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В сборник включены результаты научных исследований, полученные сотрудниками и аспирантами математических кафедр Латвийского университета, института математики и информатики Латвийского университета, а также работы специалистов других вузов и учреждений, сотрудничающих с математиками Латвийского университета. Большинство из опубликованных в сборнике результатов получены за период 1991 - 1993 гг. Включенные в сборник рукописи, как правило, не рецензируются.

The proceedings contain the results of the investigations made in 1991 - 1993 by lecturers, research workers and post-graduate students of the Department of Mathematics and the Institute of Mathematics and Computer Sciences of the Latvian University. Besides the proceedings reveal the results of the investigations made by some specialists from other universities and institutes collaborating with mathematicians of Latvian University. The works included in the proceedings generally are not reviewed by referees.

SATURA RĀDĪTĀJS

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AN OVERVIEW OF THREE WORKS OF E.GRINBERGS IN GRAPH THEORY AND COMBINATORICS

D.Zeps, J.Dambitis

Annotation. In the article two special graph constructions have been considered. The first one gives counterexample to A.Adam hypothesis, the second how to build three - connected graphs with exactly one Hamiltonian cycle. The third work shows an attempt to number types of subgraphs in the complete graph. AMS SC 05CXX

Examples of non-Adamian multigraphs.

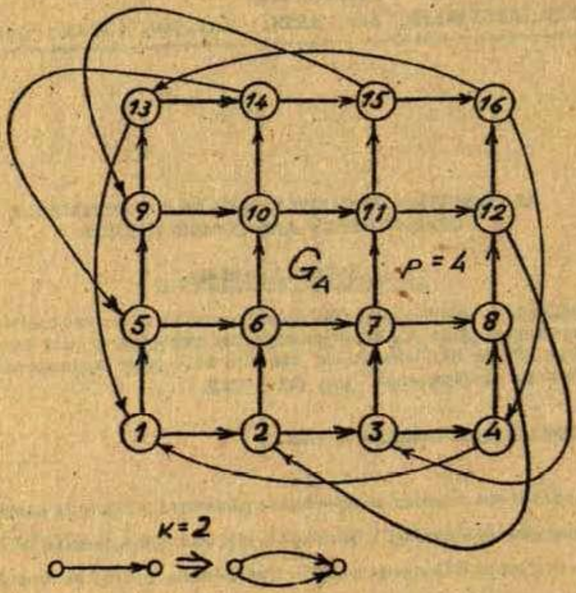
Examples of non-Adamian multigraphs at a parameter $k \geq 2$ (k is the number of parallel directed edges) were developed by E.Grinbergs in mid 70 s. The manuscript of this work was found in the archives of E.Grinbergs in 1985, after his death. In 1987 we have published this manuscript in Russiar (see Латв. матем. ежегодник, вып. 31).

It was A.Adams who first formulated hypothesis that in each oriented finite graph with oriented cycles there exists a directed edge which, when reoriented produces a graph with a smaller number of oriented cycles than there are in the original graph.

We will use c_i to denote the number of oriented cycles with a length i, and d_j for the number of bypasses with a length j for the directed edge (x. y) i.e. the number of oriented cycles for the directed edge (y, x) with a length j. If each edge is replaced by k parallel directed edges, the number of oriented cycles, when reversing arbitrary edge, grows provided

$$k-1+\sum_j k^j d_j - \sum_i k^{i-1} c_i > 0.$$

Here, E.Grinbergs defines a special class of oriented graphs G_p whose vertices are intersections of straight lines forming from $p \geq 4$ horizontal and p vertical lines. Figure 1 presents graph G_4 (p=4). Graph G_p has the so-called basic directed edges, it has been proved that only basic directed edges must define c_i and d_j.



The set of basic directed edges is formed by directed edges

$$\left\{(a, a+1)\ ;\ 1 \le a \le \left[\frac{p+1}{2}\right]\right\}$$

and following theorem is proved:

In graph G_p at $p \geq 2$ the basic directed edge $(a, a+1)$ has

$$c_p = d_{p+1} = \binom{p-1}{a-1},$$

$$c_{2p} = \binom{2p-1}{p-1} - \binom{p-1}{a-1}2^{p-1},$$

$$d_{2p+1} = \binom{2p-1}{p-2} - \binom{p-1}{a-1}2^{p-1} + \frac{1}{p}\binom{p}{a}\binom{p}{a-1}.$$

all the right side values of the equations are larger than zero at $p \geq 4$.

So, in graph G_p replacing each directed edge (a, b) by k=2 parallel directed edges (e_{i1} = (a, b), e_{i2} = (a, b), i1 ≠ i2) there is formed a graph which is an example of non-Adamian multigraph.

When analysing the relation

$$f_p = \frac{c_{2p}}{d_{2p+1}}$$

E.Grinbergs arrives at a conclusion that "not under taking special studies one cannot reject the possibility that at large p (p = 100, f_p = 1.076), when f_p is close to 1, graph G_p changes into non-Adamian multigraph by replacing not all the directed edges by two parallel directed edges".

In this work description of the properties of graphs G_p as well as two G_p modifications to obtain other non-Adamian multigraphs is given.

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The number of decomposing coverings of complete graphs.
(unpublished)

Manuscript of this work of E.Grinbergs was discovered in 1990 although it has not been signed and dated. He started to read a course of lectures in combinatorics in the earlier 70s to the students of mathematics. In the september 1971 he read course of lectures to both the students and those interested in discrete mathematics. When using the notes of his students, one can see that in this course of lectures E.Grinbergs already developed the main proofs of the theorem, but it is likely that the final results were obtained in late 70 s. The manuscript also has been found in his work dating back to 1975 and 1976. The numerical results were certainly obtained by E.Grinbergs without the help of computer.

The author get interested in the problem of the number of k - trees (k > 1) in complete graph when evaluating the number of trees in arbitrary graph and generalizing this idea to the case of k - trees. The above research was conducted at the beginning of 70 s.

Here, E.Grinbergs defines the number of different structures in complete (directed and nondirected) graph. He also defines permissible decomposing covering as a subgraph of complete graph which contains all the vertices and whose nonempty connected components belong to the given nonintersecting (two different types have no common subgraphs) s types. Trees, cycles and connected graphs with one cycle and one hanging vertex (degree of the type equals one) can serve as types.

If $l_n(h)$ is the number of all connected coverings of type h (the number of h type graphs with n labelled vertices), the exponential generating function (h type basic function) is

$$L[h]=\sum_{n=1}^{\infty} l_n[h]\frac{x^n}{n!}$$

α_i is the number of covering of i type, but

$$\{\alpha_1, \alpha_2, ..., \alpha_s\} = \{\alpha_i\}$$

denotes a vector which is the signature of covering.

Number $k=\sum_{i=1}^{s}\alpha_i$ will be called the order of covering, but $l_n\{\alpha_i\}$ will be the number of all coverings with a given signature, while $L\{\alpha_i\}$ will be the exponential generating function $l_{n,k}$ is the number of all coverings with an order k

$$l_{n,k}=\sum l\{\alpha_i\}$$

$$L_k=\sum L\{\alpha_i\}$$

is the exponential generating function. $l_n=\sum_k l_{n,k}$ will denote the number of all coverings and $L=\sum_k L_k$ exponential generating function.

The following theorem has been proved:

If the basic functions L[h] (h=1,2,...,s) for the given s covering component types are known, the below equations are valid

$$L(\alpha_j) = \prod_{i=1}^{s} \frac{(L(i))^{\alpha_i}}{\alpha_i!},$$

$$L_k = \frac{1}{k!}\left(\sum_{i=1}^{s} (L(i))\right)^k,$$

$$L = \exp\left(\sum_{i=1}^{s} (L(i))\right).$$

A method has been presented on how to find the number of trees of complete nondirected graph with v hanging vertices by means of the theorem

n/v	2	3	4	5	6	7	n^{n-2}
3	3	0	0	0	0	0	3
4	12	4	0	0	0	0	16
5	60	60	5	0	0	0	125
6	360	720	210	6	0	0	1296
7	2520	8400	5250	630	7	0	16807
8	20160	100800	109200	30240	1736	8	262144

Vertex with a degree 0 has been considered as hanging vertex in k-tree. A method on how to determine the number of 2-trees with v hanging vertices has been given.

n/v	2	3	4	5	6	7	2-tree
2	1	0	0	0	0	0	1
3	0	3	0	0	0	0	3
4	0	12	3	0	0	0	15
5	0	60	50	0	0	0	110
6	0	360	630	90	0	0	1080
7	0	2520	7560	3150	147	0	13377
8	0	20160	92400	75600	12320	224	200704

If $T_{n,k}$ denotes the number of k-trees of n vertices of complete graph, the following equations are proved:

$$T_{n,k}=n^{n-2k-1}\frac{n!}{k!}2^{-k}\sum_{j=0}^{k}(-1)^{j}(k+j)\binom{k}{j}\frac{(2n)^{k-j}}{(n-k-j)!}$$

$$T_{n,k}=\frac{n^{n-2k}}{2^{k-1}}\binom{n-1}{k-1}P_k(n).$$

where $P_k(n)$ is polynomial degree k-1 with coefficients which are whole numbers($T_{n,1}=n^{n-2}$).

Table presents the number of k-trees

n/k	1	2	3	4	5	6
2	1	1	0	0	0	0
3	3	3	1	0	0	0
4	16	15	6	1	0	0
5	125	110	45	10	1	0
6	1296	1080	435	105	15	1
7	16807	13377	5250	1295	210	21
8	262144	200704	76608	18865	3220	378

Three - connected graphs, with only one Hamiltonian cycle.

(unpublished)

Article [1] is prepared by D.Zeps from several peeces of manuscript found in the archives of E.Grinbergs.

This problem of the existence of three-connected graph with exactly one Hamiltonian cycle (or 1-H graph) E.Grinbergs called Zeps problem, because D.Zeps in 1978 asked E.Grinbergs to give him some constructions for non trivial (3-connected if possible) graphs with exactly one Hamiltonian cycle in order to check on the computer his programm for the finding of Hamiltonian cycle.

In 1979, E.Grinbergs found such three - connected graphs giving also the construction how to build arbitrary large graphs of the class.

On a combinatorical meeting in Moscow (1980.) it was told about this E.Grinbergs result, putting the questions about the minimal such graph and about the existence of a planar such graph. It is interesting to mark, that L.S.Melnikow (Novosibirsk) on this same meeting told that he had been tried early to find a triangulation with exactly one Hamiltonian cycle. In 1988 J.Kratochvil (Praque) and D.Zeps proved that such graphs don't exist [2].

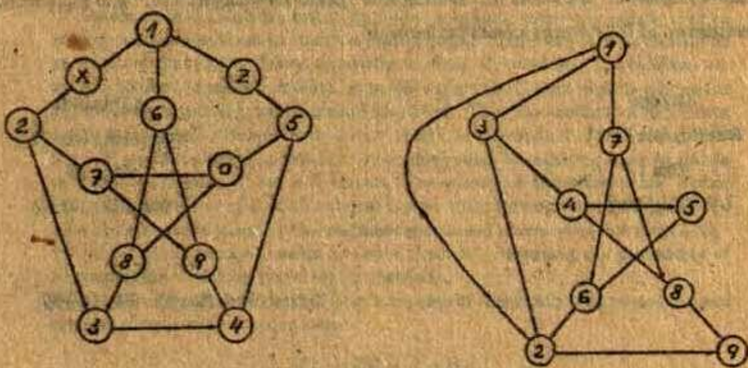
Let graph G contain s-triple <x,y,z>, if there exist three veritices x, y and z that

1) there is only one Hamiltonian path between vertices x and y,

2) there is no Hamiltonian path between vertices x and z, 3) graph is either three - connected or it becomes three - connected when adding a new vertex t and edges [t,x], [t,y] and [t,z].

Taking two graphs with s-triples <x,y,z> and <x',y',z'>, we obtain 1-H graph if edges [x,y'], [y,x'], [z,z'] are added.

In every 1-H graph it is easy to find s-triples: as x let be taken arbitrary vertex, as y and z - such vertices that the edge [x,y] goes into Hamiltonian cycle but the edge [x,z] doesn't. This gives way to construct arbitrary large 1-H graphs.



<x, 3, z>   <1,9,5>

Fig.2. Shows two modifications of Petersen graph with s-triples.
Using that with 9 vertices we get the minimal known 1-H graph up to now.

1. Э.Я.Гринберг. Трехсвязные графы с единственным гамильтоновым циклом. РФАП Латвии, Рига, 1986. (E.Grinbergs. Three-connected graphs with exactly one Hamiltonian cycle. Fund of programms, Riga, 1986.)

2. J.Kratochvil, D.Zeps. On the Number of Hamiltonian Cycles in Triangulations. J.Graph Theory 12 (1988), 191-194.

Д.Зепс, Я.Дамбитис. Обзор трёх работ Э.Гринберга по теории графов и комбинаторике.

Аннотация. В работе рассматривается две специальные конструкции графов. Первая конструкция опровергает гипотезу А.Адама, вторая позволяет строить графы с одним Гамильтоновым циклом. В третей работе дана теорема, с помощью которой перечислены разные подграфы полного графа.

D.Zeps, J.Dambītis. E.Grinberga trīs rakstu grafu teorijā un kombinatorikā apskats.

Anotācija. Rakstā aplūkotas divas speciālas grafu konstrukcijas. Pirmā konstrukcija izveido grafus A.Adama hipotēzes pretpiemērus, otrā - grafus ar vienu Hamiltona ciklu. Trešajā darbā formulēta teorēma, ar kuras palīdzību pārskaitīti pilna grafa dažādie apakšgrafi.

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ON THE PROBABILISTIC BASIS OF A COMBINATORIAL IDENTITY

J. Lapiņš A. Lorencs

Abstract

Some combinatorial identities are proved. The main result is obtained by comparing two different representations of the probability of a suitable event.

AMS Subject Classification 05A19.

Combinatorial identities involving finite or infinite sums are widely used in solving various problems in discrete mathematics. They are also applicable to problems of probability theory, especially to those of calculating probabilities according to their classical definition. A number of methods have been developed for the purpose of proving combinatorial identities. Thus the method of recurrence relations, the method based on reciprocal relations,the method of generating functions [1] and the general approach to evaluating sums by reducing them to simple or multiple integrals [2] are well known. The present authors had came across interesting identities of a combinatorial nature when solving statistical problems. Attempts to prove them by the methods mentioned above were not successful. But it turned out to be possible to prove them by computing the probability of an appropriate event by two different methods.

Let $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$ be independent identically distributed normal random vectors with expectations

$$EX = EY = \mu = (0,0)$$

and with variance matrices

$$VarX = VarY = D = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \quad |\rho| < 1.$$

Let us consider a random vector $Z = Y - X$. Then

$$EZ = EY - EX = \mu - \mu = (0,0),$$

$$VarZ = VarY + VarX = 2D = \begin{pmatrix} 2 & 2\rho \\ 2\rho & 2 \end{pmatrix}$$

and Z also is a normal random vector. We can calculate the probability p of the event $\{X_1 < Y_1, X_2 < Y_2\}$ as follows:

$$p = P\{X_1 < Y_1, X_2 < Y_2\} = P\{Z_1 > 0, Z_2 > 0\} =$$

$$= \int_0^{+\infty} dz_1 \int_0^{+\infty} \frac{1}{4\pi\sqrt{1-\rho^2}} \exp\left(-\frac{z_1^2 - 2\rho z_1 z_2 + z_2^2}{4(1-\rho^2)}\right) dz_2.$$

Using a substitution $z_1 = r\cos\varphi$, $z_2 = r\sin\varphi$

$$p = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{+\infty} \frac{1}{4\pi\sqrt{1-\rho^2}} \exp\left(-\frac{r^2(1-\rho\sin\varphi)}{4(1-\rho^2)}\right) r dr =$$

$$= \frac{\sqrt{1-\rho^2}}{2\pi} \int_0^{\frac{\pi}{2}} \frac{d\varphi}{1-\rho\sin 2\varphi} = \frac{1}{4} + \frac{1}{2\pi} \operatorname{arctg} \frac{\rho}{\sqrt{1-\rho^2}} = \frac{1}{4} + \frac{1}{2\pi} \arcsin \rho.$$

For $|\rho| < 1$ we have

$$\arcsin \rho = \sum_{t=0}^{\infty} \frac{(2t-1)!!}{(2t)!!\,(2t+1)} \rho^{2t+1}$$

(here and below

$$(2t)!! = 2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2t-2) \cdot 2t,$$

$$(2t+1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2t-1)(2t+1),$$

$$(-1)!! = 0!! = 1!! = 1,$$

$t \geq 1$) and

$$\frac{1}{1-\rho^2} = \sum_{s=0}^{\infty} \rho^{2s},$$

therefore for values of ρ, $|\rho| < 1$, the probability p can be represented in the following form

$$p = \frac{1}{4} + \frac{1-\rho^2}{2\pi} \frac{\arcsin \rho}{1-\rho^2} = \frac{1}{4} + \frac{1-\rho^2}{2\pi} \sum_{t=0}^{\infty} \sum_{s=0}^{\infty} \frac{(2t-1)!!}{(2t)!!\,(2t+1)} \rho^{2t+2s+1} =$$

$$= \frac{1}{4} + \frac{1-\rho^2}{2\pi} \sum_{j=0}^{\infty} \rho^{2j+1} \sum_{t=0}^{j} \frac{(2t-1)!!}{(2t)!!\,(2t+1)} \qquad (1)$$

Now we shall calculate the probability p in a different way, but at first let us consider some simple combinatorial identities. The first of them is given in [1] and is as follows: $\forall k \in N$

$$\sum_{j=0}^{k} C_{2j}^{j} C_{2k-2j}^{k-j} = 4^{k} \qquad (2)$$

Proof. For the values of x, $|x| < \frac{1}{4}$,

$$(1-4x)^{-\frac{1}{2}} = \sum_{j=0}^{\infty} \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(\frac{1}{2}-j\right)}{j!}(-1)^{j} 4^{j} x^{j} =$$

$$= \sum_{j=0}^{\infty} \frac{(2j-1)!!}{j!} 2^{j} x^{j} = \sum_{j=0}^{\infty} C_{2j}^{j} x^{j},$$

$$(1-4x)^{-1} = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} C_{2j}^{j} C_{2i}^{i} x^{j+i} = \sum_{k=0}^{\infty} x^{k} \sum_{j=0}^{k} C_{2j}^{j} C_{2k-2j}^{k-j},$$

and also

$$(1-4x)^{-1} = \sum_{k=0}^{\infty} 4^{k} x^{k},$$

therefore really $\forall k \in N$

$$\sum_{j=0}^{k} C_{2j}^{j} C_{2k-2j}^{k-j} = 4^{k}.$$

Let us denote by $S_m(k)$ the sum

$$\sum_{j=0}^{k} \frac{(2m+2j-1)!!\,(2m-2j+1)}{2^{j}\,(2j)!!}.$$

$\forall m, k \in N$

$$S_m(k) = \frac{(2m+2k+1)!!}{2^{k}(2k)!!} \qquad (3)$$

Proof. For $k = 0$ we have

$$S_m(0) = \sum_{j=0}^{0} \frac{(2m+2j-1)!!\,(2m-2j+1)}{2^{j}\,(2j)!!} = (2m+1)!! = \frac{(2m+2\cdot 0+1)!!}{2^{0}(2\cdot 0)!!}.$$

Let us suppose that for some $r \in N$ identity (3) fulfils for values of k, $k \leq r$, and consider the sum $S_m(r+1)$. According to the assumption

$$S_m(r) = \frac{(2m+2r+1)!!}{2^{r}(2r)!!},$$

$$S_m(r+1) = S_m(r) + \frac{(2m+2r+1)!!\,(2m-2r-1)}{2^{r+1}(2r+2)!!} =$$

$$= \frac{(2m+2r+1)!!}{2^{r}(2r)!!} + \frac{(2m+2r+1)!!\,(2m-2r-1)}{2^{r+1}(2r+2)!!} = \frac{(2m+2r+3)!!}{2^{r+1}(2r+2)!!}.$$

Hence we can conclude that identity (3) is true for every $k \in N$.
Further we shall use the following denotations:

$$W_t = \sum_{s=0}^{t} \frac{(2s-1)!!}{2^s (2s)!!} \quad ,t \in N,$$

$$V_{t,j} = \sum_{s=t+1}^{j} \frac{(2s-1)!!}{2^s (2s-2t-1)!!} \quad ,j,t \in N, j > t, V_{t,t} = 0,$$

$$U_{t,j} = \sum_{s=0}^{t} \frac{(2j-2t+2s-1)!!}{2^{j-t+s} (2s)!!} \quad ,j,t \in N, j \geq t.$$

We will show by means of mathematical induction that the quantities W_t, $V_{t,j}$ and $U_{t,j}$ satisfy the following identity: $\forall t\ j \in N,\ j \geq t$,

$$\frac{1}{(2t)!!} V_{t,j} = W_t - \frac{1}{(2j-2t-1)!!} U_{t,j}. \qquad (4)$$

Proof. For $j = t$ we have

$$\frac{1}{(2t)!!} V_{t,t} = 0 = \sum_{s=0}^{t} \frac{(2s-1)!!}{2^s (2s)!!} - \sum_{s=0}^{t} \frac{(2s-1)!!}{2^s (2s)!!} = W_t - \frac{1}{(2t-2t-1)!!} U_{t,t}.$$

Let us now assume that identity (4) is true for every j, $j \leq m$, and consider the following expression

$$\left(W_t - \frac{1}{(2m-2t+1)!!} U_{t,m+1}\right) - \left(W_t - \frac{1}{(2m-2t-1)!!} U_{t,m}\right) =$$

$$= \frac{1}{2^{m+1-t} (2m-2t+1)!!} \sum_{s=0}^{t} \frac{(2m-2t+2s-1)!!\,(2m-2t-2s+1)}{2^s (2s)!!} =$$

$$= \frac{(2m+1)!!}{2^{m+1} (2m-2t+1)!!\,(2t)!!}.$$

Here the last equality follows from identity (3). According to our assumption

$$W_t - \frac{1}{(2m-2t-1)!!} U_{t,m} = \frac{1}{(2t)!!} V_{t,m} \ ,$$

therefore

$$W_t - \frac{1}{(2m-2t+1)!!} U_{t,m+1} =$$

$$= \frac{1}{(2t)!!} \sum_{s=t+1}^{m} \frac{(2s-1)!!}{2^s (2s-2t-1)!!} + \frac{(2m+1)!!}{2^{m+1} (2m-2t+1)!!\,(2t)!!} = \frac{1}{(2t)!!} V_{t,m+1}.$$

Thus the identity (4) is true for every $j,t \in N$, $j \geq t$. Now we start to express the probability of the event $\{X_1 < Y_1, X_2 < Y_2\}$ in a different form.

$$p = P\{X_1 < Y_1,\ X_2 < Y_2\} = \int_{-\infty}^{+\infty} dy_1 \int_{-\infty}^{y_1} dx_1 \int_{-\infty}^{+\infty} dy_2 \int_{-\infty}^{y_2} \frac{1-\rho^2}{4\pi^2} \times$$

$$\times \exp\left(-\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2} - \frac{y_1^2 - 2\rho y_1 y_2 + y_2^2}{2}\right) dx_2 =$$

$$= \frac{1-\rho^2}{4\pi^2} \int_{-\infty}^{+\infty} dy_1 \int_{-\infty}^{y_1} dx_1 \int_{-\infty}^{+\infty} dy_2 \int_{-\infty}^{y_2} \exp(-\frac{x_1^2 + x_2^2 + y_1^2 + y_2^2}{2}) \times$$

$$\times \left[\sum_{j=0}^{n} \frac{\rho^j (x_1 x_2 + y_1 y_2)^j}{j!} + R_n(\rho, x_1, x_2, y_1, y_2)\right] dx_2 =$$

$$= \frac{1-\rho^2}{4\pi^2} \sum_{j=0}^{n} \frac{\rho^j}{j!} \int_{-\infty}^{+\infty} dy_1 \int_{-\infty}^{y_1} dx_1 \int_{-\infty}^{+\infty} dy_2 \int_{-\infty}^{y_2} (x_1 x_2 + y_1 y_2)^j \times$$

$$\times \exp(-\frac{x_1^2 + x_2^2 + y_1^2 + y_2^2}{2}) dx_2 + R_n(\rho),$$

where

$$R_n(\rho, x_1, x_2, y_1, y_2) =$$

$$= \frac{\rho^{n+1}}{(n+1)!} (x_1 x_2 + y_1 y_2)^{n+1} \exp(\theta \rho (x_1 x_2 + y_1 y_2)), \quad 0 < \theta < 1,$$

$$R_n(\rho) = \int_{-\infty}^{+\infty} dy_1 \int_{-\infty}^{y_1} dx_1 \int_{-\infty}^{+\infty} dy_2 \int_{-\infty}^{y_2} \frac{1-\rho^2}{4\pi^2} \times$$

$$\times \exp(-\frac{x_1^2 + x_2^2 + y_1^2 + y_2^2}{2}) \cdot R_n(\rho, x_1, x_2, y_1, y_2) \, dx_2.$$

From the inequality

$$|x_1 x_2| \le \frac{x_1^2 + x_2^2}{2}$$

we derive the following estimation of $R_n(\rho, x_1, x_2, y_1, y_2)$:

$$|R_n(\rho, x_1, x_2, y_1, y_2)| \le$$

$$\le \frac{\rho^{n+1}}{(n+1)!} \left(\frac{x_1^2 + x_2^2 + y_1^2 + y_2^2}{2}\right)^{n+1} \exp\left(|\rho| \frac{x_1^2 + x_2^2 + y_1^2 + y_2^2}{2}\right)$$

and thus

$$|R_n(\rho)| \le \int_{-\infty}^{+\infty} dy_1 \int_{-\infty}^{+\infty} dx_1 \int_{-\infty}^{+\infty} dy_2 \int_{-\infty}^{+\infty} \frac{1-\rho^2}{4\pi^2} \frac{|\rho|^{n+1}}{(n+1)!} \times$$

$$\times \left(\frac{x_1^2 + x_2^2 + y_1^2 + y_2^2}{2}\right)^{n+1} \exp\left(-(1-|\rho|)\frac{x_1^2 + x_2^2 + y_1^2 + y_2^2}{2}\right) dx_2.$$

Using the substitution $y_1 = r \cos \varphi_1$, $y_2 = r \sin \varphi_1 \cos \varphi_2$, $x_1 = r \sin \varphi_1 \sin \varphi_2 \cos \varphi_3$, $x_2 = r \sin \varphi_1 \sin \varphi_2 \sin \varphi_3$, we have in case $|\rho| < 1$

$$|R_n(\rho)| \le \frac{1-\rho^2}{4\pi^2}\frac{|\rho|^{n+1}}{(n+1)!}\int\limits_0^{+\infty}\frac{r^{2n+5}}{2^{n+1}}\times$$

$$\times\exp\Big(-\frac{r^2}{2}(1-|\rho|)\Big)dr\int\limits_0^{\pi}\sin^2\varphi_1 d\varphi_1\int\limits_0^{\pi}\sin\varphi_2 d\varphi_2\int\limits_0^{2\pi}d\varphi_3 =$$

$$= \frac{(1-\rho^2)|\rho|^{n+1}}{2^{n+2}(n+1)!}\int\limits_0^{+\infty} r^{2n+5}\exp\Big(-\frac{r^2}{2}(1-|\rho|)\Big)dr =$$

$$= \frac{(1-\rho^2)|\rho|^{n+1}(2n+4)!!}{2^{n+2}(n+1)!(1-|\rho|)^{n+3}} = (n+2)\frac{1-\rho^2}{(1-|\rho|)^2}\left(\frac{|\rho|}{1-|\rho|}\right)^{n+1}.$$

If $|\rho| < \frac{1}{2}$, then

$$\frac{|\rho|}{1-|\rho|} < 1, \quad (n+2)\frac{1-\rho^2}{(1-|\rho|)^2}\left(\frac{|\rho|}{1-|\rho|}\right)^{n+1} \underset{n\to\infty}{\longrightarrow} 0,$$

and therefore

$$|R_n(\rho)| \underset{n\to\infty}{\longrightarrow} 0.$$

Thus for values of ρ, $|\rho| < \frac{1}{2}$, we have

$$p = \frac{1-\rho^2}{4\pi^2}\sum_{j=0}^{\infty}\frac{\rho^j}{j!}\int\limits_{-\infty}^{+\infty}dy_1\int\limits_{-\infty}^{y_1}dx_1\int\limits_{-\infty}^{+\infty}dy_2\int\limits_{-\infty}^{y_2}\sum_{t=0}^{j}C_j^t x_1^t x_2^t y_1^{j-t} y_2^{j-t}\times$$

$$\times\exp\Big(-\frac{x_1^2+x_2^2+y_1^2+y_2^2}{2}\Big)dx_2 = (1-\rho^2)\sum_{j=0}^{\infty}\rho^j\sum_{t=0}^{j}\frac{1}{t!(j-t)!}K^2_{t,j-t},$$

where

$$K_{t,s} = \int\limits_{-\infty}^{+\infty}y^t\varphi(y)\,dy\int\limits_{-\infty}^{y}x^s\varphi(x)\,dx, \quad \text{and} \quad \varphi(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right).$$

Direct calculations using the method of partial integration give us, that

$$K_{2t,2s} = \frac{(2t-1)!!(2s-1)!!}{2}, \qquad K_{2t+1,2s+1} = 0,$$

$$K_{2t,2s+1} = -\frac{(2s)!!}{2\sqrt{\pi}}\sum_{i=0}^{s}\frac{(2t+2i-1)!!}{2^{t+i}(2i)!!},$$

and

$$K_{2t+1,2s} = \frac{(2s-1)!!}{2\sqrt{\pi}}\left[\sum_{i=t+1}^{t+s}\frac{(2i-1)!!}{2^i(2i-2t-1)!!} + (2t)!!\sum_{i=0}^{t}\frac{(2i-1)!!}{2^i(2i)!!}\right] =$$

$$= \frac{(2s-1)!!}{2\sqrt{\pi}}[-V_{t,j+s} + (2t)!!W_t] = \frac{(2t)!!}{2\sqrt{\pi}}U_{t,j+s}.$$

Here the last equality follows from identity (4). Since

$$\sum_{t=0}^{j} \frac{1}{(2t)!\,(2j-2t)!} K^2_{2t,2j-2t} =$$

$$= \frac{1}{4}\sum_{t=0}^{j} \frac{(2t)!\,(2j-2t)!}{(2t)!!\,(2j-2t)!!} = 4^{-j-1}\sum_{t=0}^{j} C^t_{2t} C^{j-t}_{2j-2t} = \frac{1}{4},$$

$$\sum_{t=0}^{j} \frac{1}{(2t)!\,(2j-2t+1)!} K^2_{2t,2j-2t+1} = \frac{1}{4\pi}\sum_{t=0}^{j} \frac{((2t)!!)^2}{(2j-2t)!\,(2t+1)!} U^2_{t,j},$$

and

$$\sum_{t=0}^{j} \frac{1}{(2t+1)!\,(2j-2t)!} K^2_{2t+1,2j-2t} = \frac{1}{4\pi}\sum_{t=0}^{j} \frac{((2t)!!)^2}{(2t+1)!\,(2j-2t)!} U^2_{t,j},$$

further we have that for every ρ, $|\rho| < \frac{1}{2}$,

$$p = \frac{1}{4} + \left(1-\rho^2\right)\sum_{j=0}^{\infty} \rho^{2j+1} \sum_{t=0}^{j} \left[\frac{K^2_{2t,2j-2t+1}}{(2t)!\,(2j-2t+1)!} + \frac{K^2_{2t+1,2j-2t}}{(2t+1)!\,(2j-2t)!}\right] =$$

$$= \frac{1}{4} + \frac{1-\rho^2}{2\pi}\sum_{j=0}^{\infty} \rho^{2j+1} \sum_{t=0}^{j} \frac{((2t)!!)^2}{(2t+1)!\,(2j-2t)!} \left(\sum_{s=0}^{t} \frac{(2j-2t+2s-1)!!}{2^{j-t+s}(2s)!!}\right)^2. \qquad (5)$$

Comparing formulas (1) and (5) we obtain the main result of this article, namely the following identity $\forall j \in N$

$$\sum_{t=0}^{j} \frac{((2t)!!)^2}{(2t+1)!\,(2j-2t)!} \left(\sum_{s=0}^{t} \frac{(2j-2t+2s-1)!!}{2^{j-t+s}(2s)!!}\right)^2 = \sum_{t=0}^{j} \frac{(2t-1)!!}{(2t)!!\,(2t+1)}. \qquad (6)$$

It is very easy to transform identity (6) into another equivalent forms, for example, as follows: $\forall j \in N$

$$\sum_{t=0}^{j} C^{2t+1}_{2j+1} \left(\sum_{s=0}^{t} \frac{(2j-2t+2s-1)!!\,(2t)!!}{2^{j-t+s}(2s)!!}\right)^2 = \sum_{t=0}^{j} C^{2t+1}_{2j+1} (2j-2t)!\,((2t-1)!!)^2,$$

and $\forall j \in N$

$$\sum_{t=0}^{j} C^{2t}_{2j+1} \left(\sum_{s=0}^{j-t} \frac{(2t+2s-1)!!\,(2j-2t)!!}{2^{t+s}(2s)!!}\right)^2 = \sum_{t=0}^{j} C^{2t}_{2j+1} (2t)!\,((2j-2t-1)!!)^2.$$

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J.Lapiņš, A.Lorencs. Par kādas kombinatoriskas identitātes varbūtību - teorētisko pamatojumu.

Anotācija. Darbā pierādītas dažas kombinatoriskas identitātes. Centrālais rezultāts iegūts, izsakot divu dažādu izteiksmju formā piemērotā veidā izvēlēta notikuma varbūtību.

Я.Лапиньш, А.Лоренц. О теоретико - вероятностном обосновании одного комбинаторного тождества.

Аннотация. В работе доказаны некоторые комбинаторные тождества. Основной результат получен путем представления двумя различными способами вероятности подходящего события.

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TWO FIXED POINT THEOREMS IN A METRIC SPACE WITH CLOSURE OPERATOR

Inese Galiņa

SUMMARY. In this work we prove two fixed point theorems in a metric space with closure operator. Both theorems are theorems stating the existence of common fixed point for families of mappings. On the first case it is family of continuous mappings, that satisfying the generalized condition of R.Kannan's theorem ([1]), and on the second case it is commutative family of nonexpansive mappings with invariance property.
AMS Subject Classification 54H25 , 47H40

Convexity structure of an examined space or set is of a great importance in a fixed point theory. In a standardcase that means that only subsets of vector spaces are examined. But it is also possible to define convexity in a metric space. For example, one of case is to show in a W.Takahashi work [2], where X is a convex metric space with a distance d if exists such that a mapping W from $X*X*[0;1]$ to X (i.e. $W(x,y;\lambda)$ defined for all pairs $x,y\in X$ and λ $(0\le\lambda\le1)$) and valued in X satisfying

$$d(u,W(x,y;\lambda))\le\lambda d(u,x)+(1-\lambda)d(u,y) \text{ for all } u\in X.$$

A Banach space and each of its convex subsets are Takahashi convex metric space.

But it is not only way how to define convexity in a metric space. Convexity structure in a metric space we can define also by making use of closure operators.

Further we act in a metric space X with a distance d. Let PX be the set of all subsets of X.

DEFINITION 1. A closure operator on X is a mapping $S:PX\to PX$ satisfying for each $A,B\in PX$ the following conditions :

1) $A\subset B\Rightarrow S(A)\subset S(B)$;
2) $A\subset S(A)$;
3) $S(S(A))=S(A)$.

<u>DEFINITION 2.</u> A closed operator S on X is said to be algebraic if for each $A \in PX$ and $x \in S(A)$ there exists a finite set $F \subset A$ such that $x \in S(F)$.

Let S be a closure operator on X. A subset A of X is said to be S-closed if $A=S(A)$. A space X is said to be S-compact if each centered system of S-closed subsets of X has a nonempty intersection. Note that intersection of S-closed subsets of X is S-closed. For more detailed applications of closure operators in fixed point theory see [3]. Note that approach of closure operators is more general as W.Takahashi convexity structure.

<u>1.GENERALIZATION OF R.KANNAN FIXED POINT THEOREM.</u>

First of all we prove this R.Kannan's theorem ([1]) in a metric space with closure operator.

THEOREM (R.Kannan, 1973). Let T be a continuous mapping of a closed convex bounded set K of a reflexive Banach space X into itself and let T have properties:

1) $\|Tx-Ty\| \le \frac{1}{2}(\|x-Tx\|+\|y-Ty\|)$, $x,y \in K$;

2) on K if for every closed convex subset F of K, mapped into itself by T and containing more than one elements, there exists $x \in F$ such that
$\|x-Tx\| < \sup\{\|y-Ty\| \mid y \in F\}$.

Then T has a unique fixed point in K.

In a case of one mapping this theorem is generalized by A.Liepin's ([3]) in subsymmetrical topological space with closure operator. We are interesting on common fixed points for family of mappings.

<u>THEOREM 1.</u> Suppose (X,d) is a metric space, S is an algebraic closure operator on X, $\overline{S(A)}=S(\overline{S(A)})=:S'(A)$ for each $A \in PX$ and X is S'-compact. Let each closed ball $B(x,r)$ ($x \in X$, $r \in \mathbb{R}_{++}$) be S-closed. Let F be a family of continuous selfmaps of X satisfying the following conditions:

$$1)\ \forall f,g,h \in F\ \forall x,y \in X\ \exists a \in]0;1[:$$
$$d(g(x),h(y)) \le a\,d(x,f(x))+(1-a)\,d(y,f(y));$$
$$2)\ \forall x \in X\ (\exists v \in F: v(x) \ne x)\ \exists y \in A(x):$$
$$\sup\{d(y,f(y)) \mid f \in F\} < \sup\{\sup\{d(z,f(z)) \mid z \in A(x)\} \mid f \in F\},$$

where $A(x) := \bigcap\{A \in PX \mid x \in A \;\&\; A=S'(A) \;\&\; \forall f \in F: f(A) \subset A\}$.

Then F has a unique common fixed point.

• Proof.

Using Zorn's Axiom and S'-compactness of X we conclude that there exists a minimal nonempty S'-closed and invariant under F subset M of X.

Let $a \in M$, and there exists $f \in F$ such that $f(a) \neq a$. Since $A(a) \subset M$, minimality of M implies that $M = A(a)$. By 2) there exists a point $a_1 \in A(a) = M$ such that:

$r := \sup\{d(a_1, f(a_1)) \mid f \in F\} < \sup\{\sup\{d(z, f(z)) \mid z \in A(a)\} f \in F\}$.

Consider the sets

$$A := \{x \in M \mid d(x, f(x)) \le r, \forall f \in F\} \text{ and}$$
$$A_1 := S(\bigcup\{f(A) \mid f \in F\}),$$

its are nonempty because $a_1 \in A$ and $f(a_1) \in A_1, \forall f \in F$.

• Since S is an algebraic closure operator then freely choosing $x \in A_1$ there exists a finite set $W \subset \bigcup\{f(A) \mid f \in F\}$ such that $x \in S(W)$.

Let $q := \sup\{\max\{d(f(x), y) \mid y \in W\} f \in F\}$, then $W \subset \bigcap\{B(f(x), q) \mid f \in F\}$. Since $\bigcap\{B(f(x), q) \mid f \in F\}$ is S-closed set as an intersection of S-closed sets, then $S(W) \subset \bigcap\{B(f(x), q) \mid f \in F\}$ and also then $x \in \bigcap\{B(f(x), q) \mid f \in F\}$, i.e., $d(x, f(x)) \le q, \forall f \in F$.

Choosing arbitrary $f \in F$. Then for freely nominate $\varepsilon \in \mathbb{R}_+$ there exists mapoings $g, h \in F$ such that:

$q - \varepsilon \le \max\{d(g(x), y) \mid y \in W\} = d(g(z), h(z))$, where $z \in A$.

Therefore $d(x, f(x)) \le q \le d(g(x), h(z)) + \varepsilon$. (1)

By 1), (1) and $z \in A$ it follows that

$d(x, f(x)) \le d(z, f(z)) + \frac{\varepsilon}{1-\alpha} \to_{\varepsilon \to 0} d(z, f(z)) \le r$ – it is valid for arbitrary mapping $f \in F$, consequently for others mappings of F we have analogical estimations, therefore $x \in A$ and then $A_1 \subset A$ and $f(x) \in A_1, \forall f \in F$. This implies that $f: A_1 \to A_1, \forall f \in F$.

Consider the set $A_2 := \overline{A_1} = \overline{S(\bigcup\{f(A) \mid f \in F\})}$. It is nonempty, it is S'-closed and by continuity of family F it is invariant under F. By minimality of M it follows that $M = A_2$. But $A_2 = \overline{A_1} \subset \overline{A} = A$ (f and d – continuities!) and

$\sup\{\sup\{d(x, f(x)) \mid x \in A\} f \in F\} = r < \sup\{\sup\{d(z, f(z)) \mid z \in M\} f \in F\}$ – contradiction is obtained. Therefore $f(a) = a, \forall f \in F$.

The uniqueness follows from 1): let be two fixed points $x_1, x_2 \in X$, then $\forall f, g, h \in F \exists \alpha \in]0;1[$:

$d(g(x_1), h(x_2)) = d(x_1, x_2) \le \alpha d(x_1, f(x_1)) + (1-\alpha) d(x_2, f(x_2)) =$
$= \alpha d(x_1, x_1) + (1-\alpha) d(x_2, x_2) = 0$. ▲

2. COMMON FIXED POINT FOR COMMUTATIVE FAMILY OF NONEXPANSIVE MAPPINGS WITH INVARIANCE PROPERTY.

Interesting on commutative family of nonexpansive mappings and its common fixed points Theorem of R.de Marr ([4]) w e l l known:

THEOREM (R.de Marr, 1963). Let F be a commutative family of nonexpansive mappings of a convex compact set K of a Banach space X into itself. Then F has a common fixed point.

Similar result is proved by M.R.Taskovic (1986,[5]) for commutative family of diameter-nonexpansive mappings.

DEFINITION 3. A mapping $f:E\to E$ (E is a Banach space) is said to be diameter-nonexpansive mapping if:

$$\|f(x)-f(y)\|\le\varphi(\sup\{\|x-z\| \mid z\in E\}),\forall x,y\in E,$$

where $\varphi:\mathbb{R}_{+}\to\mathbb{R}_{+}$ with property $\varphi(t)\le t,\forall t\in\mathbb{R}_{+}$.

In both cases essential invariance property of family is used.

DEFINITION 4. A family F of nonexpansive mappings f of K into itself is said to have invariance property in K if for any compact Takahashi convex subset $E\subset K$ such that $f(E)\subset E$ for each $f\in F$ there exists a compact subset $M\subset E$ such that $f(M)=M$ for each $f\in F$.

This Definition 4 is given in a work of W.Takahashi([2]) and the following theorem is proved:

THEOREM (W.Takahashi, 1970). Let K be a compact Takahashi convex metric space. If F is a family of nonexpansive mappings with invariance property in K, then the family F has a common fixed point.

In the same place he asserts that if F is a left amenable semigroup of nonexpansive mappings T of a compact Takahashi convex metric space into K, then the family F has invariance property in K. We note that our conception of S-closed set is general as conception of convex set in Takahashi convex metric space, therefore we prove similar theorem in a metric space with closure operator. But for this purpose we give two new definitions.

DEFINITION 5. A family F of mappings f of K into itself is said to have S-invariance property in K if for any S-compact and S-closed set $E\subset K$ such that $f(E)\subset E$ for each $f\in F$ there exists a S-compact subset $M \subset E$ such that $f(M)=M$ for each $f \in F$.

EXAMPLE.

$X=\mathbb{R}$, $d(x,y)=|x-y|$, S - operator of closed and convex hull. $F=\{f(x):=kx| \ x \in X,\ k \in]0;1]\}$. So that $f(E) \subset E$, then $E=[a;b], \forall a,b \in \mathbb{R}$ and $0 \in [a;b]$. In this case $M=\{0\}$.

DEFINITION 6. A S-closed set K of a metric space X with closure operator S has S-normal structure if for each S-closed subset H of K with $diamH \neq 0$ there exists a point u such that

$$diamH > \sup\{d(x,u) \,|\, x \in H\}.$$

Note that every convex compact set K of a Banach space X has s-normal structure if closure operator S is defined as closed and convex hull ([6]).

THEOREM 2. Suppose (X,d) is a S-compact metric space with closure operator S and X is a S-normal structure. Let each closed ball $B(x,r)$ ($x \in X$, $r \in \mathbb{R}_{+}$) be S-closed. If F is a family of nonexpansive mappings with S-invariance property in X, then the family F has a common fixed point.

PROOF.

Using Zorn's Axiom and S-compactness of X we conclude that there exists a minimal nonempty S-closed and invariant under F subset M of X.

Let $a \in M$, and there exists $f \in F$ such that $f(a) \neq a$. By S-invariance property in X of family F implies that exist S-compact set $M_1 \subset M$ such that $f(M_1)=M_1, \forall f \in F$. If $diamM_1 > 0$, then by S-normal structure of space X exist element $u \in S(M_1)-M_1$ such that:

$$r := \sup\{d(u,x) \,|\, x \in M_1\} < diamM_1 \le diamM.$$

Consider the set $M_0 := (\bigcap\{B(x,r) \,|\, x \in M_1\}) \cap M_1$. It is nonempty because $u \in M_0$, it is S-closed as an intersection of S-closed sets and S-compact. We prove that M_0 is invariant under F. We assume freely $z \in M_0$ and $f \in F$. and freely $x \in M_0 \subset M_1$ - since $f(M_1)=M_1$, then exist $w \in M_1$ that $x=f(w)$. Therefore $d(f(z),x)=d(f(z),f(w)) \le d(z,w) \le r$ (because $z \in M_0, w \in M_1$) (2)

These (2) can for all $z \in M_0$, all $f \in F$ and all $x \in M_0$, i.e., $f(M_0) \subset M_0, \forall f \in F$. By minimality of M obviously $M=M_0$, but $diamM_0 \le r < diamM_1 \le diamM$. The obtained contradiction completes the proof.

NOTE. If family of mappings is diameter-nonexpansive mapping, then assertion of Theorem 2 is true. In Proof then (2) motivation is following:

$$d(f(z),x)=d(f(z),f(w))\le\varphi(\sup\{d(z,u)\mid u\in M_1\})\le\sup\{d(z,u)\mid u\in M_1\}\le r.$$

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<u>И.Галина. Две теоремы о неподвижной точке в метрическом пространстве с оператором замыкания.</u>

<u>Аннотация.</u> В работе доказаны две теоремы о неподвижной точке в метрическом пространстве с оператором замыкания. В обоих теоремах рассматривается семейство отображении, которое в первом случае есть семейство непрерывных отображений, которое удовлетворяет обобщенному условию R.Kannan, и во втором случае - перестановочное семейство нерастягивающих отображений со свойством инвариантности.

УДК 517.98

<u>I.Galina. Divas nekustīgā punkta teorēmas metriskā telpā ar slēguma operatoru.</u>

<u>Anotācija.</u> Darbā pierādītas divas nekustīgā punkta teorēmas metriskā telpā ar slēguma operatoru. Abās teorēmās pierādīta kopīga nekustīgā punkta eksistence attēlojuma saimei. Pirmajā gadījumā tā ir nepārtrauktu attēlojumu saime, kas apmierina R.Kannana teorēmas vispārinātu nosacījumu, otrajā gadījumā tā ir neizstiepjošu attēlojumu komutatīva saime ar invariances īpašību.

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FIXED POINTS OF MAPPINGS ON NONCONVEX SETS

I.Kaprāne, A.Liepiņš

ABSTRACT. Existence of fixed points for nonexpansive mapping on compact metric stars is proved.
AMS Subject Clasification 54H25.

Let X be a metric space with a distance d.

DEFINITION 1 [1]. A mapping $W: X \times X \times [0,1] \longrightarrow X$ is said to be a convex structure on X if and only if for all $x,y \in [0,1]$
$d(z,W(x,y,t)) \leq t \cdot d(z,x)+(1-t)\cdot d(z,y)$, for all $z \in X$.

X together with a convex structure is called a Takahashi convex metric space.

DEFINITION 2. A nonempty subset A of a Takahashi convex metric space X is said to be a T-star if and only if:

1) A is compact;

2) there exists a point $c \in A$ (a star center of A) such that $W(x,ct) \in A$ for all $x \in A$, $t \in [0,1]$;

3) $d(W(x,c,t),W(y,c,t)) < d(x,y)$ for all x, $y \in A$ $(x \neq y)$, $t \in]0,1[$.

Let $f: X \to X$.

DEFINITION 3. A selfmap f of X is said to be nonexpansive if and only if $d(f(x),f(y)) \leq d(x,y)$ for all x, $y \in X$.

THEOREM. Let A be a T-star in a Takahashi convex metric space X, and let f be nonexpansive selfmap of A.

Then f has a fixed point.

PROOF. Let $\varepsilon \in R_+$, $d := \operatorname{diam} A$ and $t \in]0,1[$ be such that $(1-t)d < \varepsilon$.

Then $d(x,W(x,c,t)) \leq t \cdot d(x,x)+(1-t)\cdot d(x,c) =$
$=(1-t)\cdot d(x,c) \leq (1-t)\cdot d < \varepsilon$ for all $x \in A$.

For all $x,y \in A$ $(x \neq y)$ we have
$d(W(f(x),c,t),W(f(y),c,t)) < d(f(x),f(y)) \leq d(x,y)$ if $f(x) \neq f(y)$
and $d(W(f(x),c,t),W(f(y),c,t))=0 < d(x,y)$ if $f(x)=f(y)$.
By Edelstein's theorem there exists $a \in A$ such that $W(f(a),c,t)=a$.
Hence, $d(f(a),W(f(a),c,t))=d(f(a),a) < \varepsilon$.

We conclude that $\inf\{d(x,f(x)) | x \in A\}=0$. By compactness of A the result follows.

<u>DEFINITION 4.</u> X will be called a C-star if and only if:

1) X is compact;

2) there exists a continuous mapping $V: X \times [0,1] \longrightarrow X$ such that $V(x,0)=x$ for all $x \in X$ and $d(V(x,t),V(y,t)) < d(x,y)$ for all $x, y \in X$ $(x \neq y)$, $t \in]0,1[$.

<u>THEOREM.</u> Let X be a C-star and let f be nonexpansive selfmap of X.

Then f has a fixed point.

<u>PROOF.</u> For all $x \in X$ and $t \in [0,1]$ we define $T(x,t) := d(x,V(x,t))$. As continuous mapping on a compact space T is uniformaly continuous.

Let $\varepsilon \in R_+$. Hence, there exists $\delta \in R_+$ such that $|T(x,t)-T(x,0)| < \varepsilon$ for all $x \in X$ and $t \in [0,1]: t < \delta$.

Let $t \in [0,1]: t < \delta$. Then $\varepsilon > |T(x,t)-(x,0)| =$

$= |d(x,V(x,t))-d(x,V(x,0))| = |d(x,V(x,t))-d(x,x)| = d(x,V(x,t))$

for all $x \in X$.

For all $x, y \in X$ $(x \neq y)$ we have

$d(V(f(x),t),V(f(y),t)) < d(f(x),f(y) \leq d(x,y)$ if $f(x) \neq f(y)$ and $d(V(f(x),t),V(f(y),t)) = 0 < d(x,y)$ of $f(x)=f(y)$.

By Edelstein's theorem there exists $a \in X$ such that $V(f(a),t)=a$. Hence, $d(f(a),V(f(a),t)) = d(f(a),a) < \varepsilon$.

We conclude that $\inf \{d(x,f(x)) \mid x \in X\} = 0$. By compactness of X the result follows.

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<u>И.Капране, А.Лиепиньш. Неподвижные точки отображений на множествах которые не являются выпуклыми.</u>

<u>Аннотация.</u> Доказано существование неподвижных точек для нерастягивающих отображений на компактных метрических звездах.

УДК 517.98.

<u>I.Kaprāne, A.Liepiņš Attēlojumu kopās, kas nav izliektas, nekustīgie punkti.</u>

<u>Anotācija.</u> Pierādīta nekustīgo punktu eksistence neizstiepjošiem attēlojumiem kompaktās metriskās zvaigznēs.

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On orthogonal reducibility of generalized inverse.

Uldis Berķis

Summary. Theorem about sufficient conditions of the equivalence of orthogonal reducibility of a linear continuous operator and its generalized inverse is proved, an extension of the concept is used for closed, dense defined operator.
AMS subject classification 47A15, 15A21, 15A06 .

In investigation [3] the concept of orthogonally reducible linear continuous operators is studied in connection with iteration procedures. This study contains also criterion, in which the adjoint of an orthogonally reducible operator is orthogonally reducible.

We study the property of orthogonal reducibility of generalized inverse for an orthogonally reducible linear continuous operator in Hilbert space.

Let $H=(X,\mathbb{K},\langle\cdot,\cdot\rangle)$ be a Hilbert space over the field $\mathbb{K}$ of the real or complex numbers, LC(H) is the space of all linear continuous operators from H into H, the unit operator will be denoted as I, the domain of definition of A as D(A), the kernel as N(A) and the range as R(A).

Definition.[3] Operator $A\in LC(H)$ is called orthogonally reducible if $N(I-A)^{\perp}$ is invariant subspace for A.

Definition.[2] An operator $A^{+}:D(A^{+})\subset H \mapsto H$, which satisfies the conditions

$$AA^{+}A=A$$
$$A^{+}AA^{+}=A^{+}$$
$$(AA^{+})^{*}=AA^{+}$$
$$(A^{+}A)^{*}=A^{+}A$$

is called the generalized (Moore-Penrose) inverse operator for $A\in LC(H)$. According to classification, given in [5], an operator $A^r:D(A^r)\subset H\mapsto H$, which satisfies

$$AA^rA=A$$
$$A^rAA^r=A^r$$

is called the semi-inverse operator for $A\in LC(H)$.

Obviously each generalized inverse operator is a semi inverse operator. It is well known from the theory of generalized inverses in Hilbert spaces, that the generalized inverse has the uniqueness property, the semi-inverse for an operator $A\in LC(H)$ may be not unique[5], the generalized inverse is a dense defined closed operator with $D(A^+)=R(A)+R(A)^\perp$, it is continuous if and only if $R(A)$ is closed [4], in this case $D(A^+)=H$.

To study the property of orthogonal reducibility for generalized inverses, the following lemma is important.

Lemma. $A\in LC(H)$ is an orthogonally reducible operator, $y\in H$ satisfies $Ay\in N(I-A)^\perp$.
Then $y\in N(I-A)^\perp$.

Proof. Since $N(I-A)$ - closed s bspace of Hilbert space H, $y\in H$ can be written as

$$y=y_1+y_2,\ \text{where}\ y_1\in N(I-A),\ y_2\in N(I-A)^\perp. \qquad (1)$$

Since $Ay_1=y_1$ and $Ay_2\in N(I-A)^\perp$. for every $x\in N(I-A)$
$0=\langle x,Ay\rangle=\langle x,Ay_1\rangle+\langle x,Ay_2\rangle=\langle x,Ay_1\rangle=\langle x,y_1\rangle$ holds.
Hence $y_1\in N(I-A)$ and $y_1\perp N(I-A)$, which implies $y_1=0$. Therefore $y=y_2\in N(I-A)^\perp$. ∎

In the following theorem sufficient conditions for the equivalence of orthogonal reducibility of an operator and its semi-inverse are given.

Theorem 1. Let $A\in LC(H)$ be operator with closed range and $A^r\in LC(H)$ is a semi-inverse for A, A and A^r are commuting operators.
Then A is orthogonally reducible if and only if A^r is orthogonally reducible.

Proof. At first we show that under theorem assumptions $N(I-A)=N(I-A^r)$.

If $x\in H$ satisfies $Ax=x$, then from

$$A(A^rx)=AA^rx=AA^rAx=Ax=x \qquad (2)$$

$$A(A^r x)=AA^r x=A^r Ax=A^r x \qquad (3)$$

$A^r x=x$ holds.

If $A^r x=x$, then from

$$A^r(Ax)=A^r Ax=A^r AA^r x=A^r x=x \qquad (4)$$

$$A^r(Ax)=A^r Ax=AA^r x=Ax \qquad (5)$$

$Ax=x$ holds.

Since $N(I-A)=N(I-A^r)$, $N(I-A)^\perp=N(I-A^r)^\perp$ holds.

Let A be orthogonally reducible, $y\in N(I-A^r)^\perp=N(I-A)^\perp$, hence $Ay\in N(I-A)^\perp$. Since $Ay=AA^r Ay=AAA^r y$, with respect to Lemma, $AA^r y\in N(I-A)^\perp$ and, finally, $A^r y\in N(I-A)^\perp=N(I-A^r)^\perp$.

Let A^r be orthogonally reducible, $y\in N(I-A)^\perp=N(I-A^r)^\perp$, hence $A^r y\in N(I-A^r)^\perp$. Since $A^r y=A^r AA^r y=A^r A^r Ay$, with respect to Lemma, $A^r Ay\in N(I-A^r)^\perp$ and, finally, $Ay\in N(I-A^r)^\perp=N(I-A)^\perp$. ∎

Note that under theorem assumptions about $R(A)$ a generalized inverse for linear continuous orthogonally reducible operator is orthogonally reducible if A and A^+ are commuting operators. Criterion, in which this property holds, is given in [1].

The assumption $R(A)$ - closed in H, is very restrictive. For generalized inverses of operators with nonclosed range we can use the concept of orthogonal reducibility in the following sense:

<u>Theorem 2.</u> Let $A\in LC(H)$, A and A^+ are commuting in $D(A^+)$. A is orthogonally reducible if and only if

$$A^+(N(I-A^+)^\perp\cap D(A^+))\subset N(I-A^+)^\perp \qquad (6)$$

(The orthogonal complement always is assumed to be in H)

<u>Proof.</u> Since $N(I-A)\subset D(A^+)$, for $x\in H$ which satisfies $Ax=x$, (2) and (3) holds, thus $N(I-A)\subset N(I-A^+)$.

For $x\in D(A^+)$ which satisfy $A^+x=x$ (4), (5) holds. Therefore $N(I-A)=N(I-A^+)$ and $N(I-A)^\perp=N(I-A^+)^\perp$.

Let A be orthogonally reducible, $y\in D(A^+)$ and $y\in N(I-A^+)^\perp=N(I-A)^\perp$, hence $Ay\in N(I-A)^\perp$. According to proof of theorem 1, $A^+y\in N(I-A)^\perp=N(I-A^+)^\perp$ holds.

Let (6) holds, $y\in N(I-A)^\perp$. Since A^+-closed, $N(I-A^+)$ is closed in H, hence (1) with $y_1\in N(I-A^+)$ and $y_2\in N(I-A^+)^\perp$ holds, therefore $y\in D(A^+)$ with property $A^+y\in N(I-A^+)^\perp$ satisfies $y\in N(I-A^+)^\perp$. According to proof of theorem 1, for $y\in D(A^+)\cap N(I-A)^\perp$ we obtain $Ay\in N(I-A)^\perp$. Since A is continuous,

$A(cl(N(I-A)^{\perp}\cap D(A^{+})))\subset cl(A(N(I-A)^{\perp}\cap D(A^{+})))\subset N(I-A)^{\perp}$. Let P be the orthoprojektor onto $N(I-A)^{\perp}$, $P(cl(D(A^{+})))\subset cl(P(D(A^{+})))$, hence $P(D(A^{+}))$ is dense in $N(I-A)^{\perp}$. With respect to $R(I-P)\subset D(A^{+})$ ($I-P$ - orthoprojektor onto $N(I-A)$), $P(D(A^{+}))\subset D(A^{+})$ holds. Therefore $N(I-A)^{\perp}\subset cl(N(I-A)^{\perp}\cap D(A^{+}))$. ∎

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У. Беркис. Об ортогональной приводимости обобщенных обратных.

Аннотация. Рассматриваются некоторые достаточные условия для эквивалентности ортогональной приводимости линейного непрерывного оператора и его обобщенного обратного. Рассматривается возможное обобщение понятия ортогональной приводимости для плотно определенного, замкнутого оператора.

U.Berkis. Par vispārinātā inversā operatora ortogonālo reducējamību.

Anotācija. Aplūkoti pietiekamie nosacījumi lineāra nepārtraukta operatora un tā vispārinātā inversā ortogonālās reducējamības ekvivalencei, aplūkots vispārinājums slēgtam, blīvi definētam operatoram.

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Invariant sets for splitting mapping in metric space

A. Reinfelds

Summary. The theorems of existence of typical invariant sets for splitting mapping in complete metric space are proved under the assumption that given mapping satisfies some natural metric inequalities. These results generalize the classic ones developed for discrete semilinear dynamical system in finite dimensional space.

AMS Subject Classification 39B52, 54H20, 34C35

0. Introduction

The main purpose of this paper is to derive invariant manifolds results for systems of semilinear ordinary differential equations and semilinear discrete dynamical systems in Euclidean and Banach spaces to corresponding ones for mapping in complete metric space. Such question for discrete dynamical system generated by a diffeomorphism of R^n onto itself is really an old result, cf. [1 - 11]. The consideration of invariant set for mapping is of interest in itself, however they are useful for obtaining various conjugacy results [9 - 14]. Moreover, applications to some problems in biology [10], numerical analysis [11] etc. require results for mappings.

1. Main results

Let X and Y be complete metric spaces with metrics ρ_1, ρ_2 respectively, and let Λ be set. Consider mapping $T: X \times Y \times \Lambda \to X \times Y \times \Lambda$, defined by

$$T(x,y,\lambda) = (f(x,y,\lambda), g(x,y,\lambda), p(\lambda)).$$

We will make the following hypothesis:

(H1) $\rho_1(x,x') \le \alpha(\lambda)\rho_1(f(x,y,\lambda), f(x',y,\lambda))$, $\alpha(\lambda) > 0$,

(H2) $\rho_1(f(x,y,\lambda), f(x,y',\lambda)) \le \beta\rho_2(y,y')$,

(H3) $\rho_2(g(x,y,\lambda), g(x',y',\lambda)) \le \gamma\rho_1(x,x') + \delta(\lambda)\rho_2(y,y')$,

(H4) mapping $f(\cdot,y,\lambda): X \to X$ is surjective,

(H5) mapping $p: \Lambda \to \Lambda$ is bijective.

Let $\alpha = \sup_\lambda \alpha(\lambda)$, $\delta = \sup_\lambda \delta(\lambda)$ and $\Delta = \sup_\lambda \alpha(\lambda)\delta(\lambda)$.

Definition. The set $M \subset X \times Y \times \Lambda$ is *invariant with respect to mapping* T, if $T(M) \subset M$.

We consider a case when the invariant set can be represented as the graph of mapping $u\colon X \times \Lambda \to Y$ or $v\colon Y \times \Lambda \to X$ Lipschitz with respect to first variable. The corresponding mappings satisfy functional equations

$$u(f(x,u(x,\lambda),\lambda),p(\lambda)) = g(x,u(x,\lambda),\lambda), \qquad (1.1)$$

$$f(v(y,\lambda),y,\lambda) = v(g(v(y,\lambda),y,\lambda),p(\lambda)), \qquad (1.2)$$

Lipschitz conditions

$$\rho_2(u(x,\lambda),u(x',\lambda)) \le k_0\rho_1(x,x'), \qquad (1.3)$$

$$\rho_1(v(y,\lambda),v(y',\lambda)) \le l_0\rho_2(y,y') \qquad (1.4)$$

and estimates

$$\rho_2(u(f(x,y,\lambda),p(\lambda)),g(x,y,\lambda)) \le (k_0\beta + \delta)\rho_2(u(x,\lambda),y), \qquad (1.5)$$

$$\rho_1(v(y,\lambda),x) \le \alpha(1 - \alpha\gamma l_0)^{-1}\rho_1(v(g(x,y,\lambda),p(\lambda)),f(x,y,\lambda)), \qquad (1.6)$$

where

$$k_0 = 2\alpha\gamma \left(1 - \Delta + \sqrt{(1 - \Delta)^2 - 4\alpha^2\beta\gamma} \right)^{-1}$$

and

$$l_0 = 2\alpha\beta \left(1 - \Delta + \sqrt{(1 - \Delta)^2 - 4\alpha^2\beta\gamma} \right)^{-1}.$$

It should be pointed out that $\beta k_0 = \gamma l_0$.

Now we formulate the main results of the paper.

Theorem 1. *Let the hypothesis* (H1) – (H5) *hold and let there exists mapping* $u_0\colon X \times \Lambda \to Y$ *such that*

$$\rho_2(u_0(x,\lambda),u_0(x',\lambda)) \le k_0\rho_1(x,x')$$

and

$$\sup_{x,\lambda} \rho_2(u_0(f(x,u_0(x,\lambda),\lambda),p(\lambda)),g(x,u_0(x,\lambda),\lambda)) < +\infty. \qquad (1.7)$$

If $\Delta + 2\alpha\sqrt{\beta\gamma} \le 1$ ($\Delta < 1$ *if* $\beta = 0$ *and* $\gamma > 0$) *and* $\delta + \beta k_0 \le 1$, *then there exists unique mapping* $u\colon X \times \Lambda \to Y$ *satisfying* (1.1), (1.3), (1.5) *and*

$$\sup_{x,\lambda} \rho_2(u(x,\lambda),u_0(x,\lambda)) < +\infty.$$

Theorem 2. *Let the hypothesis* (H1) – (H5) *hold and let there exists mapping* $v_0\colon Y \times \Lambda \to X$ *such that*

$$\rho_1(v_0(y,\lambda),v_0(y',\lambda)) \le l_0\rho_2(y,y')$$

and

$$\sup_{y,\lambda} \rho_1(v_0(g(v_0(y,\lambda),y,\lambda),p(\lambda)),f(v_0(y,\lambda),y,\lambda)) < +\infty. \qquad (1.8)$$

If $\Delta + 2\alpha\sqrt{\beta\gamma} \le 1$ ($\Delta < 1$ *if* $\beta > 0$ *and* $\gamma = 0$) *and* $\alpha(1 + \gamma l_0) < 1$, *then there exists unique mapping* $v\colon Y \times \Lambda \to X$ *satisfying* (1.2), (1.4), (1.6) *and*

$$\sup_{y,\lambda} \rho_1(v(y,\lambda),v_0(y,\lambda)) < +\infty.$$

Theorem 3. *Let the hypothesis (H1) - (H5) hold and let there exist mappings $x_0: \Lambda \to X$ and $y_0: \Lambda \to Y$ such that*

$$f(x_0(\lambda),y_0(\lambda),\lambda) = x_0(p(\lambda)),$$

$$g(x_0(\lambda),y_0(\lambda),\lambda) = y_0(p(\lambda)).$$

If $\Delta + 2\alpha\sqrt{\beta\gamma} < 1$, then there exist unique mappings $u: X \times \Lambda \to Y$, $v: Y \times \Lambda \to X$ satisfying (1.1) - (1.6), $u(x_0(\lambda),\lambda) = y_0(\lambda)$ and $v(y_0(\lambda),\lambda) = x_0(\lambda)$.

2. Auxiliary lemmas.

Let us consider the set of mappings

$$\mathfrak{M}(k) = \left\{ u: X \times \Lambda \to Y \mid \rho_2(u(x,\lambda),u(x',\lambda)) \le k\rho_1(x,x') \right\}.$$

Lemma 1. *Let $\alpha\beta k < 1$ and $u \in \mathfrak{M}(k)$. Then mapping $\varphi: X \times \Lambda \to X \times \Lambda$, defined by $\varphi(x,\lambda) = (f(x,u(x,\lambda),\lambda),p(\lambda))$, is bijective.*

Proof. At first let us show that φ is injective. Otherwise there is $(x,\lambda) \ne (x',\lambda')$, but $\varphi(x,\lambda) = \varphi(x',\lambda')$. Hence according to (H5) and using (H1) and (H2) we get

$$\rho_1(x,x') \le \alpha\rho_1(f(x,u(x,\lambda),\lambda),f(x',u(x,\lambda),\lambda)) =$$

$$= \alpha\rho_1(f(x',u(x',\lambda),\lambda),f(x',u(x,\lambda),\lambda)) \le \alpha\beta k\rho_1(x,x').$$

Since $\alpha\beta k < 1$, it follows immediately that $\rho_1(x,x') = 0$ and $x = x'$.

The next show that φ is surjective. Let us define now a mapping $\Phi: X \to X$ by the equality

$$f(\Phi(x),u(x,\lambda),\lambda) = x_1.$$

According to (H1) and (H4), for every $x \in X$ there is unique $\Phi(x)$ and

$$\rho_1(\Phi(x),\Phi(x')) \le \alpha\rho_1(f(\Phi(x),u(x,\lambda),\lambda),f(\Phi(x'),u(x,\lambda),\lambda)) =$$

$$= \alpha\rho_1(f(\Phi(x'),u(x',\lambda),\lambda),f(\Phi(x'),u(x,\lambda),\lambda)) \le \alpha\beta k\rho_1(x,x').$$

We obtain that Φ is a contraction on X. Therefore for every $x_1 \in X$ there is unique $x \in X$ such that

$$f(x,u(x,\lambda),\lambda) = x_1.$$

Thus φ is bijective. Lemma is proved.

Next introduce the operator $\mathfrak{L}$ at $\mathfrak{M}(k)$ defined by the equality

$$(\mathfrak{L}u)(f(x,u(x,\lambda),\lambda),p(\lambda)) = g(x,u(x,\lambda),\lambda).$$

Lemma 2. *If $\Delta + 2\alpha\sqrt{\beta\gamma} \le 1$ ($\Delta < 1$ if $\beta = 0$ and $\gamma > 0$), then there exists $k_0 \ge 0$ such that $\mathfrak{L}(\mathfrak{M}(k_0)) \subset \mathfrak{M}(k_0)$.*

Proof. Taking into account (H3) we get

$$\rho_2((\mathfrak{L}u)(f(x,u(x,\lambda),\lambda),p(\lambda)),(\mathfrak{L}u)(f(x',u(x',\lambda),\lambda),p(\lambda))) =$$
$$= \rho_2(g(x,u(x,\lambda),\lambda),g(x',u(x',\lambda),\lambda)) \le (\gamma + \delta(\lambda)k)\rho_1(x,x').$$

On the other hand, we obtain

$$\rho_1(x,x') \le \alpha(\lambda)\rho_1(f(x,u(x,\lambda),\lambda),f(x',u(x,\lambda),\lambda)) \le$$
$$\le \alpha(\lambda)\rho_1(f(x,u(x,\lambda),\lambda),f(x',u(x',\lambda),\lambda)) + \alpha\beta k\rho_1(x,x').$$

It follows that

$$\rho_1(x,x') \le \alpha(\lambda)(1 - \alpha\beta k)^{-1}\rho_1(f(x,u(x,\lambda),\lambda),f(x',u(x',\lambda),\lambda)).$$

Therefore

$$\rho_2((\mathfrak{L}u)(f(x,u(x,\lambda),\lambda),p(\lambda)),(\mathfrak{L}u)(f(x',u(x',\lambda),\lambda),p(\lambda))) \le$$
$$\le (\alpha\gamma + \Delta k)(1 - \alpha\beta k)^{-1}\rho_1(f(x,u(x,\lambda),\lambda),f(x',u(x',\lambda),\lambda)).$$

If $k \ge 0$ satisfies inequality

$$0 \le (\alpha\gamma + \Delta k)(1 - \alpha\beta k)^{-1} \le k,$$

then $\mathfrak{L}(\mathfrak{M}(k)) \subset \mathfrak{M}(k)$. Such $k \ge 0$ exists, if $\Delta + 2\alpha\sqrt{\beta\gamma} \le 1$ ($\Delta < 1$ if $\beta = 0$ and $\gamma > 0$). We choose

$$k_0 = 2\alpha\gamma \left(1 - \Delta + \sqrt{(1 - \Delta)^2 - 4\alpha^2\beta\gamma}\right)^{-1}.$$

Lemma is proved.

Next let us consider the set of mappings

$$\mathfrak{N}(l) = \left\{ v\colon Y \times \Lambda \to X \mid \rho_1(v(y,\lambda),v(y',\lambda)) \le l\rho_2(y,y') \right\}$$

and let us introduce the operator $\mathfrak{K}$ at $\mathfrak{N}(l)$ by the equality

$$f(\mathfrak{K}v(y,\lambda),y,\lambda) = v(g(v(y,\lambda),y,\lambda),p(\lambda)).$$

Operator $\mathfrak{K}$ is correctly defined, because mapping $f(\cdot,y,\lambda)\colon X \to X$ is surjective and hypothesis (H1) is fulfilled.

Lemma 3. *If $\Delta + 2\alpha\sqrt{\beta\gamma} \le 1$ ($\Delta < 1$ if $\beta > 0$ and $\gamma = 0$), then there exists $l_0 \ge 0$ such that $\mathfrak{K}(\mathfrak{N}(l_0)) \subset \mathfrak{N}(l_0)$.*

Proof. According to (H1) - (H3), we get

$$\rho_1(\mathfrak{K}v(y,\lambda),\mathfrak{K}v(y',\lambda)) \le \alpha(\lambda)\rho_1(f(\mathfrak{K}v(y,\lambda),y,\lambda),f(\mathfrak{K}v(y',\lambda),y,\lambda)) \le$$
$$\le \alpha(\lambda)\rho_1(v(g(v(y,\lambda),y,\lambda),p(\lambda)),v(g(v(y',\lambda),y',\lambda),p(\lambda))) +$$
$$+ \alpha\rho_1(f(\mathfrak{K}v(y',\lambda),y,\lambda),f(\mathfrak{K}v(y',\lambda),y',\lambda)) \le$$
$$\le (\alpha(\lambda)l(\gamma l + \delta(\lambda)) + \alpha\beta)\rho_2(y,y').$$

If

$$0 \le l(\alpha\gamma l + \Delta) + \alpha\beta \le l,$$

then $\mathfrak{K}(\mathfrak{N}(l)) \subset \mathfrak{N}(l)$. Such $l \ge 0$ exists, if $\Delta + 2\alpha\sqrt{\beta\gamma} \le 1$ ($\Delta < 1$ if $\beta > 0$ and $\gamma = 0$). We choose

$$l_0 = 2\alpha\beta \left(1 - \Delta + \sqrt{(1 - \Delta)^2 - 4\alpha^2\beta\gamma}\right)^{-1}.$$

Lemma is proved.

3. Proofs of theorems

Proof of theorem 1. The set

$$\mathfrak{M} = \left\{ u \in \mathfrak{R}(k_0) \mid \sup_{x,\lambda} \rho_2(u(x,\lambda),u_0(x,\lambda)) < +\infty \right\}$$

is complete metric space, if the metric is defined by equality

$$d(u,u') = \sup_{x,\lambda} \rho_2(u(x,\lambda),u'(x,\lambda)).$$

Let us prove that $\mathfrak{L}$ is a contraction. Let

$$x_1 = f(x,u(x,\lambda),\lambda) = f(x',u'(x',\lambda),\lambda),\ \lambda_1 = p(\lambda).$$

We have

$$\rho_2((\mathfrak{L}u)(x_1,\lambda_1),(\mathfrak{L}u')(x_1,\lambda_1)) =$$
$$= \rho_2(g(x,u(x,\lambda),\lambda),g(x',u'(x',\lambda),\lambda)) \le$$
$$\le (\gamma + \delta(\lambda)k_0)\rho_1(x,x') + \delta(\lambda)\rho_2(u(x,\lambda),u'(x,\lambda)).$$

On the other hand

$$\rho_1(x,x') \le \alpha(\lambda)\rho_1(f(x,u(x,\lambda),\lambda),f(x',u(x,\lambda),\lambda)) =$$
$$= \alpha(\lambda)\rho_1(f(x',u'(x',\lambda),\lambda),f(x',u(x,\lambda),\lambda)) \le$$
$$\le \alpha(\lambda)\beta\rho_2(u'(x,\lambda),u(x,\lambda)) + \alpha\beta k_0\rho_1(x,x').$$

Therefore

$$\rho_1(x,x') \le \alpha(\lambda)\beta(1 - \alpha\beta k_0)^{-1}\rho_2(u'(x,\lambda),u(x,\lambda)).$$

We get

$$\rho_2((\mathfrak{L}u)(x_1,\lambda_1),(\mathfrak{L}u')(x_1,\lambda_1)) \le$$
$$\le ((\alpha\gamma + \Delta k_0)\beta(1 - \alpha\beta k_0)^{-1} + \delta(\lambda))\rho_2(u(x,\lambda),u'(x,\lambda)).$$

Hence

$$d(\mathfrak{L}u,\mathfrak{L}u') \le (\delta + \beta k_0)d(u,u').$$

We have

$$\rho_2((\mathfrak{L}u_0)(f(x,u_0(x,\lambda),\lambda),p(\lambda)),u_0(f(x,u_0(x,\lambda),\lambda),p(\lambda))) =$$
$$= \rho_2(g(x,u_0(x,\lambda),\lambda),u_0(f(x,u_0(x,\lambda),\lambda),p(\lambda))).$$

Therefore

$$d(\mathfrak{L}u_0,u_0) \le \sup_{x,\lambda} \rho_2(g(x,u_0(x,\lambda),\lambda),u_0(f(x,u_0(x,\lambda),\lambda),p(\lambda))).$$

Hence

$$d(\mathfrak{L}u,u_0) \le d(\mathfrak{L}u,\mathfrak{L}u_0) + d(\mathfrak{L}u_0,u_0) \le (\delta + \beta k_0)d(u,u_0) +$$
$$+ \sup_{x,\lambda} \rho_2(g(x,u_0(x,\lambda),\lambda),u_0(f(x,u_0(x,\lambda),\lambda),p(\lambda))). \qquad (3.1)$$

We obtain that $\mathfrak{L}$ is a contraction on $\mathfrak{M}$. It involves in $\mathfrak{M}$ there is unique mapping u satisfying functional equation (1.1).

From (3.1) we have

$$d(u,u_0) \le (1 - \delta - \beta k_0)^{-1} \times$$
$$\times \sup_{x,\lambda} \rho_2(g(x,u_0(x,\lambda),\lambda),u_0(f(x,u_0(x,\lambda),\lambda),p(\lambda))).$$

Let us prove (1.5). We have

$$\rho_2(u(f(x,y,\lambda),p(\lambda)),g(x,y,\lambda)) \le$$
$$\le \rho_2(u(f(x,y,\lambda),p(\lambda)),u(f(x,u(x,\lambda),\lambda),p(\lambda))) +$$
$$+ \rho_2(g(x,y,\lambda),g(x,u(x,\lambda),\lambda)) \le (\delta + \beta k_0)\rho_2(y,u(x,\lambda)).$$

Using the mathematical induction, we obtain

$$\rho_2(u(f^n(x,y,\lambda),p^n(\lambda)),g^n(x,y,\lambda)) \le (\delta + \beta k_0)^n \rho_2(y,u(x,\lambda)),$$

where (f^n,g^n,p^n) is n-th iterate of T. Thus the invariant set is asymptotically stable in sense, that arbitrary point iterate to them. Theorem is proved.

Proof of theorem 2. The set

$$\mathfrak{N} = \left\{ u \in \mathfrak{N}(l_0) \mid \sup_{y,\lambda} \rho_2(v(y,\lambda),v_0(y,\lambda)) < +\infty \right\}$$

is complete metric space, if the metric is defined by

$$d(v,v') = \sup_{y,\lambda} \rho_1(v(y,\lambda),v'(y,\lambda)).$$

Let us prove that $\mathfrak{T}$ is a contraction. We have

$$\rho_1(\mathfrak{T}v(y,\lambda),\mathfrak{T}v'(y,\lambda)) \le$$
$$\le \alpha\rho_1(f(\mathfrak{T}v(y,\lambda),y,\lambda),f(\mathfrak{T}v'(y,\lambda),y,\lambda)) =$$
$$= \alpha\rho_1(v(g(v(y,\lambda),y,\lambda),p(\lambda)),v'(g(v'(y,\lambda),y,\lambda),p(\lambda))) \le$$
$$\le \alpha\rho_1(v(g(v(y,\lambda),y,\lambda),p(\lambda)),v'(g(v(y,\lambda),y,\lambda),p(\lambda))) +$$
$$+ \alpha\rho_1(v'(g(v(y,\lambda),y,\lambda),p(\lambda)),v'(g(v'(y,\lambda),y,\lambda),p(\lambda))).$$

Therefore

$$d(\mathfrak{T}v,\mathfrak{T}v') \le \alpha(1 + \gamma l_0)d(v,v').$$

We have

$$\rho_1(\mathfrak{T}v_0(y,\lambda),v_0(y,\lambda)) \le \alpha\rho_1(f(\mathfrak{T}v_0(y,\lambda),y,\lambda),f(v_0(y,\lambda),y,\lambda)) =$$
$$= \alpha\rho_1(v_0(g(v_0(y,\lambda),y,\lambda),p(\lambda)),f(v_0(y,\lambda),y,\lambda)).$$

We get

$$d(\mathfrak{T}v_0,v_0) \le \alpha\sup_{y,\lambda} \rho_1(f(v_0(y,\lambda),y,\lambda),v_0(g(v_0(y,\lambda),y,\lambda),p(\lambda))).$$

Therefore

$$d(\mathfrak{T}v,v_0) \le d(\mathfrak{T}v,\mathfrak{T}v_0) + d(\mathfrak{T}v_0,v_0) \le \alpha(1 + \gamma l_0)d(v,v_0) +$$
$$+ \alpha\sup_{y,\lambda} \rho_1(f(v_0(y,\lambda),y,\lambda),v_0(g(v_0(y,\lambda),y,\lambda),p(\lambda))). \qquad (3.2)$$

We obtain that $\mathfrak{T}$ is a contraction on $\mathfrak{N}$. It involves in $\mathfrak{N}$ there is unique mapping v satisfying functional equation (1.2).

From (3.2) we have

$$d(v,v_0) \le (\alpha^{-1} - (1 + \gamma l_0))^{-1} \times$$
$$\times \sup_{y,\lambda} \rho_1(f(v_0(y,\lambda),y,\lambda),v_0(g(v_0(y,\lambda),y,\lambda),p(\lambda))).$$

Let us prove (1.6). We have

$$\rho_1(x,v(y,\lambda)) \le \alpha\rho_1(f(x,y,\lambda),f(v(y,\lambda),y,\lambda)) =$$
$$= \alpha\rho_1(f(x,y,\lambda),v(g(v(y,\lambda),y,\lambda),p(\lambda))) \le$$
$$\le \alpha\rho_1(f(x,y,\lambda),v(g(x,y,\lambda),p(\lambda))) + \alpha\gamma l_0\rho_1(x,v(y,\lambda)).$$

Therefore, we obtain (1.6).

By using the mathematical induction, we obtain

$$\rho_1(f^n(x,y,\lambda),v(g^n(x,y,\lambda),p^n(\lambda))) \ge (\alpha^{-1} - \gamma l_0)^n \rho_1(x,v(y,\lambda)).$$

Theorem is proved.

Proof of theorem 3. Let us endow the set

$$\mathfrak{M}_0 = \{ u \in \mathfrak{M}(k_0) \mid u(x_0(\lambda),\lambda) = y_0(\lambda) \}$$

with the metric.

$$d_2(u,u') = \sup_{x,\lambda} \rho_2(u(x,\lambda),u'(x,\lambda))(\rho_1(x,x_0(\lambda)))^{-1}.$$

Then $\mathfrak{M}_0$ become the complete metric space. Let $u \in \mathfrak{M}_0$ and let us note that $f(x,u(x,\lambda),\lambda) = x_0(p(\lambda))$, if $x = x_0(\lambda)$. It follows that

$$(\mathfrak{L}u)(x_0(p(\lambda)),p(\lambda)) = (\mathfrak{L}u)(f(x_0(\lambda),u(x_0(\lambda),\lambda),\lambda),p(\lambda))) =$$
$$= g(x_0(\lambda),u(x_0(\lambda),\lambda),\lambda) = y_0(p(\lambda)),$$

or $\mathfrak{L}u \in \mathfrak{M}_0$. We must prove that $\mathfrak{L}$ is a contraction on $\mathfrak{M}_0$. Let

$$x_1 = f(x,u(x,\lambda),\lambda) = f(x',u'(x',\lambda),\lambda)), \ \lambda_1 = p(\lambda).$$

Analogously, like in theorem 1, we obtain

$$\rho_2((\mathfrak{L}u)(x_1,\lambda_1),(\mathfrak{L}u')(x_1,\lambda_1)) \le (\delta(\lambda) + \beta k_0)\rho_2(u(x,\lambda),u'(x,\lambda)).$$

Let us estimate

$$\rho_1(x,x_0(\lambda)) \le \alpha(\lambda)\rho_1(f(x,u(x,\lambda),\lambda),f(x_0(\lambda),u(x,\lambda),\lambda)) \le$$
$$\le \alpha(\lambda)\rho_1(f(x,u(x,\lambda),\lambda),x_0(p(\lambda))) +$$
$$+ \alpha\rho_1(f(x_0(\lambda),u(x,\lambda),\lambda),f(x_0(\lambda),u(x_0(\lambda),\lambda),\lambda)) \le$$
$$\le \alpha(\lambda)\rho_1(f(x,u(x,\lambda),\lambda),x_0(p(\lambda))) + \alpha\beta k_0\rho_1(x,x_0(\lambda)).$$

Hence,

$$\rho_1(x,x_0(\lambda)) \le \alpha(\lambda)(1 - \alpha\beta k_0)^{-1}\rho_1(x_1,x_0(\lambda_1)).$$

We have

$$\rho_2((\mathfrak{L}u)(x_1,\lambda_1),(\mathfrak{L}u')(x_1,\lambda_1)) \le$$
$$\le (\delta(\lambda) + \beta k_0)d_2(u,u')\rho_1(x,x_0(\lambda)) \le$$
$$\le (\Delta + \alpha\beta k_0)(1 - \alpha\beta k_0)^{-1}d_2(u,u')\rho_1(x_1,x_0(\lambda_1)).$$

Therefore

$$d_2(\mathfrak{L}u,\mathfrak{L}u') \le (\Delta + \alpha\beta k_0)(1 - \alpha\beta k_0)^{-1}d_2(u,u').$$

If $\Delta + 2\alpha\sqrt{\beta\gamma} < 1$, then

$$(\Delta + \alpha\beta k_0)(1 - \alpha\beta k_0)^{-1} =$$
$$= \left(1 + \Delta - \sqrt{(1 - \Delta)^2 - 4\alpha^2\beta\gamma} \right)\left(1 + \Delta + \sqrt{(1 - \Delta)^2 - 4\alpha^2\beta\gamma} \right)^{-1} < 1.$$

We obtain that $\mathfrak{L}$ is a contraction on $\mathfrak{M}_0$. It involves in $\mathfrak{M}_0$ there is unique mapping u satisfying functional equation (1.1) and estimates (1.3) and (1.5). The first part of the theorem is established.

Let us prove the existence of invariant manifold given by the mapping $v: Y \times \Lambda \to X$.

The set

$$\mathfrak{N}_0 = \{ v \in \mathfrak{N}(l_0) \mid v(y_0(\lambda),\lambda) = x_0(\lambda) \}$$

becomes a complete metric space, if the metric is defined by

$$d_1(v,v') = \sup_{y,\lambda} \rho_1(v(y,\lambda),v'(y,\lambda))(\rho_2(y,y_0(\lambda)))^{-1}.$$

Using the definition of $\mathfrak{X}$, we obtain $\mathfrak{X}v(y_0(\lambda),\lambda) = x_0(\lambda)$. Therefore, $\mathfrak{X}v \in \mathfrak{R}_0$, if $v \in \mathfrak{R}_0$. It follows that

$$\rho_1(\mathfrak{X}v(y,\lambda),\mathfrak{X}v'(y,\lambda)) \le$$
$$\le \alpha(\lambda)\rho_1(f(\mathfrak{X}v(y,\lambda),y,\lambda),f(\mathfrak{X}v'(y,\lambda),y,\lambda)) =$$
$$= \alpha(\lambda)\rho_1(v(g(v(y,\lambda),y,\lambda),p(\lambda)),v'(g(v'(y,\lambda),y,\lambda),p(\lambda))) \le$$
$$\le \alpha(\lambda)\rho_1(v(g(v(y,\lambda),y,\lambda),p(\lambda)),v'(g(v(y,\lambda),y,\lambda),p(\lambda))) +$$
$$+ \alpha\gamma l_0\rho_1(v(y,\lambda),v'(y,\lambda)) \le \alpha\gamma l_0 d_1(v,v')\rho_2(y,y_0(\lambda)) +$$
$$+ \alpha(\lambda)d_1(v,v')\rho_2(g(v(y,\lambda),y,\lambda),y_0(p(\lambda))).$$

Let us estimate

$$\rho_2(g(v(y,\lambda),y,\lambda),y_0(p(\lambda))) =$$
$$= \rho_2(g(v(y,\lambda),y,\lambda),g(v(y_0(\lambda),\lambda),y_0(\lambda),\lambda)) \le$$
$$\le (\delta(\lambda) + \gamma l_0)\rho_2(y,y_0(\lambda)).$$

Therefore

$$\rho_1(\mathfrak{X}v(y,\lambda),\mathfrak{X}v'(y,\lambda)) \le (\Delta + 2\alpha\gamma l_0)d_1(v,v')\rho_2(y,y_0(\lambda)).$$

We get

$$d_1(\mathfrak{X}v,\mathfrak{X}v') \le (\Delta + 2\alpha\gamma l_0)d_1(v,v').$$

Besides

$$\Delta + 2\alpha\gamma l_0 = 1 - \sqrt{(1 - \Delta)^2 - 4\alpha^2\beta\gamma} < 1.$$

We obtain that $\mathfrak{X}$ is a contraction on $\mathfrak{R}_0$. It involves in $\mathfrak{R}_0$ there is unique mapping v satisfying functional equation (1.2) and estimates (1.4) and (1.6).

Theorem is proved.

Remark. In the case when Λ is topological space, mapping T is continuous and p is homeomorpfism the mappings u and v are continuous in both variables.

4. Example

Let us consider difference equations on $\mathbb{Z}$ in Banach space

$$x(n + 1) = A(n)x(n) + F(n,x(n),y(n)),$$
$$y(n + 1) = B(n)y(n) + G(n,x(n),y(n)),$$

where $x \in X$, $y \in Y$, $A(n)$ and $B(n)$ are bounded linear mappings, $A(n)$ is invertible, $\|B(n)\| < \|A(n)^{-1}\|^{-1}$ and mappings $F: \mathbb{Z} \times X \times Y \to X$, $G: \mathbb{Z} \times X \times Y \to Y$ satisfies Lipschitz conditions

$$|F(n,x,y) - F(n,x',y')| \le c(|x - x'| + |y - y'|),$$
$$|G(n,x,y) - G(n,x',y')| \le c(|x - x'| + |y - y'|).$$

It is easy to verify that this mapping satisfies the hypothesis (H1) - (H5), where $\alpha = ((\sup_n\|A(n)^{-1}\|)^{-1} - c)^{-1}$, $\beta = \gamma = c$,

$\delta = \sup_n\|B(n)\| + \varepsilon$, $\Delta = (\sup_n(\|A(n)^{-1}\|\ \|B(n)\|) + \varepsilon\sup_n\|A(n)^{-1}\|) \times (1 - \varepsilon\sup_n\|A(n)^{-1}\|)^{-1}$. The mapping given by formula $x_1 = A(n)x + F(n,x,y)$ for fixed y is surjective, if $\varepsilon\|A(n)^{-1}\| < 1$.

The condition $\Delta + 2\alpha\sqrt{\beta\gamma} \le 1$ reduces to the inequality $\varepsilon \le \varepsilon_1 = (4\sup_n\|A(n)^{-1}\|)^{-1}(1 - \sup_n(\|A(n)^{-1}\|\ \|B(n)\|)$. Let us note that $\varepsilon\|A(n)^{-1}\| \le 4^{-1} < 1$.

Let

$$\mu = 2\sup_n\|B(n)\| + \sup_n\|A(n)^{-1}\|^{-1}(1 - \sup_n(\|A(n)^{-1}\|\ \|B(n)\|)),$$
$$\nu = \sup_n\|A(n)^{-1}\|^{-1}(1 + \sup_n(\|A(n)^{-1}\|\ \|B(n)\|)).$$

The condition $\delta + \beta k_0 < 1$ for $\mu < 2$ is fulfilled, if $\varepsilon \le \varepsilon_1$ and for $\mu \ge 2$ if $\sup_n\|B(n)\| < 1$ and

$\varepsilon < (1 - \sup_n(\|A(n)^{-1}\|\|B(n)\|) + (\sup_n\|A(n)^{-1}\|)(\sup_n\|B(n)\| - 1)) \times (1 - \sup_n\|B(n)\|)(1 - \sup_n(\|A(n)^{-1}\|\ \|B(n)\|))^{-1}$.

The condition $\alpha(1 + \gamma l_0) < 1$ for $\nu \le 2$ is fulfilled, if $\sup_n(\|A(n)^{-1}\|\ \|B(n)\|) < \sup_n\|A(n)^{-1}\| < 1$ and

$$\varepsilon < (1 - \sup_n(\|A(n)^{-1}\|)(1 - (\sup_n(\|A(n)^{-1}\|)^{-1} \times \sup_n(\|A(n)^{-1}\|\ \|B(n)\|))(1 - \sup_n(\|A(n)^{-1}\|\ \|B(n)\|)^{-1}$$

and for $\nu > 2$, if $\varepsilon \le \varepsilon_1$.

The condition (1.7) is fulfilled if $\sup_{x,n}|G(n,x,0)| < +\infty$.

The condition (1.8) is fulfilled if $\sup_{y,n}|F(n,0,y)| < +\infty$.

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A.Reinfelds Sadalošos attēlojumu invariantās kopas metriskā telpā.

Anotācija. Pierādītas tipisku invariantu kopu eksistences teorēmas sadalošiem attēlojumiem pilnā metriskā telpā, ja dotais attēlojums apmierina raksturīgas metriskas nevienādības.

А.А.Рейнфелд. Инвариантные множества расщепляющихся отображений в полном метрическом пространстве.

Аннотация. Доказаны теоремы существования характеристических инвариантных множеств для расщепляющихся отображений в полном метрическом пространстве, если данное отображение удовлетворяет некоторым естественным неравенствам. Эти результаты являются обобщением классических результатов относительно дискретных квазилинейных динамической системы в конечномерном пространстве.
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CONTINUOUS DEPENDENCE ON PARAMETERS OF SOLUTIONS FOR BOUNDARY VALUE PROBLEM

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Abstract. Sufficient conditions for the continuous dependence of solutions to boundary value problems on parameters are given for a functional-differential equation.
AMS Subject classification 47B38.

Consider boundary value problems

$$x'=F_0x, \qquad \Phi_0x=0, \tag{1}$$

$$x'=F_kx, \qquad \Phi_kx=0, \tag{2}$$

where $F_m:AC(I_m,R^n)\to L(I_m,R^n)$, $\Phi_m:AC(I_m,R^n)\to R^n$, $I_m=[a_m,b_m]$, $-\infty<a_m<b_m<+\infty$, $m\in\{0,1,2,\dots\}$, $n,k\in\{1,2,3,\dots\}$, $AC(I_m,R^n)$ - the space of absolutely continuous functions $x:I_m\to R^n$ with the norm

$$\|x\|_{Cm}=\max\Big\{|x_i(t)|:\ (i,t)\in\{1,\dots,n\}\times I_m\Big\},$$

$L(I_m,R^n)$ - the space of Lebesque summable functions $x:I_m\to R^n$ with the norm

$$\|x\|_{Lm}=\max\Big\{\int_{I_m}|x_i(t)|dt:\ i\in\{1,\dots,n\}\Big\},$$

R^n - Euclid's space with the norm

$$\|x\|_R=\Big(\sum_{i=1}^n x_i^2\Big)^{1/2}.$$

In the work the conditions will be given, under which the sequence of solutions to the problem (2) converges to the solution of the problem (1). Our results generalize the related ones in [1,2].

Introduce the mapping r_k, $k=1,2,3,\dots$ in a following way.

Let for any $x:I_k\to R^n$ and $t\in I_k$

$$(r_k x)(t)=x[a_k+(b_k-a_k)(t-a_0)(b_0-a_0)^{-1}).$$

Denote by $S(F_m,\Phi_m)$, $m=0,1,2,\ldots$ the set of solutions to the boundary value problems (1), (2).

Suppose that $x_m\in S(F_m,\Phi_m)$ exists for any $m=0,1,2,\ldots$ and denote by $B(x_0,r)$ a neighbourhood of x_0 with radius $r>0$:

$$B(x_0,r)=\left\{x\in AC(I_0,R^n):\ \|x-x_0\|_{C_0}\le r\right\}.$$

THEOREM. Let the conditions hold:

1. boundary value problem (1) has a unique solution x_0 in $B(x_0,r)$,
2. mappings F_0 and Φ_0 are continuous on $B(x_0,r)$,
3. the map F_0 is bounded on $B(x_0,r)$, i.e. $g\in L(I_0,R^n)$ exists such that for any $x\in B(x_0,r)$ we have $|F_0x|\le g$,
4. $r_k x_k\in B(x_0,r)$ for any $k\in\{1,2,\ldots\}$,
5. $\lim\limits_{k\to\infty} a_k=a_0$, $\lim\limits_{k\to\infty} b_k=b_0$,

   $\lim\limits_{k\to\infty} \|F_0 r_k x_k - r_k F_k x_k\|_{L_0}=0$,

   $\lim\limits_{k\to\infty} \|\Phi_0 r_k x_k\|_R=0$.

Then $\lim\limits_{k\to\infty} \|x_0-r_k x_k\|_{C_0}=0$.

Proof. Suppose, without lost of generality, that

$$\|F_0 r_k x_k - r_k F_k x_k\|_{L_0}<k^{-1}$$

for any $k\in\{1,2,\ldots\}$. In view of the condition 4 the sequence $k\to r_k x_k$ is uniformly bounded. Let us show that it is equicontinuous. From

$$\frac{dr_k x_k}{dt}=\frac{b_k-a_k}{b_0-a_0}\, r_k F_k x_k$$

for any $k\in\{1,2,\ldots\}$, $t_1\in[a_0,b_0)$ and $t_2\in(t_1,b_0]$ we have the estimate

$$|(v_k x_k)(t_2)-(v_k x_k)(t_1)| = \left|\int_{t_1}^{t_2} \frac{b_k-a_k}{b_0-a_0}(v_k F_k x_k)(t)dt\right| \le$$

$$\le \frac{b_k-a_k}{b_0-a_0}\int_{t_1}^{t_2}|(v_k F_k x_k)(t)-(F_0 v_k x_k)(t)+(F_0 v_k x_k)(t)|dt \le \tag{3}$$

$$\le \frac{b_k-a_k}{b_0-a_0}\left((k^{-1},\ldots,k^{-1}) + \int_{t_1}^{t_2} g(t)dt\right).$$

Hence the equicontinuity follows. Without lost of generality suppose that the sequence $k \to v_k x_k$ converges to the element y. From (3) we have that for any $t_1 \in [a_0, b_0)$ and $t_2 \in (t_1, b_0]$

$$|y(t_2)-y(t_1)| \le \int_{t_1}^{t_2} g(t)dt$$

and it follows that $y \in AC(I_0, R^n)$. Therefore from the condition 4 we conclude that $y \in B(x_0, r)$.

From

$$\lim_{k\to\infty} \|F_0 y - v_k F_k x_k\|_{L_0} \le$$

$$\le \lim_{k\to\infty} \|F_0 y - F_0 v_k x_k\|_{L_0} + \lim_{k\to\infty} \|F_0 v_k x_k - v_k F_k x_k\|_{L_0} = 0$$

and the equality

$$(v_k x_k)(t)-(v_k x_k)(a) = \int_{a_0}^{t} \frac{b_k-a_k}{b_0-a_0} (v_k F_k x_k)(t)dt$$

we obtain

$$y(t)-y(a) = \int_a^t (F_0 y)(t)dt.$$

Hence, $y' = F_0 y$. Besides

$$\|\Phi_0 y\|_R = \lim_{k\to\infty} \|\Phi_0 y - \Phi_0 v_k x_k + \Phi_0 v_k x_k\|_R \le$$

$$\le \lim_{k\to\infty} \|\Phi_0 y - \Phi_0 v_k x_k\|_R + \lim_{k\to\infty} \|\Phi_0 v_k x_k\|_R = 0,$$

and therefore y solves the problem (1). From $y\in B(x_0,r)$ and the condition 1 we get that $y=x_0$, and hence $\lim_{k\to\infty} \|x_0-v_k x_k\|_{C_D}=0$.

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V.Ponomarjovs. Robežproblēmu atrisinājumu nepārtrauktā atkarība no parametriem.

Anotācija. Formulēti robežproblēmas atrisinājuma nepārtrauktas atkarības no parametriem pietiekamie nosacījumi funkcionāli-diferenciālajiem vienādojumiem.

В.Пономарев. Непрерывная зависимость решения краевой задачи от параметров.

Аннотация. Приводятся достаточные условия непрерывной зависимости решения краевой задачи от параметров для функционально-дифференциального уравнения. УДК 517.988.5.

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ON s-SETS, H-SETS and θ-PERFECT FUNCTIONS

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Summary. This paper is a generalization of the paper "On S-closed Spaces" of Mashhour, Allam and Zahran (1991), [6]. AMS SC 54C10

1. Introduction

Throughout the present paper, (X, τ) and (Y, θ) (or simply X and Y) denote topological spaces on which no separation axioms are assumed unless explicity stated.

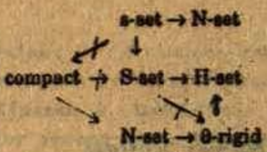
A set A is semi-open (resp. pre-open, α-open, β-open) iff $A\subset\overline{\mathring{A}}$ (resp. $A\subset\mathring{\overline{A}}$, $A\subset\mathring{\overline{\mathring{A}}}$, $A\subset\overline{\mathring{\overline{A}}}$). A set, the complement of which is semi-open (resp. pre-open, α-open, β-open) is called semi-closed (resp. pre-closed, α-closed, β-closed).

A set A is called s-set (resp. S-set) if for each semi-open cover $\{U_i : i\in I\}$ of A, there exists a finite subset I_0 of I such that $\{scl\, U_i : i\in I_0\}$ (resp. $\{\overline{U}_i : i\in I_0\}$) is a cover of A. A set A is called N-set (resp. θ-rigid, H-set) if for each open cover

$\{U_i : i\in I\}$, there exists a finite subset I_0 such that $A\subset\bigcup_{i\in I_0}\mathring{\overline{U}}_i$, (resp. $A\subset\mathring{\overline{\bigcup_{i\in I_0}U_i}}$, $A\subset\bigcup_{i\in I_0}\overline{U}_i$).

If a topological space is an H-set (resp. N-set, S-set, s-set) as being its subset, it is called quasi - H-closed (resp. N-closed, S-closed, s-closed).

The following diagram is well known in any topological space.



A set A in (X, τ) is called regular-open (resp. θ-open) if $A = \mathring{\bar{A}}$ (resp. for each $x \in A$, there exists an open set U such that $x \in U \subset \bar{U} \subset A$). τ_s will stand for the semi-regularization topology for which the family of all regular open sets is a base. All θ-open (resp. α-open) sets form a topology on X, this topology is denoted by τ_θ (resp. τ^α).

The following are well known for a topological space (X, τ).

(X, τ) is almost regular iff $\tau_s = \tau_\theta$. (X, τ) is semi-regular iff $\tau = \tau_s$. (X, τ) is regular iff $\tau = \tau_s = \tau_\theta$.

In an almost regular space (X, τ),

N-set ↔ θ-rigid ↔ H-set ↔ τ_θ -compact set (from Theorem 3.6 in [8] and the above diagram).

In a semi-regular space, it is known that compact set ↔ N-set, so in a regular space (X, τ),

τ-compact set ↔ N-set ↔ θ-rigid ↔ H-set ↔ τ_θ - compact set

In (X, τ), if every regular open set is closed, then (X, τ) is called extremely disconnected (shortly, e.d.). If (X, τ) is e.d. then it is almost regular.

τ_{co} will denote the topology which has the base the family of all clopen sets in (X, τ). For any topological space (X, τ), $\tau_{co} \subset \tau_\theta \subset \tau_s \subset \tau$. Clearly an H-set in (X, τ) is τ_{co} - compact. But the converse is not true.

__Example 1.1.__ The set of real numbers R with its usual topology $\mathcal{U}$ is a regular space. Only X and $\emptyset$ are the clopen sets of (R, $\mathcal{U}$). So each subset of R is $\mathcal{U}_{co}$ - compact. But any open interval is not an H-set.

If (X, τ) is e.d., then $\tau_s = \tau_\theta = \tau_{co}$. But the converse is not true for any space (X, τ). Clearly if X is finite then, (X, τ) is e.d. iff $\tau_s = \tau_{co} = \tau_\theta$.

__Example 1.2.__ (This is an example of Gillman and Jerison and it is the Example 7.2 in [1]). Let X be an uncountable set and choose a point $x_0 \in X$. The topology τ on X is generated by having all sets {x} open for $x \neq x_0$, and if $x_0 \in G$ then G is open if X-G is countable. It was shown that (X, τ) is not e.d. ([1], page 245). But, $\tau_s = \tau_\theta = \tau_{co}$.

The family of clopen sets $CO(X) = \{A \subset X : x_0 \in A,\ A^c \text{ is countable}\} \cup$

$\{B \subset X : B \subset X \setminus \{x_0\},\ B \text{ is countable}\}$

The family of regular open sets $RO(X) = CO(X) \cup \{D \subset X \setminus \{x_0\} : D \text{ and } D^c \text{ are not countable}\}$

If $D \subset X \setminus \{x_0\}$ and D, D^c are not countable, then $D = \cup\{\{x\}: x \in D\}$. $D \in \tau_{co}$. Hence $\tau_s = \tau_{co}$.

In [9], it was shown that s-sets and H-sets are the same in an e.d. space.

Now, we have the following diagram in an e.d. space (X, τ).

s-set ↔ S-set ↔ N-set ↔ θ-rigid ↔ H-set ↔ τ_θ-compact set ↔ τ_{co}-compact set.

__2. Inverse Images of S-Sets, H-Sets__

For a subset A of X,

$cl_\theta A = \{x \in X : \bar{U} \cap A \neq \emptyset \text{ for each open set } U \text{ containing } x\}$

θ-semiclosure of A = $scl_\theta A$ (It is called s-closure of A in [3])

$= \{x \in X : \bar{U} \cap A \neq \emptyset \text{ for each semi-open set } U \text{ containing } x\}$

The following definitions are known.

Let $f: X \to Y$ be a function.

a) If $f(A)$ is closed in Y for each regular closed set A in X, then f is called regular closed function.

b) If $f^{-1}(\overline{V}) \subset \overline{f^{-1}(V)}$ for each open set V in Y, then f is called almost-open.

c) If $f(A)$ is α-open in Y for each open set A in X, then f is called α-open.

d) If $f(A)$ is semi-closed in Y for each regular closed set A in X, then f is called weakly semi-closed [6].

e) If $f(A)$ is β-closed in Y for each closed set A in X, then f is called β-closed.

f) If $f(A)$ is pre-open in Y for each open set A in X, then f is called pre-open.

It is well known that a function f is almost-open iff it is pre-open.

We don't give the definitions of θ-perfect and s-perfect functions. They are defined in [2] and [3].

If $f: X \to Y$ is θ-perfect, then the inverse images of H-sets are H-sets (3.1.1 Corollary in [2]). If $f: X \to Y$ is s-perfect, then the inverse images of S-sets are S-sets (Proposition 3.3 in [3]).

The following example shows that, a function which satisfy the conditions of Theorem 2.2 must not necessarily be either θ-perfect or s-perfect.

<u>Example 2.1.</u> Let $X = \{a, b, c, d\}$, $\tau = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$

Let $f: X \to X$, $f(a) = b$, $f(b) = b$, $f(c) = c$, $f(d) = c$. f is pre-open and weakly semi-closed, $f^{-1}(y)$ is s-set, S-set, θ-rigid, compact.

But for the subset $A = \{a\}$ of X,

$scl_\theta A = \{a\}$, $cl_\theta A = \{a, d\}$, $f(A) = \{b\}$, $scl_\theta f(A) = \{b, c\}$, $cl_\theta f(A) = \{b, c, d\}$.

$f(cl_\theta A) = \{b, c\}$, $f(scl_\theta A) = \{b\}$, $scl_\theta f(A) \not\subset f(scl_\theta A)$ and $cl_\theta f(A) \not\subset f(cl_\theta A)$.

Hence f is neither θ-perfect (from 3.1.1 Corollary in [2]) nor s-perfect (from the definition of s-perfect function in [3]).

Theorem 2.2. Let f be a weakly semi-closed, pre-open function, and $f^{-1}(y)$ θ-rigid for each $y \in Y$. Then $f^{-1}(G)$ is an H-set for each S-set G in Y.

Proof: Let G be an S-set in Y and $\{U_i : i \in I\}$ an open cover of $f^{-1}(G)$.

For each $y \in G$, $f^{-1}(y) \subset \bigcup_{i \in I} U_i$. Since $f^{-1}(y)$ is θ-rigid there exists a finite subset I_y of I such that $f^{-1}(y) \subset \bigcup_{i \in I_y} \overline{U_i}^{\circ}$. Let $U_y = \overline{\bigcup_{i \in I_y} U_i}^{\circ}$, then U_y is a regular open set. Since f is weakly semi-closed, there exists a semi-open set V_y in Y such that $y \in V_y$, $f^{-1}(V_y) \subset U_y$ (from Theorem 3.4 in [6]). We have $G \subset \bigcup_{y \in G} V_y$. Since G is an S-set, there exists a finite number of points $y_1, y_2, \ldots, y_n$ in G such that $G \subset \bigcup_{j=1}^{n} \overline{V}_{y_j}$.

$$f^{-1}(G) \subset \bigcup_{j=1}^{n} f^{-1}(\overline{V}_{y_j}) \quad \text{(since } f \text{ is pre-open and } V_{y_j} \text{ is semi-open and from Proposition 3.2 in [4])}$$

$$\subset \bigcup_{j=1}^{n} \overline{f^{-1}(V_{y_j})}$$

$$\subset \bigcup_{j=1}^{n} (\overline{U}_{y_j}) = \bigcup_{j=1}^{n} (\overline{\overline{\bigcup_{i \in I_{y_j}} U_i}^{\circ}}) \subset \bigcup_{j=1}^{n} (\overline{\bigcup_{i \in I_{y_j}} U_i}) = \bigcup_{j=1}^{n} (\bigcup_{i \in I_{y_j}} \overline{U}_i)$$

Corollary 2.3. Let (X, τ) be an almost regular space, $f: X \to Y$ weakly semi-closed and pre-open, and $f^{-1}(y)$ τ_θ-compact for each $y \in Y$. If G is an S-set, then $f^{-1}(G)$ is an N-set.

Corollary 2.4. Let (X, τ) be a regular space, $f: X \to Y$ weakly semi-closed and pre-open, and $f^{-1}(y)$ τ_θ-compact for each $y \in Y$. If G is an S-set, then $f^{-1}(G)$ is compact.

Corollary 2.5. Let (X, τ) be an e.d. space, $f: X \to Y$ weakly semi-closed and pre-open, and $f^{-1}(y)$ τ_{∞}-compact for each $y \in Y$. If G is an S-set, then $f^{-1}(G)$ is an s-set.

Corollary 2.6. Let $f: X \to Y$ be weakly semi-closed and pre-open, and $f^{-1}(y)$ θ-rigid for each $y \in Y$. If Y is S-closed, then X is quasi H-closed.

Corollary 2.7. Let (X, τ) be an almost regular space, $f: X \to Y$ weakly semi-closed and pre-open, and $f^{-1}(y)$ τ_{θ}-compact for each $y \in Y$. I. Y is S-closed, then X is N-closed.

Corollary 2.8. Let (X, τ) be a regular space, $f: X \to Y$ weakly semi-closed and pre-open, and $f^{-1}(y)$ τ_{θ}-compact for each $y \in Y$. If Y is S-closed, then X is compact.

Corollary 2.9. Let (X, τ) be e.d. space, $f: X \to Y$ weakly semi-closed and pre-open, and $f^{-1}(y)$ τ_{∞}-compact for each $y \in Y$. If Y is S-closed, then X is s-closed.

Corollary 2.10. Let (Y, σ) be e.d. space, $f: X \to Y$ weakly semi-closed and pre-open, and $f^{-1}(y)$ θ-rigid for each $y \in Y$. If G is σ_{∞}-compact in Y then $f^{-1}(G)$ is an H-set in X.

Corollary 2.11. (Theorem 3.5 in [6]). Let be an e.d. space and $f: X \to Y$ a weakly semi-closed, almost-open surjection and $f^{-1}(y)$ be S-set for each $y \in Y$. If G is S-set in Y, then $f^{-1}(G)$ is S-set.

Corollary 2.12. (Theorem 3.7 in [6]). Let $\overline{X}$ be an e.d. space , $f: X \to Y$ be weakly semi-closed, almost-open surjection and $f^{-1}(y)$ be compact for each $y \in Y$. If Y is an S-closed space, then X is S-closed.

Lemma 2.13. If $f: X \to Y$ is regular closed, pre-open, then for each $y \in Y$ and each open set U containing $f^{-1}(y)$, there is an open set V containing y such that

$f^{-1}(\overline{V}) \subset \overline{U}$.

Proof. Let $y \in Y$, $f^{-1}(y) \subset U$ and U be an open set. $f^{-1}(y) \subset U \subset \mathring{\overline{U}} \subset \overline{U}$. Since $\mathring{\overline{U}} \in RO(X)$ and f is regular closed, from Lemma 2.2 in [5], there is an open set V in Y such that $y \in V$ and $f^{-1}(V) \subset \mathring{\overline{U}}$. Since f is pre-open, we have $f^{-1}(\overline{V}) \subset$

$f^{-1}(\overline{V}) \subset \overline{f^{-1}(V)} \subset \overline{U}$.

Theorem 2.14. If f: X→Y is regular closed, pre-open and $f^{-1}(y)$ θ-rigid for each y∈Y, then f is θ-perfect.

Proof. From (3,6) in [2] and Lemma 2.13, for each A⊂X, $cl_{\theta}\, f(A) \subset f(cl_{\theta}\, A)$. Now, from 3.4.1 Corollary in [2], f is θ-perfect.

Corollary 2.15. If f: X→Y is regular closed, pre-open and $f^{-1}(y)$ θ-rigid for each y∈Y, then the inverse images of H-sets are H-sets.

We get the similar corollaries as we do after Theorem 2.2 by accepting X or Y respectively almost regular or regular or e.d. For example:

Corollary 2.16. Let (X, τ) be an e.d. space, f: X→Y regular closed and pre-open, and $f^{-1}(y)$ τ_{co}-compact for each y∈Y. If G is an H-set, then $f^{-1}(G)$ is an s-set.

Corollary 2.17. (Theorem 2.3 in [6]) Let (X, τ) be an e.d. space, f: X→Y be a regular closed, almost-open surjection, and $f^{-1}(y)$ be S-set for each y∈Y. If G is an H-set in Y, then $f^{-1}(G)$ is S-set in X.

Corollary 2.18. (Theorem 2.4 in [6]) Let (X, τ) be an e.d. space, f: X→Y be a regular closed, almost-open surjection with compact point inverses. If Y is quasi H-closed, then X is S-closed.

Lemma 2.19. Let X be e.d. space. If f: X→Y is β-closed and α-open, then f is weakly semi-closed and pre-open.

Proof. It is known that every α-open function is pre-open. If A is a regular closed set, since X is e.d., A is open at the same time. So f(A) is an α-open,

β-closed set. We get $[f(A)]^{c}_{s} \subset f(A) \subset [f(A)]^{s}_{s}$, hence $f(A)=[f(A)]^{a}_{s}$ and f(A) is regular-open set. Since every regular open set is semi-closed, f(A) is a semi-closed set.

Corollary 2.20. (Theorem 2.8 in [6]). Let X be e.d. space and f: X→Y an

α-open, β-closed surjection and $f^{-1}(y)$ be S-set for each y∈Y. If G is S-set in Y, then $f^{-1}(G)$ is S-set in X.

It is kno n that if f: X→Y is an open bijection, then $f^{-1}(G)$ is an H-set in X, for each H-set G in Y.

<u>Theorem 2.21.</u> Let $f: (X, \tau) \to (Y, \vartheta)$ be an α-open bijection. Then $f^{-1}(G)$ is an H-set for each H-set G in Y.

<u>Proof:</u> Let $f: (X, \tau) \to (Y, \vartheta)$ be an α-open bijection and G be an H-set in

(Y, ϑ). Then $f: (X, \tau) \to (Y, \vartheta^{\alpha})$ is an open bijection. Sivaraj showed that (Y, ϑ) and (Y, ϑ^{α}) have the same H-sets in [7]. So G is an H-set in (Y, ϑ^{α}). $f^{-1}(G)$ will be an H-set in (X, τ).

<u>Corollary 2.22.</u> Let $f: (X, \tau) \to (Y, \vartheta)$ be an α-open bijection. If Y is quasi

H-closed, then X is quasi-H-closed.

<u>Corollary 2.23.</u> (Theorem 2.1 in [6]) If a function $f: (X, \tau) \to (Y, \vartheta)$ is an

α-open bijection and Y is S-closed, then X is quasi-H-closed.

<u>Corollary 2.24.</u> Let X be almost regular, $f: X \to Y$ α-open bijection. If G is an H-set in Y, then $f^{-1}(G)$ is an N-Set.

<u>Corollary 2.25.</u> Let X be regular, $f: X \to Y$ α-open bijection. If G is an H-set in Y, then $f^{-1}(G)$ is compact.

<u>Corollary 2.26.</u> Let X be e.d., $f: X \to Y$ α-open bijection. If G is an H-set in Y, then $f^{-1}(G)$ is an s-set in X.

<u>Corollary 2.27.</u> Let X be e.d., $f: X \to Y$ α-open bijection. If Y is quasi H-closed, then X is s-closed.

Again we get many corollaries as in the same way we do after Theorem 2.2.

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T.H.Jalvača. Par s-kopām, H-kopām un θ-perfektām funkcijām.

Anotācija. Šis darbs ir Mašhora, Allama un Zahrana darba "Par S-slēgtām telpām", (1991), [6] vispārinājums.

Т.Ялвач. О s-множествах, H-множествах и θ-совершенных отображениях.

Аннотация. Данная работа является обобщением работы А.С. Мешхоура, А.А. Аллама и А.М. Захрана "О S-замкнутых отображениях" [6].

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A NOTE ON FUZZY COMPACTNESS (*)

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ABSTRACT

In this paper, we give some characterizations of fuzzy compactness in Lowen's sense, emphasizing the one given in terms of ultrafilters on X, which is applied to give an easy proof of Alexander's Subbase Theorem. Strong fuzzy compactness and ultra fuzzy compactness are also characterized in terms of ultrafilters, emphasis being done in the analogies and differences between the three definitions of compactness in fuzzy topological spaces.

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PALABRAS CLAVE: t-prefilter, t*-prefilter, fuzzy-compactness, ultrafilter.

1.- PRELIMINARIES

In the sequel, I_0 and I_1 will denote the intervals $(0,1]$ and $[0,1)$, as usual. For any set A, $2^{(A)}$ will mean the family of all the finite subsets of A. We will denote c_t ($t \in I$) the fuzzy subset with constant value t.

DEFINITION 1.1.- [4] A subset $\mathcal{F} \subset I^X$ is a *prefilter* on X if and only if $\mathcal{F} \neq \emptyset$ and:

(i) for all $\mu, \nu \in \mathcal{F}$, we have $\mu \wedge \nu \in \mathcal{F}$;

(ii) if $\mu \geq \nu$ and $\nu \in \mathcal{F}$, then $\mu \in \mathcal{F}$;

(iii) $c_0 \notin \mathcal{F}$

(*) The main part of this paper was communicated in the XV Jornadas Luso-Espanholas de Matemática in 1990, Evora (Portugal)

DEFINITION 1.2.- [4] Given a prefilter $\mathcal{F}$ the *characteristic set* of $\mathcal{F}$ with respect to 0 is defined as:

$$C(\mathcal{F}) = \{\alpha \in I : \forall \nu \in \mathcal{F} \;\; \exists x \in X \text{ such that } \nu(x) > \alpha\}$$

From the definition it is easily derived that:

$$C(\mathcal{F}) = \{\alpha \in I \; ; \; c_\alpha \notin \mathcal{F}\}$$

and hence, the characteristic set of a prefilter will be either $[0, c]$ with $c \in I_1$ or $[0, c)$ with $c \in I_0$.

The number:

$$c(\mathcal{F}) = \sup C(\mathcal{F}) = \inf\{t \; ; \; c_t \in \mathcal{F}\}$$

is called the *characteristic value* of a prefilter $\mathcal{F}$

DEFINITION 1.3.- [4] A prefilter $\mathcal{F}$ is called a *prime* prefilter if for all $\mu, \nu \in I^X$ such that $\mu \vee \nu \in \mathcal{F}$ we have either $\mu \in \mathcal{F}$ or $\nu \in \mathcal{F}$.

DEFINITION 1.4.- [4] If $\mathcal{F}$ is a prefilter, then the *adherence* of $\mathcal{F}$ is defined as:

$$\text{adh}\, \mathcal{F} = \inf_{\nu \in \mathcal{F}} \bar{\nu}$$

$\bar{\nu}$ being the fuzzy topological closure of ν.
The *limit* of $\mathcal{F}$ is defined as:

$$\lim \mathcal{F} = \inf_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} \text{adh}\, \mathcal{G}$$

where $\mathcal{P}_m(\mathcal{F})$ is the set of all minimal prime prefilters finers than $\mathcal{F}$.

In [1] Lowen defined fuzzy-compactness in the following way:

DEFINITION 1.5.- [1] A fuzzy topological space (X, δ) is *fuzzy-compact* if for each family $\beta \subset \delta$ and $\alpha \in I_0$ such that $\sup_{\mu \in \beta} \mu \geq \alpha$ and for all $\epsilon \in (0, \alpha]$, there exists $\beta_\epsilon \in 2^{(\beta)}$ such that $\sup_{\mu \in \beta_\epsilon} \mu \geq \alpha - \epsilon$.

In a subsequent paper, [4], he gave the characterization below, in terms of prefilters and adherences:

PROPOSITION 1.1.- [4] *A fuzzy topological space (X, δ) is fuzzy-compact if and only if each one of this conditions is satisfied.*

(i) $\forall \mathcal{F}$ prefilter on X such that $C(\mathcal{F}) \neq \{0\}$, we have:

$$\sup_{x \in X} \operatorname{adh} \mathcal{F}(x) \geq c(\mathcal{F}).$$

(ii) $\forall \mathcal{F}$ prefilter prime on X, we have:

$$\sup_{x \in X} \operatorname{adh} \mathcal{F}(x) \geq c(\mathcal{F}).$$

DEFINITION 1.6.- [5] A prefilter $\mathcal{F}$ on X is said to be a t-*prefilter* if $c_t \notin \mathcal{F}$.

NOTE: We call the attention to the following facts:

(i) $C(\mathcal{F}) = [0, t] \iff \mathcal{F}$ is a t-prefilter and it is not a $(t + \epsilon)$-prefilter for every $\epsilon \in (0, 1 - t]$.

(ii) $C(\mathcal{F}) = [0, t) \iff \mathcal{F}$ is a $(t - \epsilon)$-prefilter for every $\epsilon \in (0, t]$ and it is not a t-prefilter.

(iii) $c(\mathcal{F}) = t \iff \forall \epsilon \in (0, t] \quad c_{t-\epsilon} \notin \mathcal{F}$ and $\forall \delta \in (0, 1 - t] \quad c_{t+\delta} \in \mathcal{F}$.

DEFINITION 1.7.- [5] Given a filter $\mathfrak{F}$ on X and $t \in [0, 1)$, we call $\omega_t(\mathfrak{F})$ the t-prefilter:

$$\omega_t(\mathfrak{F}) = \{\mu \in I^X \; ; \; \mu^{-1}(t, 1] \in \mathfrak{F}\}$$

For a t-prefilter $\mathcal{F}$ on X, we call $i_t(\mathcal{F})$ the filter:

$$i_t(\mathcal{F}) = \{\mu^{-1}(t, 1] \; ; \; \mu \in \mathcal{F}\}$$

DEFINITION 1.8.- [6] Given a filter $\mathfrak{F}$ on X and $t \in [0, 1)$, we call $\omega_{t^*}(\mathfrak{F})$ the prefilter:

$$\omega_{t^*}(\mathfrak{F}) = \langle\{\mu \in I^X \; ; \; \mu^{-1}[t, 1] \in \mathfrak{F}\}\rangle$$

For a prefilter $\mathcal{F}$ on X, we call $i_{t^*}(\mathcal{F})$ the filter:

$$i_{t^*}(\mathcal{F}) = \{\mu^{-1}[t, 1] \; ; \; \mu \in \mathcal{F}\}$$

NOTE: It is easy to see that:

$$C(\omega_t(\mathfrak{F})) = [0, t] \quad \text{and} \quad C(\omega_{t^*}(\mathfrak{F})) = [0, t)$$

DEFINITION 1.9.- [7] For a given a prefilter $\mathcal{F}$ on X, we call $\mathcal{F}^*$ the prefilter:

$$\mathcal{F}^* = \{\sup_{\epsilon \in I_1}(\mu_\epsilon - \epsilon) : \{\mu_\epsilon\}_{\epsilon \in I_0} \in \mathcal{F}^{I_0}\}$$

This prefilter has the advantage of containing the constant corresponding to the characteristic value, which will be useful in the sequel. This is expressed in the following lemma.

LEMMA 1.1.- *Let $\mathcal{F}$ be a prefilter $\mathcal{F}$ such that $c(\mathcal{F}) = t$. Then $C(\mathcal{F}^*) = [0, t)$.*

2.- SOME CHARACTERIZATIONS OF FUZZY COMPACTNESS

The results contained in [4], about fuzzy compactness, bring us to the following:

PROPOSITION 2.1.- *If (X, δ) is a fuzzy topological space, then the following conditions are equivalent:*

(i) *(X, δ) is fuzzy-compact.*

(ii) *For any prefilter $\mathcal{F}$ on X such that $C(\mathcal{F}) = [0, t]$, with $t \in (0, 1)$, we have:*

$$\sup_{x \in X} \operatorname{adh} \mathcal{F}(x) \geq t.$$

(iii) *For any prefilter $\mathcal{F}$ on X such that $C(\mathcal{F}) = [0, t)$, with $t \in (0, 1]$, we have:*

$$\sup_{x \in X} \operatorname{adh} \mathcal{F}(x) \geq t.$$

(iv) *For any ultrafilter $\mathfrak{U}$ on X and $t \in (0, 1]$, we have:*

$$\sup_{x \in X} \lim \bigcap_{s < t} \omega_s(\mathfrak{U})(x) \geq t.$$

(v) *For any ultrafilter $\mathfrak{U}$ on X and $t \in (0, 1]$, we have:*

$$\sup_{x \in X} \lim \omega_{t^-}(\mathfrak{U})(x) \geq t.$$

(vi) For any ultrafilter $\mathfrak{U}$ on X and $t \in (0,1)$, we have:

$$\sup_{x \in X} \lim \omega_t(\mathfrak{U})(x) \geq t.$$

Proof: The first thing to be done is to construct a prefilter whose characteristic set is an open interval, associated to every prefilter $\mathcal{F}$, whose characteristic set is a closed interval.

(ii) ⇒ (iii).
If $\mathcal{F}$ is a prefilter on X such that $C(\mathcal{F}) = [0,t)$, with $t \in (0,1]$, for all $s \in (0,t)$ we call $\mathcal{F}^s = (\mathcal{F}, \{c_{s'}\}_{s' \in (s,1]})$.
We have that $C(\mathcal{F}^s) = [0,s]$:

- $c_s \notin \mathcal{F}^s$ because if $c_s \in \mathcal{F}^s$, there would exist $\mu \in \mathcal{F}$ and $s' \in (s,1]$ such that $c_s \geq \mu \wedge c_{s'}$ and hence we would have that $c_s \geq \mu$, and then, $c_s \in \mathcal{F}$, but this is impossible because $C(\mathcal{F}) = [0,t)$.
- $\forall \epsilon \in (0, 1-s] \quad c_{s+\epsilon} \in \mathcal{F}^s$ because $s + \epsilon \in (s,1]$.

Therefore, $s \in (0,t) \subsetneq (0,1)$ and we can use *(ii)* to conclude

$$\sup_{x \in X} \operatorname{adh} \mathcal{F}(x) \geq \sup_{x \in X} \operatorname{adh} \mathcal{F}^s(x) \geq s \qquad \forall s \in (0,t)$$

Consequently,

$$\sup_{x \in X} \operatorname{adh} \mathcal{F}(x) \geq t$$

(iii) ⇒ (iv).
It is a straightforward consequence of the fact that $C(\bigcap_{s<t} \omega_s(\mathfrak{U})) = [0,t)$:

- $c_t \in \bigcap_{s<t} \omega_s(\mathfrak{U})$, because $c_t^{-1}(s,1] = X \quad \forall s < t$.
- $\forall \epsilon \in (0,t], \quad c_{t-\epsilon} \notin \bigcap_{s<t} \omega_s(\mathfrak{U})$ because $c_{t-\epsilon}^{-1}(t-\epsilon,1] = \emptyset \notin \mathfrak{U}$.

(iv) ⇒ (v).
It follows of the inclusion $\omega_{t^-}(\mathfrak{U}) \subset \bigcap_{s<t} \omega_s(\mathfrak{U})$

(v) ⇒ (vi).
It is a consequence of $\omega_t(\mathfrak{U}) \subset \omega_{t^-}(\mathfrak{U})$

(vi) ⇒ *(ii)*.
If $\mathcal{F}$ is a prefilter on X such that $C(\mathcal{F}) = [0,t]$, with $t \in (0,1)$, we call $\mathcal{F}^{\#}$ the maximal t-prefilter which contains $\mathcal{F}$, then $\mathcal{F}^{\#} = \omega_t i_t(\mathcal{F}^{\#})$ and we have that:

$$\sup_{x\in X} \text{adh}\, \mathcal{F}(x) \geq \sup_{x\in X} \text{adh}\, (\mathcal{F}^{\#})(x) = \sup_{x\in X} \text{adh}\, \omega_t i_t(\mathcal{F}^{\#})(x)$$

But $i_t(\mathcal{F}^{\#})$ is an usual ultrafilter, so we can use *(vi)* and we get that

$$\sup_{x\in X} \text{adh}\, \mathcal{F}(x) \geq t$$

We finish the proof just pointing out that, by proposition 1.1, *(i)* $\iff$ *(ii)* and *(iii)*. ◻

DEFINITION 2.1.- [2] A fuzzy topological space (X,δ) is *t-compact* with $t \in [0,1)$ if for each family $\beta \subset \delta$ such that $\bigcup_{\mu\in\beta} \mu^{-1}(t,1] = X$, there exists $\beta_1 \in 2^{(\beta)}$ such that $\bigcup_{\mu\in\beta_1} \mu^{-1}(t,1] = X$.

DEFINITION 2.2.- [3] A fuzzy topological space (X,δ) is *strong-fuzzy-compact* if it is t-compact, $\forall t \in [0,1)$

PROPOSITION 2.2.- *If (X,δ) is a fuzzy topological space and $t \in (0,1]$, then the following are equivalent:*

(i) (X,δ) *is* $(1-t)$*-compact.*

(ii) For any prefilter $\mathcal{F}$ on X such that $C(\mathcal{F}) = [0,t)$, we have:

$$\exists x \in X \quad \textit{such that} \quad \text{adh}\, \mathcal{F}(x) \geq t.$$

(iii) For any ultrafilter $\mathfrak{U}$ on X, we have:

$$\exists x \in X \quad \textit{such that} \quad \lim \bigcap_{s<t} \omega_s(\mathfrak{U})(x) \geq t.$$

(iv) For any ultrafilter $\mathfrak{U}$ on X, we have:

$$\exists x \in X \quad \textit{such that} \quad \lim \omega_{t^*}(\mathfrak{U})(x) \geq t.$$

Proof:

(i) $\Longleftrightarrow$ (iv). It is well known that
(X,δ) $(1-t)$-compact $\Longleftrightarrow$ $(X, i_{1-t}(\delta))$ compact, or equivalently:
Given an ultrafilter $\mathfrak{U}$, there exists some $x \in X$ such that $\mathfrak{U}$ converges to x in X endowed with the topology $i_{1-t}(\delta)$, therefore:
For any $\mu \in \delta$ such that $x \in \mu^{-1}(1-t,1]$ we have that $\mu^{-1}(1-t,1] \in \mathfrak{U}$, even more,
for any $\nu \in \delta^c$ such that $x \in (\nu^c)^{-1}(1-t,1]$ we have that $(\nu^c)^{-1}(1-t,1] \in \mathfrak{U}$.
Having into account the following equivalences:

(a) $x \in (\nu^c)^{-1}(1-t,1] \Longleftrightarrow \nu(x) < t$,

(b) $(\nu^c)^{-1}(1-t,1] \in \mathfrak{U} \Longleftrightarrow (\nu^c)^{-1}[0,1-t] \notin \mathfrak{U} \Longleftrightarrow \nu^{-1}[t,1] \notin \mathfrak{U}$,

we obtain,
for any $\nu \in \delta^c$ such that $\nu(x) < t$ we have that $\nu^{-1}[t,1] \notin \mathfrak{U}$, and so,
for any $\nu \in \delta^c$ such that $\nu^{-1}[t,1] \in \mathfrak{U}$ we have that $\nu(x) \geq t$. Hence,

$$\inf_{\substack{\nu \in \delta^c \\ \nu^{-1}[t,1] \in \mathfrak{U}}} \nu(x) \geq t \Longleftrightarrow \lim \omega_{t^*}(\mathfrak{U})(x) \geq t.$$

On the other hand, if there exists some $x \in X$ such that $\lim \omega_{t^*}(\mathfrak{U})(x) \geq t$, then for any $\nu \in \delta^c$ such that $\nu^{-1}[t,1] \in \mathfrak{U}$ we have that $\nu(x) \geq t$.
Hence, for any $\nu \in \delta^c$ such that $\nu(x) < t$, we have that $\nu^{-1}[t,1] \notin \mathfrak{U}$, and consequently, for any $\mu \in \delta$ such that $x \in \mu^{-1}(1-t,1]$ we have that $\mu^{-1}(1-t,1] \in \mathfrak{U}$ $\Longleftrightarrow$ $(X, i_{1-t}(\delta))$ compact $\Longleftrightarrow$ (X,δ) t-compact.

(iv) ⇒ (ii).
If $\mathcal{F}$ is a prefilter on X such that $C(\mathcal{F}) = [0,t]$, we call $\mathcal{F}^{\#}$ the maximal t^*-prefilter which contains $\mathcal{F}$, then $\mathcal{F}^{\#} = \omega_{t^*}(i_{t^*}(\mathcal{F}^{\#}))$ and we have that:

$$\forall x \in X \qquad \text{adh}\, \mathcal{F}(x) \geq \text{adh}\, (\mathcal{F}^{\#})(x) = \text{adh}\, \omega_{t^*}(i_{t^*}(\mathcal{F}^{\#}))(x)$$

But $i_{t^*}(\mathcal{F}^{\#})$ is an usual ultrafilter, so we can use *(v)* and we get that

$$\exists x \in X \quad \text{such that} \quad \text{adh}\, \mathcal{F}(x) \geq t$$

(ii) ⇒ (iii).
This is a consequence of the equality $C(\bigcap_{s<t} \omega_s(\mathfrak{U})) = [0,t)$:

(iii) ⇒ (iv).
This is immediate because $\omega_{t^*}(\mathfrak{U}) \subset \bigcap_{s<t} \omega_s(\mathfrak{U})$. ∎

COROLLARY 2.1.- *If (X,δ) is a fuzzy topological space then the following conditions are equivalent:*

(i) (X,δ) *is strong-fuzzy-compact.*

(ii) For any prefilter $\mathcal{F}$ *on* X *such that* $C(\mathcal{F}) = [0,t)$, *with* $t \in (0,1]$, *we have:*

$$\exists x_t \in X \quad \text{such that} \quad \text{adh}\, \mathcal{F}(x_t) \geq t.$$

(iii) For any ultrafilter $\mathfrak{U}$ *on* X *and* $t \in (0,1]$, *we have:*

$$\exists x_t \in X \quad \text{such that} \quad \lim \bigcap_{s<t} \omega_s(\mathfrak{U})(x_t) \geq t.$$

(iv) For any ultrafilter $\mathfrak{U}$ *on* X *and* $t \in (0,1]$, *we have:*

$$\exists x_t \in X \quad \text{such that} \quad \lim \omega_{t^-}(\mathfrak{U})(x_t) \geq t.$$

DEFINITION 2.3.- [3] A fuzzy topological space (X,δ) is *ultra-fuzzy-compact* if the topological space $(X, i(\delta))$ is compact.

PROPOSITION 2.3.- *Let* (X,δ) *be a fuzzy topological space. The following conditions are equivalent:*

(i) (X,δ) *is ultra-fuzzy-compact.*

(ii) There exists $x \in X$ *such that for any prefilter* $\mathcal{F}$ *on* X *such that* $C(\mathcal{F}) = [0,t]$, *with* $t \in (0,1)$, *we have:*

$$\text{adh}\, \mathcal{F}(x) \geq t.$$

(iii) There exists $x \in X$ *such that for any prefilter* $\mathcal{F}$ *on* X *such that* $C(\mathcal{F}) = [0,t)$, *with* $t \in (0,1]$, *we have:*

$$\text{adh}\, \mathcal{F}(x) \geq t.$$

(iv) There exists $x \in X$ *such that for any ultrafilter* $\mathfrak{U}$ *on* X *and* $t \in (0,1]$, *we have:*

$$\lim \bigcap_{s<t} \omega_s(\mathfrak{U})(x) \geq t.$$

(v) There exists $x \in X$ *such that for any ultrafilter* $\mathfrak{U}$ *on* X *and* $t \in (0,1]$, *we have:*

$$\lim \omega_{t^-}(\mathfrak{U})(x) \geq t.$$

(vi) There exists $x \in X$ *such that for any ultrafilter* $\mathfrak{U}$ *on* X *and* $t \in (0,1)$, *we have:*

$$\lim \omega_t(\mathfrak{U})(x) \geq t.$$

Proof:

(i)$\Longleftrightarrow$*(v)*

By definition 2.3, (X,δ) ultra-fuzzy-compact $\Longleftrightarrow$ $(X, i(\delta))$ compact.
That is equivalent to say:
For any ultrafilter $\mathfrak{U}$, there exists $x \in \bigcap_{U \in \mathfrak{U}} \overline{U} = \bigcap_{\substack{U \in \mathfrak{U} \\ U \in i(\delta)'}} U = \bigcap_{\substack{\mu^{-1}[0,t] \in \mathfrak{U} \\ \mu \in \delta, t \in I_1}} \mu^{-1}[0,t]$.
where the closure is considered in the topological space $(X, i(\delta))$.

Hence, for any ultrafilter $\mathfrak{U}$ there exists $x \in X$ such that $\forall \mu \in \delta$ and $\forall t \in I_1$ with $\mu^{-1}[0,t] \in \mathfrak{U}$ we have that $x \in \mu^{-1}[0,t]$ and equivalently, for any ultrafilter $\mathfrak{U}$ exists $x \in X$ such that $\forall \nu \in \delta^c$ and $\forall t \in I_1$ with $\nu^{-1}[1-t,1] \in \mathfrak{U}$ we have that $x \in \nu^{-1}[1-t,1]$. Finally, we can say that for any ultrafilter $\mathfrak{U}$ exists $x \in X$ such that $\forall t \in I_0$ we have that $\operatorname{adh} \omega_{t^*}(\mathfrak{U})(x) \geq t$

(ii) $\Rightarrow$ *(iii)*.

Let $\mathcal{F}$ be a prefilter on X, such that $C(\mathcal{F}) = [0,t)$, for some $t \in (0,1]$. For all $s \in (0,t)$ we call $\mathcal{F}' = \langle \mathcal{F}, \{c_{r}\}_{r \in (s,t)} \rangle$.
We have that $C(\mathcal{F}') = [0,s]$, therefore, $s \in (0,t)$ $(0,1)$ and we can use *(ii)*, to conclude

$$\exists x \in X \quad \text{such that} \quad \operatorname{adh} \mathcal{F}(x) \geq \operatorname{adh} \mathcal{F}'(x) \geq s \qquad \forall s \in (0,t)$$

Consequently,

$$\exists x \in X \quad \text{such that} \quad \operatorname{adh} \mathcal{F}(x) \geq t$$

(iii) $\Rightarrow$ *(iv)*.

It is straightforward because: $C(\bigcap_{s<t} \omega_s(\mathfrak{U})) = [0,t)$:

(iv) $\Rightarrow$ *(v)*.

It follows from the inclusion $\omega_{t^*}(\mathfrak{U}) \subset \bigcap_{s<t} \omega_s(\mathfrak{U})$

(v) $\Rightarrow$ *(vi)*.

It is a consequence of the fact: $\omega_t(\mathfrak{U}) \subset \omega_{t^*}(\mathfrak{U})$

(vi) $\Rightarrow$ *(ii)*.

If $\mathcal{F}$ is a prefilter on X such that $C(\mathcal{F}) = [0,t]$, with $t \in (0,1)$, we call $\mathcal{F}^{\#}$ the maximal t-prefilter which contains $\mathcal{F}$. Then $\mathcal{F}^{\#} = \omega_t i_t(\mathcal{F}^{\#})$ and we have that:

$$\forall x \in X \quad \operatorname{adh} \mathcal{F}(x) \geq \operatorname{adh} (\mathcal{F}^{\#})(x) = \operatorname{adh} \omega_t i_t(\mathcal{F}^{\#})(x)$$

But $i_t(\mathcal{F}^\#)$ is an usual ultrafilter, so we can use *(vi)*, and we get that

$$\exists x \in X \quad \text{such that} \quad \text{adh}\, \mathcal{F}(x) \geq t \; \blacksquare$$

OBSERVATION: If we consider topologies in Lowen's sense, those that contain all of the constants, in the results of propositions 2.1, 2.2 and 2.3 and corollary 2.1 we would get the equality.

OBSERVATION: From proposition 2.1 and 2.3 and corollary 2.2 we see inmediately that

ultra-fuzzy-compactness $\Longrightarrow$ strong-fuzzy-compactness $\Longrightarrow$ fuzzy-compactness

Here we get more than that, we see exactly which is the differance between the three definitions of compactness and why the equivalence is not true.

3.- AN EASY PROOF OF THE SUBBASE THEOREM

The above results allow us to give an easy proof of the Alexander subbase theorem. We need a previous lemma characterizing the adherence in terms of bases and subbases.

LEMMA 3.1.- *Given a fuzzy topological space (X,δ) and a prefilter $\mathcal{F}$ on X, we have that:*

(i) If β is a base generating δ;

$$\text{adh}\, \mathcal{F}(x) = \inf_{\substack{\nu \in \mathcal{F} \\ \nu^* \in \beta \\ \nu^* \supseteq \nu}} \nu^*(x)$$

(ii) If $\mathcal{F}$ is prime and Σ is a subbase generating δ;

$$\text{adh}\, \mathcal{F}(x) = \inf_{\substack{\nu \in \mathcal{F} \\ \nu^* \in \Sigma}} \nu^*(x)$$

Proof:

(i) We first prove the inequality $\leq$.

$\text{adh}\, \mathcal{F}(x) = \inf_{\nu \in \mathcal{F}} \bar{\nu}(x) = \inf_{\substack{\nu \in \mathcal{F} \\ \nu^* \in \delta}} \nu^*(x)$.

Since $\beta \subset \delta$, we can conclude: $\text{adh}\, \mathcal{F}(x) \leq \inf_{\substack{\nu \in \mathcal{F} \\ \nu^* \in \beta}} \nu^*(x)$.

Now, we only have to show that for any constant $C >$ adh $\mathcal{F}$, we have: $C > \inf_{\substack{\nu \in \mathcal{F} \\ \nu^c \in \beta}} \nu(x)$.

But adh $\mathcal{F} < C$ implies there must exist $\nu \in \mathcal{F}$ such that $\nu^c \in \delta$ and $\nu(x) < C$. Since β is a base generating δ, then there exist a family $\{\mu_j\}_{j \in J} \subset \beta$ such that $\nu^c = \sup_{j \in J} \mu_j$, and whence, $\nu = \inf_{j \in J} \mu_j^c$.

Consequently:

$$\inf_{\substack{\nu \in \mathcal{F} \\ \nu^c \in \beta}} \nu(x) \leq \inf_{j \in J} \mu_j^c(x) = \nu(x) < C$$

(ii) In the same way, the inequality $\leq$ follows easily from the fact that $\Sigma \subset \delta$.

It is only remainsto prove that for any constant $C >$ adh $\mathcal{F}$,it is verified: $C > \inf_{\substack{\nu \in \mathcal{F} \\ \nu^c \in \Sigma}} \nu(x)$.

But adh $\mathcal{F} < C$ implies there must exist $\nu \in \mathcal{F}$ such that $\nu^c \in \delta$ and $\nu(x) < C$. Since Σ is a subbase generating δ, there exists a family $\{\mu_{j,i}\}_{j \in J, i=1,2,\ldots,n_j} \subset \Sigma$ such that

$\nu^c = \sup_{j \in J} \inf_{i=1,2,\ldots,n_j} \mu_{j,i}$, or, $\nu = \inf_{j \in J} \sup_{i=1,2,\ldots,n_j} \mu_{j,i}^c$.

Then, for any $j \in J$ we have that $\sup_{i=1,2,\ldots,n_j} \mu_{j,i}^c \geq \nu \in \mathcal{F}$,

and whence $\sup_{i=1,2,\ldots,n_j} \mu_{j,i}^c \in \mathcal{F}$.

Taking into account that $\mathcal{F}$ is a prime prefilter, we draw that there must exist $i_j \in \{1, 2, \ldots, n_j\}$ such that $\mu_{j,i_j}^c \in \mathcal{F}$.

Consequently

$$\inf_{\substack{\nu \in \mathcal{F} \\ \nu^c \in \Sigma}} \nu(x) \leq \inf_{j \in J} \mu_{j,i_j}^c(x) \leq \inf_{j \in J} \sup_{i=1,2,\ldots,n_j} \mu_{j,i}^c(x) = \nu(x) < C \blacksquare$$

THEOREM 3.1.- (Alexander) (*A fuzzy topological space* (X, δ) *is fuzzy-compact if and only if given a subbase* Σ *generating* δ *we have that for each family* $\beta \subset \Sigma$ *and for every* $\alpha \in I_0$ *such that* $\sup_{\mu \in \beta} \mu \geq \alpha$ *and for each* $\epsilon \in (0, \alpha]$, $\exists \beta_\epsilon \in 2^{(\beta)}$ *such that* $\sup_{\mu \in \beta_\epsilon} \mu \geq \alpha - \epsilon$.

Proof:

The if part is trivial.

On the other hand, if (X, δ) were not fuzzy-compact, by characterization *(v)* of proposition 2.1, we know that there exists an ultrafilter $\mathfrak{U}$ and $t \in (0, 1]$ such that $\sup_{x \in X} \text{adh}\, \omega_{t^-}(\mathfrak{U})(x) < t$.

That is, we can choose $\gamma \in (0,t]$ such that $\text{adh}\, \omega_{t^*}(\mathfrak{U})(x) \leq t - \gamma \quad \forall x \in X$.
Using lemma 3.1, we can write
$\inf_{\substack{\nu \in \omega_{t^*}(\mathfrak{U}) \\ \nu^c \in \Sigma}} \nu \leq c_{t-\gamma}$, or equivalently, $\sup_{\mu \in \beta} \mu \geq c_{1-t+\gamma}$, where $\beta = \{\mu \in \Sigma \, ; \, \mu^c \in \omega_{t^*}(\mathfrak{U})\}$, we have that
$\sup_{\mu \in \beta} \mu \geq c_{1-t+\gamma}$.
Now, by the hypothesis, for any $\epsilon \in (0, 1-t+\gamma]$, there exist $\beta_\epsilon \in 2^{(\beta)}$ such that $\sup_{\mu \in \beta_\epsilon} \mu \geq 1-t+\gamma-\epsilon$.
Therefore $(\sup_{\mu \in \beta_\epsilon} \mu)^{-1}[1-t+\gamma-\epsilon, 1] = \bigcup_{\mu \in \beta_\epsilon} \mu^{-1}[1-t+\gamma-\epsilon, 1] = X$.
Since $\mathfrak{U}$ is an usual ultrafilter, there must exist $\mu \in \beta_\epsilon$ such that $\mu^{-1}[1-t+\gamma-\epsilon, 1] \in \mathfrak{U}$.
On the other hand $\mu^c \in \omega_{t^*}(\mathfrak{U})$ because $\mu \in \beta_\epsilon \subset \beta$ and we find a contradiction by choosing $\epsilon < \gamma$. ∎

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H. Gutierres, M.A. de Prada Visente. Par fazi-kompaktību.

Anotācija. Darbā sniegti fazi-topoloģisku telpu jauni kompaktības Lovena nozīmē raksturojumi. Viens no tiem dota ultrafiltru terminos, kas ļauj iegūt jaunu, vienkāršu pierādījumu Aleksandera teorēmai par priekšbāzi. Ultrafiltru terminos iegūti arī stingrās kompaktības un ultrafazi-kompaktības raksturojumi. Liela vērība pievērsta šo triju kompaktības definīciju fazi-topoloģiskām telpām līdzību un atšķirību analīzei.

Х. Гутиеррес, М.А. Де Прада Виценте. Заметка о нечеткой компактности.

Аннотация. В работе приводятся новые характеристики компактности в смысле Ловена для нечетких топологических пространств. Одна из них дана в терминах ультрафильтров, что позволяет получить новое простое доказательство теоремы Александера о предбазе.

Кроме того, получены характеристики сильной компактности и ультранечеткой компактности в терминах ультрафильтров. Значительное внимание уделяется обсуждению сходства и различия этих трех определений компактности для нечетких топологических пространств.

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The stability transformation of discrete random processes by finite state automata: Theory and applications

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Abstract. The paper presents a survey of the authors' investigations in the area of implementation of qualitative physically-based random number generators. The authors' approach is based on the stabilization of the process of physical primary random number generators by a suitable finite state automaton. The method allows to obtain discrete distributions up to a given accuracy and confidence level using the primary generators with inprecisely determined parameters which can change during the time. Special attention is paid on the problems of rate of the constructed generators. This paper is essentially an extended version of a talk given by J. Lapiņš on June, 18, 1992 at the 14th Nordic Conference on Mathematical Statistics, Røros, Norway.

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Our investigation concerns the problem of constructing "good" random number generators. The use of pseudorandom number generators does not yet have a satisfactory logical foundation, but the speed, stability and accuracy of physical random number generators is far from being perfect. If we deal with stable generators, either of discrete or continuous distributions, then no special problems arise. Various methods exist of transforming them into other stable distributions with precisely known parameters (see A. Lorenz (1974), A. Lorenc (1976a)).

Serious problems occur when the parameters of the original distribution oscillate and the action of the generator is not sufficiently stabilizable by technical means. Assuming that the random number generator emits an independent sequence of random variables $X_1, X_2, \dots, X_n, \dots$, the problem of testing homogeneity arises. In this connection there appears the problem of finding a value n(e) which guarantees with probability $p > 1-\alpha$ the rejection of the hypothesis H_0 of homogeneity of the sequence, if for a given natural number h and for each i there exist s,t such that $i \le s < t \le i+h$, and the inequality

$$|P\{X_s < a\} - P\{X_t < a\}| > \varepsilon$$

holds for some a.

Clearly, for very small ε and α the value of n will be very large. In this connection we propose the following concept:

1) We test the primary generator using a high confidence level without demanding the precise estimation of the parameters (in this situation a moderate sample size is sufficient).

2) The sequence from the primary generator is subjected to the logical transform which guarantees high accuracy of the parameters (thus without additional testing the high confidence level is preserved).

In this paper the problems of finding stabilizing transformations of discrete random processes over a finite set by means of finite state automata are considered. A.S.Davis (1961) initiated a series of articles in which a stochastic automaton (a controlled Markov chain) was represented as a connection of a random input signal source (RSS) with a finite (deterministic) state automaton. However, the theoretical attractiveness and universality of this approach to implementing a stochastic automaton depends on some ideal properties of RSSs that are difficult to realise physically. By considering the relative stability (or instability) of the RSSs employed, A.Dvoretzky and J.Wolfowitz (1951) proposed a method of stabilizing the input process by an adder modulo n.

The stabilizing properties of adders in terms of finite Abelian groups were investigated also by N.Vorobyev (1954), and in automata-theoretical terms this problem of transforming a random sequence into the uniform distribution $\pi_0=(1/n,1/n,\dots,1/n)$ was restated by A.Gill(1962). Later this stabilization method was generalized by A.Lorencs (1974 a, 1976 b) and his colleagues J.Lapiņš and I.Mētra (1973). They developed a more effective technique of transformation than the adders or Gill automata and also applied new more precise estimation methods of the stabilization rate of the output process.

1. Basic definitions and topics.

Let $B_{v,\alpha,r}$ be the set of finite nonhomogeneous Markov chains $\{X_t\}_{t\geq 0}$ of order μ $(\mu\leq v)$ over the set $R=\{0,1,\dots,r\}$ satisfying the condition

$$\forall t\,\forall(x,x_t,\dots,x_0)\in R^{t+2}\,(P\{X_{t+1}=x|X_t=x_t,\dots,X_0=x_0\}\geq\alpha), \qquad (1)$$

where α is a positive real number. Let $A=[X,Y,Z,\Delta,\lambda]$ be a finite state automaton (FSA), $X\supset R$, $Y=\{0,1,\dots,m-1\}$, $Z=\{0,1,\dots,n-1\}$, $\Delta: Z\times X\to Z$, $\lambda: Z\to Y$. For any $z_0\in Z$ and $\{X_t\}_{t\geq 0}\in B_{v,\alpha,r}$ let us define inductively two following sequences $\{Z_t\}_{t\geq 0}$ and $\{Y_t\}_{t\geq 0}$ of random variables:

$$Z_0=\Delta(z_0,X_0),\ Y_0=\lambda(Z_0).$$

If for some t $(t>0)$ $Z_0, Z_1,\dots,Z_{t-1}$ are defined then $Z_t=\Delta(Z_{t-1},X_t)$ and $Y_t=\lambda(Z_t)$. In this case the sequence $\{Z_t\}_{t\geq 0}$ of random variables is a Markov chain of order $\mu+1$ and the sequence $\{Y_t\}_{t\geq 0}$ in general will not be a Markov chain.

Let us denote for a stochastic vector $\pi=(p_0,p_1,\dots,p_{m-1})$ with all positive components

$$\sup_{\{X_t\}_{t\geq 0}\in B_{v,\alpha,r}}\ \max_{z_0\in Z}\ \sup_{t\geq 0}\ \max_{(y,y_0,\dots,y_t)\in Y^{t+2}} |P\{Y_{t+T}=y|Y_t=y_t,\dots,Y_0=y_0\}-p_y|$$

by $\Omega(A,T,\pi)$.

Definition 1. FSA A transforms Markov chains from the set $B_{V,\alpha,r}$ into the random distribution π iff $\lim_{T\to\infty} \Omega(A,T,\pi)=0$.

In this paper we concentrate our attention (except the section 5) on FSA $A=[X,Y,Z,\Delta,\lambda]$ for which $Y=Z=\{0,1,...,n-1\}$ and $\forall z\in Z(\lambda(z)=z)$. Such FSA A we describe by the triple $A=[X,Z,\Delta]$.

There are several interesting and important subclasses of the class of FSA which is possible to use as stability transformers.

a) g - circula... transformer (Gauss automaton).

Let n and g be natural numbers and the greatest common divisor (g,n) of them is equal to 1. FSA $A=[X,Z,\Delta]$ is called a g-circulant transformer if $X\subset Z=\{0,1,...,n-1\}$, and $\forall x\in X$, $z\in Z$ $(\Delta(z,x)=zg+x \pmod n)$.

b) Gill automaton (adder modulo n).

In the special case if g=1 and $X=R=\{0,1,...,r\}$, the g-circulant transformer is called a Gill automaton.

c) Linear sequential machines (linear finite state automata).

Let p be a prime number, $q=p^s$, $L_{u,v}$ be a set of all uxv matrices over Galua field GF(q).

Definition 2. FSA $A=[X,Z,\Delta]$ is a linear sequential machine (LSM) over GF(q) if there exist positive numbers k,m, and matrices $C\in L_{k,k}$, $H\in L_{k,m}$ such, that $X=L_{m,1}$, $Z=L_{k,1}$ and $\forall z\in Z$, $x\in X$ $(\Delta(z,x)=Cz+Hx)$.

In this paper:

(i) the class M of FSA transforming Markov chains from the set $B_{V,\alpha,r}$ into the uniform distribution $\pi_0=(1/n,1/n,...,1/n)$ is described;

(ii) the rate of convergence of functional $\Omega(.\,,.\,,.)$ to 0 for FSA of the class M is estimated;

(iii) the subclass of FSA having the highest rate of convergence is described;

(iv) some possible applications of the theoretical results are discussed;

(v) some possible generalizations and modifications are discussed.

2. Stability automata and convergence rate.

Let $A=[X,Z,\Delta]$. We will use the so called transition matrices of the FSA A, i.e. the $n\times n$ matrices

$$A_x=(a_{zz'}(x)),\ x\in X,$$

$$\text{where } a_{zz'}(x)=\begin{cases} 1 & \text{if } \Delta(z,x)=z', \\ 0 & \text{otherwise.} \end{cases}$$

Let us denote also by A the matrix

$$A=1/(r+1)\sum_{x\in X} A_x.$$

Theorem 1. (A.Lorenc (1976 a.b, 1978 a)) The FSA $A=[X,Z,\Delta]$ transforms Markov chains from the set $B_{V,\alpha,r}$ into some random distribution $\pi=(p_0,p_1,\ldots,p_{n-1})$ iff the following two conditions are satisfied:

(i) $\forall x\in X,\ z,z'\in Z\ (z\neq z'\Rightarrow\Delta(z,x)\neq\Delta(z'x))$,

(ii) there exists a natural number ξ such that the matrix A^{ξ} has a positive column.

Besides the only one possible positive random distribution π is $\pi_0=(1/n,1/n,\ldots 1/n)$, and $\Omega(A,T,\pi_0)\leq(r+1)^{v}(1-(r+1)^{v}\alpha^{\xi+v})^{[T/(\xi+v)]}$.

Remark 1. From conditions (i) and (ii) of Theorem 1 it follows, that for some natural number γ all elements of the matrix A^{γ} are positive.

Remark 2. For any n there exist FSA A, satisfying conditions (i) and (ii) of Theorem 1 with $\gamma=]\log_{r+1}n[$. Here and below $]a[=-[-a]$, and $[a]$ is the integer part of the number a.

Theorem 2. (A.Lorenc (1976 c), J.Lapiņš and A.Lorenc (1986)) Let p be a prime number, $r+1=q=p^s$, $A=[X,Z,\Delta]$ be a LSM over GF(q), $X=L_{1,1}$, $Z=L_{k,1}$, $\forall z\in Z$, $x\in X$ $(\Delta(z,x)=Cz+Hx)$. The LSM **A** transforms Markov chains from the set $B_{v,\alpha,q-1}$ into the random distribution $\pi_0=(1/q^k,1/q^k,...1/q^k)$ iff the matrix C is nonsingular and matrices H, CH, ... $C^{k-1}H$ are linearly independent over GF(q). Moreover functional Ω in this case satisfies the inequality

$$\Omega(A,T,\pi_0)\le q^v(1-q^{v+1}\alpha)^{k+v)[T/(k+v)]}. \quad (2)$$

The following Theorem gives us a lower bound of the transformation rate of discrete random processes into some random distribution π.

Theorem 3. Let FSA $A=[X,Z,\Delta]$ transforms Markov chains from the set $B_{v,\alpha,r}$ into the random distribution π with at least two positive components. Then

$$\Omega(A,T,\pi)\ge 1/2\left(\frac{1-(r+1)\alpha}{1-(r-1)\alpha}\right)^T. \quad (3)$$

Remark. 3. In the case $\alpha\in[r/(r+1)^2,1/(r+1)]$ the inequality $\Omega(A,T,\pi)\ge((1-(r+1)\alpha))^T/2$ is proved and it seems very credible that the last inequality holds for $\alpha\in]0,1/(r+1)]$.

3. Special subclasses of stabilizing FSA.

a) Gill automata.

The lower bound of the transformation rate given in Theorem 3 is possible to improve in the case when **A** is a Gill automaton.

Theorem .4. (A.Lorenc, J.Lapiņš (1984)) Let $A=[X,Z,\Delta]$, $\{0,1,...,r\}\subset X$, $Z=\{0,1,...n-1\}$, $r<n$, $\forall x\in X$, $z\in Z$ $(\Delta(z,x)\equiv z+x \pmod n)$, and $(r+1,n)=m>1$. Then for the transformation rate of Markov chains from the set $B_{v,\alpha,r}$ into the random distribution $\pi_0=(1/n,1/n,...,1/n)$ by FSA **A** the following inequality fulfils

$$\Omega(A,T,\pi_0)\ge((m-1)/n)(1-(r+1)\alpha)^{[(T+v)/(1+v)]} \quad (4)$$

A.Lorenc (1976 b) has also obtained more precise formulas for the transformation rate of Markov chains from the set $B_{0,\alpha,r}$ (sequences of independent random variables) and from the set $B_{1,\alpha,r}$ (simple Markov chains) into the random distribution $\pi_0=(1/n,1/n,...,1/n)$ by the Gill automaton $A=[X,Z,\Delta]$. The proofs of these results are based on the property of circulant matrices that they can all be reduced to the diagonal form with the same orthogonal transformation. The following result is an example of these results corresponding to the case $v=0$, $r=1$.

Theorem 5. (A.Lorenc (1974 b)) Let $A=[X,Z,\Delta]$, $\{0,1\}\subset X$, $Z=\{0,1,...,n-1\}$, $\forall x\in X$, $z\in Z$ $(\Delta(z,x)\equiv z+x \pmod n)$. The FSA A transforms sequences of independent random variables $\{X_t\}_{t\geq 0}\in B_{0,\alpha,1}$ into the random distribution $\pi_0=(1/n,1/n,...,1/n)$, and

$$\Omega(A,T,\pi_0)\leq(\rho/n)(\rho^{T/2}+\rho^T/(1-\rho)^T),$$

where $\rho=1-4\alpha(1-\alpha)\sin^2(\pi/n)$.

b) g - circulant transformers.

Very precise formulas are obtained in A.Lorenc (1978b) for the transformation rate of sequences of independent random variables from the set $B_{0,\alpha,r}$ into the random distribution $\pi_0=(1/n,1/n,...,1/n)$ by the g-circulant transformer $A=[X,Z,\Delta]$. Here again we have mentioned only one particular example of these results corresponding to the case $v=0$, $r=1$.

Theorem 6. (A.Lorenc (1978 b)) Let p be a prime number, k and g be positive integers, $n=p^k$, $(g,n)=1$, $A=[X,Z,\Delta]$, $\{0,1\}\subset X$, $Z=\{0,1,...,n-1\}$, $\forall x\in X$, $z\in Z$ $(\Delta(z,x)\equiv gz+x \pmod n)$. The FSA A transforms sequences of independent random variables $\{X_t\}_{t\geq 0}\in B_{0,\alpha,1}$ into the random distribution $\pi_0=(1/n,1/n,...,1/n)$, and

$$\Omega(A,T,\pi_0)\leq 1/n\sum_{j=0}^{k-1}\varphi(p^{k-j})\left(\frac{\alpha^{p^{k-j}}+(1-\alpha)^{p^{k-j}}}{\alpha^{p^{k-j-1}}+(1-\alpha)^{p^{k-j-1}}}\right)^{p^j[T/\varphi(n)]},$$

where φ is the Euler function.

c) Linear sequential machines over finite Galua fields.

The following theorem characterizes the subclass of LSM which have the highest transformation rate known at present if we transform sequences of random variables from the set $B_{0,\alpha,r}$.

Theorem 7. (J.Lapiņš and A.Lorenc (1986)) Let p be a prime number, $r+1=q=p^s$, $Q=(q^k-1)/(q-1)$, $A=[X,Z,\Delta]$ be a LSM over GF(q), $X=L_{1,1}$, $Z=L_{k,1}$, $\forall z\in Z$, $x\in X$ $(\Delta(z,x)=Cz+Hx)$, matrices $H, CH,\dots,C^{k-1}H$ are linearly independent over GF(q), and the least positive integer τ, such that C^{τ} is the unit matrix over GF(q), is equal to q^k-1. Then LSM A transforms sequences of independent random variables from the set $B_{0,\alpha,q-1}$ into the random distribution $\pi_0=(1/q^k,1/q^k,\dots,1/q^k)$, and

$$\Omega(A,T,\pi\ )\le((q^k-1)/q^k)(1-q\alpha)^{[T/Q]q^{k-1}} \quad . \tag{5}$$

Remark 4. For any prime number p and natural numbers κ and s there exist matrices C and H, satisfying conditions of Theorem 7.

Remark 5. The inequality (5) becomes an equality if T/Q is an integer.

4. Applications.

The theoretical results discussed above have a number of different applications, some of which have already been used in practice:

(i) they can be used for the implementation of physical random number generators with a high confidence level and accuracy. We note that the described LSM have excellent properties - a high stabilization rate and, at the same time, a very simple structural implementation;

(ii) they can be used for creating the recurrence type cryptosystems. The actual cryptograms of these systems have practically non-informative statistics;

(iii) they can be used as an approach to proving interesting limit theorems, which can be expressed in classical terms of probability theory. As an example we can mention the generalizations given by A. Lorenc (1986) and A. Lorenc and A. Lapiņš (1990) of the results concerning the statistical properties of elementary symmetric functions (see J.D. Smith (1974)).

5. Generalizations and modifications.

The results mentioned in previous sections are generalized in several directions.

A) The set $B^{*}_{v,\alpha,r}$ of Markov chains $\{X_t\}_{t\geq 0}$ is introduced in A.Lorenc and J.Lapiņš (1984) for which

(i) $\{X_t\}_{t\geq 0}$ is a Markov chain over the set $R=\{0,1,\dots,r\}$ $(r\geq 2)$ whose order does not exceed $v(v\geq 0)$, and

(ii) all transition probabilities of this Markov chain are less than or equal to $1-\alpha$ (α is a positive real number).

Theorem 8. (A.Lorenc and J.Lapiņš (1984)) Let $A=[X,Z,\Delta]$ be a FSA, $X=R\subset Z=\{0,1,\dots,n-1\}$, $\forall x\in X$, $z\in Z$ $(\Delta(z,x)\equiv zg+x(\mathrm{mod}\ n))$, $(g,n)=1$ $(g\geq 1)$. The FSA A transforms Markov chains from the set $B^{*}_{0,\alpha,r}$ into the random distribution $\pi_0=(1/n, 1/n,\dots, 1/n)$ iff the least prime divisor p of the number n is greater than r.

B) If we use a FSA $A=[X,Y,Z,\Delta,\lambda]$, for which the function $\lambda:Z\to Y$ is not injective, as transformer of Markov chains then it is not necessary for the FSA A to satisfy the condition (i) of Theorem 1. The study of some properties of special block matrices which are a generalization of the double stochastic matrices allow to become to the following extention.

Theorem 9. (J.Lapiņš, A.Lorenc (1985)) Let $m=kn$, $A=[X,Y,Z,\Delta,\lambda]$ be a FSA, $X=\{0,1,\dots,r\}$, $Y=\{0,1,\dots,n-1\}$, $Z=\{0,1,\dots,m-1\}$, $Z=\cup_{i=1}^{k}Z_i$, $|Z_i|=n$, and the following conditions are satisfied:

(i) $\forall i(\lambda(Z_i)\overset{\mathrm{def}}{=}\{\lambda(z)|z\in Z_i\}=Y)$,

(ii) $\forall i\forall x\in X\exists j\,(\Delta(Z_i,x)\overset{\mathrm{def}}{=}\{\Delta(z,x)|z\in Z_i\}=Z_j)$,

(iii) $\exists\xi\in N\ \forall z,z'\in Z\ \exists x_1,\dots,x_\xi\in X\ (\Delta(z,x_1\dots x_\xi)\overset{\mathrm{def}}{=}$
$\overset{\mathrm{def}}{=}\Delta(\dots\Delta(\Delta(z,x_1),x_2),\dots,x_\xi)=z')$.

Then FSA A transforms Markov chains from $B_{v,\alpha,r}$ into the random distribution $\pi_0=(1/n,1/n,\dots,1/n)$, and

$\Omega(A,T,\pi_1)\leq c(1-ck\alpha^{\xi+v})^{[T/(\xi+v)]}$,

where the number c satisfies the inequalities $(r+1)^v<c\leq n(r+1)^v$.

Remark 6. The principal result holds also by the following modification of the condition (iii):

(iii') Every set of states Z_i of the FSA **A** is a subset of one cyclic subclass of a Markov chain $\{Z_t\}_{t\geq 0}$ induced by a process $\{X_t\}_{t\geq 0}$ with μ=o.

C) A result analogue to Theorem 2 is possible to prove also in the case when the Galua field GF(q) is replaced by a finite commutative ring K with unit element. In this case the condition that matrix C is not singular is to be replaced by the condition that detC is not a divisor of zero of the ring K.

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A.Lorencs, J.Lapiņš. Diskrētu gadījumprocesu stabilizējošs pārveidojums ar galīgiem determinētiem automātiem: teorija un pielietojumi.

Anotācija. Rakstā sniegts pārskats par autoru pētījumiem kvalitatīvu fizikāli bāzētu gadījumskaitļu ģeneratoru izveides problemātikā. Autoru pieejas pamatā ir fizikālu bāzes ģeneratoru izejas procesa stabilizācija ar piemērotu galīgu determinētu automātu palīdzību. Metode ļauj iegūt diskrētus sadalījumus ar iepriekš uzdotu precizitāti un ticamību, izmantojot bāzes ģeneratorus ar laikā mainīgiem un neprecīzi noteiktiem parametriem. Īpaša uzmanība veltīta konstruējamo ģeneratoru ātrdarbības jautājumiem. Raksta pamatā ir referāts, kuru J.Lapiņš nolasīja 14. Ziemeļu Valstu Statistikas Konferencē Rērusā, Norvēģijā, 1992. gada 18. jūnijā.

А.Лоренц, Я.Лапиньш. Стабилизирующее преобразование дискретных случайных процессов при помощи конечных детерминированных автоматов: теория и приложения.

Аннотация. В работе дан обзор исследований авторов по проблематике построения качественных физических генераторов случайных чисел. Подход авторов основан на идее стабилизации выходного процесса исходных физических генераторов случайных сигналов при помощи подходящего конечного детерминированного автомата. Метод позволяет получать дискретные распределения с заданной точностью и надежностью, используя базисные генераторы, вероятностные параметры которых определены недостаточно точно и могут меняться во времени. Особое внимание уделено вопросам быстродействия получаемых генераторов. В основе работы положено сообщение, с которым на 14 конференции Северных Стран по статистике в Рерусе (Норвегия) выступил Я.Лапиньш.

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APPLICATION OF CHEBYSHEV INTERPOLATIONS TO EIGENFUNCTIONS AND EIGENFORMS OF CONIC SHELLS

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Abstract. Projector of Chebyshev interpolation, its properties and application to determining of eigenvalue and eigenfunction of simple boundary problems are considered. AMS SC 65 N 25

1. Projector of Chebyshev interpolation and its properties.

Lagrangian interpolation formulae with knot-points witch coincide with the zeroes of Chebyshev polynomials are convenient to use in discretion of boundary and eigenvalue-eigenfunction problems. As compared to finite difference or finite element methods the formulae of this type have the following advantages:

a) in case of smooth functions the approximation accuracy with these formulae increases (insaturability of the method),

b) the method is applyable to problems with complicated boundary conditions including systems of broad diversity of boundary conditions.

Chebyshev interpolations have been used previously [1] to determine eigenfrequencies and eigenforms of a conic shell its ends being fixed. The present study attempts to consider modification of the method in other cases of boundary conditions.

Let us consider projector I_N adjusting a polynomial of the order $N+1$ to any function $u(x) \in C[-1,1]$ according to

$$I_N u(x) = \left[\frac{1-x}{2} u(-1) + \frac{1+x}{2} u(1) \right] T_N(x) + (1-x^2) \sum_{k=1}^{+\infty} \frac{u(x_k)\, T_N(x)}{(x-x_k)(1-x_k^2)\, T_N'(x_k)} . \quad (1.1)$$

Then $I_N : C[-1,1] \to \mathfrak{M}_{N+1}$,

where $\mathfrak{M}_{N+1}$ is a set of polynomials of the order $N+1$, $T_N(x)$ -

Chebyshev polynomials of the 1-st type, $x_k=\cos\frac{(2k-1)\pi}{2N}$, $k=1,2,\dots,N$ – zeroes of the polynomial.

The series of discrete points

$$\tau_{N+2}=\{-1,1,x_1,\dots,x_N\} \tag{1.2}$$

is referred to as the complete Chebyshev lattice of the interval [-1,1];

$$\tau_N=\{x_1,x_2,\dots,x_N\} - \tag{1.3}$$

internal lattice of the interval [-1,1] while

$$\tau_2=\{-1,1\} \tag{1.4}$$

as the lattice of interval end points.

Operator S_{N+2} defined by

$$S_{N+2}u(x)=\{u(-1),u(1),u(x_1),\dots,u(x_N)\}^T\equiv u_{N+2} \tag{1.5}$$

and relating a vector-column u_{N+2} of dimension $N+2$ to each function $u(x)$ will be called reducing operator of the complete Chebyshev lattice τ_{N+2}.

In a similar way operators S_N and S_2 will be used:

$$S_N u(x)=\{u(x_1),u(x_2),\dots,u(x_N)\}^T\equiv u_N, \tag{1.6}$$

$$S_2 u(x)=\{u(-1),u(1)\}^T\equiv u_2. \tag{1.7}$$

The operators thus defined have the following properties.

1°. Operators I_N, S_{N+2}, S_N and S_2 are linear.

2°. Operator I_N reflects any polynomial of the order $m\leq N+1$.

□ Really, let $R_m(x)$ be an arbitrary polynomial and $m\leq N+1$. Then $R_m(x)-I_N R_m(x)=Q_{N+1}(x)$ is a polynomial the order of which does not exceed $N+1$. From (1.1) it is easy to see that $Q_{N+1}(x)$ is zero in any point of the complete lattice $(N+2)$, i.e., $Q_{N+1}(x)=0$. ■

3°. $$S_{N+2}I_N u(x)=S_{N+2}u(x)\equiv u_{N+2}. \tag{1.8}$$

Equation (1.8) is valid also in the case if S_{N+2} is substituted by S_N or S_2 and u_{N+2} - by u_N or u_2. This follows directly from (1.1) - (1.7).

4°. Let us denote $D=\frac{d}{dx}$. Then

$$S_{N+2}DI_N u(x) = B_{N+2} S_{N+2} u(x) = B_{N+2} u_{N+2}, \tag{1.9}$$

where B_{N+} is a square matrix $(N+2)\times(N+2)$ of the structure

$$B_{N+2}=\left|\begin{array}{c|c} C_{2,2} & C_{2,N} \\ \hline \mathcal{D}_{N,2} & \Delta_{N,N} \end{array}\right|, \tag{1.10}$$

where

$$C_{2,2}=\left|\begin{matrix} -(N^2+\frac{1}{2}) & \frac{(-1)^N}{2} \\ \frac{(-1)^{N+1}}{2} & N^2+\frac{1}{2} \end{matrix}\right| = \left\|\begin{matrix} -a_1 & b_1 \\ -b_1 & a_1 \end{matrix}\right\|, \tag{1.11}$$

$$a_1=N^2+\frac{1}{2}, \quad b_1=\frac{(-1)^N}{2},$$

$$C_{2,N}=\left|\begin{matrix} c_1, c_2, \dots, c_N \\ d_1, d_2, \dots, d_N \end{matrix}\right|, \tag{1.12}$$

$$c_k=\frac{2(-1)^{N+k}}{N\sqrt{1-x_k^2}(1+x_k)}, \quad d_k=\frac{2(-1)^k}{N\sqrt{1-x_k^2}(1-x_k)}, \quad k=1,\dots,N; \tag{1.13}$$

$$\mathcal{D}_{N,2}=\left\|\begin{matrix} e_1 & f_1 \\ e_2 & f_2 \\ \vdots & \vdots \\ e_N & f_N \end{matrix}\right\|, \quad \begin{matrix} e_k=(-1)^{N+k-1}\frac{N}{2}\sqrt{\frac{1-x_k}{1+x_k}}, \\ f_k=(-1)^{k-1}\frac{N}{2}\sqrt{\frac{1+x_k}{1-x_k}}, \\ k=1,2,\dots,N; \end{matrix} \tag{1.14}$$

$$\Delta_{N,N}=\| b_{lk} \|, \quad l,k=1,2,\dots,N, \tag{1.15}$$

$$b_{l,k}=\begin{cases}\dfrac{(-1)^{l+k}}{x_l-x_k}\sqrt{\dfrac{1-x_l^2}{1-x_k^2}}, & l\neq k,\\ -\dfrac{3x_l}{2(1-x_l^2)}, & l=k.\end{cases} \tag{1.16}$$

In a similar way it can be proved that

$$S_N D I_N u(x)=B_N u_N,\quad S_2 D I_N u(x)=B_2 u_2, \tag{1.17}$$

where

$$B_N=\| \mathcal{D}_{N,2} \,\vdots\, \Delta_{N,N} \|,\quad B_2=\| C_{2,2} \,\vdots\, C_{2,N} \|. \tag{1.18}$$

Let's consider the approximation error

$$\delta_N[u(x)]=u(x)-I_N u(x). \tag{1.19}$$

Operator A_N is defined to project $C[-1,1]$ on $\mathfrak{M}_{N-1}$ by

$$A_N u(x)=T_N(x)\sum_{k=1}^{N}\frac{u(x_k)}{(x-x_k)T_N'(x_k)}, \tag{1.20}$$

the error being expressed as

$$r_N[u(x)]=u(x)-A_N u(x). \tag{1.21}$$

Expression (1.20) is interpolation over points τ_N of the internal lattice of the interval [-1,1]. Errors δ_N and r_N are related to each other by

$$\delta_N[u(x)]=\frac{1-x^2}{2}\left\{r_N\left[\frac{u(x)-u(1)}{1-x}\right]+r_N\left[\frac{u(x)-u(-1)}{1+x}\right]\right\}, \tag{1.22}$$

which is proved by obtaining r_N from (1.20) - (1.21) and δ_N - from (1.1) and (1.19). So it is sufficient to consider r_N.

Let $R_N(f)$ be residual of Gaussian quadrature

$$R_N(f)=\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}dx-\frac{\pi}{N}\sum_{k=1}^{N}f(x_k),\quad x_k=\cos\frac{(2k-1)\pi}{2N}, \tag{1.23}$$

while $\rho_N[f(x)]$ - residual of approximation of function f by partial sum of Fourier series

$$\rho_N[f(x)] - \sum_{k=0}^{N-1} c_k T_k(x), \quad c_k = \frac{2\varepsilon_k}{\pi} \int_{-1}^{1} \frac{f(x)T_k(x)dx}{\sqrt{1-x^2}}, \tag{1.24}$$

$\varepsilon_k = \frac{1}{2}$, if $k=0$ and $\varepsilon_k = 1$, if $k \neq 0$.

Lemma 1. If $f(x)$ is differentiable over almost all of interval $[-1,1]$ and $f'(x)$ is a function of limited variation, then

$$\tau_N[f(x)] = \rho_N[f(x)] + \frac{2}{\pi} \sum_{k=0}^{N-1} \varepsilon_k R_N(fT_k) T_k(x), \tag{1.25}$$

where $R_N(fT_k) = \frac{\pi}{2} \sum_{j=1}^{+\infty} (-1)^{j-1} (c_{2Nj+k} + c_{2Nj-k}).$ (1.26)

▫ Substitution the integral in (1.24) by area (1.23) provides

$$\rho_N[f(x)] = f(x) - A_N f(x) - \frac{2}{\pi} \sum_{k=0}^{N-1} \varepsilon_k R_N(fT_k) T_k(x), \tag{1.27}$$

which coincides with (1.25). The proof of (1.26) is given in [2].▪

Lemma 2.

$$\|\tau_N[f(x)]\|_{L_2} = \sum_{k=N}^{+\infty} c_k^2 + \left(\frac{2}{\pi}\right)^2 \sum_{k=0}^{+\infty} \varepsilon_k^2 R_N^2(fT_k), \tag{1.28}$$

where $R_N(fT_k)$ is determined by (1.26).

▫ The proof follows from (1.25), since the right side of it is Fourier series of Chebyshev polynomials of the 1-st type.▪

Consequencies.

$$\|\tau_N[f(x)]\|_{L_2} \geq \|\rho_N[f(x)]\|_{L_2} = \sum_{k=N}^{+\infty} c_k^2.$$

Its follows from these the lemmas how residue τ_N or its norm is seen to be explicated in terms of Fourier coefficients C_m with $m \geq N$. If $N \to +\infty$, $C_m \to 0$ for smooth enough functions the sooner the smoother is the function f. Hence, the interpolation formulae are not saturated, but the $\tau_N[f(x)] \to 0$ if $N \to +\infty$. More exact expressions for residues $\tau_N[f(x)]$ and $\sigma_N[u(x)]$ or derivatives of them, asymptotic formulae if $N \to +\infty$ etc.can be examined,too. We shall not do it here.

2. Application of Chebyshev interpolations to the problem of eigenvalues and eigenfunctions of second order differential equations.

Let's consider the problem of eigenvalues and eigenfunctions

$$q_0(x)u''(x)+q_1(x)u'(x)+q_2(x)u(x)+\lambda u(x)=0 \qquad (2.1)$$

$$\alpha_1 u(-1)+\beta_1 u'(-1)=0,$$

$$\alpha_2 u(1)+\beta_2 u'(1)=0, \quad |\alpha_i|^2+|\beta_i|^2\neq 0, \; i=1,2. \qquad (2.2)$$

By approximations

$$u(x)=I_N u(x)+\sigma_N[u(x)] \qquad (2.3)$$

$$Du(x)=DI_N u(x)+D\sigma_N[u(x)]=(DI_N+D\sigma_N)u(x) \qquad (2.4)$$

$$D^2u(x)=DI_N Du(x)+D\sigma_N Du(x)=(DI_N+D\sigma_N)^2u(x). \qquad (2.5)$$

and reduction of these equations to lattice τ_{N+2} and making use of (1.9) one obtains

$$u_{N+2}=\{u(-1),u(1),u(x_1),u(x_2),\dots,u(x_N)\}^T, \qquad (2.6)$$

$$u'_{N+2}=(B_{N+2}+\varepsilon_{N+2})u_{N+2}\approx B_{N+2}u_{N+2}, \qquad (2.7)$$

$$u''_{N+2}=(B_{N+2}+\varepsilon_{N+2})^2u_{N+2}\approx B^2_{N+2}u_{N+2}, \qquad (2.8)$$

where B_{N+2} is matrix (1.10).

$$\varepsilon_{N+2}u_{N+2}=S_{N+2}D\sigma_N[u(x)].$$

Boundary conditions (2.2) in the matrix form is written as

$$\Gamma_1 u_2+\Gamma_2 u'_2=0, \qquad (2.9)$$

where

$$u_2=\{u(-1),u(1)\}^T, \quad \Gamma_1=diag(\alpha_1,\alpha_2), \; \Gamma_2=diag(\beta_1,\beta_2) \qquad (2.10)$$

Substitution of u'_2 from (2.7) and (1.10) - (1.13) into (2.9) provides

$$(\Gamma_1+\Gamma_2 C_{2,2})u_2+\Gamma_2 C_{2,N}u_N=0 \tag{2.11}$$

Reducing (2.1) to internal knot points τ_N and using (2.6) - (2.8), (1.10) - (1.18) gives

$$[Q_0\widetilde{\mathcal{D}}_{N,2}+Q_1\mathcal{D}_{N,2}]u_2+[Q_0\widetilde{\Delta}_{N,N}+Q_1\Delta_{N,N}+Q_2]u_N+\lambda u_N=0 \tag{2.12}$$

where

$$Q_i=diag(q_i(x_1),q_i(x_2),\dots,q_i(x_N)),\ i=0,1,2. \tag{2.13}$$

Matrices $\mathcal{D}_{N,2}$ and $\Delta_{N,N}$ are defined by (1.14) - (1.16) while $\widetilde{\mathcal{D}}_{N,2}$ and $\widetilde{\Delta}_{N,N}$ are the corresponding block matrices comprising B^2_{N+2}:

$$B^2_{N+2}=\left\|\begin{array}{c|c}\widetilde{C}_{2,2} & \widetilde{C}_{2,N}\\ \hline \widetilde{\mathcal{D}}_{N,2} & \widetilde{\Delta}_{N,N}\end{array}\right\|, \tag{2.14}$$

where

$$\widetilde{C}_{2,2}=C^2_{2,2}+C_{2,N}\mathcal{D}_{N,2}, \tag{2.15}$$

$$\widetilde{C}_{2,N}=C_{2,2}C_{2,N}+C_{2,N}\Delta_{N,N}, \tag{2.16}$$

$$\widetilde{\mathcal{D}}_{N,2}=\mathcal{D}_{N,2}C_{2,2}+\Delta_{N,N}\mathcal{D}_{N,2}, \tag{2.17}$$

$$\widetilde{\Delta}_{N,N}=\mathcal{D}_{N,2}C_{2,N}+\Delta^2_{N,N}. \tag{2.18}$$

Excluding vector u_2 from (2.11) and (2.12), a system of algebraic equations is obtained

$$(P_N+\lambda E)u_N=0, \tag{2.19}$$

where

$$P_N=Q_0\widetilde{\Delta}_{N,N}+Q_1\Delta_{N,N}+Q_2-(Q_0\widetilde{\mathcal{D}}_{N,2}+Q_1\mathcal{D}_{N,2})(\Gamma_1+\Gamma_2C_{2,2})^{-1}\Gamma_2C_{2,N}. \tag{2.20}$$

Thus the eigenvalues are approximated by those of matrix P_N. Since u_N is known, the values of $u(-1)$ and $u(1)$ are found first, then interpolation (1.1) is used to obtain the eigenfunction.

3. Eigenvalues and eigenfunctions of conic shell at different ways of fixing its ends.

Let us consider the problem of eigenvalues and eigenfunctions of a conic shell in the matrix form, the same notations as in [1] being used:

$$A_0 \frac{d^2V}{dx^2} + B(x)\frac{dV}{dx} + C(x)V + \lambda E V = 0, \qquad (3.1)$$

where

$$B(x) = \frac{1}{x}(A_0 + A_1 + m\tilde{A}_3) \qquad (3.2)$$

$$C(x) = \frac{1}{x}(A_5 - m\tilde{A}_4 - m^2 A_2), \qquad (3.3)$$

$A_0, A_1, A_2, \tilde{A}_3, \tilde{A}_4, A_5, B(x), C(x)$ are square matrices of dimension 5 × 5, and

$$V = V(x) = (v_1(x), v_2(x), \ldots, v_5(x))^T - \qquad (3.4)$$

the column matrix to be found under boundary conditions

$$\left[\alpha_1(x)V(x) + \beta_1(x)V'(x)\right]_{x=-1} = 0, \quad \left[\alpha_2(x)V(x) + \beta_2(x)V'(x)\right]_{x=1} = 0 \qquad (3.5)$$

where $\alpha_1(x), \alpha_2(x)$ and corresponding $\beta_1(x)$ and $\beta_2(x)$ are 5 × 5 square matrices comprised of the same selected five rows of matrices Z_1 and Z_2.

$$Z_1 = \left\| \begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{d_2}{x} & \frac{md_2}{x\sin\gamma_0} & \frac{d_2}{x\,tg\gamma_0} & 0 & 0 \\ 0 & 0 & 0 & \frac{d_2}{x} & \frac{md_2}{x\sin\gamma_0} \\ \frac{md_4}{x\sin\gamma_0} & \frac{d_4}{x} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{md_4}{x\sin\gamma_0} & \frac{d_4}{x} \\ 0 & 0 & 0 & d_3 & 0 \end{matrix} \right\|, \quad Z_2 = \left\| \begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -d_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -d_4 \\ 0 & 0 & 0 & d_3 & 0 \end{matrix} \right\| \qquad (3.6)$$

The choice of rows depends on the selected boundary conditions. E.g., if both ends of the shell are fixed, i.e., the 1-st boundary condition (see Table 1 in [1]) is satisfied at both the ends, then matrices α_1 and α_2 are formed from rows 1 - 5 of matrix Z_1, β_1 and β_2 - from rows 1-5 of matrix Z_2 ; if both ends of the shell are free, i.e., the 8-th boundary condition is satisfied at both ends, then α_1 and α_2 as well as β_1 and β_2 are made of rows 6-10 of matrices Z_1 and Z_2 respectively; if one of the ends $x=-1$ is fixed i.e., the 1-st condition is satisfied at this end, then α_1 and β_1 are made from rows 1-5 of Z_1 and Z_2, and if the other end $x=1$ is fixed being free in direction x , i.e., the 2-nd condition is satisfied at this end , then α_2 and β_2 are made from rows 2-6 of Z_1 and Z_2.

Approximations for (3.1) and (3.5) are obtained substituting function $u(x)$ by column matrix $V(x)$ in (2.3) - (2.5). Reducing equations thus obtained to lattice τ_{N+2} and employing equation similar to (1.9) where $u(x)$ is substituted by $V(x)$ a $(5N+10)$ - dimensional vector is obtained

$$V_{N+2}=\{V(-1),V(1),V(x_1),\dots,V(x_N)\}^T, \tag{3.7}$$

where

$$V(-1)=\{v_1(-1),v_2(-1),\dots,v_5(-1)\},\ V(1)=\{v_1(1),v_2(1),\dots,v_5(1)\},$$
$$V(x_k)=\{v_1(x_k),v_2(x_k),\dots,v_5(x_k)\},\ k=1,2,\dots,N; \tag{3.8}$$

$$V'_{N+2}\approx B_{N+2}V_{N+2},\quad V''_{N+2}\approx B^2_{N+2}V_{N+2}, \tag{3.9}$$

and B_{N+2} is a $(5N+10)\times(5N+10)$ matrix (1.10) wherein the elements of block matrices $C_{2,2}, C_{2,N}, \mathcal{D}_{N,2}$ and $\Delta_{N,N}$ are substituted by 5×5 diagonal square matrices the preceding element being on the diagonal, i.e.,

$$C_{2,2}=\left\|\begin{array}{cccc|cccc} -a_1 & & & & b_1 & & & \\ & -a_1 & & 0 & & b_1 & & 0 \\ & & -a_1 & & & & b_1 & \\ 0 & & & -a_1 & 0 & & & b_1 \\ \hline -b_1 & & & & a_1 & & & \\ & -b_1 & & 0 & & a_1 & & 0 \\ & & -b_1 & & & & a_1 & \\ 0 & & & -b_1 & 0 & & & a_1 \end{array}\right\|, \qquad \mathcal{D}_{N,2}=\left\|\begin{array}{cccc|cccc} e_1 & & & 0 & f_1 & & & 0 \\ & e_1 & & & & f_1 & & \\ & & e_1 & & & & f_1 & \\ 0 & & & e_1 & 0 & & & f_1 \\ \hline e_2 & & & 0 & f_2 & & & 0 \\ & e_2 & & & & f_2 & & \\ & & e_2 & & & & f_2 & \\ 0 & & & e_2 & 0 & & & f_2 \\ \hline \vdots & & & & \vdots & & & \\ \hline e_N & & & 0 & f_N & & & 0 \\ & e_N & & & & f_N & & \\ & & e_N & & & & f_N & \\ 0 & & & e_N & 0 & & & f_N \end{array}\right\|, \tag{3.10}$$

$$C_{2,N}=\left\|\begin{array}{cc|cc|c|cc} c_1 & 0 & c_2 & 0 & \dots & c_N & 0 \\ 0 & c_1 & 0 & c_2 & & 0 & c_N \\ \hline d_1 & 0 & d_2 & 0 & \dots & d_N & 0 \\ 0 & d_1 & 0 & d_2 & & 0 & d_N \end{array}\right\|,$$

$$\Delta_{N,N}=\|B_{l,k}\|, \quad B_{l,k}=\left\|\begin{array}{ccc} b_{lk} & & 0 \\ & \ddots & \\ 0 & & b_{lk} \end{array}\right\|, \quad l,k=1,2,\dots,N. \tag{3.11}$$

Boundary conditions (3.5) can be writen as

$$\Gamma_1 V_2 + \Gamma_2 V_2' = 0 \tag{3.12}$$

where

$$V_2=\{V(-1),V(1)\}^T, \quad \Gamma_1=diag(\alpha_1(-1),\alpha_2(1)), \Gamma_2=diag(\beta_1(-1),\beta_2(1)). \tag{3.13}$$

V_2' being put into (3.12) from (3.9) and (1.10) -(1.13), (3,10), yields

$$(\Gamma_1+\Gamma_2 C_{2,2})V_2+\Gamma_2 C_{2,N} V_N=0. \tag{3.14}$$

Equation (3.1) being reduced to internal knot points of τ_N, and using of (3.7) - (3.11), (1.10) -(1.18), we obtain

$$[Q_0\widetilde{\mathcal{D}}_{N,2}+Q_1\mathcal{D}_{N,2}]V_2+[Q_0\widetilde{\Delta}_{N,N}+Q_1\Delta_{N,N}+Q_2]V_N+\lambda E\dot{V}_N=0, \tag{3.15}$$

where

$$Q_0 = diag(A_0, A_0, \dots, A_0), \tag{3.16}$$

$$Q_1 = diag(B(x_1), B(x_2), \dots, B(x_N)), \tag{3.17}$$

$$Q_2 = diag(C(x_1), C(x_2), \dots, C(x_N)) - \tag{3.18}$$

are diagonal matrices with corresponding block matrices along the diagonal. Matrices $\mathcal{D}_{N,2}$ and $\Delta_{N,N}$ are defined by (3.10), (3.11) and (1.14) - (1.16) while $\tilde{\mathcal{D}}_{N,2}$ and $\tilde{\Delta}_{N,N}$ are the corresponding block matrices comprising $B^{2}_{N+2}{}^{N,2}$ according to (2.14) -(2.18).

Exclusion of vector V_2 from (3.14) and (3.15) provides a system of algebraic equations

$$(P_N + \lambda E) V_N = 0 \tag{3.19}$$

where P_N is calculated from (2.20) where the matrices are substituted according to the new formulae (3.7) - (3.18). Here P_N is a $5N \times 5N$ matrix. Further calculations are similar to the case of a single equation.

The method can be used directly to solve the boundary problem if the equation or boundary conditions (or both of them) are not homogeneous. In this case instead of (3.19) a system of unhomogeneous linear algebraic equations is obtained with respect to the vector V_N to be found.

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Т. Цирулис, В. Нейманис. Приложения формул интерполяции Чебышева для определения собственных частот и форм конических оболочек.

Аннотация. В статье рассмотрен проектор интерполяции Чебышева, его свойства и приложения. Рассматриваемый проектор применяется для определения собственных частот и функций в простейших краевых задачах. В частности рассмотрена методика нахождения собственных частот и форм конической оболочки при разных способах закрепления концов. Метод может быть без существенных изменения применен также для решения краевых задач с произвольными линейными граничными условиями. УДК 810.83:830.3.

T. Cirulis, V. Neimanis. Čebiševa interpolācijas formulu lietojumi konisku čaulu īpašfrekvenču un īpašformu noteikšanā.

Anotācija. Rakstā apskatīts Čebiševa interpolācijas projektors, tā īpašības un lietojumi vienkāršāko robežproblēmu īpašvērtību un īpašfunkciju noteikšanā. Kā pielietojums apskatīta metode īpašfrekvenču un formu noteikšanai koniskai čaulai ar dažādiem šīs čaulas galu nostiprināšanas veidiem. Bez būtiskām izmaiņām metode ir lietojama arī tieši robežproblēmu atrisināšanai pie dažādiem homogēniem vai nehomogēniem lineāriem robežnosacījumiem.

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SOME REMARKS ON B-SPLINES

M. Goldman

Abstract: B-splines were introduced by Curry and Schoenberg in 1947 [1]. The definition of this notion is based on the theory of finite differences. In this paper the definition of a B-spline and conclusion on its basic properties are made without finite differences. AMS CS 65D07.

Let $t_1,\dots,t_n \in \mathbb{R}$, $t_1<\dots<t_n$, and let m be a natural number. Let's consider the spline $s(t)=\sum_{i=1}^{n}\alpha_i(t-t_i)_+^m$, $t\in\mathbb{R}$.

Evidently, that $s(t)=0$ when $t\le t_1$, whatever were the coefficients $\alpha_1,\dots,\alpha_n$. If $t\ge t_n$, then $(t-t_i)_+^m=(t-t_i)^m$, $i=1,\dots,n$, and $s(t)=\sum_{i=1}^{n}\alpha_i(t-t_i)^m=\sum_{i=1}^{n}\alpha_i\sum_{k=0}^{m}C_m^k t^{m-k}(-1)^k t_i^k =$
$=\sum_{k=0}^{m}[(-1)^k C_m^k\sum_{i=1}^{n}\alpha_i t_i^k]t^{m-k}$.

Proceeding from this, the next statement follows.

Lemma 1. Let $\mathcal{N}(S)$ denote the set of all zeroes of the function $s(t)=\sum_{i=1}^{n}\alpha_i(t-t_i)_+^m$. Then $|\mathcal{N}(S)\cap\{t\in\mathbb{R}|t\ge t_n\}|>m$ if and only if the coefficients $\alpha_1,\dots,\alpha_n$ of spline $s(t)$ are satisfying the system of equations

$$\sum_{i=1}^{n}\alpha_i t_i^k=0,\ k=0,\dots,m. \qquad (1)$$

Basing on Lemma 1 let's prove the following theorem.

Theorem 1. If $n\le m+1$, then the inequality $|\mathcal{N}(S)\cap\{t\in\mathbb{R}|t\ge t_n\}|>m$ is possible only when $\alpha_i=0$, $i=1,\dots,n$.

Proof. Due to the Lemma 1 it is necessary to reveal that if $n\le m+1$, then the system (1) has only a trivial solution.

In the case when $n=m+1$, the number of coefficients $\alpha_1,\dots,\alpha_{m+1}$ of the spline $s(t)$ is equal to the number of equation of the system (1). As this system is homogeneous and its determinant is positive (because it is Vandermond's determinant $\det(t_i^k), i=1,\dots,m+1; k=0,\dots,m$, in which $t_1<\dots<t_{m+1}$), then $\alpha_i=0, i=1,\dots,m+1$. In the case when $n\le m$, the system of equations consisting of the first n equations of the system (1), has only a trivial solution $\alpha_1=\dots=\alpha_n=0$ (due to the same considerations as in the above mentioned case, when $n=m+1$); it is also satisfying the other equations of the system (1).

The Theorem 1 is proved.

Let now $n=m+2$ (so as $s(t)=\sum_{i=1}^{m+2}\alpha_i(t-t_i)_+^m$). According to Lemma 1 the inequality $|\mathcal{N}(s)\cap\{t\in\mathbb{R}\,|\,t\ge t_{m+2}\}|>m$ is possible if and only if

$$\sum_{i=1}^{m+2}\alpha_i t_i^k=0,\ k=0,\dots,m. \tag{2}$$

Let's find the set of all non-trivial solutions of the system (2). We shall add to the system (2) the equation

$$\sum_{i=1}^{m+2}\alpha_i t_i^{m+1}=C, \tag{3}$$

where C is an arbitrary constant, different from zero. The system of equations (2)-(3) is uniquely solvable, as its determinant $\Delta>0$:

$$\Delta=\begin{vmatrix}1 & 1 & \dots & 1\\ t_1 & t_2 & \dots & t_{m+2}\\ \vdots & & & \vdots\\ t_1^{m+1} & t_2^{m+1} & \dots & t_{m+2}^{m+1}\end{vmatrix},\ t_1<\dots<t_{m+2}. \tag{4}$$

According to Cramer's rule we find:

$$\alpha_i=(-1)^{m+i}C\,\frac{\Delta_i}{\Delta},\ i=1,\dots,m+2, \tag{5}$$

where Δ_i is the determinant, received from the determinant Δ by means of removal from it the i-column and the last line. Since $C\ne0$, $\Delta>0$ and $\Delta_i>0, i=1,\dots,m+2$, we have $\alpha_i\ne0$ for any $i=1,\dots,m+2$.

Evidently, that the set of all non-trivial solutions of the system (2) is being exhausted by the obtained solutions of the system (2)-(3), when C is running through all $\mathbb{R}\setminus\{0\}$. Really, if $\alpha_1,\dots,\alpha_{m+2}$ is any non-trivial solution of the system (2), then $\sum_{i=1}^{m+2}\alpha_i t_i^{m+1}\neq 0$, because in the opposite case $\alpha_1,\dots,\alpha_{m+2}$ could have been the solution of the system of equations $\sum_{i=1}^{m+2}\alpha_i t_i^k=0$, $k=0,\dots,m+1$, which has only the trivial solution.

Let's summarise the received result.

<u>Theorem 2</u>. The set $\mathbb{R}\setminus\mathcal{N}(S)$, where $S(t)=\sum_{i=1}^{m+2}\alpha_i(t-t_i)_+^m$, is contained in the interval (t_1,t_{m+2}) if and only if the coefficients $\alpha_1,\dots,\alpha_{m+2}$ satisfy the system of equations (2). The set of all non-trivial solutions of the system (2) is being exhausted by the solutions, given by the formulae (5) (from which it follows that all $\alpha_i\neq 0, i=1,\dots,m+2$). Now let's present the definition of a B-spline.

<u>Definition</u>. The spline $S(t)=\sum_{i=1}^{m+2}\alpha_i(t-t_i)_+^m$ is called a B-spline of the degree m on the net $t_1,\dots,t_{m+2}$, where $t_1<\dots<t_{m+2}$, if and only if $\alpha_1,\dots,\alpha_{m+2}$ is the solution of the system (2)-(3), in which $C\neq 0$ (or iff $\alpha_1,\dots,\alpha_{m+2}$ is a non-trivial solution of the system (2)).

<u>Theorem 3</u>. A function $S(t)=\sum\alpha_i(t-t_i)_+^m$, $t\in\mathbb{R}$, is a B-spline if and only if $\emptyset\neq\mathbb{R}\setminus\mathcal{N}(S)\subset(t_1,t_{m+2})$.

<u>Proof</u>. As it was established, if $S(t)$ is a B-spline, then $\mathbb{R}\setminus\mathcal{N}(S)\subset(t_1,t_{m+2})$. Let's show that $\mathbb{R}\setminus\mathcal{N}(S)\neq\emptyset$. For this we reveal that, for example, $(t_1,t_2)\subset\mathbb{R}\setminus\mathcal{N}(S)$. If $t\in(t_1,t_2)$ then $S(t)=\alpha_1(t-t_1)^m$, and as $\alpha_1\neq 0$, then $S(t)\neq 0$ for any $t\in(t_1,t_2)$. Thus, if $S(t)$ is a B-spline, then $\emptyset\neq\mathbb{R}\setminus\mathcal{N}(S)\subset(t_1,t_{m+2})$. Conversely, if $\mathbb{R}\setminus\mathcal{N}(S)\subset(t_1,t_{m+2})$, then $S(t)=0$ for all $t\geq t_{m+2}$, hence $\sum_{i=1}^{m+2}\alpha_i t_i^k=0, k=0,\dots,m$ (see Lemma 1); if

$R \setminus \mathcal{N}(S) \neq \emptyset$, then $\sum_{i=1}^{m+2} \alpha_i t_i^{m+1} \neq 0$ (see the end of proof in the Theorem 2). This means that $S(t)$ is a B-spline (see the Definition, given above).

As we have seen, a B-spline is being determined with the accuracy up to multiplier C, different from zero. B-splines are normed in different ways by means of corresponding choice of constant C.

Let's find the value of C from the condition $\int_{t_1}^{t_{m+2}} S(t)dt = 1$. We have:

$$1 = \int_{t_1}^{t_{m+2}} S(t)dt = \int_{t_1}^{t_{m+2}} \left[\sum_{i=1}^{m+2} \alpha_i (t-t_i)_+^m\right]dt = \sum_{i=1}^{m+2} \alpha_i \frac{(t-t_i)_+^{m+1}}{m+1}\Bigg|_{t_1}^{t_{m+2}} =$$

$$= \frac{1}{m+1}\sum_{i=1}^{m+2} \alpha_i (t_{m+2}-t_i)^{m+1} = \frac{1}{m+1}\sum_{i=1}^{m+2}\alpha_i \sum_{k=0}^{m+1} C_{m+1}^k t_{m+2}^{m+1-k}(-1)^k t_i^k =$$

$$= \frac{1}{m+1}\sum_{k=0}^{m+1} C_{m+1}^k t_{m+2}^{m+1-k}(-1)^k \sum_{i=1}^{m+2}\alpha_i t_i^k = \frac{(-1)^{m+1}}{m+1}\sum_{i=1}^{m+2}\alpha_i t_i^{m+1} =$$

$$= \frac{(-1)^{m+1}}{m+1} C \quad \text{(the last equality follows from (3)).}$$

From here it follows that

$$C = (-1)^{m+1}(m+1). \tag{6}$$

Substituting the obtained value of C in the formulae (5), we shall receive

$$\alpha_i = (-1)^{i+1}(m+1)\frac{\Delta_i}{\Delta},\ i = 1, \dots, m+2 \tag{7}$$

We shall use the notation $B(t; t_1, \dots, t_{m+2})$ for the B-spline, corresponding to the constant $C = (-1)^{m+1}(m+1)$. Let's regard the following three B-splines when $m \geq 2$:

$$B_m(t; t_1, \dots, t_{m+2}) = \sum_{i=1}^{m+2} \alpha_i (t-t_i)_+^m. \tag{8}$$

$$B_{m-1}(t; t_1, \dots, t_{m+1}) = \sum_{i=1}^{m+1} \beta_i (t-t_i)_+^{m-1}, \tag{9}$$

$$B_{m-1}(t;t_2,\dots,t_{m+2})=\sum_{i=2}^{m+2}\gamma_i(t-t_i)_+^{m-1}. \tag{10}$$

According to (2), (3) and (6) we have :

$$\sum_{i=1}^{m+2}\alpha_i t_i^k=0,\ k=0,\dots,m;\ \sum_{i=1}^{m+2}\alpha_i t_i^{m+1}=(-1)^{m+1}(m+1), \tag{11}$$

$$\sum_{i=1}^{m+1}\beta_i t_i^k=0,\ k=0,\dots,m-1;\ \sum_{i=1}^{m+1}\beta_i t_i^m=(-1)^m m, \tag{12}$$

$$\sum_{i=2}^{m+2}\gamma_i t_i^k=0,\ k=0,\dots,m-1;\ \sum_{i=2}^{m+2}\gamma_i t_i^m=(-1)^m m. \tag{13}$$

Lemma 2.

$$C_i:=\beta_i(t_i-t_1)+\gamma_i(t_{m+2}-t_i)=0,\ i=2,\dots,m+1 \tag{14}$$

Proof. First let's show that $\sum_{i=2}^{m+1}C_i t_i^k=0$ $k=0,\dots,m-1$

We have :

$$\sum_{i=2}^{m+1}C_i t_i^k=\sum_{i=2}^{m+1}\beta_i t_i^{k+1}-t_1\sum_{i=2}^{m+1}\beta_i t_i^k+t_{m+2}\sum_{i=2}^{m+1}\gamma_i t_i^k-\sum_{i=2}^{m+1}\gamma_i t_i^{k+1}. \tag{15}$$

From the equalities (12) and (13) it follows that

$$\sum_{i=2}^{m+1}\beta_i t_i^k=-\beta_1 t_1^k,\ \sum_{i=2}^{m+1}\gamma_i t_i^k=-\gamma_{m+2}t_{m+2}^k,\ k=0,\dots,m-1, \tag{16}$$

$$\sum_{i=2}^{m+1}\beta_i t_i^{k+1}=-\beta_1 t_1^{k+1},\ \sum_{i=2}^{m+1}\gamma_i t_i^{k+1}=-\gamma_{m+2}t_{m+2}^{k+1},\ k=0,\dots,m-2. \tag{17}$$

By replacing in the right part of the equality (15) all the addenda according to formulae (16), (17), we get :

$$\sum_{i=2}^{m+1}C_i t_i^k=0,\ k=0,\dots,m-1 \tag{18}$$

Since the determinant $\det(t_i^k), i=2,\dots,m+1; k=0,\dots,m-1$, is different from zero, the homogeneous system (18) have only a trivial solution $C_2=\dots=C_{m+1}=0$.

Lemma 3.

a)

$$\sum_{i=1}^{m+1} \beta_i (t-t_i)_+^m = (t-t_1) B_{m-1}(t; t_1, \dots, t_{m+1}) - \sum_{i=2}^{m+1} \beta_i (t_i - t_1)(t-t_i)_+^{m-1}, \quad (19)$$

b)

$$\sum_{i=2}^{m+2} \gamma_i (t-t_i)_+^m = (t-t_{m+2}) B_{m-1}(t; t_2, \dots, t_{m+2}) - \sum_{i=2}^{m+1} \gamma_i (t_i - t_{m+2})(t-t_i)_+^{m-1}. \quad (20)$$

Proof. a) $\sum_{i=1}^{m+1} \beta_i (t-t_i)_+^m = \sum_{i=1}^{m+1} \beta_i (t-t_i)_+^{m-1}(t-t_i) = t \sum_{i=1}^{m+1} \beta_i (t-t_i)_+^{m-1} -$

$$-\sum_{i=1}^{m+1} \beta_i t_i (t-t_i)_+^{m-1} = t \sum_{i=1}^{m+1} \beta_i (t-t_i)_+^{m-1} - \sum_{i=1}^{m+1} \beta_i (t_i - t_1)(t-t_i)_+^{m-1} -$$

$$-t_1 \sum_{i=1}^{m+1} \beta_i (t-t_i)_+^{m-1} = (t-t_1) \sum_{i=1}^{m+1} \beta_i (t-t_i)_+^{m-1} - \sum_{i=2}^{m+1} \beta_i (t_i - t_1)(t-t_i)_+^{m-1}.$$

b) is being proved analogically.

Lemma 4.

$$B_m(t; t_1, \dots, t_{m+2}) = \frac{m+1}{m} \frac{1}{t_{m+2} - t_1} \left[\sum_{i=1}^{m+1} \beta_i (t-t_i)_+^m - \sum_{i=2}^{m+2} \gamma_i (t-t_i)_+^m \right]. \quad (21)$$

Proof. From the equalities (12) and (13) it follows that

$$\beta_1 t_1^k + \sum_{i=2}^{m+1} (\beta_i - \gamma_i) t_i^k - \gamma_{m+2} t_{m+2}^k = 0, \; k = 0, \dots, m \quad (22)$$

Supposing that

$$\tilde{\alpha}_1 = \beta_1; \; \tilde{\alpha}_i = \beta_i - \gamma_i, \; i = 2, \dots, m+1; \; \tilde{\alpha}_{m+2} = -\gamma_{m+2}. \quad (23)$$

In these notation (22) acquires the following view :

$$\sum_{i=1}^{m+2} \tilde{\alpha}_i t_i^k = 0, \; k = 0, \dots, m. \quad (24)$$

Let

$$\tilde{c}=\sum_{i=1}^{m+2}\tilde{\alpha}_i t_i^{m+1}. \tag{25}$$

Then

$$\tilde{c}=\sum_{i=1}^{m+1}\beta_i t_i^{m+1}-\sum_{i=2}^{m+2}\gamma_i t_i^{m+1}=\sum_{i=1}^{m+1}\beta_i t_i^m t_i-\sum_{i=2}^{m+2}\gamma_i t_i^m t_i=$$

$$=\sum_{i=1}^{m+1}\beta_i t_i^m(t_i-t_1)+t_1\sum_{i=1}^{m+1}\beta_i t_i^m-\Big[\sum_{i=2}^{m+2}\gamma_i t_i^m(t_i-t_{m+2})+t_{m+2}\sum_{i=2}^{m+2}\gamma_i t_i^m\Big]=$$

$$=t_1\sum_{i=1}^{m+1}\beta_i t_i^m-t_{m+2}\sum_{i=2}^{m+2}\gamma_i t_i^m+\sum_{i=2}^{m+1}[\beta_i(t_i-t_1)+\gamma_i(t_{m+2}-t_i)]t_i^m.$$

Since, according to Lemma 2, $\sum_{i=2}^{m+1}[\beta_i(t_i-t_1)+\gamma_i(t_{m+2}-t_i)]t_i^m=0$,

we have: $\tilde{c}=t_1\sum_{i=1}^{m+1}\beta_i t_i^m-t_{m+2}\sum_{i=2}^{m+2}\gamma_i t_i^m$. But $\sum_{i=1}^{m+1}\beta_i t_i^m=\sum_{i=2}^{m+2}\gamma_i t_i^m=$

$=(-1)^m m$ (see (12) and (13)), hence,

$$\tilde{c}=(-1)^{m+1}m(t_{m+2}-t_1). \tag{26}$$

From (24), (25) and (26) it follows that the function

$\tilde{s}(t)=\sum_{i=1}^{m+2}\tilde{\alpha}_i(t-t_i)_+^m$ is a B-spline (see the Definition).

Since $$\frac{\alpha_i}{\tilde{\alpha}_i}=\frac{c}{\tilde{c}}=\frac{(-1)^{m+1}(m+1)}{(-1)^{m+1}m(t_{m+2}-t_1)}=\frac{m+1}{m}\,\frac{1}{t_{m+2}-t_1},$$

we have:

$$\sum_{i=1}^{m+2}\alpha_i(t-t_i)_+^m=\frac{m+1}{m}\,\frac{1}{t_{m+2}-t_1}\sum_{i=1}^{m+2}\tilde{\alpha}_i(t-t_i)_+^m=$$

$$=\frac{m+1}{m}\,\frac{1}{t_{m+2}-t_1}\Big[\sum_{i=1}^{m+1}\beta_i(t-t_i)_+^m-\sum_{i=2}^{m+2}\gamma_i(t-t_i)_+^m\Big] \text{ (see (23)).}$$

By this the formula (21) is proved.

Theorem 3.

$$B(t;t_1,\dots,t_{m+2})=\frac{m+1}{m}\,\frac{1}{t_{m+2}-t_1}\left[(t-t_1)B_{m-1}(t;t_1,\dots,t_{m+1})+(t_{m+2}-t)B_{m-1}(t;t_2,\dots,t_{m+2})\right] \qquad (27)$$

Proof. From the formulae (19) and (20) (Lemma 3), it follows that

$$\sum_{i=1}^{m+1}\beta_i(t-t_i)_+^m-\sum_{i=2}^{m+2}\gamma_i(t-t_i)_+^m=(t-t_1)B_{m-1}(t;t_1,\dots,t_{m+1})+$$

$$+(t_{m+2}-t)B_{m-1}(t;t_2,\dots,t_{m+2})-\sum_{i=2}^{m+1}\left[\beta_i(t_i-t_1)+\gamma_i(t_{m+2}-t_i)\right](t-t_i)_+^{m-1}.$$

Proceeding from this, in virtue of formulae (21) and (14), we get the formula (27).

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M.Goldmans. Dažas piezīmes par B-splainiem.

Anotācija. B-splainus 1947. gadā ieveda Karri un Šēnbergs, definējot tos ar galīgo diferenču teorijas palīdzību. Šajā rakstā B-splainu definīcijā un to pamatīpašību pierādījumos izmantotas tikai lineāru vienādojumu sistēmu vienkāršās īpašības, neiesaistot tur galīgo diferenču jēdzienu.

М.Гольдман. Замечание о B-сплайнах.

Аннотация. B-сплайны были введены Карри и Шенбергом в 1947 г. Определение этого понятия было основано на теории конечных разностей. В данной заметке определение B-сплайна и вывод основных его свойств осуществляется без привлечения конечных разностей; используются лишь простейшие сведения о системах линейных уравнений. УДК 517.5.

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ON PIECEWISE CONSTANT BIVARIATE APPROXIMATION

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Summary. The approximation of bivariate functions by piecewise constant functions over regular grid on rectangle is studied. It is proved that the degree of approximation can be increased by constructing the operator as the sum of three interpolation type operators on special grids. The exact error bounds on $W^{1,1}_{\infty}$ are obtained.

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0. Introduction.

Let $\Delta(M)=\{t_i(M) \mid i=\overline{0,M}\}$ for some integer $M>0$ be a uniform partition of $[0,1]$ with $t_i(M)=\frac{i}{M}$. By means of functions

$$s_i(M;t)=\begin{cases}1, & \text{when } t\in[t_{i-1}(M),t_i(M)[,\\ 0, & \text{when } t\in[0,1]\setminus[t_{i-1}(M),t_i(M)[,\end{cases}\quad i=\overline{1,M},$$

we define

$$S(M,N)=\left\{s(t,\tau)=\sum_{i=1}^{M}\sum_{j=1}^{N}c_{ij}s_i(M;t)s_j(N;\tau) \,\middle|\, c_{ij}\in R\right\}$$

as the space of piecewise constant functions on $[0,1]\times[0,1]$ over the regular rectangular grid $\Delta(M,N)=\Delta(M)\times\Delta(N)$.

Let I be $[0,1]$ or $[0,1]\times[0,1]$. By $C(I)$ we denote the space of continuous functions defined on I, by $L_p(I)$, $1\le p\le\infty$, the usual Lebesgue space of p-integrable functions on I, equipped with the norm

$$\|f\|_p=\begin{cases}\left(\int_I |f(\omega)|^p d\omega\right)^{\frac{1}{p}}, & \text{when } p<\infty,\\ \operatorname*{sup\,vrai}_{\omega\in I}|f(\omega)|, & \text{when } p=\infty.\end{cases}$$

Further, we introduce the space $L_\infty^{1,1}$ of absolutely continuous functions f on $[0,1]\times[0,1]$ with partial derivatives $f^{(1,0)}, f^{(0,1)}, f^{(1,1)} \in L_\infty([0,1]\times[0,1])$. The space $W_\infty^{1,1}$ we define by

$$W_\infty^{1,1} = \{ f \in L_\infty^{1,1} \mid \| f^{(1,0)}(\cdot,0)\|_\infty \le 1, \| f^{(0,1)}(0,\cdot)\|_\infty \le 1, \| f^{(1,1)}\|_\infty \le 1 \}.$$

A.M.Avakjan [2] has obtained the best approximation of $W_\infty^{1,1}$ by subspace $S(M,N)$

$$\sup_{f\in W_\infty^{1,1}} \inf_{s\in S(M,N)} \| f - s \|_\infty = \frac{1}{M} + \frac{1}{N} - \frac{1}{2MN}. \tag{1}$$

But usually we cannot determine the best approximation for a given $f \in L_\infty^{1,1}$. The goal may be to construct an approximation operator A, which maps $L_\infty^{1,1}$ into $S(M,N)$ and has good approximation properties. It means that Af is "good" to compare with the best approximation for any $f \in L_\infty^{1,1}$. Two of such operators, studied in 1 (see also sections 3 and 4), provide the exactness

$$\sup_{f\in W_\infty^{1,1}} \| f - Af \|_\infty = \frac{1}{M} + \frac{1}{N} - \frac{1}{4MN}. \tag{2}$$

In this paper we increase the degree of piecewise constant approximation with partition from [3].

1. Approximation operator.

Let the approximation of functions $f \in C([0,1])$ be defined by the rule

$$f \longrightarrow A(M;f,t) = \sum_{i=1}^{M} A_i(M;f)\, s_i(M;t),$$

where $A_i(M)$, $i=\overline{1,M}$, are linear functionals (there $A_i(M;f) = A_i(M)f$).

A well-known and useful principle for the transfer of univariate approximation methods to bivariate ones consists in the tensor product scheme. Using functionals $A_i(M)$ we introduce operators $A_i^t(M)$ and $A_i^\tau(M)$, which map $C([0,1]\times[0,1])$ into $C([0,1])$:

$A_i^t(M)f = A_i^t(M;f,.)$,

where $A_i^t(M;f,\tau) = A_i(M;f(.,\tau))$ for $\tau \in [0,1]$;

$A_i^\tau(M)f = A_i^\tau(M;f,.)$,

where $A_i^\tau(M;f,t) = A_i(M;f(t,.))$ for $t \in [0,1]$.

Under the assumption that

$A_i(M)A_j^\tau(N)f = A_j(N)A_i^t(M)f$ for every $f \in C([0,1]\times[0,1])$

we denote by $A_{ij}(M,N)$ functionals

$A_{ij}(M,N)f = A_i(M)A_j^\tau(N)f = A_j(N)A_i^t(M)f$, $i=\overline{1,M}$, $j=\overline{1,N}$.

Finally, we define the approximation operator

$A(M,N): C([0,1]\times[0,1]) \to S(M,N)$ according to the tensor product scheme by the equality

$$A(M,N;f,t,\tau) = \sum_{i=1}^{M}\sum_{j=1}^{N} A_{ij}(M,N;f)\, s_i(M;t)\, s_j(N;\tau)$$

(there $A(M,N;f,.,.) = A(M,N)f$, $A_{ij}(M,N;f) = A_{ij}(M,N)f$).

We assume that the functionals $A_i(M)$ use the information about functions values only on the corresponding intervals $[t_{i-1}(M), t_i(M)[$ and are exact on constants. In this case operators $A(M)$ and $A(M,N)$ define approximations of interpolation type.

This paper is devoted to the study of the approximation $f \to \bar{A}(M,N,m,n)f$, defined as a sum

$$\bar{A}(M,N,m,n) = A(mM,N) + A(M,nN) - A(M,N), \qquad (3)$$

where m and n are some positive integers.

2. Integral representation of the error and error bounds.

In this section we estimate $f - \bar{A}(M,N,m,n)f$ on $W_\infty^{1,1}$. Using the definition (3) of the operator $\bar{A}(M,N,m,n)$ we get

$$f - \bar{A}(M,N,m,n)f = (f - A(mM,N)f) + (f - A(M,nN)f) - (f - A(M,N)f). \qquad (4)$$

For $f - A(M,N)f$ we have the following results proved in [1].

Theorem 1. If the approximation $A(M)$ is exact for constant functions and functionals $A_i(M)$, $i=\overline{1,M}$, satisfy the condition

$$A_i(M)\int_0^1 g(u)h(\cdot,u)\,du=\int_0^1 g(u)A_i(M;h(\cdot,u))\,du$$

for every $g\in L_\infty([0,1])$, where

$$h(t,u)=(t-u)_+^0=\begin{cases}1, & \text{when } t\geq u,\\ 0, & \text{when } t<u,\end{cases}$$

then for $f\in L_\infty^{1,1}$ we have

$$f(t,\tau)-A(M,N;f,t,\tau)=\int_0^1 f^{(1,0)}(u,0)\kappa(M;t,u)\,du+ \tag{5}$$
$$+\int_0^1 f^{(0,1)}(0,v)\kappa(N;\tau,v)\,dv+\int_0^1\int_0^1 f^{(1,1)}(u,v)K(MN;t,\tau,u,v)\,du\,dv,$$

where

$$\kappa(M;t,u)=h(t,u)-A(M;h(\cdot,u),t),$$

$$K(M,N;t,\tau,u,v)=h(t,u)h(\tau,v)-A(M;h(\cdot,u),t)A(N;h(\cdot,v),\tau).$$

Theorem 2. For the functional $E:W_\infty^{1,1}\to R$, defined by the equality

$$Ef=\int_0^1 f^{(1,0)}(u,0)\kappa_t(u)\,du+\int_0^1 f^{(0,1)}(0,v)\kappa_\tau(v)\,dv+$$
$$+\int_0^1\int_0^1 f^{(1,1)}(u,v)K(u,v)\,du\,dv$$

with $\kappa_t,\kappa_\tau\in L_1([0,1])$ and $K\in L_1([0,1]\times[0,1])$, it holds

$$\sup|Ef|=\|\kappa_t\|_1+\|\kappa_\tau\|_1+\|K\|_1.$$

Under the assumptions made in Theorem 1, taking (4) into account, we obtain that

$$f(t,\tau)-\bar{A}(M,N,m,n;f,t,\tau)=\int_0^1 f^{(1,0)}(u,0)\kappa(mM;t,u)\,du+ \tag{6}$$
$$+\int_0^1 f^{(0,1)}(0,v)\kappa(nN;\tau,v)\,dv+\int_0^1\int_0^1 f^{(1,1)}(u,v)\bar{K}(M,N,m,n,t,\tau,u,v)\,du\,dv,$$

where

$$\bar{K}(M,N,m,n)=K(mM,N)+K(M,nN)-K(M,N).$$

From (6) we can arrive at the error bound

$$|f(t,\tau)-\bar{A}(M,N,m,n;f,t,\tau)|\leqslant \|f^{(1,0)}(\cdot,0)\|_\infty \|\kappa(mM;t,\cdot)\|_1 + \|f^{(0,1)}(0,\cdot)\|_\infty \|\kappa(nN;\tau,\cdot)\|_1 + \|f^{(1,1)}\|_\infty \|\bar{K}(M,N,m,n;t,\tau,\cdot,\cdot)\|_1. \quad (7)$$

Theorem 2 guarantees that the estimate (7) is exact on $W^{1,1}_\infty$. It means that

$$\sup_{f\in W^{1,1}_\infty} |f(t,\tau)-\bar{A}(M,N,m,n;f,t,\tau)| = \|\kappa(mM;t,\cdot)\|_1 + \|\kappa(nN;\tau,\cdot)\|_1 + \|\bar{K}(M,N,m,n;t,\tau,\cdot,\cdot)\|_1 \quad (8)$$

for any $(t,\tau)\in[0,1]\times[0,1]$.

For further investigation of the value (8) it is useful to transform the expression of $\bar{K}$:

$$\bar{K}(M,N,m,n) = K(mM,nN) + Q(M,N,m,n), \quad (9)$$

where

$$J(M,N,m,n;t,\tau,u,v) = A(mM,nN;H(\cdot,\cdot,u,v),t,\tau) + A(M,N;H(\cdot,\cdot,u,v),t,\tau) - A(mM,N;H(\cdot,\cdot,u,v),t,\tau) - A(M,nN;H(\cdot,\cdot,u,v),t,\tau),$$

$$H(t,\tau,u,v) = h(t,u)h(\tau,v).$$

For $Q(M,N,m,n)$ we also have

$$Q(M,N,m,n;t,\tau,u,v) = q(M,m;t,u)\,q(N,n;\tau,v),$$

where $q(M,m;t,u) = A(M;h(\cdot,u),t) - A(mM;h(\cdot,u),t)$.

Let $P(a,b,c,d)$ be a rectangle with sides $t=a$, $t=b$, $\tau=c$, $\tau=d$. Rectangles $P(t_{i-1}(M),t_i(M),\tau_{j-1}(N),\tau_j(N)),\ i=\overline{1,M},\ j=\overline{1,N}$ of the grid $\Delta(M,N)$ we denote by $\Delta_{ij}(M,N)$. Suppose that $(t,\tau)\in\Delta_{ij}(M,N)$. It is easy to see that for $(u,v)\notin\Delta_{ij}(M,N)$

$$\kappa(M;t,u)=0,\ \kappa(N;\tau,v)=0,\ Q(M,N,m,n;t,\tau;u,v)=0,$$

$$K(M,N;t,\tau,u,v) = \begin{cases} \kappa(M;t,u) & \text{for } (u,v)\in P(t_{i-1}(M),t_i(M),0,\tau_{j-1}(N)), \\ \kappa(N;\tau,v) & \text{for } (u,v)\in P(0,t_{i-1}(M),\tau_{j-1}(N),\tau_j(N)), \\ 0 & \text{for all other } (u,v)\notin\Delta_{ij}(M,N). \end{cases}$$

Therefore

$$\| K(M;t,.)\|_1 = \int_{t_{i-1}(M)}^{t_i(M)} |K(M;t,u)|\,du, \quad \| K(N;\tau,.)\|_1 = \int_{\tau_{j-1}(N)}^{\tau_j(N)} |K(N;\tau,v)|\,dv, \tag{10}$$

$$\| K(M,N;t,\tau,.,.)\|_1 = \frac{i-1}{M}\| K(N;\tau,.)\|_1 + \frac{j-1}{N}\| K(M;t,.)\|_1 + \tag{11}$$

$$+ \iint_{\Delta_{ij}(M,N)} |K(M,N;t,\tau,u,v)|\,du\,dv,$$

$$\| Q(M,N,m,n;t,\tau,.,.)\|_1 = \iint_{\Delta_{ij}(M,N)} |Q(M,N,m,n;t,\tau,u,v)|\,du\,dv. \tag{12}$$

The rest of the paper is devoted to further computation of these values for two concrete approximations.

3. Approximation of function values at middle points.

Let $u_i(M)$ be the middle point of an interval $[t_{i-1}(M), t_i(M)]$: $u_i(M) = \frac{2i-1}{M}$, $i=\overline{1,M}$.

We consider the univariate approximation $A(M)$ defined on the basis of functionals

$$A_i(M)f = f(u_i(M)), \quad i=\overline{1,M}. \tag{13}$$

As it was mentioned above, in this case the bivariate approximation $A(M,N)$ provides (2). The main result of this section is the exact L_∞-bound of $f - \bar{A}(M,N,m,n)f$ on $W^{1,1}_\infty$.

Theorem 3. For the approximation $\bar{A}(M,N,m,n)$ defined by (3) on the basis of (13) it holds

$$\sup_{f\in W^{1,1}_\infty} \| f-\bar{A}(M,N,m,n)f\|_\infty = \frac{1}{mM} + \frac{1}{nN} + \frac{1}{4MN} - \frac{1}{4mMN} - \frac{1}{4nMN}. \tag{14}$$

Proof. First, we notice that conditions from Theorem 1 and Theorem 2 are fulfilled. So, the estimate (8) is true and we need the norms $\| \bar{K}(M,N,m,n;t,\tau,.,.)\|_1$, $\| K(mM;t,.)\|_1$, $\| K(nN;\tau,.)\|_1$.

Let denote $\Delta_{ij}^{i'j'}(M,N,m,n) = \Delta_{(i-1)m+i',(j-1)n+j'}(mM,nN)$, $i=\overline{1,M}$, $j=\overline{1,N}$, $i'=\overline{1,m}$, $j'=\overline{1,n}$, and suppose that $(t,\tau)\in\Delta_{ij}^{i'j'}(M,N,m,n)$.

Then for $(u,v)\in\Delta_{ij}^{i'j'}(M,N,m,n)$ we have

$$\kappa(mM;t,u)=$$
$$=\begin{cases}\operatorname{sgn}(t-u_i^{i'}(M,m)), & \text{when } u \text{ is between } t \text{ and } u_i^{i'}(M,m),\\ 0, & \text{otherwise},\end{cases}\tag{15}$$

$$K(mM,nN;t,\tau,u,v)=\tag{16}$$
$$=\begin{cases}\operatorname{sgn}(t-u_i^{i'}(M,m)), & \text{when } (u,v)\in P(t,u_i^{i'}(M,m),t_j^{j'-1}(N,n),\min\{\tau,u_j^{j'}(N,n)\}),\\ \operatorname{sgn}(\tau-u_j^{j'}(N,n)), & \text{when } (u,v)\in P(t_i^{i'}(M,m),\min\{t,u_i^{i'}(M,m)\},\tau,u_j^{j'}(N,n)),\\ \frac{1}{2}(\operatorname{sgn}(t-u_i^{i'}(M,m))+\operatorname{sgn}(\tau-u_j^{j'}(N,n))), & \text{when } (u,v)\in P(t,u_i^{i'}(M,m),\tau,u_j^{j'}(N,n)),\\ 0, & \text{otherwise},\end{cases}$$

and for $(u,v)\in\Delta_{ij}(M,N)$

$$Q(M,N,m,n;t,\tau,u,v)=\tag{17}$$
$$=\begin{cases}\operatorname{sgn}((u_i(M)-u_i^{i'}(M,m))(u_j(N)-u_j^{j'}(N,n))), & \text{when } (u,v)\in P(u_i(M),u_i^{i'}(M,m),u_j(N),u_j^{j'}(N,n)),\\ 0, & \text{otherwise}\end{cases}$$

(there $u_i^{i'}(M,m)=u_{(i-1)m+i'}(mM)$, $t_i^{i'}(M,m)=t_{(i-1)m+i'}(mM)$).

From (10) and (15) it follows that

$$\|\kappa(mM;t,\cdot)\|_1=|t-u_i^{i'}(M,m)|,\quad \|\kappa(nN;\tau,\cdot)\|_1=|\tau-u_j^{j'}(N,n)|\tag{18}$$

Using (9), (11), (12), (16) and (17), we estimate

$$\|\bar K(M,N,m,n;t,\tau,\cdot,\cdot)\|_1\le\|K(mM,nN;t,\tau,\cdot,\cdot)\|_1+$$
$$+\|Q(M,N,m,n;t,\tau,\cdot,\cdot)\|_1\le G(M,N,m,n;t,\tau),$$

where $G(M,N,m,n;t,\tau)$ is the sum of the squares of four rectangles:

$$P(t,u_i^{i'}(M,m),\tau,u_j^{j'}(N,n)),$$
$$P(t,u_i^{i'}(M,m),0,u_j^{j'}(N,n)),$$
$$P(0,u_i^{i'}(M,m),\tau,u_j^{j'}(N,n)),$$
$$P(u_i(M),u_i^{i'}(M,m),u_j(N),u_j^{j'}(N,n)).$$

By summation we get

$$G(M,N,m,n;t,\tau)=|t-u_i^{i'}(M,m)|\,|\tau-u_j^{j'}(N,n)|+$$
$$+\frac{2mi+2i'-1}{2mM}|\tau-u_j^{j'}(N,n)|+\frac{2nj+2j'-1}{2Nn}|t-u_i^{i'}(M,m)|+\frac{|m+1-2i'|}{2mM}\,\frac{|n+1-2j'|}{2nN}.$$

It is easy to see that $G(M,N,m,n;\cdot,\cdot)$ has the maximal value at the point (1,1) (when $i=M$, $j=N$, $i'=m$, $j'=n$ respectively)

$$\|G(M,N,m,n;\dots)\|_\infty = G(M,N,m,n;1,1) = \\ = \frac{1}{2mM}+\frac{1}{2nN}+\frac{1}{4MN}-\frac{1}{4mMN}-\frac{1}{4nMN}.$$

Direct calculations of the norm $\|\bar{K}(M,N,m,n;1,1,\dots)\|_1$ give the equality

$$\|\bar{K}(M,N,m,n;1,1,\dots)\|_1 = G(M,N,m,n;1,1).$$

Therefore

$$\sup_{(t,\tau)\in[0,1]\times[0,1]} \|\bar{K}(M,N,m,n;t,\tau,\dots)\|_1 = \|\bar{K}(M,N,m,n;1,1,\dots)\|_1 = \\ = \frac{1}{2mM}+\frac{1}{2nN}+\frac{1}{4MN}-\frac{1}{4mMN}-\frac{1}{4nMN}.$$

Taking into account also that

$$\sup_{t\in[0,1]} \|K(mM;t,\cdot)\|_1 = \|K(mM;1,\cdot)\|_1 = \frac{1}{2mM},$$

$$\sup_{\tau\in[0,1]} \|K(nN;\tau,\cdot)\|_1 = \|K(nN;1,\cdot)\|_1 = \frac{1}{2nN}$$

(see (18)), by (8) we obtain the final result

$$\sup_{f\in W^{1,1}_\infty} \|f-\bar{A}(M,N,m,n)f\|_\infty = \|\bar{K}(M,N,m,n;1,1,\dots)\|_1 + \\ + \|K(mM;1,\cdot)\|_1 + \|K(nN;1,\cdot)\|_1 = \frac{1}{mM}+\frac{1}{nN}+\frac{1}{4MN}-\frac{1}{4mMN}-\frac{1}{4nMN}.$$

4. Approximation of function integral mean values.

Let the univariate approximation $A(M)$ be defined via

$$A_i(M)f = M\int_{t_{i-1}(M)}^{t_i(M)} f(t)\,dt, \quad i=\overline{1,M}. \tag{19}$$

It has been proved in [1] that in this case the bivariate approximation $A(M,N)$ also provides (2). Now we obtain the exact L_∞-bound of the error $f-\bar{A}(M,N,m,n)f$ on $W^{1,1}_\infty$.

Theorem 4. For the approximation $\bar{A}(M,N,m,n)$ defined by (3) on the basis of (19) it holds

$$\sup_{f\in W^{1,1}_\infty} \|f-\bar{A}(M,N,m,n)f\|_\infty = \frac{1}{mM}+\frac{1}{nN}+\frac{1}{4MN}-\frac{1}{4mMN}-\frac{1}{4nMN}. \tag{20}$$

Proof. Since conditions of Theorem 1 and Theorem 2 are fulfilled and (8) is true, it is sufficient to show the equality

$$\sup_{(t,\tau)\in[0,1]\times[0,1]} \left(\|\kappa(mM;t,\cdot)\|_1+\|\kappa(nN;\tau,\cdot)\|_1+\|\bar K(M,N,m,n;t,\tau,\cdot,\cdot)\|_1\right)= \frac{1}{mM}+\frac{1}{nN}+\frac{1}{4MN}-\frac{1}{4mMN}-\frac{1}{4nMN}. \tag{21}$$

To show (21) we use the estimate

$$\|\bar K(M,N,m,n;t,\tau,\cdot,\cdot)\|_1\le\|K(mM,nN;t,\tau,\cdot,\cdot)\|_1+\|Q(M,N,m,n;t,\tau,\cdot,\cdot)\|_1, \tag{22}$$

which follows immediately from (9).

Suppose that $(t,\tau)\in\Delta_{ij}^{i'j'}(M,N,m,n)$. Then for $(u,v)\in\Delta_{ij}^{i'j'}(M,N,m,n)$ we have

$$\kappa(mM;t,u)=\begin{cases} mM(u-t_i^{i'-1}(M,m)), & \text{when } u\in[t_i^{i'-1}(M,m),t],\\ -mM(t_i^{i'}(M,m)-u), & \text{when } u\in\,]t,t_i^{i'}(M,m)],\end{cases} \tag{23}$$

$$K(mM,nN;t,\tau,u,v)=\begin{cases} 1-mnMN(t_i^{i'}(M,m)-u)(t_j^{j'}(N,n)-v), \\ \qquad\text{when } (u,v)\in P(t_i^{i'-1}(M,m),t,t_j^{j'-1}(N,n),\tau)\\ -mnMN(t_i^{i'}(M,m)-u)(t_j^{j'}(N,n)-v),\\ \qquad\text{when } (u,v)\notin P(t_i^{i'-1}(M,m),t,t_j^{j'-1}(N,n),\tau)\end{cases} \tag{24}$$

and for $(u,v)\in\Delta_{ij}(M,N)$

$$q(M,m;t,u)=\begin{cases} -M(u-t_{i-1}(M)), & \text{when } u\in[t_{i-1}(M),t_i^{i'-1}(M,m)[,\\ M(i'-1)(u-t_i^{i'-1}(M,m))-M(m-i')(t_i^{i'}(M,m)-u), \\ \qquad\text{when } u\in[t_i^{i'-1}(M,m),t_i^{i'}(M,m)],\\ M(t_i(M)-u), & \text{when } u\in\,]t_i^{i'}(M,m),t_i(M)].\end{cases} \tag{25}$$

From (10)-(12) and (23)-(25) it follows that

$$\|\kappa(mM;t,\cdot)\|_1=\frac{1}{2}mM\left((t_i^{i'}(M,m)-t)^2+(t-t_i^{i'-1}(M,m))^2\right),$$

$$\|Q(M,N,m,n;t,\tau,\cdot,\cdot)\|_1=\frac{((i'-1)^2+(m-i')^2)((j'-1)^2+(n-j')^2)}{4m(m-1)n(n-1)MN}, \tag{26}$$

$$\|K(mM,nN;t,\tau,\cdot,\cdot)\|_1 = \frac{1}{4mnMN}\mu\left(\frac{t-t_i^{k-1}(M,m)}{mM},\frac{\tau-t_j^{l-1}(N,n)}{nN}\right)+$$
$$+\frac{(i-1)m+k-1}{mM}\|\kappa(nN;\tau,\cdot)\|_1+\frac{(j-1)n+l-1}{nN}\|\kappa(mM;t,\cdot)\|_1, \tag{27}$$

where $\mu(\alpha,\beta)=1-4\alpha\beta+4\alpha^2\beta+4\alpha\beta^2-2\alpha^2\beta^2$, $(\alpha,\beta)\in[0,1]\times[0,1]$.
In virtue of the equalities

$$\sup_{t\in[t_i^{k-1}(M,m),t_i^k(M,m)]}\|\kappa(mM;t,\cdot)\|_1=\|\kappa(mM;t_i^k(M,m),\cdot)\|_1=\frac{1}{2mM}, \tag{28}$$

$$\sup \mu(\alpha,\beta)=\mu(1,1)=3,$$

by (27) we obtain the estimate

$$\sup_{(t,\tau)\in[0,1]\times[0,1]}\|K(mM,nN;t,\tau,\cdot,\cdot)\|_1=\|K(mM,nN;1,1,\cdot,\cdot)\|_1=$$
$$=\frac{1}{2mM}+\frac{1}{2nN}-\frac{1}{4mnMN}. \tag{29}$$

Using (26) we get

$$\sup_{(t,\tau)\in[0,1]\times[0,1]}\|Q(M,N,m,n;t,\tau,\cdot,\cdot)\|_1=\|Q(M,N,m,n;1,1,\cdot,\cdot)\|_1=\frac{(m-1)(n-1)}{4mnMN}. \tag{30}$$

From (22), (29) and (30) we have the inequality

$$\sup_{(t,\tau)\in[0,1]\times[0,1]}\|\bar K(M,N,m,n;t,\tau,\cdot,\cdot)\|_1\le\|K(mM,nN;1,1,\cdot,\cdot)\|_1+\|Q(M,N,m,n;1,1,\cdot,\cdot)\|_1.$$

Besides, the equality

$$\|\bar K(M,N,m,n;1,1,\cdot,\cdot)\|_1=\|K(mM,nN;1,1,\cdot,\cdot)\|_1+\|Q(M,N,m,n;1,1,\cdot,\cdot)\|_1$$

holds because of (9) and nonnegativity of the functions $K(mM,nN;1,1,\cdot,\cdot)$ and $Q(M,N,m,n;1,1,\cdot,\cdot)$ (see (24) and (25)).
It means that

$$\sup_{(t,\tau)\in[0,1]\times[0,1]}\|\bar K(M,N,m,n;t,\tau,\cdot,\cdot)\|_1=\|\bar K(M,N,m,n;1,1,\cdot,\cdot)\|_1=$$
$$=\frac{1}{2mM}+\frac{1}{2nN}+\frac{1}{4MN}-\frac{1}{4mMN}-\frac{1}{4nMN}.$$

Taking into account also that

$$\sup_{t\in[0,1]}\|\kappa(mM;t,\cdot)\|_1=\|\kappa(mM;1,\cdot)\|_1=\frac{1}{2mM},$$

$$\sup_{\tau\in[0,1]}\|\kappa(nN;\tau,\cdot)\|_1=\|\kappa(nN;1,\cdot)\|_1=\frac{1}{2nN}$$

(see (28)), we arrive at (21).

5. Conclusion.

Analysing results (2), (14) and (20) we want to point out the following.

1) The approximation $\bar{A}(M,N,m,n)$ in case (13) is equivalent to the one in case (19) with respect to exactness on the class $W_\infty^{1,1}$.

2) If $m=1$ or $n=1$, then Theorem 3 and Theorem 4 give the equality (2) in virtue of

$$\bar{A}(M,N,m,1)=A(mM,N),\quad \bar{A}(M,N,1,n)=A(M,nN)$$

(see (3)). Indeed, we have

$$\sup_{f\in W_\infty^{1,1}}\|f-\bar{A}(M,N,m,1)f\|_\infty=\frac{1}{mM}+\frac{1}{N}-\frac{1}{4mMN},$$

$$\sup_{f\in W_\infty^{1,1}}\|f-\bar{A}(M,N,1,n)f\|_\infty=\frac{1}{M}+\frac{1}{nN}-\frac{1}{4nMN}.$$

3) In case $M=N$ the approximation $A(M,N)$ uses $L=M^2$ units of information. In terms of L we can rewrite the equality (2) in form

$$\sup_{f\in W_\infty^{1,1}}\|f-A(M,M)f\|_\infty=2L^{-\frac{1}{2}}-\frac{1}{4}L^{-1}. \tag{31}$$

But the approximation $\bar{A}(M,N,m,n)$ in case $M=N=m=n$ uses $L=2M^3$ units of information and from (14) or (20) it follows that

$$\sup_{f\in W_\infty^{1,1}}\|f-\bar{A}(M,M,M,M)f\|_\infty\overset{*}{=}\frac{9}{4}\sqrt[3]{4}\,L^{-\frac{2}{3}}-L^{-1}. \tag{32}$$

Comparing (31) with (32) we can conclude that for a fixed L the use of the approximation $\bar{A}$ is more efficient than the approximation A with respect to exactness on the class $W_\infty^{1,1}$.

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С.Асмусс. О кусочно-постоянной аппроксимации функций двух переменных.

Аннотация. Рассматривается приближение функций двух переменных кусочно-постоянными относительно регулярного разбиения прямоугольной области функциями. Установлено, что задание приближения в виде суммы трех операторов интерполяции на сетках специального вида позволяет существенно повысить порядок погрешности, для которой получены точные на классе $W_\infty^{1,1}$ оценки. УДК 517.5.

S.Asmuss. Par aproksimāciju ar gabaliem konstantām funkcijām divu argumentu gadījumā.

Anotācija. Dotajā rakstā aplūkota divargumentu funkciju aproksimācija ar gabaliem konstantām funkcijām, kas definētas uz taisnstūra apgabala regulāra režģa. Pierādīts, ka aproksimācijas kārtu var paaugstināt, veidojot aproksimācijas operatoru kā triju uz speciāliem režģiem definētu interpolācijas operatoru summu. Iegūti precīzi tuvinājuma kļūdas novērtējumi $W_\infty^{1,1}$ funkciju klasē.

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SOME QUESTIONS OF THE CALCULATION STABILITY OF HANKEL INTEGRAL FOR DISCONTINUOUS FUNCTIONS. 1.

M.Belov

Summary. This work regards the construction of the stable algorithm for the inversion of Hankel transform. This method is based on the asymptotic solution for an auxiliary problem of mathematical physics. In the first part of the work the formal asymptotic solution was obtained. The justification and the algorithm itself are subjects of the second part. AMS SC 65M,65I, 65B10.

Basic notation

1. $F(y) = H(f(x)) = \int_0^{+\infty} xf(x)\, J_\nu(yx)dx$, $(y>0, \nu>-1)$ is an integral Hankel transform (IHT);
   $f(x)$ is an original function;
   $F(x)$ is an image.
2. J_ν is Bessel function.
3. K_ν^n is the original functions class $(n = 0,1,2,\ldots,\nu>-1)$.
4. $H_t(F(y)) = \int_0^{+\infty} ye^{-ty^2} F(y)\, J_\nu(xy)dy$, $(t>0, x>0, \nu>-1)$.
5. $g(x) = x^{-\alpha} f(x)$.
6. $g_k = g^{(k)}(0)/k!$, $k = 0,1,2,\ldots$.

7. $\Upsilon_{2n+1}(x)=\sum_{s=0}^{2n+1} g_s x^{\alpha+s}$.

8. $Lu=\frac{\partial}{\partial x}(x\frac{\partial u}{\partial x})-\frac{\nu^2}{x}u.$

9. $L_x u=\frac{1}{x}Lu=\frac{1}{x}\frac{\partial}{\partial x}(x\frac{\partial u}{\partial x})-\frac{\nu^2}{x^2}u.$

10. $A_{n,x}U=\sum_{k=0}^{n}\frac{t^k}{k!}L_x^k f(x)\quad (n=0,1,2,\ldots).$

11. $z=\frac{x}{2\sqrt{t}}\qquad (x>0,\ t>0).$

12. $A_{m,z}V=\sum_{k=0}^{m} t^{\frac{k+\alpha}{2}} V_k(z)\Big|_{z=\frac{x}{2\sqrt{t}}}\qquad (m=0,1,2,\ldots).$

13. $V_k(z)=2^{\alpha+k} g_k z^{\nu} e^{-z^2}\frac{\Gamma\left[\frac{k+2+\alpha+\nu}{2}\right]}{\Gamma(\nu+1)}\,{}_1F_1\left[\frac{k+2+\alpha+\nu}{2},\ \nu+1;z^2\right]$

$(k=0,1,2,\ldots).$

14. $\Gamma(x)$ is Gamma - function.

15. ${}_1F_1(a,b;x)$, $\Phi(a,b;x)$ are degenerate hypergeometric functions.

16. $A_{m,z}A_{n,x}U=\sum_{j=0}^{m} g_j t^{\frac{j+\alpha}{2}}\sum_{k=0}^{n}\frac{\omega_{j,k}}{k!\,(2z)^{2k-j-\alpha}}\Big|_{z=\frac{x}{2\sqrt{t}}}$

$(m,n=0,1,2,\ldots).$

17. $\omega_{m,k}=4^k\left[\frac{\nu-m-\alpha}{2}\right]_k\left[-\frac{m+\alpha+\nu}{2}\right]_k .$

18. $(a)_k=\Gamma(a+k)/\Gamma(a)$ is Pohgammer's symbol.

19. $x_j>0$ is a point of discontinuity of the original $f=f(x)$.

20. $\Delta_k=L_x^k f\Big|_{x=x_j+0}-L_x^k f\Big|_{x=x_j-0}\qquad (k=0,1,2,\ldots).$

21. $\Delta_k^* = \frac{d}{dx} L_*^k f \Big|_{x=x_j+0} - \frac{d}{dx} L_*^k f \Big|_{x=x_j-0}$ $(k = 0,1,2,\ldots)$.

22. $\Omega_k = \Omega_k(x_j, p, s) = \frac{x_j}{2\,\Gamma(k+1)} \cdot \int_p^{+\infty} \frac{(t-p)^k}{t^{k+1}}\, e^{-st}\, I_\nu(t)dt$
$(k = 0,1,2,\ldots)$.

23. $\Omega_k^* = \Omega_k^*(x, x_j, p, s) = \frac{1}{2x\,\Gamma(k+1)} \int_p^{+\infty} \frac{(t-p)^k}{t^k}\, e^{-st} \Big[x_j\, I_\nu(t) -$

$- x\, I_\nu'(t)\Big]dt$ $(k = 0,1,2,\ldots)$.

24. $I_\nu(x)$ is a modified Bessel function.

25. $p = \frac{xx_j}{2t}$, $s = \frac{1}{2}\Big[\frac{x}{x_j} + \frac{x_j}{x}\Big]$ $(x>0, t>0)$.

26. $(\nu, m) = \frac{\Gamma(\frac{1}{2} + \nu + m)}{m!\,\Gamma(\frac{1}{2} + \nu - m)}$ is Hankel's symbol.

27. $(\nu, m)^* = \frac{1}{2}\,[(\nu-1, m) + (\nu+1, m)]$.

28. $z_j = \frac{x - x_j}{2\sqrt{t}}$.

29. $P_n(x,t;x_j) = \sum_{k=0}^{n} \Delta_k^*\, t^k\, \Omega_k + \sum_{k=0}^{n} \Delta_k\, t^k\, \Omega_k^*$ is a local asymptotic solution at a point of discontinuity $x = x_j$.

30. $u_n(x,t) = A_{n,x}U + (A_{2n+1,z}V - A_{2n+1,z}A_{n,x}U)\Phi_0(x) + P_n(x,t;x_j)$ is a global asymptotic solution when $\alpha \geq -2$.

31. $u_n^*(x,t) = \Phi_1(\frac{x}{\sqrt{t}})A_{n,x}U + (A_{2n+1,z}V - \Phi_1(\frac{x}{\sqrt{t}})A_{2n+1,z}A_{n,x}U) \times$
$\times \Phi_0(x) + P_n(x,t;x_j)$ is a global asymptotic solution when $\alpha < -2$.

32. $\Phi_o(x)$, $\Phi_1(x) \in C^{(\infty)}([0,+\infty[)$ is a cutting function:

$$\Phi_o(x) = \begin{cases} 0, & x \geq 2, \\ 1, & 0 \leq x \leq 1, \end{cases}$$

in the interval $1 \leq x \leq 2$ $\Phi_o(x)$ is monotonically decreasing from 1 to 0;

$$\Phi_1(x) = \begin{cases} 0, & 0 \leq x \leq 1/2, \\ 1, & x \geq 1, \end{cases}$$

in the interval $1/2 \leq x \leq 1$ $\Phi_1(x)$ is monotonically increasing from 1 to 0.

33. $$\operatorname{sgn} x = \begin{cases} 1, & x>0 \\ 0, & x=0 \\ -1, & x<0 . \end{cases}$$

34. $G_t^{(m)}$ is an operator with the rule:

$$G_t^{(m)}u(t) = 2u\left(\frac{t}{2^{1/m}}\right) - u(t) \qquad (m = 1,2,3,\dots).$$

35. G_t^n is an operator with the rule:

$$G_t^n u(t) = \begin{cases} u(t), & n=0 \\ G_t^{(n)}\, G_t^{(n-1)} \dots G_t^{(1)} u(t) & n=1,2,\dots . \end{cases}$$

36. $R_n(x,t;x_j) = (T_{2n+1}(x) - G_t^n A_{2n+1,x}^{\nu})\Phi_o(x) - G_t^n P_n(x,t;x_j)$ is a discontinuos original impact function ($n = 0,1,2,\dots$).

37. $\tilde{F} = \tilde{F}(y)$ are approximation image $F = F(y)$ values.

38. $\Delta F = \Delta F(y)$ is an image setting error ($F = \tilde{F} + \Delta F$).

39. $\tilde{f}_n(x,t) = G_t^n H_t(\tilde{F}(y)) + R_n(x,t;x_j)$ are approximate original $f = f(x)$ values ($x>0$, $t>0$, $n = 0,1,2,\dots$).

40. $\rho_n = f(x) - \tilde{f}_n(x,t)\Big|_{t=t(\varepsilon)}$ is an original approximation error.

1. Statement of the problem and solution method.

The effective instrument for the broad class of applied problems solution [6,9,13] is an integral Hankel transform:

$$F(y) = H(f(x)) = \int_0^{+\infty} xf(x)J_\nu(yx)dx \quad (y>0,\ \nu>-1), \tag{1.1}$$

where $f = f(x)$ is an original function, and $F = F(y)$ is the corresponding image.

As usual, using integral transforms, after solving of the problem in the image space the inversion problem becomes the central one. The problem is to obtain the original f, if we know the image F.

Under certain conditions [5] the original f can be obtained by means of Hankel integral too:

$$f(x) = H(F(y)) = \int_0^{+\infty} yF(y)J_\nu(xy)dy \quad (x>0). \tag{1.2}$$

However, like in the integral Laplace transform case [4,10], the problem to find the original from a known image is incorrect [12]. "Small" changes of an image (in "natural" metrics for applied problems) can cause "large" changes of the original. So, the analytic structure of the integral (1.2) itself is adapted for the "infinitely large" original $f = f(x)$ changes, according to "infinitely small" image $F = F(y)$ variations. As the result, even "insignificant" errors of the image $F(y)$ can cause a "large" error (even up to the complete loss of accuracy) for the numerical calculation of the integral (1.2).

This work regards the stable algorithm construction for the inversion of Hankel transform (IHT) in the sense of regularization method [12], when the inversion accuracy is in accordance with the error of the image setting .

The method presented in the work [3] is used for the construction of such algorithms. Using this method we can examine an auxiliary problem:

$$\begin{aligned} &Lu = x\frac{\partial u}{\partial t} \quad (x>0,\ t>0), \\ &|u|_{x=+0}| \le M\ ,\ u|_{t=+0} = f(x), \end{aligned} \tag{1.3}$$

where

$$Lu = \frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) - \frac{\nu^2}{x}u . \qquad (1.4)$$

Using IHT we can represent the problem's (1.3) solution when $\nu \geq 0$ as follows:

$$u = u(x,t) = \int_0^{+\infty} y e^{-ty^2} F(y)\, J_\nu(xy)dy, \qquad (1.5)$$

where $F(y) = H(f(x))$ is an image of the original function $f(x)$ due to Hankel.

The integral in the formula (1.5) contrary to (1.2) contains (when $t>0$) a stabilizing factor $exp(-ty^2)$ in the sense of the monography [12]. The presence of the stabilizing factor leads to the stability of the integral (1.5) calculation for "small" image F changes.

Next, for the problem (1.3) we construct an asymptotic solution when $t \longrightarrow +0$. This asymptotic solution allows to estimate the deviation of the function $u = u(x,t)$, defined by the integral (1.5), from the original function $f = f(x)$ ($t>0$ is fixed). Then we conform the choice of $t>0$ (here $t>0$ is the regularization parameter in the sense of [12]) to the error of the image F setting. Thus we obtain a stable inversion algorithm.

The asymptotic solution of the problem (1.3) contains an additional analytic information about the discontinuous structure of the original f. The implementation of this additional analytic information into the calculation algorithm allows us to obtain the stable approximations to the function $f(x)$ not only in the continuous intervals, but in the points of discontinuity, too.

So, here the asymptotic expansion is an instrument implementing the additional (a priori) analytic information about the solution structure into numerical algorithm in the most natural way.

2. Some preliminary results.

The structure and justification of the results depend on the choice of the original functions class.

Definition 1. K_ν^n is a set of such functions f: $R_+ \to C$, that:

1) in the interval $]0, T[$ (for a $\forall$ $T>0$) f has at most a finite number of points of discontinuity: $x_1 < x_2 < \ldots < x_m$, where $x_j = x_j(f)$, $m = m(f,T) < +\infty$, and all the points are the points of the first kind discontinuity and $\inf_{k,n} |x_k - x_n| \geq \Delta > 0$;

2) $f(x) = O(x^\alpha)$, when $x \to +0$, and $\alpha + \nu + 2 > 0$;

3) $f(x) = O(x^\beta)$, when $x \to +\infty$, where $\beta < -3/2$;

4) in every interval $[x_j, x_{j+1}]$, $j = 1,2,\ldots$ function f has continuous derivatives, which don't exceed the order n (in the end points of continuity), when $n = \infty$ $f(x) \in C^{(\infty)}([x_j, x_{j+1}])$, if the original $f(x)$ when $x>0$ has finite number of points of discontinuity: $0<x_1<x_2<\ldots<x_N<+\infty$, than $f(x) \in C^{(n)}([x_N, +\infty[)$;

5) $g(x) = x^{-\alpha} f(x) \in C^{(n)}([0, x_1])$ (in the end points of continuity), if when $x>0$ $f(x)$ hasn't points of discontinuity, than $g(x) \in C^{(n)}([0,+\infty[)$;

6) for any $x \geq 1$ $|f^{(k)}(x)| \leq D_k < +\infty$ $(k = 0,1,2,\ldots, n)$, where D_k don't depend on f, and

$f^{(k)}(x) = \frac{1}{2} [f^{(k)}(x+0) + f^{(k)}(x-0)]$;

7) when $x \in [0,1]$ $|g^{(k)}(x)| \leq D_k$ $(k = 0,1,2,\ldots, n)$, where $g(x) = x^{-\alpha} f(x)$ and $g^{(k)}(0) = g^{(k)}(+0)$.

Remark. It is obviously, that $K_\nu^m \subset K_\nu^n$, $m>n$.

Basing on the results [7], we have

Lemma 1. If $f(x) \in K_\nu^0$, $\nu > -1$, then for any $y>0$ the integral (1.1) is convergent, and, besides

$$F(y) = O(y^{\alpha'}),\ y \to +0, \tag{2.1}$$

$$F(y) = O(y^{\beta'}),\ y \to +\infty, \tag{2.2}$$

where $\alpha' \geq \min(\nu, -\beta - 2)$ and $\beta' \leq \max(-\frac{1}{2}, -\alpha - 2)$.■

So, for any $f(x) \in \mathring{H}_\nu^n$, $(\nu>-1,\ n \geq 0)$ there exists the image $F(y) = H(f(x))$, defined by the integral (1.1).

It is well known [1] that the problem's (1.3) solution, when $\nu \geq 0$ can be presented by corresponding Green function:

$$u = u(x,t) = \frac{1}{2t}\int_0^{+\infty} y f(y) e^{-\frac{x^2+y^2}{4t}} I_\nu\left(\frac{xy}{2t}\right)dy, \tag{2.3}$$

where $I_\nu(z)$ is modified Bessel function.

Lemma 2. Let function $u = u(x,t)$ $(x>0, t>0)$ be defined by (2.3), where $f(x) \in \mathring{H}_\nu^0$, $\nu>-1$. Then for any $x>0$

$$\lim_{t\to+0} u(x,t) = \frac{1}{2}\,[f(x+0) + f(x-0)]. \tag{2.4}$$

Remark. This lemma is true even if we have weaker requirements to the function $f(x)$. However, we shall not need such generalizations here.

Basing on [11] when $t>0$, $\tau>0$ and $\nu>-1$, we have

$$\frac{1}{2t}\, e^{-\frac{x^2+\tau^2}{4t}} I_\nu\left(\frac{x\tau}{2t}\right) = \int_0^{+\infty} y e^{-ty^2} J_\nu(xy) J_\nu(\tau y)dy. \tag{2.5}$$

Replacing (2.5) into (2.3) and changing the integration order, we have

Lemma 3. Let the function $u = u(x,t)$ $(x>0, t>0)$ be defined by formula (2.3), where $f \in \mathring{H}_\nu^0$, $\nu>-1$. Then for any $x>0$ and any $t>0$ we have

$$u = H_t(F(y)) \equiv \int_0^{+\infty} y e^{-ty^2} F(y) J_\nu(xy)dy, \tag{2.6}$$

where $F(y) = H(f(x))$ is the image of the original function f due Hankel, and the convergence of the integral (2.6) is guaranteed.∎

From lemmas 2 and 3 we have

Theorem 1. Let $f(x) \in \mathring{H}_\nu^0$, $\nu>-1$, and $F(y) = H(f(x))$. Then for any $x>0$

$$\lim_{t\to+0} H_t(F(y)) = \frac{1}{2}[f(x+0) + f(x-0)]. \tag{2.7}$$

where $H_t(F(y))$ is defined by (2.6).■

Remark. From $f(x) \in M_\nu^c, (\nu>-1)$, and $F(y) = H(f(x))$, in general, we can't conclude that $f(x) = H(F(y))$, because of the possibility that the corresponding integral (1.2) can be convergent (see lemma 1).

The formula $f(x) = H(F(y))$ is true only if we have some additional restrictions [5].

Conclusion. Thus, if $f(x) \in M_\nu^n, (n=0,1,2,\ldots,\nu\geq 0)$, than the problem's (1.3) solution can be presented as $u(x,t) = H_t(F(y))$, where $F(y)=H(f(x))$ and for any $x>0$ $\lim_{t\to+0} u(x,t) = \frac{1}{2}[f(x+0)+f(x-0)]$.

Now we can regard the construction of the asymptotic $(t\to+0)$ solution for the problem (1.3).

3. Formal asymptotic solution.

When we construct the formal asymptotic solution (FAS) for the problem (1.3), we assume that $f(x) \in M_\nu^\infty$, $\nu\geq 0$. This limitation will be weakened later.

We can also assume that original $f(x)$ when $x>0$ has only one point of discontinuity $x = x_j>0$. This limitation is not essential. The results obtained in the sequel can be also spread on the general case $f(x) \in M_\nu^c$ without difficulties.

When constructing FAS, we'll suppose, that

$$f(x) = x^a g(x) = \sum_{k=0}^{+\infty} g_k x^{k+a}, \quad g_k = \frac{g^{(k)}(0)}{k!}, \quad x \in]0,\Delta], \qquad (3.1)$$

where $0<\Delta<x_j$. This limitation will be weakened later.

In the sequel we assume that the distance between the point of interior discontinuity $x = x_j>0$ and the boundary $x=0$ is sufficiently large $(x_j \geq t^{1/4})$. So the asymptotic $(t\to+0)$—interaction between the interior point of discontinuity $x = x_j$ and the boundary $x=0$ is not regarded. This limitation is not taken off latter.

3.1. Regular asymptotic.

If PAS of the equation (1.3) when $t\to+0$ is presented as series by powers of t, we can easy obtain the so called regular asymptotic:

$$U(x,t) = \sum_{k=0}^{n} \frac{t^k}{k!} L_\nu^k f(x) , \tag{3.2}$$

where

$$L_\nu = \frac{1}{x} L = \frac{\partial^2}{\partial x^2} + \frac{1}{x}\frac{\partial}{\partial x} - \frac{\nu^2}{x^2} \tag{3.3}$$

The regular asymptotic satisfies the initial condition of the problem (1.3), but it is not defined in the point of discontinuity $x=x_j$ of the original f and in the boundary point $x=0$.

Lemma 4. When $x>0$ $(x\neq x_j)$ and $k=0,1,2,\dots$

$$L_\nu^k f = f^{(2k)}(x) + \frac{1}{x^{2k}} \sum_{l=0}^{2k-1} x^l f^{(l)}(x) A_{l,k} , \tag{3.4}$$

where

$$A_{l,k} = \sum_{m=0}^{l} \frac{(-1)^m a_{l-m,k}}{(l-m)!\, m!} , \tag{3.5}$$

$$a_{kn} = 4^n \left(\frac{\nu-k}{2}\right)_n \left(-\frac{k+\nu}{2}\right)_n . \blacksquare \tag{3.6}$$

Remark. Here and later we assume that $\sum_{l=n}^{m} = 0$ when $m<n$.

Lemma 5. When $x>0$ $(x\neq x_j)$ and $k= 0,1,2,\dots$

$$\frac{d}{dx} L_\nu^k f = f^{(2k+1)}(x) + \frac{1}{x^{2k+1}} \sum_{l=0}^{2k} x^l f^{(l)}(x) B_{l,k}, \tag{3.7}$$

where

$$B_{l,k} = \sum_{m=0}^{l} \frac{(-1)^m}{(l-m)!\, m!} (l-m-2k) a_{l-m,k} \tag{3.8}$$

and a_{kn} are defined by (3.6). $\blacksquare$

Remark. $\frac{d}{dx} L_{\nu}^{0} f = f^{(1)}(x)$, since $g_{0,0} = 0$.

Taking into account (3.1) we have

Lemma 6. When $k=0,1,2,\ldots$ we have the expansion

$$L_{\nu}^{k} f = \frac{1}{x^{2k}} \sum_{m=0}^{+\infty} \omega_{m,k}\, g_m\, x^{m+\alpha}, \quad x \in]0,\Delta[, \tag{3.9}$$

$$\omega_{m,k} = 4^{k} \left(\frac{\nu - m - \alpha}{2}\right)_k \left(-\frac{m+\alpha+\nu}{2}\right)_k . \tag{3.10}$$

From the obtained earlier results we have that if at the interior point of discontinuity $x = x_j$ $(x_j \geq t^{1/4})$ the regular asymptotic (3.2) is formally uniform, then when $x \to +0$, it is not (even formally) uniform, and, so, the construction of the asymptotic solution for the problem (1.3) can be included in the bisingular problems class [8]. In the sequal we shall use some general technical methods exposed in [8].

The special situation appears when $\alpha = \pm\nu$ $(f(x) = x^{\pm\nu} g(x))$. Then from (3.10) we have, that $\omega_{m,k} = 0$ when $m = 0,2,\ldots,2k-2$ and, so, we have

Corollary. If $\alpha = \pm\nu$ and in the expansion (3.1) $g_{2m+1} = 0$ $(m=0,1,2,\ldots)$ it is

$$g(x) = \sum_{m=0}^{+\infty} g_{2m}\, x^{2m}, \quad x \in]0,\Delta], \tag{3.11}$$

then we have

$$L_{\nu}^{k} f = \sum_{m=0}^{+\infty} \omega_{2k+2m,k}\, g_{2k+2m}\, x^{2m\pm\nu}, \quad x \in]0,\Delta[, \tag{3.12}$$

$k=0,1,2,\ldots$. ■

In this case at a point $x=0$ the regular asymptotic (3.2) is formally uniform (the bisingularity doesn't exists).

3.2. Local asymptotic solution at a boundary point $x=0$.

Since the regular asymptotic (3.2) at a point $x=0$, is unfitable we have to construct a different form of the asymptotic approximation for the problem's (1.3) solution. For that case we implement "quick" time $\tau = t/\epsilon$, where $\epsilon \to +0$ is a parameter. So, the problem (1.3) is transformed into the following problem

$$\epsilon L u = x \frac{\partial u}{\partial \tau} \qquad (x>0,\ \tau>0),$$
$$|u|_{x\to+0}| \le M, \quad u|_{\tau=+0} = f(x). \qquad (3.13)$$

At a boundary $x=0$ we can find FAS of the problem (3.13) as follows:

$$V = \epsilon^{\alpha/2} \sum_{k=0}^{+\infty} \epsilon^{k/2}\, \tilde{v}_k(\xi,\tau), \quad \xi = \frac{x}{\sqrt{\epsilon}}. \qquad (3.14)$$

Replacing (3.14) into (3.13) (using in (3.13) the argument ξ) and having made an expansion by small parameter ϵ, povers, we obtain the following problem to determine functions $\tilde{v}_k$:

$$\frac{\partial}{\partial \xi}\left[\xi \frac{\partial \tilde{v}_k}{\partial \xi}\right] - \frac{\nu^2}{\xi}\tilde{v}_k = \xi \frac{\partial \tilde{v}_k}{\partial \tau} \qquad (\xi>0,\ \tau>0),$$
$$|\tilde{v}_k|_{\xi=+0}| \le M, \qquad \tilde{v}_k|_{\tau=+0} = g_k\, \xi^{k+\alpha} \quad (k=0,1,2,\dots), \qquad (3.15)$$

where g_k are defined by (3.1).

Using formula (2.3), we obtain

Lemma 7. Let $\nu \ge 0$ and $\alpha+\nu+2 > 0$, then the problem's (3.15) solution can be presented as follows:

$$\tilde{v}_k = \frac{g_k e^{-\frac{\xi^2}{4\tau}}}{2^{\nu-\alpha-k}} \left[\frac{\xi}{\sqrt{\tau}}\right]^{\nu} \tau^{\frac{k+\alpha}{2}} \frac{\Gamma\left[\frac{k+\alpha+2+\nu}{2}\right]}{\Gamma(\nu+1)}\, {}_1F_1\left[\frac{k+\alpha+2+\nu}{2},\ \nu+1;\ \frac{\xi^2}{4\tau}\right]$$
$$(k=0,1,2,\dots). \qquad (3.16)$$

where ${}_1F_1(a,b;x)$ is a degenerate hypergeometric function. ■

Returning in (3.14) to the "old" arguments $t=\epsilon\tau$ and $x=\sqrt{\epsilon}\xi$, we have the local (at boundary $x=0$) FAS of the problem (1.3):

$$V = \sum_{k=0}^{+\infty} t^{\frac{k+\alpha}{2}}\, v_k(z), \quad z = \frac{x}{2\sqrt{t}}. \qquad (3.17)$$

where

$$v_k(z) = 2^{\alpha+k} g_k\, z^{\nu} e^{-z^2} \frac{\Gamma\left[\frac{k+2+\alpha+\nu}{2}\right]}{\Gamma(\nu+1)}\, {}_1F_1\left[\frac{k+2+\alpha+\nu}{2},\ \nu+1; z^2\right]. \qquad (3.18)$$

3.3. Asymptotic expansions conformation.

Later we'll work not with formal asymptotic series (3.2) and (3.17), but with the finite sums as follows:

$$A_{n,x}U = \sum_{k=0}^{n} \frac{t^k}{k!} L_*^k f, \tag{3.19}$$

$$A_{m,z}V = \sum_{k=0}^{m} t^{\frac{k+\alpha}{2}} v_k(z)\Big|_{z=\frac{x}{2\sqrt{t}}}, \tag{3.20}$$

where $n=0,1,2,\ldots$ and $m=0,1,2,\ldots$.

We assume, that

$$L_*^k f\Big|_{x=x_j} = \frac{1}{2}\left\{L_*^k f\Big|_{x=x_j+0} + L_*^k f\Big|_{x=x_j-0}\right\}, \tag{3.21}$$

and, so, we define the function $A_{n,x}U$ in a point of discontinuity $x=x_j$ of the original $f=f(x)$.

Lemma 8. For any $n \ge 0$ the function $A_{n,x}U$ has following properties:

1) $A_{n,x}U \in C^{(\infty)}_{t,x}$ $([0,+\infty[,]0,x_j])$ (in a point $x=x_j$ by the left continuity);

2) $A_{n,x}U \in C^{(\infty)}_{t,x}$ $([0,+\infty[,[x_j+\infty[)$ (in a point $x=x_j$ by the right continuity);

3) $\forall\, x>0$ $A_{n,x}U \in C^{(\infty)}_{t}$ $([0,+\infty[)$;

4) $\forall\, x>0$ $\lim\limits_{t\to+0} A_{n,x}U = f(x)$.■

Lemma 9. For any $m \ge 0$, when $\alpha+2+\nu>0$ and $\nu>-1$ the function

$$\Phi_m(x,t) = A_{m,z}V = \sum_{k=0}^{m} t^{\frac{k+\alpha}{2}} v_k(z)\Big|_{z=\frac{x}{2\sqrt{t}}}$$

has the following properties:

1) $\Phi_m(x,t) \in C^{(\infty)}_{t,x}$ $(t>0,x>0)$;

2) in the region $(x>0,t>0)$, the function Φ_m satisfies the equation

$$L\Phi_m = x\frac{\partial}{\partial t}\Phi_m\ ;$$

3) for any $x>0$

$$\lim_{t\to+0} \Phi_m = \sum_{k=0}^{m} g_k x^{k+\alpha};$$

4) when $\nu\geq 0$ and for any $t>0$, $\Phi_m \in C_x^{(\infty)}([0,+\infty[)$ in particular, when $\nu\geq 0$ $|\Phi_m|_{x\to+0}| \leq M < +\infty$ for any $t>0$.■

To coordinate the PAS (3.19) and (3.20) the following functions $A_{m,z}A_{n,x}U$ and $A_{n,x}A_{m,z}V$ are implemented. The function $A_{m,z}A_{n,x}U$ is defined as follows: in (3.19) $L_\nu^k f$ are expanded into series (ordinary or asymptotic) by powers $x\to+0$, then x is expressed by means of $z = \frac{x}{2\sqrt{t}}$ and in the obtained formula we keep only $n+1$ the first (main) item relative to the powers $t\to+0$.

Thus we have

$$A_{m,z} A_{n,x} U = \sum_{j=0}^{m} g_j\, t^{\frac{j+\alpha}{2}} \sum_{k=0}^{n} \frac{\omega_{j,k}}{k!\,(2z)^{2k-j-\alpha}}\Bigg|_{z=\frac{x}{2\sqrt{t}}}, \qquad (3.22)$$

where $\omega_{j,k}$ is defined by (3.10), and g_j by (3.1).

The function $A_{n,x}A_{m,z}V$ can be constructed similarly: in (3.20) the functions $U_k(z)$ are expanded into series by powers $z\to+\infty$, then z is expressed by means $x = 2\sqrt{t}\, z$ and in the obtained expression we keep only the first $n+1$ (main) item relative to the powers $t\to+0$.

Thus we have

$$A_{n,x}A_{m,z}V = \sum_{k=0}^{n} \frac{t^k}{k!\,x^{2k}} \sum_{j=0}^{m} g_j\, \omega_{j,k}\, x^{j+\alpha}. \qquad (3.23)$$

Next we have an important theorem about the regular ($A_{n,x}U$) and local ($A_{m,z}V$) PASs conformation.

Theorem 2. For any $n\geq 0$ and any $m\geq 0$ in the region ($x>0$, $t>0$)

$$A_{n,x}A_{m,z}V = A_{m,z}A_{n,x}U.■$$

$A_{n,x}U$ and $A_{m,z}V$ form the composite asymptotic expansion:

$$S_n(x,t) = A_{n,x}U + (A_{2n+1,z}V - A_{2n+1,z}A_{n,x}U)\,\Phi_0(x), \qquad (3.24)$$

where the cutting function $\Phi_0 \in C^{(\infty)}([0,+\infty[)$ is defined as follows:

$$\Psi_0(x) = \begin{cases} 0, & x \ge 2, \\ 1, & 0 \le x \le 1, \end{cases} \qquad (3.25)$$

and in the interval $1 \le x \le 2$ $\Psi_0(x)$ monotonically decreases from 1 to 0. So, when $x \ge 2$ $S_n(x,t) = A_{n,x}U$. The choice of the function Ψ_0 is not unique. However the cutting function Ψ_0 doesn't influence the "quantitative side" of the approximation, and is implemented only to obtain a smoother degree of the PAS.

Lemma 10. For any $n \ge 0$ the function $A_{2n+1,x}A_{n,x}U$ has the following properties:

1) $A_{2n+1,x}A_{n,x}U \in C^{(\infty)}_{t,x}$ $(t>0, x>0)$;

2) for any $x>0$ $A_{2n+1,x}A_{n,x}U \in C^{(\infty)}_{t}$ $([0,+\infty[)$;

3) for any $x>0$

$$\lim_{t\to+0} A_{2n+1,x}A_{n,x}U = \sum_{j=0}^{2n+1} B_j\, x^{j+\alpha},$$

where B_j are defined by (3.1). ∎

Corollary. According to Lemma 9 we have, that for any $n \ge 0$ and any $x>0$

$$\lim_{t\to+0}(A_{2n+1,x}V - A_{2n+1,x}A_{n,x}U) = 0. \qquad (3.26)$$

From (3.20), (3.18) and (3.22) we have

Lemma 11. For any $x \ge \Delta > 0$ and any $n \ge 0$ the asymptotic expansion

$$A_{2n+1,x}V - A_{2n+1,x}A_{n,x}U \sim \sum_{j=0}^{2n+1} x^{\alpha+j} B_j \sum_{k=n+1}^{+\infty} \frac{\omega_{j,k}}{k!\,x^{2k}} t^k, \quad t\to+0, \qquad (3.27)$$

that allows termwise differentiation, is true. ∎

According to Lemma 6 and formulas (3.19) and (3.22) we have

Lemma 12. For any $n \ge 0$ and when $x \in]0,\Delta[$ we have a presentation

$$A_{n,x}U - A_{2n+1,x}A_{n,x}U = \sum_{k=0}^{n} \frac{t^k}{k!\,x^{2k}} \sum_{j=2n+2}^{+\infty} B_j\, \omega_{j,k}\, x^{j+\alpha}. \;∎ \qquad (3.28)$$

Corollary. If $\alpha \ge -2$, then the function

$$A_{n,x}U - A_{2n+1,x}A_{n,x}U \in C^{(\infty)}_{t,x} \; (t\ge0, 0\le x<\Delta).$$

in particular, only when $\alpha \geq -2$ we can guarantee that the function $A_{n,x}U - A_{2n+1,x}A_{n,x}U$ is bounded when $x \to +0$ and $t>0$.

3.4. Local asymptotic solution at the point of discontinuity $x=x_j$.

The regular asymptotic $A_{n,x}U$ in the point of discontinuity $x=x_j$ of the original $f(x)$ also has a discontinuity of the first kind. Hence, we have to add to $A_{n,x}U$ a correction function $P_n=P_n(x,t;x_j)$, so, that the function $A_{n,x}U+P_n$ in the point $x = x_j$ would have the higher smoothness order.

Lemma 13. Let for any $n \geq 0$ the function $P_n=P_n(x,t;x_j)$ $(x_j>0)$ in the region $(x \geq 0, t>0)$ satisfy the equation

$$x \frac{\partial}{\partial t} P_n = LP_n + x_j\, \delta(x-x_j)\sum_{k=0}^{n} \frac{t^k}{k!} \overset{\bullet}{\Delta}_k + x_j\, \delta'(x-x_j)\sum_{k=0}^{n} \frac{t^k}{k!} \Delta_k,$$

where

$$\Delta_k = L_*^k f \Big|_{x=x_j+0} - L_*^k f \Big|_{x=x_j-0},$$

$$\overset{\bullet}{\Delta}_k = \frac{d}{dx} L_*^k f \Big|_{x=x_j+0} - \frac{d}{dx} L_*^k f \Big|_{x=x_j-0},$$

and $\delta(x)$ is Delta function of Dirac. Then in the region $(x>0, t>0)$

$$L(A_{n,x}+P_n) - x\frac{\partial}{\partial t}(A_{n,x}U+P_n) = x\frac{t^n}{n!} L_*^{n+1} f. \quad \blacksquare$$

So, the correction function P_n can be found as the solution of the following problem:

$$x \frac{\partial}{\partial t} P_n = LP_n + x_j\, \delta(x-x_j)\sum_{k=0}^{n} \frac{t^k}{k!} \overset{\bullet}{\Delta}_k + x_j\, \delta'(x-x_j)\sum_{k=0}^{n} \frac{t^k}{k!} \Delta_k,$$

$$(x>0,\ t>0), \quad |P_n|_{x=+0}| \leq M, \qquad P_n|_{t=+0}=0 \tag{3.29}$$

Lemma 14. For any $n \geq 0$ and when $\nu \geq 0$ the problem's (3.29) solution can be presented as follows:

$$P_n = P_n(x,t;x_j) = \sum_{k=0}^{n} \overset{\bullet}{\Delta}_k\, t^k\, \Omega_k(x_j,p,s) + \sum_{k=0}^{n} \Delta_k\, t^k\, \Omega_k^{\bullet}(x,x_j,p,s), \tag{3.30}$$

where $p = \frac{xx_j}{2t}$, $s = \frac{1}{2}\left(\frac{x}{x_j} + \frac{x_j}{x}\right)$, (3.31)

$$\Omega_k = \Omega_k(x_j,p,s) = \frac{x_j}{2\,\Gamma(k+1)}\int_p^{+\infty}\frac{(u-p)^k}{u^{k+1}}\,e^{-su}\,I_\nu(u)du, \qquad (3.32)$$

$$\Omega_k^* = \Omega_k^*(x,x_j,p,s) = \frac{1}{2x\,\Gamma(k+1)}\int_p^{+\infty}\frac{(u-p)^k}{u^k}\,e^{-su}\left[x_j\,I_\nu(u) - \right.$$

$$\left. - x\,I_\nu'(u)\right]du \qquad (3.33)$$

and $I_\nu(z)$ is modified Bessel function.▪

Remark. Due to the estimates

$$I_\nu(u) = \frac{e^u}{\sqrt{2\pi u}}\left[1 + O\left(\tfrac{1}{u}\right)\right], \quad u \to +\infty;$$

$$x_jI_\nu(u) - xI_\nu'(u) = \frac{e^u}{\sqrt{2\pi u}}\left[x_j - x + O\left(\tfrac{1}{u}\right)\right], \quad u \to +\infty$$

we have the convergence of the integrals in the formulae (3.32) and (3.33) for any $k \geq 0$, any $p>0$ and any $s \geq 1$.

The next theorem describes the behaviour of the functions Ω_k and Ω_k^* when $p \to +\infty$.

Theorem 3. For any $k \geq 0$, for any $\nu>-1$ and $s = \frac{1}{2}\left[\frac{x}{x_j} + \frac{x_j}{x}\right] \geq 1$ ($x>0$, $x_j>0$) when $p = \frac{xx_j}{2t} \to +\infty$ have a place the following asymptotic expansions:

$$\Omega_k = \frac{1}{2}\left(\frac{x_jt}{\pi x}\right)^{\frac{1}{2}} e^{-z_j^2}\left\{\sum_{m=0}^{M-1}\frac{(-1)^m(\nu,m)}{2^m p^m}\,\Phi(k+1,\tfrac{1}{2}-m;z_j^2) + O\left(\frac{1}{p^M}\right)\right\}, \quad (3.34)$$

$$\Omega_k^* = \frac{1}{4}\left(\frac{x_j}{\pi xt}\right)^{\frac{1}{2}} e^{-z_j^2}\left\{-2\,\mathrm{sgn}(x-x_j)\sqrt{t}\,\Phi(k+\tfrac{1}{2},\tfrac{1}{2};z_j^2) + \right.$$

$$\left. + \sum_{m=1}^{M-1}\frac{(-1)^m[(\nu,m)x_j-(\nu,m)^*x]}{2^m p^m}\,\Phi(k+1,\tfrac{3}{2}-m;z_j^2) + O\left(\frac{x+x_j}{p^M}\right)\right\} \qquad (3.35)$$

where $M \in \mathbb{N}$, $z_j = \frac{(x-x_j)}{2\sqrt{t}}$.

$$(\nu,m) = \frac{\Gamma(\frac{1}{2}+\nu+m)}{m!\Gamma(\frac{1}{2}+\nu-m)} \tag{3.36}$$

is Hankel's symbol, and

$$(\nu,m)^* = \frac{1}{2}\,[(\nu-1,m) + (\nu+1,m)] \tag{3.37}$$

and $\Phi(a,b;x)$ is a degenerate hypergeometric function.■

Remarks. 1) The estimates are uniform in respect of $\nu\in]-1,N]$, where $-1<N<+\infty$ is fixed.

2) If we replace (3.34) and (3.35) into (3.30), then when $x,\ x_j \to +\infty$ for the correction function P_n (as expected) we have the expression which coincides with the corresponding presentation of the correction function for the integral Fourier transform [2].

From Theorem 3 we have that when $\frac{t}{xx_j} \ll 1$ (if $\nu>1$, then when $\frac{\nu^2 t}{xx_j} \ll 1$), functions Ω_k and Ω_k^* can be calculated using the asymptotic formulae (3.34) and (3.35). For example, if $t\to+0$, then the asymptotic formulae (3.34) and (3.35) fit for the calculations in case $x,x_j \geq \Delta \sim 1$ ($\Delta \sim \nu^2$, if $\nu>1$ is sufficiently large). Besides, since $s_j^2 = \frac{(x-x_j)^2}{4t}$, when $t\to+0$, the asymptotic presentations of the functions Ω_k and Ω_k^* are "essentially different from zero" only in a small neighbourhood of the point of discontinuity $x=x_j$ (for example, it will be sufficient to regard the neighbourhood $|x-x_j|\leq t^{1/4}$). Out of this region the asymptotic expansions (3.34) and (3.35) can be replaced by "asymptotic zero". However, there is a question. It is possible when $t\to+0$ ($x_j \geq \Delta > 0$ is fixed), what corresponds to the condition $p = \frac{xx_j}{2t} \to +0$, that the functions Ω_k and Ω_k^* impact can't be neglected, i.e., the use of the asymptotic formulae (3.34) and (3.35) is not sufficient for the calculation of Ω_k and Ω_k^* for all $x>0$, when $x_j \geq \Delta > 0$ and $t\to+0$.

The following estimates have place in this direction

Lemma 15. Let $p = \frac{xx_j}{2t}$, $s = \frac{1}{2}\left[\frac{x}{x_j} + \frac{x_j}{x}\right]$, then

1) when $\nu \geq 0$, $x \neq x_j$

$$\Omega_k = O\left[\frac{x_j t}{(x-x_j)^2}\, e^{-\frac{(x-x_j)^2}{4t}}\right],$$

is uniform concerning to $x>0$, $x_j>0$, $t>0$ and $k=0,1,2,\ldots$;

2) when $\nu>-1$

$$\Omega_k = O\left[x_j\, e^{-\frac{(x-x_j)^2}{4t}}\right],$$

is uniform concerning to $p = \frac{xx_j}{2t} \geq \delta > 0$ and $k=0,1,2,\ldots$;

3) when $-1<\nu<0$ and $x \neq x_j$

$$\Omega_k = O\left[\frac{(xx_j)^\nu x_j t^{1-\nu}}{(x-x_j)^2}\, e^{-\frac{(x-x_j)^2}{4t}}\right],$$

is uniform concerning to $p = \frac{xx_j}{2t} \to +0$ and $k=0,1,2,\ldots$;

4) when $\nu>0$

$$\Omega_k = O\left[x_j\, e^{-\frac{(x-x_j)^2}{4t}}\right]$$

is uniform concerning to $x>0$, $x_j>0$, $t>0$ and $k=0,1,2,\ldots$. ■

Lemma 16. Let $p = \frac{xx_j}{2t}$, $s = \frac{1}{2}\left(\frac{x}{x_j} + \frac{x_j}{x}\right)$, then

1) when $\nu\geq 1$, $\nu=0$ and $x \neq x_j$

$$\Omega_k^* = O\left[\frac{x_j(x_j+x)}{(x-x_j)^2}\, e^{-\frac{(x-x_j)^2}{4t}}\right],$$

is uniform concerning to $x>0$, $x_j>0$, $t>0$ and $k=0,1,2,\ldots$;

2) when $\nu>-1$ and $x \neq x_j$

$$\Omega_k^* = O\left[\frac{x_j(x_j+x)}{(x-x_j)^2}\, e^{-\frac{(x-x_j)^2}{4t}}\right],$$

is uniform concerning to $p = \frac{xx_j}{2t} \geq \delta > 0$ and $k=0,1,2,\ldots$;

3) when $0<\nu<1$ and $x \neq x_j$

$$\Omega_k^* = \frac{1}{(x-x_j)^2} e^{-\frac{(x-x_j)^2}{4t}} \left\{O(x_j^2) + O(tp^{\nu})\right\},$$

is uniform concerning to $p = \frac{xx_j}{2t} \to +0$ and $k=0,1,2,\ldots$;

4) when $-1<\nu<0$ and $x \neq x_j$

$$\Omega_k^* = \frac{p^{\nu}}{(x-x_j)^2} e^{-\frac{(x-x_j)^2}{4t}} \left\{O(x_j^2) + O(t)\right\},$$

is uniform concerning to $p = \frac{xx_j}{2t} \to +0$ and $k=0,1,2,\ldots$.•

According to the estimates obtained in Lemmas 15 and 16 we have: if $\nu > -1$ and $x_j \geq \Delta \sim 1$ ($\Delta \sim \nu$, when $\nu > 1$), then when t $\left[\frac{(1+\nu^2)t}{xx_j} \ll 1\right]$ are small, the functions Ω_k and Ω_k^* can be calculated by asymptotic formulae (3.34) and (3.35) and the impact of Ω_k and Ω_k^* is essential only in a small neighbourhood of a point $x = x_j$ ($|x-x_j| \leq t^{1/4}$). Out of this region Ω_k and Ω_k^* are asymptotically small concerning to the main errors (see theorem 6).

If x_j is small and asymptotic formulae (3.34) and (3.35) "don't work", then the functions Ω_k and Ω_k^* can be calculated by the integral presentations (3.32) and (3.33). The principle difficulties will not appear here, because of the integrands monotony, and in (3.32) the integrands are positive when $\nu > -1$.

The only difficulty can emerge when calculating the integral (3.33) in the case, when $x \sim x_j$, i.e. when $s \sim 1$. However, we can easily show, that when $p = \frac{xx_j}{2t}$ and $s = \frac{1}{2}\left(\frac{x}{x_j} + \frac{x_j}{x}\right)$

$$\Omega_k^*(x,x_j,p,s) = -\mathrm{sgn}(x-x_j)\left(\frac{x_j}{4\pi x}\right)^{\frac{1}{2}} e^{-s_j^2}\, \Phi\left(k+\tfrac{1}{2}, \tfrac{1}{2}; s_j^2\right) +$$

$$+ \frac{1}{2x\,\Gamma(k+1)} \int_p^{+\infty} \frac{(u-p)^k}{u^k} e^{-su}\, W_{\nu}(x,x_j,u)\,du, \tag{3.38}$$

where

$$W_\nu(x,x_j,u) = x_j I_\nu(u) - x I'_\nu(u) - \frac{e^u}{\sqrt{2\pi u}}(x_j - x) = O\left(\frac{e^u}{u^{3/2}}\right), \quad (3.39)$$

$u \to +\infty$.

We can make the numerical calculation of the integral (3.38). when $x \sim x_j$, without principle difficulties.

The other presentations can be obtained for functions Ω_k and Ω_k^*. For example, in case (3.32) we can expand modified Bessel function I_ν into series and integrating termwise, we get

$$\Omega_k = \frac{x_j}{2} e^{-\xi} \sum_{m=0}^{+\infty} \frac{\Psi(k+1-2m-\nu, 1-2m-\nu; \xi)}{m!\Gamma(m+\nu+1)(2a)^{\nu+2m}}, \quad (3.40)$$

where $\xi = \frac{1}{4t}(x^2 + x_j^2)$. However, we can't say, that for the numerical calculation of the function Ω_k, the obtained series (3.40) have any advantage over the integral presentation (3.32).

The justification and the algorithm itself are subjects of the second part.

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<u>Н.Белов. Некоторые вопросы устойчивого вычисления интеграла Ганкеля для разрывных функций. I.</u>

Аннотация. Работа посвящена построению устойчивого алгоритма обращения интегрального преобразования Ганкеля Методика базируется на асимптотическом решении вспомогательной задачи математической физики. В первой части работы получено формальное асимптотическое решение. Обоснование и сам алгоритм обращения будут даны во второй части. УДК 519.651.

<u>N.Belovs. Dažas stabilas aprēķināšanas metodes Hankeļa integrāļiem ar pārtrauktām funkcijām. I.</u>

Anotācija. Darbs veltīts stabilu aprēķinu metožu iegūšanai Hankeļa integrālās transformācijas inversijai. Šīs metodes tiek veidotas, izmantojot noteiktas matemātiskās fizikas palīgproblēmas asimptotiskos atrisinājumus. Darba pirmajā daļā tiek apskatīta tikai formāla risināšanas metode. Pamatojums un pats risināšanas algoritms tiks doti otrajā daļā.

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In the Council on Mathematics of the University of Latvia

On October 4, 1991, the Council of Ministers (i.e. the government) of Latvia promulgated the law "The Statute on Conferment of Scientific Degrees"; in accordance with this law there were established two scientific degrees in Latvia: Doctor (Dr.) and Doctor Habilitatus (Dr. Habil.). The second degree is the higher one. The Statute envisaged the creation of the s.c. promotional councils (for conferment of Doctor degrees) and habilitational councils (having the right to confer both Doctor and Doctor Habil. degrees).

In order to create such councils it was necessary to give preliminarily to some institution the right to confer a number of first Doctor Habil. and Doctor degrees. This right was given to the Latvian Council of Science (LCS).

The community of Latvian mathematicians after making the analysis of the situation and discussing it decided:

1) to create three promotional councils each one of which representing a sufficiently wide area of mathematics;

2) to create only one habilitational council by including in it practically all Latvian mathematicians, who had at that time the degree of Doctor Science in Physics & Mathematics; this council hould represent the whole spectrum of mathematical sciences.

Since, according to the Statute, the council should be founded at a certain institution, in the role of such an institution the faculty of physics and mathematics of the University of Latvia was chosen.

After discussing possible candidates for the habilitational council and voting for them personally, the Scientific Council of the University (SCU) has delivered the corresponding proposition to the LSC, where it was thoroughly studied by a special commission. As the result, on June 1992, the LCS has founded the habilitational Council on Mathematics consisting of 11 members (experts and at the same time the LSC has conferred upon 9 of these experts the degree of Dr. Habil. in Mathematics. They are: A. Buikis, M.-R. Freivalds, J. Klokov, A. Lepin, A. Lorencs, B. Plotkin, U. Raitums, J. Strazdins and J. Tsarkov. (The remaining two of the experts were already earlier conferred in other areas: J. Barzdins as Dr. Habil. in Computer Science and H. Kalis as Dr. Habil. in Physics.)

At the first meeting of the newly created Council A. Buikis was elected as the chairman, M.-R. Freivalds as the vice-chairman, and I. Pagodkina as the scientific secretary of the HCM. At the second meeting the council has conferred the degree of (promoted) Dr. Math. on 13 Latvian mathematicians (having at that time the degree of Candidate in Physics & Mathematics of the former USSR) as well as given them the right to be included as members into one of the 3 promotional councils. These promotional councils, according to the Statute, were rightfull till July, 1993, and their principal aim was to nostrify (i.e. to equate) the candidate degrees of the former USSR of the Latvian mathematicians as the new Dr. degrees. On the other hand, according to the statute, the principal aim of the HCM was to confer (or not to confer) the degrees of Dr. Habil. Math. and Dr. Math. upon a doctorant after the defence of his Thesis at a special session of the Council. At the first one of such sessions, which was held on November 19, 1992, A. Sostak has successfully defended his Dr. Habil. Math. Thesis "Foundations of the Theory of Fuzzy Topological Spaces". The reviewers of the Thesis were A. Arhangel'skii V. Malykhin (both from Moscow, Russia) and U. Raitums. At the next session held on January 7, 1993 T. Cirulis successfully defended Dr. Habil. Math. Thesis "Asymptotic Methods in Mathematical Physics and Numerical Analysis"; his reviewers were A. Nikiforov (Moscow, Russia), M. Sapagovas (Vilnius, Lithuania) and H. Kalis. At last, on April 29, 1993 the successfull defence of H. Kalis's Dr. Habil. Math. Thesis took place; the title of the Thesis is "The Development and Application of Special Numerical Methods for Solution of Problems for Mathematical Physics, Hydrodynamics and Magnetohydrodynamics"; the reviewers were M. Feistauer (Prague, Czechia), M. Sapagovas (Vilnius, Lithuania) and A. Buikis.

At the end of this article, I would like to express my personal sincere hope that our HCM will successfully continue its work in future, that the atmosphere at its sessions will always be critical, practical and friendly at the same time, and that all the Thesis defended at the sessions will be of the world's quality, i.e. high quality.

The chairman of the Habilitational Council on Mathematics at the University of Latvia, professor A. Buikis

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ASYMPTOTIC METHODS IN MATHEMATICAL PHYSICS AND NUMERICAL MATHEMATICS

Dr.Hab.Math. Thesis (Summary)

T.Cīrulis

Abstract. Here we consider the most important results which made the basis for T.Cīrulis Dr. Hab. Math. Thesis. In particular, we develop new methods allowing to obtain asymptotic expansion of functions and discuss different applications of asymptotic expansions of functions for development of new calculation methods and for precisation of the existing ones in mathematical physics and numerical analysis. Most effectively the asymptotic expansion methods can be applied to realize numerical inversions of classic integral transformations where, by using additional information about the singular points of images, we succeeded in creating calculating algorithms of increased precision. AMS SC 65M, 65D32.

1. Method of gradientlines for finding asymptotic expansions of integrals.

We are interested in finding of the asymptotic expansion for the integral

$$J(z)=\int_L \Phi(zh(t))f(t)dt,\ z\to\infty,\ \alpha\leq \arg z\leq\beta, \tag{1.1}$$

in case, when the integrand function is analytic in a region $\mathcal{D}$ containing the contour L and the kernel function $\Phi(u)$ satisfies exponential asymptotics for large $|u|$. For example, the kernel $\Phi(u)$ may be $\operatorname{erf} u$, $\operatorname{si} u$, $J_\nu(u)$, $Y_\nu(u)$, $K_\nu(u)$ e.a.

In our work [1] the asymptotic solution of the integral (1.1) is obtained by means of the s.c. gradientlines method; this method generalizes the classical methods of a saddle point and of a stationary phase. The essence of the method is that the contur L is deformed into vectorlines of $grad|h(t)|$ and the following five types of points can be critical ones:

1) the roots of the equation $h(t)=0$ (i.e. the knot points of the family of gradientlines);
2) the roots of the equation $h'(t)=0$ in case $h(t)\neq 0$ (i.e. the saddle points of the family of gradientlines);
3) the singular points of function $h(t)$;
4) the singular points of function $f(t)$;
5) the end points of the contour L.

In our monograph [1] we consider, how to calculate asymptotic influence from all types of critical points and some generalizations of integral (1.1) when the critical points are dependent on parameter z.

Comparing with the saddle point method, in the gradientlines method critical points of a new type appear, namely these are the roots of equation $h(t)=0$ which are the knot points of function's $h(t)$ gradientlines family. The possibilities of the method visually can be illustrated by the example, in which the complete asymptotic expansion of the hypergeometrical function ${}_1F_2\left[\begin{matrix}\alpha;\\ \beta,\gamma;\end{matrix} -\frac{x^2}{4}\right]$ in the complex plane is obtained from the integral presentation

$$ {}_1F_2\left[\begin{matrix}\alpha;\\ \beta,\gamma;\end{matrix} -\frac{x^2}{4}\right]=\frac{2\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\left(\frac{x}{2}\right)^{1-\beta}\int_0^1 t^{2\alpha-\beta}(1-t^2)^{\gamma-\alpha-1}J_{\beta-1}(xt)\,dt \quad (Re\,\gamma>Re\,\alpha>0) \qquad (1.2)$$

and it is the following one. If $x\to\infty, |arg\,x|<\pi$ then

$$ {}_1F_2\left[\begin{matrix}\alpha;\\ \beta,\gamma;\end{matrix} -\frac{x^2}{4}\right]\sim\frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha)}\sqrt{\frac{2}{\pi x}}\left(\frac{x}{2}\right)^{1+\alpha-\beta-\gamma}\times $$
$$ \times\left\{\cos\left[x+\frac{\pi}{2}\left(\alpha-\gamma-\beta+\frac{1}{2}\right)\right]\sum_{m=0}^{+\infty}(-1)^m B_{2m}x^{-2m}+\right. $$
$$ \left.+\sin\left[x+\frac{\pi}{2}\left(\alpha-\gamma-\beta+\frac{1}{2}\right)\right]\sum_{m=0}^{+\infty}(-1)^{m+1}B_{2m+1}x^{-2m-1}\right\}+ \qquad (1.3)$$
$$ +\frac{1}{\pi^2}\Gamma(\beta)\Gamma(\gamma)\sin\pi(\beta-\alpha)\sin\pi(\gamma-\alpha)\left(\frac{x}{2}\right)^{-2\alpha}\times $$

$$\times\sum_{m=0}^{+\infty}\frac{(-1)^m}{m!}(\alpha)_m\,\Gamma(\alpha-\beta+m+1)\,\Gamma(\alpha-\gamma+m+1)\left(\frac{x^2}{4}\right)^{-m}, \qquad (1.3)$$

where

$$B_k=\frac{1}{2^k k!}\sum_{l=0}^{k}\binom{k}{l}(\gamma-\alpha)_l\left(\tfrac{3}{2}-\beta\right)_{k-l}\left(\beta-\tfrac{1}{2}\right)_{k-l}\times$$

$$\times\sum_{j=0}^{l}\binom{l}{j}\left(\beta-2\alpha+k-l+\tfrac{1}{2}\right)_j(\alpha+1-\gamma)_{l-j}\,2^j, \qquad (1.4)$$

or

$$B_k=\frac{(-1)^k}{2^k}\sum_{l=0}^{k}(\gamma-\alpha)_l\left(\beta-\tfrac{1}{2}\right)_{k-l}\binom{\beta-\frac{3}{2}}{k-l}\sum_{j=0}^{l}\binom{2\alpha-\beta-k+l-\frac{1}{2}}{j}\binom{\gamma-\alpha-1}{l-j}2^j. \qquad (1.5)$$

From formulae (1.4) and (1.5) is follows that

$$B_0=1,\ B_1=-\frac{1}{2}(\beta-\gamma)^2+(1-\alpha)(\beta+\gamma)+\frac{3\alpha^2}{2}-\alpha-\frac{3}{8}, \qquad (1.6)$$

$$B_2=\frac{9}{8}\alpha^4-\frac{7}{2}\alpha^3+\frac{43}{16}\alpha^2-\frac{\alpha}{8}+\frac{15}{128}+\frac{\gamma+\beta}{4}\left(-6\alpha^3+18\alpha^2-\frac{25}{2}\alpha+\frac{1}{2}\right)+$$
$$+\frac{(\gamma+\beta)^2}{4}(\alpha^2-3\alpha+2)+\frac{(\gamma-\beta)^2}{4}\left(-3\alpha^2+4\alpha-\frac{9}{4}\right)+\frac{1}{2}(\gamma+\beta)(\gamma-\beta)^2(\alpha-1)+$$
$$+\frac{1}{32}(\gamma-\beta)^4. \qquad (1.7)$$

As Y.L.Luke noticed in his book [10] there are not known the coefficients expressions B_k of asymptotic formulae for hypergeometric functions ${}_pF_{p+1}$; only a recursion formula for these coefficients is known. For the function ${}_1F_2$ this formula is the following one

$$2(k+1)B_{k+1}=\left[3k^2+2k(1+\gamma+\beta-3\alpha)+2B_1\right]B_k-$$
$$-\left(k+\gamma+\beta-\alpha-\tfrac{3}{2}\right)\left[\left(k+\gamma+\beta-\alpha+\tfrac{1}{2}\right)\left(k-\gamma-\beta-\alpha+\tfrac{1}{2}\right)+4\gamma\beta\right]B_{k-1} \qquad (1.8)$$
$$k=1,2,3,\ldots$$

where B_0 and B_1 are determined by the formulae (1.6). One can verify that calculating B_2 by means of the recurrent formula (1.8) it will coincide with (1.7).

The asymptotic formula (1.3) is suitable also in the case when $\alpha=\gamma$ ($\alpha=\beta$), (in this situation the formula transforms into the asymptotic formula for Bessel functions) as well as in case $\alpha=-n$ (in this situation polynomials are in the both sides of the formula (1.3)).

2. A general scheme for applications of asymptotic expansions to create and precise calculating methods.

Asymptotic expansion of a function is usually done and used in cases when it is necessary to find local approximations of the function in the neighbourhood of a definite parameter or a fixed point. In the series of our works [2-9] we discuss some new methods allowing to use the asymptotic expansions; in these methods the asymptotic e pansions are used as additional information concerning given functions in order to precise global calculation methods in mathematical physics and numerical analysis.

Definition 1. Assume that in the neighbourhood of a point x_0 the following asymptotic expansion of a function f is given

$$f(x)\sim\sum_{n=0}^{+\infty}a_n g_n(x)\,,\ x\to x_0\,, \tag{2.1}$$

where $\{g_n(x)\}$ is an asymptotic scale. Then the number sequence (a_n) will be called the information of a function f in the neighbourhood of a point x_0 in respect of the scale $\{g_n(x)\}$. The first m members $(a_0, a_1, \ldots a_m)$ of this sequence is called the partial information of the function f.

There are possible also cases when information about function in the neighbourhood of point x_0 is given by several sequences; for instance if the asymptotic expansion (2,1) has different expressions (e.g. when $x\to x_0+0$ and $x\to x_0-0$) or depending on complex values x and x_0 of $\arg x$.

In the simpliest case when the function f is sufficiently many times continuously differentiable in the neighbourhood of the point x_0 the information about f is determined by the sequence of Taylor coefficients.

The simpliest partial information about a fixed function f in a neighbourhood of point x_0 is $a_0 = f(x_0)$. This information is also the most frequently used in mathematical physics and numerical analysis. To distinguish this simpliest information which reduces to determination of the function's value in a given point, the other information, i.e. the information which is determined by other members of the sequence $(a_1, a_2, \dots)$ will be called the additional information.

The general problem setting about the use of the additional information is the following one:

Given a problem of mathematical physics or numerical mathematics

$$Au = f(x), \; x \in \mathcal{D}, \tag{2.2}$$

where A is a linear or non-linear operator, f is a given function (vector function) and u is the function (vector function) to be found. The equation (2.2) can be supplied with one or several additional conditions of a certain type, for example the boundary condition

$$Gu\big|_{\partial\mathcal{D}} = \varphi(s), \; s \in \partial\mathcal{D}, \tag{2.3}$$

where G is a definite oparator and φ is a given function.

To solve the problem (2.2)+(2.3) by means of one of the existing approximate methods suitable for the use of electronic calculators, the problem is to be discretized, since electronic computers can treat only finite information. Depending on the chosen method this information about operators A and G and the given functions f and φ can differ. Most often in classical methods the simpliest information is used: namely, the values of functions f and φ in the given points $x_k \in \mathcal{D}$ and $s_m \in \partial\mathcal{D}$. However, often a much larger information about functions f and φ is known than the one which is directly needed in the calculating algorithms.

For example, the given functions can be analytic with singular points of a certain type. It is also possible that in some points the functions or their certain order derivatives have known gaps. In some cases it is known, that the contour

$\partial\mathcal{D}$ of the region $\mathcal{D}$ has certain non-smoothness and other peculiarities which are not explicitly used in the calculating algorithms. All this forms the additional information about the given functions and one can investigate the problem how this additional information could be used for the solution of the problem. In this connection one usually has to consider severaler tasks, the most important of which are the following ones.

1° In which points and what additional information is to be taken into account when solving the problem (2.2)+(2.3) by means of the chosen method.

2° In what way this additional information is to be taken into account when developing a more precise calculation method.

3° To what extent the use of the additional information increases the precision to compare with the use of the method without this additional information. Usually the tasks 1° and 2° are being considered simultaneously by the following scheme: having discretized the problem, the chosen calculation algorithm is being considered; aproximal solution obtained with help of this algorithm is dependent on one or several parameters (the step h in the finite differences method the natural number N showing the number of knot points in the region $\mathcal{D}$, the coordinate function or the number of finite elements, e.a.).

Further, one forms the deviation, or the error between the precise and the proximate solution and looks for the asymptotic formula of this deviation when the parameters, e.g. N approaches to $+\infty$. To determine such asymptotic formulae one has usually to use different kind of additional information about given functions and operators and this in fact gives the answer to the question set in task 1°.

By adding the found asymptotic expression to the proximal solution, we obtain the solution of the task 2°. To obtain the answer on the question set in task 3° one has to estimate and to compare the corresponding errors of the both solutions.

Observe, that the above mentioned scheme is the simpliest but not the only one, which can be used in practice. However, even to realize it, one often has to solve difficult asymptotic problems which essentially restrict the applicability of the method. In section 4 we shall consider another scheme of utilization of the asymptotic expansion.

3. The revision of the integral quadratu e formula by means of asymptotic expansions.

The use of the additional information about the integrand function enables to determine the asymptotic formulae for the remainder R_N $(N\to+\infty)$ in the important quadrature

$$\int_a^b \rho(x) f(x)\,dx = \sum_{k=0}^{N-1} p_k f(x_k) + R_N . \tag{3.1}$$

In our work [3] with the help of methods of complex variable functions theory the asymptotic formulae of the remainder are found in case when the knot points x_k are uniformly situated in the interval $[a,b]$

They are different versions of triangle, trapezium and parabolic formulae, besides in the terminal points $x=a$ and $x=b$ the function f may have easily interpreted singularities. The essence of this method is that certain contour integral in the complex plane is being considered, for which the corresponding residues sum is equal to the quadratures sum, and the remainder R_N is expressed by means of an integral. The clasical methods of asymptotic expansions are being applied to this integral. As an example, we shall consider the following example from [3].

If

$$f(x)=(x-a)^{\alpha-1}(b-x)^{\beta-1}\varphi(x),\ \alpha>0,\ \beta>0,\ \varphi\in C^{m+2}[a,b] \tag{3.2}$$

then

$$\int_a^b f(x)\,dx = \frac{b-a}{N}\sum_{n=0}^{N-1} f(x_n(\delta)) + R_N(\delta,m) + O(T_N^{-m-1-\nu}) \tag{3.3}$$

where $x_n(\delta)=a+\frac{1}{N}(n+\delta)(b-a)$, $0<\delta<1$, $T_N=\pi N/(b-a)$, $\nu=\min\{\alpha,\beta,1\}$,

$$R_N(\delta,m)=\sum_{k=0}^{m}\frac{a_k(\delta)}{(2T_N)^{k+\alpha}}+\sum_{k=0}^{m}\frac{b_k(\delta)}{(2T_N)^{k+\beta}} , \tag{3.4}$$

$$a_k(\delta) = -\frac{2\Gamma(\alpha+k)}{k!}\frac{d^k}{da^k}\left[(b-a)^{\beta-1}\varphi(a)\right]\sum_{\ell=1}^{+\infty}\ell^{-k-\alpha}\cos\frac{\pi}{2}(\alpha+k-4\delta\ell),$$

$$b_k(\delta) = -\frac{2\Gamma(\beta+k)}{k!}\frac{d^k}{db^k}\left[(b-a)^{\alpha-1}\varphi(b)\right]\sum_{\ell=1}^{+\infty}\ell^{-k-\beta}\cos\frac{\pi}{2}(\beta-k+4\delta\ell). \quad (3.5)$$

From formulae (3.3)-(3.5) it is dear that to determine the asymptotic expression of the remainder R_N one needs some additional information about $f(x)$ (or $\varphi(x)$) in the terminal points $x=a$ and $x=b$ of the integration interval.

To revise the simpliest Gauss - type quadrature formula one needs quite different additional information about the integrand function: the information about the integrand function in the terminal points is not needed, but on the other hand it is necessary to know the gaps (whenever they exist) of the function or of its definite-order's derivatives in the integration interval, or also the information about the singular points of this function's analytic extension to the complex plane. All this additional information is being included in the asymptotic formula of the remainder R_N by means of Fourier coefficients. We shall mention here the following result from [4] .

If $f\in C[-1,1]$ and $f'(x)$ exists almost everywhere with bounded variation, then for the remainder R_N in the Gauss quadrature formulae

$$\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}dx = \frac{\pi}{N}\sum_{k=1}^{N}f(x_k)+R_N(f), \quad x_k=\cos\frac{\pi(2k-1)}{2N}, \quad (3.6)$$

the following formula is valid

$$R_N(f) = \pi\sum_{k=1}^{+\infty}(-1)^{k-1}C_{2Nk}(f), \quad (3.7)$$

where

$$C_n(f) = \frac{2}{\pi\varepsilon_n}\int_{-1}^{1}\frac{f(x)T_n(x)dx}{\sqrt{1-x^2}}, \quad \varepsilon_n=\begin{cases}2, & n=0,\\ 1, & n\neq 0.\end{cases} \quad (3.8)$$

In the formula (3.7) the series absolutely converges. If f is analytic in interval $[-1,1]$ the series (3.7) is also asymptotic.

Since the Fourier coefficient asymptotics for different function classes is sufficiently well studied in the literature, therefore from (3.7) one can easily get also the asymptotic formulae of the remainder $R_N(f)$.

Let us consider one more example illustrating how the asymptotic of the remainder of the interpolation formula can be found applying the asymptotic of the remainder of the quadrature formula.

Assume that

$$f(x)=T_N(x)\sum_{k=1}^{N}\frac{f(x_k)}{(x-x_k)T_N'(x_k)}+r_N[f(x)],\quad x_k=\cos\frac{\pi(2k-1)}{2N}, \tag{3.9}$$

$$f(x)=\sum_{k=0}^{N-1}c_kT_k(x)+\rho_N[f(x)], \tag{3.10}$$

where c_k are determined by means of formula (3.8). One can easily verify the following result: if in formula (3.10) are replaced by the integral (3.8) and the Gauss type quadrature formula (3.6) is applied to the integral, then the interpolation formula is obtained. Thus the remainder $r_N[f(x)]$ in the interpolation formula (3.9) can be expressed in such a form:

$$r_N[f(x)]=\rho_N[f(x)]+\sum_{k=0}^{N-1}R_N(fT_k)T_k(x) \tag{3.11}$$

where

$$\rho_N[f(x)]=\sum_{k=0}^{+\infty}c_{N+k}T_{N+k}(x), \tag{3.12}$$

$$R_N(fT_k)=\frac{\pi}{2}\sum_{m=1}^{+\infty}(-1)^{m-1}(c_{2Nm+k}+c_{2Nm-k}). \tag{3.13}$$

From formulae (3.11) -(3.13) one can observe how the remainder $r_N[f(x)]$ in the asymptotic formula can be written knowing the Fourier coefficients $c_n(f)\,(n\to\infty)$ of the asymptotic formula.

We shall mention also that with the help of formula (3.9) one can find approximation also for the derivative $f^{(k)}(x)$ and in a similar way make up of their remainders asymptotic formulae which also depend on the $c_n(f)$ asymptotics for large n.

Observe that when the smoothness of function f increases, the coefficients $C_n(f)$ faster approach zero; therefore formula (3.9) has the non-saturation property. From this point of view the interpolation type formula (3.9) is more suitable for solution of different problems, then, say, the method of finite differences or the method of finite elements whose approximation formulae are saturated. (When the smoothness of the function increases the error does not diminish).

4. Applications of asymptotic expansions for approximate inversion of integraltransformation.

Many linear (as well as linearized non-linear) problems of mathematical physics can be solved by means of classical integraltransformation. Having found first of all the image of the solution to be discovered, one has further to accomplish inversion of integraltransformation. Since the inversion of integraltransformation is an incorrect problem, for its solution it seems reasonable to use asymptotic methods with the use of certain additional information. Having constructed the solution of the problem of mathematical physics following the abovementioned scheme, in the algorithm automatically appears also additional information about the given in the original problem functions and region.

In our monograph [9] we have made a detailed study of different asymptotic methods, which can be used for approximate inversion of Laplace and Fourier integraltransformations. These methods without essential changes can be carried over the Mellin transformations and, to some extent, also over other classical integraltransformations. Taking into account the large size of [9] we shall mention here only the most important results.

1° An asymptotical interval extension method, is being worked out; this method is based on the following definition:

Definition 2. Assume that we have to calculate function $f(t)$ where $t \in [0, +\infty[$. Consider the following function

$$f(t,T)=\begin{cases} f(t), & t\in[0,T], \\ f_1(t), & t>T \end{cases} \tag{4.1}$$

where $f_1(t)$ is a rather freely chosen function satisfying some general conditions depending on the given problem. (For instance for Laplace transformation where $f(t)$ is the original to be found, $f_1(t)$ is to be chosen in such a way, that the corresponding Laplace integral exists). If $g(t,T)$ is the asymptotic expression of $f(t,T)$ when $T\to+\infty$, i.e.

$$f(t,T)\sim g(t,T),\ T\to+\infty, \tag{4.2}$$

then

$$f(t)\approx g(t,T),\ t\in[0,T]. \tag{4.3}$$

Definition 2 can be used also in the intervals $[a,b]$, $]-\infty,+\infty[$ as well as in the case of a function of many variable.

The approximate formula (4.3) has some properties, the most important of which are the following ones:

a) To determine the asymptotic formula (4.2) it is essentially to use the additional information about the given functions.

b) $g(t,T)$ is generally dependent on an arbitrary function (e.g. on $f_1(t)$ in formula (4.1)). This property is of a great importance in applications by changing the "free" function one can control the exactness of calculation directly, with the help of calculator without any theoretical consideration.

2° The asymptotic interval extension method is worked out for the inversion of Laplace transformation in case when the original or certain its derivatives in some points have gaps. In this case we start with considering the smoothed original found by means of the method mentioned in p. 1° The transition from the smoothed original to the discontinuous original to be discovered is being done using the additional information about character of gaps. Also for this method the theoretical estimation of the error is given.

3°. Some methods for discovering the original are united under the name Fourier - asymptotics methods. The essence of these methods is as follows. The original $f(t)$ is being found as a sum of a Fourier series with respect to a definite ortogonal system of functions, but to determine Fourier coefficients are being used both numerical and asymptotic methods. The main attention is paid to the two types of realization of the method:

a) $f(t)$ is expanded in Fourier series in the interval $[0,+\infty[$, for example, with respect to Jacobi polinomials $\delta>0$, $P_n^{(\alpha,\beta)}[1-2\exp(-\delta t)]$ and Fourier coefficients C_n are determined in two different ways: the first coefficients $C_0, C_1, \dots, C_N$ by means of simple numerical algorithms, but the coefficients $C_{N+1}, \dots$ with the help of asymptotics formulae for C_N when $N\to+\infty$

b) $f(t)$ is expanded in Fourier series in the interval $[0,T]$ and for the Fourier coefficients $C_n(T)$ the asymptotic formulae when $T\to+\infty$ for all $n=0,1,2,\dots$ are being found.

In all discussed methods of inversion for Laplace transformation regardless of its realization way the same additional information is needed, namely:

1) the information about those singular points of the image $F(p)$ for which $\operatorname{Re} p_k$ is the largest (several points can be with this property).

2) The information about $F(p)$ in the neighbourhood of the point $p=\infty$. This information can be replaced by additional information about the original $f(t)$ in the neighbourhood of the point $t=+\infty$.

As an example of the use of additional information developing a calculation method with the help of Laplace transformation we shall mention a problem considered in [9].

For the problem

$$\rho\frac{\partial u}{\partial t}=\operatorname{div}(\kappa \operatorname{grad} u)-qu, \quad u|_{t=0}=u_0(x), \qquad (4.4)$$

$$\left(\alpha(x)u(x)+\beta(x)\frac{\partial u}{\partial n}\right)_{\partial G}=0, \qquad (4.5)$$

where $x\in G\subset \mathbb{R}^n$, $\rho=\rho(x)$, $\kappa=\kappa(x)$, $q=q(x)$

as the additional information one has to use the first eigen-values $\lambda_1,\lambda_2,\dots,\lambda_m$ and eigen-functions of the corresponding spectral problem

$$div(\kappa\, grad\, v)-qv+\lambda\rho v=0,\ \left(\alpha v+\beta\frac{\partial v}{\partial n}\right)_{\partial G}=0. \qquad (4.6)$$

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Т.Цирулис. Асимптотические методы в математической физике и вычислительной математике (резюме хабилитационной работы по математике).

Аннотация. Рассмотрены основные результаты, которые составляют основу хабилитационной работы автора: новые методы получения асимптотических разложений функций и различные применения асимптотических разложений как дополнительная информация о функциях для создания новых или уточнения раннее известных методов в задачах математической физики или численного анализа. Особенно эффективны разработанные асимптотические методы обращения классических интегральных преобразований, где в качестве дополнительной информации используются сведения об особых точках изображения. УДК 517.44, 519.64.

T.Cīrulis. Asimptotiskās metodes matemātiskajā fizikā un skaitlošanas metemātikā (kopsavilkums habilitācijas darbam matemātikā).

Anotācija. Apskatīti svarīgākie rezultāti, kas ņemti par pamatu autora habilitācijas darbam: jaunas funkciju asimptotisko attīstījumu iegūšanas metodes un dažādi asimptotisko attīstījumu lietojumi kā papildinformācija par dotajām funkcijām, lai izveidotu matemātiskajā fizikā un skaitliskajā analīzē jaunas aprēķinu metodes vai precizētu jau esošās. Sevišķi efektīvi ir asimptotisko attīstījumā lietojumi klasisko integrālo transformāciju skaitliskās inversijas realizēšanai, kur, izmantojot papildinformāciju par attēla singulārajiem punktiem, izdodas izveidot paaugstinātas precizitātes aprēķinu algoritmus.

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FOUNDATIONS OF THE THEORY OF FUZZY TOPOLOGICAL SPACES

(Summary of Dr. hab. Math. Thesis)

A. Šostak

INTRODUCTION

The concept of a fuzzy set introduced in 1965 by L.Zadeh [Za] drew attention and evoked a steady interest both among "pure" mathematicians and among specialists using mathematical ideas and methods when dealing with problems of applied nature. Among the first to display a deep and motivated interest in fuzzy sets were general topologists. Already as early as in 1968 C.L. Chang [Ch] has made the first attempt to "engraft" the concept of a fuzzy set to the body of General Topology.

According to Chang, a fuzzy topological space is a pair $(X,\mathcal{T})$ where X is a set and $\mathcal{T}$ is a set of its fuzzy subsets satisfying the following axioms: (1Ch) $0,1 \in \mathcal{T}$; (2Ch) if $U,V \in \mathcal{T}$, then $U \wedge V \in \mathcal{T}$, and (3Ch) if $U_\gamma \in \mathcal{T}$ for all $\gamma \in \Gamma$, then also $\bigvee U_\gamma \in \mathcal{T}$. (Recall that a <u>fuzzy (sub)set</u> of a set X is just a mapping $M: X \to [0,1]$; the value $M(x)$ is interpreted as the degree to which the point x belongs to M. The family of all fuzzy sets of X is denoted I^X. Intersections and unions of fuzzy sets are defined, respectively, as their infimum ($\wedge$) and supremum ($\vee$). A usual set $A \subset X$ is identified with its characteristic function $A: X \to 2=\{0,1\}$. We do not distinguish between a constant fuzzy set $\alpha: X \to I$ and the corresponding value $\alpha \in I$. The complement of M is defined as $M^c = 1-M$.) A mapping $f: X \to Y$ where $(X,\mathcal{T}_X)$, $(Y,\mathcal{T}_Y)$ are fuzzy topological spaces is called continuous if $f^{-1}(V)$ $(:= V \circ f) \in \mathcal{T}_X$ for all $V \in \mathcal{T}_Y$. The category of Chang fuzzy topological spaces and continuous mappings between them will be denoted CFT.

Having noticed that in many cases some concrete properties of the unit interval I in the definition of a fuzzy set are unessential and, on the other hand, too restrictive, J.A.Goguen [Go_1] generalized this concept by introducing the notion of an L-fuzzy set as a mapping $M: X \to L$, where L is a complete bounded distributive lattice. By substituting in Chang's definition L-fuzzy sets for ordinary fuzzy sets, Goguen comes to the concept of an L-fuzzy topological space [Go_2]. L-fuzzy topological spaces and naturally defined continuous mappings of such spaces form a category CFT(L) called in the sequel the category of Chang L-fuzzy topological spaces. Obviously CFT(I)=CFT and CFT(2)= Top (the category of ordinary topological spaces).

At present there is a large number (at least 800) of works in which Chang (L-)Fuzzy spaces and some related objects are studied (see e.g. our survey [17]). However, the above exposed approach to the problem and the subject of Fuzzy Topology seems to be not sufficiently consistent because of the following its shortcomings. (See also criticism in [$Hö_1$],[Di],[MSY],[Ku].)

First, according to this approach one deals with a crisp structure τ of topological type on the family L^X of L-fuzzy sets of a given set X (i.e. $\tau \subset L^X$). But to be consistent, one should define an L-fuzzy topology on X as an L-fuzzy structure $\mathcal{T}$ of topological type on the family L^X (i.e. $\mathcal{T}: L^X \to L$).

Second, most authors when studying problems of fuzzy topology consider usally (L-)fuzzy topological spaces themselves. But, to our opinion, in fuzzy situation the study of (L-)fuzzy subsets in (L-)fuzzy topological spaces (in particular, in usual topological spaces) is of principal interest - this direction has no meaningful analogue in usual (crisp) topology.

Third, when studying properties of (L-)fuzzy topological spaces one usually takes up his stand on schemes based on classic (i.e. two-valued) logics (e.g. a given object either has or does not have a given topological property). However, we assume that in fuzzy situation often it is more natural to use schemes based on multivalued logics (e.g. a given object may have some topological property to a certain degree).

The principal aim of this Thesis is to develop a new consistent approach to the theory of L-fuzzy topological spaces which would be free of these defects and which would include in itself as special, in a known sense as crisp, cases the "standard" theories of L-fuzzy topological spaces (and hence also the model of classic topology).

CONTENTS OF THE THESIS

Chapter 0 (Preliminaries) contains definitions, constructions and results from the fuzzy set theory used in the main text. In particular, following [DP] , see also [Di] , we define here the fuzzy inclusion of fuzzy sets $M, N \in I^X$ by setting $M \tilde{\subset} N = \inf M^c(x) \vee N(x)$. If $M, N \in 2^X$, then, obviously, $M \tilde{\subset} N = 1$ iff $M \subset N$, otherwise $M \tilde{\subset} N = 0$. Many properties of relation $\tilde{\subset}$ are in some sense analogical to the corresponding properties of the classical inclusion. For example, $M \vee N \tilde{\subset} A \vee B \geqslant (M \tilde{\subset} A) \wedge (N \tilde{\subset} B)$ $(A, B, M, N \in I^X)$; for each mapping $f: X \to Y$ it holds $M \tilde{\subset} N \leqslant f(M) \tilde{\subset} f(N)$ e.a.[9]. Fuzzy cardinals which are needed to develop the fuzzy version of topological theory of cardinal invariants, are defined as non-increasing mappings $\varkappa: K \to I$ (K is the class of ordinary cardinals) such that $\varkappa(0)=1$, and $\varkappa(a)=0$ for some $a \in K$; a usual cardinal a is identified with the fuzzy cardinal $\tilde{a}$ such that $\tilde{a}(a)=1$ and $\tilde{a}(b)=0$ if $b>a$ An elementary arithmetics of fuzzy cardinals is worked out [16] (cf [Lu],[Wy]).

Chapter I is devoted to the general theory of fuzzy topological spaces. In Section 1.1 we introduce and discuss such fundamental concepts as a fuzzy topological space, a continuous mapping, fuzzy closure e.a.([22],[9],[17]; cf also [Hö$_1$],[Ku],[Di]).

A fuzzy topology on a set X is a mapping $\mathcal{T}: I^X \to I$ such that (1) $\mathcal{T}(0) = \mathcal{T}(1) = 1$; (2) $\mathcal{T}(U \wedge V) \geqslant \mathcal{T}(U) \wedge \mathcal{T}(V)$ for all $U, V \in I^X$, and (3) $\mathcal{T}(\vee U_\gamma) \geqslant \wedge \mathcal{T}(U_\gamma)$ for each family of fuzzy sets U_γ , $\gamma \in \Gamma$. The pair $(X, \mathcal{T})$ is called a fuzzy (topological) space. The inequality $\mathcal{T}(U) \geqslant a$ is interpreted as the statement "the openness degree of a fuzzy set U is not less than a", and the inequality $\mathcal{T}(U^c) \geqslant a$ as the statement "the closedness degree of U is not less than a". (In the Thesis we consider L-fuzzy topological spaces where L is a complete bounded distributive lattice, and a large part of the results are obtained in the context of L-fuzzy topological spaces. However, here we restrict ourselves with the case L=I because it is, to our opinion, the most important one and, on the other hand, this restriction allows us to formulate the main results more compactly and clearly: otherwise to formulate one or another result we often need to put special additional restrictions on the lattice L (such as complete distributivity, the existence of involution, separability, to be a chain e.a.)

A mapping $f: X \to Y$, where $(X,\mathcal{T}_X),(Y,\mathcal{T}_Y)$ are fuzzy topological spaces, is called continuous if $\mathcal{T}_X(f^{-1}(V)) \geq \mathcal{T}_Y(V)$ for each $V \in I^Y$. Fuzzy topological spaces and their continuous mappings form a category denoted FT.

An important class of fuzzy spaces is constituted by laminated fuzzy spaces: so we call spaces the fuzzy topology of which satisfies the following strenghtened version of axiom (1) (cf [Lo_1]):

(1^λ) $\mathcal{T}(c) = 1$ for each constant $c \in I$.

The complete subcategory of FT consisting of laminated spaces is denoted LFT. An important property of laminated fuzzy spaces is that all constant mappings between them are continuous and hence, in particular, the set of morphism between any two laminated spaces is non-empty (cf also [Lo_1],[Lo_2]).

In Section 1.2 lattice-type properties of families of fuzzy topologies are studied. In particular, we consider the constructions of supremum and infimum for families of fuzzy topologies, final and initial fuzzy topologies. Besides, with the help of this concepts we investigate operations of product, of direct sup, of a quotient space and of a subspace [26],[17].

In Section 1.3 some functors in the category FT are discussed. In particular, we consider the natural embedding functor e:Top→→CFT (↪FT), the λ-modification functor λ: FT → LFT which to every fuzzy space $(X,\mathcal{T})$ assigns the fuzzy space $(X,\mathcal{T}^\lambda)$ where $\mathcal{T}^\lambda$ is the weakest laminated fuzzy topology dominating $\mathcal{T}$, the ι-modification functor ι: FT → FT defined as $\iota(X,\mathcal{T})=(X,\iota\mathcal{T})$ where $\iota\mathcal{T} = \bigvee_\alpha(\iota\mathcal{T}_\alpha(U)\wedge\alpha)$, and $\iota\mathcal{T}_\alpha = \{U^{-1}(\beta,1]: \beta \geq 0,\ U \in I^X,\ \mathcal{T}(U) \geq \alpha\}$, and some other functors [26],[17]. The behaviour of these functors in respect of different operations is being studied. In particular, it is proved that λ and ι commute with products. The functors considered here are of double interest for us. Firstly, when restricted to subcategories of FT, they establish some useful relations between the subcategories. Specifically, they realise important embeddings of Top into the category FT allowing by the same token to find various interpretations of the category Top as a category of Fuzzy Topology. Secondly, these functors establish some schemes useful both for constructing of special examples of fuzzy spaces and in proofs of some results of Fuzzy Topology.

Section 1.4 is devoted to the study of some categorical properties of categories of Fuzzy Topology. In particular, we establish that the subcategory CFT is both reflective and coreflective in the category FT, and that the categories LFT, LCFT and λ(Top) are coreflective in FT [5],[6],[18].

Local structure of a fuzzy space is the subject of Section 1.5. An essential difference in this respect between fuzzy and classic Topology is caused by the fact that no full-bodied analogue of the concept of a point exists in fuzzy situation. The so called fuzzy points, defined as mappings $x_o^t: X \to (0,1]$, where $x_o \in X$, $t \in (0,1]$ and $x_o^t(x_o)=t$, $x_o^t(x)=0$ if $x \neq x_o$ [PL], in their properties differ essentially from the usual points. Therefore the necessity arises to consider, along with the belongness relation $x_o^t \in M$ (:= $M(x_o) \geq t$), a dual relation: the so called q-coincidence relation $x_o^t qM$ (:= $M(x_o)+t>1$) [PL]; see also [Ka], [17]. The central results of this section are characterizations of fuzzy topologies by means of their neighborhood and q-neighborhood structures [29]. In particular:

Let $(X,\mathcal{T})$ be a fuzzy space and $\mathfrak{X}$ be the set of its fuzzy points. Then the mapping $Q: \mathfrak{X} \times I^X \to I$ defined by the equality $Q(x_o^t,U) = \sup\{\mathcal{T}(V): V \leq U, V(x_o) > t^c$ (the so called Q-neighborhood structure of the space $(X,\mathcal{T})$) has the following properties ($p \in \mathfrak{X}$, $Q(p,U) := Q_p(U)$, $U,U_1,U_2 \in I^X$):

(1q) if $Q_p(U)>0$, then pqU; (2q) $\sup\{Q_p(U): U \in I^X\} = 1$;

(3q) $Q_p(U_1 \wedge U_2) \geq Q_p(U_1) \wedge Q_p(U_2)$; (4q) if $U \leq U'$, then $Q_p(U') \geq Q_p(U)$; (5q) $Q_p(U) \leq \sup_{V \leq U} Q_p(V) \wedge (\bigwedge_{rqV} Q_r(V))$.

Conversely, if X is a set and $Q: \mathfrak{X} \times I^X \to I$ satisfies the above conditions (1q)-(5q), then the mapping $\mathcal{T}: I^X \to I$ defined by the equality $\mathcal{T}(U) = \inf\{Q_p(U): pqU\}$ is a fuzzy topology on X and besides the corresponding Q-neighborhood structure is Q .

The convergence theory for fuzzy spaces is developed in Section 1.6. It is based on the concept of a fuzzy net [PL] and on its derivative concept of the convergence structure [40]. The convergence structure of a fuzzy space $(X,\mathcal{T})$ is defined as a special mapping Con: $N(x) \times I \to I^X$ where $N(x)$ is the class of all fuzzy nets in the space X. In particular, we establish here the fuzzy version of the well-known Kelly Theorem [Ke, p. 106]).

Section 1.7 is devoted to the concept of continuity defect for a mapping of fuzzy spaces. It is essentially a fuzzy concept which has no meaningful prototype in classic, "crisp" mathematics basing on two-valued logics. The defect of a mapping $f:(X,\mathcal{T}_X) \to (Y,\mathcal{T}_Y)$ at a level $\alpha \in I$ is defined as $cd_\alpha(f) = \sup\{\sup(f^{-1}(V)-\mathrm{Int}(f^{-1}(V)))(x): V \in I^Y, \mathcal{T}_Y(V) \geqslant \alpha\}$. (In case of ordinary topological spaces a defect is either 0, if the mapping is continuous, or 1, otherwise.) Of the results of this section we mention here the inequality $cd_\alpha(g \circ f) \leqslant cd_\alpha(f) + cd_\alpha(g)$ (where $f:X \to Y$, $g: Y \to Z$) and the formula $cd_\alpha(\Delta f_\gamma) = \bigvee_\gamma cd_\alpha(f_\gamma)$ where $\Delta f_\gamma : X \to \prod Y_\gamma$ is the diagonal product [E] of the family of mappings $f_\gamma : X \to Y_\gamma$, $\gamma \in \Gamma$ [17],[35],[36].

Chapter 2 of the Thesis consisting of ten sections is devoted to investigation of important concrete topological properties of fuzzy spaces and their fuzzy subsets. Here we shall restrict ourselves to the case of Chang fuzzy spaces: the study of topological properties of general fuzzy spaces can be reduced to Chang's case by means of representation of a fuzzy topology $\mathcal{T}$ direct system of its level Chang fuzzy topologies $\mathcal{T}_\alpha = \{U \in I^X : \mathcal{T}(U) \geqslant \alpha : \alpha \in (0,1]\}$. (cf [NR]). Besides, when formulating definitions and results from Chapter 2 we confine ourselves with one typical case: the exposition in full generality would essentially increase the size of the summary. Thus, in the sequel (X,τ) or just X stands for a Chang fuzzy topological space and M for its arbitrary fuzzy subset.

In Section 2.1 separation properties of fuzzy spaces are studied. By a Hausdorfness spectrum of (X,τ) we call the set $H(X) = \{b \in I: x,y \in X, x \neq y, \forall \varepsilon > 0\ \exists U,V \in \tau$ s.t. $U(x) \geqslant b-\varepsilon$, $V(y) \geqslant b-\varepsilon$, $U \tilde{\subset} V^c \geqslant b-\varepsilon\}$. Among other we prove here:

If X is the product of fuzzy spaces X_i, $i \in J$, then $\cap H(X_i) \subset H(X)$. If, besides, all spaces are laminated, then the equality holds. [7],[17].

For every fuzzy space X $H(X) = Cl(\Delta, X^2)$, where Δ is the diagonal of the space X and $Cl(M,Y)$ is the s.c. closedness spectrum of a fuzzy set M in a fuzzy space Y [7],[17]. (In fact this is a fuzzy version of the classic characterisation of Hausdorff spaces as spaces with closed diagonals.)

If mappings $f,g: X \to Y$ are continuous, then $Cl(E,X) = H(X)$, where $E = \{x \in X: f(x) = g(x)\}$ [7],[17].

Observe that the results of this section include in itself as special cases many of existing separation theories in Fuzzy Topology (see e.g. [Ro_1],[Ro_3],[PL],[SLS],[Ks] e.a.)

The property of E-regularity is studied in Section 2.2. A fuzzy space (X,τ) is called E-regular, where E is a fixed fuzzy space, if the family $\hat{C}(X,E)$ of all continuous mappings of X into finite powers of E (i.e. $\hat{C}(X,E) := \bigcup\{C(X,E^n) : n\in N\}$) separates points and closed fuzzy sets of X (i.e. for every $x\in X$, every closed $A\in I^X$ and every $\varepsilon>0$ there exists $f\in\hat{C}(X,E)$ s.t. $A(x) \geq \overline{f(A)}(f(x))-\varepsilon$ [15],[27]). It is proved that a space X is embeddable into a product E^k where k is some cardinal iff X is an E-regular K_0-space (the last property means that $\forall x,y\in X\ \exists U\in\tau$ s.t. $U(x)\neq U(y)$ - this is probably the weakest separation-type property in fuzzy topology). Some characterizations of E-regularity are obtained; in particular the characterization of E-regularity by means of convergence structures [15],[27]. In case $E=F(I)$ (Hutton's unit interval [Hu_1]) F(I)-regularity is equivalent to the complete regularity in the sense of Hutton-Katsaras [Hu_2],[Ka]. However, the role of E-regularity in Fuzzy Topology is essentially more important than the role of complete regularity in General Topology and, on the other hand, than the role of its prototype - crisp E-regularity in General Topology. One of the reasons for this is that in Fuzzy Topology along with the fuzzy interval F(I) there exists a number of other canonical objects: such as the fuzzy probabilistic interval [Lo_3], Eklund-Gähler's interval [EG], Höhle's interval [$Hö_3$] e.a. saying nothing about versions of these spaces in categories of FT(L)-type.

Section 2.3. is devoted to one of the most important topological properties, namely, to compactness. The developed approach to compactness theory is based on the concept of compactness spectrum. By the compactness spectrum of a fuzzy set M in a fuzzy space X we call the set $C(M) := \{b\in I : \forall\,\mathcal{U}\subset\tau,\ \forall\varepsilon>0\ ((M \tilde{\subset} \vee\mathcal{U} \geq b) \Rightarrow (\exists\,\mathcal{U}_0\subset\mathcal{U},\ |\mathcal{U}_0|\leq\aleph_0,\ M\tilde{\subset}\vee\mathcal{U}_0 \geq b-\varepsilon)\}$. The value $c(M)= \inf(I\setminus C(M))$ is called compactness degree of the fuzzy set M.

If X,Y are fuzzy spaces, $M\in I^X$ and a mapping $f: X\to Y$ is continuous, then $C(M)\subset C(fM)$ [9],[23].

Let $M=\prod M_i$ be the product of fuzzy sets $M_i : X_i \to I$ in the product space $X=\prod X_i$. Then $(a,b]\subset C(M)$ and $(M) \geq \inf c(M_i)$. In case all M_i are normed (i.e. $\sup M_i(x_i)=1$)

the equality $c(M)=\inf c(M_i)$ holds [9],[23].

If $M,N \in I^X$ and M is closed, then $C(N) \subset C(M \wedge N)$ and $c(N) \leq c(M \wedge N)$ [9],[23].

If $f: X \to Y$ is a closed continuous mapping, then $c(f^{-1}(N)) \geq c(f) \wedge c(N)$ for each $N \in I^Y$ where $c(f)=\inf(I \setminus \cap\{C(f^{-1}(y)) : y \in Y\})$.

If $f: X \to Y$, $g: Y \to Z$ are closed continuous mappings, then $c(g \cdot f) \geq c(g) \wedge c(f)$ [9],[23].

If $b \leq h(X) \wedge c(M) \wedge c(N)$ and $M \tilde{\subset} N^c \geq b$ (where $M,N \in I^X$ and $h(X)=\sup H(X)$), then for each $\varepsilon>0$ there exist $U,V \in \tau$ s.t. $M \tilde{\subset} U \geq b-\varepsilon$ $N \tilde{\subset} V \geq b-\varepsilon$, $U \tilde{\subset} V^c \geq b-\varepsilon$ [9],[23].

The last result and a theorem stating that under certain assumptions on separation of X it holds $ACl(M) \cap (1/2,1] = C(M) \cap (1/2,1]$ where $ACl(M) = \{b \in I: (\forall Z \supset X,\ b \in H(Z)) \Rightarrow (b \in Cl(M,Z))\}$ is the s.c. absolute closedness spectrum of M [9],[23], present an external, in a sense, description of the connection between compactness properties of a fuzzy set and the "closedness degree" to which the fuzzy set is located in the corresponding fuzzy space. These results as well as their well-known crisp prototypes essentially use separation properties of the considered spaces. However, a peculiarity of fuzzy topology is thatmany important fuzzy spaces have very weak separation properties. The next two theorems present "external" description of compactness without any restrictions on separatedness of the spaces. But first we need to define the relative closedness spectrum $RCl(M,X)$ of a fuzzy set M in a fuzzy space X: $RCl(M,X) = \{b \in I: \forall x_0 \in X, \forall \mathcal{U} \subset \tau, \varepsilon>0, \forall \xi \in (0,\varepsilon] ((\bar{M}(x_0) > b^c) \& (M \tilde{\subset} \vee \mathcal{U} \geq b)) \Rightarrow (\exists V \in \tau,\ V(x_0) \geq b-\xi,\ \exists \mathcal{U}_0 \subset \mathcal{U}, |\mathcal{U}_0| < \aleph_0 (\forall x\ \ (M(x) \geq b^c+\varepsilon) \& (V(x) \geq b-\varepsilon) \Rightarrow (\vee \mathcal{U}_0(x) \geq b-\varepsilon)))$.
Basic properties of the relative closedness spectrum are studied in Thesis; in particular, these spectra are characterized by means of fuzzy nets. An important example of a relative closedness spectrum is expressed by the equalities $RCl(F(I),F(R)) = RCl(R,F(R))=[0,1]$. (In this connection we recall that, as S. Rodabaugh showed [Ro_2] there is no non-trivial closed subset in the fuzzy real line $F(\mathbb{R})$ [GSW] at all.

<u>Theorem</u> [33]. $C(X) \cap RCl(M,X) \subset C(M) \subset RCl(M,X)$.

<u>Theorem</u> [33]. $b \in C(M)$ where $M \in I^X$ iff $b \in RCl(M,Z)$ for each space Z containing the space X.

In case when X is a usual topological space, the results of this section can be interpreted as an alternative compactness theory for real-valued mappings of the space X (cf with B. Pasynkov's theory; see e.g. [Pa] e.a.) The key for such an interpretation is contained in the next statement:

If X is a topological space, $M \in I^X$, $b \in I$ and the sets $M^{-1}[d,1]$ are compact for all $d > b^c$, then $c(M) \geq b$. Conversely, if $c(M) \geq b$ and M is upper semicontinuous, then the sets $M^{-1}[d,1]$ are compact for all $d > b^c$. In particular, if a mapping is perfect, then $c(M) = 1$, and if a mapping is upper semicontinuous and $c(M) = 1$, then the set $M^{-1}(0,1]$ is σ-compact [9],[17].

In Section 2.4 patterned after definition of compactness spectrum, we introduce the concepts of Lindelöfness spectrum $L(M)$ and countable compactness spectrum $CC(M)$ of a fuzzy set M in a fuzzy space X. Besides, we define heriditary Lindelöfness spectrum of a space X as $HL(X) := \cap\{L(M): M \in I^X\}$. Of the results of this section we cite here the following:

If X, Y are fuzzy spaces, $M \in I^X$, $N \in I^Y$, then $l(M \times N) \geq l(M) \wedge c(N)$, where $l(M) = \inf(I \setminus L(M))$; if in a fuzzy space (X, τ) every $V \in \tau$ is a countable union of closed fuzzy sets, then $L(X) = HL(X)$; if X, Y are fuzzy spaces and $w(X) \leq \aleph_0$, then $hl(X \times Y) = hl(Y)$ where $hl(Y) = \inf(I \setminus HL(Y))$; a topological space X is hereditary Lindelöf iff $hl(\omega X) = 1$ (ω is Lowen's functor [Lo_1]) [12],[13],[24].

Starting from characteri-ation of compacta as closed subspaces of Tychonoff cubes [E],[Ke], R.Engelking and S.Mrowka introduced the concept of an E-compact space where E is a fixed Hausdorff space [EM]. The fuzzy analogue of this notion is studied in Section 2.5. Observe however, that the theory of E-compactness in Fuzzy Topology is essentially different to compare with its crisp prototype specifically because we have to relinquish separation conditions of the considered spaces as well as to avoid the use of closed subspaces (see also comments when discussing sections 2.2 and 2.3.).

A fuzzy set M in a fuzzy space X is called E-compact, where E is a fixed fuzzy space, if there exist a cardinal k and a homeomorphism $h: X \hookrightarrow E^k$ such that $RCl(hM, E^k) = [0,1]$. Let $K(E)$ denote the class of all E-compact subsets of fuzzy spaces. We prove that:

The product of E-compact fuzzy sets is E-compact;

if X is an E-regular space, where E is strongly compact [GS*], and $w(X)$ k, then for each b $C(M)$ a homeomorphism $h: X$ E^k exists such that b $RCl(hM, E^k)$ and hence if $c(M) = 1$, then M $K(E)$;

and $w(X) \le k$, then for each $b \in C(M)$ a homeomorphism $h: X \hookrightarrow E^k$ exists such that $b \in RCl(hM, E^k)$ and hence if $c(M)=1$, then $M \in K(E)$;

$K(F(I)) \cap$ Top is the class of all compacta;

$K(F(R)) \cap$ Top is the class of all realcompact spaces;

if X, E are topological spaces, then $X \in K(E)$ iff $\omega X \in K(\omega E)$.

In the second part of Section 2.5 the theory of E-compact (in particular, of completely regular compact) extensions for fuzzy sets is being developed. In particular, a construction allowing to describe all E-compact extensions for a given fuzzy set M in a fuzzy space X is presented here. [44].

The connectedness type properties are the subject of Section 2.6. By the disconnectedness spectrum of a fuzzy set M in a fuzzy space (X,τ) we call the set $D(M) = \{b \in I : \exists\, U_1, U_2 \in \tau,\ M \tilde{\subset} U_i < b$ $i=1,2;\ M \tilde{\subset} U_1 \vee U_2 \ge b,\ \sup (U_1 \wedge U_2)(x) < b\}$; its complement $S(M) = I \setminus D(M)$ is called the connectedness spectrum of M and the value $s(M) = \inf D(M)$ is called the cornectedness degree of M. The main results of this section have appeared in [10],[23],[17]. Among them:

Let $M = \prod M_i$ be the product of fuzzy sets $M_i: X_i \to I$ in the product fuzzy space $X = \prod X_i$. Then $S(M) \supset \bigcap_i S(M_i)$ and $s(M) \ge$ $\ge \inf_i s(M_i)$. If besides all M_i are normed, then $S(M) = \bigcap_i S(M_i)$ and hence $s(M) = \inf_i s(M_i)$.

If X is a topological space and a · mapping $M: X \to I$ is either closed or open, then $S(M) = [0,1]$.

The concept of a fuzzy (pseudo)metric is introduced and studied in Section 2.7. The main result here is a metrization criterium of Nagata-Smirnov's type for fuzzy spaces.

In Section 2.8 we define and study fuzzy stratifiable spaces. Stratifiable spaces are, to our opinion, a successful generalization of fuzzy metrizable spaces: on one hand they have very nice topological properties similar to those of metric spaces (such as productivity, hereditarity, invariance under closed mappings a.s.) and, on the other hand, the definition of fuzzy stratifiable spaces is not based on the use of fuzzy points and this essentially simplifies the study and use of such spaces [1],[2],[4],[39].

The theory of cardinal invariants in Fuzzy Topology is being developed in Section 2.9, the main results of which are published in [29]. We define such cardinal properties of a fuzzy set M in a fuzzy space X as weight w_M, density d_M, spread s_M, extent e_M, Souslin number c_M and Lindelöf number l_M. As an example we reproduce here the definition of weight.

A family $B \subset \tau$ is called a base of M if for each $V \in \tau$ there exist families $B_V \subset B$ and $C_V \subset B^c$ such that $M \wedge V = M \wedge (\vee B_V)$ and $M \wedge V^c = M \wedge (\wedge C_V)$. We set $w(M) = \min\{\xi \in K: \exists B \subset \tau, |B| \leq \xi$ and B is a base for $M\}$. The fuzzy cardinal $w_M: K \to I$ defined by the formula $w_M(a) = \sup\{t \in I: w(M \cdot M^{-1}(t,1]) \geq a\}$ is called the weight of M.

If $M = \prod M_i$, $i \in J$, is the product of fuzzy sets $M_i: X_i \to I$, $i \in J$, in the product fuzzy space $X = \prod X_i$, then $w_M \leq (w_{Mi}) \vee |J|$. In case all M_i are normed, then $w_{Mi} \leq w_M$.

$w_M \geq d_M \geq c_M$; $w_M \geq l_M$ and $w_M \geq s_M \geq c_M$ for each fuzzy set M.

If $c_{\prod\{M_i: i \in J_0\}} \leq \aleph_0$ for every finite set $J_0 \subset J$, then $c_M \leq \aleph_0$, too. In case all M_i are normed, $\vee c_{M_i} \leq c_M$ and $\vee d_{M_i} \leq d_M$.

The last, 10th Section of this Chapter stands, in a known sense, apart. Here, starting to realize A.Archangel'skii's programme, we characterize some topological properties of a fuzzy space (X,τ) as location properties of the fuzzy topology τ in the Tychonoff cube I^X. Among other, we prove here:

A fuzzy space (X,) is a T_1-space iff τ is dense in I^X. [31].

If (X,τ) is a T_1-space, then τ is open in I^X iff (X,τ) is discrete [31].

The tightness of a fuzzy space (X,τ) does not exceed k iff τ is closed in the space $(I^X, \sup(T_r^{|X|}, T_{1k}^{|X|}))$ where $T_r^{|X|}$ is the product of |X| copies of topology $T_r = \{(a,1]: a \in I\} \cup \{I\}$ on I and $T_{1k}^{|X|}$ is the k-box product of |X| copies of the topology $T_1 = \{[0,a): a \in I\} \cup \{\emptyset\}$ on I [31].

What fuzzy topological spaces are to be considered as fuzzy analogues of a given topological space (X,T) ? A known answer to this question is provided in Section 1.2 where functors e, λ, ι e.a were defined; these functors for a given topological space (X,T) assign fuzzy topological spaces (X,eT), $(X,\lambda T)$, $(X,\iota T)$ e.a. The essential peculiarity of these functors is that they change only topological structure leaving unchanged the sets of the corresponding spaces. However, although a fuzzy space of $(X,\mu T)$ type is, in a known sense, a fuzzy copy of (X,T), it usually can not play in Fuzzy Topology the role which is played by the space (X,T) in General Topology. The objects of $(X,\mu T)$ type are, in some sense, "too pure" to fulfil in Fuzzy Topology the job, which (X,T) carries out in General Topology.

A different approach to this problem is worked out in Chapter III. Here two general schemes are developed which allow to construct, starting from a given topological space, some essentially new fuzzy spaces (cf [GSW],[Hu],[Rc],[Kl],[Lo] e.a.)

<u>Construction F(X)</u> (Section 3.1; see [20],[21]). Let $(X,\leq)$ be a linearly ordered topological space and let $Z(X)$ denote the set of all non-increasing mappings $z: X\to I$ such that $\sup_X z(x)=1$ and $\inf_X z(x)=0$. Define the equivalence relation "$\sim$" on $Z(X)$ by setting $z\sim z'$ iff $z(x^-)=z'(x^-)$ and $z(x^+)=z'(x^+)$ for each $x\in X$ where $z(x^-)=\inf\{z(t): t<x\}$, $z(x^+)=\sup\{z(t): t>x\}$. Let $F(X)$ denote the quotient space $Z(X)/\sim$ and let a fuzzy topology on it be defined by the subbase $\{r_a, l_b: a,b\in X\}$ where $r_a[z]=z(a^+)$ and $l_b[z]=1-z(b^-)$. The resulting fuzzy space will be denoted just $F(X)$ and its laminated modification $F^\lambda(X)$.

Important examples of application of this construction are spaces $F(R)$ and $F(I)$ which, up to a natural isomorphism, are just the fuzzy real line [GSW] and the Hutton unit interval[Hu_1] respectively.

The mapping $h: X\to F(X)$ defined by the equality $h(a)=[z_a]$ where $z_a=\{x: x\leq a\}$, is a homeomorphic embedding of a topological space X into the fuzzy space $F(X)$.

Let X,Y be linearly ordered topological spaces and let $f:X\to Y$ be a non-decreasing function. Assigning to each $[z]\in F(X)$ the element $\hat{f}[z]=[u]\in F(Y)$ defined by $u(y)=\inf\{z(x): f(x)\leq y\}$, we obtained a mapping $\hat{f}: F(X)\to F(Y)$. It is proved that the mappings $\hat{f}: F(X)\to F(Y)$ and $\hat{f}: F^\lambda(X)\to F^\lambda(Y)$ are continuous. Thus constructions F and F^λ can be interpreted as functors from the category ORD of all linearly ordered spaces and non-decreasing mappings into the category CFT.

The relations between properties of spaces $(X,\leq)$ and $F(X)$ are studied in Section 3.2. In particular, it is proved that:

- $w(F(X)) = w(F^\lambda(X)) = w(X)$ for each space X ;

a space X is bounded iff the spaces $F(X)$ and $F^\lambda(X)$ are strongly compact in the sense of [GSW];

if X is not bounded, then $cf(X)=l(F(X))=l(F^{\lambda}(X))$ where $cf(X) = \min\{|Y|: Y\subset X, Y$ is unbounded in $X\}$ (the s.c. cofinal character of the space X) Thus $l(F(X))\leq l(X)$ and, in particular, spaces $F(R)$ and $F^{\lambda}(R)$ are Lindelöf.

The following properties are equivalent: (1) X is metrizable; (2) $F(X)$ is stratifiable; (3) $F^{\lambda}(X)$ is stratifiable.

Construction $M_{\xi}(X)$ (Section 3.3; see [8],[14],[17]; in case of a separable metric space X see [Lo_3]). Let X be a topological space, $B(X)$ be the σ-algebra of its Borel sets, and let $M(X)$ denote the family of all probability measures $p: B(X)\to I$. Given a family ξ of lower semicontinuous functions $U: X\to I$ we define a fuzzy topology on $M(X)$ by the subbase $\{\delta_U: U\in\xi\}\subset I^{M(X)}$, where $\delta_U(p) = \int U dp$. The resulting fuzzy topological space is denoted $M_{\xi}(X)$ or just M_{ξ}.

The most important special cases of the construction M_{ξ} are M_T (i.e. $\xi=T$) and $M_{\omega T}$ (i.e. $\xi=\omega T$, the family of all lower semicontinuous mappings from (X,T) into I).

If $f: (X,T_X)\to(Y,{}_Y)$ is a continuous mapping, then by setting $\hat{f}(p)(E) = p(f^{-1}(E))$, where $p\in M(X)$ and $E\in B(Y)$ one obtains continuous mappings $\hat{f}: M_{TX}\to M_{TY}$ and $\hat{f}: M_{\omega TX}\to M_{\omega TY}$. This allows to interpret constructions M_T and $M_{\omega T}$ as functors from the category Top into the category CFT.

Let (X,T) be a topological space and let τ be a fuzzy topology on X such that $\iota\tau = T$. Define a mapping $h: (X,\tau)\to M_{\xi}(X)$ by setting $h(x)=p_x$ where p_x is the measure degenerate in x. Then h is a homeomorphic embedding iff ξ is a subbase of the fuzzy topology τ. In particular, the mappings $h: (X,T)\to M_T(X)$ and $h: (X,\omega T)\to M_{\omega T}(X)$ are homeomorphic embeddings.

We consider also relations between fuzzy topologies obtained by the above scheme and the s.c. weak topologies [Ba] on sets of probability measures. Specifically, it is proved, that if (X,T) is perfectly normal, then the topologies $\iota\tau_T$ and $\iota\tau_{\omega T}$ coincide with the weak topology on $M(X)$.

Relations between topological properties of $M_{\xi}(X)$ and those of the original space (X,T) are studied in Section 3.4.

If (X,T) is a T_1-space and $\xi\subset\omega T$, then $d(M_{\xi}(X))\leq d(X,T)$

If (X,T) is a T_0-space and $\xi \subset \omega T$, then $d(K_\xi(X)) \leq d(X,T)$ where $K_\xi(X)$ is the subspace of $M_\xi(X)$ consisting of two-valued measures [14].

The following properties are equivalent for a normal space: (1) (X,T) is countably compact; (2) $M_T(X)$ is countably compact; (3) $M_{\omega T}(X)$ is countably compact; (4) $M_T(X)$ is compact; (5) $M_{\omega T}(X)$ is compact [14].

If (X,T) is perfectly normal then the fuzzy space $M_{\omega T}(X)$ is b-Hausdorff for every $b>0$ (i.e. $\forall\, p,q \in M(X)\ \exists\, U,V \in \mathcal{T}$ such that $U(p)>b$, $V(q)>b$ and $U \wedge V \leq b$) [14].

The subject of Section 3.5 is the study of interconnections between constructions $M_\xi(X)$ and $F(X)$ [14].

If X is a linearly ordered space of countable character and without isolated points, then the spaces $F(X)$ and $M_\xi(X)$ where $\xi = \{\{x: x<b\}, \{x: x>a\}: a,b \in X\}$ are homeomorphic. (The corresponding homeomorphism φ is defined by the equality $\varphi(p) = [z_p]$ where $z_p(x) := p\{y: y \geq x\}$, i.e. to every probability measure the corresponding distribution function is assigned.) [14], [17].

If X is a linearly ordered space of countable weight and without isolated points, then the spaces $F(X)$ and $M_T(X)$, as well as the spaces $F^\lambda(X)$ and $M'_{\omega T}(X)$ are homeomorphic [14], [17].

The last, IVth Chapter stands by itself in the Thesis because of its quite specific subject of research. While in the rest of the Thesis we were interested in the category FT(L) and its subcategories CFT(L) and LCFT(L) for a fixed lattice L, and besides often we confined ourselves with the special case L=I, the subject of research in Chapter IV is the ategory GFT (General Category of Fuzzy Spaces) [17], [25], containing L-fuzzy spaces with different lattices L (cf the analogous propounding of the problem in case of Chang's spaces in S.Rodabaugh's papers [Ro_4], [Ro_5]; see also [Ku]). The objects of the category GFT are quadruples $(X,L,\mathcal{T},K)$ where X is a set, L and K are bounded sup-complete lattices and $\mathcal{T}: L^X \to K$ is a mapping satisfying the axioms which are completely analogous to the ones introduced in Section 1.1. As morphisms of GFT are taken triples (f,φ,ψ): $(X_1,L_1,\mathcal{T}_1,K_1) \to (X_2,L_2,\mathcal{T}_2,K_2)$ where $f: X_1 \to X_2$, $\varphi: L_2 \to L_1$, $\psi: K_2 \to K_1$ are mappings, φ and

ψ preserve arbitrary suprema and finite infima and, besides, $\mathcal{T}_1(\varphi \circ V \circ f) \geq \psi(\mathcal{T}_2(V))$ for every $V: X_2 \to L_2$ (the last inequality is a kind of continuity condition) [25].

In Section 4.2 fuzzy topological categories considered in Chapters I - III and some other categories (both new ones and already existing in the literature on Fuzzy Topology [Ro_4], [Ro_5] [Hu_3], [HR]) are characterized as subcategories of the category GFT. For example, FT(L) can be characterized as the subcategory GFT(L,ε_L) of the category GFT, whose objects are quadruples $(X,L,\mathcal{T},L)$ and whose morphisms are the triples $(f,\varepsilon_L,\varepsilon_L)$ ($\varepsilon_L: L \to L$ stands for the identity mapping.) Observe, that FT(L) is not complete in the category GFT. Let GCFT(L) denote the complete subcategory of the category GFT, the objects of which are of the kind $(X,L,\mathcal{T},2)$. The category of Chang L-fuzzy topological spaces CFT(L) can be characterized now as the intersection of the categories GFT(L,ε_L) and GCFT(L). Of a known interest is also the complete subcategory GFT($*$) of GFT whose objects are $(*,L,\mathcal{T},K)$ where $*$ is the one-point set. The complete subcategory of GFT($*$) constituted by objects satisfying the condition $\mathcal{T} = 1$ is isomorphic to the category LOC of locales [Jo](cf [Ro_5]).

Basic operations in category GFT are studied in Section 4.3. Notice that their definitions essentially differ from the corresponding definitions in categories of FT(L) type (cf [Ro_5]). As an example we define here the product in GFT.

Let $\{L_\gamma : \gamma \in \Gamma\}$ be a family of sup-complete distributive bounded lattices; following [Hu_3] by $L := \otimes L_\gamma$ we denote the set whose elements are subsets $a \subset \prod\{L_\gamma^+ : \gamma \in \Gamma\}$ such that (1) if $t \in a$ and $s \leq t$, then $s \in a$; (2) if $b = \prod b_\gamma \subset a$, then also $\beta = (\beta_\gamma)_\gamma \in a$ where $\beta_\gamma = \sup b_\gamma$. By setting $a \leq b$ iff $a \subset b$ $(a,b \in L)$ the set L becomes a bounded sup-complete distributive lattice. (E.g. if $L_\gamma = 2^{Z_\gamma}$ for each $\gamma \in \Gamma$, where Z_γ is some set, then $\otimes L_\gamma = 2^{\prod Z_\gamma}$ [Ek].) By the equality $\pi_{\gamma_0}(t_{\gamma_0}) = \{s \in \prod L_\gamma^+ : s_{\gamma_0} \leq t_{\gamma_0}\}$ a mapping $\pi_{\gamma_0} : L_{\gamma_0} \to L$ is defined.

Consider now a family $\{(X_\gamma, L_\gamma, \mathcal{T}_\gamma, K_\gamma) : \gamma \in \Gamma\} \subset Ob(GFT)$. The product of this family is defined as the quadruple $(X,L,\mathcal{T},K)$ where $X = \prod X_\gamma$, $L = \otimes L_\gamma$, $K = \otimes K_\gamma$ and $\mathcal{T}: L^X \to K$ is the (L,K)-fuzzy topology on X which is initial for the family $(p_\gamma, \pi_\gamma, \xi_\gamma)$, $\gamma \in \Gamma$;

here $p_\gamma : X \to X_\gamma$ stands for the usual projection and $\pi_\gamma : L_\gamma \to L$, $\xi_\gamma : K_\gamma \to K$ are defined in the same way as it is done in the previous paragraph.

It is important to observe that each assertion in which the product in the category GFT is used, contains completely different information if compared with a similar assertion in the realm of FT(L) type categories. One of the reasons for this is that the product in GFT compels the change of the lattice. Moreover, $\otimes L_\gamma = L$ where $L_\gamma = L$ for all γ iff $L = 2$ [Ek]. It follows from here that for ordinary topological spaces usual product is equivalent to the product in the category GFT, and hence also in each one of its subcategories - this is one of the evidences and displays of the invariance of General Topology in Fuzzy Topology.

However, in some cases information on operations in GFT allows to extract additional information about the categories of FT(L) type. In particular, it can be applied in investigation of subspaces of L-fuzzy spaces (i.e. of objects of categories FT(L)) on the basis of L-fuzzy sets.

The problem of algebraic characterization of fuzzy topological spaces in the category GCFT is studied in Section 4.4. [32], [37],[38]. By the Plotkin semigroup of a fuzzy space (X,L,τ) $(:=(X,L,\tau,2))$ we call the product $P(X,L,\tau)=C(X,L,\tau) \times L^X$ where $C(X,L,\tau) := \{(f,\mu) : (X,L,\tau) \to (X,L,\tau)\}$ is the endomorphism semigroup of the space (X,L,τ) with operation "." defined as $((f_1,\mu_1),U_1)\cdot((f_2,\mu_2),U_2) = ((f_2\cdot f_1,\mu_1\cdot\mu_2), U_2\cdot f_1)$. (A similar semigroup was used in Plotkin's paper [Pl] devoted to the theory of algebraic automata.) By setting $(f_1,\mu_1,U_1) \prec (f_2,\mu_2,U_2)$ iff $f_1 = f_2, \mu_1 = \mu_2$ and $U_1 \le U_2$ we introduce a partial order $\prec$ in the semigroup $P(X,L,\tau)$. We say that semigroups $P_i = P(X_i,L_i,\tau_i)$, i=1,2, are ω-isomorphic iff the isomorphism $\sigma : P_1 \to P_2$ exists such that $\sigma(C(X_1,L_1,\tau_1)\times\tau_1) = C(X_2,L_2,\tau_2)$ and $(f,\mu,U_1) \prec (f,\mu,U_2)$ iff $\sigma(f,\mu,U_1) \prec \sigma(f,\mu,U_2)$.

Laminated fuzzy spaces are homeomorphic in the category GFT iff their Plotkin semigroups are ω-isomorphic [32],[37],[38].

This approach can be used also when studying categories of GFT(L) type. We say that an ω-isomorphism σ is rigid if $\sigma(\varepsilon_{X_1},\varepsilon_L,a) = (\varepsilon_{X_2},\varepsilon_L,a)$ for every $a \in L$.

Laminated Chang L-fuzzy spaces are homeomorphic iff their Plotkin semigroups are regidly isomorphic [32],[37],[38].

In the last, Section 4.5 the compactness theory in the category GFT is being developed: Here one can trace the special features which are typical for concrete topological theories in the category GFT.

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А.Шостак. Основы теории нечетких топологических пространств (резюме хабилитационной работы по математике)

Аннотация. Изложены основные идеи, концепции и важнейшие результаты работы.

A.Šostaks. Fazi-topoloģisku telpu teorijas pamati (kopsavilkums habilitācijas darbam matemātikā).

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AUSARBEITUNG UND ANWENDUNG DER SPEZIALEN NUMERISCHEN METHODEN ZUR LOSUNG DER PROBLEME DER MATHEMATISCHEN PHYSIK, HYDRODYNAMIK UND MAGNETOHYDRODYNAMIK

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(Habilationsarbeit vom Fachbereich Mathematik zur Erlangung des Grades eines Doktors)

Summary. In this Dr.hab.math. theses "The working out and application of special numerical methods for solving of problems for mathematical physics, hydrodynamics and magnetohydrodynamics (MHD)" special finite-difference approximations are discribed. There are effective universal numerical methods (finite-difference and finite-element methods) for the solution of boundary-value problems of MHD equations for viscous incompressible flows based on nonlinear Navier-Stokes and Maxwell equations for small Reynolds and Hartman numbers. However, the presence of large parameters at first order derivatives or small parameters at second order derivatives in the system of MHD equations (large Reynolds, Hartman and others numbers) or of equations for mathematical physics cause additional difficulties for the application of general methods and they become uneffective (little speed of convergence, low precision). Thus important to work out special methods of solution, the so-called regular convergence computational methods, for the regarded problems (Doolan E.P., Miller J.J., Schilders W.H., Allen D.N., Southwell R.V., Ilhyn A.M.). Such methods must be propagated to the system of equations for mathematical physics. This give the basis for the development of special monotonous vector-difference schemes with perturbation coefficient of function-matrix.

AMS Subject Classification 65N20.

EINLEITUNG

Die Aktualität der Arbeit

Zur letzter Zeit in der MHD einer zähen inkompressiblen Flüssigkeit formiert man eine neue Richtung - die MHD der starken Felder , welche von der MHD-Technologie intensiv stimuliert wird (die MHD-Pumpen, die Beschleunigungsapparate, die Kristallisationsapparate,die Aluminiumelektrolyseapparate und andere MHD-Geräte). Die Anwesenheit eines starken Magnetfeldes führt zum den spezifischen MHD-Effekten, zum Beispiel, zur Entstehung verschiedener Geschwindigkeitstrukturen mit den Grenzschichten. Die theoretische Grundlage für die Beschreibung dieser MHD-Prozessen ist ein abgeschlossenes System von partiellen Differentialgleichungen zweiter Ordnung mit großen Parametern bei den Ableitungen erster Ordnung (die großen Werte von $Ha, S, Al, S^*, Re, Re_m, \Gamma_K, Gr$ -Zahlen). Wenn die Parameterwerten nicht groß sind, existierten für die Lösung der entsprechenden MHD-Grenzwertaufgaben, die sich auf die vollen nichtlinearen Navier-Stokes Gleichungen der Hydrodynamik stützen, effektive universale numerische Methoden (die Differenzenmethoden und die Finitelementenmethoden). Falls die Parameterwerte im MHD-Gleichungssystem bei den partiellen Ableitungen zweiter Ordnung klein oder die Parameter bei den Ableitungen erster Ordnung groß sind,zerfällt die MHD-Strömung in reguläre Gebiete und besondere Gebiete. In den regulären Gebieten kann man die Strömungsparameter durch glatte Funktionen mit endlichen Gradienten beschreiben. In den besonderen Gebieten (die MHD-Grenzschichten von Hartman und Ludford) sind die Gradienten der Lösungen sehr groß und hier finden schnelle Übergangsprozesse statt. Das führt zur Verkleinerung der Konvergenzgeschwindigkeit und der Genauigkeit der klassischen Differenzenschemen. Das Vorhandensein besonderer Grenzschichten mit großen Gradienten der Strömungsparameter erschwehrt die Anwendung von Schemen, die gleichmäßig für das ganze

Strömungsgebiet eine hohe Genauigkeit haben. Für den Aufbau der gleichmäßig konvergierenden speziellen Differenzenschemen muß ein Perturbationskoeffizient in Schemas eingeführt werden. Solche speziellen monotonen Differenzenschemas sind nicht nur für die Lösung der MHD-Gleichungen mit großen Parameter bei den Ableitungen erster Ordnung oder mit kleinen Parametern bei den Ableitungen zweiter Ordnung effektiv, sondern auch für die Lösung der Gleichungen in weiten Parameterwechselgebieten. Also können die speziellen Differenzenschemen wie eine wesentliche Ergänzung der Standartprogrammpaketen zur numerischen Lösung der Differentialgleichungen angewandt werdem. Die speziellen numerischen Methoden haben 2 hauptsächliche Eigenschaften: 1) die Anwendung eines groben Netzes, 2) die Konvergenz des Algorythmus unabhängig von Netzschritten. Das ist wichtig zur Lösung der Kontinuitätsmechanikrandwertaufgaben mit eingeschränkten EBM.

Für die linearen GDG und die Randwertaufgaben der mathematischen Physik sind solche speziellen s.g. gleichmäßige numerische Methoden schon von N.Bachvalow, A.Iljin, K.Jemeljanow, G.Schischkin, R.Kellog, D.Miller, E.Dulan und anderen Wissenschaftlern betrachtet. Die speziellen Lösungsmethoden müssen zur Lösung des MHD-Gleichungsystems und der nichtlinearen Gleichungen der mathematischen Physik übertragen und ausgebreitet werden. Dieser Forschungsobjekt kann in allernächster Zeit die effektive Algorythmen zur Lösung der Randwertaufgaben mit großen Parameter verspricht werden. Die gleichmäßige Konvergenz der speziellen Methoden kann nur für einige einfachsten Modellproblemen (eindimensionalen und linearen) theoretisch bewiesen werden.

Bei der Konstruktion des Differenzenschemas sind einige Grundforderungen zu erfüllen: 1) die Approximation, 2) die Stabilität, 3) die Genauigkeit, 4) die Effektivität. Die Bestimmung der Approximation und der Stabilität stellt die Grundaufgabe der Theorie der Differenzenschemen dar. Die Approximation kann man gewöhnlich mit der Entwicklung der Taylorreihe bis zur Gliedern einschließlich zweiter Ordnung von τ und h erforscht werden. Wenn die Lösung ein Grenzschichtcharakter (die Exponentialart) hat, ist diese Methode nicht genaue, denn die Approximationsfehler nicht nur von den Netzschrittsgraden abhängt, aber auch von den Ableitungen höchster Ordnungen der Problemslösungen, welche unbekannt sind. Die Stabilität des Differenzenschemas behandelt man wie die stetige Abhängigkeit der diskreten Lösung von den Anfangsdaten (die Randbedingungen, die Anfangsbedingungen, die rechten Seiten und die Koeffizienten der Gleichungen). Zeitweilig haben wir keine genügende Theorie für die Stabilitätsforschung der nichtlinearen Differenzenschemas., Deshalb kann die Stabilität nur für die linearen oder linearisierten Differenzenschemas mit der Hilfe des Maximumprinzips oder lokalen Kriteriums von Neiman geprüft werden. In diesem Beitrag werden verschiedene Approximationsweisen der Modelle der linearen eindimensionalen und zweidimensionalen Randwertaufgaben mit Rücksicht auf die analytischen Lösungen dieser Probleme betrachtet. Die praktisch wichtigsten Problemberecnnungresultaten (auch für die mehrdimensionale und nichtlineare Probleme), welche mit den spezialen Methoden gewonnen sind, werden mit anderen numerischen, analytischen oder physikalischen Resultaten verglichen. Die numerische Untersuchung gab die Möglichkeit zum Schluß viele spezifische MHD-Strömungseigenschaften ohne teuere physikalische Experimente im Institut für Physik festzustellen. Im vorliegenden Beitrag basieren alle Ergebnisse der Aufgabestellung und der Lösung neuen MHD-Randwertaufgaben auf den spezialen Methoden. Kurz betrachten wir die Hauptprinzipien der Konstruktion der spezialen Differenzenschemen. Im Falle der Approximation der eindimensionalen Randwertaufgaben im inregulären Netz kann die Integro-Interpolationsmethode von G.Marchuk und A.Samarski angewendet werden. In der Umgebung des 3-Punkts Netzschablons kann die Differentialgleichung zweiter Ordnung mit den Ableitungen erster Ordnung mit Hilfe der Exponentialtransformation in eine seltstkonjugierte Form verwandelt werden. Wenn die Koeffiziente der Differentialgleichung und die rechte Seite stückweise konstante Funktionen sind, bekommen wir das genaue Differenzenschema. Wenn die Koeffizienten veränderlich sind, können in der Netzschablonsumgebung die Mittelwerte von ihnem genommen werden. Im mehrdimensionalem Fall kann die Diskretisation von einem nach dem anderen einzelnen Veränderlichen und die Approximation von entsprechenden eindimensionalen Differentialoperatoren mit den Differenzenoperatoren zweiter Ordnung angewendet werden. Ähnlich kann man die Approximation der Differentialsysteme mit den Differentialgleichungen zweiter Ordnung durchführen. Die Funktionen-Matrizen in den Koeffizienten der Vektorsdifferenzenschemas werden durch die Funktionwerte aus dem Matrixspektrum erzählt mit Hilfe der Lagransch-Silvester Interpolationspolynome. Die konstruirten spezialen Differenzenschemas sind monoton, sie beschreiben den Grenzschichtcharakter gut und sind gleichmäßig auf Lösungen des linearen Modellproblems konvergent. Die monotonen Schemas geben die Möglichkeit viele praktische Probleme mit befriedigender Genauigkeit in groben Netzen und mit kleinen ERM zu lösen. Mit den neuen Methoden kann man praktisch wichtige Problemen der mathematischen Physik, Hydrodynamik und Magnetohydrodynamik in weiten Parameterswechselgebieten zu lösen. Zwischen solchen Problemen sind auch das Problem von Hant über die Rechnung der Magnetfeldsinduktion und der Geschwindigkeit in einer freien Verschiebunströmung abhängig von elektrischen Strömleitungsverhältnissen auf den Elektroden.

Das Prinzip der durchgehenden Rechnung erfordert die Einheitlichkeit des Differenzenschemas in den Grenzschichten von Hartman und den regulären Gebieten(die Kerne). In diesem Fall werden die Grenzschichten in der Rechnung automatisch wiedergegeben. Spezialen Lösungsmethoden der Randwertaufgaben können bei der Projektierung und der Ausarbeitung der neuen MHD-Technik und Technologie verwendet werden.

Die Geschichte der Arbeit

Im Jahre 1964 beginnt man unter der Leitung von Akademie-mitglied N.Janenko die numerische Modellierung der zähen inkompressiblen Flüssigkeit, basierend auf das volle nichtlineare Navier-Stokes Gleichungssystem. Während des zweiten mechanischen A'llunionskongreß in Moskau, war eine Arbeitsgruppe von den Mathematikern und Mechanikern organisiert. Der Autor nahm an dieser Arbeitsgruppe teil. Vom Jahre 1966 bis 1984 fanden regelmäßig jedes zweite Jahr die A'llunionsseminare über die numerische Methoden der zähen Flüssigkeit statt. Der Autor der Arbeit hat im Jahre 1963 zum erstenmal die numerichen Resultate über die Bewegung eines starren Zylinders in einer allseitig unendlich ausgedehnten zähen inkompressiblen elektroleitenden Flüssigkeit mit einem Magnetfeld bekommen; im Jahre 1971 hat er die Dissertation des Kandidaten der Wissenschaften "Über einige Lösungsmethoden des MHD-Differentialgleichungsystems" verteidigt, und 1991 - die Doktordissertation "Die Berechnung der zähen inkompressiblen elektroleitenden Flüssigkeitströmung mit den spezialen numerischen Methoden im Falle des großen Parameters". Zum erstenmal war für die neuen MHD Randwertaufgaben die Lösbarkeit bewiesen und wurden systematische numerische Untersuchungen dieser Aufgaben in weiten Parameterswechselgebieten begonnen (die Hauptarbeiten: [1-19]). Die wichtigste Forschungsmethode war die Anwendung der Differenzenschemas. Es waren originalen Annäherungsmethoden zur Lösung den Randwertaufgaben von den Raumströmungen und in unbeschränkten Gebieten ausgearbeitet. Die spezialen numerischen Methoden wurden in die Ausarbeitung der Vertragsarbeiten zwischen dem Rechenzentrum der Universität und dem Institut für Physik der Akademie der Wissenschaften eingeführt. Die Hauptresultate der Forschungen wurden in verschiedenen Konferenzen und Tagungen berichtet: in 6 Rigaer A'llunionsseminaren über die Magnetohydrodynamik (1972-1990); in 8 A'llunionsseminaren-Schulen über die numerischen Methoden der zähen Flüssigkeit unter der Leitung von N.Janhenko (1972-1986); auf der 6.Internationalen Konferenz über die numerischen Methoden der Hydrodynamik (Tbilissi, 1978); der 6. A'llunionstagung über die theoretische und angewandte Mechanik (Taschkent, 1986); in der internationalen Seminar-Schule über die mathematischen Modelle, analytische und numerische Methoden der Übertragungstheorie (Minsk, 1986); im internationalen Symposium IUTAM über die flüssigen Metalle in der MHD (Riga, 1988); an der internationalen Konferenz über die numerischen Methoden der Flüssigkeitsdynamik (Novosibirsk, 1990); in 5 internationalen Seminaren (Karls Universität in Prag, 1976-1988) u.a. Die wissenschaftlichen Arbeiten des Autors sind hauptsächlich in den wissenschaftlichen Zeitschriften "Magnetohydrodynamik" und "Das mathematische Jahrbuch Lettland's" veröffentlicht (siehe den Abschnitt "Literatur").

Weiter werden die Ideen der Werke des Autors betrachtet und die Übersicht der Arbeiten seit dem Jahre 1990, welche in der Doktordissertation [11-13,15,16] nicht dargestellt sind, durchgeführt.

1. DIE WICHTIGSTEN MATHEMATISCHEN MODELLE

Die Strömung einer zähen inkompressiblen elektroleitenden Flüssigkeit kann durch das Navier-Stokes Gleichungssystem

$$D\vec{v}/Dt = -\rho^{-1}\operatorname{grad} p + \nu\Delta\vec{v} + \vec{F}^e + \vec{F} \quad (1.1)$$

und die Maxwell's Gleichung

$$\operatorname{rot}\vec{E} = -\partial\vec{B}/\partial t, \quad \operatorname{rot}\vec{B} = \mu\vec{j} \quad (1.2)$$

beschrieben werden, welche von dem Om's Gesetz

$$\vec{j} = \sigma(\vec{E} + \vec{v}\times\vec{B}) \quad (1.3)$$

und den Kontinuitätsgleichungen

$$\mathrm{div}\,\vec{v} = \mathrm{div}\,\vec{j} = \mathrm{div}\,\vec{B} = 0 \tag{1.4}$$

ergänzt werden müssen.

Im Falle der Wärmenkonvektion sei es

$$\vec{F} = \vec{g}\,(1-\beta(T-T_0)),$$

wobei T_0 die Gleichgewichttemperatur ist.

Die Energiegleichung (die Wärmeleitungsgleichung) kann in der Form

$$\rho c_p\, DT/Dt = \lambda \Delta T + \sigma^{-1} \vec{j}^{\,2} \tag{1.5}$$

dargestellt werden.

Wenn in (1.2),(1.3) die Vektoren $\vec{E}, \vec{j}$ eliminiert sind, kann die Induktionsgleichung in der Form

$$\partial \vec{B}/\partial t = \mathrm{rot}(\vec{v} \times \vec{B}) + \nu_m \Delta \vec{B} \tag{1.6}$$

geschrieben werden.

Das stationäre elektrische Feld wird mit der Hilfe des skalaren Potenzials φ ($\vec{E} = -\mathrm{grad}\,\varphi$) beschrieben.

Für die zwidimensionalen laminaren MHD-Strömungen kann man von den Bewegungsgleichungen (1.1) in den kartesischen Koordinaten (x, y) den Druck p ausschließen und eine parabolische Gleichung für die Wirbelfunktion ω in der ohnedimensionale Form

$$\partial\omega/\partial t + u\,\partial\omega/\partial x + v\,\partial\omega/\partial y = Re^{-1}\Delta\omega + f \tag{1.7}$$

bekommen, wobei f die z-Rotorkomponente des Elektromagnetkrafts $\vec{F}^e$ ($f = \mathrm{rot}\,\vec{F}^e$) ist.

2. DIE SPEZIALEN DIFFERENZENMETHODEN ZUR LÖSUNG DER PROBLEMEN DER MATHEMATISCHEN PHYSIK

Die Ausarbeitung der spezialen numerischen Methoden zur Lösung der Problemen der mathematischen Physik ist zweckmäßig mit den einfachsten eindimensionalen Modellgleichungen, einschließlich mit Couchy and Randwertproblemen für GDG, zu beginnen [3,5]. Die gewonnenen Algorythmen verallgemeinern die Differenzenschemas von A.Iljin, G.Schischkin, K.Jemeljanov, N.Bachvalow, A.Samarski, G.Marchuk.

2.1. Die Lösung der Anfangs- und Randwertsaufgaben für gewöhnliche Differentialgleichungen

In der Literatur sind viele Algorythmen und Methoden (implizite und explizite) zur Lösung des Couchy Problems den GDG bekannt, zum Beispiel, die Methoden von Eiler, Adams, Runge-Kutta u.a. [18]. In expliziten Methoden führt die Stabilitätsforderung zu starken Beschränkungen für den Netzschritt, aber die impliziten Methoden mit EHM sind schwehr realisierbar. In der Arbeit [5] betrachtete spezialen Methoden wenden teilweise die Mangel der klassischen Methoden ab.

Die Differenzenmethoden werden im gleichmäßigen Netz des Segments [0,1]

$$\omega_h = \{ x_k = kh, \; k = \overline{1, N-1}, \; Nh = 1 \} \qquad (2.1)$$

gebildet. Für die GDG erster Ordnung in einem Elementarsegment $\mathcal{J}_k = [x_k, x_{k+1}]$ des Netzes (2.1) ist das explizite Differenzenschema mit der erhöhenden Genauigkeit gebildet. Bei der Lösung des Anfangswertproblems im Segment $\mathcal{J}_k$ für die linearen GDG erster Ordnung

$$y'(x) = a(x)\, y(x), \; y(x_k) = y_k , \qquad (2.2)$$

bildet man das spezielle absolut konvergierende stabile explizite Differenzenschema in der Form

$$(y_{k+1}-y_k)/h=\gamma(a_k)a_k y_k , \qquad (2.3)$$

wobei $a=a(x)$ im Segment $\mathfrak{I}_k$ eine glatte Funktion und

$$\gamma(a)=(ah)^{-1}(\exp(ah)-1) \qquad (2.4)$$

- ein Perturbationskoeffizient des Schemas ist .
Die Annäherung zur Lösung y_{k+1} ist gleich mit der genauen Lösung des Problems (2.2) $y(x_{k+1})$, wenn der Koeffizient a konstant oder "einfrieren" ist, d.h., $a(x)=a_k, x\in\mathfrak{I}_k$.
Wenn der Wert a_k ein Mittelwert der Funktion $a(x)$ im Segment $\mathfrak{I}_k$ ist

$$a_k=\frac{1}{h}\int_{x_k}^{x_{k+1}} a(x)\,dx , \qquad (2.5)$$

dann ist das Differenzenschema (2.3) genau entsprechend der Lösung des Differentialproblems (2.2) mit veränderlichem Koeffizient a. Wenn $a_k=a(x_k)$, $\gamma\equiv 1$ sind, dann bekommen wir die explizite Methode von Eiler, welche nur bei einer Bedingung $h\le 2/|a_k|$ $(a_k\le 0)$ stabil ist.

Sei $a_k=(a(x_k)+a(x_{k+1}))/2$, dann vergrößert die Genauigkeit des Algorithmus (2.3) über eine Ordnung im Vergleich mit der Methode von Eiler. Das Differenzenschema (2.3) erhaltet seine Form für die Lösung des m-Gleichungssystems (2.2), wo $a(x)$, a_k die quadratischen $m\times m$ Matrizen und $y'(x), y(x), y_k, y_{k+1}$ - die m-dimensionalen Vektoren sind, wobei

$$\gamma(a)=(ah)^{-1}(\exp(ah)-E) \qquad (2.6)$$

- die Matrix-Funktion der Perturbationskoeffizienten und E - eine Einheitsmatrix der m-Ordnung sind. Die Matrix-Funktion $\gamma(a)$ kann man mit der Hilfe der Funktionswerte $\gamma=\gamma(\lambda)$ aus dem Matrixspektrum und durch die Anwendung des Interpolationspolinoms von Lagransch-Silvester berechnen. Das explisite Vektordifferenzenschema (2.3) ist absolut stabil, wenn alle Eigenwerten $\lambda_i \; (i=\overline{1,m})$ der Matrix a nichtpositiv sind. Dieses Schema ist für den Matrizen a mit stückweise-konstanten in den Segmenten $\mathcal{J}_k \; (k=\overline{1,N-1})$ Elementen genau.

Um das Anfangswertproblem der nichthomogenen linearen Gleichung

$$y'(x)=a(x)y(x)+f(x),\; y(x_k)=y_k \tag{2.7}$$

im Segment $\mathcal{J}_k$ zu lösen, muß man solches diskrete Anfangswertproblem

$$(y_{k+1}-y_k)/h=\gamma(a_k)(a_k y_k+f_k) \tag{2.8}$$

verwenden. Das Problem (2.8) hat mindestens die Genauigkeit der zweiten Ordnung, wenn

$$a_k=\frac{1}{h}\int_{x_k}^{x_{k+1}} a(x)dx,\quad f_k=\frac{1}{h}\int_{x_k}^{x_{k+1}} f(x)dx .$$

Der Algorithmus (2.8) erhaltet seine Form auf für die Lösung des m-Gleichungssystems (2.7), wenn a, a_k - die Matrizen und $y', y, f, y_k, y_{k+1}, f_k$ - die Vektoren sind.

Im Falle des nichtlinearen Anfangswertproblems

$$y'(x)=F(y(x)),\; y(x_k)=y_k \tag{2.9}$$

wird man im Segment $\mathcal{J}_k$ die Methode von Newton benutzt. Der Algorithmus kann man in folgender Form bestellen

$$(y_{k+1}-y_k)/h=\gamma(a_k)F_k\,, \tag{2.10}$$

wo

$$a_k=\frac{\partial F}{\partial y}\Big|_{x=x_k},\quad F_k=F(y_k). \tag{2.11}$$

Das speziale explizite Schema (2.10) ist stabil, und integriert man das linearisierte von Newton Anfangswertproblem genau. Die Netzmethode wird im Grunde für die genauen Differenzenschemas für linearisierten Differentialgleichungen mit konstanten Koeffizienten gebraucht [3],[8]. Für die spezialen Differenzenapproximationen des Differentialoperators in einem nichtregulären Netz verwendet man die Integro-Interpolationsmethode. Die Differenschemes sind monoton, gut beschreiben die Grenzschichten und sind gleichmäßig konvergent, wenn der Koeffizient bei der Ableitung zweiter Ordnung zum Null, oder der Koeffizient bei der Ableitung erster Ordnung nach Unendlichkeit strebt.

Als Beispiel betrachten wir die Grenzwertaufgabe

$$\begin{cases} Lu = u'' + au' + bu = f(x) \\ u(0) = u_a, \quad u(1) = u_b \end{cases} \tag{2.12}$$

und die Approximation in einem nichtregulären Netz, welches die Netzpunkte $x_k \in [0,1]$, $k = \overline{1, N-1}$, $(x_0 = 0, x_N = 1)$ enthält. Hier sind $u_a, u_b \in \mathbb{K}$ und a, b, f - stückweise konstanten Funktionen im Segment $[0,1]$, welche in jedem Untersegmenten $[x_{k-1}, x_k], [x_k, x_{k+1}]$ die konstanten Werten $a_k^-, b_k^-, f_k^-; a_k^+, b_k^+, f_k^+$ abnehmen. In diesem Fall kann ein genaues Differenzenschema

$$\begin{cases} \Lambda u_k \equiv B_k(u_{k+1} - u_k) - A_k(u_k - u_{k-1}) - D_k u_k = \varphi_k \\ u_0 = u_a, \; u_N = u_b, \quad k = \overline{1, N-1} \end{cases} \tag{2.13}$$

konstruierte werden, wo u_k, φ_k die Netzfunktionswerten sind, welche $u(x_k), f(x_k)$ approximieren; aber A_k, B_k, D_k sind die Schemaskoeffizienten. Das Differenzenschema (2.13) ist monoton bei $A_k > 0, B_k > 0, D_k \geq 0$.

Abhängig vom Koeffizienten b in der Gleichung (2.12) kann man das genaue Differenzenschema (2.13) in zwei Weisen bekommen.

1. Sei $b = 0$, dann wird das Schema mit der Integro-Interpolationsmethode gewonnen, wenn man die Gleichung (2.12) in einer selbstkonjugierten Form geschrieben ist. Dann

$$B_k = (\hbar_k h_k^+)^{-1} s(-a_k^+ h_k^+), \; A_k = (\hbar_k h_k^-)^{-1} s(a_k^- h_k^-),$$

$$D_k = 0, \; \varphi_k = \hbar_k^{-1} [h_k^- r(h_k^- a_k^-) f_k^- + h_k^+ r(-h_k^+ a_k^+) f_k^+], \tag{2.14}$$

wo

$$S(z)=z(e^{z}-1)^{-1}>0,\ r(z)=z^{-1}(1-S(z)),$$

$$h_k^+=x_{k+1}-x_k,\ h_k^-=x_k-x_{k-1},\ \hbar_k=(h_k^++h_k^-)/2. \qquad (2.15)$$

Wenn $\varphi_k=f_k=f(x_k)$, dann hat das Differenzenschema (2.13) die zweite Approximationsordnung.

Im Falle des regulären Netzes bekommen wir das Differenzenschema von Iljin

$$\begin{cases} \gamma u_{x\bar{x}} + a u_{\overset{\circ}{x}} = f(x),\ x\in\omega_h \\ u(0)=u_a,\ u(1)=u_b \end{cases}, \qquad (2.16)$$

wobei

$$\gamma=(ah/2)\,\mathrm{cth}(ah/2)$$

- der Perturbationskoeffizient des Schemas ist.

Das Schema (2.16) ist genau , wenn a, f die konstanten Koeffizienten sind.

2. Wenn $b\neq 0$ dann kann man das genaue Schema mit der analytischen Lösung der Gleichungen (2.12) in jedem elementaren Untersegment $[x_{k-1},x_k]$ und $[x_k, x_{k+1}]$ bekommen durch die Fixierung der Netzfunktionswerten u_{k-1}, u_k, u_{k+1} am Ende der Segments. Das Differenzenschema ist monoton, wenn $b\leq 0$ ist. Im spezialen Falle $a=0$ bekommen wir den Algorithmus von N.Bachvalow in der Form $(b>0)$

$$\begin{cases} \gamma u_{x\bar{x}} + bu = f(x),\ x\in\omega_h \\ u(0)=u_a,\ u(1)=u_b \end{cases},$$

wo

$$\gamma=\left(\frac{\varkappa h/2}{\sin(\varkappa h/2)}\right)^2,\ \varkappa=\sqrt{b}.$$

Analogisch kann man das genaue Differenzenschema bekommen, wenn $f(x)$ eine stückweise-lineare Funktion in jeden Segment $[x_{k-1}, x_k]$, $k=\overline{1,N}$ ist [8].

Im Falle des Gleichungssystem (2.12), wenn u, u', f, u_a, u_b die m-dimensionalen Vektoren und a, b - die Matrizen der m-Ordnung sind das Differenzenschema (2.13) seine Form erhält, wenn $u_k, u_{k\pm1}, \varphi_k, f_k^-, f_k^+$ - die Vektoren, $z, a_k^+, a_k^-, B_k, A_k, D_k$ - die Matrizen,

$$s(z) = z(\exp(z) - E)^{-1}, \; r(z) = z^{-1}(E - s(z))$$

- die Matrixfunktionen sind (E ist die Einheitsmatrix der m-Ordnung). Wenn die Matrizen a_k^+, a_k^- symmetrisch sind, dann haben sie alle realen Eigenwerte λ und die Matrizen A_k, B_k haben positive Eigenwerte $(\hbar_k h_k^\pm)^{-1} s(\mp\lambda^\pm h_k^\pm)$, d.h. das Differenzenschema (2.13), (2.14) monoton ist.

In den Anwendungen spielt wichtige Rolle die Differentialgleichung

$$(\nu u')' + a u' = f(x), \tag{2.17}$$

wo $\nu > 0$. Das entsprechende monotone Differenzenschema wird in der Form (2.13) geschrieben [9], wo

$$B_k = (\hbar_k h_k^+)^{-1} \nu_k^+ s(-\alpha_k^+ h_k^+) > 0, \; A_k = (\hbar_k h_k^-)^{-1} \nu_k^- s(\alpha_k^- h_k^-) > 0,$$

$$D_k = 0, \; \varphi_k = f(x_k), \; \alpha = a\nu^{-1}. \tag{2.18}$$

Das Schema (2.13) ist genau, wenn die Koeffizienten ν, a stückweise Konstanten in jeden elementaren Segmente sind, und $f = const$.

Im Falle des Systems (2.17) ist ν eine positivdefinite Matrix. Wenn die Koeffizienten ν, a, b der Gleichungen (2.12), (2.17) stetig sind, dann kann man die Approximation dieser Koeffizienten mit stückweise-glatten Funktionen verwenden, zum Beispiel, der Koeffizient $a(x)$ im Segment $[x_{k-1}, x_k]$ kann mit dem Mittelwerte $a_k^- = (h_k^-)^{-1} \int_{x_k}^{x_{k+1}} a(x)\,dx$ approximiert werden.

2.2. Die speziellen Differenzenschemas zur Lösung der Randwertaufgaben für partielle Differentialgleichungen des elliptischen Typus

Wir betrachten die Randwertaufgaben der partiellen Differentialgleichung des elliptischen Typus

$$\begin{cases} Lu = f(x_1,x_2), & (x_1,x_2) \in \Omega \\ u = \mu(x_1,x_2), & (x_1,x_2) \in \partial\Omega \end{cases} \qquad (2.19)$$

im Rechteck

$$\Omega = \{(x_1,x_2): 0 < x_1 < l_1, 0 < x_2 < l_2\}$$

mit der Grenze $\partial\Omega$, wo

$$Lu \equiv \nu^{(1)}\partial^2 u/\partial x_1^2 + \nu^{(2)}\partial^2 u/\partial x_2^2 + a_1\partial u/\partial x_1 + a_2 \partial u/\partial x_2 + bu$$

- der Differentialoperator 2. Ordnung ist,
f, μ - stetige Funktionen sind, $b \le 0,\ \nu^{(1)} > 0,\ \nu^{(2)} > 0$.
Ersetzt man die Ableitungen im regulären Netz

$$\omega_h = \{(x_1^{(i_1)}, x_2^{(i_2)}): x_m^{(i_m)} = i_m h_m, i_m = \overline{0,N_m}, h_m N_m = l_m, m = 0,1\} \qquad (2.20)$$

durch die Differenzen, dann bekommt das monotone Differenzenschema zweiter Approximationsordnung

$$\begin{cases} \Lambda y = \varphi(x_1,x_2), & (x_1,x_2) \in \omega_h \\ y = \mu(x_1,x_2), & (x_1,x_2) \in \partial\omega_h \end{cases} , \qquad (2.21)$$

wo $\partial\omega_h = \omega_h \cap \partial\Omega$, $y = y(x_1,x_2)$, $\varphi = \varphi(x_1,x_2)$ - die Netzfunktionen sind, welche die stetige Funktionen in Punkten des Netzes ω_h approximieren.

$$\Lambda y \equiv \nu^{(1)} \delta_1 y_{x_1\bar{x}_1} + \nu^{(2)} \delta_2 y_{x_2\bar{x}_2} + a_1 y_{\overset{\circ}{x}_1} + a_2 y_{\overset{\circ}{x}_2} + by$$

- der Differenzenoperator mit 5-Punktschablon .

$\gamma_m = (R_m/2)\, cth(R_m/2)$ – die perturbationskoeffizienten des Schemas und $R_m = a_m h_m / \nu^{(m)}$ – die Netz-Reinolds-Zahl sind.

Das Differenzenschema (2.21) verallgemeinert das eindimensionale Schema (2.16) im Falle der zwei Dimensionen, wenn die Approximation der eindimensionalen Differentialausdrücken

$$L_m u \equiv \nu^{(m)} \partial^2 u / \partial x_m^2 + a_m \partial u / \partial x_m$$

mit den Differenzenausdrücken

$$\Lambda_m y \equiv \nu^{(m)} \gamma_m y_{x_m \bar{x}_m} + a_m y_{\overset{\circ}{x}_m}$$

angewendet werden $(m = 1, 2)$.

Das Differenzenschema in den zentralen Differenzen $(\gamma_1 = \gamma_2 = 1)$ ist monoton nur dann, wenn

$$\max_m |R_m| \leq 2. \qquad (2.22)$$

Das Schema (2.21) kann man in der vektorialen Form verallgemeinern, wobei a_1, a_2, b – die quadratischen Matrizen, u, f, y, φ – die Vektoren und γ_1, γ_2 – die Matrixfunktionen sind. Im Falle der konstanten Koeffizienten kann man beide Probleme (2.19), (2.21) analytisch mit der Hilfe der Fourier-Reihen lösen (die stetigen und diskreten Varianten) [17]. Die Lösungen sind für die Approbation der numerischen Algorythmen gültig.

Wenn man das zweidimensionale Analog der Gleichung (2.17) betrachtet

$$Lu \equiv \frac{\partial}{\partial x_1}\left(\nu^{(1)} \frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\nu^{(2)} \frac{\partial u}{\partial x_2}\right) + a_1 \frac{\partial u}{\partial x_1} + a_2 \frac{\partial u}{\partial x_2} = f(x_1, x_2), \qquad (2.23)$$

dann bekommt man mit der Approximation im nichtregulären Netz (die Netzpunkten $(x_1^{(i)}, x_2^{(j)})$) die Differenzengleichungen

$$\Lambda y_{ij} \equiv B_{ij}(u_{i+1,j} - y_{ij}) - A_{ij}(y_{ij} - y_{i-1,j}) + \tilde{B}_{ij}(y_{i,j+1} - y_{ij}) - \tilde{A}_{ij}(y_{ij} - y_{i,j-1}) = f_{ij}, \quad (2.24)$$

wo y_{ij}, f_{ij} die Netzfunktionswerte sind, welche die Werten $u(x_1^{(i)}, x_2^{(j)}), f(x_1^{(i)}, x_2^{(j)})$ approximieren, $A_{ij}, B_{ij}, \tilde{A}_{ij}, \tilde{B}_{ij}$ sind die Koeffizienten (das Differenzenschema ist monoton, wenn die Koeffizienten positiv sind),

$$B_{ij} = (\hbar_i h_{i+1})^{-1} S(-\alpha_{i+1/2,j} h_{i+1}) \nu^{(1)}_{i+1/2,j} > 0,$$

$$A_{ij} = (\hbar_i h_i)^{-1} S(\alpha_{i-1/2,j} h_i) \nu^{(1)}_{i-1/2,j} > 0,$$

$$\tilde{B}_{ij} = (\bar{g}_j g_{j+1})^{-1} S(-\tilde{\alpha}_{i,j+1/2} g_{j+1}) \nu^{(2)}_{i,j+1/2} > 0,$$

$$\tilde{A}_{ij} = (\bar{g}_j g_j)^{-1} S(\tilde{\alpha}_{i,j-1/2} g_j) \nu^{(2)}_{i,j-1/2} > 0.$$

Hier

$$\alpha = a_1/\nu^{(1)}, \tilde{\alpha} = a_2/\nu^{(2)}, h_i = x_1^{(i)} - x_1^{(i-1)}, \nu^{(1)} > 0, \nu^{(2)} > 0,$$

$$g_j = x_2^{(j)} - x_2^{(j-1)}, \hbar_i = (h_i + h_{i+1})/2, \bar{g}_j = (g_j + g_{j+1})/2,$$

und die Größen mit den Indexen $i \pm 1/2, j \pm 1/2$ bezeichnen die entsprechenden Mittelwerten der Netzfunktionen in den Segmenten

$$(x_1^{(i-1)}, x_1^{(i)}), \ (x_1^{(i)}, x_1^{(i+1)}),$$

$$(x_2^{(j-1)}, x_2^{(j)}), \ (x_2^{(j)}, x_2^{(j+1)}).$$

Die Algorythmen werden für n-Veränderlichen und für das System verallgemeinert, wobei $\nu^{(1)}, \nu^{(2)}, a_1, a_2, b$ - die Matrizen, u, f, φ - die Vektoren und S, γ_1, γ_2 - die Matrixfunktionen sind [6,9].

2.3. Die Lösungsmethoden für die Anfangs-Randwertaufgaben der Gleichungen des parabolischen Typus

Wir betrachten das nichtstationäre Problem

$$\begin{cases} \partial u/\partial t = Lu + f(x,t),\ x\in(0,l),\ t>0 \\ u|_{t=0} = \varphi(x),\ x\in[0,l] \\ u|_{x=0} = u|_{x=l} = 0,\ t\geq 0, \end{cases} \tag{2.25}$$

wo $Lu \equiv \nu\partial^2 u/\partial x^2 + a\,\partial u/\partial x$ - der Differentialausdruck und u, f, ν, a - die gegebenen Funktionen sind $(\nu>0)$.

Bei der Ausarbeitung der spezialen Methoden ist wichtig der Fall, wenn der Parameter $|a|$ groß oder ν -klein Parameter ist. Aus der monotonen Approximation (2.16) in einem regulären Netz (2.1) folgt die gerade Methode wie eines Anfangswertproblem für GDG

$$\begin{cases} du_i/dt = \gamma\nu u_{x_i\bar{x}_i} + a u_{\overset{\circ}{x}_i} + f_i(t) \\ u_0(t) = u_N(t) = 0,\ u_i(0) = \varphi_i, \end{cases} \tag{2.26}$$

wo

$$u_i(t) = u(x_i,t),\ f_i(t) = f(x_i,t),\ \varphi_i = \varphi(x_i),\ i=\overline{1,N-1}.$$

Das entsprechende 6-punkten Schablon-Differenzenshemas wird in der Form enthalten

$$\begin{cases} (y^{n+1}(x) - y^n(x))/\tau = \Lambda(\sigma y^{n+1}(x) + (1-\sigma)y^n(x)) + \bar{f}(x),\ x\in\omega_h \\ y^0(x) = \varphi(x),\ y^n(0) = y^n(l) = 0,\ n\geq 0, \end{cases} \tag{2.27}$$

wobei $\Lambda y \equiv \nu\gamma y_{x\bar{x}} + a y_{\overset{\circ}{x}}$ - der Differenzenausdruck, $0\leq\sigma\leq 1$ - der Schemasparameter ($\sigma=0$ - das explizite Schema, $\sigma=0{,}5$ - das Schema von Krank-Nikolson), $\bar{f}(x)$ - die Approximation der Funktion der rechten Seite $f(x,t)$ nach der $n+1/2$ -Schichten sind. Wenn die Koeffisienten ν, a konstant sind, dann kann man die Lösungen dieser Problemem analytisch mit der Fourier

Methode bekommen. Zu diesem Zweck wird das biortonormierende Operators $(-L)$, $(-\Lambda)$ – Eigenfunktionsystem

$$X_k(x)=\sqrt{\tfrac{2}{\ell}}\exp\left(-\frac{ax}{2\nu}\right)\sin\left(\frac{k\pi x}{\ell}\right),$$

$$\overline{X}_k(x)=\sqrt{\tfrac{2}{\ell}}\exp\left(\frac{ax}{2\nu}\right)\sin\left(\frac{k\pi x}{\ell}\right), \qquad (2.28)$$

benutzt.

Die Eigenwerten dieser Operatoren sind verschiedene

$$\lambda_k = a^2/(4\nu)+\nu(k\pi/\ell)^2,\ k=1,2,\dots$$

$$\lambda_k^h=\frac{a}{h}\left(ch\,\frac{ah}{2\nu}-\cos\frac{k\pi h}{\ell}\right)/sh\,\frac{ah}{2\nu}, \qquad (2.29)$$

$$k=\overline{1,N-1}\,.$$

Weil die Approximation des Operators Λ monoton ist, dann ist für das Schema (2.27) die Stabilitätsungleichung $\sigma\geq\sigma_0=0{,}5-(\tau\lambda^h_{N-1})^{-1}$ – erfüllt, aber wenn $\sigma=0{,}5$ ist, hat das Schema auch die zweite Approximationsordnung in der Zeit. Im Falle $\sigma=0$ kann man ein absolut stabilen Differenzenschema (2.27) mit zwei Parametern γ und $\tau=\tilde{\gamma}$ konstruieren, wo

$$\tilde{\gamma}=(1-\exp(-2c))/(2c),\ c=\gamma h^{-2}\nu.$$

Das folgt aus den Estimationen

$$\gamma\geqslant ah/(2\nu),\ \tilde{\gamma}\leqslant(2c)^{-1}.$$

Das explizite absolut stabile Schema kann man auch für die zweidimensionale Wärmeleitungsgleichung (2.25) bekommen, wenn

$$L=L_1+L_2\ ,\quad x=(x_1,x_2)\ ,$$

$$L_m u=\nu^{(m)}\partial^2 u/\partial x_m^2+a_m\,\partial u/\partial x_m\ ,\ \nu^{(m)}>0,$$

$$m=1,2.$$

Dann sind die Differenzengleichungen in der Form

$$(y^{n+1}(x)-y^n(x))/\tilde{\gamma}=\Lambda y^n(x)+\overline{f}(x),\ n\geqslant 0, \qquad (2.30)$$

wo
$$\Lambda = \Lambda_1 + \Lambda_2 , \quad x = (x_1, x_2) \in \omega_{h_1 h_2} ,$$
$$\Lambda_m y \equiv \nu^{(m)} \gamma_m y_{x_m \bar{x}_m} + a_m y_{\overset{\circ}{x}_m} , \quad m = 1, 2 ,$$
$$\gamma = (1 + \exp(-2C))/(2C), \quad C = (\nu^{(1)} \gamma_2 h_1^{-2} + \nu^{(2)} \gamma_2 h_2^{-2}).$$

Um das Anfangs-Randwertproblem der zweidimensionalen Wärmeleitungsgleichung zu lösen, werden die alternierenden Richtungsmethoden verwändet. Zum Beispiel, eine von diesen Methoden (die Funktionen sind nach der $n+1/2$ -Schicht ausgeschlossen) ist in der Form

$$(E - \tfrac{\tau}{2}\Lambda_1)(E - \tfrac{\tau}{2}\Lambda_2) \frac{y^{n+1}(x) - y^n(x)}{\tau} = \Lambda y^n(x) + \bar{f}(x), \quad (2.31)$$

wo E der Einheitsoperator ist.

3. Die Differenzenschemen zur Lösung der eindimensionalen Randwertaufgaben der Hydrodynamik und Magnetohydrodynamik

Im Abschnitt 2 werden die für die Lösung der GDG ausgearbeiteten monotonen Differenzenschemas für die Lösung der entsprechenden eindimensionalen Randwertprobleme der Hydrodynamik [9] und MHD [7,10] übertragen. Im beliebigen orthogonalen kurvelinieen Koordinatensystem (q_1, q_2) verwendet man bei der Differentialoperatorsapproximation mit den 5-Punktenshablons Differenzen die entsprechenden eindimensionalen Approximationen mit den 3-Punktenschablons Differenzen [9]. Wie im Falle des eindimensional Modelles des stationären Gleichungsystems (1.21-1.23) betrachten wir das System der GDG

$$\begin{cases} (b_1 u')' + a_1 u' + c w' + d H' = f_1 \\ (b_2 w')' + a_2 w' = f_2 \\ (b_3 H')' + a_3 H' = f_3 , \end{cases} \quad (3.1)$$

wo die Koeffizienten und die Funktionen von der Variable x

abhängig sind, $b_1>0, b_2>0, b_3>0$, $u'=du/dx$, u.s.w. Das System (3.1) kann man in der vektorialen Form (2.17) beschreiben, wo

$$\nu=\begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix}, \quad a=\begin{pmatrix} a_1 & c & d \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix},$$

die Matrizen sind, aber die Vektoren $\vec{u}$, $\vec{f}$ haben die Koordinaten u, W, H und f_1, f_2, f_3. Das entsprechende Vektordifferenzenschema hat die Form (2.13), (2.18), wo A_K, B_K, d_K – die Matrizen, aber u_K, φ_K – die Vektoren sind. Das Schema ist monoton, wenn die Matrizen $z= =\alpha h = a\nu^{-1}h$ reale Eigenwerte haben, aber die Eigenwerte der Matrizenfunktionen $s(z)$ positiv sind, d.i. $A_K>0$, $B_K>0$. Wenn $s(z)$ mit Hilfe der Lagransch-Silvester Interpolationsmethode berechnet wird, bekommen wir

$$s(z)=\begin{pmatrix} s(\lambda_1) & c b_2^{-1}\delta s(\lambda_1,\lambda_2)h & d b_3^{-1}\delta s(\lambda_1,\lambda_3)h \\ 0 & s(\lambda_2) & 0 \\ 0 & 0 & s(\lambda_3) \end{pmatrix}, \tag{3.2}$$

wo $\lambda_m = a_m b_m^{-1} h \; (m=1,2,3)$, $s(\lambda)=\lambda/(e^{\lambda}-1)$,

$$\delta s(\lambda,\bar\lambda) = (s(\lambda)-s(\bar\lambda))/(\lambda-\bar\lambda).$$

Darum

$$B_K=\frac{1}{h_K h_K^+}\begin{pmatrix} s(-\lambda_1^+)b_1^+ & -c^+\delta s(-\lambda_1^+,-\lambda_2^+)h_K^+ & -d^+\delta s(-\lambda_1^+,-\lambda_3^+)h_K^+ \\ 0 & s(-\lambda_2^+)b_2^+ & 0 \\ 0 & 0 & s(-\lambda_3^+)b_3^+ \end{pmatrix},$$

$$A_K=\frac{1}{h_K h_K^-}\begin{pmatrix} s(\lambda_1^-)b_1^- & c^-\delta s(\lambda_1^-,\lambda_2^-)h_K^- & d^-\delta s(\lambda_1^-,\lambda_3^-)h_K^- \\ 0 & s(\lambda_2^-)b_2^- & 0 \\ 0 & 0 & s(\lambda_3^-)b_3^- \end{pmatrix}, \tag{3.3}$$

mit $\lambda_m^+ = a_m^+ h_k^+ / b_m^+$, $\lambda_m^- = a_m^- h_k^- / b_m^-$ $(m=1,2,3)$.

Wenn $\bar{\lambda} \to \lambda$, dann $\delta s(\lambda,\bar{\lambda}) \to s'(\lambda)$.

Im Falle der konstanten Koeffizienten und des regulären Netzes kann der Differenzenausdruck $\Lambda \vec{u}_k$ in folgender Form geschrieben werden

$$\Lambda \vec{u}_k = h^{-2}[s(-z)\nu(\vec{u}_{k+1} - \vec{u}_k \pm (\vec{u}_{k+1} + \vec{u}_{k-1})/2 -$$
$$- s(z)\nu(\vec{u}_k - \vec{u}_{k-1} \pm (\vec{u}_{k+1} + \vec{u}_{k-1})/2] = (s(-z)+s(z)) \cdot$$
$$\cdot \nu \vec{u}_{x\bar{x}_k}/2 + (s(-z)-s(z)) h^{-1} \nu \vec{u}_{\overset{\circ}{x}_k} = \gamma(z)\nu \vec{u}_{x\bar{x}_k} + a\vec{u}_{\overset{\circ}{x}_k},$$

wo $z = a\nu^{-1}h$, $s(-z)-s(z) = z$, $(s(-z)+s(z))/2 = \gamma(z) =$
$=(z/2)\,\mathrm{cth}(z/2)$ - der Perturbationskoeffizient des Schemas ist. Mit der Berechnung $\gamma(z)$ aus dem Spektrum der Matrix z bekommen wir

$$\gamma(z) = \begin{pmatrix} \gamma_1 & c b_2^{-1} \delta\gamma(\lambda_1,\lambda_2) h & d b_3^{-1} \gamma(\lambda_1,\lambda_3) h \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \gamma_3 \end{pmatrix}, \qquad (3.4)$$

wo
$\gamma_m = \gamma(\lambda_m)$, $m=1,2,3$, $\delta\gamma(\lambda,\bar{\lambda}) = (\gamma(\lambda)-\gamma(\bar{\lambda}))/(\lambda-\bar{\lambda})$.

Damit können die Differenzengleichungen zur Lösung des Systems (3.1) im regulären Netz durch folgende Gleichungen dargestellt werden

$$\begin{cases} b_1\gamma_1 u_{x\bar{x}} + a_1 u_{\overset{\circ}{x}} + c w_{\overset{\circ}{x}} + d H_{\overset{\circ}{x}} + ch\,\delta\gamma(\lambda_1,\lambda_2) W_{x\bar{x}} + dh\,\delta\gamma(\lambda_1,\lambda_3) H_{x\bar{x}} = f_1 \\ b_2\gamma_2 W_{x\bar{x}} + a_2 W_{\overset{\circ}{x}} = f_2 \\ b_3\gamma_3 H_{x\bar{x}} + a_3 H_{\overset{\circ}{x}} = f_3 . \end{cases} \qquad (3.5)$$

In der Arbeit [10] wird die Überlegenheit der Differenzengleichungen (3.5) mit den Perturbationskoeffizienten γ_m, δ_j im Vergleich mit klassichen Approximationen bezeigt, wenn der Parameter C groß ist. Die Gleichungen (3.5) unterscheiden sich von den standarten Differenzengleichungen von A.Iljin mit den Koeffizienten δ_j, welche im Falle der konstanten Koeffizienten b_m, a_m, f_m $(m=1,2,3)$, c, d und der Randbedingung erster Art der Konstruktion genaue Differenzenschemas sichern. Das genaue Differenzenschema (2.13),(2.28),(3.3) kann man auf das nichtreguläre Netz übertragen, wenn die Koeffizienten stückweise konstant sind. Im Falle der veränderlichen Koeffizienten hat das Differenzenschema mindestens die zweite Approximationsordnung, und sie konvergiert gleichmäßig, wenn a_1, a_2, a_3 oder $C, d \to \infty$.
In den Arbeiten [15,16,17] wurden andere eindimensionale hydrodynamische und MHD Modellproblemen betrachtet. Wir merken den Fall an, wenn die Matrix in der Form

$$a = \begin{pmatrix} a_1 & c & d \\ 0 & a_2 & e \\ 0 & 0 & a_3 \end{pmatrix}$$

ist, oder die zweite Gleichung (3.1) in der Form $(b_2 w')' + a_2 w' + eH' = f_2$ ist. Dann hat die Matrix $S(z)$ (3.2) folgende Form

$$S(z) = \begin{pmatrix} S(\lambda_1) & c b_2^{-1} \delta S(\lambda_1,\lambda_2) h & d b_3^{-1} \delta S(\lambda_1,\lambda_3) h \\ 0 & S(\lambda_2) & e b_3^{-1} \delta S(\lambda_2,\lambda_3) h \\ 0 & 0 & S(\lambda_3) \end{pmatrix}.$$

4. Die Eigenartigkeiten der Realisierung der numerischen Methoden zur Lösung der Randwertaufgaben

Die Differenzenschemas können nach folgenden Merkmalen klassifiziert werden: 1) die Approximationsweise des Hauptdifferentialoperators im Raum (die einseitigen und zentralen Differenzen, die monotonen und genauen Approximationen); 2) die Approximationsweise des Differentialoperators in der Zeit (explizite, implizite und halbexplizite Approximation); 3) die Approximationen der Grenzbedingungen; 4) die Lösungsmethoden des Differenzenschemaverfahrens (die Stationärisierungsmethode, die alternativen-Richtungsmethoden, die iterativen Methoden). In der letzten Zeit rechnet man die Problemen der numerischen Hydrodynamik und MHD bei genügend großen Parametern (Re, Ha, Gr - Zahlen). Deshalb müssen spezialen Netzmethoden und andere numerischen Verfahren ausgearbeitet werden, welche mit folgenden Eigenarten sich charaktisieren:
1) die Anwendung eines nicht regulären Netzes mit der Möglichkeit des Netz dicht zusammenpressen in den Gebieten mit großen Lösungsgradienten; 2) die monotone Approximation zweiter Ordnung der konvektiven Gliedern; 3) die effektive Anwendung des impliziten Schemas, was die Lösung der mehrdimensionalen Problemen stufenweise auf die Lösung der eindimensionalen Problemen reduziert; 4) die Verbesserung der Approximationsmethoden der Grenzbedingungen, zum Beispiel, für die Wirbelfunktion; 5) die Lösung des Differenzengleichungssystems mit der verhöhenen Genauigkeit.

Effektiv sind die impliziten Vektordifferenzenschemas und die lokalen Relaxationslösungsmethoden für das Gleichungssystem in der $(n+1)$ Zeitschicht. Der optimale Relaxationsparameter ist in jedem Netzpunkt der Differenzengleichungen mit konstanten Koeffizienten zu bestimmen. Die gute Modellgleichung für die Bestimmung des optimalen Relaxationsparameters ist die lineare Gleichung (2.19)($\varepsilon=0$)

und seine Differenzenanalog (2.21) [11,13]. Wir betrachten den Iterationsprozeß mit dem Relaxationsparameter ω^r $(1 \le \omega^r \le 2)$ in der Form

$$y_{ij}^{(k+1)} = (1-\omega^r) y_{ij}^{(k)} + \frac{\omega^r}{2(\gamma_1+\gamma_2)} \Big[(\gamma_1 - R_1/2) y_{i-1,j}^{(k+1)} + (\gamma_1 + R_1/2) y_{i+1,j}^{(k)} + (\gamma_2 - R_2/2) y_{i,j-1}^{(k+1)} + (\gamma_2 + R_2/2) y_{i,j+1}^{(k)} - h^2 \varphi_{ij} \Big], \quad i = \overline{1, N_1-1}, \; j = \overline{1, N_2-1}, \tag{4.1}$$

wo $y_{ij}^{(k)}$ - die Netzfunktionswerten in der k -ten Iteration $k = 0,1,2,\dots$, $h_1 = h_2 = h$, $y_{ij}^{(0)}$ die gegebene Anfangsannäherung ist.

Die Differenz $z_{ij}^{(k)} = y_{ij}^{(k)} - y_{ij}$ hat die homogenen Gleichungen (4.3) $(\varphi_{ij} = 0)$, wo der optimale Relaxationsparameter ω_0^r in der Form ist [13,18]

$$\omega_0^r = 2/(1 + \sqrt{1 - \eta_0^2}), \tag{4.2}$$

wobei

$$\eta_0 = (\gamma_1+\gamma_2)^{-1} \Big[\sqrt{\gamma_1^2 - R_1^2/4} \cos\frac{\pi}{N_1} + \sqrt{\gamma_2^2 - R_2^2/4} \cos\frac{\pi}{N_2} \Big] < 1.$$

Wenn $\gamma_m = 1$ ist, dann ist die Formel (4.4) nur dann gültig, wenn $|R_m| \le 2$ $(m = 1,2)$ ist. Diese Formel verallgemeinert die in der Literatur bekannten Formeln von Jung, Rassel, Takemitsu, Strikverda, Botte und Veldmen. Die Berechnungsresultaten zeigen, daß die Differenzenschemus von Iljin den Vorzug im Vergleich zu den zentralen und einseitigen Differenzenmethoden haben, Wenn die Reinoldszahlwerten R_1, R_2 des Netzes wachsen, dann streben die Parameterwerten ω_0^r zu Null (dib Methode von Seidel).

Zur Lösung der Anfangs-Randwertproblemen der mathematischen Physik und MHD spielt wichtige Rolle die Stabilitätsbedingungen und die Genauigkeit der Differenzenschema [1]

Hier kann man sehen, daß die absolut stabilen impliziten Differenzenschemas (die Methoden von Krank-Nikolson, Djufort-Frankel, Pismen-Rekford, Duglas-Rekford) die Genauigkeit für die Lösung der nichtstationären Aufgaben nicht garantieren. Bei der Auflösung der hydrodynamischen und MHD Problemen ist es notwendig die beiden Spektren (der stetigen und diskreten Aufgaben) zu vergleichen. Das hilft die verschiedenen Stabilitäts-Prozessen (numerische und hydrodynamische) miteinander zu vergleichen und die stabilen Algorithmen auszuarbeiten.

Für die Gleichung (1.7) ist das explizite Differenzenschema mit der monotonen Approximation (2.30) in der Form

$$(\omega^{n+1}-\omega^n)/\tau+u\omega^n_{\overset{\circ}{x}}+v\omega^n_{\overset{\circ}{y}}=Re^{-1}(\gamma_1\omega^n_{x\bar{x}}+\gamma_2\omega^n_{y\bar{y}})+f, \quad (4.3)$$

wo $\gamma_m=(R_m/2)\,cth(R_m/2)$, $m=1,2$,

$R_1=|u|h_1 Re$, $R_2=|v|h_2 Re$, $n\geq 0$.

Die Stabilitätsanalyse basiert auf dem Maximumprinzip, aus dem die Stabilitätsungleichungen folgen

$$\gamma_m \geq R_m/2 \quad (m=1,2), \quad (4.4)$$

$$\tau\leq \frac{1}{2}Re/(\gamma_1 h_1^{-2}+\gamma_2 h_2^{-2}). \quad (4.5)$$

Für das klassische Schema mit der zentralen Differenzen $(\gamma_1=\gamma_2=1)$ folgt von (4.4) die wesentliche Ungleichung $R_m\leq 2$. Für das monotone Schema ist die Ungleichung (4.4) automatisch erfüllt. Die Einchränkungen (4.5) in bezug auf den Zeitschritt τ können auch fortgenommen werden, wenn im monotonen Schema (2.30) $\tau=\tilde{\gamma}$ ist, und bekommen wir eine explizites absolut stabiles Differenzenschema (4.3).

Die Gleichungen (1.7) des parametrischen Schemas mit zwei Schichten sind in der Form

$$(\omega^{n+1}-\omega^n)/\tau=Re^{-1}[\sigma\Lambda\omega^{n+1}+(1-\sigma)\Lambda\omega^n]+f,\; n\geq 0; \quad (4.6)$$

wo

$$\Lambda = \Lambda_1 + \Lambda_2 ,$$

$$\Lambda_1 \omega = \gamma_1 \omega_{x\bar{x}} - Re\, u \omega^{o}_{x} , \quad \Lambda_2 \omega = \gamma_2 \omega_{y\bar{y}} - Re\, v \omega^{o}_{y} ,$$

$$0 \leq \sigma \leq 1 .$$

Aus dem Maximumprinzip folgt die Ungleichung (4.4) und die Ungleichung

$$\tau \leq \frac{1}{2} Re / \left((1-\sigma)(\gamma_1 h_1^{-2} + \gamma_2 h_2^{-2}) \right) . \qquad (4.7)$$

Die stationäre Lösung der Differenzengleichungen $(\sigma \neq 0)$ kann mit der absolut stabilen Iterationsmethode von Seidel oder mit der obene Relaxationsmethode mit den optimalen Parametern ω_0^r (4.2) realisiert werden. Für das Schema der zentralen Differenzen gilt nur die untere Relaxationsmethode $(\omega^r < 1)$.

ZUSAMMENFASSUNG

Aus der Basis der numerischen Untersuchungen im Gebiet der mathematischen Physik, Hydrodynamik und MHD der zähen inkompressiblen elektroleitenden Flüssigkeiten kann man folgende Schlüssen ziehen.

1. Die konstruierten Differenzenschemas für die Lösung der Grenzwertprobleme können in weiten Parameterwechselgebisten verwendet werden. Die Differenzenschemas sind monoton, sie beschreiben gut die Grenzschichten und approximieren gleichmäßig zumindest mit der Genauigkeit der zweiten Ordnung die entsprechenden Grenzproblemen.

2. Die Effektivität der enthalteten spezialen Methoden ist für die Lösung der Randwertproblemen den Gleichungen mit großen Parametern bei den Ableitungen erster Ordnung begründet. Die optimalen Koeffizienten der Relaxationsmethode und die Stabilitätsbedingungen sind für die monotonen Differenzenschemen bekommen.

3. Die ausgearbeitenen Methoden zusammen mit den experimentalen Untersuchungen ermöglichen die mathematischen Modelle zu schaffen und die Eigenschaften dieser Modelle abhängig von den Parametern zu bekommen.

WICHTIGSTE BEZEICHNUNGEN

MHD - die Magnetohydrodynamik.

GDG - die gewöhnlichen Differentialgleichungen.

t - der Zeitparameter.

$\vec{v}, \vec{\omega}$ - die Geschwindigkeit und Wirbel-Vektoren.

$V_x = u,\ V_y = v,\ V_z = w$ - die Geschwindigkeitsvektorkomponenten in kartesischen Koordinaten.

$\omega = \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} = -\Delta\psi$ - die z-Wirbelvektorkomponente oder die Wirbelfunktion.

$p, \Pi = p + \rho|\vec{v}|^2/2$ - der Flüssigkeitsdruck und der volle Druck.

$\vec{F}, \vec{f}, \vec{F}^e$ - die äußeren Kraftvektoren und der elektromagnetische Kraftvektor $\vec{F}^e = \rho^{-1}(\vec{j} \times \vec{B})$.

$\vec{B}, \vec{H}, \vec{E}$ - die Magnetfeldsinduktions-Intensivitäts- und Elektrischfeldsvektoren.

$\vec{j}, \vec{g}$ - die Strömungsdichte- und Schwerkraftbeschleunigungsvektoren.

I, T, Q - die elektrische Strömung, die Mediumtemperatur und die Quellefunktion der Wärme.

$\rho, \sigma, \nu, \eta, \beta, c_p, \lambda, \mu, \nu_m, \chi$ - die Koeffizienten der Dichte, des Elektroleitvermögens, der kinematischen und dynamischen Zähigkeiten, der Wärmeverbreiterung, der Wärmekapazität des Wärmeleitvermögens, der Magnetpermabilität, der Magnetviskosität und des Temperaturleitvermögens ($\eta = \nu\rho$, $\nu_m = (\mu\sigma)^{-1}$, $\chi = \lambda(\rho c_p)^{-1}$)

V_0, l_0, B_0, T_0 - die charakterisierenden Größen der Geschwindigkeit, der Länge, der Magnetfeldinduktion und Temperatur.

$Re = \frac{V_0 L_0}{\nu}, Re_m = \frac{V_0 L_0}{\nu_m}$ - Reynolds- und Reynoldsmagnetzahlen.

$Ha = B_0 L_0 \sqrt{\sigma/\eta}$, $Al = B_0^2/(\mu \rho V_0^2)$ - Hartmann- und Alsvenzahlen.

$S = Al \cdot Re_m$, $Gr = \beta g L_0^3 (T - T_{max}) \nu^{-2}$ - Stjuart- und Grasgofzahlen.

$S^* = \Gamma_k^2 Re\, Re_m$ - der Elektrowirbelströmungparameter.

$\Gamma_k = (V_\theta)_{max} / V_0$ - der Strömungsrotationkoeffizient .

$Pr = \nu/\chi$, $Pe = Gr \cdot Pr$ - Prantl- und Peklezahlen.

α_0 - der Winkel zwischen der Richtung des äußeren Magnetfeldes und der Ox -Koordinatenachse.

L , $\nabla = grad, div, \Delta$ - Differential-, Gradient-, Divergenz- und Laplasoperatore.

$D/Dt = \partial/\partial t + (\vec{V} \cdot grad)$ - die volle Ableitung zur Zeit.

$\partial/\partial n$ - der Normalableitungsoperator in bezug auf die äußere Grenznormale.

$q_{x\bar{x}}, q_{\overset{\circ}{x}}$ - die zentralen Differenzen zweiter und erster Ordnung.

τ, h - die gleichmäßigen Netzschritten zur Zeit und im Raum.

$R_h = qh/\nu$ - Reynoldszahl des "Netzes" mit der Geschwindigkeitskomponente q.

γ - der Differenzenschemaperturbationskoeffizient:

1) $\gamma = R_h/2 \; cth\, R_h/2$ - das Differenzenschema von Iljin,

2) $\gamma = R_h/2 + (1 + R_h/2)^{-1}$ - das Differenzenschema von Samarski,

3) $\gamma = 1 + R_h/2$ - das Differenzenschema gegen den Strom, oder die Upwind-Differenzen,

4) $\gamma = 1$ - das klassiche Schema mit zentralen Differenzen.

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H.Kalis. Speciālu skaitlisko metožu izstrāde un pielietošana matemātiskās fizikas, hidrodināmikas un magnētiskās hidrodināmikas problēmu risināšanā.

Anotācija. Galvenie habilitācijas darba rezultāti hab. Dr.mat. grāda iegūšanai ir speciālu diferenču shēmu konstruēšana matemātiskās fizikas robežproblēmu risināšanai plašā parametru izmaiņas diapazonā. Diferenču shēmas ir monotonas, labi apraksta robežslāņus un vienmērīgi vismaz ar otrās kārtas precizitāti aproksimē attiecīgās nepārtrauktās robežproblēmas. Parādīta iegūto speciālo metožu efektivitāte, risinot robežproblēmas, kuru vienādojumos ir lieli parametri pie pirmās kārtas atvasinājumiem vai mazi parametri pie augstākās (otrās) kārtas atvasinājumiem. Noteikti optimālie relaksācijas metodes koeficienti un stabilitātes nosacījumi monotonām diferenču shēmām. Izstrādātās skaitliskās metodes kopā ar eksperimentāliem pētījumiem deva iespēju radīt nesaspiežamu elektrovadošu viskozu šķidrumu plūsmu matemātiskos modeļus un izpētīt to īpašības atkarībā no parametriem.

Х.Калис. Разработка и применение специальных численных методов для решения задач математической физики, гидродинамики и магнитной гидродинамики.

Аннотация. Основным результатом хабилитационной работы для соискания степени хаб.доктора по математике является разработка специальных монотонных конечно-разностных схем для решения системы уравнений математической физики при наличии малых параметров при старших производных или больших параметров при младших производных. УДК 538.4.

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LATVIAN MATHEMATICAL SOCIETY

The Latvian Mathematical Society was founded on January 15, 1993(registered on March 5, 1993). This is a voluntary organisation uniting 66 members - about a half of all working mathematicians in the country.

Among the main purposes of the LMS are:
to support researches in mathematics and promote mathematical education in the country;
to contact with international professional mathematical organizations and to represent interests of Latvian mathematicians.

The Latvian Mathematical Society is headed by the Board elected by the Common Meeting of the members of the Society. At present the Board consists of 7 members: Dr. Hab. Math. Uldis Raitums (the Chief of the Society), Dr. Math. A. Reinfelds (the Vice-Chief of the Society), Dr. Hab. Math. A. Šostak (the Secretary of the Society), Dr. Hab. Math. J. Carkov, Dr. Hab. Math. I. Strazdiņš, Dr. Math. J. Cīrulis (the Cashier of the Society) and Dr. Math. A. Andžans.

The LMS establishes business-like cooperation with similar organizations in other countries of Europe and in the world. A useful cooperation is being established with the Estonian Mathematical Society, Polish Mathematical Society, Finnish Mathematical Society, Mathematical Society of Denmark and some others. This cooperation envisages, in particular, exchange of actual information concerning the activities of our societies, important events in mathematical life in the countries; exchange of information about the main directions of scientific interest and researches of working mathematicians in the countries; the exchange of scientific and popular literature published with the support of the Societies as well, as with other mathematical publications printed in the countries.

The LMS organizes lectures devoted to different actual directions of research in Mathematics. In particular, on May 19, 1993 Dr. Math. A. Reinfelds presented a lecture in which he has considered some problems of the qualitative theory of dynamic systems. The LMS is planning to publish the contents of these lectures in a special series of preprints.

In future, the Journal "Transactions of the Latvian Mathematical Society" is planned to be published, too.

The LMS starts the work on the Latvian-English-German-Russian vocabulary of mathematical terminology. This work is coordinated by professor I. Strazdiņš.

U. Raitums, the Chairman of the Latvian Mathematical Society

A. Šostaks, the Secretary of the Latvian Mathematical Society

MATEMĀTIKA

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