_0(x, y)$ from the integrated equation (37) at the points $x = 0$ and $x = -\delta$. We will demand that the conjugation condition (9_2) be fulfilled, using (23) once more. This gives

$$\frac{1}{\delta}\frac{\partial U_0}{\partial x}\bigg|_{x=0} = \frac{\beta_0}{\delta\beta}\frac{\partial U}{\partial x}\bigg|_{x=0} = -C_1\frac{\beta_0\mu}{\beta\delta}(1 - \mu_1)\Phi(y).$$

In its turn, from (33), with due regard for (41), we derive:

$$\frac{1}{\delta}\frac{\partial U_0}{\partial x}\bigg|_{x=-\delta} = \frac{1}{\delta^2}(b_1 e - b_2 e^{-1})u_0(y)$$
$$-C_1\frac{1 + \mu_1}{\delta^2}(a_1 e - a_2 e^{-1})\Phi(y) - \frac{1}{\delta^2}(d_1 e - d_2 e^{-1}).$$

Then

$$\frac{1}{\delta}\frac{\partial U_0}{\partial x}\bigg|_{x=-\delta}^{x=0} = -\frac{B_1 e - B_2 e^{-1}}{\delta^2 K_1}u_0(y) + \frac{D_1 e - D_2 e^{-1}}{\delta^2 K_1}$$
$$+C_1\left(\frac{(1 + \mu_1)(A_1 e - A_2 e^{-1})}{\delta^2 K_1} - \frac{\beta_0\mu}{\beta\delta}(1 - \mu_1)\right)\Phi(y).$$

Let us substitute this expression into equation (37), having divided preliminarily the function $\Phi(y)$ from (18) into two parts:

$$\Phi(y) = \Phi_0 - \Phi_1\cosh(\rho y),$$

where

$$\Phi_1 = ((\beta b)^{-1}\sinh(1) + \cosh(1) - \sinh(1))^{-1},$$

$$\Phi_0 = ((\beta b)^{-1}\sinh(1) + \cosh(1))\Phi_1.$$

Then equation (37) can be written in the short form

$$\frac{d^2 u_0}{dy^2} - \lambda^2 u_0 = -D_3 - C_1\Theta_{3,0} + C_1\Theta_{3,1}\cosh(\rho y). \tag{42}$$

Here

$$\lambda^2 = \frac{B_1 e - B_2 e^{-1}}{\delta^2 K_1} = \frac{\beta_0^0\delta(e - 1)^2}{e\delta^2 K_1},$$

$$D_3 = \frac{D_1 e - D_2 e^{-1}}{\delta^2 K_1} = \frac{2\beta_0^0 \delta}{e\delta^2 K_1},$$

$$\Theta_{3,i} = \Theta_3 \Phi_i, \quad i = 0, 1,$$

where

$$\Theta_3 = \frac{(A_1 e - A_2 e^{-1})(1+\mu_1)}{\delta^2 K_1} - \frac{\beta_0 \mu}{\beta\delta}(1-\mu_1)$$
$$= \frac{\beta_0^0 \delta(e - e^{-1} - 2)}{\delta^2 K_1}(1+\mu_1) - \frac{\beta_0 \mu}{\beta\delta}(1-\mu_1).$$

From (7) we derive the following boundary condition

$$u_0'(0) = 0$$

for equation (42). Thus the general solution of this equation appears to be

$$u_0(y) = C_3 \cosh(\lambda y) + C_1\theta_3 + C_1\theta_4 \cosh(\rho y) + d_3. \tag{43}$$

Here

$$\theta_3 = \frac{\Theta_{3,0}}{\lambda^2}, \quad \theta_4 = \frac{\Theta_{3,1}}{\rho^2 - \lambda^2}, \quad \left(\rho = \frac{1}{b}\right), \quad d_3 = \frac{D_3}{\lambda^2}.$$

The expression (43) is valid if $\lambda \neq d^{-1} = \rho$. In case $\lambda = \rho$, instead of (43) we have:

$$u_0(y) = C_3 \cosh(\lambda y) + C_1\theta_3 + C_1\Theta_{3,1}\frac{y \sinh(\lambda y)}{2\lambda} + d_3. \tag{43'}$$

In analysis that follows only the general case (i.e. solution (43)) is considered in detail.

6. CONJUGATION OF THE PARTIAL SOLUTIONS

Now in our solutions we have at hand 3 free constants: in formula (23) (solution for the fin), in formulas (36), (40), (for the wall upper part), and in formulas (33), (41), and (43) (for the wall lower part).

The first requirement to be set is that at the point $x = 0$, $y = b$ of the contact between the wall upper part and the fin the temperatures coincide. Then from (23), (34), and (40) we obtain:

$$C_1(1+\mu_1)\Phi(b) = C_2 b_0 \cosh(\kappa(1-b)) + \theta_2 b_0 - d_0. \tag{44}$$

The next two demands will be associated with the contact line between the upper and the lower parts of the wall. We will require that the mean temperatures and the mean fluxes coincide.

From (40) and (43) it follows:

$$C_2 \cosh(\kappa(1-b)) + \theta_2 = C_3 \cosh(\lambda b) + C_1(\theta_4 \cosh(1) + \theta_3) + d_3. \tag{45}$$

In its turn, the coincidence of the fluxes (i.e. that of the derivatives) gives the equation

$$-C_2\kappa \sinh(\kappa(1-b)) = C_3\lambda \sinh(\lambda b) + C_1\theta_4 d^{-1} \sinh(1). \tag{46}$$

It is easy to eliminate C_3 from this equation system.

From (46) we have:

$$C_3 = -\frac{C_1\theta_5 + C_2\kappa \sinh(\kappa(1-b))}{\lambda \sinh(\lambda b)}, \tag{47}$$

where

$$\theta_5 = \theta_4 d^{-1} \sinh(1).$$

Substituting thus obtained expression for C_3 into equation (45) we derive C_2:

$$C_2 = \frac{C_1(\theta_3 - \theta_4 \cosh(1) - \theta_5\lambda^{-1} \coth(\lambda b)) + d_3 - \theta_2}{\cosh(\kappa(1-b)) + \kappa\lambda^{-1} \sinh(\kappa(1-b)) \coth(\lambda b)}. \tag{48}$$

Now it remains to substitute the expression C_2 into equation (44) so that C_1 is found. It is readily seen that the denominators in equations (47) and (48) are not zeros. It is possible also, at the last step to act otherwise: namely, to derive C_1 from equation (44) and substitute it into equations (48). Thus all the free constants can be obtained and the approximate analytical solution to the problem found uniquely.

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TUVINĀTS DIVDIMENSIJU ANALĪTISKS ATRISINĀJUMS GARENISKAI TAISNSTŪRA RIBAI

MARGARITA BUIĶE un ANDRIS BUIĶIS

Šajā rakstā ir iegūts tuvināts analītisks atrisinājums stacionārai siltuma apmaiņas problēmai regulārai sistēmai ar garenisku taisnstūra ribu.

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A CERTAIN MATHEMATICAL MODEL OF THE SURFACE CHEMICAL REACTIONS TAKING INTO ACCOUNT THE GRAVITATION

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ABSTRACT

The certain technological process determining by chemical reactions on the surface of the uniformly moving glass fibre material taking into account the effect of gravitation is considered. There is carried out the mathematical model of this process, which under some assumptions is reduced to boundary value problem for self-similar system of ordinary differential equations.

INTRODUCTION

The mathematical model of chemical reactions which are taking place on the surface of the uniformly moving vertically imbedded glass fibre material, if it is pulled through bathes filled with acid solution, was constructed and partly numerically investigated in the article [1].

The urgency of this investigation was caused by necessity to minimize substance of alkaline metal oxides in the glass fibre material and thus magnify its thermal strength. The simplest scheme of the mentioned surface reactions in the acid solution is the reaction between some alkaline metal oxide, which is situated on the surface of glass fibre material, and acid. As a result of this reaction is the forming of the alkaline metal salt in the solution.

The generalization of this mathematical model, which was reduced to the

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boundary value problem for self-similar system of second order ordinary differential equations with boundary conditions at the endpoints of the positive semiaxis, was considered in the paper [2]. There was investigated the boundary value problem

$$f_j'' + \beta_j \eta f_j' = \frac{1 + \sigma\eta^2}{\frac{1}{\rho_0} + \sum_{i=1}^{k} \alpha_i f_i} \sum_{i=1}^{k} \alpha_i f_i' f_j',$$

$$f_j'(0) = A_j f_1(0)(\lambda_{k+1} f_j(0) + \lambda_j),$$

$$f_j(\infty) = m_j^*,$$

where $j = 1, \ldots, k$; $\sigma, \beta_j, A_j > 0$; $\alpha_j, \lambda_{k+1} < 0$; $\lambda_j \in R$;

$$m_j^* \in [0,1], \quad \sum_{j=1}^{k} m_j^* \leq 1.$$

In particular, the sufficient conditions for solvability of this boundary value problem was formulated in the paper [2].

The circumstance that the effect of gravitation was not taken into account, when this mathematical model was constructed, is causing certain objections. We offer in this paper perfection of the mentioned mathematical model taking into account the effect of gravitation in the situation which is implied by vertically imbedded uniformly moving glass fibre material.

THE MATHEMATICAL MODEL

Let x_1, x_2 respectively are the spatial coordinates in the lengthwise and the normal directions of the glass fibre material;

u_1, u_2 are the velocity components of the acid solution flow in the directions corresponding to axes x_1, x_2;

ρ_i, m_i, D_i respectively are the density, the mass concentration and the diffusion coefficient of acid ($i = 1$) and alkaline metal salt ($i = 2$) in the solution flow, ρ_0 is the density of water,

$$\rho = \left(\frac{\rho_0 - \rho_1}{\rho_0 \rho_1} m_1 + \frac{\rho_0 - \rho_2}{\rho_0 \rho_2} m_2 + \frac{1}{\rho_0} \right)$$

is the density of the solution,

ν is the coefficient of viscosity,

g is the acceleration of gravity and introduce the function

$$\varphi = \frac{\rho^* - \rho}{\rho},$$

where ρ^* is the initial density of the acid solution ($x_1 = 0$).

Granting this and retaining simplifying assumptions as it was done in [5], the equations of hydrodynamics (the equations of flow continuity and momentum conservation) for the boundary layer we can write in the following form

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0,$$

$$u_1 \frac{\partial u_1}{\partial x_1} + u_2 \frac{\partial u_2}{\partial x_2} = \nu \frac{\partial^2 u_1}{\partial x_2^2} + g\varphi,$$

but the differential equations of substances transport appear like this

$$\rho \left(u_1 \frac{\partial m_j}{\partial x_1} + u_2 \frac{\partial m_j}{\partial x_2} \right) = D_j \left(\frac{\partial}{\partial x_1} \left(\rho \frac{\partial m_j}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left(\frac{\partial m_j}{\partial x_2} \right) \right), \quad j = 1, 2.$$

With respect to the boundary conditions we must say that

$$u_1(x_1, 0) = v_0, u_2(x_1, 0) = 0,$$

$$u_1(x_1, \infty) = u_2(x_1, \infty) = 0,$$

$$\varphi(x_1, \infty) = 0,$$

$$m_1(0, x_2) = m_1^*, m_2(0, x_2) = 0,$$

where v_0 is the velocity of the glass fibre material and m_1^* is the initial mass concentration of the acid.

The boundary conditions which are determined by the surface chemical reactions are discussed in the [1, 2] and have the following form

$$\frac{\partial m_j(x_1, 0)}{\partial x_2} = A_j m_1(x_1, 0)(\lambda_3 m_j(x_1, 0) + \lambda_j), \quad j = 1, 2,$$

where physically evaluated coefficients A_j, $\lambda_2 > 0$, $\lambda_1, \lambda_3 < 0$.

THE SELF-SIMILAR PROBLEMS

Introducing the flow function Φ by the equalities

$$u_1 = \frac{\partial \Phi}{\partial x_2}, \quad u_2 = -\frac{\partial \Phi}{\partial x_1},$$

choosing new variables as it was done in [4]

$$\eta = \left(\frac{g}{4\nu^2 x_1} \right)^{\frac{1}{4}} x_2, \quad h = \frac{\Phi}{(4x_1)^{\frac{3}{4}} g^{\frac{1}{4}} \nu^{\frac{1}{2}}},$$

using designations $f_j = m_j(\eta), j = 1,2$ and assuming inequalities

$$\frac{\partial^2 m_j}{\partial x_1^2} \ll \frac{\partial^2 m_j}{\partial x_2^2}, \quad j = 1,2,$$

now we obtain the following system of ordinary differential equations

$$f_j''' + 3\frac{\nu}{D_j} h f_j' = \rho\left(1 + \frac{\nu\eta^2}{8g^{\frac{1}{2}}x_1^{\frac{3}{2}}}\right)\left(\frac{\rho_0 - \rho_1}{\rho_0\rho_1} f_1' f_j' + \frac{\rho_0 - \rho_2}{\rho_0\rho_2} f_2' f_j'\right), j = 1,2,$$

$$h''' + 3hh'' - 2(h')^2 = \varphi.$$

As far as $v_0 \neq 0$ similarly succeding [1] we can also to use the new variables

$$\eta = x_2\left(\frac{v_0}{\nu x_1}\right)^{\frac{1}{2}}, h = \Phi(\nu v_0 x_1)^{-\frac{1}{2}}.$$

Then the obtained system of ordinary differential equations has the following view

$$f_j''' + \frac{\nu}{2D_j} h f_j' = \rho\left(1 + \frac{\nu\eta^2}{4v_0 x_1}\right)\left(\frac{\rho_0 - \rho_1}{\rho_0\rho_1} f_1' f_j' + \frac{\rho_0 - \rho_2}{\rho_0\rho_2} f_2' f_j'\right), j = 1,2,$$

$$h''' + \frac{1}{2} h h'' = \frac{g x_1}{v_0^2}\varphi\rho\rho^*$$

The boundary conditions in both cases are reducing to the following form

$$f_j'(0) = \bar{A}_j f_1(0)(\lambda_3 f_j(0) + \lambda_j), \quad j = 1,2,$$

$$h(0) = 0, h'(0) = \tilde{\mu},$$

$$f_1(\infty) = m_1^*, f_2(\infty) = 0,$$

$$h'(\infty) = 0.$$

Of course, the positive factor $\bar{A}_j$ is different in these cases, and note that $\tilde{\mu} = 1$ in the second case.

As well as in both cases we are obtaining for any fixed $x_1 \in (0, +\infty)$ self-similar system, which does not satisfies the classical Bernstein Nagumo Opial conditions providing an a priori estimate for the derivative of the bounded solution the solvability of these boundary value problems causes a mathematical interest.

GENERALIZATION

The boundary value problem which generalizes the mathematical model of the surface chemical reactions taking into account the effect of gravitation can be written analogously as it is done in the paper [2]. Similarly we derive the boundary value problem

$$f_j'' + \beta_j h f_j' = \rho(\gamma_{1j} + \gamma_{2j}\eta^2) \sum_{i=1}^{k} \alpha_i f_i' f_j',$$

$$h''' + ahh'' - b(h')^2 = R(\rho),$$

$$f_j'(0) = A_j f_1(0)(\lambda_{k+1} f_j(0) + \lambda_j),$$

$$h(0) = 0, h'(0) = \mu,$$

$$f_j(\infty) = m_j^*, h'(\infty) = 0,$$

where $A_j, \beta_j, \gamma_{1j}, \gamma_{2j}, \beta_0, a > 0$; $b \geq 0$; $\mu, \lambda_j \in R$; $\alpha_j, \lambda_{k+1} < 0$; $j = 1, \ldots, k$;

$$m_j^* \in [0,1], \sum_{j=1}^{k} m_j^* \leq 1; R \in C([0, +\infty)),$$

$$\rho = \left(\sum_{i=1}^{k} \alpha_i f_i + \beta_0 \right)$$

We note that for solvability proving of this boundary value problem is not possible to employ a straightforward application of the method of a priori estimates as it was done in the paper [3] because the boundary value problem

$$f'' = \alpha,$$

$$f'(0) = f(\infty) = 0$$

has not a solution for any $\alpha \in R$, $\alpha \neq 0$.

If we are able to show that the component of solution h has a sublinear growth, the proof of the existence theorem becomes possible using the procedures from the paper [2].

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PAR KĀDU VIRSMAS ĶĪMISKO REAKCIJU MATEMĀTISKO MODELI, IEVĒROJOT GRAVITĀCIJAS IETEKMI

JĀNIS CEPĪTIS

Aplūkots kāda tehnoloģiskā procesa, kuru nosaka virsmas ķīmiskās reakcijas uz vertikāli iegremdēta un ar vienmērīgu ātrumu vilkta stikla šķiedras auduma, matemātiskais modelis gravitācijas ietekmē.

Pie zināmiem nosacījumiem iegūta robežproblēma automodeļu parasto diferenciālvienādojumu sistēmai.

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SOME REMARKS TO SOLVABILITY OF SYSTEMS WITH DISCONTINUOUS NONLINEARITIES

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ABSTRACT

The solvability problem for the systems of the type

$$(\alpha_i(u_1,u_2)u_i')' + \beta_i(u_1,u_2) = f_i, \tag{1}$$

$$u_i(a_j) = A_{ij}, \quad i,j = 1,2, \tag{2}$$

is considered. Unboundedness of the lowest coefficients β_i has been shown to create generally speaking the non-existence of the solution determined in the so-called mushy region by one and the same unknown function.

Keywords. Elliptic systems, discontinuous nonlinearities, solvability.
AMS subject classification. 35J45

Discontinuous nonlinearities (DN) appear in modelling physical processes with phase transitions. A typical example of such processes is that of crystallization. In a case of a single equation one can determine all DN in the same fashion, i.e. by one and the same unknown function in the so-called mushy region [1]. It has not been understood well enough yet what the right way to define the solution for systems with DN is.

Let $\Omega = (a_1, a_2)$, $f_i \in L_2(\Omega)$, $g \in C(\mathbb{R})$, $Q_0 = \{\, (t,\tau) \in \mathbb{R} \times \mathbb{R} \quad \tau = g(t) \,\}$, $Q = (\mathbb{R} \times \mathbb{R}) \setminus Q_0$ and let the coefficients $\alpha_k : Q \to \mathbb{R}$, where $\alpha_3 = \beta_1$, $\alpha_4 = \beta_2$,

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be continuous functions in Q with finite limits

$$\alpha_k^{\mp}(t,\tau) := \lim_{s \to \tau \mp 0} \alpha_k(t,s), \quad k = 1, \ldots, 4.$$

By analogy with a single equation (see, for example, [1]) we use the following definition of the solution.

DEFINITION. The pair $u = (u_1, u_2) \in H^1(\Omega) \times H^1(\Omega)$ is said to be the solution of (1)–(2) if the boundary condition (2) holds and if a such function $\varphi \in L_2(\Omega)$, with values from [0,1], exists that

$$\int_\Omega [\alpha_i^*(u,\varphi) u_i' \eta_i' - \alpha_{i+2}^*(u,\varphi)\eta_i + f_i \eta_i]\, dx = 0 \quad \forall \eta \in H_0^1, \quad i = 1,2, \tag{3}$$

where

$$\alpha_k^*(u,\varphi) = \alpha_k^+(u)\varphi + \alpha_k^-(1-\varphi), \quad k = 1, \ldots, 4. \tag{4}$$

REMARK 1. The values of function φ are essential only in the mushy region $\Omega_0 := \{ x \in \Omega \quad u_2(x) = g(u_1(x)) \}$, because $\alpha_k^* = \alpha_k^+ = \alpha_k^- = \alpha_k$ outside Ω_0.

REMARK 2. The solvability of (1)–(2) in a case $\beta_1 = \beta_2 = 0$ under some additional assumptions on α_i $(i = 1,2)$ by appropriate smoothing technique of DN has been stated in [2].

To construct the desirable counterexample when one of β_i is not zero we choose

$$\Omega = (0,1), \quad A_{11} = A_{21} = 0, \quad A_{12} = A_{22} = 1, \quad \beta_1 = 0,$$
$$f_1 = f_2 = 0, \quad g(t) = t, \quad w = u_2 - u_1,$$

$$\alpha(w) = \alpha_1(u) = \alpha_2(u) = \begin{cases} 1 & \text{if } w < 0, \\ 2 & \text{if } w > 0, \end{cases}$$

$$\beta(w) = \beta_2(u) = \begin{cases} -1 & \text{if } w > 0, \\ \pi^2 w - 1 & \text{if } w < 0. \end{cases}$$

Then (2), (3) can be rewritten as follows

$$\int_\Omega \alpha^*(w,\varphi) u_1' \eta_1'\, dx = 0 \quad \forall \eta_1 \in H_0^1 \tag{5}$$

$$\int_\Omega (\alpha^*(w,\varphi) u_2' \eta_2' - \beta(w)\eta_2)\, dx = 0 \quad \forall \eta_2 \in H_0^1 \tag{6}$$

$$u_i(0) = 0, \quad u_i(1) = 1, \quad i = 1,2. \tag{7}$$

LEMMA 1. *Suppose that* (u_1, u_2) *is the solution of the system* (5)–(7). *Then* $w = u_2 - u_1 \leq 0$ *in* Ω.

Proof. Let $\Omega_+ = \{ x \in \Omega \quad w(x) > 0 \}$. If $w(x_1) > 0$ for some $x_1 \in \Omega$ then $\operatorname{meas} \Omega_+ > 0$ because of continuity of $w \in H_0^1$.

Difference of the equalities (5) and (6) with $\eta_1 = \eta_2 = \eta$ yields

$$\int_\Omega (\alpha^*(w, \varphi) w' \eta' - \beta^*(w, \varphi) \eta)\, dx = 0 \quad \forall \eta \in H_0^1. \tag{8}$$

Let us choose $\eta = \max(0, w)$. Such η is of class H_0^1 and $\eta' = w'$ in Ω_+ and $\eta = 0$ outside Ω_+, see [3]. As $\alpha^* = 2$ and $\beta^* = -1$ in Ω_+ the equality (8) implies

$$\int_{\Omega_+} (2w'^2 + w)\, dx = 0$$

what contradicts to assumption $w(x_1) > 0$.

This completes the proof.

THEOREM 1. *The system* (5)–(7) *has no solution* (u_1, u_2).

Proof. Let $\Omega_- = \{ x \in \Omega : w(x) < 0 \}$ and $\Omega_0 = \{ x \in \Omega \quad w(x) = 0 \}$. As $\beta^* = -1$ in Ω_0 then taking into account Lemma 1 and the well-known fact that $w' = 0$ almost everywhere in Ω_0, see, for example [3], from the equality (8) we obtain

$$\int_{\Omega_-} (w'\eta' - (\pi^2 w - 1)\eta)\, dx + \int_{\Omega_0} \eta\, dx$$

$$\int_\Omega (w'\eta' - \pi^2 w \eta)\, dx + \int_\Omega \eta\, dx = 0 \quad \forall \eta \in H_0^1. \tag{9}$$

Let η be the function $sin \pi x$. Then the former integral in (9) is zero but the latter is positive, a contradiction.

This completes the proof.

REMARK 3. This counterexample, see also [4], is rather a specific one because of unboundedness of coefficient β_2 and therefore a lack of *a priori* estimate $\|u\|_{H^1} \leq const$.

Is there a counterexample for analogous system with bounded coefficient β_2?

Below we shall show some systems for which a counterexample does not exist or in other words we shall formulate requirements on coefficient β which

guarantee the solvability of the following system

$$\int_\Omega \alpha_1^*(w,\varphi)u'\eta_1' \, dx = 0 \quad \forall \eta_1 \in H_0^1 \tag{10}$$

$$\int_\Omega (\alpha_2^*(w,\varphi)v'\eta_2' - \beta^*(w,\varphi)\eta_2) \, dx = 0 \quad \forall \eta_2 \in H_0^1 \tag{11}$$

$$u(0) = v(0) = 0, \quad u(1) = v(1) = 1. \tag{12}$$

As above let $\alpha_1^\pm$, $\alpha_2^\pm$ and $\beta^\pm$ be the right and left limits of the coefficients α_1, α_2 and β respectively when $w = v - u \to 0$.

LEMMA 2. *Let* 0 *be the point of the segment with endpoints* $\beta^\pm$ *Then the system* (10)-(12) *has a solution.*

Proof. Proof consists of the direct verification that functions $u(x) = v(x) = x$ with $\varphi(x) = \varphi_0 = \beta^-/(\beta^- - \beta^+)$ satisfy relationships (10)-(12). Observing that the function $\gamma(\varphi) = \beta^+\varphi + \beta^-(1-\varphi)$ is monotone and $\gamma(0) = \beta^-$, $\gamma(1) = \beta^+$, $\gamma(\varphi_0) = 0$ we see easily that $\varphi_0 \in [0,1]$ as it is required by Definition.

REMARK 4. Notice that boundedness of the coefficient β is not required in previous Lemma.

LEMMA 3. *If* $\beta(w)$ *is bounded when* $w > 0$ *and* $\beta^+ > 0$ *then the system* (10)-(12) *possesses a solution.*

Proof. Let us consider the system

$$\int_\Omega \alpha_{12}(w)u'\eta_1' \, dx = 0 \quad \forall \eta_1 \in H_0^1 \tag{13}$$

$$\int_\Omega (\alpha_{22}(w)v'\eta_2' - \beta_2(w)\eta_2) \, dx = 0 \quad \forall \eta_2 \in H_0^1 \tag{14}$$

where $\alpha_{12}(w)$, $\alpha_{22}(w)$ and $\beta_2(w)$ are equal with $\alpha_1(w)$, $\alpha_2(w)$ and $\beta(w)$ when $w > 0$ and α_1^+, α_2^+ and β^+ when $w \le 0$ respectively. Due to continuity and boundedness of the coefficients the system (12)-(14) has a solution, for example, by Schauder's principle.

Now our aim is to state that $w \ge 0$. Note that this implies Lemma 3.

Really if $w(x_1) < 0$ for some $x_1 \in \Omega$ then due to continuity of w there is interval $\Omega_1 = (x_2, x_3)$ where $w < 0$ and $w(x_2) = w(x_3) = 0$.

Let us choose functions η_1, η_2 as follows: $\eta_1 = \alpha_2^+ w$, $\eta_2 = \alpha_1^+ w$ in Ω_1 and $\eta_1 = \eta_2 = 0$ outside this interval. Then the difference of the equalities (13)–(14) yields

$$\int_{\Omega_1} (\alpha_2^+ \alpha_1^+ w'^2 - \alpha_1^+ \beta^+ w)\, dx = 0$$

with a positive integrand.

This contradiction completes the proof.

LEMMA 4. *If $\beta(w)$ is bounded when $w < 0$ and $\beta^- < 0$ then the system* (10)–(12) *possesses a solution.*

Proof. Proof is analogous to the one as for Lemma 3.

From Lemmas 2–4 it follows:

THEOREM 2. *The system* (10)–(12) *is solvable for any bounded and continuous function $w \mapsto \beta(w)$, if $w \neq 0$.*

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DAŽAS PIEZĪMES PAR SISTĒMU AR PĀRTRAUKTĀM NELINEARITĀTĒM ATRISINĀMĪBU

ANDREJS CIBULIS

Pierādīts, ka jaunāko koeficientu neierobežotība, vispārīgi runājot, izraisa atrisinājuma neeksistenci, ja to definē ar vienas un tās pašas nezināmās funkcijas palīdzību tā saucamajā divfāzu zonā.

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TILING RECTANGLES WITH PENTOMINOES

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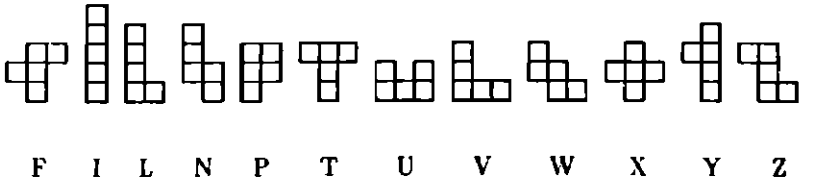
ABSTRACT

In this note we give some new results concerning the unsolved problem of determining all rectangles packable with congruent pentominoes.

Introduction. Solomon W. Golomb introduced the word *polyomino* in his 1954 paper [1]. He defined a polyomino (a generalized domino) to be a finite set of rook-wise connected cells in an infinite chessboard. In a case when a polyomino consists exactly of n cells (unit squares) it is called a *n-omino*. Players of the computer game TETRIS™ are familiar with tetrominoes. There are 5 tetrominoes there. Three next ones in sequence are 12 pentominoes, see Figure 1, where they are conventionally named after letters of the alphabet. It is this wealth of pentomino puzzles that has attracted the laymen and the professional mathematicians, alike to polyominoes.



F I L N P T U V W X Y Z

Figure 1. 12 pentominoes.

We are interested in the problem of packing a rectangle with copies of a given pentomino. The works [2, 3, 4] are the classic references in this context.

Our results have been obtained jointly by man and computer. A com-

puter program written for the Pentium 200 succeeded in finding some new $\mathcal{P}$-*rectangles* (i. e. a rectangles packable with congruent copies of pentominoes $\mathcal{P}$). The main idea of our algorithm has been tied with computer analysis of the shapes of the boundaries which arise when first k columns ($k = 1, 2, 3, \ldots$) are covered by pentominoes.

$\mathcal{P}$-rectangle is said to be *prime* (rectangle) if it cannot be divided into smaller $\mathcal{P}$-rectangles. A box-packing theorem which asserts that prime sets are always finite with an important implication for polyomino and polycube enthusiasts was proved in [4]. However, the problem of characterizing the rectangles (or boxes) that can be packed with polyominoes (or respectively) polycubes is difficult and solved only in a very few cases.

Pentominoes which cannot tile rectangles. There is not much to tell about these pentominoes. By simple checking one can state that copies of pentominoes F, N, T, U, V, W, X and Z respectively cannot tile any rectangle.

I-*rectangles.* For I-pentominoes the necessary condition that at least one side length, of $\mathcal{P}$-rectangles is divided by 5 is also sufficient. Therefore all I-rectangles are as follows: $5k \times m$, where k and m are arbitrary positive integers.

P-*rectangles.* There are only two prime P-rectangles: 5×2 and 15×7. All the other P-rectangles one can obtain from rectangles of these two types and represent them as follows: $5 \times 2n$ and $5k \times m$ where $n \geq 1$, $k \geq 2$ and $m \geq 6$ or $m = 2, 4$.

L-*rectangles.* There are also only two prime L-rectangles 5×2 and 15×7. The P-rectangle 15×7 and L-rectangle 15×9 are mentioned in [2] but rectangle 15×7 shown in Figure 2 we did not succeed in finding in literature. L-rectangles are only those ones: $5 \times 2n$ and $5k \times m$ where $n \geq 1$, $k \geq 2$ and $m \geq 6$ or $m = 2, 4$.

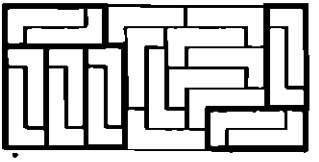


Figure 2. L-rectangle 15×7.

Y-*rectangles.* In 1970 C. J. Bouwkamp and D. A. Klarner [3] wrote *that the problem of determining all prime rectangles for the Y is far from solved. ...Unless small rectangles that can be packed with Y's have been overlooked the following rectangles are primes for the Y:*

$(5\times10), (10\times16), (10\times24), (15\times16), (15\times22), (15\times24), (20\times22)$ and (22×25), (*)

and pointed out that *the most important open problem in this area is to determine whether the Y packs some rectangle with an odd area.*

In a comparatively short time after this problem was posed J. Haselgrove stated that 15×15 is in fact Y-rectangle. He wrote [5]: *The computer runs showed that it is not possible to fill rectangles of area less that 225, or of dimensions* 9×25, *but that the* 15×15 *square can be filled as figure shows... I do not know whether the* 15×15 *solution* (see Figure 3) *is unique apart from the obvious H-symmetries.*

Now we know that the solution is not unique. Moreover, in 1996 the problem of packing the square 15×15 with Y-pentominoes was posed as a contest problem in the supplement *Krōnis Visam* of Latvia newspaper *Neatkarīgā Rīta Avīze* (September 21). Two of contestants M. Opmanis and A. Litvinovs found the solutions of this difficult problem. The solution of M. Opmanis has been published in the above-mentioned newspaper (December 14) and shown in Figure 4.

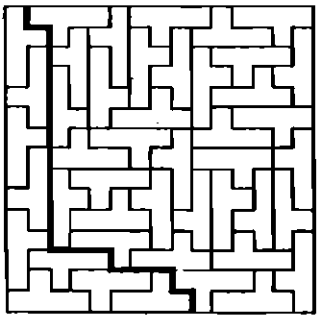


Figure 3. Solution of J. Haselgrove.

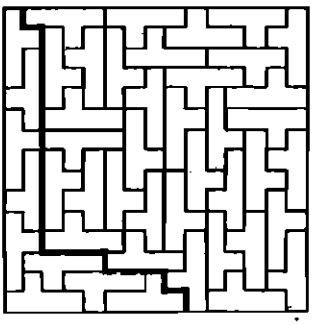


Figure 4. Solution of M. Opmanis.

In 1974 J. Bitner [6] by means of computer analysed the Y-rectangles with dimension $5n \times 12$. By exhaustive searching he stated that no solutions exist for $n < 10$. Hence $n = 10$ yields the smallest such region. Moreover, he succeeds in finding such a 50×12 solution which one can easily extend to solutions for $5n \times 12$, $n \geq 11$, periodically repeating the darkened region as in Figure 5. So $12 \times b$ is Y-rectangle only when $b = 5n$, $n \geq 10$.

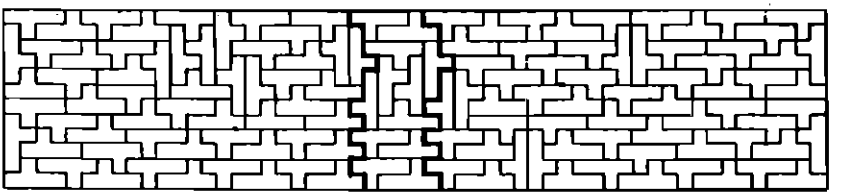


Figure 5. Y-rectangle 55×12.

Precaution of C. J. Bouwkamp and D. A. Klarner [3] is not superfluous. After ten years K. Scherer [7] stated that some of Y-rectangles, see (*), are

not primes and presented the new revised list (5×10), (10×14), (10×16), (15×14), (15×15), (15×16), (15×22), (20×9), (20×11), (25×22), (25×27), (30×9), (35×27) and $(5n \times 12)$, $n = 10, \ldots, 19$.

By computer analysis and some theoretical arguments we found the following (and as it seems) new Y primes (at least in [1–10] these Y-rectangles cannot be found):

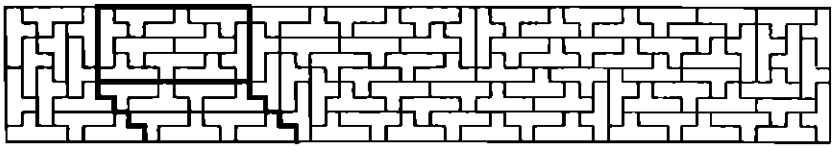


Figure 6. Y-rectangle 55×9.

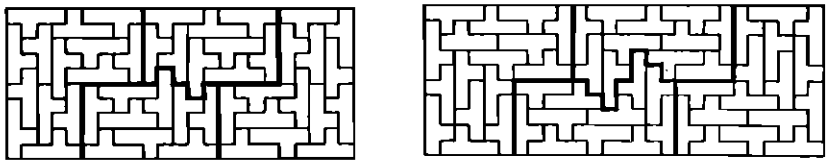


Figure 7. Y-rectangles 23×10 and 27×10.

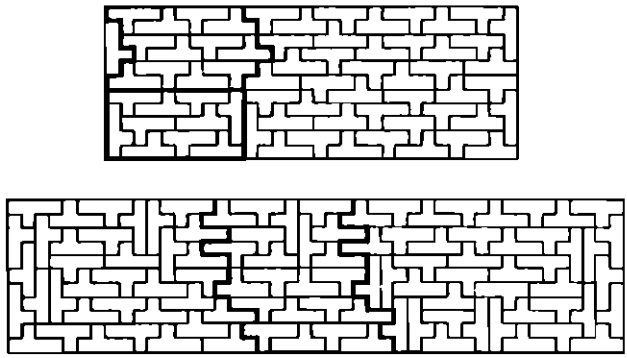


Figure 8. Y-rectangles 30×11 and 45×11.

Remark. The rectangle 35×27, see [7], is not prime. It can be composed of the three smaller Y-rectangles: 35×11, 20×16 and 15×16.

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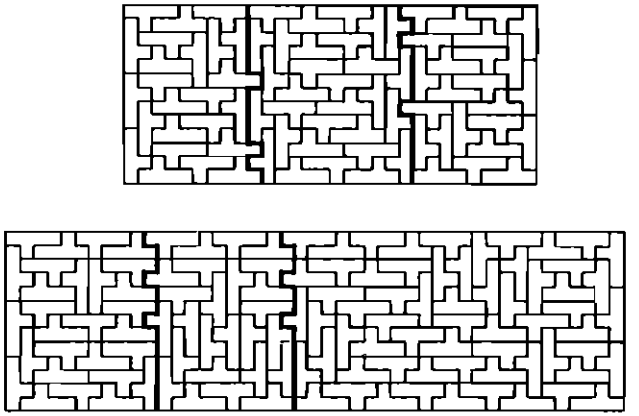

Figure 9. Y–rectangle 30 x 13 and 45 x 13.

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TAISNSTŪRU PĀRKLĀŠANA AR PENTAMINO

ANDREJS CIBULIS un ILVARS MIZNIKS

Aplūkota līdz šim neatrisināta problēma par visu taisnstūru atrašanu, kuri pārklājami ar vienādiem pentamino. Ir atrasti vairāki jauni pirmtaisnstūri.

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APPLICATION OF DM-METHOD FOR NUMERICAL SOLVING OF NONLINEAR PARTIAL DIFFERENTIAL EQUATION

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ABSTRACT

A variant of straight line method for numerical solving of nonlinear problems of partial differential equations with initial and boundary conditions is considered in this paper. The corresponding system of ordinary differential equations has been solved by means of the degenerate matrix method which is simply programmable and provides high precision even for stiff equations. Also problems with conditions of the periodicity instead of boundary conditions are considered. The solution is approximated by global nonsaturated interpolations with nodes as zeroes of Chebyshev's polynomials.

1. INTRODUCTION

For solving nonlinear problems of partial differential equations by means of the straight line method one of the difficulties is to solve a corresponding system of ordinary differential equations. Often it is a stiff one and needs very laborious calculations by means of traditional methods. We recommend new method for the numerical solving of an initial value's problem of nonlinear systems of differential equations [1] which in further we will call the DM-method (degenerate matrix method). This method is based on the global nonsaturated approximations of unknown functions and uses the special degenerate matrices together with the iteration process for solving equations received after a discretization of the problem. The calculation's algorithm for the DM-method can be represented in the matrix form, simply programmed

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on computers for a system of any degree, and gives very high precision even for stiff equations. Therefore, the DM-method extends possibilities of the straight line method for partial differential equations. The paper is organized as follows. In the Sect. 2 we introduce formally the mathematical basis of the DM-method. In the Sect. 3 we give two algorithms for solving partial differential equation $u'_t + (uu^{(3)}_{xxx})' = 0$ with the initial and boundary value's conditions for the solution using a nonsaturated interpolation with respect to the coordinate variable x in the matrix form. There are given some numerical results of the calculation. In the Sect. 4 we consider similar problems if the solution complies with a condition of the periodicity in x.

2. DM-METHOD FOR SOLVING NONLINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

To obtain a numerical solution of the system

$$\frac{dy_k}{dt} = f_k(t, y_1, y_2, \ldots, y_m),\ y_k(a) = \alpha_k,\ k = 1, 2, \ldots, m \tag{2.1}$$

in the interval $t \in [a, b]$ we will apply the degenerate matrix method (DM-method) which has the following mathematical basis. Let us chosen nodes

$$a = t_0, t_1, , \ldots, t_{N+1} = b \tag{2.2}$$

and $q_{N+2}(t)$ be a polynomial which has these nodes as zeroes. Contracting system (2.1) on nodes (2.2) and replacing the vector of derivatives

$$\mathbf{f}'_{N+2} = \{f'(t_0), f'(t_1), \ldots, f'(t_{N+1})\}^T \tag{2.3}$$

by approximately equal to

$$\mathbf{f}'_{N+2} \approx \Delta_{N+2}\mathbf{f}_{N+2}, \tag{2.4}$$

where $\mathbf{f}_{N+2} = (f(t_0), f(t_1), \ldots, f(t_{N+1}))^T$, we can convert (2.1) into following matrix form

$$\Delta_{N+2}\mathbf{Y}[N+2, m] = \mathbf{F}[N+2, m]. \tag{2.5}$$

Here Δ_{N+2} is the $(N+2) \times (N+2)$ interpolation's matrix for derivatives which has elements [3]

$$\delta_{ik} = \frac{q'_{N+2}(t_i)}{(t_i - t_k)q'_{N+2}(t_k)} \text{ if } i \neq k,\ \delta_{kk} = \frac{q''_{N+2}(t_k)}{2q'_{N+2}(t_k)},\quad (i, k = 0, 1, \ldots, N+1). \tag{2.6}$$

$\mathbf{Y}[N+2, m]$ is the $(N+2) \times m$ matrix which has elements

$$y_{ik} = y_k(t_i), i = 0, 1, 2, \ldots, N+1, \quad k = 1, 2, \ldots, m, \tag{2.7}$$

as approximate values of the unknown functions.
$\mathbf{F}[N+2, m]$ is the $(N+2) \times m$ matrix also with the elements

$$f_{ik} = f_k(t_i, y_1(t_i), y_2(t_i), \ldots, y_m(t_i)). \tag{2.8}$$

Always the interpolation's matrix for derivatives $\mathbf{\Delta}_{N+2}$ is degenerate and the rank $\mathbf{\Delta}_{N+2} = N+1$. As the matrix $\mathbf{\Delta}_{N+2}$ has not an inverse, we define the following pseudoinverse matrix.

DEFINITION 2.1. Matrix $\mathbf{B}_{N+2}$ is called a pseudoinverse matrix for $\mathbf{\Delta}_{N+2}$ if the equality

$$\mathbf{B}_{N+2}\mathbf{\Delta}_{N+2} = \mathbf{E}_{N+2} - \mathbf{I}_{N+2} \tag{2.9}$$

holds, where $\mathbf{E}_{N+2}$ is the identity matrix and all elements of the matrix $\mathbf{I}_{N+2}$ are zeroes except the elements of the first column consisting of the digit one.

The matrix $\mathbf{\Delta}_{N+2}$ has not the unique pseudoinverse matrix $\mathbf{B}_{N+2}$. Elements b_{ik} of $\mathbf{B}_{N+2}$ have the general representation

$$b_{ik} = b_{ik}^{(0)} + c_i/q'_{N+2}(t_k);$$

$$b_{ik}^{(0)} = \frac{1}{q'_{N+2}(t_k)} \int_{t_0}^{t_i} \frac{q_{N+2}(t)}{t - t_k} dt, \quad (i, k = 0, 1, \ldots, N+1), \tag{2.10}$$

where c_i are arbitrary constants depended only on i. The norms

$$\|\mathbf{B}_{N+2}\| := \max_i \sum_{k=0}^{N+1} |b_{ik}|$$

are always satisfying the inequality

$$\|\mathbf{B}_{N+2}\| \geq t_{N+1} - t_0. \tag{2.11}$$

Further we choose the nodes t_i and the corresponding pseudoinverse matrix $\mathbf{B}_{N+2}$ to produce the two properties:

1) the approximation of function is nonsaturated;
2) $\|\mathbf{B}_{N+2}\| = t_{N+1} - t_0$, i.e., the norm is minimally possible.

These two properties hold true if we choose

$$t_i = a + 0.5(b-a)(1 + x_i), \; x_i = -\cos\frac{i\pi}{N+1} \tag{2.12}$$

and elements b_{ik} for the pseudoinverse matrix $\mathbf{B}_{N+2}$ in the form $b_{ik} = (b-a)g_{ik}$, where

$$g_{ik} = \frac{1}{2p'_{N+2}(x_k)} \int_{-1}^{x_i} \frac{p_{N+2}(x)}{x - x_k} dx,\ p_{N+2}(x) = (1-x^2)U_N(x). \tag{2.13}$$

$U_N(x)$ is the Chebyshev's polynomial of the second kind. The elements g_{ik} do not depend on the interval $[a, b]$ and

$$\|\mathbf{G}_{N+2}\| = \max_{0 \le i \le N+1} \sum_{k=0}^{N+1} |g_{ik}| = 1. \tag{2.14}$$

for any N. Multiplying the equation (2.5) from the left side by the matrix $\mathbf{B}_{N+2} = h\mathbf{G}_{N+2}, h = b - a$, we obtain this equation into matrix form

$$\mathbf{Y}[N+2, m] = h\mathbf{G}_{N+2}\mathbf{F}[N+2, m] + \mathbf{A}[N+2, m], \tag{2.15}$$

where $h = b - a$ and $\mathbf{A}[N+2, m]$ is the $(N+2) \times m$ matrix; its elements

$$y_{ik} = y_k(t_i); f_{ik} = f_k(t_i, y_1(t_i), \ldots, y_m(t_i)); a_{ik} = y_k(t_0) = \alpha_k, \tag{2.16}$$

$(i = 0, 1, \ldots, N+1,\ k = 1, 2, \ldots, m)$, i.e., they are the initial values of unknown functions $y_k(t)$. By means of (2.13) the elements g_{ik} of the degenerate matrix $\mathbf{G}_{N+2}$ can be represented in the following exact and explicit form [3] which is very suitable for calculations on computers.

$$g_{i0} = \frac{1}{4(N+1)} \left\{ \frac{2N^2 + 4N + 1}{N(N+1)(N+2)} + (-1)^i \left[\frac{(N+1)x_i}{N(N+2)} - \frac{1}{N+1} \right] \right\} \tag{2.17}$$

$$g_{i,N+1} = \frac{(-1)^{N+1}}{4(N+1)} \left\{ \frac{1}{N(N+1)(N+2)} + (-1)^i \left[\frac{(N+1)x_i}{N(N+2)} + \frac{1}{N+1} \right] \right\} \tag{2.18}$$

$$g_{ik} = \frac{1}{(N+1)\sin\theta_k} \sum_{n=0}^{N} (-1)^n s[i, n] \sin(n+1)\theta_k, (k = 1, 2, \ldots, N), \tag{2.19}$$

where $x_i = -\cos(i\pi/(N+1)), \theta_k = \pi k/(N+1)$,

$$s[i, n] = (-1)^n \left[\frac{2}{(1-n)(n+1)(n+3)} \right.$$

$$\left. - \frac{\cos(n+1)\theta_i}{2(n+1)} + \frac{\cos(n-1)\theta_i}{4(n-1)} + \frac{\cos(n+3)\theta_i}{4(n+3)} \right] \tag{2.20}$$

if $n \geq 2$, $i = 0, 1, 2, \ldots, N+1$,

$$s[i,0] = (1+x_i)^2(2-x_i)/3; \;\; s[i,1] = -(1-x_i^2)^2/2.$$

It follows that g_{ik} can be calculated with any desired precision. If $h = b - a$ is small enough, the equation (2.15) can be solved iteratively. Due to the equality $\|\mathbf{G}_{N+2}\| = 1$, the sufficient condition for convergence of the iteration process is

$$\max_{a \leq t \leq b} \left\| \frac{D(f_1, f_2, \ldots, f_m)}{D(y_1, y_2, \ldots, y_m)} \right\| < \frac{1}{h}, \tag{2.21}$$

where $\frac{D(f_1,\ldots,f_m)}{D(y_1,\ldots,y_m)}$ is the Jacobi matrix, and $y_i = y_i(t)$. This condition (2.21) guarantees that the nonlinear operator $h\mathbf{G}_{N+2}\mathbf{F}[N+2, m]$ in (2.15) is a contracting operator mapping the set of matrices $(N+2) \times m$ into it. For the large interval $[a, b]$ it is necessary to divide it into small parts $a = h_0 < h_1 < \quad < h_n = b$, and to solve the system (2.1) in each subinterval $[h_l, h_{l+1}]$, $l = 0, 1, \ldots, n-1$, separately. In addition, the solution at the points $t = h_{l+1}$ must be chosen as an initial value for the solution in the next interval $[t_{l+1}, t_{l+2}]$. Matrix $\mathbf{G}_{N+1}$ does not change, but the new nodes t_k, $k = 0, 1, \ldots, N+1$ may be calculated by (2.12) with $a = h_l, b = h_{l+1}$ and $h = h_{l+1} - h_l$. Thus, the global solution for system (2.1) can be found by using the aforementioned multistep procedure involving iterative loops.

3. NUMERICAL SOLVING OF PARTIAL DIFFERENTIAL EQUATION WITH INITIAL AND BOUNDARY CONDITIONS

We consider the solving of the equation

$$u'_t + (u^n u^{(3)}_{xxx})'_x = 0 \tag{3.1}$$

with $n = 1$ as an example of an application of the DM-method.
Let us be given the problem

$$u'_t + (uu^{(3)}_{xxx})'_x = 0, u(0,x) = f(x), x \in [0,1], \tag{3.2}$$

$$u'_x = u^{(3)}_{xxx} = 0 \text{ at } x = 0 \text{ and } x = 1. \tag{3.3}$$

We choose the nodes

$$x_m = 0.5(1+\tau_m), \;\; \tau_m = -\cos\frac{\pi m}{N+1} \tag{3.4}$$

and approximate derivatives in variable x by means of the interpolation's matrix Δ_{N+2} with elements (2.6). This is a global nonsaturated approximation

on the interval $[0, 1]$. Further we will use for simplicity the notations

$$u(x) = u(t, x); \mathbf{u}_N = (u(x_1), u(x_2), \ldots, u(x_N))^T \tag{3.5}$$

Similary we denote

$$\mathbf{u}_N^{(k)} = \left(u^{(k)}(x_1), u^{(k)}(x_2), \ldots, u^{(k)}(x_N)\right)^T$$

The derivatives have an approximation

$$u'(x_m) = \sum_{k=0}^{N+1} \delta_{mk} u(x_k), \tag{3.6}$$

where

$$\delta_{mk} = \frac{2(-1)^{m+k}}{t_m - t_k} \text{ if } m \neq k \text{ and } \delta_{mm} = -\frac{t_m}{1 - t_m^2} \text{ if } m = k \tag{3.7}$$

for $m, k = 1, 2, \ldots, N$ and

$$\delta_{0,k} = -4\delta_{k,0} = \frac{4(-1)^{k+1}}{1 + \tau_k};\ \delta_{N+1,k} = -4\delta_{k,N+1} = \frac{4(-1)^{N+k+1}}{1 - \tau_k} \tag{3.8}$$

$$\delta_{N+1,0} = -\delta_{0,N+1} = (-1)^{N+1}; \delta_{N+1,N+1} = -\delta_{0,0} = \frac{1}{3}(2N^2 + 4N + 3).$$

If we add to system (3.6) the equalities $u'(x_0) = u'(x_{N+1})$ having arisen from (3.3), then we have for the vector $\mathbf{u}_N$ the following representation in the matrix form:

$$\mathbf{u}_N' = \mathbf{S}_N \mathbf{u}_N. \tag{3.9}$$

The elements of $\mathbf{S}_N$ are

$$s_{mk} = \delta_{mk} + \frac{(-1)^m \alpha_k}{1 + \tau_m} + \frac{(-1)^{N+m} \beta_k}{1 - \tau_m}, \tag{3.10}$$

where

$$\alpha_k = \frac{6(-1)^{k+1}}{N(N+2)(N^2+2N+3)} \left[\frac{N^2 + 2N}{1 + \tau_k} - \frac{3\tau_k}{1 - \tau_k^2}\right]$$

$$\beta_k = \frac{6(-1)^{k+N}}{N(N+2)(N^2+2N+3)} \left[\frac{N^2 + 2N}{1 - \tau_k} + \frac{3\tau_k}{1 - \tau_k^2}\right], \quad k, m = 1, 2, \ldots, N. \tag{3.11}$$

Analogously we obtain that

$$\mathbf{u}''_N = \mathbf{S}_N^{(1)}\mathbf{u}'_N = \mathbf{S}_N^{(1)}\mathbf{S}_N\mathbf{u}_N, \tag{3.12}$$

where elements of $\mathbf{S}_N^{(1)}$ are $s_{mk}^{(1)} = \delta_{mk}$ since $u'(x_0) = u'(x_{N+1}) = 0$, and

$$\mathbf{u}_N^{(3)} = \mathbf{S}_N\mathbf{S}_N^{(1)}\mathbf{S}_N\mathbf{u}_N. \tag{3.13}$$

Finally we have for the vector $\mathbf{u}_N$ the system of ordinary nonlinear differential equations

$$\frac{d\mathbf{u}_N}{dt} + \mathbf{S}_N^{(1)}\left(\mathbf{u}_N * (\mathbf{S}_N\mathbf{S}_N^{(1)}\mathbf{S}_N\mathbf{u}_N)\right) = 0, \tag{3.14}$$

$$\mathbf{u}_N\,|_{t=0} = (f(x_1), f(x_2), \quad ., f(x_N))^T \tag{3.15}$$

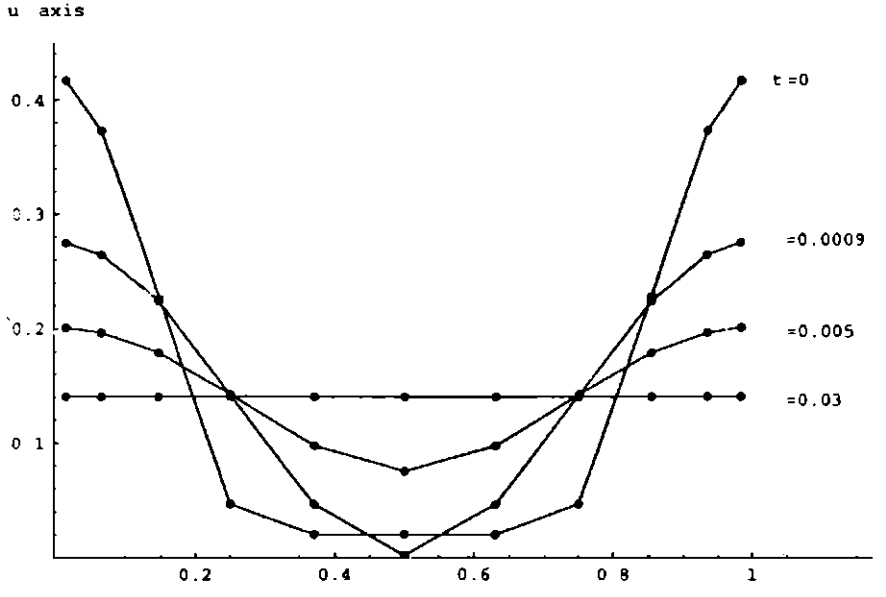

Figure 3.1. The graphs of the solution of the problem (3.2)–(3.3) for several moments of the time obtained by the method (3.14)–(3.15)

where * denotes the operation for the direct product of matrices. This system can be solved by means of DM-method developed in Sect. 2.

REMARK. Instead of (3.12) we can use the system

$$\frac{d\mathbf{u}_N}{dt} + (\mathbf{S}_N\mathbf{u}_N) * \left(\mathbf{S}_N\mathbf{S}_N^{(1)}\mathbf{S}_N\mathbf{u}_N\right) + \mathbf{u}_N * \left((\mathbf{S}_N^{(1)}\mathbf{S}_N)^2\mathbf{u}_N\right) = 0 \tag{3.16}$$

Table 3.1.
Numerical solution of the problem (3.2)–(3.3) using the method (3.14)–(3.15)

t/x	0.017	0.067	0.146	0.250	0.370	0.5
0.000	0.4168	0.3728	0.2274	0.0468	0.0200	0.0200
0.00093	0.2748	0.2641	0.2236	0.1417	0.0461	0.0017
0.005	0.2007	0.1962	0.1787	0.1423	0.0972	0.0750
0.030	0.1404	0.1404	0.1403	0.1402	0.1400	0.1399

which also follows from the identity

$$(uu^{(3)})' = uu^{(4)} + u'u^{(3)}$$

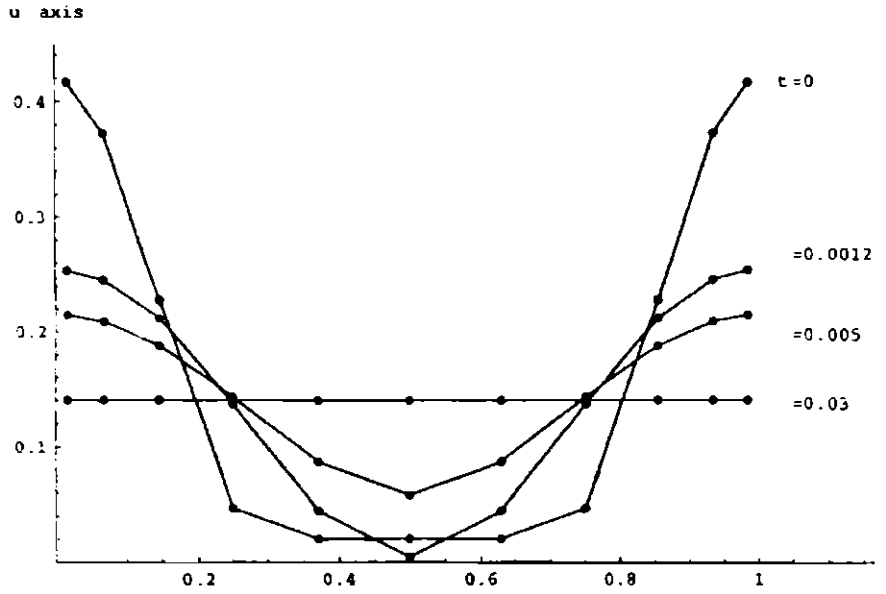

Figure 3.2. The graphs of the solution of the problem (3.2)–(3.3) for several moments of the time obtained by the method (3.16)

As an example of efficiency of the above described algorithms, here we present some results for numerical solution of the problem (3.2)–(3.3) with the initial condition $u(0,x) = f(x)$, where

$$f(x) = 0.2(1 + \cos(3\pi x/10)), \text{ if } 0 \le x \le 3/10; f(x) = 0.2, \text{ if } 0.3 \le x \le 0.7;$$

$$f(x) = 0.2(1 + \cos(7\pi x/10)), \text{ if } 0.7 \le x \le 1. \tag{3.17}$$

All calculations were carried out by the computer system "Mathematica" version 2.2 using 16-digit precision arithmetic with $N = 11$ nodes for x and

Table 3.2.
Numerical solution of the problem (3.2)–(3.3) using the method (3.16)

t/x	0.017	0.067	0.146	0.250	0.370	0.5
0.000	0.4168	0.3728	0.2274	0.0468	0.0200	0.0200
0.0012	0.2539	0.2455	0.2117	0.2372	0.0445	0.0039
0.005	0.2147	0.2092	0.1879	0.1435	0.0872	0.0579
0.030	0.1408	0.1408	0.1407	0.1405	0.1402	0.1401

for t in each step of size $s = 0.00001$. Obtained numerical values of the solution $u(t,x)$ for several values of time t are shown in Tables 3.1 and 3.2, but the corresponding graphs are illustrated in Fig.3.1 and 3.2. Computed results show that in both cases at the beginning of the process the solution tends to zero in the middle of the interval $[0,1]$, but later it tends to the constant equal with 0.14, i.e., to the direct value of integral of $u(0,x)$ on $[0,1]$.

4. PROBLEMS WITH CONDITIONS OF PERIODICITY

The problem (3.2)-(3.3) considered in the Sect. 3 can be formulated as follows. It is necessary to find the solution of equation and initial condition (3.2) and following conditions instead of (3.3):

$$u(t,-x) \equiv u(t,x); u(t,x+2) \equiv u(t,x), \tag{4.1}$$

i.e., the evenness and the periodicity in x of the solution. Substituting $z = \pi x$ and $\tau = \pi^4 t$ leads to the problem

$$u'_\tau + (uu^{(3)}_{zzz})'_z = 0, \tag{4.2}$$

$$u\,|_{\tau=0} = f(\frac{z}{\pi}) \equiv g(z), \tag{4.3}$$

$$u(\tau,-z) \equiv u(\tau,z);\ \ u(\tau,z+2\pi) \equiv u(\tau,z). \tag{4.4}$$

If we seek the solution of (4.2)–(4.4) in the form

$$u(\tau,z) = \sum_{n=0}^{\infty} c_n(\tau)\cos nz, \tag{4.5}$$

then $c_n(\tau)$ decrease rapidly for a smooth solution as n tends to infinity. After the substitution $\theta = \cos z$ we get instead (4.5) the Fourier series with Chebyshev's polynomials of the first kind in the variable θ. Using results of the article [2] we conclude that the contracting of the problem (4.2)–(4.4) at the nodes

$$\theta_k = -\cos\frac{(2k-1)\pi}{2N},\ \ k = 1,2,\ldots,N \tag{4.6}$$

gives a nonsaturated approximation. Therefore, we can apply the following algorithm for numerical solving the problem (4.2)–(4.4):

1) the substituting $\theta = \cos z$ in the equation (4.2);

2) the contracting equation after this substitution at nodes $\theta_k, k = 1, 2, \ldots$ on interval $(-1, 1)$, and replacing of derivatives by means of interpolation's matrices for corresponding derivatives;

3) the solving of the system of ordinary differential equations by means of DM-method.

After the mentioned substitution $\theta = \cos z$ in (4.2) we get

$$u'_\tau + u[(1-\theta^2)^2 u^{(4)}_\theta - 6\theta(1-\theta^2)u^{(3)}_\theta + (7\theta^2 - 4)u^{(2)}_\theta - \theta u'_\theta] +$$

$$+(1-\theta^2)u'_\theta[-u'_\theta - 3\theta u^{(2)}_\theta + (1-\theta^2)u^{(3)}_\theta] = 0. \tag{4.7}$$

For derivatives in this case we have the interpolation's matrix $\mathbf{Q}_N$ with elements [3]

$$\sigma_{im} = \frac{(-1)^{i+m}\sqrt{1-\theta_m^2}}{(\theta_i - \theta_m)\sqrt{1-\theta_i^2}}, \quad i \neq m, \tag{4.8}$$

$$\sigma_{mm} = \frac{\theta_m}{2(1-\theta_m^2)}, \quad \theta_i = -\cos\frac{(2i-1)\pi}{2N}, \quad i, m = 1, 2, \ldots, N. \tag{4.9}$$

After these manipulations we obtain the system of differential equations in the following matrix form

$$\frac{d\mathbf{u}_N}{d\tau} + \mathbf{u}_N * (\mathbf{A}_N \mathbf{u}_N) + \mathrm{diag}\{1-\theta_m^2\}[(\mathbf{Q}_N \mathbf{u}_N) * (\mathbf{B}_N \mathbf{u}_N)] = 0, \tag{4.10}$$

where

$$\mathbf{A}_N = \mathrm{diag}\{(1-\theta_m^2)^2\}\mathbf{Q}_N^4 - 6\mathrm{diag}\{\theta_m(1-\theta_m^2)\}\mathbf{Q}_N^3 +$$

$$+\mathrm{diag}\{(7\theta_m^2 - 4)\}\mathbf{Q}_N^2 + \mathrm{diag}\{\theta_m\}\mathbf{Q}_N, \tag{4.11}$$

$$\mathbf{B}_N = \mathrm{diag}\{1-\theta_m^2\}\mathbf{Q}_N^3 - 3\mathrm{diag}\{\theta_m\}\mathbf{Q}_N^2 - \mathbf{Q}_N, \quad \tau = \pi^4 t, \quad s = \cos\pi x. \tag{4.12}$$

Analogously it is possible to find the algorithm for numerical solution of the problem

$$u'_t + u^{(4)}_{xxxx} + u^{(2)}_{xx} + 0.5(u'_x)^2 = 0, \tag{4.13}$$

$$u\,|_{t=0} = f(x),\ u(t,-x) \equiv u(t,x);\ u(t,x+T) \equiv u(t,x). \tag{4.14}$$

Substituting $x = \mu z, t = \mu^4\tau, \mu = T/(2\pi)$ leads to

$$u'_\tau + u^{(4)}_{zzzz} + \mu^2 u^{(2)}_{zz} + 0.5\mu^2(u'_z)^2 = 0, \tag{4.15}$$

$$u\,|_{\tau=0} = g(x) \equiv f(\mu z);\ u(\tau,-z) \equiv u(\tau,z);\ u(\tau,z+2\pi) \equiv u(\tau,z). \tag{4.16}$$

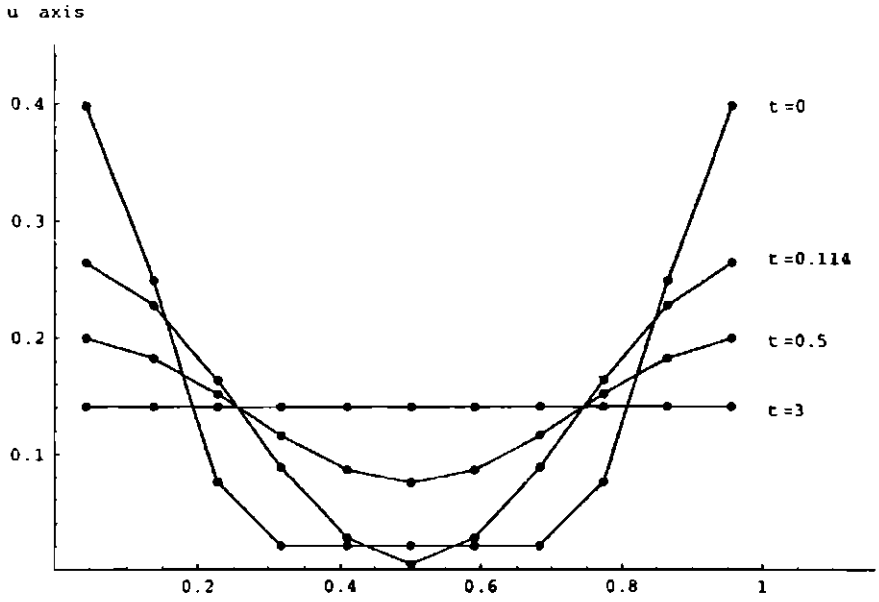

Figure 4.3. The graphs of the solution of the problem (3.2)–(3.3) for several moments of the time using (4.9)–(4.10)

Table 4.3.
Numerical solution of the problem (3.2)–(3.3) using the method (4.9)–(4.10)

τ/x	0.045	0.136	0.227	0.3318	0.409	0.5
0.000	0.3978	0.2485	0.0753	0.0200	0.0200	0.0200
0.114	0.2640	0.2274	0.1631	0.0880	0.0269	0.0043
0.500	0.1994	0.1820	0.1513	0.1155	0.0856	0.0744
3.000	0.1405	0.1404	0.1403	0.1402	0.1401	0.1400

Substituting $\theta = \cos z$ and contracting (4.14) at nodes (4.6) we have for the unknown vector $\mathbf{u}_N$ = the system

$$\frac{d\mathbf{u}_N}{d\tau} + \mathbf{C}_N\mathbf{u}_N + 0.5\mu^2\mathrm{diag}\{1-\theta_m^2\}\,[(\mathbf{Q}_N\mathbf{u}_N) * (\mathbf{Q}_N\mathbf{u}_N)] = 0, \qquad (4.17)$$

$$\mathbf{C}_N = \mathrm{diag}\{(1-\theta_m^2)^2\}\mathbf{Q}_N^4 - \mathrm{diag}\{6\theta_m(1-\theta_m^2)\}\mathbf{Q}_N^3 +$$

$$+\mathrm{diag}\{7\theta_m^2 - 4 + \mu^2(1-\theta_m^2)\}\mathbf{Q}_N^2 - \mathrm{diag}\{(1+\mu^2)\theta_m\}\mathbf{Q}_N. \qquad (4.18)$$

Numerical calculations by this method showed the following results for the problem (3.2)–(3.3) with the initial function (3.17). Also here all calculations were carried out by the computer system "Mathematica" version 2.2 using 16-digit precision arithmetic with $N = 11$ nodes for x and for τ in each

step of size $s = 0.002$. Obtained numerical values of the solution $u(\tau, x)$ at the points $x = (2i - 1)/22$, $i = 1, 2, \ldots, 6$ for several values of time τ are shown in Table 4.1 but the corresponding graphs are illustrated in Figure 4.1. Computed results show also that the beginning of the process the solution tends to zero in the middle of the interval $[0, 1]$, but later it tends at the all set of points x to the constant equal 0.1401 – the mean value of the solution which does not depend on τ and is equal to the area under the each curve $u = u(\tau, x), 0 \leq x \leq 1$, $\tau = const$.

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DM-METODES LIETOJUMI NELINEĀRU PARCIĀLO DIFERENCIĀLVIENĀDOJUMU SKAITLISKAJAI ATRISINĀŠANAI

TEODORS CĪRULIS un OJĀRS LIETUVIETIS

Tiek aplūkotas nelineāru parciālo diferenciālvienādojumu problēmas ar sākuma un robežnosacījumiem, lietojot taišņu metodi. Parasto diferenciālvienādojumu atrisināšanai lieto deģenerēto matricu metodi, kas ir vienkārši programmējama un dod augstu precizitāti pat stingrām sistēmām. Atsevišķi tiek aplūkotas problēmas ar periodiskiem robežnosacījumiem, lietojot nepiesātinātās interpolācijas ar Čebiševa polinomu nullēm kā mezglu punktiem.

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THE EXACT FINITE–DIFFERENCE APPROXIMATION FOR DIFFERENTIAL EQUATION OF THE SECOND ORDER WITH CONSTANT PIECE–WISE COEFFICIENTS

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ABSTRACT

The exact finite–difference scheme for solving the boundary–value problem of a differential equations of the second order with constant piece–wise coefficients is developed.

We shall consider differential equations of the second order in the following form

$$L_1\mathbf{u} \equiv \partial(\lambda\partial\mathbf{u}/\partial x)/\partial x - a\partial\mathbf{u}/\partial x = \mathbf{f}, \tag{1}$$

$$L_2\mathbf{u} \equiv \partial(\lambda\partial\mathbf{u}/\partial x)/\partial x - b^2\mathbf{u} = \mathbf{f}, \tag{2}$$

$$L_3\mathbf{u} \equiv \partial(\lambda\partial\mathbf{u}/\partial x)/\partial x + b^2\mathbf{u} = \mathbf{f}, \tag{3}$$

where $x \in (0,l)$, $\lambda > 0$. Let the coefficients λ, a, b, $\mathbf{f}$ be piece–wise continuous in the interval $(0,l)$ and a nonuniform grid contains the discontinuity points of coefficients. The function $\mathbf{u}$ and the flux–function $\lambda\partial\mathbf{u}/\partial x$ are continuous. We assume that the boundary conditions can be written as

$$\nu_0\lambda_1\partial\mathbf{u}/\partial x - \alpha_0\mathbf{u} = -\alpha_0\Phi_0, \quad x = 0, \tag{4}$$

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$$\nu_1 \lambda_N \partial \mathbf{u}/\partial x + \alpha_1 \mathbf{u} = \alpha_1 \Phi_1, \quad x = l, \tag{5}$$

where $\alpha_0 \geq 0$, $\alpha_1 \geq 0$, or $\alpha_0 = \alpha_1 = \infty$ – in accordance with Dirichlet boundary condition.

Equations (1,2,3) are the steady-state forms of corresponding heat transfer equations. In case of the problem with time dependence, the right-side function $\mathbf{f}$ can be considered in the form $\mathbf{f} = \partial \mathbf{u}/\partial t$.

1. The approximation of differential boundary–value problem (1,4,5) is based on the conservation law approach of the method of finite volumes [1]. To derive a difference equation associated with the j-th grid point x_j we integrate the self-adjoint form of differential equation (1)

$$\partial(\lambda^* \partial \mathbf{u}/\partial x)\partial x = \mathbf{G}(x) \tag{6}$$

in the intervals $(x_{j-0.5}, x_{j+0.5})$:

$$\mathbf{W}_{j+0.5} - \mathbf{W}_{j-0.5} = \int_{x_{j-0.5}}^{x_j} \mathbf{G}_j \, dz + \int_{x_j}^{x_{j+0.5}} \mathbf{G}_{j+1} \, dz, \tag{7}$$

where

$$\mathbf{W}_{j\pm 0.5} = \mathbf{W}|_{x=x_{j\pm 0.5}}, \quad x_{j\pm 0.5} = (x_j + x_{j\pm 1})/2, \quad h_j = x_j - x_{j-1},$$

$$\mathbf{W}(x) = \lambda^* \partial \mathbf{u}/\partial x, \quad \lambda^* = J\lambda, \quad \mathbf{G} = J\mathbf{f}, \quad J(x) = \exp\left(-\int_{x_j}^{x} \lambda^{-1} a \, dt\right)$$

This is the integral form of the conservation law on the interval $(x_{j-0.5}, x_{j+0.5})$. In classical formulation of the finite volumes method, it is assumed that the flux terms $\mathbf{W}_{j\pm 0.5}$ in (7) are approximate with the difference expressions. Then the corresponding difference scheme is not exact for given functions $\mathbf{G}_j$ in case of piece–wise constant functions of λ, a, $\mathbf{f}$. Here we have the possibility to make the exact difference scheme [2]. Therefore, we integrate equation (6) from $x_{j-0.5}$ to $x \in (x_{j-1}, x_j)$, divide this expression by λ^* and integrate from x_{j-1} to x_j. We obtain $\mathbf{W}_{j-0.5} = A_j^-(\mathbf{u}_j - \mathbf{u}_{j-1}) - A_j^- \mathbf{B}_j$, where $(A_j)^{-1} \quad \int_{x_{j-1}}^{x_j} (\lambda^*)^{-1} \, dx$, $\mathbf{B}_j = \int_{x_{j-1}}^{x_j} (\lambda^*)^{-1} \, dx \int_{x_{j-0.5}}^{x} \mathbf{G}_j \, d\xi$, and $\mathbf{u}_j$, $\mathbf{u}_{j-1}$ represents the value of function $\mathbf{u}$ at x_j, x_{j-1}. Similarly,

$$\mathbf{W}_{j+0.5} = A_{j+1}^+(\mathbf{u}_{j+1} - \mathbf{u}_j) - A_{j+1}^+ \mathbf{B}_{j+1},$$

where $(A_{j+1}^+)^{-1} = \int_{x_j}^{x_{j+1}} (\lambda^*)^{-1} \, dx$. To derive a 3-point exact difference equation associated with the central grid point x_j we want to apply equation (7) in the form

$$h_j \Lambda_1 \mathbf{u}_j \equiv A_{j+1}^+(\mathbf{u}_{j+1} - \mathbf{u}_j) - A_j^-(\mathbf{u}_j - \mathbf{u}_{j-1}) = \mathbf{R}_j^- + \mathbf{R}_j^+, \quad j = \overline{1, N-1} \tag{8}$$

where $\hbar_j = 0.5(h_j+h_{j+1})$, and $\mathbf{R}_j^- = \int_{x_{j-1}}^{x_j}(1 - A_j^- \int_x^{x_j}(\lambda^*)^{-1}d\xi)\mathbf{G}_j\,dx$, $\mathbf{R}_j^+ = \int_{x_j}^{x_{j+1}}(1 - A_{j+1}^+ \int_{x_j}^{x}(\lambda^*)^{-1}d\xi)\mathbf{G}_{j+1}\,dx$.

If the values of parameters λ, a of equation (1) in intervals (x_{j-1}, x_j), (x_j, x_{j+1}) are equal to $\lambda_{j-0.5}, a_{j-0.5}$ and $\lambda_{j+0.5}, a_{j+0.5}$, respectively, then $A_j^- = \lambda_{j-0.5}g(-\beta_j)/h_j > 0$, $A_{j+1}^+ = \lambda_{j+0.5}(\beta_{j+1})/h_{j+1} > 0$, $\mathbf{R}_j^- = \int_{x_{j-1}}^{x_j}[1 - \exp(-\beta_j(1-(x_j-x)/h_j))]/[1-\exp(-\beta_j)]f(x)\,dx$, $\mathbf{R}_j^+ = \int_{x_j}^{x_{j+1}}[1-\exp(\beta_{j+1}(1-(x-x_j)/h_{j+1}))]/[1-\exp(\beta_{j+1})]f(x)dx$, where $g(s) = s(\exp(s)-1)^{-1}$ is a real positive function with $\beta_j = (\lambda_{j-0.5})^{-1}a_{j-0.5}h_j$, $\beta_{j+1} = (\lambda_{j+0.5})^{-1}a_{j+0.5}h_{j+1}$, $g(s) = 1 - s/2 + O(s^2)$.

If the values of the parameter $\mathbf{f}$ in intervals (x_{j-1}, x_j), (x_j, x_{j+1}) are equal to $\mathbf{f}_{j-0.5}$ and $\mathbf{f}_{j+0.5}$, then $\mathbf{R}_j^- = h_j r(-\beta_j)\mathbf{f}_{j-0.5}$ and $\mathbf{R}_j^+ = h_{j+1}r(\beta_{j+1})\mathbf{f}_{j+0.5}$, respectively, where $r(s) = s^{-1}(1 - g(s))$ is a real positive function with the expression $r(s) = 0.5 - s/12 + O(s^2)$.

For the approximation of the boundary condition (4) we apply the integral form of the conservation law to the half interval $(0, x_{0.5})$. Due to boundary condition at $x = 0$ the value $\mathbf{W}_0$ is known and we can easily derive the following 2–point exact difference equation associated with grid point $x_0 = 0$

$$h_1\Lambda_1\mathbf{u}_0 \equiv A_1^+(\mathbf{u}_1 - \mathbf{u}_0) - \alpha_0(u_0 - \Phi_0) = \mathbf{R}_0^+, \tag{9}$$

where $A_1^+ = \lambda_{0.5}(h_1)^{-1}g(\beta_1) > 0$, $\beta_1 = \lambda_{0.5}^{-1}a_{0.5}h_1$, $\mathbf{R}_0^+ = h_1 r(\beta_1)f_{0.5}$.

Similarly in advance for the approximation of the boundary condition (5) we obtain 2–point exact difference equation associated with grid point $x_N = l$ in the following form

$$h_N\Lambda_1\mathbf{u}_N \equiv -\alpha_1(\mathbf{u}_N - \Phi_1) - A_N^-(\mathbf{u}_N - \mathbf{u}_{N-1}) = \mathbf{R}_N^-, \tag{10}$$

where $A_N^- = \lambda_{N-0.5}h_N^{-1}g(-\beta_N) > 0$, $\beta_N \quad \lambda_{N-0.5}^{-1}a_{N-0.5}h_N$ and $\mathbf{R}_j^-$ $h_N r(-\beta_N)\mathbf{f}_{N-0.5}$.

We see that the difference equations (8,9,10) are exact approximations for solving boundary-value problem (1,4,5).

In case of uniform grid ($h_j = h_{j+1} = h$) and constant coefficients λ, a, $\mathbf{f}$ we have the Il'yn difference equations [3] in the form

$$\Lambda_1\mathbf{u}_j \equiv \gamma_I(\beta)\delta^2\mathbf{u}_j - a\delta\mathbf{u}_j = \mathbf{f}, \quad j = \overline{1, N-1}, \tag{11}$$

where $\delta\mathbf{u}_j = (\mathbf{u}_{j+1} - \mathbf{u}_{j-1})/(2h)$, $\delta^2\mathbf{u}_j = (\mathbf{u}_{j+1} - 2\mathbf{u}_j + \mathbf{u}_{j-1})/h^2$ denote the central difference expressions for the derivatives $\partial\mathbf{u}/\partial x$, $\partial^2\mathbf{u}/\partial x^2$ at the point $x_j = jh$ of uniform grid with step h; $\gamma_I(s) = 0.5s\coth(0.5s)$ is the Il'yn perturbation coefficient for the monotone difference scheme with $\gamma_I(s) = 1+s^2/12+O(s^4)$. In this case $\beta_j = \beta_{j+1} = \beta$, $g(\pm\beta_j) = \gamma_I(\beta) - \pm\beta_j/2$, $r(\beta_j) + r(-\beta_j) = 1$.

Difference equations (9,10) on the boundary have the forms

$$h\Lambda_1\mathbf{u}_0 \equiv \lambda g(\beta)(\mathbf{u}_1 - \mathbf{u}_0)/h - \alpha_0(\mathbf{u}_0 - \Phi_0) = \lambda\mathbf{f}(1 - g(\beta))/a, \tag{12}$$

$$-h\Lambda_1 \mathbf{u}_N \equiv \lambda g(-\beta)(\mathbf{u}_N - \mathbf{u}_{N-1})/h + \alpha_1(\mathbf{u}_N - \Phi_1) = \lambda \mathbf{f}(1 - g(-\beta))/a. \quad (13)$$

2. The exact approximation of the differential boundary-value problem (2,4,5) with piece-wise constant coefficients λ, b is based on the analytic solutions $u^1(x)$, $u^2(x)$ of differential equation (2) in corresponding intervals (x_{j-1}, x_j) and (x_j, x_{j+1}) in following form

$$u^1(x) = C_1^- \cosh(\omega_j x) + C_2^- \sinh(\omega_j x) - \omega_j \lambda_{j-0.5}^{-1} R_j^-, \quad x \in (x_{j-1}, x_j),$$

$$u^2(x) = C_1^+ \cosh(\omega_{j+1} x) + C_2^+ \sinh(\omega_{j+1} x) + \omega_{j+1} \lambda_{j+0.5}^{-1} R_j^+, \quad x \in (x_j, x_{j+1}),$$

where $\omega_j = b_{j-0.5}\lambda_{j-0.5}^{-1}$, $\omega_{j+1} = b_{j+0.5}\lambda_{j+0.5}^{-1}$,

$$R_j^- = \int_x^{x_j} \sinh(\omega_j(x - \xi)) f(\xi)\, d\xi,$$

$$R_j^+ = \int_{x_j}^{x} \sinh(\omega_{j+1}(x - \xi)) f(\xi)\, d\xi.$$

The functions u^1, u^2 and the flux-functions $\lambda_{j-0.5}\partial u^1/\partial x$, $\lambda_{j+0.5}\partial u^2/\partial x$ must be continuous in the point x_j, and with $u^1(x_{j-1}) = \mathbf{u}_{j-1}$, $u^2(x_{j+1}) = \mathbf{u}_{j+1}$, $u^1(x_j) = u^2(x_j) = \mathbf{u}_j$ we have the following 3-point exact difference equation

$$h_j \Lambda_2 \mathbf{u}_j \equiv A_j \mathbf{u}_{j-1} - C_j \mathbf{u}_j + B_j \mathbf{u}_{j+1} = \mathbf{R}_j^- + \mathbf{R}_j^+, \quad j = \overline{1, N-1} \quad (14)$$

where $A_j = \lambda_{j-0.5}\omega_j / \sinh(\omega_j h_j) > 0$, $B_j = \lambda_{j+0.5}\omega_{j+1} / \sinh(\omega_{j+1} h_{j+1}) > 0$, $C_j = A_j \cosh(\omega_j h_j) + B_j \cosh(\omega_{j+1} h_{j+1})$,

$$\mathbf{R}_j^- = \int_{x_{j-1}}^{x_j} \sinh(\omega_j(\xi - x_{j-1}))/\sinh(\omega_j h_j) f(\xi)\, d\xi,$$

$$\mathbf{R}_j^+ = \int_{x_j}^{x_{j+1}} \sinh(\omega_{j+1}(x_{j+1} - \xi))/\sinh(\omega_{j+1} h_{j+1}) f(\xi)\, d\xi.$$

For the exact approximation of the boundary condition (4) we have the following solution $u(x)$ of differential equation (2) in the interval $(0, x_1)$:

$$u(x) = C_1 \cosh(\omega_1 x) + C_2 \sinh(\omega_1 x) + \omega_1 \lambda_{0.5}^{-1} R_0^+,$$

where $R_0^+ = \int_0^x \sinh(\omega_1(x - \xi)) f(\xi)\, d\xi$.

From the conditions (4) and $u(0) = \mathbf{u}_0$, $u(x_1) = \mathbf{u}_1$ it follows that

$$h_1 \Lambda_2 \mathbf{u}_0 \equiv B_0 \mathbf{u}_1 - (B_0 \cosh(\omega_1 h_1) + \alpha_0)\mathbf{u}_0 + \alpha_0 \Phi_0 = \mathbf{R}_0^+. \quad (15)$$

Similarly, for the approximation of the boundary condition (5) we obtain 2-point exact difference equation

$$h_N \Lambda_2 \mathbf{u}_N \equiv A_N \mathbf{u}_{N-1} - (A_N \cosh(\omega_N h_N) + \alpha_1)\mathbf{u}_N + \alpha_1 \Phi_1 = \mathbf{R}_N^-. \quad (16)$$

In case of uniform grid with mesh step h and constant coefficients λ, b, $\mathbf{f}$ we have the Bahvalov difference equations [4] in the form

$$\Lambda_2 \mathbf{u}_j \equiv \gamma_B(s)\lambda\delta^2 \mathbf{u}_j - b^2 \mathbf{u}_j = \mathbf{f}, \quad j = \overline{1, N-1}, \quad (17)$$

where $\gamma_B(s) = (s/\sinh(s))^2$, $(s = \omega h/2 = bh\lambda^{-0.5}/2)$ is the Bahvalov perturbation coefficient for the monotone difference scheme with $\gamma_B(s) = (1 + s^2/6 + O(s^4))^{-1}$

Difference equations (15,16) on the boundaries have the forms

$$h\Lambda_2 \mathbf{u}_0 \equiv B_0 h(\mathbf{u}_1 - \mathbf{u}_0)/h - \alpha_0(\mathbf{u}_0 - \Phi_0) - \lambda\omega \tanh(\omega h/2)\mathbf{u}_0$$

$$= \tanh(\omega h/2)\mathbf{f}/\omega, \quad (18)$$

$$-h\Lambda_2 \mathbf{u}_N \equiv A_N h(\mathbf{u}_N - \mathbf{u}_{N-1})/h + \alpha_1(\mathbf{u}_N - \Phi_1) + \lambda\omega \tanh(\omega h/2)\mathbf{u}_N$$

$$= -\tanh(\omega h/2)\mathbf{f}\omega. \quad (19)$$

3. The exact difference scheme for the differential boundary-value problem (3,4,5) is in the same forms (14,15,16) or (17,18,19), where the functions b, ω, $\cosh(iz)$, $\sinh(iz)$, $\tanh(iz)$ are replaced with functions ib, $i\omega$, $\cos(z)$, $i\sin(z)$. $i\tan(z)$; $i = (-1)^{0.5}$, $i^2 = -1$. Therefore, the exact difference scheme can be obtained replacing the hyperbolical functions by the trigonometrical functions ones.

4. For the differential equation

$$L_4 \mathbf{u} \equiv \partial(\lambda \partial \mathbf{u}/\partial x)/\partial x - b\mathbf{u} = \mathbf{f}, \quad (20)$$

the finite-difference scheme is in the same form, where the function b is replaced with $|b|^{0.5}$, $(b > 0)$ or $i|b|^{0.5}$, $(b < 0)$.

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PRECĪZĀ DIFERENČU SHĒMA OTRĀS KĀRTAS DIFERENCIĀLVIENĀDOJUMU AR GABALIEM KONSTANTIEM KOEFICIENTIEM APROKSIMĀCIJAI

HARIJS KALIS

Dažu 2. kārtas diferenciālvienādojumu robežproblēmām ir konstruētas precīzās diferenču shēmas gabaliem konstantu koeficientu gadījumā.

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ON THE NUMERICAL SIMULATIONS OF THE LIQUID TRANSFER IN FLEECE

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ABSTRACT

The numerical method for the liquid transport in fleece is based on the explicit difference scheme.

The mathematical modelling of the liquid transfer in the wet–absorbing and wet–giving porous media material with nonlinear characteristics, for example, in fleece causes additional difficulties. The nonlinear time–dependent one dimensional in x–coordinate–space mass transfer equation is of the following form [1]:

$$\partial s/\partial t = \partial(k\partial p/\partial x)/\partial x, \tag{1}$$

where $s = s(x,t)$ is the content of water in the material ($0 \le s \le 1$, $s = 1$ – for the satured flow), $k = k(s)$ is the hydraulic conductivity (the permeability), $p = p(s)$ is the capillary pressure, $x \in (0, L)$, $t > 0$. We assume that the capillary pressure $p(s)$ is not depended on the history of the transfer process, active hysteresis between wetting and drying processes and gravity can be neglected.

The function k is defined as power function on s:

$$k = k(s) = \tilde{k}s^{\beta}, \quad (\tilde{k} = const \ge 0.0001, \quad \beta = const \ge 3). \tag{2}$$

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The function p is defined as a polynomial function on s in a segment $[0, \tilde{s}]$ and as a power function on a segment $[\tilde{s}, 1]$ by the following setting:

$$p(s) = -P(s), \quad s \in [0, \tilde{s}]; \quad p(s) = -p_{00}(1 - s^{\alpha}), \quad s \in [\tilde{s}, 1], \tag{3}$$

where $p_{00} = \tilde{p}/(1 - \tilde{s}^{\alpha})$. $\tilde{p} = const = -p(\tilde{s}) \in [0.01, 0.1]$, $\alpha = const \geq 10$, $\tilde{s} = const \in [0.1, 0.9]$. If the function $p(s) \in C^1[0, 1]$, then the second order polinomial $P(s) = P_2(s)$ has the form

$$P_2(s) = a_{22}(1 - s/\tilde{s})^2 + a_{11}(1 - s/\tilde{s}) + \tilde{p}, \tag{4}$$

where $a_{22} = p_l - \tilde{p} - a_{11}$. $a_{11} = p_{00}\alpha\tilde{s}^{\alpha}$, $p_l = const \in [0.1, 0.6]$. $P_2(s)$ is a quadratic polinomial under the following assumptions: $P_2(0) = p_l$, $P_2(\tilde{s}) = \tilde{p}$, $P_2'(\tilde{s}) = -p'(\tilde{s})$.

If the function $p(s) \in C^2[0, 1]$, then the third order polinomial $P(s) = P_3(s)$ has the form

$$P_3(s) = a_{33}^{*}(1 - s/\tilde{s})^3 + a_{22}^{*}(1 - s/\tilde{s})^2 + a_{11}(1 - s/\tilde{s}) + \tilde{p}, \tag{5}$$

where $a_{33}^{*} = p_l - \tilde{p} - a_{22}^{*} - a_{11}$, $a_{22}^{*} = -a_{11}(\alpha - 1)/2$. $P_3(s)$ is a cubic polinomial under the following assumptions: $P_3(0) = p_l$, $P_3(\tilde{s}) = \tilde{p}$. $P_3'(\tilde{s}) = -p'(\tilde{s})$, $P_3''(\tilde{s}) = -p''(\tilde{s})$.

We can define the "diffusion" coefficient $\nu(s) = k(s)p'(s)$ and the integral variable $\sigma = \int_0^s \nu(\xi)\,d\xi$, then the corresponding models of equation (1) are in the following form

$$\partial s/\partial t = \partial(\nu \partial s/\partial x)/\partial x, \tag{6}$$

$$\partial s/\partial t = \partial^2 \phi/\partial x^2 \tag{7}$$

The functions ν and ϕ are defined by the following expressions:

1) For $p(s) \in C^1[0, 1]$ -

$$\nu(s) = \begin{cases} \tilde{k}s^{\beta}\tilde{s}^{-1}(a_{11} + 2a_{22}(1 - s/\tilde{s})), & s \in [0, \tilde{s}] \\ \tilde{k}\alpha p_{00}s^{\beta+\alpha-1}, & s \in [\tilde{s}, 1], \end{cases} \tag{8}$$

$$\phi(s) = \begin{cases} \tilde{k}\tilde{s}^{-1}[a_{33}s^{b_1}/b_1 - 2a_{22}\tilde{s}^{-1}s^{b_2}/b_2], & s \in [0, \tilde{s}] \\ \tilde{k}\tilde{s}^{\beta}[a_{33}/b_1 - 2a_{22}/b_2] + \tilde{k}\alpha p_{00}[s^{b_3} - \tilde{s}^{b_3}]/b_3, & s \in [\tilde{s}, 1], \end{cases} \tag{9}$$

where

$$a_{33} = a_{11} + 2a_{22}, \quad b_1 = \beta + 1, \quad b_2 = \beta + 2, \quad b_3 = \beta + \alpha.$$

2) For $p(s) \in C^2[0, 1]$

$$\nu(s) = \begin{cases} \tilde{k}s^{\beta}\tilde{s}^{-1}(a_{11} + 2a_{22}^{*}(1 - s/\tilde{s}) + 3a_{33}^{*}(1 - s/\tilde{s})^2), & s \in [0, \tilde{s}] \\ \tilde{k}\alpha p_{00}s^{\beta+\alpha-1}, & s \in [\tilde{s}, 1], \end{cases} \tag{10}$$

$$\phi(s) = \begin{cases} \tilde{k}\bar{s}^{-1}[a_{44}^* s^{b_1}/b_1 - a_{55}^* \bar{s}^{-1} s^{b_2}/b_2 + 3a_{33}^* \bar{s}^{-2} s^b/b], & s \in [0, \bar{s}] \\ \tilde{k}\bar{s}^{\beta}[a_{44}^*/b_1 - a_{55}^*/b_2 + 3a_{33}^*/b] + \tilde{k}\alpha p_{00}[s^{b_3} - \tilde{s}^{b_3}]/b_3, & s \in [\bar{s}, 1], \end{cases} \tag{11}$$

where

$$a_{44}^* = a_{11} + 2a_{22}^* + 3a_{33}^*, \quad a_{55}^* = 2a_{22}^* + 6a_{33}^*, \quad b = \beta + 3.$$

The values of parameters $\tilde{k}$, β, $\bar{s}$, $\tilde{p}$, p_t, α are different in every layer of fleece. We will consider the following boundary conditions:

1) homogeneous boundary conditions of the first order

$$s|_{x=0} = s|_{x=L} = 0, \tag{12}$$

2) boundary conditions of the second order

$$\begin{aligned} k\partial p/\partial x|_{x=0} = \nu\partial s/\partial x|_{x=0} = \partial\phi/\partial x|_{x=0} = -Q_0, \\ k\partial p/\partial x|_{x=L} = \nu\partial s/\partial x|_{x=L} = \partial\phi/\partial x|_{x=L} = 0, \end{aligned} \tag{13}$$

where the flux function (the intensity of the liquid source) $Q_0 \neq 0$ on the finite interval of time $(0, t_0)$. If $t \geq t_0$, then $Q_0 = 0$: on the surface $x = 0$ liquid is poured for a duration of t_0 sec. with the intensity Q_0, then there is a break.

The initial condition is in the form $s|_{t=0} = s^0(x)$, $x \in (0, L)$, where $s^0(x)$ is a given function.

The nonlinear mathematical model is the degenerate case ($k(0) = \nu(0) = 0$) of the porous medium type equations [2].

1. The numerical solution will be found on the discrete set of the points $x_j = jh$, $j = \overline{0, N}$, $Nh = L$ at the time $t = t_n = n\tau$, $n > 0$.

The approximation of differential equation (6) is based on the conservation law approach of the method of finite volumes [3]. We shall refer to the endpoints of the interval about the point x_j as $x_{j\pm0.5} = (x_j + x_{j\pm1})/2$. This interval $(x_{j-0.5}, x_{j+0.5})$ is referred to as the control volume associated with the grid point x_j. To derive a difference equation associated with the j-th grid point x_j we integrate the differential equation (6) on the intervals $(x_{j-0.5}, x_{j+0.5})$:

$$\mathbf{W}_{j+0.5} - \mathbf{W}_{j-0.5} = \int_{x_{j-0.5}}^{x_{j+0.5}} \mathbf{f}\, dx, \tag{14}$$

where

$$\mathbf{W}_{j\pm0.5} = \mathbf{W}|_{x=x_{j\pm0.5}}, \quad \mathbf{W}(x) = \nu\partial s/\partial x, \quad \mathbf{f} = \partial s/\partial t.$$

This is the integral form of the conservation law to the interval $(x_{j-0.5}, x_{j+0.5})$. In the classical formulation for the finite volumes method [3] it is assumed that the flux terms $\mathbf{W}_{j\pm0.5}$ in (14) are approximated by the difference expressions. Then the corresponding difference scheme is not exact for the given functions

$\mathbf{f}$ in the case of piece–wise constant function ν. Here we have the possibility to make the difference scheme exact.

Therefore we integrate equation (6) from $x_{j-0.5}$ to $x \in (x_{j-1}, x_j)$. We get

$$\mathbf{W} - \mathbf{W}_{j-0.5} = \int_{x_{j-0.5}}^{x} \mathbf{f}\, d\xi.$$

After dividing this expression by ν and integrating from x_{j-1} to x_j we obtain

$$\mathbf{u}_j - \mathbf{u}_{j-1} = (A_j)^{-1}\mathbf{W}_{j-0.5} + \mathbf{B}_j,$$

where

$$(A_j)^{-1} = \int_{x_{j-1}}^{x_j} (\nu)^{-1}\, dx, \quad \mathbf{B}_j = \int_{x_{j-1}}^{x_j} (\nu)^{-1}\, dx \int_{x_{j-0.5}}^{x} \mathbf{f}\, d\xi,$$

and $\mathbf{u}_j$, $\mathbf{u}_{j-1}$ represent the value of function s at x_j, x_{j-1}, respectively.

Hence

$$\mathbf{W}_{j-0.5} = A_j(\mathbf{u}_j - \mathbf{u}_{j-1}) - A_j\mathbf{B}_j.$$

Similarly determining the flux term $\mathbf{W}_{j+0.5}$ by integrating equation (6) on the intervals $(x_{j+0.5}, x)$, $x \in (x_j, x_{j+1})$ and on (x_j, x_{j+1}) one obtains

$$\mathbf{W}_{j+0.5} = A_{j+1}(\mathbf{u}_{j+1} - \mathbf{u}_j) - A_{j+1}\mathbf{B}_{j+1},$$

where $\mathbf{u}_{j+1}$ is the value of function s at x_{j+1}.

To derive a 3–point exact difference equation in the case of piece–wise constant function ν associated with the central grid point x_j we want to apply equation (14) in the form

$$h\Lambda\mathbf{u}_j \equiv A_{j+1}(\mathbf{u}_{j+1} - \mathbf{u}_j) - A_j(\mathbf{u}_j - \mathbf{u}_{j-1}) = \mathbf{R}_j, \quad j = \overline{1, N-1}, \tag{15}$$

where

$$\mathbf{R}_j = A_{j+1}\mathbf{B}_{j+1} - A_j\mathbf{B}_j + \int_{x_{j-0.5}}^{x_{j+0.5}} \mathbf{f}\, dx.$$

If the values of parameters ν, $\mathbf{f}$ on the intervals (x_{j-1}, x_j), (x_j, x_{j+1}) are equal to $\nu_{j-0.5}, \mathbf{f}_{j-0.5}$ and $\nu_{j+0.5}, \mathbf{f}_{j+0.5}$ respectively, then

$$\begin{aligned} &A_j = \nu_{j-0.5}/h, A_{j+1} = \nu_{j+0.5}/h, \\ &\mathbf{R}_j = h(\mathbf{f}_{j+0.5} + \mathbf{f}_{j-0.5})/2. \end{aligned} \tag{16}$$

For the boundary conditions of the second order (13) we apply the integral form of the conservation law to the half interval $(x_0, x_{0.5})$ in the form

$$\mathbf{W}_{0.5} - \mathbf{W}_0 = \int_{x_0}^{x_{0.5}} \mathbf{f}\, dx,$$

where $\mathbf{W}_0 = \mathbf{W}|_{x=x_0} = -Q_0$, $\mathbf{W}_{0.5} = \mathbf{W}|_{x=x_{0.5}}$.

Like above, we integrate this equation from $x_{0.5}$ to $x \in (x_0, x_1)$ and from x_0 to x_1. Now we can easily derive the following 2–point exact difference equation associated with grid point $x_0 = 0$

$$h/2\Lambda \mathbf{u}_0 \equiv A_1(\mathbf{u}_1 - \mathbf{u}_0) + Q_0 = \mathbf{R}_0, \tag{17}$$

where $A_1 = \nu_{0.5}/h$, $\mathbf{R}_0 = h/2\mathbf{f}_{0.5}$, and $\mathbf{u}_0, \mathbf{u}_1$ represent the value of function s at x_0, x_1, respectively.

Similarly in advance, we integrate the equation (6) from $x_{N-0.5}$ to x_N in order to approximate the boundary condition (13) at $x = L$. We obtain 2–point exact difference equation associated with grid point $x_N = L$ in the following form

$$h/2\Lambda \mathbf{u}_N \equiv -A_N(\mathbf{u}_N - \mathbf{u}_{N-1}) = \mathbf{R}_N, \tag{18}$$

where $A_N = \nu_{N-0.5}/h$, $\mathbf{R}_N = h/2\mathbf{f}_{N-0.5}$ and $\mathbf{u}_{N-1}$, $\mathbf{u}_N$ represent the values of function s at x_{N-1}, x_N, respectively.

Therefore, the difference scheme (15,17,18) is monotone ($A_j > 0, A_{j+1} > 0$) and has a unique solution. The finite–difference equations can be solved by factorisation method for tri–diagonal matrix (Thomas algorithm [3]).

For the differential equation (1) the finite–dfiference equations (15,17,18) are in the same form, where $\mathbf{u}_j = p_j$, $\nu = k$. For the differential equation (7) these equations are also in the same form, where $\mathbf{u}_j = \phi_j$, $\nu = 1$.

The nonlinear differential equations (1,6), where the functions ν, k are depending on s, can be approximated in the form (15), where $\nu_{j\pm0.5}$, $k_{j\pm0.5}$ denote the discrete approximation of ν, k on the corresponding intervals, $(x_{j-1}, x_j), (x_j, x_{j+1})$, for example $\nu_{j\pm0.5} = (\nu(s_j) + \nu(s_{j\pm1}))/2$.

The discretization (truncation) error of the difference equations is locally of the second order in space.

By integrating differential equations (1,6,7) in the interval $(0, L)$ we get

$$\frac{d}{dt}\Omega(t) = \mathbf{W}_N - \mathbf{W}_0, \tag{19}$$

where $\Omega(t) = \int_0^L s(x,t)\,dx$ is the common water contents in the materials. In the case of $\mathbf{W}_N = \mathbf{W}_0 = 0$ we have $\Omega(t) = const$. Therefore, from the boundary conditions (13) it follows that

1) $\Omega(t) = Q_0 t$ for $t < t_0$ and $\Omega(t) = \Omega(t_0) = Q_0 t_0 = const$ for $t \geq t_0$ if $s^0(x) = 0$,

2) $\Omega(t) = \Omega(0) = 0.5$ if $s^0(x) = (1 - \cos(2\pi x))/2, Q_0 = 0$.

2. Let us consider the nonlinear time-depending equations (1,6,7) $\mathbf{f} = \partial s/\partial t$ and suppose the parameters ν, k to be the functions depending on x, t, s. The corresponding discretized version for difference scheme is [4]

$$(s_j^{n+1} - s_j^n)/\tau = \sigma\Lambda \mathbf{u}_j^{n+1} + (1 - \sigma)\Lambda \mathbf{u}_j^n, \tag{20}$$

where u_j^n, s_j^n denotes the discrete approximation of u, s at the grid point x_j and at the time moment $t = t_n = n\tau$, $n = 0, 1, \ldots$, $\sigma \in (0, 1)$ is the weight parameter of the schemes, Λu_j are the difference expressions of (15), u is equal to p, s, ϕ respectively for equations (1,6) and (7), $j = \overline{0, N}$. If $\sigma \neq 0.5$ then truncation error for (20) is locally of the first order in time. In case of $\sigma = 0.5$ it is of the second order. Investigating the stability of difference scheme, we apply the maximum principle [4], requiring the corresponding coefficients of difference equations (20) to be positive, that is in the form:

$$\tau \leq h((1-\sigma)K)^{-1}, \tag{21}$$

where $K = \max(A_j + A_{j+1})$, $j = \overline{1, N-1}$.

The stability condition (21) has the following form in the case of constant coefficients

$$\nu k_0(1-\sigma) \leq 1, \tag{22}$$

where $k_0 = 2\tau/h^2$. The stability conditions (21,22) has the form $k_0 \leq 1/\nu_{max}$ or $\tau \leq h^2/(2\nu_{max})$ in the case of $\sigma = 0$ (the explicit scheme), where $\nu_{max} = \nu(1) = \tilde{k}\alpha p_{00}$.

The system of nonlinear algebraic equations (20) can be solved by Newton or Gauss-Seidel iterations methods in every time level. It is easier to realize explicit schemes with $\sigma = 0$ than the implicit ones.

3. The results of numerical experiments were obtained for equations (1,6,7), where $L = 1$, $\tilde{s} = 0.5$, $\tilde{k} = 0.0017$, $\tilde{p} = 0.1$, $p_l = 0.6$, $\alpha = 10$, $\beta = 3.5$, $N = 20$, $\tau = 0.1$, $t = 10; 100$, $s^0(x) = \sin(\pi x)$, $s^0(x) = 0$, $s^c(x) = (1 - \cos(2\pi x))/2$, $x \in [0, 1]$, $Q_0 = 0.001; 0.00005; 0$.

The values of functions $k, P_2, P_3, \nu_2, \nu_3, \phi_2, \phi_3$ are shown in the following table:

s	$k.M$	p_2	p_3	$\nu_2.M$	$\nu_3.M$	$\phi_2.M$	$\phi_3.M$
0.1	0.0537	0.4202	0.3557	0.0860	0.1030	0.0020	0.0025
0.2	0.6080	0.2802	0.2077	0.7300	0.6560	0.0364	0.0370
0.3	2.510	0.1802	0.1319	2.010	1.200	0.1770	0.1340
0.4	6.880	0.1202	0.1040	2.760	0.8210	0.4230	0.2450
0.5	15.00	0.1000	0.1000	0.0294	0.0294	0.6090	2.530
0.6	28.40	0.0995	0.0995	0.2870	0.2870	0.6210	2.540
0.7	48.80	0.0973	0.0973	1.970	1.970	0.7100	2.630
0.8	77.90	0.0893	0.0893	10.50	10.50	1.230	3.145
0.9	118.0	0.0652	0.0652	45.60	45.60	3.650	5.560
1.0	170.0	0.0	0.0	170.0	170.0	13.20	15.10

where

$$M = 10^{-5},\ \nu_2;\ \phi;\ p = \nu;\ \phi;\ p|_{P=P_3},\ \nu_3;\ \phi;\ p = \nu;\ \phi;\ p|_{P=P_3}.$$

The numerical experiment shows, that the explicit scheme is more stable for equation (6) than for equations (1,7). The numerical results for equations (1,6) are a little smaller dependent of the form polymomial $P(x)$.

3.1. The numerical values of functions s for (12), $s^0(x) = \sin(\pi x)$ can be seen in the following tables (in the first table $t = 10$, while in the second one $t = 100$):

x	ν_2	p_2	ϕ_2	ν_3	p_3	ϕ_3
0.05	0.1629	0.1651	0.1631	0.1600	0.1625	0.1612
0.10	0.3132	0.3158	0.3157	0.3086	0.3083	0.3281
0.15	0.4485	0.4415	0.4430	0.4511	0.4475	0.5002
0.20	0.5905	0.5887	0.5907	0.5952	0.5925	0.5232
0.25	0.7443	0.7392	0.7400	0.7444	0.7392	0.7398
0.30	0.8499	0.8536	0.8531	0.8499	0.8536	0.8530
0.35	0.9010	0.9033	0.9030	0.9010	0.9033	0.9030
0.40	0.9277	0.9291	0.9289	0.9277	0.9291	0.9289
0.45	0.9412	0.9422	0.9421	0.9412	0.9422	0.9421
0.50	0.9454	0.9462	0.9461	0.9454	0.9462	0.9461

x	ν_2	p_2	ϕ_2	ν_3	p_3	ϕ_3
0.05	0.2140	0.2199	0.2224	0.1857	0.1920	0.2799
0.10	0.3381	0.3352	0.3348	0.3075	0.3073	0.4999
0.15	0.4697	0.4307	0.4205	0.4938	0.4291	0.4997
0.20	0.6995	0.6914	0.7235	0.7279	0.7454	0.5004
0.25	0.7688	0.7992	0.8002	0.7961	0.8048	0.7539
0.30	0.8247	0.8318	0.8314	0.8277	0.8334	0.8153
0.35	0.8445	0.8497	0.8491	0.8460	0.8503	0.8406
0.40	0.8560	0.8602	0.8596	0.8560	0.8605	0.8542
0.45	0.8623	0.8659	0.8654	0.8629	0.8660	0.8613
0.50	0.8642	0.8677	0.8672	0.8648	0.8678	0.8635

Here ν_2, p_2, ϕ_2 and ν_3, p_3, ϕ_3 denotes the calculations of functions ν, p, ϕ corresponding to $P(s) = P_2(s)$ (3,4,8,9) and $P(s) = P_3(s)$ (3,5,10,11).

The results are symmetrical with respect to the point $x = 0.5$, therefore, it is sufficient to analyse the left side of the interval (0,0.5) only. The results of s are too diferent in the interval (0.1,0.3), what is in a correspondence with values of functions P_2 and P_3 in the interval (0.3,0.4).

3.2. As the test of the linear differential equation (6) with $\nu = const > 0$ we consider the solutions in the following form:

1) $s(x,t) = \exp(-\pi^2\nu t)\sin(\pi x)$ for (12) and $s^0(x) = \sin(\pi x)$,

2) $s(x,t) = (1 - \cos(2\pi x)\exp(-4\pi^2\nu t))/2$ for (13), $\Omega(t) = 0.5$, $Q_0 = 0$ and $s^0(x) = (1 - \cos(2\pi x))/2$.

The corresponding solutions of the discrete equations (15,20) with $\sigma = 0$ also can be written in the following analytic form:

1) $s_j^n = (1 - \tau\nu\lambda_1)^n \sin(\pi x_j)$,

2) $s_j^n = (1 - (1 - \tau\nu\lambda_2)^n \cos(2\pi x_j))/2$,
where $\lambda_1 = 4h^{-2}\sin^2(\pi h/2)$, $\lambda_2 = 4h^{-2}\sin^2(\pi h)$, $n = t_n/\tau$, $x_j = jh$.

By simulating the solutions of the test-problem at the moment of time $t = 100 (N = 20, h = 0.05, n = 1000, \tau = 0.1, \nu = 10^{-5})$ the following results have been obtained:

		(12)		(13)	
j	x	$s(x)$	s_j^n	$s(x)$	s_j^n
1	0.05	0.15490	0.15490	0.04273	0.04288
3	0.15	0.44954	0.44953	0.21739	0.21748
5	0.25	0.70018	0.70016	0.50000	0.50000
7	0.35	0.88227	0.88226	0.78261	0.78252
9	0.45	0.97801	0.97799	0.95727	0.95712
10	0.50	0.99020	0.99018	0.98080	0.98065

3.3. The numerical values of functions s for (6,13), $s^0(x) = \frac{1-\cos(2\pi x)}{2}$, $Q_0 = 0$ can be seen in the following table:

	$P_2(s)$			$P_3(s)$		
x	$t = 100$	$t = 500$	$t = 1000$	$t = 100$	$t = 500$	$t = 1000$
0.05	0.0264	0.1364	0.2256	0.0304	0.1545	0.1640
0.10	0.1256	0.2695	0.3048	0.1373	0.2460	0.2554
0.15	0.2598	0.3471	0.3590	0.2504	0.3150	0.3276
0.20	0.3671	0.4141	0.4081	0.3551	0.3943	0.4282
0.25	0.5934	0.5377	0.4738	0.5981	0.5667	0.6085
0.30	0.7541	0.6873	0.6110	0.7548	0.6931	0.6786
0.35	0.8008	0.7271	0.6819	0.8010	0.7295	0.7046
0.40	0.8222	0.7450	0.7048	0.8223	0.7465	0.7179
0.45	0.8326	0.7537	0.7147	0.8327	0.7548	0.7246
0.50	0.8358	0.7564	0.7176	0.8358	0.7574	0.7267

The integrals $\Omega(100)$, $\Omega(500)$, $\Omega(1000)$ calculated with a help of the Simpson rule have the following values:

1) 0.5009; 0.4999; 0.4865 for $P_2(s)$,

2) 0.5006; 0.4999; 0.5001 for $P_3(s)$.

In the case of Dirichlet boundary condition (12) there integrals are also constant with the following values for $P_3(x)$: 0.4989; 0.5067; 0.4999.

3.4. The numerical values of functions s for (6,13), $s^0(x) = \frac{1-\cos(2\pi x)}{2}$, $Q_0 = 0.001; 0.0005$, $P = P_3$ can be seen in the following table.

The values of the function s are equal zero if $s > 0.5$. The integrals $\Omega(t_0)$, $\Omega(1000)$ have the following values:

1) 0.1388; 0.1380 for $Q_0 = 0.001$,

2) 0.2429; 0.2397 for $Q_0 = 0.0005$.

x	$Q_0 = 0.001$ $t_0 = 138$	$t = 1000$	$Q_0 = 0.0005$ $t = 480$	$t = 1000$
0.00	0.9850	0.6560	0.9850	0.7628
0.05	0.9400	0.6398	0.9652	0.7603
0.10	0.8447	0.5211	0.9393	0.7521
0.15	0.4626	0.3938	0.9015	0.7354
0.20	0.0202	0.3287	0.8320	0.6997
0.25	0.0000	0.2700	0.6198	0.5749
0.30	0.0000	0.2008	0.0486	0.3854
0.35	0.0000	0.0772	0.0005	0.2890
0.40	0.0000	0.0007	0.0000	0.1881
0.45	0.0000	0.0000	0.0000	0.0327
0.50	0.0000	0.0000	0.0000	0.0000

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PAR ŠĶIDRUMA TRANSPORTA SKAITLISKO MODELĒŠANU PORAINĀ MATERIĀLĀ

HARIJS KALIS un JĀNIS RUDZĪTIS

Skaitliskā metode šķidruma pārnesei porainā vidē pamatojas uz atklātas shēmas lietošanu.

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DRIFT – DIFFUSION APPROXIMATION FOR NUMERICAL MODELLING OF TRANSIENT CHARGE TRANSFER IN 2–D VERTICAL GaAs FIELD–EFFECT PHOTOTRANSISTOR

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ABSTRACT

Drift – diffusion approximation for GaAs semiconductor is proposed. Transient characteristics, potential energy distributions in a vertical buried–gate GaAs field–effect phototransistor have been calculated.

1. INTRODUCTION

Numerical simulation of steady–state processes in vertical silicon field–effect transistors was undertaken by a number of authors [1, 2]. Transient photoprocesses were usually investigated in terms of asymptotic models [3], neglecting nonlinear effects and a two–dimensional character of carrier and field distributions in the structure. Both of these factors can be conveniently accounted for in terms of a drift–diffusion approach. In a two–dimensional case, effective

numerical methods of solving non-steady-state problems have been developed using this approach. In particular, methods involving semi-implicit differential schemes proved very efficient when applied to bipolar and unipolar silicon structures [4–6].

Applicability of the diffusion–drift approximation to GaAs structures has been demonstrated in [7] where experimental values of the mobility of electrons and holes, bandgap narrowing and effective free carriers lifetimes were used to describe carrier transport. Recombination was accounted for in terms of Shockley–Hall–Read and Auger models. Steady-state characteristics of bipolar GaAs structures were obtained and it was found that, for the transistors considered, experimental and calculated values of the gain coefficient were fairly close. Therefore, in the calculations below we assume, that Shockley–Hall–Reed recombination is dominant, except in heavily doped regions, where Auger recombination prevails. That is, radiative recombination will be neglected.

In the present paper, we describe results of numerical simulation of the photocarrier transport in vertical GaAs field-effect transistors with the source and the gate biased.

2. IDENTIFICATION OF THE PROBLEM

Calculations of the non-steady-state photocarrier transport are carried out in a diffusion-drift approximation using a system of equations comprising continuity equations for electrons and holes and Poisson's equation:

$$\operatorname{div} \vec{J}_n = q\left(R - G + \frac{\partial n}{\partial t}\right), \tag{1}$$

$$\operatorname{div} \vec{J}_p = -q\left(R - G + \frac{\partial p}{\partial t}\right), \tag{2}$$

$$\nabla(k\nabla\varphi) = -\frac{q}{\kappa_0}\left(p - n + N_d - N_a\right), \tag{3}$$

where n and p are electron and hole concentrations, respectively; φ is the electrostatic potential; κ is the dielectric constant of a semiconductor; N_d and N_a are concentrations of ionized donor and acceptor impurities, respectively. Other symbols are either commonly used or will be defined later.

The electron, $\vec{J}_n$, and hole, $\vec{J}_n$, current densities will be expressed [6, 8] as

$$\vec{J}_n = q\mu_n\left(-n\left(\nabla\varphi + \nabla\varphi_g^c\right) + \frac{k_B T}{q}\nabla n\right), \tag{4}$$

$$\vec{J}_p = -q\mu_p\left(-p\left(\nabla\varphi + \nabla\varphi_g^h\right) + \frac{k_B T}{q}\nabla p\right), \tag{5}$$

where μ_n, μ_p are electron and hole mobilities, respectively; φ_g^e, φ_g^h are the potentials of bandgap narrowing due to doping with donors and acceptors, respectively; T is the absolute temperature.

The net current density, including the displacement current $\vec{J}_d$, is:

$$\vec{J}_d = -\kappa\kappa_0 \frac{\partial}{\partial t} \nabla\varphi, \tag{6}$$

$$\vec{J} = \vec{J}_n + \vec{J}_p + \vec{J}_d. \tag{7}$$

The mobilities μ_n and μ_p are assumed field–dependent:

$$\mu_{n(p)} = \mu^0_{n(p)} F_{n(p)}(E), \tag{8}$$

where μ_n^0 and μ_p^0 are the mobilities in a weak electrostatic field E, and $F_{n(p)}(E)$ is a field–dependent term.

In formula (8) and elsewhere below, indices n and p refer to electrons and holes, respectively. The mobilities in a weak electrostatic field μ_n^0 and μ_p^0, are calculated in terms of the momentum relaxation time using Boltzmann's distribution function:

$$\mu^0_{n(p)} = \frac{4q}{3\sqrt{\pi} m^*_{n(p)}} \int_0^\infty \tau_{n(p)}(\varepsilon) \varepsilon^{3/2} \exp(-\varepsilon)\, d\varepsilon, \tag{9}$$

where τ_n, τ_p are momentum relaxation times for electrons and holes, respectively; m_n^*, m_p^* are effective masses of electrons and holes, respectively.

The momentum relaxation time $\tau_{n(p)}$ is defined as a sum of reciprocal momentum relaxation times for individual scattering mechanisms involved:

$$\tau_{n(p)}(\varepsilon) = \frac{1}{\sum_i \nu^i_{n(p)}(\varepsilon)}, \quad \nu^i_{n(p)}(\varepsilon) = \frac{1}{\tau^i_{n(p)}(\varepsilon)}, \tag{10}$$

where $\tau^i_{n(p)}$ are momentum relaxation times for scattering on phonons and ionized impurities and $\nu^i_{n(p)}$ are respective collision frequencies. Summation in formula (10) is to be performed over all types i of scattering mechanisms involved. In the present instance these include scattering by the deformation potential, nonpolar and polar optical phonons, piezoelectric scattering, and scattering on ionized impurities. Corresponding expressions for collision frequencies are well known [9–11] and have been calculated taking into account the correlation effects in scattering on the doping atom centers [12].

The collision frequences are written taking into account electron – electron and hole – hole scattering with the correction coefficients γ. These coefficients are derived in the form [13, 14] and are taken equal to: $\gamma^d = \gamma^{no} = \gamma^{pe} = 0.88$, $\gamma^{po} = 1$, $\gamma_n^N = 0.632$, $\gamma_p^N = 0.295$.

Values of the effective masses of electrons and holes used are $m_n^* = 0.067m_0$, $m_p^* = 0.45m_0$, where m_0 is the free electron mass.

Some details of the mobility calculation are to be mentioned. An expression for the screening length λ was taken from [15]:

$$\lambda = \left(\frac{q^2}{\kappa\kappa_0 k_B T} \left(n + p + \frac{N_d + N_a}{T_i/T} \right) \right)^{-1/2} \tag{11}$$

The last term in (11), which is a ratio of the impurity concentration to the normalized ion screening temperature T_i, gives finite values of the screening length even for small values of n and p encountered in compensated semiconductors. Hence, formula (9) can be used in mobility calculations for the compensated regions of a semiconductor structure as well. In the present work, the same as in [15], the considered range of T_i values is $7000\text{K} < T_i < 9000\text{K}$. After substituting formula (10) into (9), the electron and hole mobilities can be calculated using the method of Gauss, with integration intervals being chosen such as to ensure smoothness of the integrand function. In a wide range of dopant concentrations from 10^{15}cm^{-3} to 10^{19}cm^{-3}, the calculated mobility values did not differ by more that 25% from experimental data [7, 16].

Calculations of the mobility in semiconductor structures will be performed in the following way. In the temperature range from 50 to 400K and the range of dopant and free carrier concentrations accessible with the state-of-the-art technology, the mobilities μ_n^0 and μ_p^0 are calculated with formula (9). These data are presented in the form of tables. In calculations of factors for density values of conduction currents at specified values of impurity and free carrier concentrations and temperature, mobility values in a weak electrostatic field are obtained by interpolation of μ_n^0 and μ_p^0 values from the tables.

Final mobility values, i.e. with account taken of the field dependence, are calculated with formula (8). The field-dependence function $F_{n(p)}(E)$, taken from [17], is rewritten in the form

$$F_{n(p)}(E) = \frac{1 + \frac{V_s}{\mu^0_{n(p)} E} \left(\frac{E}{E_N} \right)^4}{1 + \left(\frac{E}{E_N} \right)^4} \tag{12}$$

where the saturated velocity $V_S = 8 \cdot 10^6\text{cm/s}$ and the critical field $E_N = 4 \cdot 10^3\text{V/cm}$ are assumed.

Potentials φ_g^e and φ_g^h for the bandgap narrowing due to high doping levels are written in the form proposed in [18]:

$$\varphi_g^{e(h)} = C_g(\ln(N_{d(a)}/N_0) + ((\ln(N_{d(a)}/N_0))^2 + 1/2)^{1/2}), \tag{13}$$

where $N_0 = 10^{17}\text{cm}^{-3}$, $C_g = 9 \cdot 10^{-3}\text{V}$. This expression, though derived for silicon, gives values in fair agreement with experimental data (e.g., see a compilation in [19]).

The recombination term in formulae (1) and (2) is written in the form

$$R = (np - n_0 p_0)\left(\frac{1}{\tau_{n_0}(p + \sqrt{n_0 p_0}) + \tau_{p_0}(n + \sqrt{n_0 p_0})} + C_n n + C_p p\right), \quad (14)$$

where n_0 and p_0 are equilibrium concentrations of electrons and holes. Formula (14) is taking into account the recombination via impurity centres according to Shockley–Hall–Reed statistics and band-to-band Auger recombination. Values of the effective lifetime of charge carriers, $\tau_{n_0} = 4 \cdot 10^{-7}$s and $\tau_{p_0} = 6 \cdot 10^{-7}$s, are taken from [20] and Auger recombination coefficients, $C_n = 2 \cdot 10^{-31}\text{cm}^6\text{s}^{-1}$ and $C_p = 10^{-29}\text{cm}^6\text{s}^{-1}$, from [21].

It is further assumed in the calculations that the structure is illuminated with monochromatic light from the source side. In this case, the generation term may be written in the form

$$G = F_0 \alpha \exp(-\alpha x), \quad (15)$$

where F_0 is the incident light intensity, α is the absorption coefficient, taken equal to $3 \cdot 10^4\text{cm}^{-1}$, and x is a spatial coordinate.

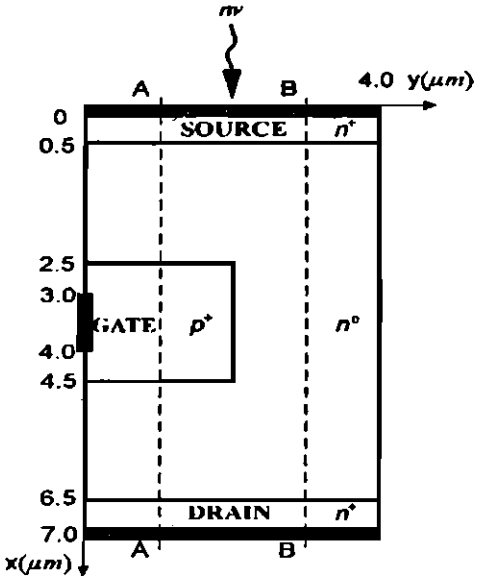

Figure 1. Schematic drawing of the structure.

The boundary conditions are defined in a way usual for the case of ohmic contacts and the absence of leakage currents. Accordingly, equilibrium values of the chemical potential f_0 and the free carrier concentration are assumed.

The expressions for the potential and the electron and hole concentrations are: $\varphi_k = V_k + (f_0)_k$, $n_k = (n_0)_k$, $p_k = (p_0)_k$, where V_k is the voltage drop between a k-th contact and the drain contact. Conditions at the free surfaces are: $(\nabla\varphi, \vec{\nu}) = 0$, $(\vec{J}_n, \vec{\nu}) = 0$, $(\vec{J}_p, \vec{\nu}) = 0$, where $\vec{\nu}$ is a unit vector of the external normal.

The system of nonlinear differential equations (1)-(3) is solved using an absolutely stable, monotonic and conservative differential scheme proposed in [6]. The computation time of one non-steady-state characteristic for a spatial network consisting of 47 × 27 points is from 60 to 300 min (IBM PC/AT 386/387, clock rate 20 MHz).

3. CALCULATION RESULTS

The structure considered is schematized in Fig.1, indicating overall dimensions, gate depth and dimensions of the contact regions.

The gate region is located in the depth of the structure, at a distance of 2μm from both the source and the drain contacts. The channel and the gate have the same width of 2μm. The concentration of the donor impurities at n^+ was given $N_d = 1.01\ \ 10^{18}\text{cm}^{-3}$, the concentration of acceptor - $N_a = 10^{16}\text{cm}^{-3}$ The concentrations of the donor and acceptor impurities at n^0 - layer were given $N_d = 1.05 \cdot 10^{16}\text{cm}^{-3}$, $N_a = 10^{16}\text{cm}^{-3}$ and at p^+ layer - $N_d = 10^{16}\text{cm}^{-3}$, $N_a = 1.01\ \ 10^{18}\text{cm}^{-3}$. Regarding layer thicknesses, length and width of the channel and impurity concentration, the structure in Fig.1 corresponds to vertical field-effect transistors whose fabrication and electrophysical properties have been reported in [22,23].

The photoprocesses in the structure under consideration were calculated in a range of light intensities and gate voltages with the source voltage fixed at $V_{DS} = 4\text{V}$ The gate voltage V_{DC} is fixed at 3.4V.

It is assumed that prior to an initial moment of time $t = 0$ the voltage $V_{DS} = 4\text{V}$ has been applied between the drain and the source and that in the currents through the structure corresponding to this voltage and the bias voltage V_{DS} have reached their steady-state values.

Calculations of the non-steady-state processes were carried out for a case where a light source of intensity F_0 was turned on at the moment of time $t = 0$, the structure being illuminated from the source side. Calculations were performed with intensity value $F_0 = 10^{22}\text{cm}^{-2}\text{s}^{-1}$

Consider now in more detail the evolution of the photocarrier transport in the structure for the case of high intensity of the light source $F_0 = 10^{22}\text{cm}^{-2}\text{s}^{-1}$ and value of the gate voltage $V_{DC} = 3.4\text{V}$, corresponding high potential barriers for electrons in the gate region.

The calculations for in the case of a high potential barrier for electrons in the gate region have revealed the following features of the charge transport in the structure. In the moments immediately following the turn-on of illumination, excess electrons and holes will be accumulating in the upper part of the structure. Then, under the effect of the high electrostatic field directed

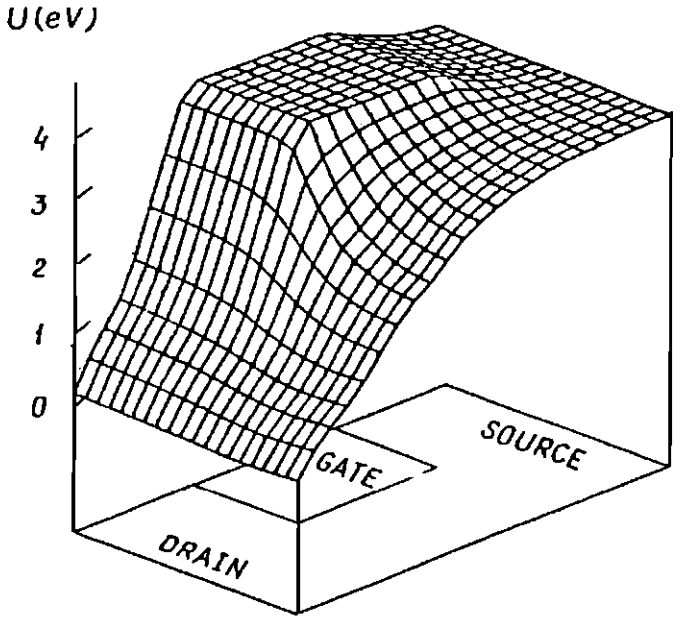

Figure 2. Distribution in the structure of the potential energy profile in the dark conduction.

along the source region of the structure, separation of the electrons and holes takes place. The holes are stored in regions of high potential energy in the above-gate part of the structure, the electrons in the upper part of the channel where the potential energy is lower.

High concentration of the photogenerated holes in the upper part of the gate causes changes in the space-charge density and lowering of the potential energy in this region which causes a flow of holes away from this region and lowering of the hole concentrations in the upper part of the gate.

In the channel, the excess electrons are driven by the electrostatic field into the drain region. The excess electrons accumulated in the lower portion of the channel significantly affect the space charge density, causing a considerable increase of the potential energy in this region, see figs.2 and 3, which favours accumulation of photogenerated holes in the region of the channel adjacent the gate. As indicated by the calculations, the hole contribution to the current density in the left portion of the channel is twice that in the above-gate part of the structure, i.e., the holes entering the gate region come predominantly from the channel side. The high potential energy in the lower portion of the channel is responsible for yet another effect, namely, an emergence in the gate of an electrostatic field gradient and associated enhancement of the hole drift, leading to large gate currents, comparable to the source current.

Higher potential energy in the lower portion of the channel also causes a

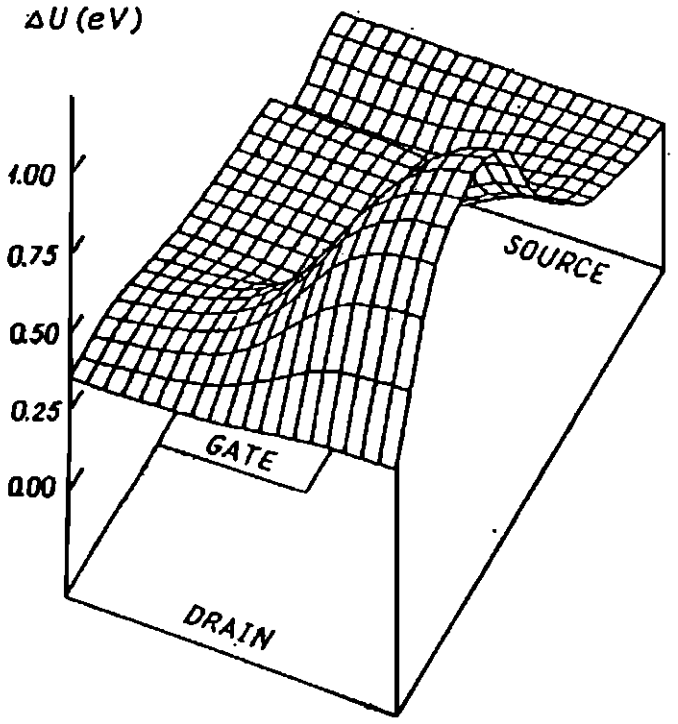

Figure 3. Distribution in the structure of excess values of the potential energy under illumination.

reduction in the concentration of excess carriers in the channel and so lowers the source current. Reduced drain current means lower current through the structure and adversely affects the gain of a transistor.

To conclude, we note the following characteristic features of the carrier transport in the vertical field-effect phototransistor with biased gate. Throughout the range of voltages and light intensities investigated the structure shows response time of the order of a few nanoseconds which is consensus with experimentally observed features [22]. The diffusion-drift approximation in combination with a semi-implicit differential scheme proposed in [6] were used for numerical simulation of the transport of photogenerated carriers in the structures investigated in [22]. Calculated values of the photocurrent rise-time and the gain of the structures are fairly close to those measured in [22]. Such agreement of calculated and experimental data warrants a suggestion that the diffusion-drift approximation can be applied for describing photocarrier transport in the type of structure studied in [22], which is characterised with rather extensive active regions, of about 1 μm, in the channel and the gate.

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DIFŪZIJAS DREIFA TUVINĀJUMS LĀDIŅU PĀRNESES FOTO–KINĒTISKO PROCESU SKAITLISKAI MODELĒŠANAI 2-D VERTIKĀLĀ GaAs LAUKA TRANZISTORĀ

VLADIMIRS KOROĻKOVS, REINHARDS MAHNKE, JĀNIS RIMŠĀNS, JURIJS SKRIĻŠ un TIMURS TABAROVS

Izstrādāta uzdevuma nostādne lādiņu fotokinētikas aprakstam GaAs pusvadītāju struktūrās. Veikti fotokinētisko raksturlielumu potenciālās enerģijas, koncentrāciju un fotostrāvu skaitliski aprēķini vertikālā lauka tranzistorā.

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ON MAXIMAL EXTENSIONS OF SOME EXTREMAL PROBLEMS

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ABSTRACT

The paper describes a family of linear elliptic operators for which the largest family of analogical operators, which preserve the weak closure of the set of feasible states for every fixed right hand side, coincides with the G–closure of the original family of operators.

Keywords. Elliptic systems, optimal control problems, extensions.
AMS subject classification. 49J20.

1. INTRODUCTION

The paper considers the question of maximal extensions of optimal control problems for elliptic systems. The formal setting of the problem is as follows.

Given a family of elliptic operators

$$\mathcal{L}: [H_0^1(\Omega)]^m \to [H^{-1}(\Omega)]^m, \quad \Omega \subset \mathbb{R}^n, \quad n \geq 2,$$
$$\mathcal{L}\overline{u} := \operatorname{div} A(\cdot)\nabla\overline{u}, \quad A \in \mathcal{A},$$

where $\mathcal{A}$ is a given set of measurable uniformly positive definite and uniformly bounded symmetric $(n \times n)(m \times m)$-matrices. And we seek for the largest set $\tilde{\mathcal{A}}$ of matrices of the same kind such that for every weakly continuous functional $\mathcal{I}: [H_0^1(\Omega)]^m \to \mathbb{R}$ and for every fixed $\overline{f} \in [H^{-1}(\Omega)]^m$ there is

$$\inf\{\mathcal{I}(\overline{u}) \mid \overline{u} \in [H_0^1(\Omega)]^m, \quad \operatorname{div} A\nabla\overline{u} = \overline{f} \text{ in } \Omega \text{ for some } A \in \tilde{\mathcal{A}}\}$$
$$\inf\{\mathcal{I}(\overline{u}) \mid \overline{u} \in [H_0^1(\Omega)]^m, \quad \operatorname{div} A\nabla\overline{u} = \overline{f} \text{ in } \Omega \text{ for some } A \in \mathcal{A}\}. \tag{1}$$

[1]This work was partly supported by Latvian Council of Science under grant 96. 0248

It is clear that the passage from $\mathcal{A}$ to the G-closure $G\mathcal{A}$ (see, for instance, Zhikov et al. [2]) of the set $\mathcal{A}$ is an admissible extension of $\mathcal{A}$. It is known (see for instance, Raitums [1]) that for the case of a single equation, i. e. $m = 1$, there exist larger than $G\mathcal{A}$ sets $\tilde{\mathcal{A}}$ for which the relationship (1) holds.

In this paper, we give a simple example for $m = n = 2$ which shows that for the case of systems with $m = n$ the set $G\mathcal{A}$ is, in general, the maximal set, i. e. if for a set $\tilde{\mathcal{A}}$ the relationship (1) holds then $\tilde{\mathcal{A}} \subset G\mathcal{A}$.

This result shows that for the optimal material layout problems described by systems of elliptic equations one can not expect suitable extensions larger than the extension via G-closure.

2. PRELIMINARIES AND THE STATEMENT OF THE PROBLEM

Let $n = 2$ and $\Omega \subset \mathbb{R}^2$ be a bounded domain with a uniformly Lipschitz boundary $\partial\Omega$. For the given positive constants $0 < \nu < \mu$ denote by $\mathcal{K}(\nu, \mu)$ the set of all $(2 \times 2)(2 \times 2)$-matrices $A \in [L_2(\Omega)]^{(2\times2)(2\times2)}$ such that

(i) $A = (a_{ij}^{pq})$, $i, j, p, q = 1, 2$;

(ii) $a_{ij}^{pq} = a_{ji}^{qp}$, $i, j, p, q = 1, 2$;

(iii) $|a_{ij}^{pq}(x)| \le \mu$ a. e. $x \in \Omega$, $i, j, p, q = 1, 2$;

(iv) $\sum_{i,j,p,q=1}^{2} a_{ij}^{pq}(x)\xi_p^i\xi_q^j \ge \nu \sum_{i,p=1}^{2} \xi_p^i\xi_p^i \ \forall \xi_p^i \in \mathbb{R}$, $i, p = 1,2$, a. e. $x \in \Omega$;

(v) $\int_\Omega (a_{ij}^{12} - a_{ij}^{21})\, dx = 0$, $i \ne j$.

Matrices $A \in \mathcal{K}(\nu, \mu)$ we shall consider as block matrices

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad A_{ij} = (a_{ij}^{pq}), \quad p, q = 1, 2, \quad i, j = 1, 2,$$

and, by definition, for $\overline{u} = (u_1, u_2)$

$$A\nabla\overline{u} = \begin{pmatrix} A_{11}\nabla u_1 + A_{12}\nabla u_2 \\ A_{21}\nabla u_1 + A_{22}\nabla u_2 \end{pmatrix},$$

$$\operatorname{div} A\nabla\overline{u} = \begin{pmatrix} \operatorname{div}(A_{11}\nabla u_1 + A_{12}\nabla u_2) \\ \operatorname{div}(A_{21}\nabla u_1 + A_{22}\nabla u_2) \end{pmatrix}$$

The unit matrix we shall denote by E, i. e.

$$E = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$$

Denote

$$H := [H_0^1(\Omega)]^2, \quad H^* = [H^{-1}(\Omega)]^2$$

Define the set $\mathcal{A}_0$ of matrices,

$$\begin{aligned} \mathcal{A}_0 := \{ A \in \mathcal{K}(\nu,\mu) \mid A(x) = \theta_1(x)k_1E + \cdots + \theta_N(x)k_NE, \\ \theta_i \in L_2(\Omega), \quad \theta_i(x) = 0 \text{ or } 1 \text{ a. e. } x \in \Omega, \quad i = 1,\ldots,N, \\ \theta_1(x) + \cdots + \theta_N(x) = 1 \text{ a. e. } x \in \Omega \}, \end{aligned} \tag{2}$$

where $0 < \alpha = k_1 < k_2 < \quad < k_N = \beta$ are given positive constants such that $\nu < \alpha < \beta < \mu$.

In the context of the optimal material layout problems matrices k_iE correspond to physical properties of given N materials, but the functions θ_i are characteristic functions of sets occupied by ith material respectively.

For the set $\mathcal{A}_0$ its G–closure is well known and

$$\begin{aligned} G\mathcal{A}_0 = \{ A \in \mathcal{K}(\nu,\mu) \mid A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \ a \in [L_2(\Omega)]^{2\times 2}, \ a \text{ symmetric}, \\ \lambda_1(a)(x) \le \lambda_2(a)(x), \\ \lambda_1(a)(x)\lambda_2(a)(x) - \lambda_1(a)(x)(\alpha+\beta) + \alpha\beta \le 0 \text{ a. e. } x \in \Omega \}. \end{aligned} \tag{3}$$

Here and what follows for a symmetric matrix a by $\lambda_1(a)(x)$ and $\lambda_2(a)(x)$ we denote the eigenvalues of the matrix $a(x)$ arranged in the order of increasing.

Let us denote, for the given set $\mathcal{A} \in \mathcal{K}(\nu,\mu)$ and the given $\overline{f} \in H^*$, by $Z(\mathcal{A},\overline{f})$ the set $Z(\mathcal{A},\overline{f}) = \{ \overline{u} \in H \mid \operatorname{div} A\nabla\overline{u} = \overline{f} \text{ in } \Omega \text{ for some } A \in \mathcal{A} \}$, i. e. $Z(\mathcal{A},\overline{f})$ is the set of all feasible states for the equations

$$\operatorname{div} A\nabla\overline{u} = \overline{f} \text{ in } \Omega, \quad \overline{u} \in H,$$

with $A \in \mathcal{A}$.

PROPOSITION 1. *Let the set $\mathcal{A}_0$ is defined by* (1). *Then for every set $\mathcal{A} \subset \mathcal{K}(\nu',\mu')$ with $0 < \nu' \le \nu < \mu \le \mu'$ such that*

$$\begin{aligned} &\inf \{ \mathcal{I}(\overline{u}) \mid \overline{u} \in H, \ \operatorname{div} A\nabla\overline{u} = \overline{f} \text{ in } \Omega \text{ for some } A \in \mathcal{A} \} \\ &\inf \{ \mathcal{I}(\overline{u}) \mid \overline{u} \in H, \ \operatorname{div} A\nabla\overline{u} = \overline{f} \text{ in } \Omega \text{ for some } A \in \mathcal{A}_0 \} \end{aligned} \tag{4}$$

for every fixed $\overline{f} \in H^*$ and every fixed weakly continuous functional $\mathcal{I}: H \to \mathbb{R}$ there is $\mathcal{A} \subset G\mathcal{A}_0$.

3. PROOF OF PROPOSITION 1

Since the set $Z(G\mathcal{A}_0,\overline{f})$ is closed in the weak topology of H then it sufficies to prove that from

$$Z(\mathcal{A},\overline{f}) \subset Z(G\mathcal{A}_0,f) \text{ for every } \overline{f} \in H^* \tag{5}$$

it follows that $\mathcal{A} \subset G\mathcal{A}_0$.

Let the set $\mathcal{A}$ satisfies the relationship (5) and let $B \in \mathcal{A}$. Then for every $\overline{u} \in H$ exists a matrix $A(\overline{u}) \in G\mathcal{A}_0$ such that

$$\operatorname{div} B\nabla\overline{u} = \operatorname{div} A(\overline{u})\nabla\overline{u} \text{ in } \Omega. \tag{6}$$

In the first step we shall show that the matrix B must be of the type

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad B_{12} = B_{21} = 0.$$

Indeed, from (6) and the representation (2) for $G\mathcal{A}_0$ it follows (considering $\overline{u} = (0, v)$, $\overline{u} = (v, 0)$) that

$$\operatorname{div} B_{12}\nabla v = 0 \text{ in } \Omega, \quad \operatorname{div} B_{21}\nabla v = 0 \text{ in } \Omega \quad \forall v \in H_0^1(\Omega).$$

It gives that

$$B_{12} = \begin{pmatrix} 0 & \gamma \\ -\gamma & 0 \end{pmatrix}, \quad B_{21} = \begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix}$$

with some constants γ and δ and from assumption (v) in the definition of $\mathcal{K}(\nu, \mu)$ we have that $\gamma = \delta = 0$.

In the second step we shall establish that $B_{11} = B_{22}$. Indeed, from the representation (2) for $G\mathcal{A}_0$ and from (6) we obtain that

$$\operatorname{div} B_{11}\nabla v = \operatorname{div} B_{22}\nabla v \text{ in } \Omega \quad \forall v \in H_0^1(\Omega).$$

But, by virtue of the assumption (ii), matrices B_{11} and B_{22} are symmetric, hence $B_{11} = B_{22}$.

Denote, for simplicity, $B_{11} = b = (b_{ij})$, $i, j = 1, 2$. It remains to show that

$$\lambda_1(b)(x)\lambda_2(b)(x) - \lambda_1(b)(x)(\alpha + \beta) + \alpha\beta \leq 0 \text{ a. e. } x \in \Omega.$$

Let $x_0 \in \Omega$ be a Lebesgue point for the matrix b and its eigenvalues $\lambda_1(b)$ and $\lambda_2(b)$. Without loosing generality we can assume that at the point x_0

$$b(x_0) = \begin{pmatrix} \lambda_1(b)(x_0) & 0 \\ 0 & \lambda_2(b)(x_0) \end{pmatrix}$$

(the description (2) of the set $G\mathcal{A}_0$ is invariant with respect to rotations of the system of coordinates).

Let $v = \psi(t)$ be a function such that

$$\psi(t) = \begin{cases} t - 2s, & 2s \leq t \leq 2s + 1, \\ 2s + 2 - t, & 2s + 1 \leq t < 2s + 2, \end{cases} \quad s = 0, \mp 1, \mp 2, \cdots.$$

Define

$$u_k(x) = \varphi(x)\frac{1}{k}\psi(kx_1), \quad v_k(x) = \varphi(x)\frac{1}{k}\psi(kx_2), \quad x \in \Omega, \quad k = 1, 2, \ldots$$

where φ is a smooth function with $\operatorname{supp}\varphi \subset \Omega$ and $\varphi(x) = 1$ in some neighbourhood Ω_0 of the point x_0.

Denote by a_k matrices which correspond to A_{11} ($\overline{u}_k = (u_k, v_k)$) according to the relationship (6). Now, from relationships

$$\begin{array}{ll} \operatorname{div} b\nabla u_k & \operatorname{div} a_k \nabla u_k \text{ in } \Omega, \\ \operatorname{div} b\nabla v_k & \operatorname{div} a_k \nabla v_k \text{ in } \Omega, \end{array}$$

$$k = 1, 2, \ldots,$$

it follows that

$$\begin{gathered} \int_\Omega \zeta\varphi(b_{11} - a_{k_{11}})\,dx = \varepsilon_k', \quad \int_\Omega \zeta\varphi(b_{22} - a_{k_{22}})\,dx = \varepsilon_k'', \quad k = 1, 2, \ldots, \\ \varepsilon_k' \to 0, \quad \varepsilon_k'' \to \infty, \end{gathered} \tag{7}$$

for every $\zeta \in C^1(\overline{\Omega})$, $\operatorname{supp}\zeta \subset \Omega$ (elements ε_k' and ε_k'' depend on φ but not on ζ).

Since the set $G\mathcal{A}_0$ is closed and convex then we can pass to the limit as $k \to \infty$ in relationship (7) what gives the existing of a matrix $A_* \in G\mathcal{A}_0$, $A_{*11} = a$, such that

$$\int_\Omega \zeta(b_{11} - a_{11})\,dx = 0, \quad \int_\Omega \zeta(b_{22} - a_{22})\,dx = 0 \quad \forall \zeta \in C^\infty, \quad \operatorname{supp}\zeta \subset \Omega_0.$$

Arbitrarily of ζ gives that

$$b_{11}(x) = a_{11}(x),\ b_{22}(x) = a_{22}(x) \text{ a. e. } x \in \Omega_0.$$

From here and relationships

$$\begin{aligned} &a_{11}(x) \geq \lambda_1(a)(x) \\ &a_{11}(x)a_{22}(x) = \lambda_1(a)(x)\lambda_2(a)(x), \end{aligned}$$

which are valid for every symmetric positive matrix, it follows that

$$\begin{aligned} &\lambda_1(b)(x)\lambda_2(b)(x) - \lambda_1(b)(x)(\alpha + \beta) + \alpha\beta \\ &b_{11}(x)b_{22}(x) - b_{11}(x)(\alpha + \beta) + \alpha\beta + [b_{11}(x) - \lambda_1(b)(x)](\alpha + \beta) \\ &a_{11}(x)a_{22}(x) - a_{11}(x)(\alpha + \beta) + \alpha\beta + [b_{11}(x) - \lambda_1(b)(x)](\alpha + \beta) \\ \leq\ &\lambda_1(a)(x)\lambda_2(a)(x) - \lambda_1(a)(x)(\alpha + \beta) + \alpha\beta + [b_{11}(x) - \lambda_1(b)(x)](\alpha + \beta) \\ \leq\ &[b_{11}(x) - \lambda_1(b)(x)](\alpha + \beta) \text{ a. e. } x \in \Omega_0. \end{aligned}$$

But this inequality means that at the point x_0 (here $\lambda_1(b)(x) = b_{11}(x)$)

$$\lambda_1(b)(x)\lambda_2(b)(x) - \lambda_1(b)(x)(\alpha + \beta) + \alpha\beta \leq 0,$$

what is enough for the inclusion $B \in GA_0$.

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DAŽU EKSTREMĀLU PROBLĒMU MAKSIMĀLIE PAPLAŠINĀJUMI

ULDIS RAITUMS

Rakstā aprakstīta viena eliptisku operatoru tāda saime, ka jebkurš tās maksimālais paplašinājums, kurš saglabā sākotnējās saimes sasniedzamo atrisinājumu kopas slēgumu vājajā topoloģijā, sakrīt ar sākotnējās saimes G -slēgumu.

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PARTIAL DECOUPLING OF IMPULSIVE DIFFERENTIAL EQUATIONS

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ABSTRACT

A reduction theorem for system of impulsive differential equations in a Banach space is proven. It is asumed that given system admits an invariant set.

Keywords. Impulsive differential equations in a Banach space, dynamical equivalence in the large.

AMS subject classification. 32A37, 34C20.

1. INTRODUCTION

Impulsive differential equations provide an adequate mathematical model of evolutionary processes that suddenly change their state at certain moments. V. Lakshmikantham, D. D. Bainov and P. S. Simeonov [4] as well as A. M. Samoilenko and N. A. Perestyuk [11] have published monographs dedicated to this subject.

The equivalence problem in the theory of ordinary differential equations were explored by many mathematicians [2, 3]. A. Reinfelds [5-11] and L. Sermone [5, 6, 13-14] and D. D. Bainov, S. I. Kostadinov and Nguyen Van Minh [1] began to discuss the equivalence problem of impulsive differential equations. In the present paper a partial decoupling theorem for systems of impulsive differential equations in a Banach space is proven assuming that the system splits into two parts, has invariant manifold and has not additional restrictions on the right hand side as in the [11].

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2. STATEMENT OF THE THEOREMS

Let $\mathbf{U}$ be a Banach space. Consider two systems of impulsive differential equations

$$du/dt = P(t,u),\ \Delta u\big|_{t=\tau_i} = S_i(u(\tau_i - 0)) \tag{1}$$

and

$$du/dt = Q(t,u),\ \Delta u\big|_{t=\tau_i} = T_i(u(\tau_i - 0)) \tag{2}$$

that satisfy the conditions of the existence and uniquenesses theorem. We assume that maximum interval of the existence of the solutions is $\mathbb{R}$. Let $\phi(\cdot, s, u_0)\colon \mathbb{R} \to \mathbf{U}$ and $\psi(\cdot, s, u_0)\colon \mathbb{R} \to \mathbf{U}$ be the solutions of the above systems, respectively. Suppose that there is a function $e\colon \mathbf{U} \to \mathbb{R}$ such that

$$\max\{|P(t,u) - Q(t,u)|, \sup_i |S_i(u) - T_i(u)|\} \le e(u).$$

DEFINITION. Two systems of impulsive differential equations (1) and (2) are *dynamically equivalent in the large* if there exists a map $H\colon \mathbb{R} \times \mathbf{U} \to \mathbf{U}$ and a positive constant c such that:

(i) $H(t,\cdot)\colon \mathbf{U} \to \mathbf{U}$ is a homeomorphism;

(ii) $H(t, \phi(t,s,u_0)) = \psi(t,s,H(s,u_0))$ for all $t \in \mathbb{R}$;

(iii) $\max\{|H(t,u) - u|, |H^{-1}(t,u) - u|\} \le ce(u)$.

Let $\mathbf{U} = \mathbf{X} \times \mathbf{Y}$ $\mathcal{L}(\mathbf{X})$ and $\mathcal{L}(\mathbf{Y})$ be the Banach spaces of linear bounded operators. Consider the following system of impulsive differential equations

$$\begin{cases} dx/dt & A(t)x + f(t,x,y), \\ dy/dt & B(t)y + g(t,x,y), \\ \Delta x\big|_{t=\tau_i} & x(\tau_i + 0) - x(\tau_i - 0) \\ & C_i x(\tau_i - 0) + p_i(x(\tau_i - 0), y(\tau_i - 0)), \\ \Delta y\big|_{t=\tau_i} & y(\tau_i + 0) - y(\tau_i - 0) \\ & D_i y(\tau_i - 0) + q_i(x(\tau_i - 0), y(\tau_i - 0)), \end{cases} \tag{3}$$

where:

(i) the maps $A\colon \mathbb{R} \to \mathcal{L}(\mathbf{X})$ and $B\colon \mathbb{R} \to \mathcal{L}(\mathbf{Y})$ are locally integrable in the Bochner sense;

(ii) the maps $f\colon \mathbb{R} \times \mathbf{X} \times \mathbf{Y} \to \mathbf{X}$ and $g\colon \mathbb{R} \times \mathbf{X} \times \mathbf{Y} \to \mathbf{Y}$ are locally integrable in the Bochner sense with respect to t for fixed x and y, and, in addition, they satisfy the estimates

$$|f(t,x,y) - f(t,x',y')| \le \varepsilon(|x - x'| + |y - y'|),$$
$$|g(t,x,y) - g(t,x',y')| \le \varepsilon(|x - x'| + |y - y'|);$$

(iii) $i \in \mathbb{Z}$, $C_i \in \mathcal{L}(\mathbf{X})$, $D_i \in \mathcal{L}(\mathbf{Y})$, the maps $p_i: \mathbf{X} \times \mathbf{Y} \to \mathbf{X}$, $q_i: \mathbf{X} \times \mathbf{Y} \to \mathbf{Y}$ satisfy the estimates

$$|p_i(x,y) - p_i(x',y')| \le \varepsilon(|x - x'| + |y - y'|),$$
$$|q_i(x,y) - q_i(x',y')| \le \varepsilon(|x - x'| + |y - y'|);$$

(iv) the maps $(x,y) \mapsto (x + C_i x + p_i(x,y), y + D_i y + q_i(x,y))$, $x \mapsto x + C_i x$ are homeomorphisms;

(v) the moments τ_i of impulse effect form a strictly increasing sequence

$$< \tau_{-2} < \tau_{-1} < \tau_0 < \tau_1 < \tau_2 <$$

where the limit points may be only $\pm\infty$.

Let $U(t,\tau)$ and $V(t,\tau)$ be the Cauchy evolutionary operators of the linear impulsive systems

$$\begin{cases} dx/dt & A(t)x, \\ \Delta x\big|_{t=\tau_i} & C_i x(\tau_i - 0), \end{cases}$$

and

$$\begin{cases} dy/dt & B(t)y, \\ \Delta y\big|_{t=\tau_i} & D_i y(\tau_i - 0) \end{cases}$$

respectively. Let the operators satisfy the estimates

$$\nu = \max\left\{\sup_t \int_{-\infty}^{t} |V(t,\tau)||U(\tau,t)|\,d\tau + \sup_t \sum_{\tau_i \le t} |V(t,\tau_i)||U(\tau_i - 0,t)|,\right.$$

$$\left.\sup_t \int_t^{+\infty} |V(\tau,t)||U(t,\tau)|\,d\tau + \sup_t \sum_{t < \tau_i} |V(\tau_i - 0,t)||U(t,\tau_i)|\right\} < +\infty,$$

$$\mu = \sup_t \left\{\int_t^{+\infty} |U(t,\tau)|\,d\tau + \sum_{t < \tau_i} |U(t,\tau_i)|\right\} < +\infty.$$

Let $\Phi(\cdot,s,x,y) = (x(\cdot,s,x,y), y(\cdot,s,x,y)): \mathbb{R} \to \mathbf{X} \times \mathbf{Y}$ be the solution of system (3), where $\Phi(s+0,s,x,y) = (x,y)$. At the break points τ_i the values for all solutions are taken at $\tau_i + 0$ unless otherwise are specified. For short, we shall use the notation $\Phi(t) = (x(t), y(t))$.

Let us give a sufficient conditions for the existence of a map $v: \mathbb{R} \times \mathbf{Y} \to \mathbf{X}$, whose graph is an invariant set.

THEOREM 1. [8]. *Let* $4\varepsilon\nu \le 1$, $2\varepsilon\mu < 1 + \sqrt{1 - 4\varepsilon\nu}$, $\sup_{t,y} |f(t,0,y)| < +\infty$ *and* $\sup_{t,y} |p_i(0,y)| < +\infty$ *Then there exists a piecewise continuous map* $v: \mathbb{R} \times \mathbf{Y} \to \mathbf{X}$ *with the following properties*:

(i) $v(t, y(t,s,v(s,y),y)) = x(t,s,v(s,y),y)$ *for all* $t \in \mathbb{R}$;

(ii) $|v(t,y) - v(t,y')| \le l|y - y'|$;

(iii) $$\int_{-\infty}^{\infty} |V(s,t)||x(t,s,x,y) - v(t,y(t,s,x,y))|\,dt + \sum_{\tau_i < t} |V(s,\tau_i)||x(\tau_i - 0, s, x, y) - v(\tau_i - 0, y(\tau_i - 0, s, x, y))| \le \nu(1 - \varepsilon(1+l)\nu)^{-1}|x - v(s,y)|,$$

where $l = (2\varepsilon\nu)^{-1}(1 - 2\varepsilon\nu - \sqrt{1 - 4\varepsilon\nu})$.

THEOREM 2. [10]. *Let* $4\varepsilon\nu < 1$, $f(t,0,0) = 0$, $g(t,0,0) = 0$, $p_i(0,0) = 0$ *and* $q_i(0,0) = 0$. *Then there exists a piecewise continuous map* $v: \mathbb{R} \times \mathbf{Y} \to \mathbf{X}$ *with the following properties*:

(i) $v(t, y(t,s,v(s,y),y)) = x(t,s,v(s,y),y)$ *for all* $t \in \mathbb{R}$;

(ii) $|v(t,y) - v(t,y')| \le l|y - y'|$;

(iii) $$\int_{-\infty}^{\infty} |V(s,t)||x(t,s,x,y) - v(t,y(t,s,x,y))|\,dt + \sum_{\tau_i < t} |V(s,\tau_i)||x(\tau_i - 0, s, x, y) - v(\tau_i - 0, y(\tau_i - 0, s, x, y))| \le \nu(1 - \varepsilon(1+l)\nu)^{-1}|x - v(s,y)|,$$

where $l = (2\varepsilon\nu)^{-1}(1 - 2\varepsilon\nu - \sqrt{1 - 4\varepsilon\nu})$.

Now we formulate the main result of the paper.

THEOREM 3. *Let* $4\varepsilon\nu < 1$ *and let there exists a map* $v: \mathbb{R} \times \mathbf{X} \times \mathbf{Y} \to \mathbf{Y}$ *satisfying* (i)–(iii). *Then there is a map* $\theta: \mathbb{R} \times \mathbf{X} \times \mathbf{Y} \to \mathbf{Y}$, *which is Lipschitzian with respect to the second variable, such that systems* (3) *and*

$$\begin{cases} dx/dt & A(t)x + f(t, x, \theta(t,x,y)), \\ dy/dt & B(t)y + g(t, v(t,y), y), \\ \Delta x\big|_{t=\tau_i} & C_i x(\tau_i - 0) + p_i(x(\tau_i - 0), \theta(\tau_i - 0, x(\tau_i - 0), y(\tau_i - 0))), \\ \Delta y\big|_{t=\tau_i} & D_i y(\tau_i - 0) + q_i(v(\tau_i - 0, y(\tau_i - 0)), y(\tau_i - 0)) \end{cases} \tag{4}$$

are dynamically equivalent in the large.

The system (4) splits into two parts. The second part of them does not contain the variable x. This result allows one to replace the given system by a much simpler one.

Let $\Psi(\cdot, s, x, y) = (x_0(\cdot, s, x, y), y_0(\cdot, s, y)): \mathbb{R} \to \mathbf{X} \times \mathbf{Y}$ be a solution of system (4), where $\Psi(s+0, s, x, y) = (x, y)$. For short, we shall use the notation $\Psi(t) = (x_0(t), y_0(t))$.

3. PROOF OF THE MAIN RESULT

Step by step we shall prove the theorem. Let $\mathbf{PC}(\mathbb{R} \times \mathbf{X} \times \mathbf{Y}, \mathbf{X})$ be a set of maps that are continuous for $(t, x, y) \in [\tau_i, \tau_{i+1}) \times \mathbf{X} \times \mathbf{Y}$ and have discontinuities of the first kind for $t = \tau_i$.

Proof. *Step* 1. The space

$$\mathcal{L}_1 = \left\{ \lambda \in \mathbf{PC}(\mathbb{R} \times \mathbf{X} \times \mathbf{Y}, \mathbf{Y}) \;\middle|\; \sup_{t,x,y} \frac{|\lambda(t,x,y)|}{|x - v(t,y)|} < +\infty \right\}$$

equipped with the norm

$$\|l\| = \sup_{t,x,y} \frac{|\lambda(t,x,y)|}{|x - v(t,y)|}$$

is a Banach space. The set

$$\mathcal{L}_1(l) = \{\lambda \in \mathcal{L}_1 \mid |\lambda(t,x,y) - \lambda(t,x',y)| \le l|x - x'|\}$$

is a closed subset of the Banach space $\mathcal{L}_1$. In $\mathcal{L}_1(l)$, there exists a unique solution of the functional equation

$$\theta(s,x,w) = \int_{-\infty}^{s} V(s,\tau)(g(\tau,\eta(\tau),y(\tau) + \theta(\tau,\eta(\tau),y(\tau)))$$

$$-g(\tau,v(\tau,y(\tau)),y(\tau)))\,d\tau$$

$$+ \sum_{\tau_i \le s} V(s,\tau_i)(q_i(\eta(\tau_i - 0),y(\tau_i - 0) + \theta(\tau_i - 0,\eta(\tau_i - 0),y(\tau_i - 0)))$$

$$-q_i(v(\tau_i - 0,y(\tau_i - 0)),y(\tau_i - 0))),$$

$$\eta(t) = U(t,s)w + \int_s^t U(t,\tau)f(\tau,\eta(\tau),y(\tau) + \theta(\tau,\eta(\tau),y(\tau)))\,d\tau$$

$$+ \sum_{s < \tau_i \le t} U(t,\tau_i)p_i(\eta(\tau_i - 0),y(\tau_i - 0) + \theta(\tau_i - 0,\eta(\tau_i - 0),y(\tau_i - 0))).$$

Let $H(s,x,y) = (x, y + \theta(s,x,y))$. We get for all $t \in \mathbb{R}$ that

$$H(t,\Psi(t,s,x,y)) = \Phi(t,s,H(s,x,y)).$$

Step 2. In $\mathcal{L}_1$ there exists a unique solution of the functional equation

$$\lambda_1(s,x,y)$$

$$\int_{-\infty}^{s} V(s,\tau)(g(\tau, v(\tau, y(\tau)+\lambda_1(\tau,\Phi(\tau))), y(\tau)+\lambda_1(\tau,\Phi(\tau))) - g(\tau,\Phi(\tau)))\,d\tau$$

$$+\sum_{\tau_i \le s} V(s,\tau_i)(q_i(v(\tau_i-0, y(\tau_i-0)+\lambda_1(\tau_i-0,\Phi(\tau_i-0))), y(\tau_i-0)$$

$$+\lambda_1(\tau_i-0,\Phi(\tau_i-0))) - q_i(\Phi(\tau_i-0))).$$

Let $H_1(s,x,y) = (x, h_1(s,x,y)) = (x, y+\lambda_1(s,x,y))$. We get for all $t \in \mathbb{R}$ that

$$h_1(t,\Phi(t,s,x,y)) = y_0(t,s,H_1(s,x,y)).$$

Step 3. In $\mathcal{L}_1$ there exists a unique trivial solution of the functional equation

$$\lambda_2(s,x,y) = \int_{-\infty}^{s} V(s,\tau)(g(\tau, v(\tau, y(\tau)+\lambda_2(\tau,\Psi(\tau))), y(\tau)$$

$$+\lambda_2(\tau,\Psi(\tau))) - g(\tau, v(\tau, y(\tau)), y(\tau)))\,d\tau$$

$$+\sum_{\tau_i \le s} V(s,\tau_i)(q_i(v(\tau_i-0, y(\tau_i-0)+\lambda_2(\tau_i-0,\Psi(\tau_i-0))), y(\tau_i-0)$$

$$+\lambda_2(\tau_i-0,\Psi(\tau_i-0))) - q_i(v(\tau_i-0, y(\tau_i-0)), y(\tau_i-0))).$$

We notice that the map $\alpha\colon \mathbb{R}\times\mathbf{X}\times\mathbf{Y}\to\mathbf{Y}$, where

$$\alpha(s,x,y) = \theta(s,x,y) + \lambda_1(s,x,y+\theta(s,x,y))$$

also satisfies the functional equation and $\alpha \in \mathcal{L}_1$. It follows that the identity

$$H_1(s, H(s,x,y)) = (x,y)$$

holds true.

Step 4. The space

$$\mathcal{L}_2 = \left\{\lambda \in \mathbf{PC}(\mathbb{R}\times\mathbf{X}\times\mathbf{Y}\times\mathbf{X},\mathbf{Y}) \,\middle|\, \sup_{t,x,y,w} \frac{|\lambda(t,x,y,w)|}{\max(|x-v(t,y)|, |x-w|)} < +\infty\right\}$$

equipped with the norm

$$\|l\| = \sup_{t,x,y,w} \frac{|\lambda(t,x,y,w)|}{\max(|x-v(t,y)|, |x-w|)}$$

is a Banach space. The set

$$\mathcal{L}_2(l) = \{\lambda \in \mathcal{L}_2 \mid |\lambda(t,x,y,w) - \lambda(t,x,y,w')| \le l|w-w'|\}$$

is a closed subset of $\mathcal{L}_2$. In $\mathcal{L}_2(l)$ there exists a unique solution of the functional equations

$$\eta(t) = U(t,s)w + \int_s^t U(t,\tau)g(\tau,\eta(\tau),y(\tau)+\lambda_3(\tau,\Phi(\tau),\eta(\tau)))\,d\tau$$

$$+ \sum_{s<\tau_i\le t} U(t,\tau_i)q_i(\eta(\tau_i-0),y(\tau_i-0)+\lambda_3(\tau_i-0,\Phi(\tau_i-0),\eta(\tau_i-0))),$$

$$\lambda_3(s,x,y,w) = \int_{-\infty}^s V(s,\tau)(g(\tau,\eta(\tau),y(\tau)+\lambda_3(\tau,\Phi(\tau),\eta(\tau)))-g(\tau,\Phi(\tau)))\,d\tau$$

$$+ \sum_{\tau_i\le s} V(s,\tau_i)(q_i(\eta(\tau_i-0),y(\tau_i-0)+\lambda_3(\tau_i-0,\Phi(\tau_i-0),\eta(\tau_i-0)))-q_i(\Phi(\tau_i-0))).$$

We notice that the map $\alpha\colon \mathbb{R}\times\mathbf{X}\times\mathbf{Y}\times\mathbf{X}\to\mathbf{Y}$, where

$$\alpha(s,x,y,w) = \lambda_1(s,x,y) + \theta(s,w,h_1(s,x,y))$$

also satisfies the functional equation, $\alpha\in\mathcal{L}_2(l)$ and $\alpha(s,x,y,x)=0$. It follows that the identity

$$H(s,H_1(s,x,y)) = (x,y)$$

holds true.

We get that $H(s,\cdot)$ is homeomorphism establishing dynamical equivalence of systems (3) and (4) in the large. It is easy to verify that if the system (3) of differential equations is autonomous and without impulse effect, then the maps v, H and H_1 are independent of $s\in\mathbb{R}$. Let us note that in our case $e(x,y) = a+\varepsilon|x|$, where a is some positive constant. Thus the proof of the theorem is complete.

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IMPULSĪVA DIFERENCIĀLVIENĀDOJUMA DAĻĒJA SADALĪŠANA

ANDREJS REINFELDS

Pierādīta redukcijas tipa teorēma impulsīvo diferenciālvienādojumu sistēmai Banaha telpā, ja dotai sistēmai eksistē invarianta kopa.

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STRONG CLOSURE OF GRAPHS ASSOCIATED WITH FAMILIES OF ELLIPTIC OPERATORS

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ABSTRACT

In this paper we present an example of a decomposable family $\mathfrak{A}$ of operators corresponding to the systems of elliptic equations for which there does not exist a family $\mathfrak{B}$ consisting of the same type operators such that the graph associated with $\mathfrak{B}$ coincides with the strong closure of the graph associated with $\mathfrak{A}$ when the number of unknown functions is greater than or equal to the number of independent variables provided that the latter is greater than one. The operators of the family $\mathfrak{A}$ correspond to isotropic systems when one and the same process is observed with different external forces.

Keywords. Strong closure, $E(\nu_1, \nu_2)$-extension, elliptic operator.
AMS subject classification. Primary 49J45, secondary 49J20.

1. INTRODUCTION

The purpose of the present paper is to show that for an arbitrary decomposable family of the second order elliptic operators of divergent type which correspond to elliptic systems there can not exist a family consisting of the same type operators which preserves the strong closure of the set of feasible states for every fixed right-hand side (see definition below and Section 3) when the number m of unknown functions is greater than or equal to the number n of independent variables ($n \geq 2$). This statement is proved by constructing a concrete family. It must be noted that for $m = n = 2$ such an example was constructed in [10], but it did not include the so-called rotations (see Section 3) and was adapted to the case which corresponds to the problems of optimal layout of two materials. Let us also note that the technique of this paper is essentially different from the used one in [10] and can not be applied for that

case, and vice versa.

Now let us introduce basic notations and definitions which are encountered in the paper. We shall consider a family $\mathfrak{A} = \{A_\chi\}_{\chi \in \Lambda}$ of operators from the Cartesian product $\left[H_0^1(\Omega)\right]^m$ of Sobolev spaces into the Cartesian product $\left[H^{-1}(\Omega)\right]^m$ of their dual spaces where Ω is a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary and

$$A_\chi = \operatorname{div} \sum_{p=1}^{r} \chi_p \mathcal{A}_p \nabla$$

$$\Lambda = \{\chi \mid \chi = (\chi_1, \ldots, \chi_r), \sum_{p=1}^{r} \chi_p = 1, \chi_p \text{ is measurable},$$

$$\chi_p(x) = 0 \text{ or } 1 \text{ a.e. } x \in \Omega\}$$

$\mathcal{A}_p, p = 1, \ldots, r$ are fixed $mn \times mn$–matrices which belong to the set $E(\nu_1, \nu_2)$, $\nu_1, \nu_2 > 0$ consisting of all $mn \times mn$–matrices $\mathcal{A}$ satisfying the conditions

$$\nu_1 \xi_{ij} \xi_{ij} \leq \mathcal{A}_{ijkl}(x) \xi_{ij} \xi_{kl} \leq \nu_2 \xi_{ij} \xi_{ij}$$
$$\mathcal{A}_{ijkl}(x) = \mathcal{A}_{klij}(x) \text{ a.e. } x \in \Omega, \text{ for all } \xi \in \mathbb{R}^{m \times n}$$
$$1 \leq i, k \leq m; 1 \leq j, l \leq n.$$

For an $mn \times mn$-matrix $\mathcal{A}$ and an $m \times n$–matrix ξ we define $(\mathcal{A}\xi)_{ij} = \mathcal{A}_{ijkl}\xi_{kl}$. Here and further on, summation over repeated indices is assumed.

The natural number r is called *a number of materials*. It is due to the fact that $\mathcal{A}_p(x)$ usually describes property of p–th material at a point $x \in \Omega$, and the sum $\sum_{p=1}^{r} \chi_p \mathcal{A}_p$ describes medium divided into r regions occupied by different materials. Throughout the sequel we assume that $r \geq 2$.

Put $V = \left[H_0^1(\Omega)\right]^m$ and $V^* = \left[H^{-1}(\Omega)\right]^m$

For a family $\mathfrak{C} = \{A\}$ of operators A from V into V^* we set

$$\operatorname{graph}(\mathfrak{C}) = \{(u, Au) \mid u \in V, A \in \mathfrak{C}\}$$

and

$$F(\mathfrak{C}, u) = \{Au \mid A \in \mathfrak{C}\}, u \in V.$$

It is clear that $\operatorname{graph}(\mathfrak{C}) \subset V \times V^*$ The space $V \times V^*$ will be considered with the strong topology.

Now we give the main definition of the paper which will be used henceforth.

A family $\mathfrak{B} = \{\operatorname{div} \mathcal{A}_\gamma \nabla\}_{\gamma \in \Upsilon}$ is said to be *an $E(\nu_1, \nu_2)$-extension of* $\mathfrak{A}$ if

$$\mathfrak{A} \subset \mathfrak{B}, \{\mathcal{A}_\gamma\}_{\gamma \in \Upsilon} \subset E(\nu_1, \nu_2)$$

and the closure of $\operatorname{graph}(\mathfrak{A})$ coincides with $\operatorname{graph}(\mathfrak{B})$.

Let $K \subset \mathcal{L}(V, V^*)$ where $\mathcal{L}(V, V^*)$ is the space of continuous linear mappings of V into V^*. For an arbitrary family $\mathfrak{C} \subset K$ we admit the following definition. A family $\mathfrak{D}$ is said to be *a K-extension of* $\mathfrak{C}$ if

$$\mathfrak{C} \subset \mathfrak{D}, \mathfrak{D} \subset K$$

and the closure of graph $(\mathfrak{C})$ coincides with graph $(\mathfrak{D})$.

A family $\mathfrak{C} = \{\operatorname{div} \mathcal{A}_\gamma \nabla\}_{\gamma \in \Upsilon}$ where $\mathcal{A}_\gamma \in E(\nu_1, \nu_2)$, $\gamma \in \Upsilon$ is called *decomposable* if for any measurable characteristic function ρ from inclusions $\operatorname{div} \mathcal{A}_1 \nabla \in \mathfrak{C}$ and $\operatorname{div} \mathcal{A}_2 \nabla \in \mathfrak{C}$ it follows that $\operatorname{div}(\rho \mathcal{A}_1 + (1-\rho)\mathcal{A}_2)\nabla \in \mathfrak{C}$. It is evident that the family $\mathfrak{A}$ is decomposable. This feature is of great significance in our future considerations because even if $m = n = 1$ there is an example of a family $\mathfrak{C} = \{\operatorname{div} \mathcal{A}_\gamma \nabla\}_{\gamma \in \Upsilon}$ with $\mathcal{A}_\gamma \in E(\nu_1, \nu_2), \gamma \in \Upsilon$ for which does not exist an $E(\nu_1, \nu_2)$-extension [11]. On the other hand, if $m = n = 1$, as $E(\nu_1, \nu_2)$-extension of the family $\mathfrak{A}$ may be chosen the $\mathfrak{A}$ itself since for this case graph $(\mathfrak{A})$ is closed.

A significance of our example may be explained from the viewpoint of optimal control problems in the following way. Consider the optimal control problem

$$\begin{aligned} &J(u) \to \min \\ &u \in V, A \in \mathfrak{A}, Au = f \end{aligned} \tag{1}$$

with the cost functional J which is not weakly continuous. For instance, the functional defined as

$$J(u) = \int_\Omega (\nabla u - \nabla v)^2 \, dx$$

where $v \in V$ is given it is not weakly continuous (such functionals where considered in [7] for $m = 1$).

If an extension of (1) (we refer to [4, 2] concerning the notion of the extension of an optimal control problem) is searched in the analogous form

$$\begin{aligned} &J(u) \to \min \\ &u \in V, B \in \mathfrak{B}, Bu = f \end{aligned} \tag{2}$$

with a larger than $\mathfrak{A}$ family $\mathfrak{B}$ of operators of the same kind, then the family $\mathfrak{B}$ must preserve the strong closure of the set of feasible states for every fixed right-hand side f. The result obtained in this paper shows that for optimal control problems governed by elliptic systems when $m \geq n \geq 2$ and with non weakly continuous cost functionals the extensions of the form (2) may not exist.

To underline the distinction between the families of operators which correspond to a single equation and the families corresponding to the systems of equations when $m \geq n \geq 2$, for $m = 1$, $n \geq 2$ we mention the following result which directly follows from [6] (cf. [7]).

THEOREM 1. *Let $\mathfrak{A}$ be as defined above. Let $m = 1, n \geq 2$. Then an $E(\nu_1, \nu_2)$-extension of the family $\mathfrak{A}$ is $\overline{\text{co}}\,\mathfrak{A}$ where $\overline{\text{co}}$ stands for the closed convex hull of a set in the space $\mathcal{L}(V, V^*)$.*

REMARK 2. In the recent paper [5] the more general result than this theorem was proved (see also [4]). From this result it follows that $\overline{\text{co}}\,\mathfrak{A}$ is extension of the family $\mathfrak{A}$ if $m \leq n - 1$.

In the case of a family $\mathfrak{C}$ of arbitrary linear continuous operators from V into V^* we present the following theorems showing the cases of existence of K-extensions.

THEOREM 3. *Let K be a closed set in the strong operator topology of $\mathcal{L}(V, V^*)$ and let $\mathfrak{C} \subset K$. If the strong closure of the set $F(\mathfrak{C}, u)$ is compact for every $u \in V$, then the sequential closure of the set $\mathfrak{C}$ in the strong operator topology of $\mathcal{L}(V, V^*)$ is the K-extension of $\mathfrak{C}$.*

THEOREM 4. *Let K be a closed set in the weak operator topology of $\mathcal{L}(V, V^*)$ and let $\mathfrak{C} \subset K$. Assume that K is bounded in the space $\mathcal{L}(V, V^*)$. If the weak closure of the set $F(\mathfrak{C}, u)$ coincides with the strong closure of this set for any $u \in V$, then the sequential closure of the set $\mathfrak{C}$ in the weak operator topology of $\mathcal{L}(V, V^*)$ is the K-extension of $\mathfrak{C}$.*

REMARK 5. Here the terms of strong and weak operator topology of $\mathcal{L}(V, V^*)$ mean the same as in [1].

REMARK 6. The theorems 3 and 4 are consequences of more general results of [11].

2. AUXILIARY RESULTS

To construct the example, we need the following auxiliary results.

THEOREM 7. (Vainikko and Kunisch [9]) *Suppose that $n \geq 2$. Let a function $\sigma \in L^\infty(\Omega)$ and a function $v \in C_0^\infty(\Omega)$ satisfy the equation $\operatorname{div} \sigma \nabla v = 0$ in $H^{-1}(\Omega)$. Then $\sigma = 0$ on $\{x \in \Omega \mid \nabla v(x) \neq 0\}$.*

THEOREM 8. (Nečas [3], cf. [8]) *Let Ω be an open set in $\mathbb{R}^n$ with Lipschitz boundary. Then there exists $c = c(\Omega) > 0$ such that for every $g \in L^2(\Omega)$*

$$\|g\|_{L^2(\Omega)/\mathbb{R}} \leq c \|\operatorname{grad} g\|_{[H^{-1}(\Omega)]^n}$$

where $L^2(\Omega)/\mathbb{R}$ is the factor space.

3. MAIN RESULT

The main result of the paper is the following.

THEOREM 9. *Let $m \geq n \geq 2$. Let $\mathfrak{A}$ be the family as in Introduction for which $\mathcal{A}_p = \alpha_p \mathcal{I}$, $p = 1, \ldots, r$ where $\mathcal{I}$ is the identical $mn \times mn$-matrix, $0 < \alpha_1 < \quad < \alpha_r$ are positive constants. Then there does not exist an $E(\nu_1, \nu_2)$-extension of the family $\mathfrak{A}$.*

Proof. Let us prove this theorem for $m = n \geq 2$ since the case $m > n \geq 2$ can be obviously reduced to the case $m = n \geq 2$. To do this, let us suppose that our assertion is not true. Then there will exist a family $\mathfrak{B} = \{\operatorname{div} \mathcal{A}_\gamma \nabla\}_{\gamma \in \Upsilon}$ which is an $E(\nu_1, \nu_2)$-extension of $\mathfrak{A}$. Hence

$$F(\mathfrak{B}, u) = \operatorname{cl} F(\mathfrak{A}, u) \text{ for all } u \in V \tag{3}$$

where cl denotes the strong closure in V^* Since

$$\operatorname{cl} F(\mathfrak{A}, u) \subset F\left(\{\operatorname{div} \mathcal{A}_\theta \nabla\}_{\theta \in \Gamma}, u\right)$$

where

$$\Gamma = \{\theta \mid \theta = (\theta_1, \ldots, \theta_r), \sum_{p=1}^{r} \theta_p = 1, \theta_p \text{ is measurable},$$
$$0 \leq \theta_p(x) \leq 1 \text{ a.e. } x \in \Omega\}$$
$$\mathcal{A}_\theta = \sum_{p=1}^{r} \theta_p \mathcal{A}_p$$

then for every $\gamma \in \Upsilon$ and every $u \in V$ there exists $\theta = \theta(u, \gamma)$ such that $\operatorname{div} \mathcal{A}_\gamma \nabla u = \operatorname{div} \mathcal{A}_\theta \nabla u$. Placing successively $u^1 = (v, 0, \ldots, 0), \ldots, u^n = (0, 0, \ldots, v)$ in this equation, from arbitrariness of $v \in H_0^1(\Omega)$ we see that it can be assumed that the tensor $\mathcal{B} = \mathcal{A}_\gamma$ has the form $(B_{ijkl}) = b_\gamma$ if $j = l$ (indices j, l are fixed) for some $n \times n$-matrix b_γ and (B_{ijkl}) is a zero-matrix if $j \neq l$. Now set

$$u_1 = (\xi_1, x)\varphi, \ldots, u_n = (\xi_n, x)\varphi$$

where $\varphi \in C_0^\infty(\Omega), \varphi(x) = 1$ in some open ball $U \subset\subset \Omega$ and the vectors $\xi_1, \ldots, \xi_n \in \mathbb{R}^n$ are linearly independent. Put $u^0 = (u_1, \ldots, u_n)$ and $u^* = (u_1, \ldots, u_1)$. According to Theorem 1, we have

$$\operatorname{cl} F(\mathfrak{A}, u^*) = F\left(\{\operatorname{div} \mathcal{A}_\theta \nabla\}_{\theta \in \Gamma}, u^*\right). \tag{4}$$

Further on we shall prove that

$$\operatorname{cl} F(\mathfrak{A}, u^0) \subset \left\{\operatorname{div} \mathcal{A}_\theta \nabla u^0 \mid \theta \in \Gamma, \forall p\ \theta_p(x) \in \{0; 1\} \text{ on } U\right\}. \tag{5}$$

From the assumptions of the theorem it follows that there exists θ^* $(\theta_1^*, \ldots, \theta_r^*) \in \Gamma$ such that

$$\alpha_1 < \sum_{p=1}^{r} \alpha_p \theta_p^* < \alpha_2 \text{ on } M \tag{6}$$

where M is a subset of U with meas $M > 0$. To establish this, choose $q > 1$ such that $q\alpha_1 < \alpha_2$. Let $0 < \varepsilon < (\alpha_2 - \alpha_1)/(q\alpha_r - \alpha_1)$ and let θ_1^* be a function which satisfies

$$0 < 1 - q\varepsilon \leq \theta_1^*(x) \leq 1 - \varepsilon \text{ a.e. } x \in M.$$

Let us define the function θ_1^* on $\Omega \setminus M$ and other functions $\theta_2^*, \ldots, \theta_r^*$ on Ω in such a way that $\theta^* = (\theta_1^*, \ldots, \theta_r^*) \in \Gamma$. Then it remains to prove (6). Since $\alpha_2 - q\alpha_1 > 0$, we have

$$\begin{aligned} \alpha_1 < \alpha_1 + \varepsilon(\alpha_2 - q\alpha_1) &= (1 - q\varepsilon)\alpha_1 + \varepsilon\alpha_2 \\ &\leq \alpha_1\theta_1^* + \alpha_2 \sum_{p=2}^{r} \theta_p^* \leq \sum_{p=1}^{r} \alpha_p \theta_p^* \quad \text{on } M. \end{aligned}$$

Since $\varepsilon < (\alpha_2 - \alpha_1)/(q\alpha_r - \alpha_1)$, we find that

$$\begin{aligned} \sum_{p=1}^{r} \alpha_p \theta_p^* \leq \alpha_1(1 - \varepsilon) + \alpha_r \sum_{p=2}^{r} \theta_p^* &\leq (1 - \varepsilon)\alpha_1 + q\varepsilon\alpha_r \\ &= \alpha_1 + \varepsilon(q\alpha_r - \alpha_1) < \alpha_2 \end{aligned} \quad \text{on } M.$$

We thus obtain (6). From (3), (4) it follows that there exists $B \in \mathfrak{B}$ such that $Bu^* = \operatorname{div} A_{\theta^*} \nabla u^*$ Using (3), (5), we have $Bu^0 = \operatorname{div} A_{\theta^0} \nabla u^0$ where $\theta^0 = (\theta_1^0, \ldots, \theta_r^0) \in \Gamma$, $\sum_{p=1}^{r} \alpha_p \theta_p^0(x) \in \{\alpha_1, \ldots, \alpha_r\}$ on U. Hence

$$\begin{aligned} \operatorname{div} b_\gamma \nabla u_1 &= \operatorname{div} \sigma_1 \nabla u_1 \\ \operatorname{div} b_\gamma \nabla u_1 &= \operatorname{div} \sigma_2 \nabla u_1 \end{aligned} \quad \text{in } H^{-1}(\Omega)$$

where we set $\sigma_1 = \sum_{p=1}^{r} \alpha_p \theta_p^*, \sigma_2 = \sum_{p=1}^{r} \alpha_p \theta_p^0$. Subtracting two last equations, we obtain $\operatorname{div}(\sigma_1 - \sigma_2)\nabla u_1 = 0$. Thus we infer from Theorem 7 that $\sigma_1(x) = \sigma_2(x) \in \{\alpha_1, \ldots, \alpha_r\}$ on M. But this is impossible since the function σ_1 satisfies (6).

It remains to prove (5). For this, let us assume that there exist $\theta \in \Gamma, \theta = (\theta_1, \ldots, \theta_r)$ and $p' \in \{1, \ldots, r\}$ such that

$$0 < \alpha_{p'-1} + \varepsilon \leq \sum_{p=1}^{r} \alpha_p \theta_p \leq \alpha_{p'} - \varepsilon \text{ on } M_1 \subset U$$

with meas $M_1 > 0$, $0 < \varepsilon < (\alpha_{p'} - \alpha_{p'-1})/2$ ($\alpha_{p'-1} = 0$ if $p' = 1$) and

$$\operatorname{div} \mathcal{A}_\theta \nabla u^0 \in \operatorname{cl} F\left(\mathfrak{A}, u^0\right)$$

Then there is a sequence $\{\chi^k\}$ such that $\operatorname{div} \mathcal{A}_{\chi^k} \nabla u^0 \to \operatorname{div} \mathcal{A}_\theta \nabla u^0$ in V^* Without lack of generality we can suppose that $\chi^k \overset{*}{\rightharpoonup} \bar{\theta}$ in $[L^\infty(\Omega)]^r$ Hence we have $\operatorname{div}(\omega_1 - \omega_2)\nabla u_1 = 0$ where we put $\omega_1 = \sum_{p=1}^{r} \alpha_p \theta_p, \omega_2 = \sum_{p=1}^{r} \alpha_p \bar{\theta}_p$. Using Theorem 7, we obtain that $\omega_1 = \omega_2$ on U. Setting $\rho_k = \sum_{p=1}^{r} \alpha_p \chi_p^k$, we have

$$\nabla u^0 \operatorname{grad}(\rho_k - \omega_1) + (\rho_k - \omega_1) \operatorname{div} \nabla u^0 \to 0 \text{ in } V^* \tag{7}$$

Since $\rho_k \overset{*}{\rightharpoonup} \omega_2$ in $L^\infty(\Omega)$, then $\rho_k \overset{*}{\rightharpoonup} \omega_2$ in $L^\infty(U)$. This implies that $\rho_k \overset{*}{\rightharpoonup} \omega_1$ in $L^\infty(U)$. Using compactness of the embedding $L^2(U) \hookrightarrow H^{-1}(U)$, we infer from (7) that

$$\nabla u^0 \operatorname{grad}(\rho_k - \omega_1) \to 0 \text{ in } \left[H^{-1}(U)\right]^n$$

Taking into account the form of ∇u^0 on U, we have

$$\operatorname{grad}(\rho_k - \omega_1) \to 0 \text{ in } \left[H^{-1}(U)\right]^n$$

By virtue of Theorem 8, it follows that

$$\rho_k - \omega_1 - |U|^{-1} \int_U (\rho_k - \omega_1)\, dx \to 0 \text{ in } L^2(U).$$

Therefore, $\rho_k \to \omega_1$ in $L^2(U)$. This gives that $\omega_1(x) \in \{\alpha_1, \ldots, \alpha_r\}$ a.e. $x \in M_1$. This is a contradiction with

$$\alpha_{p'-1} + \varepsilon \leq \omega_1 \leq \alpha_{p'} - \varepsilon \text{ on } M_1, \operatorname{meas} M_1 > 0.$$

REMARK 10. The result that we have just proved includes the so-called *rotations* of initial materials in the following sense ($m \geq n \geq 2$). For an $n \times n$-matrix Q with entries q_{ij} from $L^\infty(\Omega)$ satisfying

$$Q(x) Q^t(x) = I \text{ a.e. } x \in \Omega \tag{8}$$

where t is the symbol of transposition, let us define an operator

$$\mathbf{Q} \quad [L^\infty(\Omega)]^{mn \times mn} \to [L^\infty(\Omega)]^{mn \times mn}$$

by the rule

$$(\mathbf{Q}\mathcal{A})_{ij'kl'} = q_{jj'} q_{ll'} \mathcal{A}_{ijkl}.$$

It is clear that $Q\mathcal{I} = \mathcal{I}$. Hence if we consider a family $\mathfrak{D}$ consisting of operators $\operatorname{div} \sum_{p=1}^{r} \chi_p Q_p \mathcal{A}_p \nabla$ for arbitrary $\chi \in \Lambda$ and for all matrices Q_p, $p = 1, \ldots, r$ satisfying (8), we see that $\mathfrak{D} = \mathfrak{A}$.

The next result immediately follows from Theorem 9.

COROLLARY 11. Let $\mathfrak{A}$ be the same family as in Theorem 9 and $f \in V^*$ Define a set $Z(\mathfrak{A}, f)$ of feasible states as follows

$$Z(\mathfrak{A}, f) = \{u \mid A_\chi u = f, \chi \in \Lambda\}.$$

Then there does not exist a family $\mathfrak{B} = \{\operatorname{div} \mathcal{A}_\gamma \nabla\}_{\gamma \in \Gamma}$ such that

$$\mathfrak{A} \subset \mathfrak{B}, \{\mathcal{A}_\gamma\}_{\gamma \in \Gamma} \subset E(\nu_1, \nu_2)$$

and the closure of the set $Z(\mathfrak{A}, f)$ coincides with the set $Z(\mathfrak{B}, f)$ for every fixed $f \in V^*$

The importance of this corollary for optimal control problems was explained in Introduction.

Now for the fixed $\varepsilon > 0$ let us consider a family $\mathfrak{A}_\varepsilon = \{A_\varepsilon\}$ of all operators of the form $A_\varepsilon = \operatorname{div} \sum_{p=1}^{r} \chi_p Q_p (\mathcal{A}_p + \varepsilon \mathcal{S}_p) \nabla$ for arbitrary $\chi \in \Lambda$ and matrices Q_p, $p = 1, \ldots, r$ satisfying (8) where $\mathcal{A}_p$, $p = 1, \ldots, r$ are the isotropic matrices (for instance, as in Theorem 9) and $\mathcal{S}_p$, $p = 1, \ldots, r$ are matrices which make the matrices $\mathcal{A}_p + \varepsilon \mathcal{S}_p$, $p = 1, \ldots, r$ anisotropic. Let us recall that $\mathfrak{A}$ is a family of all operators of the form $A = \operatorname{div} \sum_{p=1}^{r} \chi_p \mathcal{A}_p \nabla$ for arbitrary $\chi \in \Lambda$. It is clear that for every $p = 1, \ldots, r$ $\mathcal{A}_p + \varepsilon \mathcal{S}_p \in E\left(\nu_1', \nu_2'\right)$ for some suitable $\nu_1', \nu_2' > 0$ and for small ε.

We have

PROPOSITION 12. *If for some $u \in V$ the strong closure of the set $F(\mathfrak{A}, u)$ is not convex, then there exists $h > 0$ such that for every $\varepsilon \in (0, h)$ the strong closure of $F(\mathfrak{A}_\varepsilon, u)$ is not convex.*

Proof. Assume the contrary. Then for every $h > 0$ there exists $\varepsilon \in (0, h)$ such that the set $\operatorname{cl} F(\mathfrak{A}_\varepsilon, u)$ is convex. Hence we can extract a sequence $\{\varepsilon\}$ of such ε's such that $\varepsilon \to 0$ and

$$\operatorname{cl} F(\mathfrak{A}_\varepsilon, u) = F(\overline{\operatorname{co}}\, \mathfrak{A}_\varepsilon, u). \tag{9}$$

Let us prove that $\operatorname{cl} F(\mathfrak{A}, u) = F(\overline{\operatorname{co}}\, \mathfrak{A}, u)$. Indeed, if $f \in F(\overline{\operatorname{co}}\, \mathfrak{A}, u)$, then there is an operator $B \in \overline{\operatorname{co}}\, \mathfrak{A}$ for which $Bu = f$. It is evident that there is

a sequence $\{B_\varepsilon\}$ such that $B_\varepsilon \in \overline{\text{co}}\,\mathfrak{A}_\varepsilon$ for all $\varepsilon \in \{\varepsilon\}$ and $B_\varepsilon u \to Bu$. By (9), we see that there exists a sequence $\left\{A_\varepsilon^k\right\} \subset \mathfrak{A}_\varepsilon$ such that $A_\varepsilon^k u \underset{k}{\to} B_\varepsilon u$. One can also show that $A_\varepsilon^k u \underset{\varepsilon}{\to} A^k u$ for some $A^k \in \mathfrak{A}$. Hence, after extracting subsequences from the sequences $\{B_\varepsilon u\}$, $\left\{A_\varepsilon^k\right\}$, $\left\{A^k\right\}$, we have $A^k u \to Bu = f$ (the subsequence is not relabelled). This completes the proof.

This proposition shows that in case of anisotropic matrices $\mathcal{A}_p + \varepsilon \mathcal{S}_p$, $p = 1, \ldots, r$ as $E\left(\nu_1', \nu_2'\right)$–extension of the family $\mathfrak{A}_\varepsilon$ can not be chosen $\overline{\text{co}}\,\mathfrak{A}_\varepsilon$ provided that ε is sufficiently small and $\mathcal{A}_p$, $p = 1, \ldots, r$ are the same as in Theorem 9. But there still remains open the question: Is there an $E\left(\nu_1', \nu_2'\right)$–extension of the family $\mathfrak{A}_\varepsilon$?

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PAR ELIPTISKU OPERATORU SAIMJU GRAFIKU STIPRO SLĒGUMU

OĻEGS ZAICEVS

Šajā rakstā mēs konstruējam dekompozējamu operatorsaimi $\mathfrak{A}$, kuras operatori atbilst eliptiskām sistēmām ar nezināmo funkciju skaitu, ne mazāku par neatkarīgo mainīgo skaitu (neatkarīgo mainīgo skaits ir lielāks par vienīnieku), kurai necksistē analoģiska saime $\mathfrak{B}$ tāda, ka saimes $\mathfrak{B}$ grafiks sakrīt ar saimes $\mathfrak{A}$ grafika stipro slēgumu. Saimes $\mathfrak{A}$ operatori atbilst izotropām sistēmām, kur viens un tāds pats process tiek apskatīts dažādu ārējo spēku iespaidā.

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THE REALIZATION OF COUPLING CONDITIONS FOR A MULTILAYERED NONWOVEN

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ABSTRACT

In several practical problems it is important to simulate the liquid transport in the layered nonwoven. We analyze a situation, when the liquid is transported by capillary pressure. We assume that the capillary properties between layers can be very different. In this situation it is important to spend a special attention to the model and to the coupling conditions.

1. THE MATHEMATICAL MODEL

Our aim is to simulate the liquid flow in the multi–layered nonwoven. We can consider the following experiment. Three rectangular nonwoven–layers lay upon each other. The layers are numbered from above. The liquid is poured on the first layer. We want to describe the spread of the liquid in this structure. We assume that the process is symmetrical, therefore we just need to analyze one quarter of the domain.

The base of the model is the Darcy's law [1]

$$\vec{q} = -\bar{K}(w)\,\mathrm{grad}\,(p(w)) \tag{1}$$

It means that the liquid transport in the given porous material can be described, when the functions $K(w)$ and $p(w)$ are known. These functions have to fit the experimental data or have to be obtained from the literature. In our numerical experiments we have used for the layer l the permeability

function $K_l(w)$ defined by

$$K_l(w) = k_l w^{n_{kl}}, \tag{2}$$

where k_l represents the hydraulic conductivity at full saturation and n_{kl} is a parameter.

We use the following notation:

$$f_l(w) = -p_{0l}(1 - w^{n_{pl}}), \tag{3}$$

where $0 \leq w \leq 1$ is the saturation, p_{0l} is the capillary fringe and n_{pl} is a parameter. In the reality, there exists a minimal value of saturation $w_{min} > 0$, but we assume that $w_{min} = 0$ and $p_l(w_{min})$ is finite.

In our case the capillary pressure $p_l(w)$ can be defined as follows [3]:

$$p_l(w) = \begin{cases} f_l(w) & , w \geq w_{0l} \\ -h_{ml} + \frac{h_{ml} + f_l(w_{0l})}{w_{0l}^2} w(2w_{0l} - w) + \frac{f_l'(w_{0l})}{w_{0l}} w(w - w_{0l}) & , w < w_{0l}. \end{cases} \tag{4}$$

The parameters h_{ml}, p_{0l}, w_{0l}, n_{pl}, k_l and n_{kl} are estimated from appropriate experiments.

By using of (1) and the balance of mass can be obtained the following mathematical problem (the problem is written for L layers):

$$m_l \frac{\partial w_l}{\partial t} \quad \nabla \cdot (K_l(w_l) \nabla \Phi_l(w_l)) \text{ in } \Omega_l, l = 1, \ldots, L, \tag{5}$$

$$K_l(w) \nabla \Phi_l(w) = 0 \text{ on } \partial\Omega \backslash \partial\Omega_0, \tag{6}$$

$$K_1(w) \nabla \Phi_1(w) \quad \begin{cases} q_0, & t < t_0, \\ 0, & t > t_0 \end{cases} \text{ on } \partial\Omega_0, \tag{7}$$

$$p_l(w_l) \quad p_{l+1}(w_{l+1}) \text{ on } \Gamma_l = \bar{\Omega}_l \bigcap \bar{\Omega}_{l+1}, \tag{8}$$

$$K_l(w_l) \frac{\partial \Phi_l}{\partial x} = K_{l+1}(w_{l+1}) \frac{\partial \Phi_{l+1}}{\partial x} \text{ on } \Gamma_l. \tag{9}$$

Here

$\Omega = \bigcup_{l=1}^{L} \Omega_l$	is the whole domain,
$\partial\Omega$	is the outer boundary of the domain Ω,
$\partial\Omega_0$	is the part of the upper surface, where the liquid is poured,
$\Phi_l(w) = p_l(w) + x$	is the piesometric head for the layer l,
m_l	is the porosity,
q_0	is the intensity of the liquid-source.

The liquid source works only during some given time-interval. Similarly the problem can be written in the 2-D case and in other coordinate systems.

The mathematical model is nonlinear with a nonlinear degeneracy. It leads to a finite speed of propagation of disturbances from the rest [4].

2. THE NUMERICAL METHOD

2.1. The numerical approximation

The problem for nonlinear parabolic equation (6) can be solved analytically only in some special cases [5]. Therefore we use a finite difference method. Similar problem is solved in [2, 3]. There are allowed full saturated zones and it leads to an elliptic-parabolic problem. We assume here that $w < 1$ and therefore we have a parabolic problem. The main difficulty in our case is connected with condition (8), which can not be always fulfilled.

The numerical solution will be found on the set of the discrete points. We approximate the domain Ω with the grid:

$$\bar{\omega}_h = \{(i\ \ h_x, j\ \ h_y, k\ \ h_z)\ \ i = 0, \ldots, n_x, j = 0, \ldots, n_y, k = 0, \ldots, n_z\},$$

where

h_x, h_y, h_z are steps of a grid;

n_x, n_y, n_z means the number of grid points in the corresponding direction.

Similarly the grid can be defined for the time variable:

$$\bar{\omega}_\tau = \{t_\xi : t_\xi = \xi \cdot \tau, \xi = 0, \ldots, n_T\},$$

where

τ is the time step,

n_T means the number of the time-steps.

A grid function $f(x_i, y_j, z_k, t_n)$ will be farther denoted as f^n_{ijk}. For the approximation of the differential equation we use the Richtmayer-Morton full implicit three layer difference scheme [6] and we do not give here formulas for the interior points and the points on the outer boundary.

2.2. The coupling conditions

When a porous structure consists of different layers, then the coupling conditions are needed to be approximated on the interior boundary (the boundary between two layers). Formally the flux and the capillary pressure on the interior boundary must be continuous (conditions (8) (9)). Because the sets of values for capillary pressure in different nonwovens can be significantly different, then the condition (8) is not always fulfilled in our model (we assume that the capillary pressure can take only finite values). Therefore, we have to analyze two variants.

<u>The flux is zero</u>

When the flux between two layers is not possible (for example in the case, when the suction of the moist material is greater than the suction of the dry material), then we have to approximate two separate conditions. We assume that $i = i_0$ corresponds to the highest point on the layer l and $i_0 + 1$

corresponds to the lowest point on the layer $l+1$. Then we can write:

$$1.5m_l\frac{w^{n+1}_{i_0jk}-w^n_{i_0jk}}{\tau}-0.5m_l\frac{w^n_{i_0jk}-w^{n-1}_{i_0jk}}{\tau}+2K^{n+1}_{i_0-\frac{1}{2}jk}\frac{\Phi^{n+1}_{i_0jk}-\Phi^{n+1}_{i_0-1jk}}{h_x^2}$$

$$-K^{n+1}_{i_0j+\frac{1}{2}k}\frac{\Phi^{n+1}_{i_0j+1k}-\Phi^{n+1}_{i_0jk}}{h_y^2}+K^{n+1}_{i_0j-\frac{1}{2}k}\frac{\Phi^{n+1}_{i_0jk}-\Phi^{n+1}_{i_0j-1k}}{h_y^2}$$

$$-K^{n+1}_{i_0jk+\frac{1}{2}}\frac{\Phi^{n+1}_{i_0jk+1}-\Phi^{n+1}_{i_0jk}}{h_z^2}+K^{n+1}_{i_0jk-\frac{1}{2}}\frac{\Phi^{n+1}_{i_0jk}-\Phi^{n+1}_{i_0jk-1}}{h_z^2}=0, \qquad (10)$$

$$1.5m_{l+1}\frac{w^{n+1}_{i_0+1jk}-w^n_{i_0+1jk}}{\tau}-0.5m_{i_0+1}\frac{w^n_{i_0+1jk}-w^{n-1}_{i_0+1jk}}{\tau}$$

$$-2K^{n+1}_{i_0+1+\frac{1}{2}jk}\frac{\Phi^{n+1}_{i_0+2jk}-\Phi^{n+1}_{i_0+1jk}}{h_x^2}-K^{n+1}_{i_0+1j+\frac{1}{2}k}\frac{\Phi^{n+1}_{i_0+1j+1k}-\Phi^{n+1}_{i_0+1jk}}{h_y^2}$$

$$+K^{n+1}_{i_0+1j-\frac{1}{2}k}\frac{\Phi^{n+1}_{i_0+1jk}-\Phi^{n+1}_{i_0+1j-1k}}{h_y^2}-K^{n+1}_{i_0+1jk+\frac{1}{2}}\frac{\Phi^{n+1}_{i_0+1jk+1}-\Phi^{n+1}_{i_0+1jk}}{h_z^2}$$

$$+K^{n+1}_{i_0+1jk-\frac{1}{2}}\frac{\Phi^{n+1}_{i_0+1jk}-\Phi^{n+1}_{i_0+1jk-1}}{h_z^2}=0. \qquad (11)$$

The flux is nonzero

If the flux between two layers is possible, then both conditions (8) and (9) have to be fulfilled. We can obtain the approximation of (9), when we add equations (10) and (11). Let $L_1(i_0,j,k,n+1)$ and $L_2(i_0+1,j,k,n+1)$ be the left hand sides of the terms (10) and (11), then the approximation of (9) can be written as:

$$L_1(i_0,j,k,n+1)+L_2(i_0+1,j,k,n+1)=0. \qquad (12)$$

The equation (8) means that the capillary pressure has to be continuous on the coupling boundary. Because this equality cannot be performed always in our model, we propose instead of using the equality (8) the minimization of the functional:

$$R(w^{n+1}_{i_0jk},w^{n+1}_{i_0+1,j,k})=0.5(p_l(w^{n+1}_{i_0jk})-p_{l+1}(w^{n+1}_{i_0+1,j,k}))^2 \qquad (13)$$

under condition (9). Here we have an optimization problem with constraints so it can be solved by Lagrange method [7]. Besides of the restriction (8) we use now additional conditions:

$$w^{n+1}_{i_0jk}\geq 0, \qquad (14)$$

$$w^{n+1}_{i_0+1,j,k} \geq 0. \tag{15}$$

Corresponding to the Lagrange method, we can minimize instead of (13) the functional

$$F(w^{n+1}_{i_0jk}, w^{n+1}_{i_0+1,j,k}) = R(w^{n+1}_{i_0jk}, w^{n+1}_{i_0+1,j,k}) + \lambda(L_1(i_0, j, k, n+1)$$

$$+L_2(i_0+1, j, k, n+1)) - \alpha_1 w^{n+1}_{i_0jk} - \alpha_2 w^{n+1}_{i_0+1,j,k} \tag{16}$$

without restrictions. Here λ, α_1 and α_2 are additional parameters.

We can see that L_1 does not depend on $w^{n+1}_{i_0+1,j,k}$ and L_2 does not depend on $w^{n+1}_{i_0jk}$. The most favourable conditions are as follows:

$$L_1(i_0, j, k, n+1) + L_2(i_0+1, j, k, n+1) = 0, \tag{17}$$

$$\frac{\partial R}{\partial w^{n+1}_{i_0jk}} + \lambda \frac{\partial L_1}{\partial w^{n+1}_{i_0jk}} - \alpha_2 = 0, \tag{18}$$

$$\frac{\partial R}{\partial w^{n+1}_{i_0+1jk}} + \lambda \frac{\partial L_2}{\partial w^{n+1}_{i_0+1jk}} - \alpha_1 = 0, \tag{19}$$

$$\alpha_1 w^{n+1}_{i_0+1jk} = 0, \tag{20}$$

$$\alpha_2 w^{n+1}_{i_0jk} = 0. \tag{21}$$

These conditions can be abbreviated:

$$L_1(i_0, j, k, n+1) + L_2(i_0+1, j, k, n+1) = 0, \tag{22}$$

$$(w^{n+1}_{i_0jk} \frac{\partial L_2}{\partial w^{n+1}_{i_0+1jk}} \cdot p_l'(w^{n+1}_{i_0jk})$$

$$+w^{n+1}_{i_0+1jk} \frac{\partial L_1}{\partial w^{n+1}_{i_0jk}} \cdot p'_{l+1}(w^{n+1}_{i_0+1jk})) \cdot (p_l(w^{n+1}_{i_0jk}) - p_{l+1}(w^{n+1}_{i_0+1jk})) = 0. \tag{23}$$

Here we have the modified coupling conditions. These equations must be solved at every point on the coupling boundary.

3. THE TREATMENT OF THE COUPLING CONDITIONS

We can linearize the conditions (22) and (23) by Newton method, but in this case the functions $p(w)$ and $K(w)$ have to be two times continuously differentiable. In our case only the first derivative of the function $p(w)$ is continuous.

We can give another interpretation of the conditions (22) and (23). From the equation (23) it follows, that two cases have to be analyzed. Either

$$p_l(w^{n+1}_{i_0jk}) = p_{l+1}(w^{n+1}_{i_0+1jk}), \tag{24}$$

or

$$w_{i_0jk}^{n+1}\frac{\partial L_2}{\partial w_{i_0+1jk}^{n+1}}\cdot p_l'(w_{i_0jk}^{n+1})+w_{i_0+1jk}^{n+1}\frac{\partial L_1}{\partial w_{i_0jk}^{n+1}}\ p_{l+1}'(w_{i_0+1jk}^{n+1})=0. \tag{25}$$

The formula (24) can be used only when both capillary pressure can be equal. For us it is easier to estimate, when the equality (8) is not possible. Further we assume the suction of the $(l+1)$–layer is greater than the suction of the l–layer (the opposite case can be handled similarly). In the framework of our model it means that

$$p_{l+1}(0)<p_l(0). \tag{26}$$

The equation (8) and (24) cannot be applied, when

$$p_{l+1}(w_{i_0+1jk}^{n+1})<p_l(0)$$

and the l–layer on the coupling boundary is dry:

$$w_{i_0jk}^{n+1}=0. \tag{27}$$

Now we can analyze this case more closely. If the capillary pressure in the $(l+1)$ layer is smaller than the minimal pressure in the l–layer, then the liquid must be sucked from the l layer. However this is impossible. Therefore, in this case the condition (25) has to be applied. From (25) and (27) it follows:

$$w_{i_0+1jk}^{n+1}\frac{\partial L_1}{\partial w_{i_0jk}^{n+1}}=0, \tag{28}$$

because

$$p_{l+1}'(w_{i_0+1jk}^{n+1})\neq 0.$$

Then $w_{i_0+1jk}^{n+1}$ can be estimated from the equation (25) where $w_{i_0jk=0}^{n+1}$ is fixed.

In the case, when the equation (8) can be applied, the equation (22) and the condition (24) can be handled together. For $w_{i_0+1jk}^{n+1}$ and $w_{i_0jk=0}^{n+1}$ then we have two separate equations:

$$c_{i_0jk}\ Y_{i_0jk}+c_{i_0+1jk}\ Y_{i_0+1jk}+s_{i_0jk}\ Y_{i_0-1jk}+o_{i_0jk}\ Y_{i_0j+1k}+v_{i_0jk}\ Y_{i_0j-1k}$$

$$+u_{i_0jk}\ Y_{i_0jk+1}+d_{i_0jk}\ Y_{i_0jk-1}+n_{i_0+1jk}\ Y_{i_0+1jk}+o_{i_0+1jk}\ Y_{i_0+1j+1k}$$

$$+v_{i_0+1jk}\ Y_{i_0+1j-1k}+u_{i_0+1jk}\ Y_{i_0+1jk+1}+d_{i_0+1jk}\ Y_{i_0+1jk-1}$$

$$+L_1(i_0,j,k,n+1)+L_2(i_0+1,j,k,n+1)=0 \tag{29}$$

and

$$p_l'(w_{i_0jk}^{n+1})\cdot Y_{i_0jk}-p_{l+1}'(w_{i_0+1jk}^{n+1})\cdot Y_{i_0+1jk}=-(p_l(w_{i_0jk}^{n+1})-p_{l+1}(w_{i_0+1jk}^{n+1})). \tag{30}$$

Here is $Y_{ijk}=w_{ijk}^{n+1}-w_{ijk}^{n}$ and c_{ijk}, s_{ijk}, o_{ijk}, v_{ijk}, u_{ijk}, n_{ijk}, d_{ijk} are coefficients of the linearized equation (22).

These are the formulas for the interior points of the coupling boundary. Algorithmically these formulas can be realized in the following way. On every Newton iteration on the coupling boundary the condition

$$p_{l+1}(w_{i_0+1jk}^{n+1})<p_l(0) \tag{31}$$

must be tested. When it is true, then we put

$$\tilde{w}_{i_0jk}^{n+1}=0, \qquad Y_{i_0jk}=0$$

and Y_{i_0+1jk} have to be estimated from the equation (29).

In the other case the equations (29) and (30) must be solved. In such a way we can get the linearized system of equations. The matrix of the system is not symmetrical. The linear system was solved with the BiCGSTAB method. The linearization must be done at every time step. The calculations show that the restriction for the time step is connected with the nonlinearity of the system.

Another iteration method can be constructed by using of the symmetrical part of the Jacobi matrix. In this case the linear system can be solved by cg–methods, but this procedure leads to a reduction of the time–step.

4. NUMERICAL EXAMPLE

We solve the problem (6)–(9) in the case of three layers. Every material is described with 6 parameters. The first layer has the best permeability and the worst suction properties. The third layer has exactly opposite properties. The parameters of the middle layer lay between the parameters of the first and the third layer. We choose the parameters in such a way that $p_1(0)$, $p_2(0)$ and $p_3(0)$ are different. Then it follows that the coupling condition (8) can not be realized exactly and here the functional (13) must be minimized.

In the example the liquid source on the top layer works two times: at $0<T<15$ and $50<T<70$. The calculation is done until the 200–th second.

In Fig. 1 is shown the distribution of the humidity on the vertical section $y=0$ by $T=15, 47.5, 70$ and 200 seconds. We can see the liquid transport in all three layers. Until $T=15$ the liquid was poured on the top surface and

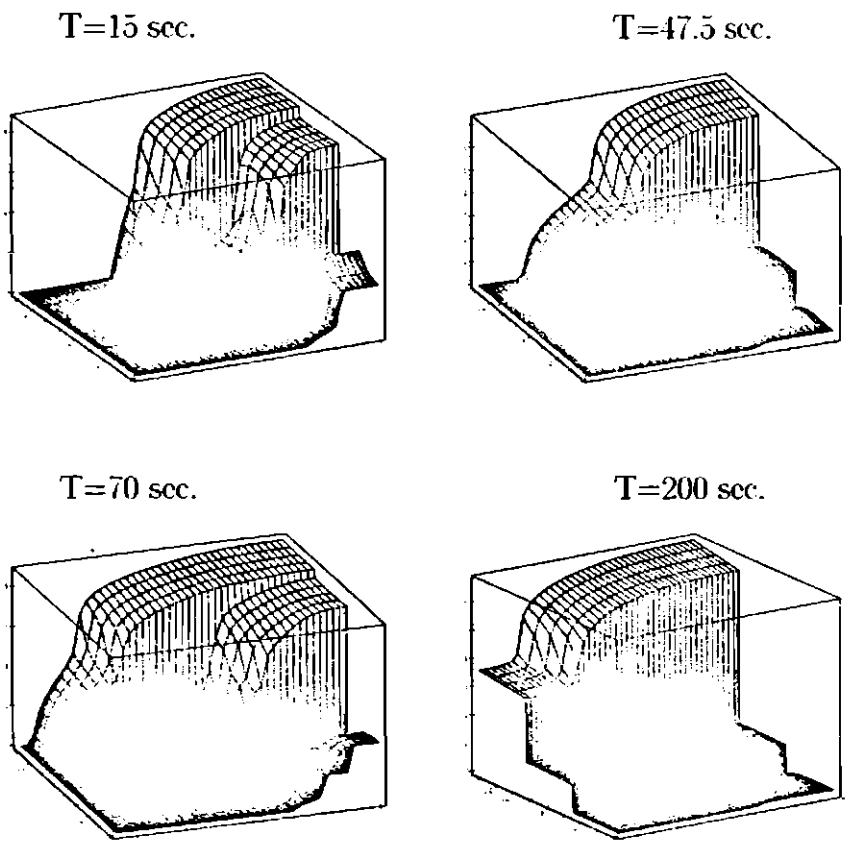

Figure 1. The humidity distribution in the vertical section $y = 0$ (the first layer is on the front)

there can be observed a nonzero saturation at every layer. At $T = 17.5$ the liquid source does not work and the saturation of the first layer goes down. Because of the small capillary suction in the first layer the wet part is localized here. At $T = 70$ the liquid source was working the second time and the first layer is wet again. After 200 seconds the most liquid is in the third layer and the first layer is relatively dry.

5. CONCLUSIONS

A modification of the coupling condition for the capillary pressure is developed, which allows to simulate unsaturated flows in porous materials with

significantly different suction properties. The condition is implemented in the 3D code for simulations of the liquid transport in multilayered nonwoven–layers.

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SAISTĪBAS NOSACĪJUMU REALIZĀCIJA DAUDZSLĀŅU TEKSTILMATERIĀLAM

AIVARS ZEMĪTIS

Daudzas praktiskas problēmas ir saistītas ar šķidruma transportu daudzslāņu tekstilmateriālos. Tiek analizēta situācija, kad šķidruma transports notiek uz kapilārā spiediena pamata. Aplūkotajā gadījumā būtiska uzmanība jāvelta modelim un saistības nosacījumiem.