Relaxation of polar order in suspensions with Quincke effect

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The Quincke effect—spontaneous rotation of dielectric particles in a liquid with low conductivity under the action of an electric field—is considered. The distribution functions for the orientation of particle rotation planes are introduced and a set of nonlinear kinetic equations is derived in the mean field approximation considering the dynamics of their orientation in the flow induced by rotating particles. As a result the nonequilibrium phase transition to the polar order, if the concentration of the particles is sufficiently high, is predicted and the condition of the synchronization of particle rotations is established. Two cases are considered: the layer of the Quincke suspension with one free boundary and the ensemble of the particles rolling on the solid wall under the action of a torque in an electric field. It is shown that in both cases the synchronization of particle rotations occurs due to the hydrodynamic interactions. In the limit of small spatial nonhomogeneity a set of nonlinear partial differential equations for the macroscopic variables—the concentration and the director of the polar order—is derived from the kinetic equation. Its properties are analyzed and compared with available recent experimental results.

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I. INTRODUCTION

Active systems with energy supplies on the level of their single elements have been recently studied in many works (for reviews, see Refs. [1–3]). A characteristic feature of such systems is the development of macroscopic flow due to hydrodynamic interactions [4,5]. An interesting example is a suspension of dielectric particles in a liquid with low conductivity. Due to the polarization of particles caused by the accumulation of free charges on their interface the suspension has a negative low-frequency polarizability, and if the electric field strength is above a threshold value the particles start to spontaneously rotate. This phenomenon, known as the Quincke effect (from work by Quincke [6]), has been studied for more than a century. Its description is based on the model of leaky dielectrics [7] (for a review, see Ref. [8]). Different phenomena have been predicted theoretically and confirmed experimentally: the negative effective viscosity effect [9,10], the increase of the conductivity of a suspension at the electric field strength above the threshold value of spontaneous rotation of dielectric particles [11,12], the chaotic dynamic of particles [13–16], and others. The role of electrohydrodynamic interactions in the Quincke effect has been considered in Refs. [5,17]. Nevertheless the basic laws responsible for the collective behavior of particles in such suspensions remain to be formulated. Here we describe the synchronization of the particle rotations due to their induced flow and formulate a set of macroscopic equations for the slow variables of the system: the concentration and the direction of polar order.

In Sec. II, the model of Quincke suspension, which considers the antisymmetric viscous stress, is introduced. In Sec. III, the kinetic equation for the distribution function of the planes of particle rotation is derived and the establishment of the polar order due to the flow induced by the rotating particles is illustrated by the numerical solution of the kinetic equation. In Sec. IV, it is shown that description of the suspension developed in Secs. II and III is applicable to the ensemble of particles rolling near the solid wall under the action of a torque, if their hydrodynamic interactions is properly taken into account. The continuum model of the suspension of active rotating particles is developed in Sec. V, where the equations for the macroscopic variables—the surface density and the director of the polar order—are derived on the basis of the kinetic equation. Some consequences of the derived macroscopic equations are considered in Sec. VI. In particular it is illustrated that the set of coupled equations for the particle concentration and the director of the polar order in the limit of small perturbations may be reduced to the Klein-Gordon equation in the unstable case.

II. MODEL

The key issue is the equation for the dielectric particle polarization $\vec{p}$ in a liquid with a low conductivity derived in Ref. [18]:

$$\frac{d\vec{p}}{dt} = \vec{\Omega} \times \vec{p} - \frac{1}{\tau}(\vec{p} - \chi \vec{E}), \tag{1}$$

where $\tau = (2\varepsilon_1 + \varepsilon_2)/[4\pi(2\sigma_1 + \sigma_2)]$ is the Maxwell charge relaxation time, $\chi = \chi^0 - \chi^\infty$, where $\chi^0 = \varepsilon_1(\sigma_2 - \sigma_1)a^3/(2\sigma_1 + \sigma_2)$ and $\chi^\infty = \varepsilon_1(\varepsilon_2 - \varepsilon_1)a^3/(2\varepsilon_1 + \varepsilon_2)$ are low- and high-frequency polarizibilities of the spherical particle with the radius a ($\varepsilon_{1,2}$ and $\sigma_{1,2}$ are the dielectric permeability and the conductivity of the liquid and the particle respectively). The angular velocity of the particle $\vec{\Omega}$, if the rotational inertia of the particle is neglected, reads

$$\vec{\Omega} = \vec{\Omega}_0 + \alpha_r \vec{p} \times \vec{E}. \tag{2}$$

Here $\vec{\Omega}_0 = 1/2\vec{\nabla} \times \vec{v}$ is the local angular velocity of the fluid and α_r is the rotational mobility of the particle. If the rotational inertia is taken into account, after corresponding scaling, the famous Lorentz equations are obtained. The chaotic dynamics of particle due to the Quincke effect was experimentally confirmed in Refs. [14,15].

In the case of a quiescent fluid ($\vec{\Omega}_0 = 0$), Eqs. (1) and (2) predict a pitchfork bifurcation to a steady state of spontaneous

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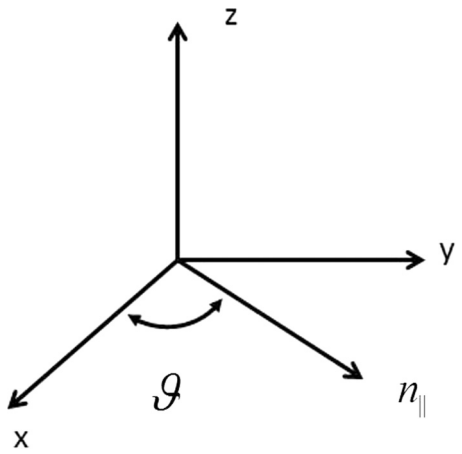

FIG. 1. Sketch of orientation of plane of particle rotation.

rotation in electric field $\vec{E} = (0,0,E)$, when

$$p_z^0 = -\frac{1}{\alpha_r \tau E},$$
$$p_\parallel^0 = \pm\frac{1}{\alpha_r \tau E}\sqrt{\frac{E^2}{E_c^2} - 1},$$
$$\Omega = \pm\frac{1}{\tau}\sqrt{\frac{E^2}{E_c^2} - 1}, \tag{3}$$

where $E_c^2 = -\frac{1}{\alpha_r \tau \chi}$ ($\chi < 0$). In this state the particle rotates clockwise ($p_\parallel^0 < 0$) or anticlockwise ($p_\parallel^0 > 0$) in the plane $(\vec{n}_\parallel, z)$, where $\vec{n}_\parallel = \cos(\vartheta)\vec{e}_x + \sin(\vartheta)\vec{e}_y$ (see Fig. 1).

The direction of the vector $\vec{n}_\parallel$, which determines the plane where the particles rotate, remains undetermined. The particular direction may be chosen due to the hydrodynamic interactions. As an example, we consider a dielectric suspension in the liquid layer confined between two plates with $z = 0$ and $z = d$ with the upper plate free to move in the y-axis direction. The conservation law of the angular momentum ($l_i = e_{ikl} x_k v_l$):

$$\rho \frac{dl_i}{dt} = \frac{\partial}{\partial x_k}(e_{iml} x_m \sigma_{lk}) + e_{ikl}\sigma_{kl} \tag{4}$$

shows that, if there is a volume density of torques acting on the suspension, its stress tensor σ_{ik} contains an antisymmetric part σ_{ik}^a, which is determined by the volume density of torques $\vec{T}$:

$$\sigma_{ik}^a = \tfrac{1}{2} e_{ikl} T_l. \tag{5}$$

As a result the boundary condition on the free plate reads (η is the viscosity of suspension)

$$\eta \frac{dv_y}{dz} + \sigma_{yz}^a = 0. \tag{6}$$

III. RELAXATION OF POLAR ORDER

Equations (1) and (2) in the shear flow $\vec{v} = [0, v_y(z), 0]$ for $\vec{p} = [p_\parallel \cos(\vartheta), p_\parallel \sin(\vartheta), p_z]$ give the set of equations

$$\frac{dp_\parallel}{dt} = \frac{1}{2}\frac{dv_y}{dz} p_z \sin(\vartheta) - \alpha_r E p_\parallel p_z - \frac{1}{\tau} p_\parallel; \tag{7}$$

$$\frac{dp_z}{dt} = -\frac{1}{2}\frac{dv_y}{dz} p_\parallel \sin(\vartheta) + \alpha_r E p_\parallel^2 - \frac{1}{\tau}(p_z - \chi E); \tag{8}$$

$$p_\parallel \frac{d\vartheta}{dt} = \frac{1}{2}\frac{dv_y}{dz} p_z \cos(\vartheta). \tag{9}$$

Here to describe the flow induced by rotating particles in the geometry considered in the Sec. II it is convenient to introduce the angle γ ($\vartheta = \pi/2 + \gamma$), which the plane of the particle rotation makes with a symmetry-breaking yz plane (see Fig. 1). It is checked numerically that at separation of slow (characteristic time scale $|dv_y/dz|^{-1}$) and fast time (scale τ) scales the equations for the change of orientation angles $\gamma^\mp (\mp p_\parallel > 0)$ may be obtained by taking in Eq. (9) $p_\parallel = p_\parallel^0$ and $p_z = p_z^0$, which gives

$$\frac{d\gamma^\mp}{dt} = \mp\frac{1}{2\sqrt{E^2/E_c^2 - 1}}\frac{dv_y}{dz}\sin(\gamma^\mp). \tag{10}$$

We see that under the action of the flow the orientation of the plane of the particle rotation changes similarly to the orientation of the compass needle in the applied magnetic field. In this case the effective applied field is proportional to the shear rate. According to Eq. (10) the particle rotating clockwise ($p_\parallel^0 < 0$) goes to $\gamma = 0$ and the particle rotating anticlockwise ($p_\parallel^0 > 0$) goes to $\gamma = \pi$.

Contribution of a single particle to the antisymmetric stress σ_{yz}^a according to the relation (5) is $1/2 p_\parallel^0 E \cos(\gamma)$. Introducing distribution functions $f_\pm(\gamma,t)$ ($f_-(\gamma,t)d\gamma$; ($p_\parallel^0 < 0$) and $f_+(\gamma,t)d\gamma$; ($p_\parallel^0 > 0$) are the numbers of rotating particles per unit volume with $\vec{n}_\parallel(\pi/2 + \gamma)$, where γ is in the interval $[\gamma, \gamma + d\gamma]$) and the contribution of all particles of the suspension to the antisymmetric stress is given by

$$\sigma_{yz}^a = -\frac{1}{2}|p_\parallel^0| \int_0^{2\pi} \cos(\gamma') f_-(\gamma',t) d\gamma' + \frac{1}{2}|p_\parallel^0| \int_0^{2\pi} \cos(\gamma') f_+(\gamma',t) d\gamma'. \tag{11}$$

As a result for the shear rate of induced flow we have

$$\frac{dv_y}{dz} = \frac{1}{2\eta}|p_\parallel^0| E \int_0^{2\pi} \cos(\gamma')[f_-(\gamma',t) - f_+(\gamma',t)] d\gamma'. \tag{12}$$

By taking into account Eq. (10), the set of kinetic equations for the distribution functions $f_\mp$ is obtained as

$$\frac{\partial f_\mp}{\partial t} \mp \frac{\partial}{\partial \gamma}\left(\frac{1}{\alpha_r 4\eta\tau}\sin(\gamma) f_\mp \int_0^{2\pi} \cos(\gamma')[f_-(\gamma',t) - f_+(\gamma',t)] d\gamma'\right) = D_r \frac{\partial^2 f_\mp}{\partial \gamma^2}, \tag{13}$$

where the random noise is accounted for by introduction of a rotational diffusion coefficient D_r.

Eq. (13) in the mean field approximation describes the kinetics of the transition to the polar order due to the flow

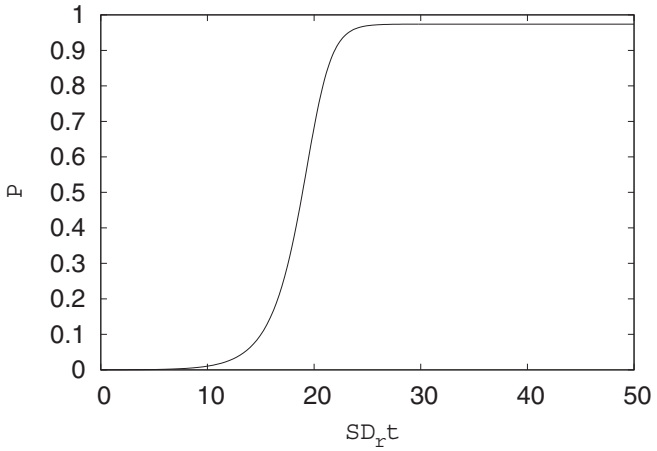

FIG. 2. Relaxation of polar order parameter P. $S = 20$.

induced by the rotating particles of an ensemble. Eq. (13) for given initial conditions is solved numerically by representing $f_{\mp}$ by the Fourier series

$$f_{\mp} = \sum_{k=-l}^{l} f_{\mp}^{(k)} \exp(ik\gamma).$$

The behavior of the solution depends on the value of the parameter $S = n/(4\alpha_r \eta D_r \tau) = 2n/n_c$ (n is the number of particles per unit volume, and n_c is its critical value). If $S > 2$ the polar order is established with particles, depending on initial conditions, on average rotating clockwise or anticlockwise around the x axis. This is illustrated in Fig. 2, where the time evolution of the order parameter of the suspension $P = \int_0^{2\pi} \cos(\gamma')[f_-(\gamma',t) - f_+(\gamma',t)]d\gamma'/n$ is shown for the initial values $\int_0^{2\pi} \cos(\gamma') f_-(\gamma',t)d\gamma'/n = 10^{-4}$ and $\int_0^{2\pi} \cos(\gamma') f_+(\gamma',t)d\gamma'/n = 0$. The distribution functions $f_{\mp}$ for the final state are shown in Fig. 3 and characterize the alignment of angular velocity of the particle rotations clockwise around the x axis.

In the steady state the alignment of the planes of particle rotation is described by the distribution function

$$f = \frac{n}{2\pi I_0(\xi)} \exp[\xi \cos(\gamma)], \tag{14}$$

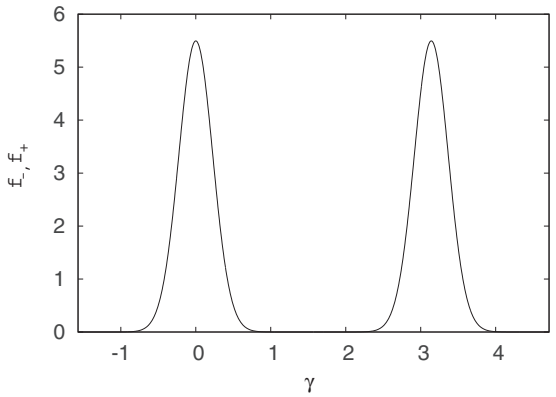

FIG. 3. Steady-state orientation distribution functions f_- ($p_\parallel < 0$ on the left) and f_+ ($p_\parallel > 0$ on the right). $S = 20$.

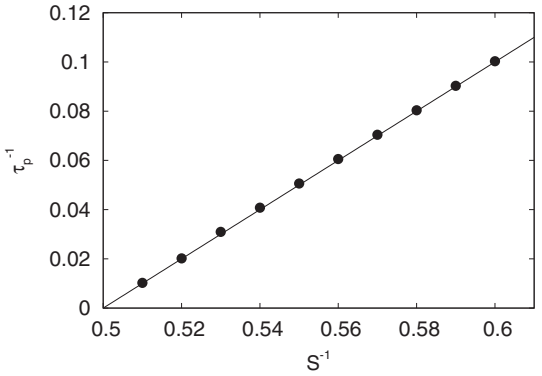

FIG. 4. Relaxation rate of the polar order in subcritical region of the parameter S. Black circles represent numerical data and the solid line represents theoretical dependence.

where $\xi = n < \cos(\gamma) > /4\alpha_r \eta D_r \tau$ and the effective field ξ is determined by the condition of self-consistency

$$\xi = S \frac{I_1(\xi)}{I_0(\xi)}. \tag{15}$$

Here and further $I_n (n = 0,1,\dots)$ are the modified Bessel functions of the first kind [19].

Equation (15) is similar to the equation for the order parameter in the Weiss model of ferromagnetism, which predicts spontaneous symmetry breaking at the critical interaction strength. In the present case the role of interactions is played by the induced flow and if it prevails over the noise, described by the rotational diffusion coefficient D_r and causing randomizing action on the orientation of planes of particle rotations, the spontaneous order in the system is established. The critical value of the interaction strength is easily obtained from Eq. (15) by using the series of modified Bessel functions, which give

$$\xi = \frac{S}{2}\xi - \frac{S}{16}\xi^3 + O(\xi^5). \tag{16}$$

From Eq. (16) we see that spontaneous order ($\xi \neq 0$) arises if $S > 2$. Corresponding to this the numerical solution of Eq. (13) shows the establishment of the polar order at $S > 2$ and its decay at $S < 2$. For the decay of the order parameter in subcritical conditions the critical slowing down of the order parameter relaxation is characteristic. The dependence of the rate τ_p^{-1} of the exponential decay of the polar order on the parameter S near the bifurcation value $S_c = 2$ obtained numerically is shown in Fig. 4. Numerical data match the theoretical dependence $\tau_p^{-1} = (S_c/S - 1)/2$ very well. Dependence of the order parameter $P = \langle \cos(\gamma) \rangle = I_1(\xi)/I_0(\xi)$ on the parameter S is shown in Fig. 5.

IV. ENSEMBLE OF ROLLING PARTICLES

The model may be applied to the situation recently studied in Ref. [20]. In Ref. [20] dielectric particles were settled on one of the plates and due to their rotation in the electric field the particles were rolling along it. This case is similar to the one considered above, except for the more complicated character of hydrodynamic interactions. The hydrodynamics of the liquid due to the particle rolling near a solid wall was considered

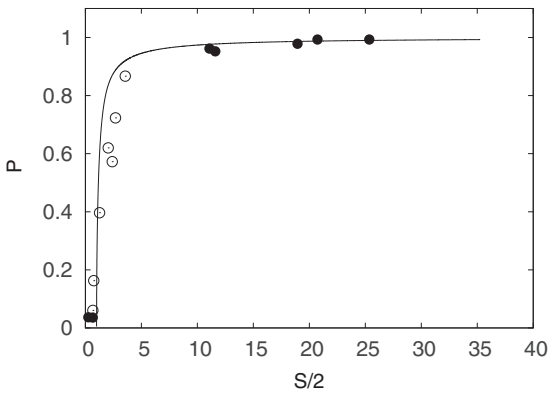

FIG. 5. Polar order parameter as a function of parameter S. Solid and open circles are experimental data from Ref. [20] with the critical concentration rescaled by 1.67.

in Refs. [21,22]. The results obtained are summarized by the mobility matrix and the coefficients $\alpha_s,\tilde{\alpha}_s$ characterizing the behavior of the particle near the solid wall under the action of shear (the z axis is perpendicular to the wall):

$$\begin{pmatrix} v_x/a \\ v_y/a \\ \Omega_x \\ \Omega_y \end{pmatrix} = \begin{pmatrix} \alpha_t & 0 & 0 & \tilde{\alpha}_t \\ 0 & \alpha_t & -\tilde{\alpha}_t & 0 \\ 0 & -\tilde{\alpha}_t & \alpha_r & 0 \\ \tilde{\alpha}_t & 0 & 0 & \alpha_r \end{pmatrix} \cdot \begin{pmatrix} aF_{ex} \\ aF_{ey} \\ T_{ex} \\ T_{ey} \end{pmatrix} + \begin{pmatrix} \alpha_s \\ \alpha_s \\ \tilde{\alpha}_s \\ \tilde{\alpha}_s \end{pmatrix} \begin{pmatrix} \frac{\partial v_x}{\partial z} \\ \frac{\partial v_y}{\partial z} \\ -\frac{\partial v_y}{\partial z} \\ \frac{\partial v_x}{\partial z} \end{pmatrix}, \quad (17)$$

where a is the radius of the particle and $\vec{F}_e$ and $\vec{T}_e$ are external force and torque acting on the particle. In the frame of the mean field model the shear rate near the given particle is created by other particles of the ensemble. By applying the Blake solution for the flow created by the rotlet near a solid wall at $(0,0,d)$ [23] [$\vec{r} = (x,y,z-d)$ and $\overline{\vec{r}} = (x,y,z+d)$]

$$v_i = \frac{1}{8\pi\eta}\left[\frac{e_{ijk}T_{ej}r_k}{r^3} - \frac{e_{ijk}T_{ej}\overline{r}_k}{\overline{r}^3} + 2he_{kj3}T_{ej}\left(\frac{\delta_{ik}}{\overline{r}^3} - \frac{3\overline{r}_i\overline{r}_k}{\overline{r}^5}\right) + 6e_{kj3}T_{ej}\frac{\overline{r}_i\overline{r}_k\overline{r}_3}{\overline{r}^5}\right] \quad (18)$$

for the shear rate induced by the rolling particles of the ensemble we have [$\vec{\rho} = (x,y)$ and $f(\vec{\rho},\vartheta')$ is the surface density of particles rolling in the direction $\vec{n}_\parallel(\vartheta') = \cos(\vartheta')\vec{e}_x + \sin(\vartheta')\vec{e}_y$]

$$\frac{\partial\vec{v}}{\partial z}(\vec{\rho}) = -\frac{3Ep_\parallel}{4\pi\eta}\int_{|\vec{\rho}-\vec{\rho}'|>2a} \times \frac{(\vec{\rho}-\vec{\rho}')[(\vec{\rho}-\vec{\rho}')\cdot\vec{n}_\parallel(\vartheta')]f(\vec{\rho},\vartheta')d\vec{\rho}'d\vartheta'}{|\vec{\rho}-\vec{\rho}'|^5}. \quad (19)$$

Relation Eq. (17), in the absence of an external force ($\vec{F}_e = 0$), and the polarization relaxation equation Eq. (1) allow us to obtain the relations for the particle rolling velocity and equation for the orientation angle of the plane of particle rotation [$\langle\vec{n}_\parallel\rangle(\vec{\rho}) = \int_0^{2\pi}\vec{n}_\parallel(\vartheta')f(\vec{\rho},\vartheta')d\vartheta'/n_s$ is the mean value of rolling direction determined by the distribution function $f(\vec{\rho},\vartheta')$ and n_s is the surface concentration of particles]

$$\vec{v} = \frac{a}{\tau}\frac{\tilde{\alpha}_t}{\alpha_r}\sqrt{\frac{E^2}{E_c^2}-1}\vec{n}_\parallel + \frac{3n_s}{8\eta\tau}\frac{\alpha_s}{\alpha_r}\sqrt{\frac{E^2}{E_c^2}-1}\langle\vec{n}_\parallel\rangle; \quad (20)$$

$$\frac{d\vartheta}{dt} = -\frac{3\tilde{\alpha}_s n_s}{8a\eta\alpha_r\tau}\frac{\partial}{\partial\vartheta}(-\vec{n}_\parallel\langle\vec{n}_\parallel\rangle). \quad (21)$$

Apart from the term in Eq. (20) responsible for the augmentation of the rolling velocity in the presence of other particles, Eqs. (20) and (21) are close to those derived in Ref. [20], if the terms due to electrostatic interactions are neglected. We point out that taking the second term on the right side of Eq. (20) into account, if the concentration of particles is large, leads to the Burgers equation for the concentration of particles, which describes the formation of a shock wave with the characteristic profile of concentration field similar to that observed in experiment [20]. This will be studied in detail elsewhere. Henceforth the second term on the right side of Eq. (20) is omitted.

The kinetic coefficients $\alpha_t,\tilde{\alpha}_t,\alpha_r,\alpha_s,\tilde{\alpha}_s$ as functions of the thickness of the lubrication layer δ between the particle and a solid wall ($\delta = d/a - 1$) were calculated in Refs. [21,22]. For estimates the following relations at scaling $\alpha_r = \overline{\alpha}_r/(8\pi\eta a^3)$; $\tilde{\alpha}_t = \overline{\alpha}_t/(8\pi\eta a^3)$ are useful:

$$\lim_{\delta\to 0}\frac{\alpha_s}{\overline{\alpha}_r} = 0.54; \lim_{\delta\to 0}\frac{\overline{\alpha_t}}{\overline{\alpha}_r} = 0.25; \lim_{\delta\to 0}\frac{\tilde{\alpha}_s}{\overline{\alpha}_r} = 0.79.$$

As a result the kinetic equation for the distribution function reads

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial\vec{\rho}}(v_r\vec{n}_\parallel f) = \frac{\partial}{\partial\vartheta}\left[\frac{2\alpha n_s}{\tau}\frac{\partial}{\partial\vartheta}(-\vec{n}_\parallel\langle\vec{n}_\parallel\rangle)f\right] + D_r\frac{\partial^2 f}{\partial\vartheta^2}. \quad (22)$$

Here the following notations are introduced: the rolling velocity of single particle $v_r = \frac{a}{\tau}\frac{\tilde{\alpha}_t}{\alpha_r}\sqrt{\frac{E^2}{E_c^2}-1}$ and $2\alpha = \frac{3\pi a^2\tilde{\alpha}_s}{\overline{\alpha}_r}$.

V. CONTINUUM MODEL

We now proceed with the derivation of the macroscopic equations for the slow variables—the concentration n_s and the unit vector $\vec{h} = \cos(\varphi)\vec{e}_x + \sin(\varphi)\vec{e}_y$ along the mean rolling direction of the particles. The kinetic Eq. (22) may be written in the form [24]

$$\frac{\partial f}{\partial t} + v_r\vec{\nabla}\cdot(\vec{n}_\parallel f) = \frac{1}{\tau}\frac{\partial}{\partial\vartheta}\left(f\frac{\partial\mu}{\partial\vartheta}\right), \quad (23)$$

which states that the flux of the particles in the orientation space j due to the noise and interactions described in the mean field approximation is given by

$$j = -\frac{1}{\tau}f\frac{\partial\mu}{\partial\vartheta}, \quad (24)$$

where μ is the chemical potential in the orientation space:

$$\mu = D_r\tau\ln(f) - 2\alpha\int_0^{2\pi}\vec{n}_\parallel\cdot\vec{n}'_\parallel f(\vartheta')d\vartheta'. \quad (25)$$

The relation (24) shows that in the spatially homogeneous case the condition of equilibrium corresponds to a constant

chemical potential in the orientation space. This allows us to easily obtain stationary spatially uniform solution of Eq. (23) for the distribution function from the condition $\mu = const$. As a result we have

$$f_0 = \frac{n_s}{2\pi I_0(\xi)} \exp(\xi \vec{n}_\parallel \cdot \vec{h}). \tag{26}$$

The effective field ξ is determined by the condition of self-consistency (15) as in Sec. III. The parameter S in this case reads $S = 2\alpha n_s/D_r\tau = 2n_s/n_{sc}$. This means that the polar order parameter as a function of the relative concentration of particles n/n_c or n_s/n_{sc} is the same in both cases. The order parameter of the ensemble of rolling particles P calculated according to Eq. (26) is compared with experimental data from Ref. [20] in Fig. 5.

We look for the solution of Eq. (23) in the form $f = f_0 + f_1$ and find f_1 as the series in a small parameter $\varepsilon = \tau/\tau_h$, where the characteristic time of the dynamics of spatial nonhomogeneity with the characteristic scale L is $\tau_h = L/v_r$. The equation for f_1 reads

$$\frac{\partial f_1}{\partial t} + v_r(\vec{n}_\parallel \cdot \vec{\nabla}) f_1 = \frac{1}{\tau}\hat{K} f_1, \tag{27}$$

where the operator $\hat{K}$ reads

$$\hat{K} f_1 = \frac{\partial}{\partial\vartheta}\left(D_r\tau \frac{\partial f_1}{\partial\vartheta} - 2\alpha f_1 \frac{\partial}{\partial\vartheta}\int_0^{2\pi} \vec{n}_\parallel \cdot \vec{n}'_\parallel f_0(\vartheta')d\vartheta' - 2\alpha f_0 \frac{\partial}{\partial\vartheta}\int_0^{2\pi} \vec{n}_\parallel \cdot \vec{n}'_\parallel f_1(\vartheta')d\vartheta' \right). \tag{28}$$

The operator $\hat{K}$ is non-self-adjoint and has two zero modes:

$$\psi_1 = \frac{\partial f_0}{\partial n_s}; \psi_2 = \frac{\partial f_0}{\partial\varphi}.$$

The zero modes $\psi_{1,2}$ have clear physical meaning. If $f_0 = f_0(\vartheta - \varphi)$ is the solution of equation $\mu = \text{const}$. then by the symmetry $f_0(\vartheta - \varphi - \delta\varphi)$ is also the solution. This means that $\partial f_0/\partial\varphi$ is the zero mode of the operator $\hat{K}$. Similar argument applies for $\partial f_0/\partial n_s$.

By looking for the solution of Eq. (27) as $f_1 \sim \exp(\lambda t)$ and representing f_1 and λ by the series in the small parameter ε

$$f_1 = f_1^{(0)} + f_1^{(1)} + \cdots; \lambda = \lambda_0 + \lambda_1 + \cdots$$

for the first-order term of zero-mode ($\lambda_0 = 0$) perturbation $f_1^{(0)} = \delta n_s \psi_1 + \delta\varphi\psi_2$ we obtain

$$\lambda_1 f_1^{(0)} + v_r\vec{\nabla}\cdot\left(\vec{n}_\parallel f_1^{(0)}\right) = \frac{1}{\tau}\hat{K} f_1^{(1)}. \tag{29}$$

The adjoint operator $\hat{K^+}$ reads

$$\hat{K^+} g = D_r\tau\frac{\partial^2 g}{\partial\vartheta^2} + 2\alpha\frac{\partial g}{\partial\vartheta}\frac{\partial}{\partial\vartheta}\int_0^{2\pi}\vec{n}_\parallel\cdot\vec{n}'_\parallel f_0(\vartheta')d\vartheta' - 2\alpha\int_0^{2\pi}\vec{n}_\parallel\cdot\vec{n}'_\parallel\frac{\partial}{\partial\vartheta'}\left(f_0'(\vartheta')\frac{\partial g(\vartheta')}{\partial\vartheta'}\right)d\vartheta'. \tag{30}$$

It has two modes with a zero eigenvalue:

$$\tilde{\psi}_1 = 1; \tilde{\psi}_2 = \vartheta I_0(\xi) - \int_0^{\vartheta}\exp(-\xi\cos(\vartheta'))d\vartheta'.$$

As a result from the condition of solubility of Eq. (29) we have [$\vec{h}_\perp = -\sin(\varphi)\vec{e}_x + \cos(\varphi)\vec{e}_y$]

$$\frac{\partial\delta n_s}{\partial t} + v_r\frac{\partial M}{\partial n_s}(\vec{h}\cdot\vec{\nabla})\delta n_s + v_r M(\vec{h}_\perp\cdot\vec{\nabla})\delta\varphi = 0; \tag{31}$$

$$c_1(\xi)\frac{\partial\delta\varphi}{\partial t} + v_r\vec{e}_z(\vec{h}\times\vec{\nabla}\delta n_s)\left(c_2(\xi) + n_s\frac{\partial\xi}{\partial n_s}c_3(\xi)\right) + v_r(\vec{h}\cdot\vec{\nabla}\delta\varphi)c_4(\xi) = 0. \tag{32}$$

Here

$$\begin{aligned} M &= \frac{n_s I_1(\xi)}{I_0(\xi)}; \\ n_s\frac{\partial\xi}{\partial n_s} &= \xi\left[1 - \xi\left(\frac{I_0(\xi)+I_2(\xi)}{2I_1(\xi)} - \frac{I_1(\xi)}{I_0(\xi)}\right)\right]^{-1}; \\ c_1(\xi) &= \frac{n_s\left[I_0^2(\xi)-1\right]}{I_0(\xi)}; \quad c_2(\xi) = \frac{I_0^2(\xi)-1}{\xi I_0(\xi)}; \\ c_3(\xi) &= \frac{I_1(\xi)}{\xi} - \frac{I_0^2(\xi)-1}{\xi^2 I_0(\xi)} - \frac{I_1(\xi)\left[I_0^2(\xi)-1\right]}{\xi I_0^2(\xi)}; \\ c_4(\xi) &= n_s\left(I_1(\xi) - \frac{I_0^2(\xi)-1}{\xi I_0(\xi)}\right). \end{aligned} \tag{33}$$

After tedious and straightforward transformations the set of Eqs. (31) and (32) may be put in the following form (x axis is along the unperturbed direction of vector $\vec{h}$ and the y axis is perpendicular to it):

$$\begin{aligned} \frac{\partial\delta\xi}{\partial t} + \frac{a(\xi)}{\xi}\frac{\partial\delta\xi}{\partial x} + a(\xi)\frac{\partial\delta\varphi}{\partial y} &= 0; \\ \frac{\partial\delta\varphi}{\partial t} + b(\xi)\frac{\partial\delta\varphi}{\partial x} - c(\xi)\frac{\partial\delta\xi}{\partial y} &= 0. \end{aligned} \tag{34}$$

Here

$$a(\xi) = v_r\left(1 - \frac{I_0(\xi)I_2(\xi)}{I_1^2(\xi)}\right)^{-1}; \quad b(\xi) = v_r\left(\frac{I_1(\xi)I_0(\xi)}{I_0^2(\xi)-1} - \frac{1}{\xi}\right); \quad c(\xi) = v_r\left(\frac{I_0(\xi)+I_2(\xi)}{2\xi I_1(\xi)} - \frac{I_1(\xi)I_0(\xi)}{\xi\left[I_0^2(\xi)-1\right]}\right). \tag{35}$$

VI. WAVE EQUATIONS

Equations (34) show that longitudinal perturbations of ξ and φ propagate in the x direction with velocities $v_1 = a(\xi)/\xi$ and $v_2 = b(\xi)$ respectively. In the general case Eqs. (34) may be put in more symmetric form by transformation to the set of coordinates moving in the x axis direction with velocity $c_0 v_r = [b(\xi) + a(\xi)/\xi]/2$. As a result Eqs. (34) in the moving set of coordinates read

$$\begin{aligned} \frac{\partial\delta\xi}{\partial t} + v_0 v_r\frac{\partial\delta\xi}{\partial x} + a(\xi)\frac{\partial\delta\varphi}{\partial y} &= 0; \\ \frac{\partial\delta\varphi}{\partial t} - v_0 v_r\frac{\partial\delta\varphi}{\partial x} - c(\xi)\frac{\partial\delta\xi}{\partial y} &= 0. \end{aligned} \tag{36}$$

Here $v_0 = [a(\xi)/\xi - b(\xi)]/2$.

It may be shown that this set of the first-order partial differential equations may be reduced to the second-order

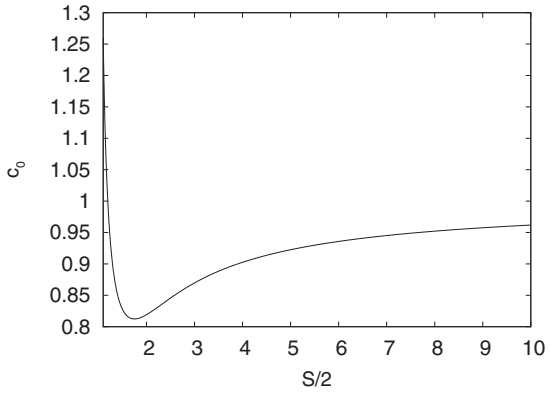

FIG. 6. Dimensionless velocity c_0 as a function of parameter S.

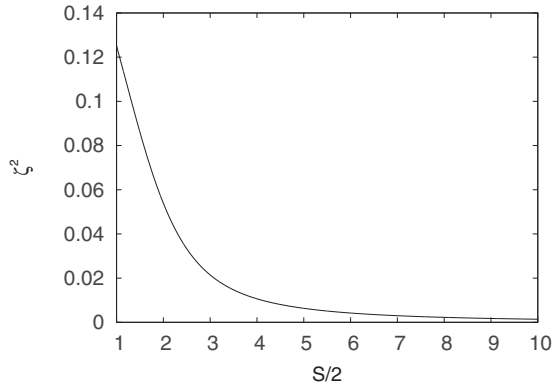

FIG. 8. Parameter ζ^2 as a function of parameter S.

partial differential equation [$\zeta^2 v_r^2 = a(\xi)c(\xi)$]

$$\frac{\partial^2\delta\xi}{\partial t^2} - (v_0 v_r)^2\frac{\partial^2\delta\xi}{\partial x^2} + (\zeta v_r)^2\frac{\partial^2\delta\xi}{\partial y^2} = 0. \tag{37}$$

Dimensionless velocity c_0 has the following asymptotic at small and large ξ $c_0(\xi) = 1/\xi + O(\xi)$ and $c_0(\xi) \to 1$ respectively. It as the function of the parameter S is shown in Fig. 6. The corresponding asymptotic behavior of the dimensionless velocity v_0 is as follows: $v_0(\xi) = 1/\xi + O(\xi)$ at small ξ and $v_0(\xi) \to 0$ at large ξ. Figure 7 shows the dimensionless velocity v_0 as a function of the parameter S.

Estimating the second derivative with the respect to y in Eq. (37) as $-\chi^2\delta\xi$, where $\chi = \pi/L_d$ in the case of zero boundary condition for the layer of particles with width L_d we obtain the Klein-Gordon equation [25] in the unstable case

$$\frac{\partial^2\delta\xi}{\partial t^2} - (v_0 v_r)^2\frac{\partial^2\delta\xi}{\partial x^2} - (\zeta v_r\chi)^2\delta\xi = 0. \tag{38}$$

The properties of this equation were studied in Ref. [26]. The parameter ζ^2 in dependence on the parameter S is shown in Fig. 8. It has the following asymptotic $\zeta^2(\xi) \to 0$ at $\xi \to \infty$ and $\zeta^2(\xi) = 1/8 + O(\xi^2)$ at $\xi \to 0$.

Equation (38) is put in dimensionless form by the introduction of the following scaling $t = L/v_r\tilde{t}; x = L\tilde{x}$ (tildes are omitted in the rest of the discussion). The solution of the Eq. (38) for given initial conditions $\delta\xi(x,0) = f(x)$ and $\partial\delta\xi(x,0)/\partial t = -v_0 f'(x)$, which corresponds to $\varphi(x,0) = 0$,

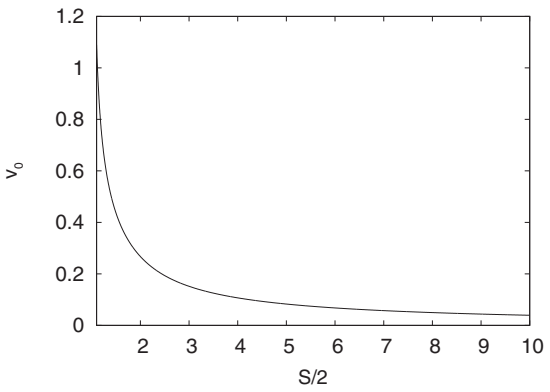

FIG. 7. Dimensionless velocity v_0 as a function of parameter S.

is obtained by the construction of the Green's function and reads ($b = \zeta\pi L/L_d$)

$$\delta\xi(x,t) = f(x - v_0 t) + bt\int_0^1 f(x - v_0 t + 2z v_0 t) \times\sqrt{\frac{1-z}{z}}I_1(2bt\sqrt{z(1-z)})dz. \tag{39}$$

Solution (39) at $f(x) = \exp(-x^2)$ and $b/v_0 = 0.1$ for three time moments corresponding to $v_0 t = 7,8,9$ is shown in Fig. 9. Formation of the long tail of the particle concentration similar to that observed in experiments [20] may be seen. The given time moments in the case of the width of the ensemble of rolling particles 5 mm [15] at a large value for the concentration background $n_s/n_{sc} \simeq 25$, when $\zeta/v_0 \simeq 1$, correspond to 73, 84, and 95 s respectively and have the same order of magnitude as the observation time in the experiment [20]. These values are obtained by taking according to the experiment [20] the rolling velocity v_r equal to 1 mm/s. The spatial scale of the tail in dimensional units at longer time $v_0 t = 100$ for the given parameters is approximately 15 mm, which also is reasonable estimate. The value of the concentration on the peak is increasing due to the instability. Its value is stabilized in the nonlinear model. In order to derive the nonlinear model of wave propagation we remark that the set of Eqs. (31) and (32) is a linear version of the nonlinear

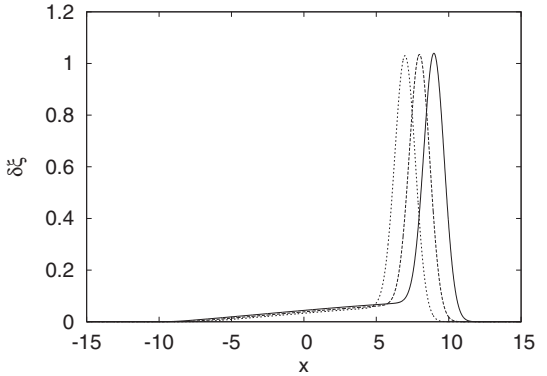

FIG. 9. Effective field ξ perturbation $\delta\xi$ for several moments of time. Dotted line, $v_0 t = 7$; dashed line, $v_0 t = 8$; and solid line, $v_0 t = 9$.

equations, which are obtained from the condition of solubility of the equation for f_1:

$$\frac{\partial f_0}{\partial t} + v_r \vec{\nabla} \cdot (\vec{n}_{\parallel} f_0) = \frac{1}{\tau} \hat{K} f_1, \tag{40}$$

where $f_0 = f_0[n_s(\vec{\rho},t), \vec{h}(\vec{\rho},t)]$:

$$\frac{\partial n_s}{\partial t} + v_r \vec{\nabla} \cdot (M\vec{h}) = 0; \tag{41}$$

$$\vec{e}_z \cdot \vec{h} \times \frac{\partial \vec{h}}{\partial t} c_1(\xi) + v_r \vec{e}_z \cdot (\vec{h} \times \vec{\nabla} n_s) \left(c_2(\xi) + n_s \frac{\partial \xi}{\partial n_s} c_3(\xi) \right) + v_r c_4(\xi) \vec{e}_z \cdot \vec{\nabla} \times \vec{h} = 0. \tag{42}$$

The study of this nonlinear model, including the two-dimensional case, is pending for future publication.

VII. CONCLUSIONS

To summarize, a mean field model of the synchronization of dielectric particle rotations caused by the Quincke effect has been developed. The set of kinetic equations for the distribution functions of the orientation of the particle rotation planes is derived considering (in the mean field approximation) its dynamics in the flow induced by the rotating particles. Its stationary solution shows the spontaneous symmetry breaking and establishment of the polar order, if the concentration of the particles is greater than the critical, which depends on the random noise. It turns out that the model is similar to the Weiss model of the ferromagnetism, where the role of interactions is played by the shear rate of a flow induced by the rotating particles. The similarity of two particular cases—the Quincke suspension in the layer with one free boundary and the ensemble of particles rolling near a solid wall under the action of the torque in the electric field—is illustrated and dependence of the polar order in these cases on the parameters, which characterize the hydrodynamic interactions, is found. The continuum model of the Quincke suspension is developed on the basis of the derived kinetic equations and the set of equations for the macroscopic variables—the concentration and the director of the polar order—is derived. The properties of its solutions are studied and it is shown that they have the features similar to the observed in the experiment.

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