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Contents

Satura rādītajs

Mathematical model for gravitational cascade separation of pourable materials at different stages of a classifier

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We consider the gravitational cascade separation process as an absorbing Markov chain. We receive here a new method for calculating a degree of fractional extraction for any narrow class of pourable material in cascade classifier at different stages and a few other interesting results.

Key words: Cascade separation, pourable materials, Markov chains
Mathematics Subject Classification (2000): 60J20

The modern commercial production and international market are associated with the transfer of vast bulks of diversified pourable materials, often transported open, without any package. To such materials, potash and phosphate fertilizers could be assigned, as well as various structural materials (sand, crushed stones, gravel), coal, corn, etc. A distinguishing feature of these materials is that they contain a considerable amount of fine dust. When such materials are loaded into vehicles or ships, a vast amount of dust is brought into the atmospheric air and further into water thus leading to the environment pollution. To remove dust from large flows of pourable materials, special apparatuses - separators - are employed, where technical and design parameters should therefore be properly chosen.

1. Describing of method of gravitational separation

Let us describe the method and the classifier for gravitational cascade separation.

Given a granular material with grain sizes ranging from a_0 to a_n, our purpose is to separate this material into n components along predetermined boundaries $a_0 < a_1 < a_2 < \cdots < a_{n-1} < a_n$. The first component must contain only grains of size from a_0 to a_1, the second one, tespectively, from a_1 to a_2, and the n-the component from a_{n-1} to a_n. We will call a *narrow class* the portion of material with grains sized between two neighboring separation boundaries. Obviously the ideal separation is not possible.

We consider the cascade method of separating pourable materials in a gravitational classifier. The classifier consists of z stages counted top-down. The air flow is fed from below, and the initial material is fed to one of the stages numbered i^*. Heavier grains fall down (we call them *coarse product*), while light-weight ones go up (we call them *fine product*). The structure of each stage of the classifier is different.

Figure 1 presents a sketch of the cascade classifier, schematically showing the movement of the material in the separation process.

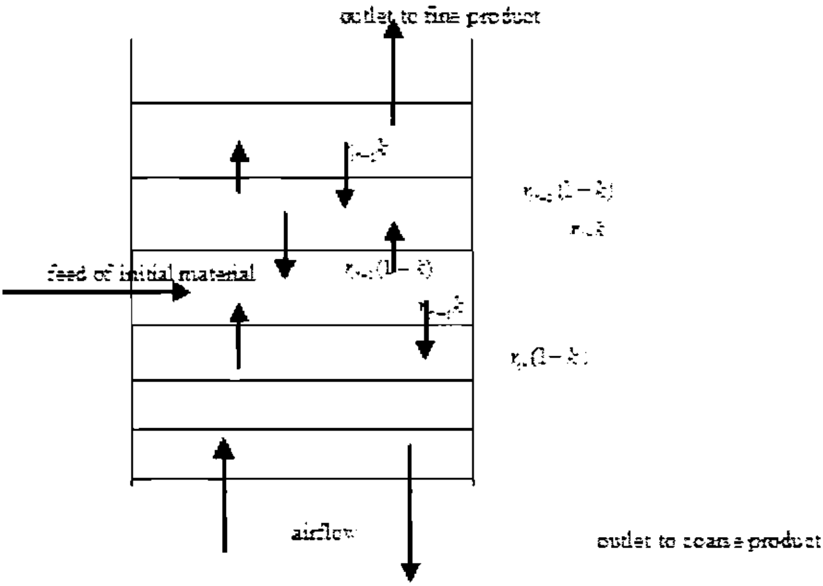

Fig. 1.

2. Previous results

The separation process is considered at discrete points of time (acts of separation).

DEFINITION 1. Let us call the function $k_i = \frac{r^*_{i,j}}{r_{i,j}}$ an *upward coefficient of separation* for the narrow class j, where $r_{i,j}$ is the quantity of the material of class j situated at the stage number i at the present moment, and $r^*_{i,j}$ is the quantity of the material of class j passing to the stage number $i-1$ within one moment of time. Then $1 - k_i$ is the *downward coefficient of separation* for the narrow class j, that is, for the grains of class j passing from stage i to stage $i+1$.

Definition 2. Let us call function $F_{f,j} = \frac{r_{f,j}}{r_{s,j}}$ the *degree of fractional extraction* to a fine product for a narrow class j, where $r_{f,j}$ is the quantity of the narrow class j output from the classifier to the fine product, and $r_{s,j}$ is the quantity of the narrow class j in the initial material.

As stated above, the process is considered at discrete points of time (acts) with equal spacing. The material is fed to the classifier with the same time interval and equally portioned. (Scheme of the material separation inside the classifier is shown in Fig. 1).

It is known that the separation results for any narrow class in each stage of the apparatus are independent of the presence of grains of other classes [6]. For a classifier with identical structure of the stages we have proved in [1], that the degree of fractional extraction is

$$F_j = \begin{cases} \dfrac{1-\sigma^{z+1-i^*}}{1-\sigma^{z+1}}, & \text{if } k \neq 0,05; \\ \dfrac{z+1-i^*}{z+1}, & \text{if } k \neq 0.5, \\ 0, & \text{if } k = 0 \end{cases}$$

where $\frac{1-k}{k}$, i^* is the number of the stage of material feed and k is the coefficient of separation of narrow class j for each stage of a classifier.

To calculate the quantity of the material of the narrow class j output to the fine product, we use the formula $r_f = F_f r_s$, where r_s is the quantity of the narrow class j in the initial product.

Using the formula $r_c = r_s - r_j$, we can obtain the quantity of the narrow class j in the coarse product.

3. Absorbing Markov chain

The principle of cascade separation of the pourable material of the narrow class j is presented in Fig. 1. This process is like a random walk of one particle of the narrow class j with upward transition probability k_i (coefficient of separation of narrow class j in stage number i) and downward transition probability $1-k_i$. It has two absorbing states. Hence, the motion of a particle of the narrow class j, in a classifier with z stages, can be represented by absorbing Markov chain with the following transition matrix

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
k_1 & 0 & 1-k_1 & 0 & 0 & 0 & \dots & 0 \\
0 & k_2 & 0 & 1-k_2 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & 0 & k_{z-1} & 0 & 1-k_{z-1} & 0 \\
0 & 0 & \dots & 0 & 0 & k_z & 0 & 1-k_z \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 1
\end{pmatrix}$$

This matrix contains $z+2$ rows and columns. The first and last states are outputs to the fine and coarse products. All other states are probabilities of transition of the particle among the stages of the classifier.

It is convenient to consider the canonical form of this matrix in an aggregated version. We unite all ergodic (absorbing) sets, and all transient sets. We have z transient states (according to the number of stages of the classifier), and two ergodic states (related to fine and coarse products). Thus, the canonical form is:

$$\left(\begin{array}{cc|ccccccc}
1 & 0 & 0 & 0 & \dots & \dots & \dots & 0 & 0 \\
0 & 1 & 0 & 0 & \dots & \dots & \dots & 0 & 0 \\
\hline
k_1 & 0 & 0 & 1-k_1 & 0 & \dots & \dots & \dots & 0 \\
0 & 0 & k_2 & 0 & 1-k_2 & 0 & \dots & \dots & 0 \\
0 & 0 & 0 & k_3 & 0 & 1-k_3 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \dots & 0 & k_{z-2} & 0 & 1-k_{z-2} & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & k_{z-1} & 0 & 1-k_{z-1} \\
0 & 1-k_z & 0 & 0 & 0 & \dots & 0 & k_z & 0
\end{array}\right)$$

Here the region O consists only of zeroes, the $z \times z$ submatrix Q is related to the process as long as the particle stays in the classifier (in the transient states), the $z \times 2$ submatrix R is related to the transition from the classifier to one of the two products (the transition from transient into ergodic states), and the 2×2 matrix I deals with the process after the particle has reached one of the two products (the ergodic set).

It follows from [4, 5] that in an absorbing Markov chain the probability of reaching one of these states tends to 1. Then we can say that the probability of a particle to reach one of the two products tends to 1.

From [4, 5] we obtain for any absorbing chain that Q^n tends to zero when n tends to ∞, and $I-Q$ is reversible, so that $(I-Q)^{-1} = \sum_k^{\infty} Q^k$.

For an absorbing Markov chain we define the *fundamental matrix* to be $N = \sum_k^{\infty} Q^k$.

We define n_j to be the function giving the total number of separation acts (moments of time) with the particle in the stage j during the process, i.e. in the transient state number j. Hence, according to [4, 5], we assert that the expectation of a particle started from the stage i to be in the stage j during n_j separation acts is an ij-coordinate of the matrix N, i.e.

$$\{E_i(n_j)\} = N. \tag{1}$$

This establishes the fact that the expectation of the total number of separation acts of the particle in a given stage is always finite, and that these expectations are simply given by N.

Each particle is fed to the classifier through the stage number i^*, and using (1) we can obtain the expectation of the total number of separation acts of the particle in the given stage.

We introduce the following notation:

$$N_2 = N(2N_{dg} - I) - N_{sq}$$

is $z \times z$ matrix, where N_{ng} results from N by setting off-diagonal entries equal to zero, and N_{sq} results from N by squaring each entry (this, of course, will generally be the same as N^2, but $D^2 = D_{sq}$ for any diagonal matrix D):

$$B = NR$$

is $z \times 2$ matrix;

$$\tau = N\xi,$$

ξ is a column vector with all entries equal to 1,

$$\tau_2 = (2N - I)\tau - \tau_{sq}.$$

Following [4, 5] for absorbing Markov chain, we can assert that the variance of particle started from the stage i to be in the stage j during n_j separation is an ij-coordinate of N_2 , i.e.,

$$\{D_i(n_j)\} = N_2. \tag{2}$$

Let T be a function giving the total number of separation acts (including the initial position), during which acts the particle remains in the classifier before its output to the fine or coarse product. If an absorbing Markov chain starts in the transient state, then T gives the total number of steps needed to reach an ergodic set.

According to [4, 5], we can assert that a for particle started in the stage number i, the expectation of the number of separation acts until it reaches the fine or coarse product in an i_{th}-coordinate of the vector τ, and the variance of the latter is i_{th}-coordinate of τ_2, i.e.

$$\{E_i(T)\} = \tau, \tag{3}$$

$$\{D_i(T)\} = \tau_2. \tag{4}$$

The particle is fed to the classifier through the stage number i^*, and hence, using (3) and (4) we obtain the expectation and variance of the total number of separation acts during which the particle is in the classifier.

Let b_{ij} be the probability for the particle starting in the stage i to reach the fine or coarse product ($j = 1$ for the fine product, and $j = 2$ for the coarse product).

Hence, using the results of [4, 5], we can obtain that for absorbing Markov chains the following relationship holds

$$\{b_{ij}\} = B = NR. \tag{5}$$

Thus, we can control the probability of reaching the necessary product by the particle by changing the place of its feeding into the classifier. Then the probability of reaching the fine product is the degree of fractional extraction for a narrow class j, when the material feeding at the stage number i.

REFERENCES

[1] Barsky, E., Buikis, M., Mathematical model for gravitational cascade separation of pourable materials at identical stages of a classifier. In: Buikis, A., Ciegis, R., Fits, A. D., (eds.). *Progress in Industrial Mathematics at ECMI 2002*, **5**, Springer, Berlin, 2004, 229-233.

[2] Seader, J. D., Henley, E. J., *Separation Process Principles*. John Wiley & Sons, New York, 1998.

[3] Khouri, F. M. *Predicting the Performance of Multistage Separation Processes*. CRC Press, Boca Raton, FL, 2000.

[4] Kemeny, J. G., Snell, J. L., *Finite Markov Chains*. Von Nostran Company, Princeton, N. J., 1960.

[5] Kemeny, J. G., Snell, J. L., Knapp, A.W., *Denumerable Markov Chains*. Springer, Berlin, 1976.

[6] Kayser, F., Zickzag-Sichter ein wingsichter nach neuen prinzip. *Chemie Ingeneur Technic*, **4** (1963).

Beramu materiālu kaskādveida separēšanas procesa analīze ar absorbējošu Markova ķēžu palīdzību lietojot klasifikatoru ar atšķirīgām atdalīšanas pakāpēm

Kopsavilkums

Mēs apskatām gravitācijas kaskādveida separācijas procesu kā absorbējošu Markova ķedi. Tiek piedāvāta jauna frakcionālās atdalīšanas pakapes aprēķināšanas metode katrai šaurai beramu materialu klasei klasifikatora ar atšķirıgaın atdalīšanas pakāpēm, kā arī vairāki citi interesanti rezultāti.

Geoeconomical and geopolitical risks: contemporary problems and solutions

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We consider some geoeconomical and geopolitical risks associated with the economical and political interactions between different countris. For this aim, the theory of differential game is applied. Some examples are given.

Key words: geoeconomics. geopolitics, geoeconomical and geopolitical risks, differential game theory

Mathematics Subject Classification (2000): 91A23, 91A80, 91B30

The beginning of the 21^{st} century was marked with grand changes in the world geoeconomics and geopolitics. The turbulent process of changes affected not only the developed but also the developing countries, not only the traditional but also the new spheres of human activities. The re-structuring of the traditionally preserved systems in these spheres, the disintegration of enormous regional world structures and the origin of new states, the formation of strategic unions, and so on – all that raised points before the science about rationalizing these phenomena, about revealing of the deep reasons and consequences of such serious changes, with the purpose of getting ready for management of the world economic system and including the national socioeconomic structures into the globalization process.

There is a very important issue in this situation [1] for seeking completely new approaches toward the problem about the geopolitical and geoeconomical risks because the strategic solutions in directions of state development in the 21^{st} century get complicated by the necessity to operate with trends, not easily predicted, in the sphere of economics which passed the state boundaries long ago (in the currency-financial, credit, investment, innovation-reproduction, social and cultural spheres).

It is necessary to take into account a great deal of factors by regulating the geopolitical and geoeconomical risks, which influence simultaneously the forming vector of the national development strategy and hereby using all the newest accessible scientific approaches in the sphere of the knowledge theory, methodology, the newest ideas and ways for mathematical and logistics formalization.

We must mention the risk situations. associated by temporal, territorial and inter-state prognoses of originating of the new type interanclave social work division; the appearance of pulsating economic boundaries which do not coincide with the national (state) ones; forming of stray world international reproduction nuclei and world income; the role of the high geoeconomical technologies, etc., as some of the most important problems of the localization of the geopolitical and geoeconomical risks.

The risks in a geoeconomics environment can affect and even affect more often whole regions, conglomerations of states, nations and peoples. Therefore the risk regulating in the globalized economy is a very important and responsible process. The slightest mistake in the prognosis or realization of a scheme for globalization, not sufficiently developed, can lead to leveling of already achieved results or to destroying progressive trends.

Effective activity in this direction is expected if the indicated problems are dealt with by a single authoritative international body, possessing, *firstly*, the ability to react quickly to the slight signals of the international society, to have effective logistics, geoeconomical and geopolitical equipment and, *secondly*, to have highly intellectual, information and communication capacity.

The problems, connected with a deep analysis of the global processes, relating to different spheres of the contemporary world space: geoeconomical, geopolitical and geostrategical [1] are particularly important. An electronic interface system between the above-mentioned spaces, for accepting solutions, for processing of risk situations on the basis of a reliable geoeconomical; prognosis must be developed to meet these aims. The electronic (computer) system for monitoring "geoeconomical climate" must be based on a corresponding information basis, on a permanently kept reliable information, on the formulating of a methodology for situational strategic data combination, on the assignment and solution of different strategic tasks, on the development of model situational strategic variants (combinations). The accuracy of the geoeconomical prognosis will depend on the accuracy of the imitation of the strategic tasks, which the particular national economy sets as a long-term purpose because the initial circumstances for operating in the geoeconomical space are different for the different economic systems. But in all cases the general methodological scheme for development of the geoeconomical prognosis must be more heuristic than systematic, more logistics than operational.

The methodological approach, connected with the idea for volumetric division of the global space into a number of spaces, which on different segments of the dynamic development have one or another hierarchic position, is taken for initial research basis. The leading space into the globalization regime is the geoeconomical space, which puts on a second place both the geopolitical space and the military-strategic space. On the other hand, there can be marked three important autonomous management spheres in the nucleus of the geoeconomical space:

- Commodity - monetary;

- Organizational - economic;
- Contract - interpolational.

The management technology of the risks of the commodity – monetary sphere is based on the prerequisite that the world market is a combination of interconnected simple (single) commodity markets, united in a final (closed) system which is in a dynamic equilibrium. The more accurately we can mould the world market, the more accurately we can guarantee the whole geoeconomical prognosis in this sphere. Every single commodity market represents an organically connected unit of the world market, one of the knots of this gigantic net, placed into the world economic (market) medium. A well-organized information system must exist, such a system that all the elementary single markets to be constantly in the range of vision of the analytic observer.

A combination of definite commodity markets can be aggregated to a separate system. A main market can be separated in each group. It influences all single commodity markets in a given group, it effects the trends for their development. Apart from the mutual influence of separate markets to each other, as well as the mutual influence of groups of similar markets, the whole market system is influenced by the so called global group in the structure of the currency-credit market and the market of the labour force. Such a model of a world market guarantees the opportunity to follow the fluctuations and the regularity of change of all separate markets, to give a quality and after that a quantity evaluation of all other interconnected markets (to separate the zone of distribution of the price impulse, to determine its distribution speed, the link of the contract prices with "the market of the medium" (the credit conditions, the rate of exchange, etc.), to evaluate quantitatively the influence of "the market of the medium" of the contract prices.

The organizational-economic sphere has its own peculiarities, the main one being the process of division of production-technological chains and exporting of separate units outside the national boundaries. In such a way, the organizational-economic structures are arranged into different interconnected chains. Generally speaking, the system of world economic connections is formed by single economic structures (organizational – functional model of the world economic sphere). Also the model of collaboration in the world economic field predetermines the division process of production-technological chains and exporting of their separate units outside the national boundaries. And a definite organizational-functional structure: scientific-research, project-designing, investing, production, foreign trade and contract, stands behind such an unit.

The contract system must be the first one within sight of the geoeconomical regulating. This is achieved in the foreign trade practice via the preparation system of memoranda. The foreign trade (the foreign economic) contracts represent an important independent level of the foreign economic information. A pressing task is to form that information level in a given system, to classify the level by dividing it into priority directions, etc. The quantity evaluation of the motivations comes from here (for example, according to the type of the difference between the world price and the price, actually achieved in the negotiation process and fixed in the contract, etc.).

A dangerous loss of control over the contract system appears with the liberalization of the foreign economic activity because the contract prices often do not reflect the world prices and therefore a system of pseudo-world prices is formed, creating a precedent of a big risk of backdoor contracts.

Therefore the following must be done, the contracts signed at all levels, including the interstate level, must be transparent. The national economic interests are reflected in one or another way in the signed contracts. In this way the very serious problem appears for establishing of correlations in the world prices and additional non-price stimuli, regulating the partners relations. Not only the value law regulates the contemporary economic world but a large system of hidden motivations does that, as well. In other words we have to deal with a peculiar parallel foreign economic system of world economic links. The contracts, dictated by the national interests, form new economic boundaries, establishing in this way the outlines of the integration and desintegration processes at a national, regional, production, financial, integration, political, military-political level and some other levels.

The geoeconomical and geopolitical risks are intrinsically characteristic of the contemporary world economic structure and therefore it is very important to learn to regulate them or at least to insure against serious losses and catastrophes. These risks have already been treated in the developed countries as success or failure due to using ready schemes for reducing the losses caused by sudden but prognosticable actions of the governments and the political state leaders. It became obvious that serious analytical research of the foreign economic and political activities and of the foreign trade is necessary for making prognosis and evaluation of the geoeconomical and geopolitical risks. Furthermore, the military political methods at an interstate level and a posteriori media for defence of the companies seem to be a late reflex which does not provide effective support to the capitals and the reputation of the state structures and the business structures.

As it seems, a serious intellectual-information research is required in this direction because one and the same political event or action of the government structures affects the activity of the companies to a different extent according to the sphere of their activity and the preventing actions, motivated by the results of the prognoses and of the economic analysis [12]. For example, the restriction of the general economic relations between Iran and the United States, caused by the events of the Islam revolution in 1979 and by the political considerations in connection with the orthodox clergy that came into power, did not hinder the sale, not made public, by the United States in Iran of highly technological military production which almost led to the impeachment of the President of the United States, Ronald Raegan.

The difficulties related to the regulating or insurance of the geopolitical and geoecomical risks are connected first with the sphere of activity of the company and the branch belonging. For example, a building company, which has short-term contracts, is sufficient to have a prognosis for the political risk for a period up to one year, while the big gas-oil or air-space companies with a long production cycle of the products and realization of the services must have a quality prognosis for the perspectives within 10 - 12 years.

The reading and evaluation of the geopolitical and geoecomical risks represent the ground on which the whole foreign policy of the state is based. It can be considered that the level of the successes of the state structures in this sphere represents an indicator of civilization of the foreign economic course, which is led.

In the contemporary controversial and dynamically changing world, the actions of the government structures can have for the state and the citizens trivial or, to the contrary, catastrophic consequences. For achieving effectiveness of the management process of the geopolitical and geoecomical risks, the strategic government actions must be based on reliable prognoses about the object of risk protection and its interactivities with the encirclement, having a short-term or long-term character.

That is particularly important for protection from the state risk (the default risk), or from the macrorisks, whose analysis and evaluation are necessary for preventing (or minimization) of possible financial and material losses, caused by false political solutions of the state structures (for example, about the investment strategy), realized under the pressure of the opposition forces, or appearing as a consequence of deliberate actions of rival encirclement.

The political unstability, originating from the unskilful interaction of the national governments with the opposition or with the regions, or from the unability of the state to fulfil its duty commitments in connection with the foreign investors as a consequence of the aggressive actions of the encirclement can be found in the basis of the state political risk. Every government strives for reaching maximal benefit by minimal political compromises, depending on the encirclement, forming the precedent of the risk.

In fact, all political risks can be generally divided into two types: political micro and macro-risks. The political unstability of the state structure or the non-productive actions of the opposition in the country, in which considerable investments have been made, is in the basis of the political macro-risk of the state, which is always discussed at a national level. In this respect, particular attention must be paid to the analysis of the real and potential ability of the partner-state to fulfil its commitments in connection with the foreign investors by the development of prognoses.

In relation to the political micro-risks, affecting the interests of the economic agents on the world market, it must be taken into account in advance that the interests of the business structures and of the executive bodies of the authorities are different. Namely, the governments always try to achieve maximal economic benefit by minimal political price, while the business structures strive to receive global and stable economic and political prosperity. Therefore, it is important for each economic agent to prognosticate "negotiating own and rival force" on the background of the economic and political situation.

Four criteria have been used for the evaluation of the political macro-risk since the time of the famous analytics V. Coplin [10, 11] and V. Overholt [18]:

- Evaluation of the position or orientation by the state factors in that or another sphere of activity;

- following this position firmly, proved by the national idea for stable development;
- authority and influence power on this sphere of state and political leaders;
- the importance of the risk aspect for increasing the state stability or the authority of the state factors, taken into consideration by the partner-state.

These criteria are used especially successfully in combination with scenario methods [12, 14, 15] and they have achieved a definite logical formalization lately. In particular, I. Walter I. [12] developed two general formulae for evaluation of the political (state) risk, corresponding to each state scenario to the same extent.

The first of them:

$$Y(t) + M(t) = A(t) + X(t),$$

where $Y(t)$ – is the val national product (VNP) of the country – candidate for economic partner interaction; $M(t)$ – is the import of that country; $A(t)$ – is the internal consumption, including investment in the production and non-production sphere; $X(t)$ – is the export. The economic point of this equation is in the fact that by growth of VNP and preserving the internal consumption, radical changes occur at the level and in the structure of $M(t)$ and $M(t)$.

The second, relating to the same country candidate for economic interaction, looks like:

$$X(t) - M(t) - DS(t) + FDJ(t) + U(t) - K0(t) = DR(t) - NRB(t),$$

where

$$DR(t) = X(t) + FDJ(t) + U(t); \quad NRB(t) = M(t) + DS(t) + K0(t);$$

$X(t), M(t)$ are taken from the first formula; $DS(t)$ – are payments on foreign loans: $FDJ(t)$ – is a general volume of direct short investments; $U(t)$ – is a general volume of gratis credits; $K0(t)$ – is a general sum of the investments in the bank system of the country: $DR(t)$ – is the quantity of the foreign currency, circulating at a given time in the country, equated to a single currency equivalent; $NRB(t)$ is a state duty, determined for being paid at a given moment.

The point of this economic-mathematical expression is in that the negative balance in the left part can have as a consequence default (refusal for payment the foreign duty and a requirement for its restructuring), motivation of the economy toward an increased foreign duty or toward finding other international sources for financing. With other words, the situation from the type $DR(t) < NRB(t)$ presupposes state political risk because the solution for default or attracting of new foreign investments lies exclusively within the competence of state and political structures.

Using this economic-mathematical formalism helps for the development of an optimal scenario for economic interaction between the states, distancing from political risks. It is necessary to use "information-intuitive thinking", based on experience and analogy, formalized into some kind of model for making solution, so that this scenario will not depend too much on the possible state cataclysms.

The priorities of the modulated scenario solutions is expressed in the possibility for finding of hidden correlations, capable to drastically change the effectiveness of the actions in the field of political risks insurance.

The non-risks management scenarios of systems, exposed to political (geopolitical) and economic (geoeconomical) risks, are generated especially effectively with the help of the developed in [2, 9] market competitive strategies (MCS). The idea and implementation of that instrument is based on the theory of the games and the economic conflict in the tract of G. Stakelberg [3, 21]. In the final, adapted for practical usage form, MCS are formalized in the following way:

DEFINITION 1. *Let us represent some dynamic market factor* (t) *into the form of a sum of its non-negative components*

$$K(t) = \sum_{(i)} K_i(t).$$

If we enter base strategies $Xi(t) = Ki(t)/K(t)$, *numerically equal to the part of the contribution of each component in* $K(t)$, *given at the beginning and the end of the management interval* $[0, T]$ *as*

$$Xi(0) = X_{i0}, \quad Xi(T) = X_{iT},$$

then MCS: $Xi^N(t)$, *realizing the expressed trend in deferential form* $K(t)$ *toward decrease,*

$$dK(t)/dt = a\ K(t)(1 - \sum_{(i)} Xi^N(t)),$$

are represented (Fig. 1) in the way

$$Xi^N(t) = \begin{cases} Xi0, & \textit{if } 0 \le t \le \tau i; \\ \max(Xi0, X_{iT}), & \textit{if } T \ge t > \tau i. \end{cases}$$

Here

$$\tau i = T - 1/a\ (1 - Xi^N(T))$$

is the temporary co-ordinate for switching the strategies;

$$a = M_1 \sum (1/(1 - Xi^N(T)))^2/T/(M_1 - 1)/\sum (1/(1 - Xi^N(T)))$$

is rate of the trend realization: $M_1 = const$, $1 < M_1 < T$, is scenario constant, depending on the expected dynamics of the prognosticable factor $K(t)$ of $[0, T]$. If for $[0, T]$ are not prognosed sharp changes of $K(t)$, then it is completely possible to put $M_1 = T/2$. Otherwise this affix must be given specially.

Analogically, MCS: $Xi^N(t)$, realizing the expressed trend in diferential form $K(t)$ toward increase

$$dK(t)/dt = b\ K(t) \sum Xi^N(t)$$

and corresponding to the base strategies $Xi(t)$, are determined in the way:

$$Xi^N(t) = \begin{cases} Xi0 & , \quad if \quad t \leq \mu_i \ ; \\ \min(Xi0, X_{iT}) & , \quad if \quad T \geq t > \mu_i \ . \end{cases}$$

where

- $\mu_i = T - 1/b\ Xi^N(T)$ is a temporary co-ordinate for switching the strategies (fig. 2);
- $b = M_2 \sum(1/Xi^N(T))^2/T/(M_2 - 1)/\sum(1/Xi^N(T))$ – is rate of the trend realization;
- M_2 is analogical const of M_1.

If $Xi(0) = X_{i0}$, $Xi(T) = X_{iT}$ are known, then we can easily define τ, μ_i, $Xi^N(t)$ and via them by using simple quantum on the switching points of the strategies to solve the deferential equation of the trends and to define $K(t)$, after which every component of the process $Ki(t)$ is determined according to the formulae: $Ki(t) = K(t) \cdot Xi^N(t)$.

In the problems for prognostic economic factors, however, $Xi(T) = X_{iT}$ are more often a priori not known and then they are determined by some systematic functional, connecting all or separate co-ordinates of $K(t)$, i.e. some of their known proportions or combinations in the future.

The situation with previously unknown terminal values of the market competitive strategies is typical for vertically structured hierarchical managements of a regulating market, suggested by the German economist G. Stakelberg [21].

For using this idea we will practically make a combination of the abovementioned formulae in the form of two aggregates

$$DR(t) = Y(t) + FDJ(t) + U(t),$$

$$NRB(t) = A(t) + DS(t) + K0(t)$$

and we will introduce the following strategies:

$$X1(t) = Y(t)/DR(t) \ ; \ X2(t) = FDJ(t)/DR(t) \ ; \ X3(t) = U(t)/DR(t) \ ;$$

$$Y1(t) = A(t)/NRB(t) \ ; \ Y2(t) = DS(t)/NRB(t) \ ; \ Y3(t) = K0(t)/NRB(t),$$

with which based on a famous methodology (see, for example, [2] or [9] we can turn to market competitive strategies (MCS).

In this case, if $DR(t)$, $NRB(t)$ decrease and

$$(dDR(t))/dt = a1 \cdot DR(t)(1 - \sum MCS(t)) \ ;$$

$$dNRB(t)/dt = b1 \cdot NRB(t)(1 - \sum MCS(t)),$$

the market competitive strategies $X1(t)$, $X2(t)$, $X3(t)$: $Y1(t)$, $Y2(t)$, $Y3(t)$ have a configuration, given on Fig. 1. Otherwise (on Fig. 2):

$$(dDR(t))/dt = a2 \cdot DR(t)(\sum MCS(t)) \ ;$$

$$dNRB(t)/dt = b2 \cdot NRB(t)(\sum MCS(t)).$$

If the interval $[0, T]$ represents an interval for prognosing of the model riskless scenario of the economic interaction between two countries, and $MCS(T)$ are known, then the technical parameters (the switching points of the strategies) of this scenario τ_i, μ_i are determined with the expressions:

$$\tau_i = T - 1/(a1(1 - MCSi(T))), \quad or \quad \tau_i = T - 1/(b1(1 - MCSi(T))),$$

$$\mu_i = T - 1/(a2\, MCSi(T)), \quad or \quad \mu_i = T - 1/(b2\, MCSi(T)),$$

where the quantities "$a1$", "$b1$", "$a2$", "$b2$" are equal to:

$$a1(\text{or } b1) = M \sum (1/(1 - MCSi(T)))^2/T/(M-1)/\sum(1/(1 - MCSi(T))),$$

$$a2(\text{or } b2) = M \sum (1/MCSi(T))^2/T/(M-1)/\sum(1/(MCSi(T))),$$

and Ç is the above-mentioned scenario constant.

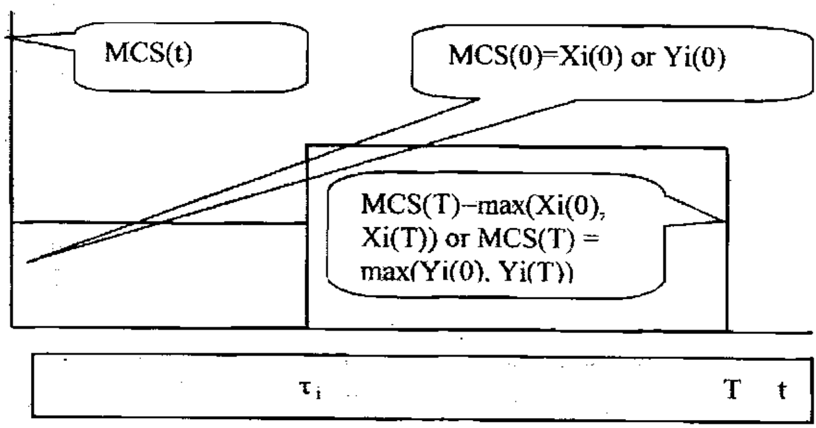

Fig. 1. General look of MCS for a decrease trend

Based on the above descriptions, the scheme for development of a model riskless scenario of the economic interaction between two countries can be the following:

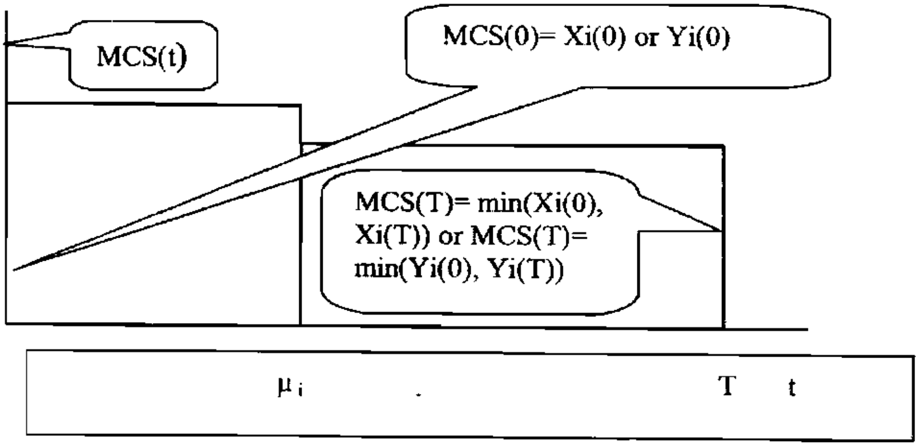

Fig. 2. General look of MCS for an increase trend

1. If $DR(T) < NRB(T)$, then the economic situation in the country candidate for economic interaction is unstable to such an extent that it can motivate the political circles toward default or other firm economic - political solutions: i. e. restraining from active economic links is recommended for avoiding the risk;

2. If $DR(T) > NRB(T)$, then that represents a signal for economic prosperity of the partner-country and opportunity for activating the economic links with it. If, besides, $DR(T) > DR(0)$, then there is a trend of increase for $DR(t)$.

   In this case if $DR(0) > NRB(0)$, active economic interaction can be recommended immediately after realizing this fact.

   More interesting is the situation when $DR(0) < NRB(0)$, showing that a moment of time t from the interval $[0, T]$ exists, in which $DR(t) = NRB(t)$, and to whose coming, restraining from active economic interactions is recommended and immediately after it the economic transactions must be accompanied by active insurance of the deals.

3. In the case $DR(T) > NRB(T)$, but $DR(T) < DR(0)$ (trend toward decrease for $DR(t)$) and $DR(0) > NRB(0)$, close collaboration can be recommended.

   The same cannot be said about the situation $DR(0) < NRB(0)$, because there exists point t from the interval $[0, T]$ in which $DR(t) = NRB(t)$, and to whose arrival, restraining from active economic interactions is recommended and immediately after it the economic transactions must be accompanied by active insurance of the deals.

4. Principally, there exist four variants of "postponed" for time t economic interaction:

- DR(increasing trend) = NRB(increasing trend);
- DR(increasing trend) = NRB(decreasing trend);
- DR(decreasing trend) = NRB(decreasing trend);
- DR(decreasing trend) = NRB(increasing trend).

To process successfully these active situations, we will give the formulae for calculating of the trajectories $DR(t)$, $NRB(t)$:

(A) Increasing trend:

$$DR(t) = DR(0)\exp\{a1 \sum_{i=1,2,3} [MCSi(0)\cdot t, \text{ if } t \le \mu_i; \quad MCSi(0)\cdot\mu_i + MCSi(T)\cdot(t-\mu_i), \text{ if } t > \mu_i]\};$$

(B) Decreasing trend:

$$DR(t) = DR(0)\exp\{a_2\,[t - \sum_{i=1,2,3} (MCSi(0)\cdot t, \text{ if } t \le \tau_i; \quad MCSi(0)\cdot\tau_i + MCSi(T)\cdot(t-\tau_i), \text{ if } t > \tau_i)]\};$$

(C) Increasing trend:

$$NRB(t) = NRB(0)\exp\{b1 \sum_{i=1,2,3} [MCSi(0)\cdot t, \text{ if } t \le \mu_i; \quad MCSi(0)\cdot\mu_i + MCS_i(T)\cdot(t-\mu_i), \text{ if } t > \mu_i]\};$$

(D) Decreasing trend:

$$NRB(t) = NRB(0)\exp\{b2\,[t - \sum_{i=1,2,3} MCSi(0)\cdot t, \text{ if } t \le \tau_i; \quad MCSi(0)\cdot\tau_i + MCSi(T)\cdot(t-\tau_i), \text{ if } t > \tau_i]\}.$$

We will look at the following pattern example. The economic – political stabilization of a certain country requires the reviving its economic relations with a hypothetic country (HC). For choosing of an appropriate moment of time from the time interval $[2002, 2007] \equiv [0, T]$, ($T = 5$ years) recommendations are required made by the top managers. The current and prognostic data, expressed in symbolic units, are given in Table 1.

It follows according to the above data that as a whole the economy of HC after the prognosis for 2007 is in a favorable collaboration situation but such a responsible recommendation can be given only after analysis of the situation $DR(t) = NRB(t)$ and after determining the corresponding moment of time for the beginning of the collaboration, having in mind that

$$DR(t) = DR(0)\,exp\{a1 \sum_{i=1,2,3} [MCSi(0)\cdot t, \text{ if } t \le \mu_i; \quad MCSi(0)\cdot\mu_i + MCSi(T)\cdot(t-\mu_i), \text{ if } t > \mu_i]\},$$

Factors	Moment		Strategies	Moment		MCS
	$t=0$	$t=T$		$t=0$	$t=T$	
Y	60	86,7	$X1$	0,66	0,66	0,66
FDJ	20	14,44	$X2$	0,22	0,11	(0,22; 0,11)
U	10	28.8	$X3$	0,12	0.23	0,12
A	23,3	73.3	$Y1$	0,22	0.66	0,22
DS	11,7	24.4	$Y2$	0,11	0,22	0,11
K	70	12,22	$Y3$	0.67	0,12	(0,67; 0,12)
DR	90	130				
NRB	105	110				

Table 1. The current and prognostic data (the numbers in the brackets represent the coordinates in the left and right curves of MCS - see Figures 1 and 2)

$$NRB(t) = NRB(0)\,exp\{b1 \sum_{i=1,2,3} [\, MCSi(0)\cdot t \text{ , if } t \le \mu_i \text{ ;}$$
$$MCSi(0)\cdot \mu_i + MCSi(T)\cdot(t-\mu_i) \text{ , if } t > \mu_i,]\},$$

from where we get :

$$a1 \sum_{i=1,2,3} [\, MCSi(0)t, \text{ if } t \le \mu_i\text{: } MCSi(0)\mu_i + MCSi(T)(t-\mu_i), \text{ if } t > \mu_i\,]$$

$$-b1 \sum_{i=1,2,3} [\, MCSi(0)t \text{ , if } t \le \mu_i\text{: } MCSi(0)\mu_i + MCSi(T)(t-\mu_i), \text{ if } t > \mu_i\,]$$

$= \ln(NRB(0)/DR(0))$.

We find with simple iterations that the favorable collaboration moment of time is $t = 3$, and the real scenario interval of a successful riskless interaction is $[2005, 2007]$.

Let us discuss another situation, connected with generating of a riskless management scenario of the political risk following the type of the modified method BERI [11, 22], which requires a systematic functional for determining of the MCS.

In this particular case, we will characterize the political risk with the following parameters:

- S – regime stability, subject to the following gradation: $S = 1$ (unstable regime), $S = 3$ (medium stability), $S = 5$ (stable);
- B – lack of riots, subject to the following gradation: $B = 1$ (strong riots), $B = 3$ (slight riots), $B = 5$ (there are no riots);
- I – lack of restrictions of foreign investments, subject to the following gradation: $I = 1$ (strong restrictions), $I = 3$ (medium restrictions), $I = 5$ (there are no restrictions);

- N - lack of restrictions of foreign trade, subject to the following gradation: $N = 1$ (strong restrictions), $N = 3$ (slight restrictions), $N = 5$ (there are no substantial restrictions).

If a total index is used:

$$P = S + B + I + N,$$

then the trend for the growth of each component is sequenced with an analogical trend for P. We introduce basic strategies

$$X1(t) = S(t)/P(t)\ ,\ X2(t) = B(t)/P(t)\ ,$$

$$X3(t) = I(t)/P(t)\ ,\ X4(t) = N(t)/P(t),$$

which are known at the initial moment of the interval of management:

$$S(0) = 3\ ,\ B(0) = 1\ ,\ I(0) = 3\ ,\ N(0) = 1,$$

and we will make a prognosis for the dynamic of the indicator P in a regime of its growth under the condition that it is recommended to provide minimum of the functional at the terminal point of the changing interval [2003, 2005]:

$$F = [N(T) - 3.5]^2,$$

i. e. at a certain free level of the foreign economic business.

The methodology of minimization of that and of similar functionals in the space MCS has been developed, for example in [2, 9], and, therefore, we can give the ready solution:

$$S(T) = 4,2\ ;\ B(T) = 2,8\ ;\ I(T) = 3,3\ ;\ N(T) = 3,5\ ,$$

which proves that similar achievement in the sphere of regulating the political risk is provided by increasing the regime stability, removing the restrictions of the foreign investments and decreasing the rate of law breaking. When a single system of parameter gradation is provided, then the received solution gives completely a concrete reference point.

REFERENCES

[1] Кочетов, Е. Г., *Геоэкономика.* Изд. БЕК, Москва, 2002.

[2] Боков, В. В., *Методология стратегического управления внешнеэкономической деятелностю.* Автореферат диссертации доктора экономических наук, Москва, 2000.

[3] Ваисборд, Е. М., Юковскии, В. И., *Введение в Дифференциалние Игри Несколких Лиц и их Приложения.* Совецкое радио, Москва, 1984.

[4] Мазун, Л., *Управление Политическими Рисками в Корпорациях США.* Страховое дело, но.1, Москва, 1997, 50–52.

[5] Ненов, Т., *Управление на риска.* Свиштов, Булгариа, 2000 (Nenov, T., *Management of the Risk.* Svishtov, Bulgaria, 2000).

[6] Турбина, К. Е., *Инвестиционные процессы и страхование инвестиции от политических рисков.* АНКИЛ, Москва, 1995.

[7] Турбина, К. Е., *Политические риски в инвестиционном процессе,* Управление риском, Но.1, Москва, 1997, 25–33.

[8] Чернов, В. А., *Анализ коммерческого риска.* Под редакции М. И. Баканова, Финанси и статистика, Москва, 1998.

[9] Bokov, V., Bilchev, S., Gladyshev, V., Model of controllable macroeconomic equilibrium and optimal equipment of economic system by labor force. In: *Proceedings of the* 24^{th} *Spring Conference of the Union of Bulgarian Mathematicians,* Svishtov, 1995, 318–323.

[10] Coplin, W., Kegley, Ch. W., (Eds). *Analizing International Relations, A Multimethod Introduction.* Praeger, New York, 1976.

[11] Coplin, W., O'Leary, M., *Systematic Political Risk Analysis for Planners, Planning Rev.* January 1983, 14–17.

[12] Gladwin, T., Walter, I., *Multinationals under Fire: Lessons in the Management of Conflict.* Wiley, New York, 1980.

[13] Robertson, D., Kellow, A., (Eds.), *Globalization and the Environment: Risk Assessment and the WTO.* Cheltenham, Elgar, 2001.

[14] Grossman, H. J., Van Huyck, J. B., Nominal sovereign, debt, risk shifting and reputation. *Journal of Economics and Business,* **45** (1993), no. 3/4, 341–352.

[15] Hibbs, D., Political parties and macroeconomic policy. *American Political Science Review.* **71** (1977), 1467–1482.

[16] Krayenbuehl, T. E., *Country Risk.* Woodhead - Faulkner, Cambridge, 1985.

[17] Loscher, G., *Das Politische Risiko bei Auslandsinvestitionen.* GBI-Verlag, Munchen, 1994.

[18] Overholt, W. H., *Political Risk.* Euromoney Publishing, London, 1983.

[19] Collins, J. M., Ruefli, T. M., Strategic risk: an ordinal approach. *Management Science,* **38** (1992), no. 12, 1707–1731.

[20] Soros, G., *The International Debt Problem.* Morgan Stanley, New York, 1983.

[21] von Stackelberg, H., *Marktform und Gleichgewicht.* Springer, Berlin, 1934 (English translation: *The Theory of Market Economy.* Oxford University Press, Oxford, 1952).

[22] Ting, W., *Multinational Risk Assessment and Management.* Quorum Books, New York, 1988.

Ģeoekonomiskie un ģeopolitiskie riski: mūsdienu problēmas un risinājumi

Kopsavilkums

Tiek apskatīti ģeoekonomiskie un ģeopolitiskie riski, kurus izraisa valstu ekonomiskā un politiska mijiedarbība. Šī mērķa sasniegšanai izmanto diferencialo spēļu teoriju un tiek iztirzāti daži ilustrējoši piemeri.

Boundedness of weak solutions of a conductive-radiative heat transfer problem

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In this paper the question about boundedness of the weak solutions of a nonlinear elliptic boundary value problem is considered. As result, a simple formula is obtained, which shows dependence of the lower and upper bounds of the weak solutions from the parameters of the problem.

Key words: boundary value problem, boundedness of solutions, conductive-radiative heat transfer, elliptic equation, integral boundary conditions

Mathematics Subject Classification (2000): 35B50, 35J65

1. Introduction

It is well known that elliptic boundary value problems with sufficiently smooth coefficients have bounded weak solutions. The methods developed in [3], [4] are applicable to general nonlinear elliptic problems with unbounded coefficients.

Although methods proposed in [3], [4] work well in general situations, the obtained results are not as strong as it is necessary for some specific applications. It is especially in the case when one wants to understand the dependence of the upper and lower bounds of weak solutions on various parameters of the problem. The estimates of bounds given by general methods are inaccurate and have complicated dependence on the parameters of the problem.

In this paper we deal with a nonlinear elliptic boundary value problem, which describes combined conductive-radiative heat transfer in a physical system. Our goal is to get accurate estimates for the upper and lower bounds for the weak solutions of the problem. Since the equation and the boundary conditions of the problem in our case include only bounded coefficients, the desired goal could be achieved by methods similar to those ones which were used to prove the maximum principle for elliptic boundary value problems [3].

Let $\Omega = [0, l] \times \Pi \subset \mathbb{R}^3$ be a bounded cylindrical Lipschitz domain without holes, i.e. $\Pi \subset \mathbb{R}^2$ is a bounded Lipschitz domain homeomorphic to the unit ball $B(0,1) \subset \mathbb{R}^2$. Let $\Sigma_s = [0, l] \times \partial\Pi$ be the lateral surface and let Σ_{ht} be some surface in $\mathbb{R}^3$, which is measurable with respect to Lebesgue surface measure. Suppose that the boundary value problem is given by the integral equality

$$\int_\Omega (k_1(\nabla(u+\phi)\cdot\nabla\psi) + k_2(u+\phi)_{x_1}\psi)\,dv + \int_{\Sigma_s} G_1(|u+\phi|^3(u+\phi))\psi\,ds = \int_{\Sigma_s} G_2(|\lambda|^3\lambda)\psi\,ds \quad \forall\psi\in\dot{V}_5,$$

where the functional space $\dot{V}_5$, the linear operators $G_1 : L_{5/4}(\Sigma_s) \mapsto L_{5/4}(\Sigma_s)$, $G_2 : L_{5/4}(\Sigma_{ht}) \to L_{5/4}(\Sigma_s)$ and the parameters k_1, k_2, λ, ϕ will be introduced in the next section. In this paper we show that if $u \in \dot{V}_5$ is a weak solution of the boundary problem and if the parameters λ, ϕ have appropriate properties, then the following simple estimate holds:

$$0 \le (\phi+u) \le \max\{\|\lambda\|_{L_\infty(\Sigma_{ht})}, \|\phi\|_{L_\infty(\partial\Omega\setminus\Sigma_s)}\}.$$

2. Preliminaries

In this section we give formulation for the boundary value problem and list some results. It is important to note that here we use essentially the methodology from the paper [5], which deals with similar mathematical models of conductive-radiative heat transfer.

Let

$$\Omega_\infty := \{x\in\mathbb{R}^3 : (x_2, x_3)\in\Pi\},$$
$$\Omega := \{x\in\mathbb{R}^3 : 0 < x_1 < l, (x_2,x_3)\in\Pi\},$$

where l is a positive constant and $\Pi \subset \mathbb{R}^2$ is a bounded Lipschitz domain homeomorphic to the unit ball $B(0,1)\subset\mathbb{R}^2$. Let

$$\{\Omega_i \subset \mathbb{R}^3 : i = 1,..,n\}$$

be a finite set of bounded Lipschitz domains such that

$$\overline{\Omega}_\infty \cap \overline{\Omega}_i = \emptyset \quad i\in\{1,...,n\},$$
$$\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset \quad i,j\in\{1,...,n\}, i\ne j.$$

We introduce the abbreviations

$$
\begin{aligned}
&\Sigma_s := [0,l] \times \partial\Pi, \\
&\Sigma_{in} := \partial\Omega \backslash \Sigma_s, \\
&\Sigma_{in}^1 := \{x \in \Sigma_{in} : x_1 = 0\}, \\
&\Sigma_{in}^2 := \{x \in \Sigma_{in} : x_1 = l\}, \\
&\Sigma_{ht} := \bigcup_{i \in \{1,\dots,n\}} \partial\Omega_i, \\
&\Sigma_r := \Sigma_s \cup \Sigma_{ht}, \\
&\Omega_r := (\bigcup_{i \in \{1,\dots,n\}} \Omega_i) \cup \Omega_\infty,
\end{aligned}
$$

and use also the following notation further in the text.

Let $A \subset B$. Then we denote by $\chi(A, B)$ the indicator function $\chi : B \to \{0, 1\}$ of the subset A.

We denote by $\mathfrak{L}(X, Y)$ the space of the linear bounded operators that map a Banach space X into a Banach space Y. Let I stand for the identity operator.

We denote by $L_p(\cdot)$ the standard Lebesgue spaces $(1 \le p \le \infty)$ and by $W_2^1(\cdot)$ the standard Sobolev spaces. Let $\dot{V}_5$, V_5 be the Banach spaces:

$$
\begin{aligned}
(V_5, \|\cdot\|_{V_5}) &:= (\{u \in W_2^1(\Omega) : u \mid \Sigma_s \in L_5(\Sigma_s)\}, \|\cdot\|_{W_2^1(\Omega)} + \|\cdot\|_{L_5(\Sigma_s)}), \\
(\dot{V}_5, \|\cdot\|_{\dot{V}_5}) &:= (\{u \in V_5 : u \mid_{\Sigma_{in}} = 0\}, \|\cdot\|_{W_2^1(\Omega)} + \|\cdot\|_{L_5(\Sigma_s)}).
\end{aligned}
$$

We suppose that there is a boundary value problem defined on Ω:

$$
\int_\Omega (k_1(\nabla(u+\varphi)\cdot\nabla\psi) - k_2(u+\varphi)_{x_1}\psi)\,dv + \int_{\Sigma_s} G_1(|u+\varphi|^3(u+\varphi))\psi\,ds
= \int_{\Sigma_s} G_2(|\lambda|^3\lambda)\psi\,ds \quad \forall\psi \in \dot{V}_5, \quad (1)
$$

where

$$
\begin{aligned}
&k_1 = const > 0, k_2 = const, \\
&\varphi \in V_5, \varphi \ge 0 \text{ a.e. on } \Sigma_{in}, \lambda \in L_5(\Sigma_{ht}), \\
&G_1 \in \mathfrak{L}(L_{5/4}(\Sigma_s), L_{5/4}(\Sigma_s)), G_2 \in \mathfrak{L}(L_{5/4}(\Sigma_{ht}), L_{5/4}(\Sigma_s)).
\end{aligned}
$$

We assume that the boundary value problem (1) is derived from the conductive-radiative heat transfer problem:

$$\int_{\Omega}(k_1(\nabla(u+\phi)\cdot\nabla\psi)+k_2(u+\phi)_{x_1}\psi)\,dv+\int_{\Sigma_s} g\psi\,ds=0 \quad \forall\psi\in \dot{V}_5, \tag{2}$$

$$\rho(x)-(1-\epsilon(x))\int_{\Sigma_r} k(x,y)\rho(y)\,ds(y)=\epsilon(x)\sigma|q(x)|^3 q(x) \text{ a.e. } x\in\Sigma_r, \tag{3}$$

$$g(x)=\rho(x)-\int_{\Sigma_r} k(x,y)\rho(y)\,ds(y) \text{ a.e. } x\in\Sigma_s, \tag{4}$$

$$q\,|_{\Sigma_s}=u+\phi, \tag{5}$$

$$q\,|_{\Sigma_{ht}}=\lambda, \tag{6}$$

where

$$k_1=const>0, k_2-const, \sigma=const>0,$$
$$\phi\in V_5, \phi\geq 0 \text{ a.e. on } \Sigma_{in}, \lambda\in L_5(\Sigma_{ht}),$$
$$\epsilon\in L_\infty(\Sigma_r), \epsilon_0\leq\epsilon\leq 1 \text{ a.e. on } \Sigma_r, \epsilon_0=const>0.$$

Here the function $k:\Sigma_r\times\Sigma_r\mapsto\mathbb{R}$ from (3), (4) is defined by a mapping

$$((x,y)\in\Sigma_r\times\Sigma_r)\mapsto k(x,y):=w(x,y)\theta(x,y),$$
$$w(x,y):=\frac{\cos(\nu(x),(y-x))\cos(\nu(y),(x-y))}{\pi|x-y|^2},$$
$$\theta(x,y):=\begin{cases}1, & \text{if } \{z\in\mathbb{R}^3: z=\tau x+(1-\tau)y, 0<\tau<1\}\cap\Omega_r=\emptyset\\ 0, & \text{otherwise}\end{cases}.$$

where $\nu(\cdot)$ denotes the outward normal of the surface Σ_r. (Note that$\nu(\cdot)$ exists almost everywhere on Σ_r since it is a Lipschitch surface).

The analytical properties of the function $k(x,y)$ and the fact that Σ_r is Lipschitz surface allow us by involving the Gauss formula) to get the following estimate:

$$0\leq\int_{\Sigma_r} k(x,y)\,ds(y)=\lim_{\delta\to 0}\int_{\Sigma_r\setminus B(x,\delta)} k(x,y)\,ds(y)=$$
$$\lim_{\delta\to 0}\int_{S^+(x,\delta)\cap Q(x,\delta)}\frac{\cos(\nu(x),(y-x))}{\pi\delta^2}\,ds(y)\leq 1 \text{ a.e. } x\in\Sigma_r, \tag{7}$$

where

$$B(x,\delta):=\{z\in\mathbb{R}^3:|z-x|\leq\delta\},$$
$$S^+(x,\delta):=\{z\in\mathbb{R}^3:|z-x|=\delta,(\nu(x)\cdot(z-x))\geq 0\},$$
$$Q(x,\delta):=\{z\in\mathbb{R}^3: z=x+\tau y, y\in\Sigma_r\setminus B(x,\delta),\tau\geq 0\}.$$

The cylindrical shape of the domain Ω_∞ and the positive distance between the surfaces Σ_s, Σ_{ht} guaranty that there exists a constant $0\leq c_1<1$ such that for every $\delta>0$ and for a.e. $x\in\Sigma_s$ the following estimate holds:

$$\text{mes}(S^+(x,\delta)\cap Q(x,\delta))\leq 2c_1\delta^2.$$

This yields the existence of a much stronger local estimate on the surface Σ_s than (7). There exists a constant $c_2 < 1$ such that:

$$0 \leq \int_{\Sigma_r} k(x, y)\, ds(y) \leq c_2 \text{ a.e. } x \in \Sigma_s. \tag{8}$$

The estimate (7) implies that the mapping

$$(u \in L_p(\Sigma_r)) \mapsto K(u) := \int_{\Sigma_r} k(x, y) u(y)\, ds(y)$$

defines the operator $K \in \mathfrak{L}(L_p(\Sigma_r), L_p(\Sigma_r))$ and

$$\|K\|_{\mathfrak{L}(L_p(\Sigma_r), L_p(\Sigma_r))} \leq 1 \tag{9}$$

for every constant $1 \leq p \leq \infty$ ([2]).

Suppose that the mappings

$$\begin{aligned}
&(u \in L_{5/4}(\Sigma_r)) \mapsto E_1(u) := (1-\epsilon)u,\\
&(u \in L_{5/4}(\Sigma_r)) \mapsto E_2(u) := \epsilon u,\\
&(u \in L_{5/4}(\Sigma_s)) \mapsto P_1(u) := \begin{cases} u(x) & x \in \Sigma_s \\ 0 & x \in \Sigma_{ht} \end{cases};\\
&(u \in L_{5/4}(\Sigma_{ht})) \mapsto P_2(u) := \begin{cases} u(x) & x \in \Sigma_{ht} \\ 0 & x \in \Sigma_s \end{cases}
\end{aligned}$$

define the operators $E_1 \in \mathfrak{L}(L_{5/4}(\Sigma_r), L_{5/4}(\Sigma_r))$, $E_2 \in \mathfrak{L}(L_{5/4}(\Sigma_r), L_{5/4}(\Sigma_r))$, $P_1 \in \mathfrak{L}(L_{5/4}(\Sigma_s), L_{5/4}(\Sigma_r))$ and $P_2 \in \mathfrak{L}(L_{5/4}(\Sigma_{ht}), L_{5/4}(\Sigma_r))$. As the estimate (9) holds, it allows us to exclude the variables g, ρ, q from the systems (2), (3), (5), (6), (4) and to obtain the boundary value problem (1). Then the operators G_1, G_2 from (1) will have the following form ([5]):

$$(u \in L_{5/4}(\Sigma_s)) \mapsto G_1(u) := (\sigma(I - K)(\sum_{i=0}^{\infty}(E_1 K)^i) E_2 P_1(u)) \,|_{\Sigma_s},$$

$$(u \in L_{5/4}(\Sigma_{ht})) \mapsto G_2(u) := (\sigma(I - K)(\sum_{i=0}^{\infty}(E_1 K)^i) E_2 P_2(u)) \,|_{\Sigma_s}.$$

Furthermore, if we denote

$$H_1 := I - \frac{G_1}{\sigma}, H_2 := \frac{G_2}{\sigma},$$

then the following properties will hold ([5]):

$$G_1 \in \mathfrak{L}(L_p(\Sigma_s), L_p(\Sigma_s)),\ G_2 \in \mathfrak{L}(L_p(\Sigma_{ht}), L_p(\Sigma_s)) \text{ for all } 5/4 \le p \le \infty, \tag{10}$$

$$H_1 \in \mathfrak{L}(L_p(\Sigma_s), L_p(\Sigma_s)),\ H_2 \in \mathfrak{L}(L_p(\Sigma_{ht}), L_p(\Sigma_s)) \text{ for all } 5/4 \le p \le \infty, \tag{11}$$

$$\|H_1\|_{\mathfrak{L}(L_p(\Sigma_s), L_p(\Sigma_s))} \le 1 \text{ for all } 5/4 \le p \le \infty. \tag{12}$$

$$G_1(\chi(\Sigma_s, \Sigma_s)) - G_2(\chi(\Sigma_{ht}, \Sigma_{ht})) \ge 0 \text{ a.e. on } \Sigma_s, \tag{13}$$

$$\text{if } u \in L_1(\Sigma_s) \text{ and } u \ge 0 \text{ a.e. on } \Sigma_s, \text{ then } H_1(u) \ge 0 \text{ a.e. on } \Sigma_s, \tag{14}$$

$$\text{if } u \in L_1(\Sigma_{ht}) \text{ and } u \ge 0 \text{ a.e. on } \Sigma_{ht}, \text{ then } H_2(u) \ge 0 \text{ a.e. on } \Sigma_s. \tag{15}$$

REMARK 1. The form

$$((u, \psi) \in \dot{V}_5 \times \dot{V}_5) \to$$
$$\int_\Omega (k_1(\nabla(u+\phi) \cdot \nabla\psi) + k_2(u+\phi)_{x_1}\psi)\, dv + \int_{\Sigma_s} G_1(|u+\phi|^3(u+\phi))\psi\, ds$$

defines a pseudomonotone operator. The estimate (8) guaranties that this operator is also coercitive. Therefore, for fixed $\lambda \in L_5(\Sigma_{ht})$ the boundary value problem (1) has at least one weak solution $u \in \dot{V}_5$.

3. Lower estimate

THEOREM 3.1. *Suppose $\lambda \in L_\infty(\Sigma_{ht})$ and $0 \le \lambda$ a.e. on Σ_{ht}. If $u \in \dot{V}_5$ is the solution of* (1), *then*

$$0 \le (\phi + u) \text{ a.e. on } \Omega.$$

Proof. Suppose that inequality $0 \le (\phi + u)$ does not hold a.e. on Ω.

Then the function

$$\psi := \min\{\phi + u, 0\}$$

must not be equal to zero. Furthermore, as $\phi \ge 0$ a.e. on Σ_{in} and $u \in \dot{V}_5$, then $\psi \in \dot{V}_5$.

From the properties of the function $\psi \in \dot{V}_5$ it follows that

$$\int_\Omega k_1(\nabla(u+\phi) \cdot \nabla\psi)\, dv = \int_\Omega k_1|\nabla\psi|^2\, dv > 0. \tag{16}$$

In addition, the following equality can be obtained:

$$\begin{aligned}
&\int_\Omega k_2(u+\phi)_{x_1}\psi\, dv \\
&= \int_\Omega k_2\psi_{x_1}\psi\, dv = \int_\Omega (k_2\frac{\psi^2}{2})_{x_1}\, dv - \int_\Omega (k_2)_{x_1}\frac{\phi^2}{2}\, dv \\
&= \int_{\Sigma^2_{in}} k_2\frac{\psi^2}{2}\, ds - \int_{\Sigma^1_{in}} k_2\frac{\psi^2}{2}\, ds - \int_\Omega (k_2)_{x_1}\frac{\phi^2}{2}\, dv = 0.
\end{aligned} \tag{17}$$

Further, since $0 \le \lambda$ a.e. on Σ_{ht} and $\psi \le 0$ a.e. on Σ_s, from the condition (15) it follows that:

$$-\int_{\Sigma_s} G_2(|\lambda|^3\lambda)\psi\, ds \ge 0. \tag{18}$$

In order to obtain an estimate for the integral

$$\int_{\Sigma_s} G_1(|u+\phi|^3(u+\phi))\psi\, ds$$

we define the function

$$\gamma := \max\{\phi + u, 0\}.$$

The properties of ψ and γ yield that

$$|u+\phi|^3(u+\phi) = |\psi|^3\psi + |\gamma|^3\gamma \tag{19}$$

and

$$\psi\gamma = 0 \tag{20}$$

a.e. on Σ_s.

As the operator G_1 is linear, the equality (19) implies that

$$\int_{\Sigma_s} G_1(|u+\phi|^3(u+\phi))\psi\, ds = \int_{\Sigma_s} G_1(|\psi|^3\psi)\psi\, ds + \int_{\Sigma_s} G_1(|\gamma|^3\gamma)\psi\, ds. \tag{21}$$

From the estimate (12) it follows that

$$\begin{aligned}
&\int_{\Sigma_s} G_1(|\psi|^3\psi)\psi\, ds\\
&\quad = \sigma\Big(\int_{\Sigma_s} |\psi|^5\, ds - \int_{\Sigma_s} H_1(|\psi|^3\psi)\psi\, ds\Big)\\
&\quad \ge \sigma\|\psi\|^5_{L_5(\Sigma_s)} - \sigma\|H_1(|\psi|^3\psi)\|_{L_{5/4}(\Sigma_s)}\|\psi\|_{L_5(\Sigma_s)}\\
&\quad \ge \sigma\|\psi\|^5_{L_5(\Sigma_s)} - \sigma\|H_1\|_{\mathfrak{L}(L_{5/4}(\Sigma_s),L_{5/4}(\Sigma_s))}\|\psi\|^4_{L_5(\Sigma_s)}\|\psi\|_{L_5(\Sigma_s)}\\
&\quad = \sigma(1 - \|H_1\|_{\mathfrak{L}(L_{5/4}(\Sigma_s),L_{5/4}(\Sigma_s))})\|\psi\|^5_{L_5(\Sigma_s)} \ge 0.
\end{aligned} \tag{22}$$

If we take into account the equality (20), the property (14) and the fact that $\gamma \ge 0$, $\psi \le 0$ a.e. on Σ_s, we get the estimate

$$\begin{aligned}
&\int_{\Sigma_s} G_1(|\gamma|^3\gamma)\psi\, ds\\
&\quad = \sigma\Big(\int_{\Sigma_s} |\gamma|^3\gamma\psi\, ds - \int_{\Sigma_s} H_1(|\gamma|^3\gamma)\psi\, ds\Big)\\
&\quad = -\sigma\int_{\Sigma_s} H_1(|\gamma|^3\gamma)\psi\, ds \ge 0.
\end{aligned} \tag{23}$$

The estimates (22), (23) and the formula (21) imply that

$$\int_{\Sigma_s} G_1(|u+\phi|^3(u+\phi))\psi\, ds \geq 0. \tag{24}$$

Now, if we take into account the obtained estimates (16), (17), (18), (24), then we get

$$\int_\Omega (k_1(\nabla(u+\phi)\cdot\nabla\psi + k_2(u+\phi)_{x_1}\psi)\, dv + \int_{\Sigma_s} G_1(|u+\phi|^3(u+\phi))\psi\, ds$$
$$- \int_{\Sigma_s} G_2(|\lambda|^3\lambda) > 0.$$

But this contradicts the equality (1), which must be also valid. □

4. Upper estimate

THEOREM 4.1. *Suppose* $\phi\,|_{\Sigma_{in}} \in L_\infty(\Sigma_{in})$, $\lambda \in L_\infty(\Sigma_{ht})$ *and* $\lambda \geq 0$ *a.e. on* Σ_{ht}. *If* $u \in \dot{V}_5$ *is the solution of* (1), *then*

$$(\phi+u) \leq \max\{\|\lambda\|_{L_\infty(\Sigma_{ht})}, \|\phi\|_{L_\infty(\Sigma_{in})}\} \text{ a.e. on } \Omega.$$

Proof. To prove the theorem we will use the technique similar to the one which was used in the proof of the previous result. Suppose that the inequality

$$(\phi+u) \leq \max\{\|\lambda\|_{L_\infty(\Sigma_{ht})}, \|\phi\|_{L_\infty(\Sigma_{in})}\}$$

does not hold a.e. on Ω.

We fix a constant

$$k := \max\{\|\lambda\|_{L_\infty(\Sigma_{ht})}, \|\phi\|_{L_\infty(\Sigma_{in})}\} \tag{25}$$

and functions

$$\psi := \max\{\phi+u-k, 0\},$$
$$\gamma := \min\{\phi+u-k, 0\}+k.$$

As k is defined by the formula (25), then $\psi \in \dot{V}_5$ and ψ could not be equal to zero.

Then we have the estimates (see (16), (17)):

$$\int_\Omega k_1(\nabla(u+\phi)\cdot\nabla\psi)\, dv > 0, \tag{26}$$

$$\int_\Omega k_2(u+\phi)_{x_1}\psi\, dv = 0. \tag{27}$$

In order to estimate the integral

$$\int_{\Sigma_s} (G_1(|u+\varphi|^3(u+\varphi)) - G_2(|\lambda|^3\lambda))\psi\, ds$$

we introduce the sets

$$A := \{x \in \Sigma_s : \psi(x) > 0\},$$
$$B := \{x \in \Sigma_s : \psi(x) \leq 0\}.$$

The properties of ψ and γ yield that

$$u + \phi = \psi + \gamma$$

a.e. on Σ_s. Furthermore, as all conditions of the Theorem 3.1 are satisfied, then $(\varphi + u) \geq 0$ a.e. on Σ_s and therefore

$$|\phi + u|^3(\varphi + u) = (\varphi + u)^4 = \psi^4 + 4\psi^3\gamma + 6\psi^2\gamma^2 + 4\psi\gamma^3 + \gamma^4 \tag{28}$$

a.e. on Σ_s.

In addition the following estimates hold:

$$\gamma = k \text{ a.e. on the } A, \tag{29}$$
$$\gamma \leq k \text{ a.e. on the } B. \tag{30}$$

As the operator G_1 is linear, the equality (28) implies that

$$\begin{aligned}
&\int_{\Sigma_s} G_1(|u+\phi|^3(u+\varphi))\psi\, ds \\
&\quad = \int_{\Sigma_s} G_1(\psi^4)\psi\, ds + \int_{\Sigma_s} 4G_1(\psi^3\gamma)\psi\, ds + \int_{\Sigma_s} 6G_1(\psi^2\gamma^2)\psi\, ds \\
&\qquad + \int_{\Sigma_s} 4G_1(\psi\gamma^2)\psi\, ds + \int_{\Sigma_s} G_1(\gamma^4)\psi\, ds.
\end{aligned} \tag{31}$$

Let us estimate the expression

$$\int_{\Sigma_s} G_1(\gamma^4)\psi\, ds - \int_{\Sigma_s} G_2(|\lambda|^3\lambda)\psi\, ds.$$

By (29), (30) and (14) we get that $H_1(\gamma^4) \leq H_1(k^4)$ a.e. on Σ_s. Therefore

$$\begin{aligned}
&\chi(A,\Sigma_s)G_1(\gamma^4)\psi \\
&\quad = \sigma\chi(A,\Sigma_s)(\gamma^4\psi - H_1(\gamma^4)\psi) = \sigma\chi(A,\Sigma_s)(k^4\psi - H_1(\gamma^4)\psi) \\
&\quad \geq \sigma\chi(A,\Sigma_s)(k^4\psi - H_1(k^4)\psi) = k^4\chi(A,\Sigma_s)G_1(\chi(\Sigma_s,\Sigma_s))\psi
\end{aligned} \tag{32}$$

a.e. on Σ_s.

In addition, the following equality

$$\chi(B, \Sigma_s) G_1(\gamma^4)\psi = 0 \tag{33}$$

holds a.e. on Σ_s.

Next, if we take into account (15), we get

$$\begin{aligned} &G_2(|\lambda|^3\lambda)\psi \\ &\quad = \sigma H_2(|\lambda|^3\lambda)\psi \leq \sigma H_2(\|\lambda\|^4_{L_\infty(\Sigma_{ht})})\psi \\ &\quad = \|\lambda\|^4_{L_\infty(\Sigma_{ht})} G_2(\chi(\Sigma_{ht}, \Sigma_{ht}))\psi \end{aligned} \tag{34}$$

a.e. on Σ_s.

By means of (32), (33), (34) we get:

$$\begin{aligned} &\int_{\Sigma_s} (G_1(\gamma^4) - G_2(|\lambda|^3\lambda))\psi\, ds \\ &\quad \geq \int_{\Sigma_s} (k^4 G_1(\chi(\Sigma_s, \Sigma_s)) - \|\lambda\|^4_{L_\infty(\Sigma_{ht})} G_2(\chi(\Sigma_{ht}, \Sigma_{ht})))\psi\, ds. \end{aligned} \tag{35}$$

As $k = \max\{\|\lambda\|_{L_\infty(\Sigma_{ht})}, \|\phi\|_{L_\infty(\Sigma_{in})}\}$, then from (10), (35) it follows that

$$\begin{aligned} &\int_{\Sigma_s} (G_1(\gamma^4) - G_2(|\lambda|^3\lambda))\psi\, ds \geq \\ &\quad \int_{\Sigma_s} k^4 (G_1(\chi(\Sigma_s, \Sigma_s)) - G_2(\chi(\Sigma_{ht}, \Sigma_{ht})))\psi\, ds \geq 0. \end{aligned} \tag{36}$$

Next, we have the estimate (see (22))

$$\int_{\Sigma_s} G_1(\psi^4)\psi\, ds \geq 0. \tag{37}$$

Let us now estimate the integral

$$\int_{\Sigma_s} 4G_1(\psi^3\gamma)\psi\, ds.$$

Again, (29), (30) and (14) imply that $H_1(\psi^3\gamma) \leq H_1(\psi^3 k)$ a.e. on Σ_s. Therefore

$$\begin{aligned} &G_1(\psi^3\gamma)\psi \\ &\quad = \sigma(\gamma\psi^4 - H_1(\psi^3\gamma)\psi) = \sigma(k\psi^4 - H_1(\psi^3\gamma)\psi) \\ &\quad \geq \sigma k(\psi^4 - H_1(\psi^3)\psi)) \end{aligned} \tag{38}$$

a.e. on A.

In addition, as $\psi = 0$ a.e. on B, it follows that

$$G_1(\psi^3\gamma)\psi = \sigma k(\psi^4 - H_1(\psi^3)\psi) = 0 \tag{39}$$

a.e. on B.

Now, by means of the formulas (38), (39) we get:

$$\begin{aligned}
&\int_{\Sigma_s} 4G_1(\psi^3\gamma)\psi\, ds \\
&\quad \geq \int_{\Sigma_s} 4\sigma k(\psi^4 - H_1(\psi^3)\psi)\, ds \\
&\quad \geq 4\sigma k\|\psi\|^4_{L_4(\Sigma_s)} - 4\sigma k\|H_1(\psi^3)\|_{L_{4/3}(\Sigma_s)}\|\psi\|_{L_4(\Sigma_s)} \\
&\quad \geq 4\sigma k\|\psi\|^4_{L_4(\Sigma_s)} - 4\sigma k\|H_1\|_{\mathcal{L}(L_{4/3}(\Sigma_s),L_{4/3}(\Sigma_s))}\|\psi\|^3_{L_4(\Sigma_s)}\|\psi\|_{L_4(\Sigma_s)} \\
&\quad = 4\sigma k(1 - \|H_1\|_{\mathcal{L}(L_{4/3}(\Sigma_s),L_{4/3}(\Sigma_s))})\|\psi\|^4_{L_4(\Sigma_s)} \geq 0.
\end{aligned} \tag{40}$$

By using similar considerations we can get:

$$\int_{\Sigma_s} 6G_1(\psi^2\gamma^2)\psi\, ds \geq 0, \tag{41}$$

$$\int_{\Sigma_s} 4G_1(\psi\gamma^3)\psi\, ds \geq 0. \tag{42}$$

The estimates (36), (37), (40), (41), (42) and the formula (31) imply that:

$$\int_{\Sigma_s} (G_1(|u+\varphi|^3(u+\varphi)) - G_2(|\lambda|^3\lambda))\psi\, ds \geq 0. \tag{43}$$

Now, if we take into account the obtained estimates (26), (27), (43), we get

$$\int_\Omega (k_1(\nabla(u+\varphi)\cdot\nabla\psi) + k_2(u+\varphi)_{x_1}\psi)\, dv + \int_{\Sigma_s} G_1(|u+\varphi|^3(u+\varphi))\psi\, ds - \int_{\Sigma_s} G_2(|\lambda|^3\lambda)\psi\, ds > 0.$$

But once again, this contradicts the equality (1), which must be also valid. ☐

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REFERENCES

[1] Birgelis, K., *Optimal control in models with conductive-radiative heat transfer.* Mathematical Modelling and Analysis, **8** (2003), no. 1, 1–12.

[2] Dunford, N., Schwartz, J.T., *Linear Operators. I.* Interscience, New York, 1957.

[3] Gilbarg, D., Trudinger, N. S., *Elliptic Partial Differential Equations of Second Order.* Springer-Verlag, Berlin, 1983.

[4] Ladyzenskaja, O. A., Uralceva, N. N., *Linear and Quasilinear Elliptic Equations.* Academic Press, New York and London, 1968.

[5] Laitinen, M., Tiihonen, T., *Conductive-radiative heat transfer in grey materials.* Report B6/2000, University of Jyväskylä, 2000, Dept. of Mathematical Information Technology.

Vispārināto atrisinājumu ierobežotība kādai siltumpārneses problēmai ar siltuma vadīšanu-izstarošanu

Kopsavilkums

Šajā rakstā tiek pētīts jautājums par kādas nelineāras eliptiskas robežproblēmas vispārināto atrisinājumu ierobežotību. Tā rezultātā tiek iegūta vienkārša sakarība, kas atspoguļo to, kāda ir vispārināto atrisinājumu apakšējās un augšējās robežas atkarība no problēmas parametriem.

Remarks on lemniscatic functions

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A set of formulae is provided for the functions $\mathrm{sl}\, t$ and $\mathrm{cl}\, t$ (known as the lemniscatic functions), which solve the differential system $\frac{x'}{1-x^2} = y$, $\frac{y'}{1+y^2} = -x$, as well as the Emden - Fowler equation $x'' = -2x^3$. We discuss similarity of the theory of the lemniscatic functions and that for elementary trigonometric functions and produce a set of formulae which are similar to those for $\sin t$ and $\cos t$. The addition theorem for $\mathrm{sl}\, t$ is given in various forms, some of them seem to be new. The theory of the Jacobian elliptic functions is used.

Key words: Lemniscatic functions, Jacobian elliptic functions, addition formulae

Mathematics Subject Classification (2000): primary 34A05, **secondary** 26A99, 33E05

Let us recall that the usual trigonometric functions can be introduced by considering the differential system

$$\begin{cases} x' = y, \\ y' = -x, \\ x(0) = 0, \quad y(0) = 1. \end{cases} \tag{1}$$

Multiply the first equation by $2x$, the second one by $2y$ and sum up the both equations. One gets

$$d(x^2 + y^2) = 0$$

or

$$x^2(t) + y^2(t) = 1, \tag{2}$$

if taking into account the initial conditions in (1).

The relation (2) shows that the functions x and y define a unit circle.

It follows from (1) that

$$x'' = -x, \tag{3}$$

$$x(0) = 0, \quad x'(0) = 1.$$

Since the equation (3) is autonomous, any function $x(\alpha + t)$, where α is a constant, is also a solution of (3). The functions $x(t)$ and $y(t)$ are linearly independent solutions of (3) because the Wronskian

$$\det\begin{pmatrix} x(t) & y(t) \\ x'(t) & y'(t) \end{pmatrix} = \det\begin{pmatrix} x(t) & y(t) \\ y(t) & -x(t) \end{pmatrix} = -x^2(t) - y^2(t) = -1 \neq 0.$$

Then by properties of linear second order differential equations

$$x(\alpha + t) = C_1 x(t) + C_2 y(t), \tag{4}$$

where C_1 and C_2 are some constants to be found. Set $t = 0$. Then $x(\alpha) = C_1 x(0) + C_2 y(0) = C_2$. Since

$$x'(\alpha + t) = y(\alpha + t) = C_1 x'(t) + C_2 y'(t) = C_1 y(t) - C_2 x(t),$$

one obtains that

$$y(\alpha) = C_1 y(0) - C_2 x(0) = C_1.$$

Thus

$$x(\alpha + t) = y(\alpha) x(t) + x(\alpha) y(t). \tag{5}$$

The relation (5) is the so called *addition theorem* for the function $x(t)$ or simply the usual formula for a sine of two arguments $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$.

Any other important property of $\sin t$ and $\cos t$ can be derived from the differential system (1) (see [4], for example).

1. Nonlinear sine-like functions

We wish to use now the scheme of the previous section in order to treat the nonlinear differential system

$$\begin{cases} \dfrac{x'}{1 + x^2} = y, \\ \dfrac{y'}{1 + y^2} = -x, \\ x(0) = 0, \quad y(0) = 1. \end{cases} \tag{6}$$

Multiply the first equation by $2x$, the second one by $2y$ and sum up the both equations. One gets then

$$d\Big(\ln\big[(1 + x^2)(1 + y^2)\big]\Big) = 0$$

or

$$\ln\big[(1 + x^2)(1 + y^2)\big] = const,$$

which, in its turn, gives

$$(1 + x^2)(1 + y^2) = 2,$$

taking into account the initial conditions in (6). The latter expression may be rewritten as

$$x^2(t) + x^2(t)y^2(t) + y^2(t) = 1. \tag{7}$$

It follows from (7) that

$$x^2(t) = \frac{1 - y^2(t)}{1 + y^2(t)} \tag{8}$$

and

$$y^2(t) = \frac{1 - x^2(t)}{1 + x^2(t)}. \tag{9}$$

The relation (7) defines a closed planar curve and provides an analogue of the unit circle (2).

We cannot use arguments of the previous section to deduce an addition theorem for the functions $x(t)$ and $y(t)$, defined by (6), because the differential system of (6) is nonlinear and does not allow the representation (4).

2. Lemniscatic functions

Let us rewrite the differential equations in (6) in the form

$$\begin{cases} x' = y\,(1 + x^2), \\ y' = -x\,(1 + y^2) \end{cases} \tag{10}$$

and differentiate the system (10). One obtains by using the relations (10) and (9) that

$$\begin{aligned} x'' &= y'(1 + x^2) + y \cdot 2xx' = -x(1 + x^2)(1 + y^2) + 2xy \cdot x' \\ &= -2x\,[1 - y\,x'] = -2x\,\left[1 - y^2(1 + x^2)\right] = -2x\,\left[1 - \tfrac{1-x^2}{1+x^2}(1 + x^2)\right] \\ &= -2\,x^3. \end{aligned}$$

It follows similarly, by virtue of (10) and (8), that

$$\begin{aligned} y'' &= -x'(1 + y^2) - x \cdot 2yy' = -y(1 + x^2)(1 + y^2) - 2xy \cdot y' \\ &= -2y\,[1 + x\,y'] = -2y\,\left[1 - x^2(1 + y^2)\right] = -2y\,\left[1 - \tfrac{1-y^2}{1+y^2}(1 + y^2)\right] \\ &= -2\,y^3. \end{aligned}$$

So it turns out that $x(t)$ and $y(t)$ are solutions of the same nonlinear second order differential equation

$$u'' = -2\,u^3, \tag{11}$$

subject to the initial conditions $x(0) = 0$ and $y(0) = 1$ respectively.

Solutions of (11) satisfy the relations

$$u'^2 + u^4 = const.$$

Taking into account the initial conditions one gets that $x(t)$ and $y(t)$ satisfy the equality

$$u'^2 + u^4 = 1.$$

Then

$$\frac{du}{dt} = \pm\sqrt{1-u^4}$$

and the functions $x(t)$ and $y(t)$ can be expressed in the form

$$t = \int_0^{x(t)} \frac{ds}{\sqrt{1-s^4}} \tag{12}$$

and

$$t = \int_{y(t)}^{1} \frac{ds}{\sqrt{1-s^4}} \tag{13}$$

for $t \in [0, A]$, where $A := \int_0^1 \frac{ds}{\sqrt{1-s^4}}$. The functions defined by the integral relations (12) and (13) are known as *the lemniscatic functions* [5, § 22.8]. So $x(t)$ and $y(t)$ can be identified with $\operatorname{sl} t$ and $\operatorname{cl} t$ respectively (the notation $\operatorname{sl} t$ and $\operatorname{cl} t$ for the lemniscatic functions was introduced by C.F. Gauss).

REMARK 1. *The usual* $\sin t$ *and* $\cos t$ *functions can be introduced in the same manner, namely, as* $t = \int_0^{\sin t} \frac{ds}{\sqrt{1-s^2}}$ *and* $t = \int_{\cos t}^{1} \frac{ds}{\sqrt{1-s^2}}$.

3. Jacobian elliptic functions

Let us remind basic properties of the Jacobian elliptic functions. The main three of them are $\operatorname{sn}(t;k)$, $\operatorname{cn}(t;k)$ and $\operatorname{dn}(t;k)$. They can be introduced as respective solutions of the (nonlinear) differential system

$$\begin{cases} x_1' = x_2 x_3, \\ x_2' = -x_1 x_3, \\ x_3' = -k^2 x_1 x_2, \quad 0 < k^2 < 1, \end{cases} \tag{14}$$

subject to the initial conditions

$$x_1(0) = 0, \quad x_2(0) = 1, \quad x_3(0) = 1.$$

The functions $\operatorname{sn}(t;k)$ and $\operatorname{cn}(t;k)$ are periodic $4K$-periodic and $\operatorname{dn}(t;k)$ is $2K$-periodic, where

$$K(k) = \int_0^1 \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}}.$$

The Jacobian elliptic functions satisfy the following basic relations [1, Ch. VII, § 1]:

$$\operatorname{sn}^2 t + \operatorname{cn}^2 t = 1, \quad k^2 \operatorname{sn}^2 t + \operatorname{dn}^2 t = 1, \tag{15}$$

which, in turn, imply

$$\operatorname{dn}^2 t - k^2 \operatorname{cn}^2 t = k_1^2,$$

where

$$k_1 = \sqrt{1 - k^2}.$$

The functions $\operatorname{cn}(t;k)$ and $\operatorname{dn}(t;k)$ are even and $\operatorname{sn}(t;k)$ is odd.

The addition theorems for the Jacobian elliptic functions are known, namely [3, P. 753-765]:

$$\begin{aligned}
\operatorname{sn}(u+v) &= (\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v + \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u)\,(1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v)^{-1},\\
\operatorname{cn}(u+v) &= (\operatorname{cn} u \operatorname{cn} v - \operatorname{sn} u \operatorname{dn} u \operatorname{sn} v \operatorname{dn} v)\,(1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v)^{-1},\\
\operatorname{dn}(u+v) &= (\operatorname{dn} u \operatorname{dn} v - k^2 \operatorname{sn} u \operatorname{cn} u \operatorname{sn} v \operatorname{cn} v)\,(1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v)^{-1}.
\end{aligned}$$

Other useful relations involving the Jacobian elliptic functions are:

$$\operatorname{sn}(t+K) = \frac{\operatorname{cn} t}{\operatorname{dn} t}, \quad \operatorname{cn}(t+K) = -k_1 \frac{\operatorname{sn} t}{\operatorname{dn} t}, \quad \operatorname{dn}(t+K) = k_1 \frac{1}{\operatorname{dn} t},$$

$$\operatorname{sn}(t+2K) = -\operatorname{sn} t, \quad \operatorname{cn}(t+2K) = -\operatorname{cn} t,$$

$$(\operatorname{sn} t)' = \operatorname{cn} t \operatorname{dn} t, \quad (\operatorname{cn} t)' = -\operatorname{sn} t \operatorname{dn} t, \quad (\operatorname{dn} t)' = -k^2 \operatorname{sn} t \operatorname{cn} t.$$

4. Relations between the Jacobian elliptic functions and the lemniscatic ones

Other nine Jacobian elliptic functions are introduced as some ratios involving the basic functions sn, cn and dn . functions above. In what follows we use also the function $\operatorname{sd}(t;k) = \frac{\operatorname{sn}(t;k)}{\operatorname{dn}(t;k)}$. It is known ([5, § 22.8]) that the lemniscatic functions can be expressed (at least in some neighborhood of $t = 0$) as [5, § 22.8]

$$\operatorname{sl} t = k \frac{\operatorname{sn} \frac{t}{k}}{\operatorname{dn} \frac{t}{k}}, \quad \operatorname{cl} t = \operatorname{cn} \frac{t}{k}, \quad k = \frac{1}{\sqrt{2}}. \tag{16}$$

In the sequel we derive the relations (16) on the whole real line $\mathbb{R}$ using only the definitions (11), (14) of the lemniscatic functions and properties of the Jacobian elliptic functions.

PROPOSITION 1. *$\operatorname{sl} t = k \operatorname{sd} \frac{t}{k}$ and $\operatorname{cl} t = \operatorname{cn} \frac{t}{k}$ for $k = \frac{1}{\sqrt{2}}$.*

Proof. Notice that $k = k_1 = \frac{1}{\sqrt{2}}$. Consider the functions $h(t) := k \operatorname{sd} t = k\frac{x_1(t)}{x_3(t)}$ and $g(t) := \operatorname{cn} t = x_2(t)$. It follows from (14) and (15) that

$$\begin{aligned}
h' &= \left(k\frac{x_1}{x_3}\right)' = k\frac{x_1' x_3 - x_1 x_3'}{x_3^2} = k\frac{(x_2 x_3)x_3 - x_1(-k^2 x_1 x_2)}{x_3^2}\\
&= k\frac{x_2(x_3^2 + k^2 x_1^2)}{x_3^2} = k x_2\left(1 + \frac{k^2 x_1^2}{x_3^2}\right) = k\, g(1 - h^2),
\end{aligned}$$

$$g' = -x_1x_3 = -\frac{x_1}{x_3}x_3^2 = -\frac{x_1}{x_3}(k^2x_2^2 + k^2) = -k^2\frac{x_1}{x_3}(1 + x_2^2) = -k\,h(1 + g^2).$$

The functions h and g satisfy also

$$h(0) = k\frac{x_1(0)}{x_3(0)} = 0, \quad g(0) = x_2(0) = 1.$$

Then the functions $h\left(\frac{t}{k}\right) = k\,\mathrm{sd}\,\frac{t}{k}$ and $g\left(\frac{t}{k}\right) = \mathrm{cn}\,\frac{t}{k}$ are solutions of the Cauchy problem (6). Solutions of the initial value problem (6) are unique since the right sides of the differential equations in (6) are polynomials and satisfy the Lipschitz condition in any bounded domain containing $\{(x, y) : |x| \le 1, |y| \le 1\}$. Hence the proof. □

The well known properties of $\mathrm{sl}\,t$ and $\mathrm{cl}\,t$ follow from the basic relations (16).

COROLLARY 1. *The function* $\mathrm{sl}\,t$ *is odd and the function* $\mathrm{cl}\,t$ *is even.*

COROLLARY 2. *The functions* $\mathrm{sl}\,t$ *and* $\mathrm{cl}\,t$ *are periodic with the minimal period of* $4A$, *where*

$$A = \int_0^1 \frac{ds}{\sqrt{1 - s^4}}. \tag{17}$$

COROLLARY 3. *The reduction formulae*

$$\mathrm{sl}(t + A) = \mathrm{cl}(t) \quad \textit{and} \quad \mathrm{cl}(t + A) = -\,\mathrm{sl}(t) \tag{18}$$

are valid.

REMARK 2. *Various reduction formulae can be derived for the functions* $\mathrm{sl}\,t$ *and* $\mathrm{cl}\,t$ *likely as in the case of the elementary functions* $\sin t$ *and* $\cos t$. *A constant* A *serves as the substitution for* $\pi/2$.

PROPOSITION 2. *The following relations are valid for any* $t \in \mathbb{R}$:

$$\mathrm{sl}'(t) = \mathrm{cl}(t)(1 + \mathrm{sl}^2(t)), \quad \mathrm{cl}'(t) = -\,\mathrm{sl}(t)(1 + \mathrm{cl}^2(t)),$$

$$\mathrm{sl}'^2(t) + \mathrm{sl}^4(t) = 1, \quad \mathrm{cl}'^2(t) + \mathrm{cl}^4(t) = 1, \quad \mathrm{sl}^2(t) + \mathrm{sl}^2(t)\,\mathrm{cl}^2(t) + \mathrm{cl}^2(t) = 1,$$

$$\mathrm{sl}(t + A) = \mathrm{cl}(t), \quad \mathrm{cl}(t + A) = -\,\mathrm{sl}(t), \ \textit{where}\ A = \int_0^1 \frac{ds}{\sqrt{1 - s^4}}.$$

Proof. Proofs can be found in [6, Propositions 7.4, 7.5 and 7.6, Corollary 7.3]. □

5. Summation results

The addition theorem for the lemniscatic functions was obtained by L. Euler in the integral form (for historical remarks one may consult [2, Sec. 2.3]). Let us mention that various forms of the sum formulae can be obtained directly from those for the Jacobian elliptic functions.

PROPOSITION 3.

$$\operatorname{sl}(\alpha+\beta)=\frac{\operatorname{sl}(\alpha)\operatorname{cl}(\beta)+\operatorname{cl}(\alpha)\operatorname{sl}(\beta)}{1-\operatorname{sl}(\alpha)\operatorname{sl}(\beta)\operatorname{cl}(\alpha)\operatorname{cl}(\beta)}\,. \tag{19}$$

PROPOSITION 4.

$$\operatorname{cl}(\alpha+\beta)=\frac{\operatorname{cl}(\alpha)\operatorname{cl}(\beta)-\operatorname{sl}(\alpha)\operatorname{sl}(\beta)}{1+\operatorname{sl}(\alpha)\operatorname{sl}(\beta)\operatorname{cl}(\alpha)\operatorname{cl}(\beta)}\,. \tag{20}$$

For the proofs one may consult [6].

The alternative forms of addition theorems are given below. Investigations of the sum formula for $\operatorname{sl} t$ go back to Fagnano and L. Euler [2, §§ 2.1, 2.2, 2.3]. The sum formula was obtained rather in the form

$$\operatorname{sl}(\alpha+\beta)=\frac{\operatorname{sl}(\alpha)\sqrt{1-\operatorname{sl}^4(\beta)}+\operatorname{sl}(\beta)\sqrt{1-\operatorname{sl}^4(\alpha)}}{1+\operatorname{sl}^2(\alpha)\operatorname{sl}^2(\beta)}\,.$$

It was derived from the integral relation (12) and therefore is applicable in some vicinity of zero. The formulae (19) and (20) are applicable for any t and are similar to those for the functions $\sin t$ and $\cos t$.

PROPOSITION 5.

$$\operatorname{sl}(\alpha+\beta)=\frac{\operatorname{sl}(\alpha)\operatorname{sl}'(\beta)-\operatorname{sl}'(\alpha)\operatorname{sl}(\beta)}{1+\operatorname{sl}^2(\alpha)\operatorname{sl}^2(\beta)}\,.$$

PROPOSITION 6.

$$\operatorname{sl}(\alpha+\beta)=-\frac{\operatorname{cl}'(\alpha)\operatorname{cl}(\beta)+\operatorname{cl}(\alpha)\operatorname{cl}'(\beta)}{1+\operatorname{cl}^2(\alpha)\operatorname{cl}^2(\beta)}\,.$$

PROPOSITION 7.

$$\operatorname{cl}(\alpha+\beta)=\frac{\operatorname{sl}'(\alpha)\operatorname{cl}(\beta)+\operatorname{sl}(\alpha)\operatorname{cl}'(\beta)}{1+\operatorname{sl}^2(\alpha)\operatorname{cl}^2(\beta)}\,.$$

The proofs are given in [7].

6. Formulae

We summarize here the main relations for the lemniscatic functions indicating also their counterparts in the theory of elementary trigonometric functions. Proofs are omitted since the formulae are obtained from the basic summation relations using the same type arguments as those used in the theory of elementary trigonometric functions.

$$\mathrm{sl}(\alpha \pm \beta) = \frac{\mathrm{sl}(\alpha)\,\mathrm{cl}(\beta) \pm \mathrm{cl}(\alpha)\,\mathrm{sl}(\beta)}{1 \mp \mathrm{sl}(\alpha)\,\mathrm{sl}(\beta)\,\mathrm{cl}(\alpha)\,\mathrm{cl}(\beta)}$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\mathrm{sl}(\alpha \pm \beta) = \frac{\mathrm{sl}(\alpha)\,\mathrm{sl}'(\beta) \pm \mathrm{sl}'(\alpha)\,\mathrm{sl}(\beta)}{1 + \mathrm{sl}^2(\alpha)\,\mathrm{sl}^2(\beta)}$$

$$\sin(\alpha \pm \beta) = \sin\alpha\sin'\beta \pm \sin'\alpha\sin\beta$$

$$\mathrm{sl}(\alpha \pm \beta) = \mp\frac{\mathrm{cl}(\alpha)\,\mathrm{cl}'(\beta) \pm \mathrm{cl}'(\alpha)\,\mathrm{cl}(\beta)}{1 + \mathrm{cl}^2(\alpha)\,\mathrm{cl}^2(\beta)}$$

$$\sin(\alpha \pm \beta) = \mp(\cos\alpha\cos'\beta \pm \cos'\alpha\cos\beta)$$

$$\mathrm{cl}(\alpha \pm \beta) = \frac{\mathrm{cl}(\alpha)\,\mathrm{cl}(\beta) \mp \mathrm{sl}(\alpha)\,\mathrm{sl}(\beta)}{1 \pm \mathrm{sl}(\alpha)\,\mathrm{sl}(\beta)\,\mathrm{cl}(\alpha)\,\mathrm{cl}(\beta)}$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\mathrm{cl}(\alpha \pm \beta) = \frac{\mathrm{sl}'(\alpha)\,\mathrm{cl}(\beta) \pm \mathrm{sl}(\alpha)\,\mathrm{cl}'(\beta)}{1 + \mathrm{sl}^2(\alpha)\,\mathrm{cl}^2(\beta)}$$

$$\cos(\alpha \pm \beta) = \sin'\alpha\cos\beta \pm \sin\alpha\cos'\beta$$

$$\mathrm{sl}(\alpha) \pm \mathrm{sl}(\beta) = \frac{2\,\mathrm{sl}\left(\frac{\alpha\pm\beta}{2}\right)\mathrm{sl}'\left(\frac{\alpha\mp\beta}{2}\right)}{1 + \mathrm{sl}^2\left(\frac{\alpha+\beta}{2}\right)\mathrm{sl}^2\left(\frac{\alpha-\beta}{2}\right)}$$

$$\sin\alpha \pm \sin\beta = 2\sin\left(\frac{\alpha\pm\beta}{2}\right)\sin'\left(\frac{\alpha\mp\beta}{2}\right)$$

$$\mathrm{sl}(\alpha) \pm \mathrm{sl}(\beta) = \frac{-2\,\mathrm{cl}'\left(\frac{\alpha\pm\beta}{2}\right)\mathrm{cl}\left(\frac{\alpha\mp\beta}{2}\right)}{1 + \mathrm{cl}^2\left(\frac{\alpha+\beta}{2}\right)\mathrm{cl}^2\left(\frac{\alpha-\beta}{2}\right)}$$

$$\sin\alpha \pm \sin\beta = -2\cos'\left(\frac{\alpha\pm\beta}{2}\right)\cos\left(\frac{\alpha\mp\beta}{2}\right)$$

$$\operatorname{cl}(\alpha) + \operatorname{cl}(\beta) = \frac{2\operatorname{cl}\left(\frac{\alpha+\beta}{2}\right)\operatorname{sl}'\left(\frac{\alpha-\beta}{2}\right)}{1+\operatorname{cl}^2\left(\frac{\alpha+\beta}{2}\right)\operatorname{sl}^2\left(\frac{\alpha-\beta}{2}\right)}$$

$$\cos\alpha - \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin'\left(\frac{\alpha-\beta}{2}\right)$$

$$\operatorname{cl}(\alpha) - \operatorname{cl}(\beta) = \frac{2\operatorname{cl}'\left(\frac{\alpha+\beta}{2}\right)\operatorname{sl}\left(\frac{\alpha-\beta}{2}\right)}{1+\operatorname{cl}^2\left(\frac{\alpha+\beta}{2}\right)\operatorname{sl}^2\left(\frac{\alpha-\beta}{2}\right)}$$

$$\cos\alpha - \cos\beta = 2\cos'\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\begin{aligned}\operatorname{sl}(\alpha) \pm \operatorname{sl}(\beta) &= 2\operatorname{sl}\left(\frac{\alpha\pm\beta}{2}\right)\operatorname{cl}\left(\frac{\alpha\mp\beta}{2}\right)\frac{1+\operatorname{sl}^2\left(\frac{\alpha\mp\beta}{2}\right)}{1+\operatorname{sl}^2\left(\frac{\alpha+\beta}{2}\right)\operatorname{sl}^2\left(\frac{\alpha-\beta}{2}\right)}\\ &= 2\operatorname{sl}\left(\frac{\alpha\pm\beta}{2}\right)\operatorname{cl}\left(\frac{\alpha\mp\beta}{2}\right)\frac{1+\operatorname{cl}^2\left(\frac{\alpha\pm\beta}{2}\right)}{1+\operatorname{cl}^2\left(\frac{\alpha+\beta}{2}\right)\operatorname{cl}^2\left(\frac{\alpha-\beta}{2}\right)}\end{aligned}$$

$$\sin\alpha \pm \sin\beta = 2\sin\left(\frac{\alpha\pm\beta}{2}\right)\cos\left(\frac{\alpha\mp\beta}{2}\right)$$

$$\begin{aligned}\operatorname{cl}(\alpha) + \operatorname{cl}(\beta) &= 2\operatorname{cl}\left(\frac{\alpha+\beta}{2}\right)\operatorname{cl}\left(\frac{\alpha-\beta}{2}\right)\frac{1+\operatorname{sl}^2\left(\frac{\alpha+\beta}{2}\right)}{1+\operatorname{sl}^2\left(\frac{\alpha+\beta}{2}\right)\operatorname{cl}^2\left(\frac{\alpha-\beta}{2}\right)}\\ &= 2\operatorname{cl}\left(\frac{\alpha+\beta}{2}\right)\operatorname{cl}\left(\frac{\alpha-\beta}{2}\right)\frac{1+\operatorname{sl}^2\left(\frac{\alpha-\beta}{2}\right)}{1+\operatorname{sl}^2\left(\frac{\alpha-\beta}{2}\right)\operatorname{cl}^2\left(\frac{\alpha+\beta}{2}\right)}\end{aligned}$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\begin{aligned}\operatorname{cl}(\alpha) - \operatorname{cl}(\beta) &= -2\operatorname{sl}\left(\frac{\alpha+\beta}{2}\right)\operatorname{sl}\left(\frac{\alpha-\beta}{2}\right)\frac{1+\operatorname{cl}^2\left(\frac{\alpha+\beta}{2}\right)}{1+\operatorname{cl}^2\left(\frac{\alpha+\beta}{2}\right)\operatorname{sl}^2\left(\frac{\alpha-\beta}{2}\right)}\\ &= -2\operatorname{sl}\left(\frac{\alpha+\beta}{2}\right)\operatorname{sl}\left(\frac{\alpha-\beta}{2}\right)\frac{1+\operatorname{cl}^2\left(\frac{\alpha-\beta}{2}\right)}{1+\operatorname{cl}^2\left(\frac{\alpha-\beta}{2}\right)\operatorname{sl}^2\left(\frac{\alpha+\beta}{2}\right)}\end{aligned}$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\operatorname{sl}(2\alpha) = \frac{2\operatorname{sl}(\alpha)\operatorname{cl}(\alpha)}{1 - \operatorname{sl}^2(\alpha)\operatorname{cl}^2(\alpha)}$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\operatorname{sl}(2\alpha) = \frac{2\operatorname{sl}(\alpha)\operatorname{cl}(\alpha)}{\operatorname{sl}^2(\alpha) + \operatorname{cl}^2(\alpha)}$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\operatorname{sl}(2\alpha) = \frac{2\operatorname{sl}(\alpha)\operatorname{sl}'(\alpha)}{1 + \operatorname{sl}^4(\alpha)}$$

$$\sin 2\alpha = 2\sin\alpha\sin'\alpha$$

$$\operatorname{sl}(2\alpha) = \frac{2\operatorname{sl}(\alpha)\operatorname{cl}(\alpha)\big(1 + \operatorname{sl}^2(\alpha)\big)}{1 + \operatorname{sl}^4(\alpha)}$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\operatorname{sl}(2\alpha) = \frac{-2\operatorname{cl}(\alpha)\operatorname{cl}'(\alpha)}{1 + \operatorname{cl}^4(\alpha)}$$

$$\sin 2\alpha = -2\cos\alpha\cos'\alpha$$

$$\operatorname{sl}(2\alpha) = \frac{2\operatorname{sl}(\alpha)\operatorname{cl}(\alpha)\big(1 + \operatorname{cl}^2(\alpha)\big)}{1 + \operatorname{cl}^4(\alpha)}$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\operatorname{cl}(2\alpha) = \frac{\operatorname{cl}^2(\alpha) - \operatorname{sl}^2(\alpha)}{1 + \operatorname{sl}^2(\alpha)\operatorname{cl}^2(\alpha)}$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$\operatorname{cl}(2\alpha) = \frac{\operatorname{sl}'(\alpha)\operatorname{cl}(\alpha) + \operatorname{sl}(\alpha)\operatorname{cl}'(\alpha)}{1 + \operatorname{sl}^2(\alpha)\operatorname{cl}^2(\alpha)}$$

$$\cos 2\alpha = \sin'\alpha\cos\alpha + \sin\alpha\cos'\alpha$$

$$\begin{aligned}\operatorname{sl}(\alpha)\operatorname{sl}(\beta) &= \frac{\operatorname{cl}(\alpha - \beta) - \operatorname{cl}(\alpha + \beta)}{2}\,\frac{1 + \operatorname{cl}^2(\alpha)\operatorname{sl}^2(\beta)}{1 + \operatorname{cl}^2(\alpha)}\\ &= \frac{\operatorname{cl}(\alpha - \beta) - \operatorname{cl}(\alpha + \beta)}{2}\,\frac{1 + \operatorname{sl}^2(\alpha)\operatorname{cl}^2(\beta)}{1 + \operatorname{cl}^2(\beta)}\end{aligned}$$

$$\sin\alpha\sin\beta = \frac{\cos(\alpha - \beta) - \cos(\alpha + \beta)}{2}$$

$$\begin{aligned}\operatorname{cl}(\alpha)\operatorname{cl}(\beta) &= \frac{\operatorname{cl}(\alpha+\beta)+\operatorname{cl}(\alpha-\beta)}{2}\,\frac{1+\operatorname{cl}^2(\alpha)\operatorname{sl}^2(\beta)}{1+\operatorname{sl}^2(\beta)}\\ &= \frac{\operatorname{cl}(\alpha+\beta)+\operatorname{cl}(\alpha-\beta)}{2}\,\frac{1+\operatorname{sl}^2(\alpha)\operatorname{cl}^2(\beta)}{1+\operatorname{sl}^2(\alpha)}\end{aligned}$$

$$\cos\alpha\cos\beta = \frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2}$$

$$\begin{aligned}\operatorname{sl}(\alpha)\operatorname{cl}(\beta) &= \frac{\operatorname{sl}(\alpha+\beta)+\operatorname{sl}(\alpha-\beta)}{2}\,\frac{1+\operatorname{sl}^2(\alpha)\operatorname{sl}^2(\beta)}{1+\operatorname{sl}^2(\beta)}\\ &= \frac{\operatorname{sl}(\alpha+\beta)+\operatorname{sl}(\alpha-\beta)}{2}\,\frac{1+\operatorname{cl}^2(\alpha)\operatorname{cl}^2(\beta)}{1+\operatorname{cl}^2(\alpha)}\end{aligned}$$

$$\sin\alpha\cos\beta = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2}$$

$$\frac{1+\operatorname{sl}^2(\beta)}{1+\operatorname{sl}^2(\alpha)\operatorname{sl}^2(\beta)} = \frac{1+\operatorname{cl}^2(\alpha)}{1+\operatorname{cl}^2(\alpha)\operatorname{cl}^2(\beta)} \qquad \frac{1+\operatorname{sl}^2(\alpha)}{1+\operatorname{sl}^4(\alpha)} = \frac{1+\operatorname{cl}^2(\alpha)}{1+\operatorname{cl}^4(\alpha)}$$

$$\frac{1+\operatorname{sl}^2(\beta)}{1+\operatorname{cl}^2(\alpha)\operatorname{sl}^2(\beta)} = \frac{1+\operatorname{sl}^2(\alpha)}{1+\operatorname{sl}^2(\alpha)\operatorname{cl}^2(\beta)} \qquad \frac{1+\operatorname{cl}^2(\beta)}{1+\operatorname{sl}^2(\alpha)\operatorname{cl}^2(\beta)} = \frac{1+\operatorname{cl}^2(\alpha)}{1+\operatorname{cl}^2(\alpha)\operatorname{sl}^2(\beta)}$$

REMARK 3. *The last four formulae seem to have no analogues in the elementary trigonometry. They can be proved easily by using the relations*

$$\operatorname{sl}^2(\alpha) = \frac{1-\operatorname{cl}^2(\alpha)}{1+\operatorname{cl}^2(\alpha)}, \quad \operatorname{cl}^2(\alpha) = \frac{1-\operatorname{sl}^2(\alpha)}{1+\operatorname{sl}^2(\alpha)},$$

which follow from the identity $\operatorname{sl}^2(\alpha)+\operatorname{sl}^2(\alpha)\operatorname{cl}^2(t)+\operatorname{cl}^2(t)=1$.

REFERENCES

[1] Chandrasekharan, K., *Elliptic Functions.* Springer, Berlin–Heidelberg, 1985.

[2] Hellegouarch, Y., *Invitation to the Mathematics of Fermat-Wiles.* Academic Press, London, 2002.

[3] Korn, G., Korn, T., *Mathematical Handbook.* McGraw Hill, New York, 1968.

[4] Tricomi, F., *Differential Equations.* Blackie & Son Ltd, 1961.

[5] Whittaker, E.T., Watson, G.N., *A Course of Modern Analysis. Part II.* Cambridge Univ. Press, 1927.

[6] Gritsans, A., Sadyrbaev, F., Lemniscatic functions in the theory of the Emden-Fowler differential equation. In: Lepin, A., Klokov, Y., Sadyrbaev, F., (eds.), *Proceedings Institute of Mathematics and Computer Science.* University of Latvia, Riga, **3** (2003), 5–27.

[7] Gritsans, A., Sadyrbaev, F., Trigonometry of lemniscatic functions. In: Lepin, A., Klokov, Y., Sadyrbaev, F., (eds.), *Proceedings Institute of Mathematics and Computer Science*. University of Latvia, Riga, **4** (2004), 22–29.

Piezīmes par lemniskātiskajām funkcijām

Kopsavilkums

Uzrādīta virkne formulu, kas saista lemniskātiskās funkcijas $\operatorname{sl} t$ *un* $\operatorname{cl} t$, *kuras apmierina diferenciālvienādojumu sistēmu* $\frac{x'}{1+x^2} = y$, $\frac{y'}{1+y^2} = -x$, *kā arī Emdena–Faulera vienādojumu* $x'' = -2x^3$. *Apskatīta lemniskātisko funkciju teorijas līdzība ar elementāro trigonometrisko funkciju teoriju, uzradot formulas lemniskātiskām funkcijām un to analogus funkcijām* $\sin t$ *un* $\cos t$. *Atrasti vairāki ekvivalenti saskaitīšanas teorēmas formulējumi funkcijai* $\operatorname{sl} t$, *no kuriem daži šķiet līdz šim nav tikuši apskatīti. Izklāsta tiek izmantota Jakobi eliptisko funkciju teorija.*

On a sorting problem

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A network sorting problem originated by an All-Union Math Olympiad is solved and generalized.

Key words: mathematical induction, network, sorting
Mathematics Subject Classification (2000): 68P10

On the 18th All-Union Mathematical Olympiad the following problem composed by A. Andžāns was proposed (see [1, p. 6]).

PROBLEM 1. *Let's consider a network of roads consisting of n sequential branches of parallel segments containing $k_1, k_2, \ldots, k_n$ segments correspondingly (see Fig. 1 with $n = 3$, $k_1 = 3$, $k_2 = 5$, $k_3 = 4$).*

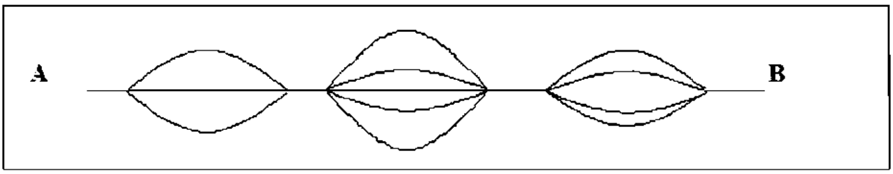

Fig. 1. An example of a network

The roads are narrow so overtaking is impossible, but the speed is allowed to vary. Movement is allowed only from the left to the right. There are N cars approaching the network at A. For what maximum value of N is it possible for the cars to leave the network at B in the reversed order?

Solution. It is easy to see that there are $W = k_1 \times k_2 \times \ldots \times k_n$ ways to traverse the network. Clearly the inequality $N \leq W$ must hold: for $N > W$ there will be two cars traversing the network in the same way, and these cars will not change their mutual position. On the other hand, the task is solvable for W cars. Clearly this holds for $n = 1$. If it holds for $n = m$, consider the network with $m+1$ branches. Divide the cars into k_1 groups, each consisting of $k_2 \times k_3 \times \ldots k_{m+1}$ consecutive cars, and at first drive these groups into separate segments of the first branch. Then apply the inductive hypothesis and drive out these groups in the reverse order, reversing each of them in the last n branches. We have proved that $N = W$ for the networks for the special type described above.

It is clear that the inequality $N \leq W$ (where W is the number of ways a network can be traversed) holds for any network with one entry and one exit, if the condition of one-way movement and non-overtaking is preserved (see, e.g., Fig. 2).

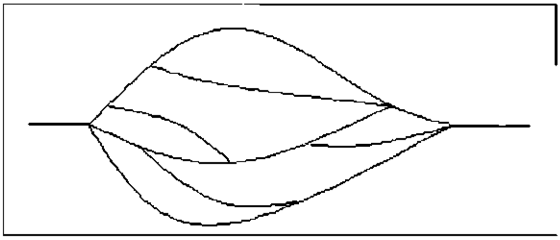

Fig. 2. A more sophistical network

Since 1986, the following question has been popular in international olympiad circles: *Does the equality $N = W$ hold for each such network?*

In this note we answer this question affirmatively and mention some generalizations of this problem.

THEOREM 1. *For each one-way non-overtaking network with one entry and one exit which can be traversed in W ways, it is possible to rearrange cars in any order while passing through this network if the number of cars doesnot exceed W.*

Proof. If there are no splitting points points (splitting points are points from which leave more than 1 road) in the network, then $W = 1$ and only one car must be rearranged. Suppose that P is the first splitting point with x exits from P. Clearly, $W = W_1 + W_2 + \ldots + W_x$, where W_i, $1 \leq i \leq x$, is the number of ways to traverse the network leaving P through the x-th exit. Now an easy induction on the number of splitting points in the network solves the problem. □

The rest of the note deals with the following generalization of the problem. Let us assume that there are N types of cars (say, Ford, Volkswagen, etc.). The initial order of them can be arbitrary. At the exit, the cars of the same type must follow each other in a row (the mutual order of the cars of the same type is not important). What is the largest value of N for which any order of the groups at the exit can be achieved?

THEOREM 2. *For each one-way non-overtaking network with one entry and one exit which can be traversed in W ways, any order of groups can be achieved for an arbitrary initial sequence of cars if the number of groups doesnot exceed W.*

THEOREM 3. *For each one-way non-overtaking network with one entry and n exits which can be traversed in $W(i)$ ways to the i-th exit, $i = 1, 2, \ldots, n$, an arbitrary order of groups in each exit can be achieved for an arbitrary initial sequence of cars if less or equal than $W(i)$ groups must appear at the i-th exit, $i = 1, 2, \ldots, n$.*

Theorems 2 and 3 are proved similarly to Theorem 1: in Theorem 2 the induction on the number of cars in the groups can be used and in Theorem 3 induction on the number of exits can be used.

THEOREM 4. *For each one-way non-overtaking network with n entries and one exit, which can be traversed in $W(i)$ ways from the i-th entry, $i = 1, 2, \ldots, n$, arbitrary order of cars at the exit can be achieved if there are no more than $W(i)$ cars at the i-th entry, $i = 1, 2, \ldots, n$.*

This theorem was proved by using the isomorphism between networks with n entries and one exit and networks with one entry and n exits (Theorem 3). Algorithms of arranging cars are opposite and conflicting for the two types of networks. Therefore a generalization for networks with n entries and n exits is impossible by using this method of arranging cars. We have found also an example (see Fig. 3) showing that Theorem 4 cannot be generalized for groups of cars. The two sequences show the initial positions - the cars are labeled with numbers, which indicates to which group the car belongs. We can easily see that the finishing sequence cannot be 4;4;3;3;2;2;1 (group number 1 finishes the first) by examining possible movements.

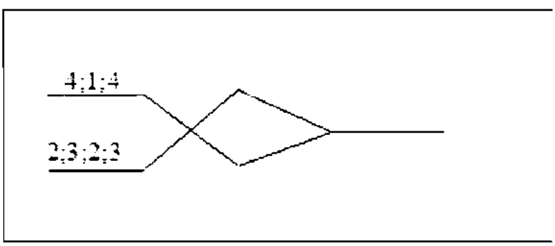

Fig. 3. A conterexample to the featured generalization of Theorem 4

REFERENCES

[1] *Задачи Отборочных Математических Олимпиад.* Под ред. Вавилова, В., Резиченко, С., МГУ, Москва., 1992.

Par kādu kārtošanas uzdevumu

Kopsavilkums

Rakstā pierādītas vairākas minimaksa teorēmas par speciāla tipa kārtošanas tīkliem. Aplūkotās problēmas radušās, vispārinot 1984. gada Vissavienības matemātikas olimpiādes uzdevumu.

Effective method of approximation of a nonlinear parabolic boundary value problem

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The approximation of a nonlinear parabolic problem is based on the finite volume method [1]. In the general case the exact finite–difference scheme approximating the problem in two–point mesh is built. The corresponding integrals are approximated using different quadrature formulae. This procedure allows one to reduce the problem described by a partial differential equation of parabolic type to an initial value problem for a system of two nonlinear ordinary differential equations of the order depending on the quadrature formulae used. It allows one to obtain the solution of the problem on the boundary of the region. Both the non–singular and singular cases are investigated and two applications of this method are presented. Numerical solutions of the corresponding algorithms are obtained using *Maple V* and *Mathematica* routines for stiff systems of ordinary differential equations.

Key words: Boundary value problem, nonlinear PDE of parabolic type, finite volume method, finite difference scheme

Mathematics Subject Classification (2000): **primary** 35K55, **secondary** 35K60, 65M06, 65M60

1. The formulation of the problem

We consider the boundary value problem for a parabolic equation in the form

$$\frac{\partial u}{\partial t} = \frac{1}{p(x)}\frac{\partial}{\partial x}\left(p(x)f'(u)\frac{\partial u}{\partial x}\right) + F(u),\ x \in (0,l), t > 0, \tag{1}$$

$$u(0,x) = u_0(x), \tag{2}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = f_1(u_1), \tag{3}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=l} = f_2(u_2), \tag{4}$$

where $u = u(t, x)$ is the unknown function,

f_1, f_2, F, f are nonlinear functions and $f'(u) > 0$,

$u_1 = u_1(t) \equiv u(t, 0)$, $u_2 = u_2(t) \equiv u(t, l)$, $f'(u) \equiv \mathrm{d}f(u)/\mathrm{d}u$.

Usually a full flux $p(x)f'(u)\dfrac{\partial u}{\partial x}$ is specified in the boundary conditions. In this case we consider that such the form of boundary conditions is already divided by $p(x)f'(u)$, $p(x) \neq 0$, i.e. nonlinear functions f_1 and f_2 already contain the factor $p(x)f'(u)$ as a denominator. The case when $p(0) = 0$ will be discussed shortly.

There, two cases should be considered:

1. **Non–singular case**, when $p(0) \neq 0$, $p(x) > 0$.
2. **Singular case**, when $p(0) = 0, p(x) \geq 0$. There we assume that $\dfrac{\mathrm{d}}{\mathrm{d}x} \ln p(x) = O\left(\dfrac{1}{x}\right)$ when $x \to 0$.

Let us write the equation (1) in an open form taking all the derivatives:

$$\frac{\partial u}{\partial t} = f'(u)\left(\frac{p'(x)}{p(x)}\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\right) + f''(u)\left(\frac{\partial u}{\partial x}\right)^2 + F(u), \tag{5}$$

where $p'(x) \equiv \mathrm{d}p/\mathrm{d}x$.

In the singular case it can be easily shown that for the solution to be bounded, the boundary condition (3) should be in the form of $f_1(u_1) \equiv 0$:

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0. \tag{6}$$

In the singular case, the behavior of the equation (1) changes when $x \to 0$.

$$\left.\frac{\partial u}{\partial t}\right|_{x=0} = (1 - \alpha)f'(u_1)\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=0} + F(u_1), \tag{7}$$

where $\alpha = \mathit{const}$. is determined by $x(\ln p(x))' \xrightarrow[x\to 0]{} \alpha$.

The problem (1 – 4) could describe many of the physical processes. One of the models could be considered in this case is a heat transfer problem. In this case, the unknown function $u(t, x)$ describes the dimensionless temperature distribution in a one–space–dimensional domain, the function $f'(u)$ is the nonlinear conductivity, $F(u)$ describes the heat sources (for example, arising from chemical reactions, or a dissipative function), and the nonlinear functions f_1, f_2 in the boundary conditions describe the radiation from heaters and convection. The nonlinear function $p(x)$ can be interpreted as follows: $p(x) = x^\alpha$. Then, for different values of α we obtain the problem in different coordinates:

- $\alpha = 0$ — cartesian coordinate system,

- $\alpha = 1$ – cylindrical coordinate system (cylindrical domain) with axial symmetry,
- $\alpha = 2$ – spherical coordinate system (spherical domain) with radial symmetry.

The problem (1 – 3) in general case is nonlinear and cannot be solved analytically and therefore it has to be approximated to solve it numerically.

2. Consistency conditions

We assume the initial condition (2) to be consistent with the boundary conditions (3), (4) in the form

$$\begin{cases} u_0'(0) = f_1(u_1(0)) \\ u_0'(l) = f_2(u_2(0)) \end{cases} . \tag{8}$$

Knowing the function u_0 in the initial condition (2), from (8) we can obtain the following relations:

$$\begin{cases} u_1(0) = \mu_1,\ \dot{u}_1(0) = \mu_3 \\ u_2(0) = \mu_2,\ \dot{u}_2(0) = \mu_4 \end{cases} , \tag{9}$$

where $\mu_i = const.$, $i = \overline{1,4}$ and $\dot{u}_{1,2} \equiv \dfrac{\partial u_{1,2}}{\partial t}$.

Let us show an example of it.

2.1. Non–singular case We assume that at the initial moment of time $t = 0$ the function $u_0(x)$ takes the following values at the both ends of the interval $[0, l]$:

$$u_0(0) = \nu_1, \quad u_0(l) = \nu_2$$

We assume the function u_0 to have the form of a cubic polynomial:

$$u_0(x) = Ax^3 + Bx^2 + Cx + \nu_1. \tag{10}$$

Applying this expression to the consistency conditions (8), we obtain that $C = f_1(\nu_1)$ and the following system of two linear algebraic equations for the coefficients A, B:

$$\begin{cases} Al^2 + Bl = \dfrac{\nu_2 - \nu_1}{l} - f_1(\nu_1) \\ 3Al^2 + 2Bl = f_2(\nu_2) - f_1(\nu_1) \end{cases} ,$$

which solves to

$$\begin{cases} A = \dfrac{1}{l^2}\left[f_1(\nu_1) + f_2(\nu_2) + \dfrac{2}{l}(\nu_1 - \nu_2)\right] \\ B = \dfrac{1}{l}\left[\dfrac{3}{l}(\nu_2 - \nu_1) - 2\,f_1(\nu_1) - f_2(\nu_2)\right] \end{cases} .$$

Applying (10) to the open form of the equation (5) and taking in account that $u_1(0) = u_0(0)$, $u_2(0) = u_0(l)$, we obtain:

$$\begin{cases} u_1(0) = \nu_1 \\ u_2(0) = \nu_2 \\ \dot{u}_1(0) = f'(\nu_1)\left[\frac{p'(0)}{p(0)}C + 2B\right] + f''(\nu_1)C^2 + F(\nu_1) \\ \dot{u}_2(0) = f'(\nu_2)\left[\frac{p'(0)}{p(0)}\left(3Al^2 + 2Bl + C\right) + 6Al + 2B\right] + \\ \qquad f''(\nu_2)\left(3Al^2 + 2Bl + C\right)^2 + F(\nu_2) \end{cases}. \tag{11}$$

2.2. Singular case In this case the initial value of $u_0(0)$ is meaningless. Reviewing the physical interpretation given in the section 1, it is clear that it is physically impossible to measure the temperature in the center of a cylinder or sphere. The same considerations are right also for other usage cases.

Assuming that at the initial moment of time $u_0(l) = \nu$ it is easily visible that the expression

$$u_0(x) = \nu - \frac{l^2 - x^2}{2l} f_2(\nu), \tag{12}$$

satisfies the consistency conditions (8). Applying it to the open form of the equation (5) and taking in account the behavior of the equation (1) when $x \to 0$ (7), we obtain

$$\begin{cases} u_1(0) = \nu - \frac{l}{2} f_2(\nu) \\ u_2(0) = \nu \\ \dot{u}_1(0) = \frac{1-\alpha}{l} f'(\nu - \frac{l}{2} f_2(\nu)) f_2(\nu) + F(\nu - \frac{l}{2} f_2(\nu)) \\ \dot{u}_2(0) = f'(\nu) f_2(\nu) \frac{lp'(l) + p(l)}{lp(l)} + f''(\nu) f_2^2(\nu) + F(\nu) \end{cases}. \tag{13}$$

3. Finite Volume method for two–point scheme

Let us rewrite the equation (1) in the form

$$p(x)G(t,x) = W'(t,x), \tag{14}$$

where $G(t,x) \equiv \frac{\partial u}{\partial t} - F(u)$ and $W(t,x) \equiv p(x) f'(u) \frac{\partial u}{\partial x}$.

For the numerical approximation, we select only two grid points $x_1 = 0$ and $x_2 = l$.

We will separate both the non–singular and singular cases and review the Finite Volume method for each of them separately.

3.1. Non–singular case By integrating the equation (14) from x_1 to $x = l/2$, we obtain the integral form of the conservation law within the interval $[0, l/2]$:

$$W_{0.5} - W_0 = \int_0^{l/2} p(x)G(t,x)\,\mathrm{d}x, \tag{15}$$

where $W_{0.5} \equiv W(t, l/2)$, $W_0 \equiv W(t,0)$.

The function W_0 is known from the boundary condition (3): $W_0 = p(0)f'(u_1)f_1(u_1)$. To obtain an expression for $W_{0.5}$, we integrate the equation (14) from $x = l/2$ to $x \in (0, l)$ and then from x_1 to x_2, thus obtaining

$$W_{0.5} = \frac{f(u_2) - f(u_1)}{a(l)} - \frac{1}{a(l)} \int_0^l \frac{\mathrm{d}x}{p(x)} \int_{l/2}^x p(\xi)G(t,\xi)\,\mathrm{d}\xi,$$

where $a(x) \equiv \int_0^x \dfrac{\mathrm{d}\xi}{p(\xi)}$. Note that in singular case this integral is divergent.

Applying it to (15), we obtain

$$\frac{f(u_2) - f(u_1)}{a(l)} - W_0 = R_1 \equiv \int_0^{l/2} p(x)G(t,x)\,\mathrm{d}x + \frac{1}{a(l)} \int_0^l \frac{\mathrm{d}x}{p(x)} \int_{l/2}^x p(\xi)G(t,\xi)\,\mathrm{d}\xi.$$

Let us integrate the second part of the integral R_1 by the partial integration.

$$\frac{1}{a(l)} \int_0^l \frac{\mathrm{d}x}{p(x)} \int_{l/2}^x p(\xi)G(t,\xi)\,\mathrm{d}\xi =$$
$$\frac{1}{a(l)} \int_0^l \left\{ \frac{\mathrm{d}}{\mathrm{d}x} \left(\int_0^x \frac{\mathrm{d}\xi}{p(\xi)} \right) \int_{l/2}^x p(\xi)G(t,\xi)\,\mathrm{d}\xi \right\} \mathrm{d}x =$$
$$\int_{l/2}^l p(x)G(t,x)\,\mathrm{d}x - \frac{1}{a(l)} \int_0^l a(x)p(x)G(t,x)\,\mathrm{d}x.$$

Summing both the integrals contained in R_1, we obtain the following difference equation associated with the point $x_1 = 0$:

$$\frac{f(u_2) - f(u_1)}{a(l)} - p(0)f'(u_1)f_1(u_1) = R_1 \equiv \int_0^l \left(1 - \frac{a(x)}{a(l)}\right) p(x)G(t,x)\,\mathrm{d}x. \tag{16}$$

By integrating the equation (14) from $x = l/2$ to x_2, we obtain the integral form of the conservation law within the interval $[l/2, l]$:

$$W_1 - W_{0.5} = \int_{l/2}^l p(x)G(t,x)\,\mathrm{d}x, \tag{17}$$

where from (4) $W_1 \equiv W(t,l) = p(l)f'(u_2)f_2(u_2)$.

As the value of $W_{0.5}$ was calculated above, applying it to (17) we obtain

$$\frac{f(u_1) - f(u_2)}{a(l)} + W_1 = R_2 \equiv \int_{l/2}^l G(t,x)\,\mathrm{d}x - \frac{1}{a(l)} \int_0^l \frac{\mathrm{d}x}{p(x)} \int_{l/2}^x p(\xi)G(t,\xi)\,\mathrm{d}\xi.$$

The second part of the integral R_2 is calculated above. Summing both the parts, we obtain the difference equation associated with the point $x_2 = l$:

$$\frac{f(u_1) - f(u_2)}{a(l)} + p(l)f'(u_2)f_2(u_2) = R_2 \equiv \int_0^l \frac{a(x)}{a(l)} p(x)G(t,x)\,\mathrm{d}x. \qquad (18)$$

The two–point finite–difference scheme (16), (18) is exact for a given function G. Summing both the equations yields

$$p(l)f'(u_2)f_2(u_2) - p(0)f'(u_1)f_1(u_1) = \int_0^l p(x)G(t,x)\,\mathrm{d}x,$$

which can be used instead of one of the equations (16), (18). This equation also follows directly from (14) and the boundary conditions (3), (4).

3.2. Singular case In this case we have to deal with a different equation (7) at the point x_1. To avoid such a situation, we introduce a small value $\varepsilon > 0$. Then, by integrating the equation (14) from $x = \varepsilon$ to $x = l/2$, we obtain the integral form of the conservation law within the interval $[\varepsilon, l/2]$:

$$W_{0.5} - W_\varepsilon = \int_\varepsilon^{l/2} p(x)G(t,x)\,\mathrm{d}x, \qquad (19)$$

where $W_\varepsilon \equiv W(t,\varepsilon)$.

The value of $W_{0.5}$ was calculated above. Though all zeros in the integrals should be changed to ε, and we obtain

$$W_{0.5} = \frac{f(u_2) - f(u_\varepsilon)}{a_\varepsilon(l)} - \frac{1}{a_\varepsilon(l)} \int_\varepsilon^l \frac{\mathrm{d}x}{p(x)} \int_{l/2}^x p(\xi)G(t,\xi)\,\mathrm{d}\xi,$$

where $a_\varepsilon(x) \equiv \int_\varepsilon^x \dfrac{\mathrm{d}\xi}{p(\xi)}$ and $u_\varepsilon = u(t,\varepsilon)$.

Applying it to (19), we obtain

$$f(u_2) - f(u_\varepsilon) - a_\varepsilon(l)W_\varepsilon = a_\varepsilon(l)\int_\varepsilon^{l/2} G(t,x)\,\mathrm{d}x + \int_\varepsilon^l \frac{\mathrm{d}x}{p(x)} \int_{l/2}^x p(\xi)G(t,\xi)\,\mathrm{d}\xi.$$

The right–hand–side integral can be calculated as shown above, thus yielding

$$f(u_2) - f(u_\varepsilon) - a_\varepsilon(l)W_\varepsilon = \int_\varepsilon^l \left(\int_x^l \frac{\mathrm{d}\xi}{p(\xi)}\right) p(x)G(t,x)\,\mathrm{d}x.$$

Assuming that when $x \to 0$ all the integrals exist and are finite, we obtain the following difference equation associated with the point $x_1 = 0$:

$$f(u_2) - f(u_1) = R_3 \equiv \int_0^l \left(\int_x^l \frac{\mathrm{d}\xi}{p(\xi)}\right) p(x)G(t,x)\,\mathrm{d}x. \qquad (20)$$

To obtain the difference equation associated with the point $x_2 = l$, we integrate the equation (14) from $x_1 = 0$ to $x_2 = l$, and taking in account the boundary conditions (3), (6) we obtain

$$p(l)f'(u_2)f_2(u_2) = R_4 \equiv \int_0^l p(x)G(t,x)\,\mathrm{d}x. \tag{21}$$

The difference scheme (20), (21) is exact for a given function G.

Note that $a(x)$ in the singular case is divergent. Both the equations (20), (21) were obtained by using this integral, but do not contain it.

4. Approximation of integrals

We use the quadrature rule of interpolating type for all integrals R_1, R_2, R_3 and R_4 considering only two–point integration formulae involving points $x_1 = 0$ and $x_2 = l$. There many of different approximation methods can be used: involving only the function G, involving first order partial derivative with respect to x of G, second order derivative of G, etc. We will show the one involving second order derivatives of function G, since all other methods are similar and can be easily produced guided by our example.

Here we would like to remark that in the general case expressions for the derivatives of G with the order higher than 3^{rd} are complicated and very inconvenient to manipulate. They should be used only in case when the approximation given below does not satisfy the requirement of the necessary precision.

As the technique of approximation and calculation of the derivatives is the same for the non–singular and singular cases, we will show only the example of approximation of the integral R_1.

Let us denote $G_1 \equiv G(t,0)$, $G_2 \equiv G(t,l)$ and $G_1^{(n)} \equiv \left.\dfrac{\partial^n G}{\partial x^n}\right|_{x=0}$; $G_2^{(n)} \equiv \left.\dfrac{\partial^n G}{\partial x^n}\right|_{x=l}$. Then from the boundary conditions (3), (4) follows

$$G'_k = f_k(u_k)\dot{u}_k - F'(u_k)f_k(u_k),\ k = 1,2,$$

where $\dot{u} \equiv \dfrac{\partial u}{\partial t}$ and $F'(u) \equiv \dfrac{\mathrm{d}F}{\mathrm{d}u}$.

$$G''_k = \frac{\partial}{\partial t}\frac{\partial^2 u_k}{\partial x^2} - F''(u_k)\left(\frac{\partial u_k}{\partial x}\right)^2 - F'(u_k)\frac{\partial^2 u_k}{\partial x^2},\ k = 1,2.$$

Let us use the equation (5) to find $\dfrac{\partial^2 u}{\partial x^2}$:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{f'(u)}\left[\dot{u} - f'(u)\frac{p'(x)}{p(x)}\frac{\partial u}{\partial x} - f''(u)\left(\frac{\partial u}{\partial x}\right)^2 - F(u)\right].$$

Then, by taking the partial derivative with respect to t from this expression, we obtain

$$\frac{\partial}{\partial t}\frac{\partial^2 u}{\partial x^2} = \frac{1}{f'(u)}\left[\frac{\partial^2 u}{\partial t^2} - \frac{p'(x)}{p(x)}\left(f''(u)\frac{\partial u}{\partial t}\frac{\partial u}{\partial x} - f'(u)\frac{\partial^2 u}{\partial t\partial x}\right) - f'''(u)\frac{\partial u}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 - 2f''(u)\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial t\partial x} - F'(u)\frac{\partial u}{\partial t} - \frac{f''(u)}{f'(u)}\frac{\partial u}{\partial t}\left[\frac{\partial u}{\partial t} - f'(u)\frac{p'(x)}{p(x)}\frac{\partial u}{\partial x} - f''(u)\left(\frac{\partial u}{\partial x}\right)^2 - F(u)\right]\right].$$

Inserting those expressions in the G_k'' gives us the necessary expression for G_k''.

Please note that in the singular case when $k = 1$ the boundary condition (6) should be used instead of (3) and the equation (7) instead of (5).

Now, let us show the approximation of the integral R_1.

Substituting $\xi = x/l$, we move to non–dimensional form and thus have two grid points $\xi_1 = 0$, $\xi_2 = 1$. Therefore the integral R_1 can be expressed with a non–dimensional integral I_1 as follows:

$$R_1 = lI_1, \quad I_1 = \int_0^1 \left(1 - \frac{\tilde{a}(\xi)}{\tilde{a}(1)}\right)\tilde{p}(\xi)g(\xi)\,\mathrm{d}\xi,$$

where $g(\xi) = G(t, \xi l)$ at the fixed moment of time t, $\tilde{a}(\xi) \equiv a(\xi l)$, $\tilde{p}(\xi) \equiv p(\xi l)$.

Denoting $g_1 \equiv g(0)$ and $g_2 \equiv g(1)$, we approximate the integral I_1 with the expression

$$I_1 = A_1 g_1 + A_2 g_2 + B_1 g_1' + B_2 g_2' + C_1 g_1'' + C_2 g_2'' + \frac{1}{6!}E_0 g^{(6)}(\tilde{\eta}),$$

where $\tilde{\eta} \in (0, 1)$.

We postulate that just this approximation integrates polynomials with an order as high as possible.

Presuming $g(\xi) = \xi^i$, $i = \overline{0, 5}$, after the calculation of the right–hand and left–hand sides of the approximating expression, we obtain the system of linear algebraic equations in the form ($g^{(6)} \equiv 0$)

$$I_1 = A_1 0^i + A_2 + i(B_1 0^{i-1} + B_2) + i(i-1)(C_1 0^{i-2} + C_2),\ i = \overline{0, 5}.$$

Defining $0^0 = 1$, we can solve this system for the coefficients A_k, B_k, C_k; $k = 1, 2$. There we do not define 0^i, $i < 0$, because in this case 0^i is always multiplied by 0, and we assume this expression to be 0. The coefficient E_0 by the error term can be calculated using the same linear algebraic equation when $i = 6$.

Moving backwards from the non–dimensional integral I_1 to R_1, one should take in account that $g^{(n)} = l^n G^{(n)}$.

Then the integral R_1 is approximated using the following expression of Hermitian interpolation:

$$R_1 = l\left[A_1G_1 + A_2G_2 + l\left(B_1G_1' + B_2G_2'\right) + l^2\left(C_1G_1'' + C_2G_2''\right) + r_6\right], \quad (22)$$

with the error term $r_6 \equiv E_0\dfrac{l^6}{6!}\dfrac{\partial^6 G(t,\eta)}{\partial x^6}$, $\eta \in (0,l)$.

5. System of ODEs for two–point scheme

Using the following difference equations:

1. in non–singular case: (16) and (18).
2. in singular case: (20) and (21)

and the right–hand side integrals' approximations with neglected error terms r_6, the approximate numerical solution for the functions $u_1(t)$, $u_2(t)$ at every time step $t > 0$ in both the non–singular and singular cases can be found by solving the following stiff system of two nonlinear ODEs with initial conditions (9) (in particular, (11) for the non–singular case and (13) for the singular case):

$$\begin{cases} a_1^1\ddot{u}_1 + a_2^1\ddot{u}_2 + b_1^1\dot{u}_1 - b_2^1\dot{u}_2 + c_1^1\dot{u}_1^2 + c_2^1\dot{u}_2^2 = d^1 \\ a_1^2\ddot{u}_1 + a_2^2\ddot{u}_2 - b_1^2\dot{u}_1 + b_2^2\dot{u}_2 + c_1^2\dot{u}_1^2 + c_2^2\dot{u}_2^2 = d^2 \end{cases}, \quad (23)$$

where the coefficients a_i^j, b_i^j, c_i^j, d^j, $i,j = 1,2$ can be determined directly from the integrals' approximation formulae and the difference equations.

This system can be easily solved by anyone of computer algebra packages with such abilities (for example, *Mathematica* or *Maple V*), as well as by the routines of *Fortran* or *C/C++* provided by many of independent software manufacturers (in particular, *NAG*) or documented in [2, 3].

During the numerical experiments it was discovered that the time needed to solve such a system of two nonlinear stiff ODEs is much more less than solving the problem straight–forward using the approximation by the finite–difference scheme or using Fourier series when it is possible to build the analytical solution.

The next section shows the example of use of this method and its efficiency in solving heat transfer problems in different domains.

6. Examples and numerical results

There, we will show two simple examples of usage of this method for the calculation of the temperature distribution in a thin plate (non–singular case) [4] and in a cylinder (singular case) [5].

6.1. Temperature distribution in a thin plate There we consider $p(x) \equiv 1$, the equation (1) to be linear: $f(u) \equiv u$, $F(u) \equiv 0$ and the functions f_1, f_2 in the boundary conditions (3), (4) in the form

$$f_1(u_1) = Bi_1(u_1^4 - \theta_b^4) + B_1(u_1 - \theta_1),$$
$$f_2(u_2) = Bi_2(\theta_t^4 - u_2^4) + B_2(\theta_2 - u_2),$$

where $B_{1,2}$ are Biot numbers, $Bi_{1,2}$ are radiation Biot numbers, $\theta_{b,t}$ are the dimensionless temperatures of heaters on the bottom and top of the plate respectively, $\theta_{1,2}$ are the dimensionless temperatures of the air on the bottom and top of the plate respectively.

There, two symmetry cases are possible:

1. $l = 1$, $f_2(T_2) = -f_1(T_1)$
2. $l = 0.5$, $f_2(T_2) \equiv 0$

We will review the 2^{nd} symmetry case ($l = 0.5$).

Assuming that at the initial moment of time $t = 0$ the temperature of the bottom surface of the plate is $u_0(0) = T_*$, and defining the initial condition in the form that satisfies (8):

$$u_0(x) = T_* - (x^2 - x)f_1(T_*),$$

from (11) we obtain

$$\begin{cases} u_1(0) = T_* \\ u_2(0) = T_* + 0.25 f_1(T_*) \\ \dot{u}_1(0) = -2f_1(T_*) \\ \dot{u}_2(0) = -2f_1(T_*) \end{cases} \tag{24}$$

The difference equations (16) and (18) have the form

$$\frac{u_2 - u_1}{l} - f'(u_1)f_1(u_1) = R_1 \equiv \int_0^l \left(1 - \frac{x}{l}\right) G(t,x)\,\mathrm{d}x$$

$$\frac{u_1 - u_2}{l} + f'(u_2)f_2(u_2) = R_2 \equiv \int_0^l \frac{x}{l}\, G(t,x)\,\mathrm{d}x$$

Approximating the integrals R_1, R_2 as it is described in the previous section, we obtain the following system of two nonlinear stiff ODEs:

$$\begin{cases} \frac{1}{1680}\ddot{u}_1 + \frac{1}{2240}\ddot{u}_2 + \left(\frac{5}{28} + \frac{13}{840}f_1'(u_1)\right)\dot{u}_1 + \frac{1}{14}\dot{u}_2 \\ \qquad = 2(u_2 - u_1) - f_1(u_1) \\ \frac{1}{2240}\ddot{u}_1 + \frac{1}{1680}\ddot{u}_2 + \left(\frac{1}{14} + \frac{1}{105}f_1'(u_1)\right)\dot{u}_1 + \frac{5}{28}\dot{u}_2 \\ \qquad = 2(u_1 - u_2) \end{cases} \tag{25}$$

During the numerical experiments, the following two cases were considered:

1. Linear boundary conditions: $\theta_1 = \theta_2 = 1$, $Bi_1 = Bi_2 = 0$, $B_1 = B_2 = 0.9$, $T_* = 0.3$ at the moments of dimensionless time $t = 0.1i$, $i = \overline{1,10}$;

2. Nonlinear boundary conditions: $\theta_b = \theta_t = 1$, $Bi_1 = Bi_2 = 0.3$, $B_1 = B_2 = 0$, $T_* = 0.3$ at the moments of dimensionless time $t = 0.2i$, $i = \overline{1,10}$

The numerical results obtained by solving the Cauchy problem (25), (24) are compared to the values of u^* obtained by the Fourier series in linear boundary conditions case and by the explicit finite difference method with the space step $h = 0.02$ and time step $\tau = h^2/6$ in nonlinear boundary conditions case. Comparison of the values of temperature obtained by different numerical methods can be seen in the tables 1 and 2.

From the results, it is visible that this method gives the precision in at least 4–5 decimal places. This precision can be heightened using higher derivatives of G in the integrals' approximation formulae.

Table 1. Linear boundary conditions

t	u_1^*	u_2^*	u_1	u_2
0.1	.403253	.264501	.403235	.264493
0.2	.489467	.370702	.489445	.370687
0.3	.563203	.461590	.563178	.461570
0.4	.626290	.539352	.626263	.539329
0.5	.680264	.605883	.680237	.605858
0.6	.726444	.662805	.726416	.662779
0.7	.765953	.711506	.765927	.711479
0.8	.799756	.753173	.799731	.753147
0.9	.828677	.788822	.828653	.788797
1.0	.853421	.819322	.853399	.819298

Table 2. Nonlinear boundary conditions

t	u_1^*	u_2^*	u_1	u_2
0.2	.416928	.343957	.416926	.343957
0.4	.528474	.458858	.528470	.458856
0.6	.630758	.566903	.630751	.566898
0.8	.719987	.664204	.719977	.664196
1.0	.793779	.747526	.793766	.747515
1.2	.851813	.815307	.851799	.815295
1.4	.895552	.867933	.895539	.867919
1.6	.927435	.907229	.927422	.907216
1.8	.950106	.935694	.950096	.935683
2.0	.965945	.955852	.965936	.955842

6.2. Temperature distribution in a cylinder Let us review the axial-symmetric heat transport problem in a cylinder. There we consider $p(x) = x$, $f(u) \equiv u$. $F(u) \equiv 0$, and the function f_2 in the boundary condition (4) in the form

$$f_2(u_2) = Bi_2(\theta_h^4 - u_2^4) + B_2(\theta_a - u_2),$$

where B_2 is the Biot number. Bi_2 is the radiation Biot number, θ_h is the dimensionless temperature of a heater, θ_a is the dimensionless temperature of the air.

The dimensionless parameter (radius of the cylinder) $l = 1$.

Assuming that at the initial moment of time $t = 0$ the temperature of the surface of the cylinder is $u_0(0) = T_*$, and defining the initial condition in the form that satisfies (8):

$$u_0(x) = T_* - \frac{l^2 - x^2}{2l} f_2(T_*),$$

like in (13) we obtain

$$\begin{cases} u_1(0) = T_* - \frac{1}{2} f_2(T_*) \\ u_2(0) = T_* \\ \dot{u}_1(0) = 2f_2(T_*) \\ \dot{u}_2(0) = 2f_2(T_*) \end{cases} \tag{26}$$

The difference equations have the form

$$\begin{aligned} u_2 - u_1 = R_3 &\equiv \textstyle\int_0^l x \ln \frac{l}{x}\, G(t,x)\,dx \\ f_2(u_2) = R_4 &\equiv \frac{1}{l} \textstyle\int_0^l xG(t,x)\,dx \end{aligned} .$$

Approximating the integrals R_3, R_4 as it is described in the previous section, we obtain the following system of two nonlinear stiff ODEs:

$$\begin{cases} \frac{319}{235200}\ddot{u}_1 - \frac{107}{44100}\ddot{u}_2 + \frac{106}{735}\dot{u}_1 + \left(\frac{311}{2940} - \frac{46}{1575} f_2'(u_2)\right)\dot{u}_2 = \\ \qquad\qquad u_2 - u_1 \\ \frac{1}{560}\ddot{u}_1 + \frac{1}{210}\ddot{u}_2 + \frac{1}{7}\dot{u}_1 + \left(\frac{5}{14} - \frac{1}{15} f_2'(u_2)\right)\dot{u}_2 = f_2(u_2) \end{cases} \tag{27}$$

During the numerical experiments, the following two cases were considered:

1. Linear boundary conditions: $\theta_a = 1$, $Bi_2 = 0$, $B_2 = 0.9$, $T_* = 0.3$ at the moments of dimensionless time $t = 0.1i$, $i = \overline{1,10}$;

2. Nonlinear boundary conditions: $\theta_h = 1$, $Bi_2 = 0.3$, $B_2 = 0$, $T_* = 0.3$ at the moments of dimensionless time $t = 0.2i$, $i = \overline{1,10}$

The numerical results obtained by solving the Cauchy problem (27), (26) are compared to the values of u^* obtained by the Fourier series in the linear boundary conditions case and by the explicit finite difference method with the space step $h = 0.02$ and time step $\tau = h^2/6$ in nonlinear boundary conditions case. Comparison of the values of temperature obtained by different numerical methods can be seen in the tables 3 and 4.

From the results it is visble that the method gives the precision of at least 4-5 decimal places. It can be hightened using the same method as in the non- singular case.

This method has an advantage comparing to the traditional approximation with the finite–difference scheme.

To correctly approximate the problem with the finite–difference scheme in the center of the cylinder, it is necessary to take the half of the space step by the zero point. It makes the scheme more complicated.

This method doesn't have such a disadvantage as it is visible from the construction of the difference scheme and the results.

Table 3. Linear boundary conditions

t	u_1^*	u_2^*	u_1	u_2
0.1	.110448	.400983	.110403	.400970
0.2	.228024	.482962	.228003	.482952
0.3	.331824	.553024	.331813	.553017
0.4	.422010	.613458	.422002	.613452
0.5	.500089	.665695	.500083	.665691
0.6	.567633	.710867	.567628	.710864
0.7	.626053	.749935	.626050	.749933
0.8	.676581	.783724	.676579	.783722
0.9	.720281	.812947	.720280	.812946
1.0	.758077	.838222	.758076	.838221

Table 4. Nonlinear boundary conditions

t	u_1^*	u_2^*	u_1	u_2
0.2	.270010	.416660	.270009	.416659
0.4	.386560	.527399	.386557	.527388
0.6	.497798	.628336	.497788	.628316
0.8	.600100	.715880	.600082	.715851
1.0	.690086	.788060	.690059	.788024
1.2	.765647	.844957	.765613	.844919
1.4	.826416	.888230	.826380	.888190
1.6	.873533	.920264	.873498	.920232
1.8	.909019	.943541	.908986	.943513
2.0	.935160	.960229	.935131	.960206

REFERENCES

[1] Morton, K. V., *Numerical Solutions of Convection-Diffuson Problems*. Oxford U.P., Chapman & Hall, London, Inc., 1996.

[2] Press, W. H., Teukolsky, S. A., Wetterling, W. T., Flannery, B. P., *Numerical Recipes in Fortran. The Art of Scientific Computing*. Cambridge University Press, 1996.

[3] Press, W. H., Teukolsky, S. A., Wetterling, W. T., Flannery, B. P., *Numerical Recipes in C. The Art of Scientific Computing, Second Edition*. Cambridge University Press. 1999. Reprint v. 2.09

[4] Kalis, H., Lasis, A., Effective methods for calculation of heat losses in plate. In: *Proceedings of the International Colloquium Modelling for Saving Resources*, Riga, May 17–18, 2001. University of Latvia, 2001, 204–209.

[5] Kalis, H., Lasis, A., Simple algorithms for calculation of the axial-symmetric heat transport problem in a cylinder. *Math. Model. Anal.*, **6** (2001), no. 2, 262 269.

Efektīva aproksimācijas metode nelineārai paraboliskai robežproblēmai

Kopsavilkums

Nelineārās paraboliskās problēmas aproksimācija bazējas uz galīgo tilpumu metodi [1]. Problēmas aproksimācijai vispārīgā gadījuma var uzbūvēt precīzo divu punktu šablona diferenču shēmu. Atticcīgie integrāļi tiek aproksimeti ar dažadam kvadratūru formulam. Tas ļauj reducēt problēmu, kuru apraksta paraboliskā tipa parciālais diferenciālvienādojums, uz Koši problēmu divu nelinearo parasto diferenciālvienādojumu sistēmu, kuru kārta ir atkarīga no izmantotām kvadratūru formulām. Tādejādi var iegūt problēmas atrisinājumu uz apgabala robežas. Ir apskatīts gan singulārs, gan nesingulārs gadījums, kā arī paradīti divi metodes lietojuma piemēri. Attiecīgo algoritmu skaitliskie atrisinājumi ir iegūti, izmantojot programmpakešu Maple V *un* Mathematica *procedūras parasto diferenciālvienādojumu stiegro sistēmu risināšanai.*

Λ-convex functions and lamination closed sets for the scalar case

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The paper considers functions $L: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, which are convex with respect to the cone $\Lambda = \{(z,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \mid \langle z,\xi\rangle = 0\}$, and subsets M of the set $\mathcal{M}$ of all strictly convex, coercive and continuously differentiable functions $F: \mathbb{R}^n \to \mathbb{R}$. It is shown that if L is Λ-convex and M consists of all $F \in \mathcal{M}$ such that $F(\cdot) + F^*(\cdot) \geq L(\cdot,\cdot)$ then M is convex and closed with respect to lamination.

Key words: convex functions, homogenization, laminate

Mathematics Subject Classification (2000): 35B27, 49J45

1. Introduction

Λ-convexity of functions $L: \mathbb{R}^m \to \mathbb{R}$ for a linear partial differential operator $\mathbb{A}$ with constant coefficients

$$\mathbb{A}: C^\infty(\mathbb{R}^n;\mathbb{R}^m) \to C^\infty(\mathbb{R}^n;\mathbb{R}^d),$$

appears as a necessary condition for the sequential weak lower semicontinuity of functionals

$$I = \int_\Omega L(w(x))\,dx$$

on the spaces

$$\mathcal{W} = \{w \in L_p(\Omega;\mathbb{R}^m) \mid \mathbb{A}w(x) = 0 \text{ in the sense of distributions}\}.$$

(For the definition of the characteristic cone Λ see e.g. Fonseca and Müller [4] or Murat [5]).

Apart from the most interesting cases $\mathbb{A} = (\mathrm{curl},\dots,\mathrm{curl})$ or $\mathbb{A} = (\mathrm{div},\dots,\mathrm{div})$ specifically interesting is the case of Λ-convex functions for $\mathbb{A} = (\mathrm{curl},\mathrm{div})$, which arises in relaxation of optimal design problems governed by potential elliptic operators. If the state equation is given as

$$\mathrm{div}\left[\sum_{s=1}^{s_0} \chi_{\Omega_s}(x)F_s'(\nabla u(x)) - f(x)\right] = 0 \text{ in } \Omega, \quad u \in W_p^1(\Omega), \quad u\big|_{\partial\Omega} = 0, \quad (1)$$

where $F_s:\mathbb{R}^n\to\mathbb{R}$, $s=1,\dots,s_0$, are strictly convex, coercive and continuously differentiable functions and the partition $\{\Omega_1;\dots;\Omega_{s_0}\}$ of Ω plays the role of the control, then the relaxation procedure leads to the set of functions

$$\begin{aligned}&\{\tilde F:\mathbb{R}^n\to\mathbb{R}\mid \tilde F \text{ is strictly convex, coercive, continously differentiable}\\&\text{and } \tilde F(z)+\tilde F^*(\xi)\ge L(z,\xi)\ \ \forall(z,\xi)\in\mathbb{R}^n\times\mathbb{R}^n\}\end{aligned}\tag{2}$$

with L being the (curl, div)-quasiconvex envelope of the function L_0,

$$L_0(z,\xi)=\min_s\{F_s(z)+F_s^*(\xi)\},\ (z,\xi)\in\mathbb{R}^n\times\mathbb{R}^n,$$

see, for instance, Raitums [7]. Here by F^* we denote the conjugate function to F.

For this scalar case the function L in (2) as (curl, div)-quasiconvex is Λ-convex with respect to the cone Λ,

$$\Lambda=\{(z,\xi)\in\mathbb{R}^n\times\mathbb{R}^n\mid\langle z,\xi\rangle=0\}.\tag{3}$$

Simple analytical descriptions for $\mathbb{A}$-quasiconvex functions are not known, nevertheless some interesting properties of sets of the type (2) can be obtained using only Λ-convexity of the function L.

One can easily expect that Λ-convexity is very close related to the procedure of lamination. For instance, the proof of Λ-convexity for integrands of sequentially weakly lower semicontinuous functionals on $W_p^1(\Omega;\mathbb{R}^m)$ is based on laminated structures, i.e. structures that depend only on one direction, see Dacorogna [2]. And we will show that for every given coercive Λ-convex, where Λ is defined by (3), function L the set of the kind (2) is convex and closed with respect to lamination (the precise formulation is given in Section 2).

2. Preliminaries

Let $n\ge 2$ be integer and let $\mathcal{M}$ be set of all functions $F:\mathbb{R}^n\to\mathbb{R}$, which satisfy the following hypotheses:

H1. F is continuous and continuously differentiable on $\mathbb{R}^n$;

H2. F is strictly convex;

H3. F is coercive;

H4. For every $F\in\mathcal{M}$ there exist a constant c_F and a continuous strictly increasing function $\gamma_F:\mathbb{R}\to\mathbb{R}$ such that

$$F(z)\le c_F+\gamma_F(|z|)|z|\quad\forall z\in\mathbb{R}^n.$$

REMARK 2.1. The hypothesis H4 can be deduced from H1 – H3.

Let $K \subset \mathbb{R}^n$ be unit cube $(0,1)^n$ and let, for $1 \le p \le \infty$, $H_p^\# = \{v \in L_p(K;\mathbb{R}^n) \mid v = \nabla u,\ u \in W_p^1(K),\ u \text{ is } K- \text{ periodic}\}$.

DEFINITION 2.1. A function $\hat{F}\colon \mathbb{R}^n \to \mathbb{R}$ is said to be laminate (in direction $e \in \mathbb{R}^n$, $|e| = 1$, constructed from $\{F_1 : \dots : F_N\} \subset \mathcal{M}$ in proportions $\{\lambda_1 : \dots : \lambda_N\}$ respectively) if there exist $R \in SO(n)$ and functions

$$\begin{aligned}
&\sigma = (\sigma_1, \dots, \sigma_N) \in L_\infty((0,1);\mathbb{R}^N),\\
&\sigma_i(t) = 0 \text{ or } 1,\ \int_0^1 \sigma_i(t)\,dt = \lambda_i,\ i = 1,\dots,N,\\
&\sigma_1(t) + \cdots + \sigma_N(t) = 1 \text{ a.e. } t \in (0,1),
\end{aligned}$$

such that

$$R^{-1}e = (1,0,\dots,0),$$

$$\hat{F}(z) = \inf_{v \in H_\infty^\#} \int_K \sum_{i=1}^N \sigma_i(x_1) F_i(R(R^{-1}z + v(x))\,dx \ \forall z \in \mathbb{R}^n. \tag{4}$$

We recall, see, for instance, Dal Maso [3], that if the functions $F_1, \dots, F_N$ have the standard p-growth ($1 < p < \infty$) at infinity then $\hat{F}$ is the standard homogenized (or Γ-limit) integrand as $\varepsilon \to 0$ for the family of integrands

$$F_\varepsilon(x,\cdot) = \sum_{i=1}^N \tilde{\sigma}_i \left(\frac{1}{\varepsilon}\langle x, e\rangle\right) F_i(\cdot),\ 0 < \varepsilon,$$

where $\tilde{\sigma}$ is the $(0,1)$-periodic extension of σ to the whole $\mathbb{R}$.

We will show in Section 3 that the function $\hat{F}$ defined by (4) belongs to $\mathcal{M}$, has the representation

$$\hat{F}(z) = \min_{\overline{\alpha} \in \mathbb{R}^N,\ \langle \overline{\alpha}, \overline{\lambda}\rangle = 0} \sum_{i=1}^N \lambda_i F_i(z + \alpha_i e) \tag{5}$$

and the adjoint function $\hat{F}^*$ has the representation

$$\begin{aligned}
\hat{F}^*(\xi) = \min\{ \sum_{i=1}^N \lambda_i F_i^*(\xi + a^i) \mid a^i \in \mathbb{R}^n,\ \langle a^i, e\rangle = 0, i = 1, \dots, N;\\
\sum_{i=1}^N \lambda_i a^i = 0\}.
\end{aligned}$$

Here and in what follows by F^* we denote the adjoint to F function,

$$F^*(\xi) = \sup_{z \in \mathbb{R}^n} [\langle z, \xi\rangle - F(z)],\ \xi \in \mathbb{R}^n,$$

and $\overline{\lambda} = (\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N$.

Consider the differential operator $\mathbb{A} = (\mathrm{curl}, \mathrm{div})$,

$$\mathbb{A}: C^\infty(\mathbb{R}^n; \mathbb{R}^n) \times C^\infty(\mathbb{R}^n; \mathbb{R}^n) \to C^\infty(\mathbb{R}^n; \mathbb{R}^{\frac{n(n-1)}{2}}) \times C^\infty(\mathbb{R}^n).$$

The characteristic cone Λ corresponding to $\mathbb{A}$

$$\Lambda = \{(z, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \mid \langle z, \xi \rangle = 0\},$$

DEFINITION 2.2. A continuous function $L: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is said to be Λ-*convex* if

$$L(\lambda z^1 + (1-\lambda)z^2, \lambda\xi^1 + (1-\lambda)\xi^2) \leq \lambda L(z^1, \xi^1) + (1-\lambda)L(z^2, \xi^2)$$

whenever $(z^2, \xi^2) - (z^1, \xi^1) \in \Lambda$ and $0 < \lambda < 1$.

DEFINITION 2.3. A set $M \subset \mathcal{M}$ is said to be closed with respect to lamination if for every finite number of functions $F_1, \ldots, F_N \in M$ the set M contains all laminates $\hat{F}$ constructed from $\{F_1; \ldots; F_N\}$ with various directions $e \in \mathbb{R}^n$ and proportions $\{\lambda_1; \ldots; \lambda_N\}$ according to formula (5).

Our main results are the following.

THEOREM 2.1. *Let $M \subset \mathcal{M}$ be set such that*

(i) *M is closed with respect to lamination;*

(ii) *there exist functions $F_-, F_+ \in \mathcal{M}$ such that $F_- \leq F \leq F_+$ $\forall F \in M$.*

Then the function $L: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$,

$$L(z, \xi) = \inf_{F \in M} [F(z) + F^*(\xi)], \ (z, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$$

is Λ-convex.

THEOREM 2.2. *Let $L: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be function such that*

(i) *L is continuous;*

(ii) *L is coercive;*

(iii) *L is Λ-convex.*

Then the set $M \subset \mathcal{M}$ (if not empty),

$$M = \{F \in \mathcal{M} \mid F(z) + F^*(\xi) \leq L(z, \xi) \ \forall (z, \xi) \in \mathbb{R}^n \times \mathbb{R}^n\}$$

is

(a) *closed with respect to lamination;*

(b) $\mathrm{co}\, M = M$;

(c) $\mathrm{co}\{F^* \mid F \in M\} = \{F^* \mid F \in M\}$.

3. Properties of laminates

A finite number of functions from $\mathcal{M}$ is equi-coercive and uniformly bounded from below, hence, the function $\hat{F}$ is well defined by (4) on the whole $\mathbb{R}^n$ and is bounded from below. Taking in (4) $v = 0$ we have that $\hat{F}$ satisfies H4. Since the convex envelope for a finite number of functions from $\mathcal{M}$ is coercive too and since the mean value of $v \in H_p^{\#}$ over K is equal to zero, then by Jensen's inequality it follows from (4) that $\hat{F}$ is also coercive. Furthermore, from convexity of the integrand in (4) with respect to the pair $(z, v(\cdot))$ it follows immediately that the function $\hat{F}$ is convex.

To prove the representation formula (5), let us consider the functions

$$\varphi_\alpha(t) = \sum_{i=1}^{N} \alpha_i \sigma_i(t)$$

with $\overline{\alpha} = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N$ such that

$$\langle \overline{\alpha}, \overline{\lambda} \rangle = 0.$$

Then, by construction, the elements v_α

$$v_\alpha(x) = \varphi_\alpha(x_1)e^1 = (\varphi_\alpha(x_1), 0, \ldots, 0)$$

belong to $H_\infty^{\#}$.

The functional

$$I(v) = \int_K \sum_{i=1}^{N} \sigma_i(x_1) F_i(R(R^{-1}z + v(x)))\, dx, \quad I\colon H_\infty^{\#} \to \mathbb{R},$$

is Frechet differentiable and

$$I'(v)\delta v = \int_K \langle \sum_{i=1}^{N} \sigma_i(x_1) R^{-1} F_i'(R(R^{-1}z + v(x))), \delta v(x) \rangle\, dx,$$

where by F' and I' we denote the derivatives of F and I respectively.

Let $V_\sigma \subset H_\infty^{\#}$ be set such that

$$V_\sigma = \{ v_\alpha \in H_\infty^{\#} \mid \overline{\alpha} \in \mathbb{R}^N, \langle \overline{\alpha}, \overline{\lambda} \rangle = 0 \},$$

where the relationship between σ and $\overline{\lambda}$ is given in Definition 2.1.

The functional I on V_σ has the representation

$$\begin{aligned} I(v_\alpha) &= \int_0^1 \sum_{i=1}^{N} \sigma_i(t) F_i(R(R^{-1}z + v_\alpha(t)))\, dt \\ &= \sum_{i=1}^{N} \lambda_i F_i(R(R^{-1}z + \alpha_i e^1) = \sum_{i=1}^{N} \lambda_i F_i(z + \alpha_i e) \end{aligned}$$

and, obviously, attains its minimum over V_σ on a unique v_{α_o}. From the Euler's equation we have

$$I'(v_{\alpha_o})v_\alpha = \int_K \langle \sum_{i=1}^{N} \sigma_i(x_1) R^{-1} F_i'(R(R^{-1}z + v_{\alpha_o}(x)), v_\alpha(x)\rangle \, dx = 0 \quad \forall v_\alpha \in V_\sigma. \tag{6}$$

From here and the fact that the integrand in $I'(v_{\alpha_o}) = (I_1', \dots, I_n')$ depends only on x_1 it follows ($v = \nabla u$ and u is K-periodic for $v \in H_\infty^\#$)

$$\begin{aligned} I'(v_{\alpha_o})v &= \int_K \sum_{j=1}^{n} I_j'(x_1) u_{x_j}(x)\, dx \\ &= \int_0^1 I_1'(x_1)\Big(\int_0^1 \cdots \int_0^1 u_{x_1}(x)\, dx_2 \cdots dx_n\Big)\, dx_1 \\ &= \int_0^1 I_1'(x_1)\psi(x_1)\, dx_1, \end{aligned} \tag{7}$$

for some integrable function ψ with

$$\int_0^1 \psi(x_1)\, dx_1 = 0.$$

Furthermore, the integrand $I_1'(\cdot)$ is constant in every $\operatorname{supp}\sigma_i$, $i = 1, \dots, N$. From here and (6) - (7) it follows immediately that

$$I'(v_{\alpha_o})v = 0 \quad \forall v \in H_\infty^\#.$$

Since I is convex and $I'(v_{\alpha_o})v = 0$ for all $v \in H_\infty^\#$, the element $v_{\alpha_o} \in V_\sigma \subset H_\infty^\#$ gives the minimum of I over $H_\infty^\#$. This way,

$$\hat{F}(z) = \min_{v\alpha \in V_\sigma} = \min_{\overline{\alpha} \in \mathbb{R}^N,\, \langle \overline{\alpha}, \overline{\lambda}\rangle = 0} \sum_{i=1}^{N} \lambda_i F_i(z + \alpha_i e), \tag{8}$$

what gives the representation (5).

Let us denote

$$\begin{aligned} &A_\lambda = \{\overline{\alpha} \in \mathbb{R}^N \mid \langle \overline{\alpha}, \overline{\lambda}\rangle = 0\}, \\ &\mathcal{F}\colon \mathbb{R}^n \times A_\lambda \to \mathbb{R},\ \mathcal{F}(z, \overline{\alpha}) = \sum_{i=1}^{N} \lambda_i F_i(z - \alpha_i e). \end{aligned}$$

The function $\mathcal{F}$ is continuously differentiable with respect to the pair $(z, \overline{\alpha})$. Let $z \in \mathbb{R}^n$ be fixed and let $\overline{\alpha}_0$ be minimizer of $\mathcal{F}(z, \cdot)$ over A_λ, i.e.

$$\mathcal{F}(z, \overline{\alpha}_0) = \min_{\overline{\alpha} \in A_\lambda} \mathcal{F}(z, \overline{\alpha}) = \hat{F}(z).$$

In these notations

$$\begin{aligned}
&\hat{F}(z+\delta z) - \hat{F}(z) - \langle \mathcal{F}'_z(z,\overline{\alpha}_0), \delta z\rangle \\
= \; & \min_{\overline{\alpha}\in A_\lambda} \mathcal{F}(z+\delta z, \overline{\alpha}) - \mathcal{F}(z,\overline{\alpha}_0) - \langle \mathcal{F}'_z(z,\overline{\alpha}_0), \delta z\rangle \\
\le \; & \mathcal{F}(z+\delta z, \overline{\alpha}_0) - \mathcal{F}(z,\overline{\alpha}_0) - \langle \mathcal{F}'_z(z,\overline{\alpha}_0), \delta z\rangle = o(|\delta z|),
\end{aligned}$$

i.e. the function $\hat{F}$ is upper semidifferentiable on $\mathbb{R}^n$. This property, together with convexity of $\hat{F}$, gives that $\hat{F}$ is continuously differentiable on $\mathbb{R}^n$, see Ball *et al* [1].

From coercivity properties of $F_1, \ldots, F_N \in \mathcal{M}$ it follows that for a fixed ball $B_\rho = \{z \in \mathbb{R}^n \mid |z| \le \rho\}$ there exists a convex bounded closed subset $A_0 \subset A_\lambda$ such that for $z \in B_\rho$ it holds

$$\hat{F}(z) = \min_{\overline{\alpha}\in A_0} \mathcal{F}(z,\overline{\alpha}).$$

Let $z', z'' \in B_\rho$, $z' \neq z''$, be fixed. Then from strict convexity of $F_1, \ldots, F_N$ it follows that for some $d > 0$

$$\mathcal{F}(\frac{1}{2}z' + \frac{1}{2}z'', \frac{1}{2}\alpha' + \frac{1}{2}\alpha'') \le \frac{1}{2}\mathcal{F}(z',\alpha') + \frac{1}{2}\mathcal{F}(z'',\alpha'') - d \quad \forall \alpha', \alpha'' \in A_0.$$

Indeed, since A_0 is a compact and if

$$z' + \alpha'_i e = z'' + \alpha''_i e, \quad i = 1, \ldots, N,$$

then from $\langle \overline{\alpha}', \overline{\lambda}\rangle = 0$ and $\langle \overline{\alpha}'', \overline{\lambda}\rangle = 0$ it should follow that $z' = z''$.

That gives

$$\begin{aligned}
\frac{\hat{F}(z') + \hat{F}(z'')}{2} &= \min_{\overline{\alpha}'\in A_0, \overline{\alpha}''\in A_0} \left[\frac{\mathcal{F}(z',\overline{\alpha}') + \mathcal{F}(z'',\overline{\alpha}'')}{2}\right] \\
&\ge \min_{\overline{\alpha}'\in A_0, \overline{\alpha}''\in A_0} \left[\mathcal{F}\left(\frac{z'+z''}{2}, \frac{\overline{\alpha}'+\overline{\alpha}''}{2}\right) + d\right] \\
&= \min_{\overline{\alpha}\in A_0} \left[\mathcal{F}\left(\frac{z'+z''}{2}, \overline{\alpha}\right) + d\right] = \hat{F}\left(\frac{z'+z''}{2}\right) + d.
\end{aligned}$$

Therefore, $\hat{F}$ is strictly convex.

This way, we have proved the following result.

PROPOSITION 3.1. *Every laminate $\hat{F}$ (in direction e constructed from $\{F_1; \ldots; F_N\} \subset \mathcal{M}$ in proportions $\{\lambda_1; \ldots; \lambda_N\}$) has the representation formula* (5) *and belongs to $\mathcal{M}$.*

Analogous result for the conjugate functions $\hat{F}^*$ is as follows.

PROPOSITION 3.2. *If $\hat{F}$ is a laminate (in direction e constructed from $\{F_1; \ldots; F_N\} \subset \mathcal{M}$ in proportions $\{\lambda_1; \ldots; \lambda_N\}$) then*

(i) $\hat{F}^*$ *has the representation*

$$\hat{F}^*(\xi) = \min\Big\{\sum_{i=1}^{N}\lambda_i F_i^*(\xi - a^i) \mid a^i \in \mathbb{R}^n, \\ \langle a^i, e\rangle = 0,\ i = 1,\dots,N; \sum_{i=1}^{N}\lambda_i a^i = 0\Big\} \tag{9}$$

(ii) $\hat{F}$ *is coercive;*

(iii) $\hat{F}$ *is continuously differentiable.*

Proof. Let A_λ be as above and let

$$D_\lambda = \Big\{\hat{d} = \{d^1;\dots;d^N\} \subset \mathbb{R}^n \mid \langle d^i, e\rangle = 0,\ i = 1,\dots,N;\ \sum_{i=1}^{N}\lambda_i d^i = 0\Big\}.$$

For $\hat{d} \in D_\lambda$ simple calculations give

$$\begin{aligned}\hat{F}^*(\xi) &= \sup_{z\in\mathbb{R}^n}[\langle z, \xi\rangle - \min_{\overline{\alpha}\in A_\lambda}\sum_{i=1}^{N}\lambda_i F_i(z + \alpha_i e)] \\ &= \sup_{z\in\mathbb{R}^n}\sup_{\overline{\alpha}\in A_\lambda}\Big\{\sum_{i=1}^{N}\lambda_i[\langle z + \alpha_i e, \xi + d^i\rangle - F_i(z + \alpha_i e]\Big\} \\ &= \sup_{z^i\in\mathbb{R}^n, i=1,\dots,N}\ \inf_{\hat{d}\in D_\lambda}\Big\{\sum_{i=1}^{N}\lambda_i[\langle z^i, \xi + d^i\rangle - F_i(z^i)]\Big\} \\ &= \inf_{\hat{d}\in D_\lambda}\sum_{i=1}^{N}\lambda_i F_i^*(\xi + d^i)\end{aligned}$$

because the convexity and coercitivity of $F_1,\dots,F_N$ ensure the exchange of sup inf to inf sup.

Since $\hat{F} \in \mathcal{M}$ is strictly convex, coercive and continuously differentiable, the the same properties has the adjoint function $\hat{F}^*$. ☐

4. Proof of main theorems

Let $M \subset \mathcal{M}$ be fixed subset closed with respect to lamination and let there exist functions $F_-, F_+ \in \mathcal{M}$ such that

$$F_-(z) \le F(z) \le F_+(z) \quad \forall z \in \mathbb{R}^n \quad \forall F \in M. \tag{10}$$

The relationship (10) ensure that all conjugate functions F^*, $F \in M$, are uniformly bounded from below,

$$F_+^*(\xi) \le F^*(\xi) \le F_-^*(\xi) \quad \forall \xi \in \mathbb{R}^n \quad \forall F \in M.$$

Hence, the function $L\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$,

$$L(z,\xi) = \inf_{F \in M} \left[F(z) + F^*(\xi)\right], \quad (z,\xi) \in \mathbb{R}^n \times \mathbb{R}^n,$$

is well defined on the whole $\mathbb{R}^n \times \mathbb{R}^n$. Besides that,

$$F_-(z) + F_+^*(\xi) \le L(z,\xi) \le F_+(z) + F_-^*(\xi) \quad \forall (z,\xi) \in \mathbb{R}^n \times \mathbb{R}^n,$$

i.e. the function L is coercive.

Let us fix two points $(z',\xi'), (z'',\xi'') \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$(\delta z, \delta\xi) = (z'' - z', \xi'' - \xi') \in \Lambda, \text{ i.e. } \langle \delta z, \delta\xi \rangle = 0.$$

Let $\varepsilon > 0$, $\lambda \in (0,1)$ be fixed and let $F_1, F_2 \in M$ be such that

$$\begin{aligned} L(z',\xi') &\ge F_1(z') + F_1^*(\xi') - \varepsilon, \\ L(z'',\xi'') &\ge F_2(z'') + F_2^*(\xi'') - \varepsilon. \end{aligned}$$

Denote

$$(z_0, \xi_0) = (\lambda z' + (1-\lambda) z'', \lambda\xi' + (1-\lambda)\xi'').$$

Then

$$\begin{aligned} (z',\xi') &= (z_0,\xi_0) - (1-\lambda)(\delta z, \delta\xi), \\ (z'',\xi'') &= (z_0,\xi_0) + \lambda(\delta z, \delta\xi). \end{aligned}$$

Consider the laminate $\hat{F}$ (in direction $\delta z/|\delta z|$, constructed from $\{F_1; F_2\}$, in proportions $\{\lambda; (1-\lambda)\}$)

$$\begin{aligned} \hat{F}(z_0) + \hat{F}^*(\xi_0) &= \min_{\tau \in \mathbb{R}} [\lambda F_1(z_0 - (1-\lambda)\tau e) + (1-\lambda) F_2(z_0 + \lambda\tau e)] \\ &+ \min_{t \in \mathbb{R}, \langle a, e\rangle = 0} [\lambda F_1^*(\xi_0 - (1-\lambda) t a) + (1-\lambda) F_2^*(\xi_0 + \lambda t a)] \end{aligned}$$

(if $\delta z = 0$, we choose e such that $\langle e, \delta\xi \rangle = 0$).

A special choice

$$\tau = |\delta z|, \ t = |\delta\xi|, \ a = \delta\xi / |\delta\xi|,$$

gives

$$\begin{aligned} L(z_0,\xi_0) &\le \hat{F}(z_0) + \hat{F}^*(\xi_0) \le \lambda F_1(z_0 - (1-\lambda)\delta z) + (1-\lambda) F_2(z_0 + \lambda \delta z) \\ &+ \lambda F_1^*(\xi_0 - (1-\lambda)\delta\xi) + (1-\lambda) F_2^*(\xi_0 + \lambda\delta\xi) \\ &= \lambda[F_1(z') + F_1^*(\xi')] + (1-\lambda)[F_2(z'') + F_2^*(\xi'')] \\ &\le \lambda L(z',\xi') + (1-\lambda) L(z'',\xi'') + 2\varepsilon. \end{aligned}$$

From here and the arbitrarity of $\varepsilon > 0$ it follows that L is Λ-convex, what completes the proof of Theorem 2.1.

Now, let us assume that we have a function $L\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, which is defined on the whole $\mathbb{R}^n \times \mathbb{R}^n$, is continuous, coercive and Λ-convex. Assume further that the set M,

$$M = \{F \in \mathcal{M} \mid F(z) + F^*(\xi) \geq L(z,\xi) \quad \forall (z,\xi) \in \mathbb{R}^n \times \mathbb{R}^n\}.$$

is not empty.

Given $F_1, F_2 \in M$, consider the laminate $\hat{F}$ (in direction e constructed from $\{F_1; F_2\}$ in proportions $\{\lambda; (1-\lambda)\}$). Then

$$\begin{aligned}
& \hat{F}(z) + \hat{F}^*(\xi) \\
= & \min_{\tau\in\mathbb{R},\,\langle a,e\rangle=0} \Big\{\lambda F_1(z-\tau(1-\lambda)e) + \lambda F_1^*(\xi-(1-\lambda)a) \\
+ & (1-\lambda)F_2(z+\tau\lambda e) + (1-\lambda)F_2^*(\xi+\lambda a)\Big\} \\
\geq & \min_{\tau\in\mathbb{R},\,\langle a,e\rangle=0} \{\lambda L(z-\tau(1-\lambda)e, \xi-(1-\lambda)a) + (1-\lambda)L(z+\lambda\tau e, \xi+\lambda a)\}
\end{aligned}$$

Since the difference of arguments for L is equal to $(\tau e, a)$ and, by construction, $(\tau e, a) \in \Lambda$, it follows that

$$\begin{aligned}
\hat{F}(z) + \hat{F}^*(\xi) \geq & \min_{\tau\in\mathbb{R},\,\langle a,e\rangle=0} L\Big(\lambda(z-\tau(1-\lambda)e) + (1-\lambda)(z+\tau\lambda e), \\
& \lambda(\xi-(1-\lambda)a) + (1-\lambda)(\xi+\lambda a)\Big) \\
= & L(z,\xi).
\end{aligned}$$

Therefore, every laminate constructed from $F_1, F_2 \in M$ belongs to M, too.

For a finite number of functions $F_1, \ldots, F_N \in M$ the laminate in direction e in proportions $\lambda_1, \ldots, \lambda_N$ respectively is

$$\begin{aligned}
\hat{F}(z) &= \min_{(\bar{\alpha},\bar{\lambda})=0} \sum_{i=1}^{N} \lambda_i F_i(z+\alpha_i e) \\
&= \min_{\alpha\in\mathbb{R}} \left\{\lambda_1 F_1(z+\alpha e) + (1-\lambda_1)\hat{F}_0\left(z-\alpha\frac{\lambda_1}{1-\lambda_1}e\right)\right\},
\end{aligned}$$

where

$$\begin{aligned}
& \hat{F}_0\left(z-\alpha\frac{\lambda_1}{1-\lambda_1}e\right) \\
= & \min\left\{\sum_{i=2}^{N} \frac{\lambda_i}{1-\lambda_1} F_i\left(z-\alpha\frac{\lambda_1}{1-\lambda_1}e+\beta_i e\right) \mid \sum_{i=2}^{N} \frac{\lambda_i}{1-\lambda_i}\beta_i = 0\right\}.
\end{aligned}$$

From here, by using simple induction argument over the number N, we have that $\hat{F} \in M$, too. That gives that the set M is closed with respect to lamination.

LEMMA 4.1. *Let $F_1, F_2 \in \mathcal{M}$. Then for every fixed $(z,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda \in [0,1]$ there exists a laminate $\hat{F}$ (in a direction e constructed from $\{F_1; F_2\}$ in proportions $\{\lambda; (1-\lambda)\}$) such that*

$$\hat{F}(z) + \hat{F}^*(\xi) \le \lambda F_1(z) + (1-\lambda)F_2(z) + [\lambda F_1 + (1-\lambda)F_2]^*(\xi).$$

Proof. For fixed proportions $\{\lambda; (1-\lambda)\}$ it holds

$$
\begin{aligned}
&\min_{e\in\mathbb{R}^n}[\hat{F}(z) + \hat{F}^*(\xi)]\\
= \ &\min_{e\in\mathbb{R}^n}\Big\{\min_{t\in\mathbb{R}}[\lambda F_1(z+t(1-\lambda)e) + (1-\lambda)F_2(z-t\lambda e)]\\
+ \ &\min_{(a,e)=0}\Big[\lambda \sup_{b^1\in\mathbb{R}^n}\Big(\langle b^1, \xi + (1-\lambda)a\rangle - F_1(b^1)\Big)\\
+ \ &(1-\lambda)\sup_{b^2\in\mathbb{R}^n}\Big(\langle b^2, \xi - \lambda a\rangle - F_2(b^2)\Big)\Big]\Big\} \qquad (11)\\
\le \ &\lambda F_1(z) + (1-\lambda)F_2(z) + \inf_{a\in\mathbb{R}^n}\ \sup_{b^1,b^2\in\mathbb{R}^n}\Big\{\langle \lambda b^1 + (1-\lambda)b^2, \xi\rangle\\
+ \ &\lambda(1-\lambda)\langle b^1 - b^2, a\rangle - [\lambda F_1(b^1) + (1-\lambda)F_2(b^2)]\Big\}.
\end{aligned}
$$

Since the mapping

$$(b^1, b^2) \to \lambda F_1(b^1) + (1-\lambda)F_2(b^2)$$

is strictly convex and coercive, we can exchange inf sup to sup inf in the right hand side of (11). The inner infimum over $a \in \mathbb{R}^n$ gives that $b^1 = b^2$. Therefore,

$$
\begin{aligned}
\min_{e\in\mathbb{R}^n}[\hat{F}(z) + \hat{F}^*(\xi)] \ &\le \ \lambda F_1(z) + (1-\lambda)F_2(z)\\
&+ \ \sup_{b\in\mathbb{R}^n}[\langle b, \xi\rangle - \lambda F_1(b) - (1-\lambda)F_2(b)]\\
&= \ \lambda F_1(z) + (1-\lambda)F_2(z) + [\lambda F_1 + (1-\lambda)F_2]^*(\xi).
\end{aligned}
$$

□

From Lemma 4.1 and the fact that the set M is closed with respect to lamination, it follows immediately that M contains all convex combinations of its elements.

Exactly the same reasoning as in Lemma 4.1, only now expressing F_1, F_2 as conjugate to F_1^*, F_2^* functions respectively, gives that for every given $F_1, F_2 \in \mathcal{M}$ and $\lambda \in [0,1]$ and fixed $(z,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ there exists a laminate $\hat{F}$ (in some direction e constructed from $\{F_1; F_2\}$ in proportions $\{\lambda; (1-\lambda)\}$) such that

$$\hat{F}(z) + \hat{F}^*(\xi) \le [\lambda F_1^* + (1-\lambda)F_2^*]^*(z) + \lambda F_1^*(\xi) + (1-\lambda)F_2^*(\xi).$$

Therefore,

$$\mathrm{co}\{F^* \mid F \in M\} = \{F^* \mid F \in M\},$$

what completes the proof of Theorem 2.2.

We conclude with the following remarks.

REMARK 4.1. For the vectorial case of linear elliptic equations the passage from the initial set of given operators to its convex hull does not, in general, preserve the weak closure of the set of solutions of the corresponding family of equations, see, for instance, Raitums [6]. Therefore, the last two statements in Theorem 2.2 about convexity properties of the set M are a specific feature of the scalar case.

REMARK 4.2. There is no a one-to-one relationship between Λ-convex functions L and closed with respect to lamination sets M. To illustrate this feature we present a simple example of a set $M_0 \subset \mathcal{M}$ such that

(i) M_0 is closed with respect to lamination;

(ii) the set

$$\{\tilde{F} \in \mathcal{M} \mid \tilde{F}(z) + \tilde{F}^*(\xi) \geq \inf_{F \in M} [F(z) + F^*(\xi)] \quad \forall (z, \xi) \subset \mathbb{R}^n \times \mathbb{R}^n\}$$

is larger than M_0.

Let $n = 2$ and let M_0 consist of functions

$$F(z) = \frac{1}{2}\langle Az, z\rangle, \; z \in \mathbb{R}^n; \quad A \in \mathcal{A},$$

where

$$\mathcal{A} = \{A \in \mathbb{R}^{2\times 2} \mid A \,\text{is symmetric}, \, \lambda_1(A) \cdot \lambda_2(A) = 1, \; \alpha \leq \lambda_1(A) \leq 1\}$$

and $0 < \alpha < 1$ is fixed.

The set $\mathcal{A}$ is G-closed, hence the set M_0 is closed with respect to lamination.

At the same time, the function F_0,

$$F_0(z) = \frac{1}{2}\alpha |z|^2$$

does not belong to M_0, but

$$F_0(z) + F_0^*(\xi) = \frac{1}{2}\alpha |z|^2 + \frac{1}{2}|\xi|^2 \geq \frac{1}{2}\langle A_0 z, z\rangle + \frac{1}{2}\langle A_0^{-1}\xi, \xi\rangle,$$

where $A_0 \in \mathcal{A}$ is such that $\lambda_1(A_0) = \alpha$, $\lambda_2(A_0) = 1/\alpha$ and the eigenvector, which corresponds to $\lambda_1(A_0)$, is parallel to z.

REFERENCES

[1] Ball, J., Kirchheim, B., Kristensen, J., Regularity of qusiconvex envelopes. *Calc. Var. Partial Differential Equations*, **11** (2000), 333–359.

[2] Dacorogna, B., *Direct Methods in the Calculus of Variation*. Springer Verlag, Berlin etc., 1989.

[3] Dal Maso, G., *An Introduction to Γ-convergence.* Birkhäuser, Boston, 1993.

[4] Fonseca, I., Müller, S., A-quasiconvexity, lower semicontinuity, and Young measures. *SIAM J. Math. Anal.*, **30** (1999), 1355–1390.

[5] Murat, F., Compacitè per compensation: condition necessaire et suffisante de continuitè faible sous une hypothèse de rang constant. *Ann. Sc. Norm. Supper Pisa Cl. Sci. IV*, **8** (1981), 68-102.

[6] Raitums, U., The maximum principle and the convexification of optimal control problems. *Control Cybernet.*, **23** (1994), 745–760.

[7] Raitums, U., Relaxation of quasilinear elliptic systems via A-quasiconvex envelopes. *ESAIM Control Optim. Calc. Var.*, **7** (2002), 309–334.

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Λ-izliektas funkcijas un attiecībā pret laminēšanu slēgtas kopas skalārajā gadījumā

Kopsavilkums

Rakstā aplūkotas funkcijas $L\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, *izliektas attiecībā pret konusu* $\Lambda = \{(z,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \mid \langle z,\xi\rangle = 0\}$, *un stingri izliektu, koercitīvu un nepārtraukti atvasinamu funkciju* $F\colon \mathbb{R}^n \to \mathbb{R}$ *kopas* M. *Rakstā paradīts, ka tad, ja* L *ir* Λ-*izliekta un* M *sastāv no tādām funkcijām* F, *ka* $F(\cdot) + F^*(\cdot) \geq L(\cdot,\cdot)$, *tad kopa* M *ir izliekta un slēgta attiecībā pret laminēšanu.*

Some remarks on the category SET(L)

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This paper considers some intrinsic properties of the category SET(L) of L-subsets of sets with a fixed basis L. We recall its definition and then dwell upon some special objects and morphisms as well as some standard constructions (e.g., product and coproduct of objects and morphisms) in this category.

Key words: L-fuzzy set, category of L-fuzzy sets, functor, special morphism, special object, standard construction

Mathematics Subject Classification (2000): primary 03E72, **secondary** 04A72, 18B99

1. Introduction

Since the inception of the notion of a fuzzy set in [4] interests of many researchers have been directed to the study of different mathematical structures involving fuzzy sets and their generalization L-fuzzy sets (see [1, 2]) or just L-sets for short. Among many other problems some authors considered the category SET(L) of all L-subsets of all sets X where L is a *fixed* lattice (see [1]). In particular in [3] are considered the relations between the category SET(L) and the topoi theory. However, as far as we know up to now there has been no paper where the intrinsic structure of the category SET(L) would be studied. The purpose of our work is to start the systematic study of the intrinsic properties of this category as well as its relations to other categories. We begin by recalling its definition, mentioning some of its basic properties and describing subcategories and related functors. Later on we consider some special morphisms (monomorphism, epimorphism, section, retraction etc.) and some special objects (initial and final objects, subobject and quotient object of an arbitrary object) in the category SET(L). Finally we discuss some standard constructions in this category, i.e., product and coproduct of objects and morphisms, pullback and pushout diagrams.

A logical continuation of this paper would be an investigation of a more general category SET(CLAT) whose objects are L-sets with *different* lattices L. This will be the subject of our forthcoming paper.

2. Definition and basic properties

In this section we will discuss some basic properties of the category SET(L). Let us start by recalling its definition (see [1]).

Suppose L is a lattice (L, $\leqslant$), i.e., a partially ordered set such that for every two points $a, b \in L$ the join $a \vee b$ and the meet $a \wedge b$ are defined and besides that there exist two points $0_L, 1_L \in L$ such that $0_L \leqslant a \leqslant 1_L$ for all $a \in L$. We assume that $0_L \neq 1_L$, i.e., L has at least two elements. Then the category SET(L) can be defined as follows.

1. The objects of SET(L) are all L-subsets of sets, i.e., mappings $X\colon \tilde{X} \to L$ where $\tilde{X}$ is an arbitrary set (maybe empty). Henceforth, the objects of SET(L) will be denoted by X, Y or Z and arbitrary sets by $\tilde{X}$, $\tilde{Y}$ or $\tilde{Z}$. By saying that an object $X \in \mathtt{Obj}\,\mathrm{SET}(L)$ is given we will always mean that X is a mapping $X\colon \tilde{X} \to L$.

2. Given two objects $X, Y \in \mathtt{Obj}\,\mathrm{SET}(L)$ the set of morphisms from X to Y $\mathtt{Mor}_{\mathrm{SET}(L)}(X, Y)$ consists of all mappings $f\colon \tilde{X} \to \tilde{Y}$ such that $X \leqslant Y \circ f$.

We will continue by recalling a trivial subcategory of the category SET(L), namely, the category SET whose objects are arbitrary sets and morphisms are arbitrary mappings between sets.

PROPOSITION 1. *The category* SET *is isomorphic to a full subcategory of the category* SET(L).

Now let us see whether the category SET(L) is connected.

PROPOSITION 2. *The category* SET(L) *is not connected.*

Proof. To prove the proposition we have to find such two objects $X, Y \in \mathtt{Obj}\,\mathrm{SET}(L)$ that the set $\mathtt{Mor}_{\mathrm{SET}(L)}(X, Y)$ is empty. It is easy to see that these objects can be every two mappings $X \equiv a \in L$ and $Y \equiv b \in L$ where $a \nleqslant b$ and $\tilde{X}$, $\tilde{Y}$ are arbitrary nonempty sets. ❑

Notice that unlike the category SET, the full subcategory $\mathrm{SET}_0(L)$ of the category SET(L) consisting of all non-void L-sets is not connected.

Lastly we will show some examples of functors related to the category SET(L).

EXAMPLE 1. $F\colon \mathrm{SET}(L) \rightsquigarrow \mathrm{SET}$: Each object $X \in \mathtt{Obj}\,\mathrm{SET}(L)$ will correspond to the set $\tilde{X} \subset \mathtt{Obj}\,\mathrm{SET}$ and each morphism $f \in \mathtt{Mor}\,\mathrm{SET}(L)$ will correspond to the same morphism $f \in \mathtt{Mor}\,\mathrm{SET}$. Obviously, F is a functor.

The next example is more complicated.

EXAMPLE 2. $F\colon \mathrm{SET}(L) \rightsquigarrow \mathrm{SET}$: Each object $X \in \mathtt{Obj}\,\mathrm{SET}(L)$ will correspond to the set $\tilde{X} \times \{X\} \in \mathtt{Obj}\,\mathrm{SET}$ where X is considered as a point from the set $L^{\tilde{X}}$ of all mappings $h\colon \tilde{X} \to L$. Each morphism $f \in \mathtt{Mor}_{\mathrm{SET}(L)}(X, Y)$ will correspond to the mapping $F(f) : \tilde{X} \times \{X\} \to \tilde{Y} \times \{Y\}$, $F(f)(\tilde{x}, X) = (f(\tilde{x}), Y)$. Obviously, F is a functor.

Clearly $F(\mathrm{SET}(L))$ is a subcategory of the category SET (not full) and besides that the categories SET(L) and $F(\mathrm{SET}(L))$ are isomorphic. Thus, the following proposition holds.

PROPOSITION 3. *The category* SET(L) *is isomorphic to a subcategory of the category* SET.

EXAMPLE 3. $F\colon \mathrm{SET}(L_1) \rightsquigarrow \mathrm{SET}(L_2)$: Suppose two categories SET(L_1) and SET(L_2) with isomorphic lattices L_1 and L_2 are given. This means that there exists an isomorphism $\varphi\colon L_1 \to L_2$. Then the functor F can be defined as follows. Each object $X\colon \tilde{X} \to L_1$ from $\mathtt{Obj}\,\mathrm{SET}(L_1)$ will correspond to the object $\varphi \circ X\colon \tilde{X} \to L_2$ from $\mathtt{Obj}\,\mathrm{SET}(L_2)$ and each morphism $f \in \mathtt{Mor}\,\mathrm{SET}(L_1)$ will correspond to the same morphism $f \in \mathtt{Mor}\,\mathrm{SET}(L_2)$ because $X \leqslant Y \circ f$ implies $\varphi \circ X \leqslant \varphi \circ Y \circ f$. Obviously, F is a functor.

With the help of this example and the fact that if $\varphi\colon L_1 \to L_2$ is an isomorphism then $\psi = \varphi^{-1}\colon L_2 \to L_1$ is also an isomorphism it is possible to construct a functor $G\colon \mathrm{SET}(L_2) \rightsquigarrow \mathrm{SET}(L_1)$. One can easily verify that $G \circ F = e_{\mathrm{SET}(L_1)}$ and $F \circ G = e_{\mathrm{SET}(L_2)}$ where e is an identity functor. Thus, the following proposition holds.

PROPOSITION 4. *If two lattices L_1 and L_2 are isomorphic then their correspondent categories* SET(L_1) *and* SET(L_2) *are also isomorphic.*

In the following two sections we will consider some special morphisms and objects in the category SET(L).

3. Special morphisms

Suppose we have two objects $X, Y \in \mathtt{Obj}\,\mathrm{SET}(L)$ and an arbitrary morphism $f \in \mathtt{Mor}_{\mathrm{SET}(L)}(X, Y)$. In the following three subsections we consider some necessary and sufficient conditions for f to be a special morphism in the category SET(L).

3.1. Monomorphisms, epimorphisms, and bimorphisms

THEOREM 1. *A morphism $f\colon X \to Y$ is a monomorphism iff f is injective.*

Proof. We will prove the necessity first and therefore assume that f is a monomorphism. Then for each object $Z \in \texttt{Obj}\,\mathrm{SET}(L)$ and every two morphisms $g, h \in \texttt{Mor}_{\mathrm{SET}(L)}(Z, X)$ such that $f \circ g = f \circ h$ it follows that $g = h$.

If f is not injective and there exist two points $x_1, x_2 \in \tilde{X}$, $x_1 \neq x_2$ such that $f(x_1) = f(x_2) = y_0 \in \tilde{Y}$ then let $\tilde{Z} = \{z_0\}$ and $Z(z_0) = 0_L$. In this case every two mappings $g, h\colon \tilde{Z} \to \tilde{X}$ will be morphisms and thus we can take $h(z_0) = x_1$ and $g(z_0) = x_2$. It is easy to see that $f \circ g = f \circ h$ but $g \neq h$.

The sufficiency is obvious. □

THEOREM 2. *A morphism $f\colon X \to Y$ is an epimorphism iff f is surjective.*

Proof. We will prove the necessity first and therefore assume that f is an epimorphism. Then for each object $Z \in \texttt{Obj}\,\mathrm{SET}(L)$ and every two morphisms $g, h \in \texttt{Mor}_{\mathrm{SET}(L)}(Y, Z)$ such that $h \circ f = g \circ f$ it follows that $g = h$.

If f is not surjective and there exists such a point $y_0 \in \tilde{Y}$ that $f^{-1}(y_0) = \varnothing$ then let us take $\tilde{Z} = \{z_1, z_2\}$ and $Z(z_1) = Z(z_2) = 1_L$. In this case every two mappings $g, h\colon \tilde{Y} \to \tilde{Z}$ will be morphisms and thus we can take $h(\tilde{Y}) = \{z_1\}$ and $g(\tilde{Y}\setminus\{y_0\}) = \{z_1\}$ but $g(y_0) = z_2$. It is easy to see that $g \circ f = h \circ f$ but $g \neq h$.

The sufficiency is obvious. □

From the last two theorems and the definition of bimorphism (notice that a morphism is said to be a bimorphism provided that it is both a monomorphism and an epimorphism) the following theorem can be derived.

THEOREM 3. *A morphism $f\colon X \to Y$ is a bimorphism iff f is bijective.*

3.2. Sections, retractions, and isomorphisms

This section is devoted to section, retraction and isomorphism in the category $\mathrm{SET}(L)$. Notice that although in the category SET these concepts are equivalent to accordingly monomorphism, epimorphism and bimorphism, in the category $\mathrm{SET}(L)$ they are quite different.

THEOREM 4. *A morphism $f\colon X \to Y$ is a section iff the following conditions are fulfilled:*

1. *f is injective;*
2. *$X = Y \circ f$;*
3. *for each $y \in \tilde{Y}$ there exists such a point $x \in \tilde{X}$ that $Y(y) \leqslant X(x)$.*

Proof. We will prove the necessity first and therefore assume that f is a section. This implies the existence of such a morphism $g \in \texttt{Mor}_{\mathrm{SET}(L)}(Y, X)$ that $g \circ f =$

e_X where e is an identity morphism which is just an identity mapping in the category SET(L). Let us prove that all conditions of the theorem are fulfilled.

The first condition follows immediately from the existence of g. In order to prove the second condition we will notice that the definition of morphism in the category SET(L) implies $X \leqslant Y \circ f$ and $Y \circ f \leqslant X \circ g \circ f$. Then, replacing $g \circ f$ by e_X, we will have that $Y \circ f \leqslant X$ and therefore $X = Y \circ f$. The third condition follows from the fact that $Y(y) \leqslant X \circ g(y)\ \forall y \in \tilde{Y}$ and thus the necessary point x will be just $g(y)$.

Now let us prove the sufficiency and therefore assume that all conditions of the theorem are fulfilled. Then there exists a mapping g such that

$$g(y) = \begin{cases} f^{-1}(y), & y \in f(\tilde{X}) \\ x_y \in \tilde{X} \text{ such that } Y(y) \leqslant X(x_y), & y \in \tilde{Y} \backslash f(\tilde{X}). \end{cases}$$

If $g \in \mathtt{Mor}_{\mathrm{SET}(L)}(Y, X)$ then the sufficiency is proved because it is easy to see that $g \circ f = e_X$, therefore let us show that $Y(y_0) \leqslant X \circ g(y_0)\ \forall y_0 \in \tilde{Y}$. If $y_0 \in f(\tilde{X})$ then $g(y_0) = x_0$ where $f(x_0) = y_0$. From the second condition of the theorem it follows that $X(x_0) = Y \circ f(x_0) = Y(y_0)$ and then $X \circ g(y_0) = X(x_0) = Y(y_0)$. In case of $y_0 \in \tilde{Y} \backslash f(\tilde{X})$ the necessary inequality follows from the definition of g. □

THEOREM 5. *A morphism $f : X \to Y$ is a retraction iff for each $y \in \tilde{Y}$ there exists such a point $x \in \tilde{X}$ that $f(x) = y$ and $X(x) = Y(y)$.*

Proof. We will prove the necessity first and therefore assume that f is a retracion. This implies the existence of such a morphism $g \in \mathtt{Mor}_{\mathrm{SET}(L)}(Y, X)$ that $f \circ g = e_Y$. Let us prove that the condition of the theorem is fulfilled. Notice that from the condition it follows that f must be surjective, therefore the first thing we need to prove is that $\forall y \in \tilde{Y}\ f^{-1}(y) \neq \varnothing$. The statement follows immediately from the existence of g. Further, suppose an arbitrary point $y_0 \in \tilde{Y}$ is given. Then there exists such a point $x_0 \in \tilde{X}$ that $x_0 = g(y_0)$ and $f(x_0) = y_0$. From the fact that f and g are morphisms we will get that $X(x_0) \leqslant Y \circ f(x_0) = Y(y_0)$ and $Y(y_0) \leqslant X \circ g(y_0) = X(x_0)$. Two last inequalities will give us the necessary equality $X(x_0) = Y(y_0)$.

Now let us prove the sufficiency and therefore assume that the condition of the theorem is fulfilled. Then the necessary mapping can be defined as follows:

$$g(y) = x \in \{z \in \tilde{X} \mid f(z) = y \text{ and } X(z) = Y(y)\}.$$

If $g \in \mathtt{Mor}_{\mathrm{SET}(L)}(Y, X)$ then the sufficiency is proved because one can easily see that $f \circ g = e_Y$, therefore our task now is to show that $Y(y_0) \leqslant X \circ g(y_0)\ \forall y_0 \in \tilde{Y}$. Suppose an arbitrary point $y_0 \in \tilde{Y}$ is given. Then $X \circ g(y_0) = X(x_0)$ where $x_0 \in \tilde{X}$. From the definition of g it follows that $X(x_0) = Y(y_0)$ which implies the necessary inequality. □

THEOREM 6. *A morphism $f : X \to Y$ is an isomorphism iff the following conditions are fulfilled:*

1. *f is bijective;*
2. $X = Y \circ f$.

Proof. We will prove the necessity first and therefore assume that f is an isomorphism. Then f is both a section and a retraction which implies the existence of such a morphism $g \in \mathtt{Mor}_{\mathrm{SET}(L)}(Y, X)$ that $f \circ g = e_Y$ and $g \circ f = e_X$. Let us prove that all conditions of the theorem are fulfilled.

The first condition follows immediately from the existence of g. The second condition follows from the fact that f is a section.

Now let us prove the sufficiency and therefore assume that all conditions of the theorem are fulfilled. Then the necessary mapping g can be obtained in the following way:

$$g(y) = f^{-1}(y), \quad y \in \tilde{Y}.$$

If $g \in \mathtt{Mor}_{\mathrm{SET}(L)}(Y, X)$ then the sufficiency is proved because it is obvious that $g \circ f = e_X$ and $f \circ g = e_Y$, therefore we need to prove that $Y(y_0) \leqslant X \circ g(y_0)\ \forall y_0 \in \tilde{Y}$. Suppose an arbitrary point $y_0 \in \tilde{Y}$ is given. Then $X \circ g(y_0) = X(x_0)$ where $f(x_0) = y_0$. From the second condition of the theorem it follows that $X(x_0) = Y \circ f(x_0) = Y(y_0)$ which implies the necessary inequality. □

The last theorem shows the necessary and sufficient conditions for two given objects $X, Y \in \mathtt{Obj}\,\mathrm{SET}(L)$ to be isomorphic.

THEOREM 7. *Two objects $X, Y \in \mathtt{Obj}\,\mathrm{SET}(L)$ are isomorphic iff for every $a \in L$, $|X^{-1}(a)| = |Y^{-1}(a)|$.*

Proof. We will prove the necessity first and therefore assume that the objects X and Y are isomorphic. This implies the existence of an isomorphism $f\colon X \to Y$. Suppose an arbitrary point $a \in L$ is given. In order to prove that $|X^{-1}(a)| = |Y^{-1}(a)|$ we have to show that there exists a bijective mapping $\varphi\colon X^{-1}(a) \to Y^{-1}(a)$. Let us prove that $\varphi = f_a = f|_{X^{-1}(a)}$. The fact that f is an isomorphism implies $X(x_0) = Y \circ f(x_0)\ \forall x_0 \in \tilde{X}$ and therefore $f_a(X^{-1}(a)) \subset Y^{-1}(a)$, besides that f_a is injective. Let us show that f_a is also surjective. Suppose an arbitrary point $y_0 \in Y^{-1}(a)$ is given. From the properties of isomorphism we derive that there exists such a point $x_0 \in \tilde{X}$ that $f(x_0) = y_0$ and $X(x_0) = Y \circ f(x_0) = Y(y_0) = a \in L$. But then $x_0 \in X^{-1}(a)$ and f_a is indeed surjective which implies the necessary equality.

Now let us prove the sufficiency and therefore assume that $|X^{-1}(a)| = |Y^{-1}(a)|\ \forall a \in L$. This gives us a family of bijective mappings $\{f_a\}_{a \in L}$ where $f_a : X^{-1}(a) \to Y^{-1}(a)$. The necessary mapping f could be obtained as follows: $f(x) = f_a(x)$, $x \in X^{-1}(a)$. It is easy to see that $\bigcup_{a \in L} X^{-1}(a) = \tilde{X}$ and also $\bigcup_{a \in L} Y^{-1}(a) = \tilde{Y}$ therefore f is bijective. The definition of f implies $X(x) = Y \circ f(x)\ \forall x \in \tilde{X}$. From the previous theorem it follows that f is an isomorphism and thus the objects X and Y are isomorphic. □

3.3. Constant morphisms and equalizers

THEOREM 8. *A morphism $f\colon X \to Y$ is a constant morphism iff $f(\tilde{X}) = \{y_0\} \subset \tilde{Y}$.*

Proof. Let us prove the necessity first and therefore assume that f is a constant morphism. Then for each object $Z \in \mathtt{Obj}\,\mathrm{SET}(L)$ and every two morphisms $g, h \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(Z, X)$ it follows that $f \circ g = f \circ h$.

If there exist two points $x_1, x_2 \in \tilde{X}$, $x_1 \neq x_2$ such that $f(x_1) = y_1 \neq y_2 = f(x_2)$ then let us take $\tilde{Z} = \{z_0\}$ and $Z(z_0) = 0_L$. In this case every two mappings $g, h\colon \tilde{Z} \to \tilde{X}$ will be morphisms and thus we can take $g(z_0) = x_1$ and $h(z_0) = x_2$. It is easy to see that $f \circ g \neq g \circ h$.

The sufficiency is obvious. □

THEOREM 9. *A morphism $f\colon X \to Y$ is an equalizer iff the following conditions are fulfilled:*

1. *f is injective;*
2. *$X = Y \circ f$.*

Proof. We will prove the necessity first and therefore assume that f is an equalizer. This implies the existence of such an object $Z \in \mathtt{Obj}\,\mathrm{SET}(L)$ and such two morphisms $g, h \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(Y, Z)$ that the following properties are satisfied:

1. $h \circ f = g \circ f$;
2. for each object $W \in \mathtt{Obj}\,\mathrm{SET}(L)$ and each morphism $m\colon W \to Y$ such that $h \circ m = g \circ m$ there exists a unique morphism $k\colon W \to X$ such that $f \circ k = m$.

Let us prove that all conditions of the theorem are fulfilled.

If f is not injective and there exist two points $x_1, x_2 \in \tilde{X}$, $x_1 \neq x_2$ such that $f(x_1) = f(x_2) = y_0 \in \tilde{Y}$ then let us take $\tilde{W} = \{w_0\}$ and $W(w_0) = 0_L$. In this case every two mappings $k_1, k_2\colon \tilde{W} \to \tilde{X}$ and also $m\colon \tilde{W} \to \tilde{Y}$ will be morphisms and thus we can take $m(w_0) = y_0$, but $k_1(w_0) = x_1$ and $k_2(w_0) = x_2$. Then obviously $g \circ m = h \circ m$ and also $f \circ k_1 = f \circ k_2 = m$ but $k_1 \neq k_2$ which contradicts the definition of equalizer.

Suppose an arbitrary point $x_0 \in \tilde{X}$ is given. We have to prove that $X(x_0) = Y \circ f(x_0)$. The fact that f is a morphism implies $X(x_0) \leqslant Y \circ f(x_0)$. Further, let us take $\tilde{W} = f(\tilde{X})$ and $W \equiv Y|_{f(\tilde{X})}$. Then the mapping m such that $h \circ m = g \circ m$ can be defined as follows: $m(x) = x, \forall x \in f(\tilde{X})$. If so then the morphism k will be just f^{-1}. The fact that f^{-1} is a morphism implies $X \circ f^{-1} \circ f(x_0) = X(x_0) \geqslant W \circ f(x_0) = Y \circ f(x_0)$. The last inequality leads to the necessary equality $X(x_0) = Y \circ f(x_0)$.

Now let us prove the sufficiency and therefore assume that all conditions of the theorem are fulfilled. We will try to find the necessary object $Z \in$ $\texttt{Obj}\,\mathrm{SET}(L)$ and two morphisms $g, h \in \texttt{Mor}_{\mathrm{SET}(L)}(Y, Z)$. Let $\tilde{Z} = \{z_1, z_0\}$ and $Z(z_0) = Z(z_1) = 1_L$. In this case every two mappings $g, h\colon \tilde{Y} \to \tilde{Z}$ will be morphisms and thus we can take $h(y) = z_1, \forall y \in \tilde{Y}$, but $g(y) = z_1, y \in f(\tilde{X})$ and $g(y) = z_0, y \in \tilde{Y} \backslash f(\tilde{X})$. It is obvious that $h \circ f = g \circ f$.

If such an object $W \in \texttt{Obj}\,\mathrm{SET}(L)$ and such a morphism $m\colon W \to Y$ are given that $h \circ m = g \circ m$ then there exists a unique mapping $k\colon \tilde{W} \to \tilde{X}$, $k = f^{-1} \circ m$ such that $f \circ k = m$. Because of $X \circ k = X \circ f^{-1} \circ m = Y \circ f \circ f^{-1} \circ m = Y \circ m \geqslant W$ k is a morphism. □

4. Special objects

In this section we will consider some special objects in the category $\mathrm{SET}(L)$, i.e, initial and final objects and also subobject and quotient object of an arbitrary object.

Let us recall basic definitions. An object X is called initial provided that for each object Y there exists a unique morphism $f\colon X \to Y$. The same way an object X is called final provided that for each object Y there exists a unique morphism $f\colon Y \to X$. Suppose we have two objects X and Y and two morphisms $f, q\colon X \to Y$ where f is a monomorphism and q is an epimorphism. The pair (f, X) is called a subobject of Y and the pair (q, Y) a quotient object of X.

The proofs of the following statements are trivial and therefore are omitted.

PROPOSITION 5. *A mapping $I\colon \varnothing \to L$ is an initial object in the category* $\mathrm{SET}(L)$.

PROPOSITION 6. *A mapping $F\colon \{*\} \to L$, $F(*) = 1_L$ is a final object in the category* $\mathrm{SET}(L)$.

Suppose we have two objects $X, Y \in \texttt{Obj}\,\mathrm{SET}(L)$.

THEOREM 10. *If the following conditions are fulfilled:*

1. $\tilde{X} \subset \tilde{Y}$;
2. *for each $x \in \tilde{X}$ it follows that $X(x) \leqslant Y(x)$;*

then there exists a monomorphism $f\colon X \to Y$ and hence (f, X) is a subobject of Y.

THEOREM 11. *If the following conditions are fulfilled:*

1. $\tilde{Y} \subset \tilde{X}$;

2. *for each* $y \in \tilde{Y}$ *it follows that* $X(y) \leqslant Y(y)$;

3. *for each* $x \in \tilde{X} \setminus \tilde{Y}$ *there exists such a point* $y \in \tilde{Y}$ *that* $X(x) \leqslant Y(y)$;

then there exists an epimorphism $q\colon X \to Y$ *and hence* (q, Y) *is a quotient object of* X.

The following example shows one way of obtaining a quotient object for an arbitrary object $X \in \mathtt{Obj}\,\mathrm{SET}(L)$.

EXAMPLE 4. Suppose an object $X \in \mathtt{Obj}\,\mathrm{SET}(L)$ is given. In order to find a quotient object (q, Y) for X we will firstly construct the necessary object Y. Let $\tilde{Y} = \{[x]_a \,|\, a \in X(\tilde{X})\}$ where $[x]_a = \{x \,|\, x \in \tilde{X}, X(x) = a \in L\}$ and $Y([x]_a) = a$, $\forall [x]_a \in \tilde{Y}$. Then the mapping $q\colon \tilde{X} \to \tilde{Y}$ can be defined as follows: $q(z) = [x]_{X(z)}$, $\forall z \in \tilde{X}$. One can easily verify that (q, Y) is a quotient object of X.

Now suppose the lattice L is complete, i.e., for every subset $A \subset L$ the join $\bigvee A$ and the meet $\bigwedge A$ are defined, and fix an arbitrary object $X \in \mathtt{Obj}\,\mathrm{SET}(L)$. By Y_{min} and Y_{max} we will denote such objects from $\mathtt{Obj}\,\mathrm{SET}(L)$ that $\tilde{Y}_{min} = \{*\}$, $Y_{min}(*) = \vee\{a \,|\, a \in X(\tilde{X})\}$ and $|\tilde{Y}_{max}| = |\tilde{X}|$, $Y_{max}(y) = 1_L$, $\forall y \in \tilde{Y}_{max}$. Obviously, there exist such morphisms $q_{min}\colon X \to Y_{min}$ and $q_{max}\colon X \to Y_{max}$ that (q_{min}, Y_{min}) and (q_{max}, Y_{max}) are the quotient objects of X and besides that the following proposition holds.

PROPOSITION 7. *If* $\vee\{a \,|\, a \in X(\tilde{X})\} \in X(\tilde{X})$ *then for each quotient object* (q, Y) *of* X *there exist such morphisms* $f_1\colon Y_{min} \to Y$ *and* $f_2\colon Y \to Y_{max}$ *that* (f_1, Y_{min}) *is a subobject of* Y *and* (f_2, Y) *is a subobject of* Y_{max}.

Thus we can say that (q_{min}, Y_{min}) and (q_{max}, Y_{max}) are in a way the "smallest" and the "biggest" quotient objects of X.

Notice that for an arbitrary object $Y \in \mathtt{Obj}\,\mathrm{SET}(L)$ the existence of such two morphisms $f_1\colon Y_{min} \to Y$ and $f_2\colon Y \to Y_{max}$ that (f_1, Y_{min}) is a subobject of Y and (f_2, Y) is a subobject of Y_{max} does not imply the existence of an epimorphism $q\colon X \to Y$.

If we will suppose that $\vee\{a \,|\, a \in X(\tilde{X})\} \notin X(\tilde{X})$ then (q_{max}, Y_{max}) will be still the "biggest" quotient object of X, however (q_{min}, Y_{min}) will no longer be its "smallest" quotient object.

PROPOSITION 8. *If such an object* $X \in \mathtt{Obj}\,\mathrm{SET}(L)$ *is given that* $\vee\{a \,|\, a \in X(\tilde{X})\} \notin X(\tilde{X})$ *then it has no "smallest" quotient object.*

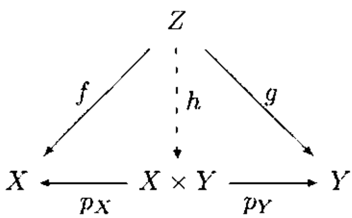
Proof. Suppose there exists such a quotient object (q'_{min}, Y'_{min}) of X that for each quotient object (q, Y) of X there exists a monomorphism $f\colon Y'_{min} \to Y$ and therefore (f, Y'_{min}) is a subobject of Y. It is obvious that then $\tilde{Y}'_{min} = \{y_0\}$ and $Y'_{min}(y_0) \geqslant \vee\{a \,|\, a \in X(\tilde{X})\}$. From the fact that (e_X, X) is a quotient object of X we will immediately get a contradiction. $\square$

5. Standard constructions

This section is devoted to some standard constructions in the category $\mathrm{SET}(L)$ which are product and coproduct of objects and morphisms, pullback and pushout. Let us start by considering product of objects.

5.1. Products of objects

Suppose we have two arbitrary objects $X, Y \in \mathtt{Obj}\,\mathrm{SET}(L)$. We will try to find their product denoted by $X \times Y$. According to the definition we have to find such an object $X \times Y \in \mathtt{Obj}\,\mathrm{SET}(L)$ and such two morphisms $p_X \in \mathtt{Mor}_{\mathrm{SET}(L)}(X \times Y, X)$ and $p_Y \in \mathtt{Mor}_{\mathrm{SET}(L)}(X \times Y, Y)$ that for each object $Z \in \mathtt{Obj}\,\mathrm{SET}(L)$ and every two morphisms $f \in \mathtt{Mor}_{\mathrm{SET}(L)}(Z, X)$ and $g \in \mathtt{Mor}_{\mathrm{SET}(L)}(Z, Y)$ there exists a unique morphism $h \in \mathtt{Mor}_{\mathrm{SET}(L)}(Z, X \times Y)$ such that the diagram



commutes.

We will start by constructing the object $X \times Y$. Let $\widetilde{X \times Y} = \tilde{X} \times \tilde{Y} = \{(x, y) \mid x \in \tilde{X}, y \in \tilde{Y}\}$ and $X \times Y(x, y) = X(x) \wedge Y(y), \forall (x, y) \in \widetilde{X \times Y}$. Then the necessary mappings p_X and p_Y can be obtained in the following way: $p_X(x, y) = x$ and $p_Y(x, y) = y$, $\forall (x, y) \in \widetilde{X \times Y}$. Every two given morphisms f and g will correspond to the mapping $h(z) = (f(z), g(z)), \forall z \in \tilde{Z}$.

Let us prove that $p_X \in \mathtt{Mor}_{\mathrm{SET}(L)}(X \times Y, X)$. For each point $(x_0, y_0) \in \widetilde{X \times Y}$ it follows that $X \circ p_X(x_0, y_0) = X(x_0) \geqslant X(x_0) \wedge Y(y_0) = X \times Y(x_0, y_0)$. The same way one can prove that $p_Y \in \mathtt{Mor}_{\mathrm{SET}(L)}(X \times Y, Y)$.

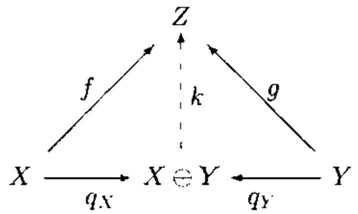
Now let us prove that $h \in \mathtt{Mor}_{\mathrm{SET}(L)}(Z, X \times Y)$. For each point $z_0 \in \tilde{Z}$ it follows that $X \times Y \circ h(z_0) = X \times Y(f(z_0), g(z_0)) = X \circ f(z_0) \wedge Y \circ g(z_0)$. The fact that f and g are morphisms implies $X \circ f(z_0) \geqslant Z(z_0)$ and $Y \circ g(z_0) \geqslant Z(z_0)$. But then $X \circ f(z_0) \wedge Y \circ g(z_0) \geqslant Z(z_0)$ and h is indeed a morphism.

One can easily verify that the above-mentioned diagram really commutes. The necessary product of objects is found.

Now let us consider coproduct of objects.

5.2. Coproducts of objects

Suppose we have two arbitrary objects $X, Y \in \mathtt{Obj}\,\mathrm{SET}(L)$. We will try to find their coproduct denoted by $X \oplus Y$. According to the definition we have to find such an object $X \oplus Y \in \mathtt{Obj}\,\mathrm{SET}(L)$ and such two morphisms $q_X \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X, X \oplus Y)$ and $q_Y \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(Y, X \oplus Y)$ that for each object $Z \in \mathtt{Obj}\,\mathrm{SET}(L)$ and every two morphisms $f \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X, Z)$ and $g \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(Y, Z)$ there exists a unique morphism $k \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X \oplus Y, Z)$ such that the diagram



commutes.

Firstly we will construct the object $X \oplus Y$. Let $\widetilde{X \oplus Y} = \tilde{X} \bigcup \tilde{Y}$. One can easily assume that $\tilde{X} \bigcap \tilde{Y} = \varnothing$, otherwise the sets $\tilde{X} \times \{1\}$ and $\tilde{Y} \times \{2\}$ can be used instead of the original ones. The mapping $X \oplus Y : \widetilde{X \oplus Y} \to L$ will be as follows:

$$X \oplus Y(x) = \begin{cases} X(x), & x \in \tilde{X} \\ Y(x), & x \in \tilde{Y}. \end{cases}$$

Then the necessary mappings q_X and q_Y will be such that $q_X(x) = x$, $\forall\, x \in \tilde{X}$ and $q_Y(y) = y$, $\forall\, y \in \tilde{Y}$. Every two given morphisms f and g will correspond to the mapping

$$k(x) = \begin{cases} f(x), & x \in \tilde{X} \\ g(x), & x \in \tilde{Y}. \end{cases}$$

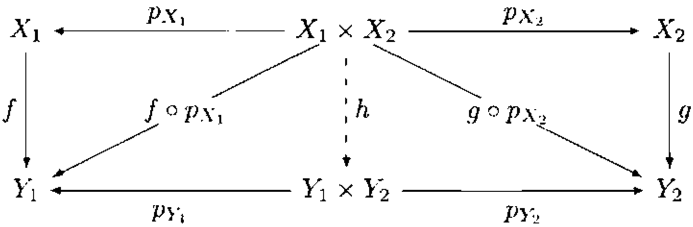
It is clear that $q_X \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X, X \oplus Y)$ and $q_Y \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(Y, X \oplus Y)$. Let us prove that $k \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X \oplus Y, Z)$. For each point $x_0 \in \widetilde{X \oplus Y}$ it follows that either $x_0 \in \tilde{X}$ or $x_0 \in \tilde{Y}$. Suppose $x_0 \in \tilde{X}$. Then $Z \circ k(x_0) = Z \circ f(x_0) \geqslant X(x_0) = X \oplus Y(x_0)$. The case $x_0 \in \tilde{Y}$ is the same. But then k is indeed a morphism.

It is easy to verify that the above-mentioned diagram really commutes. The necessary coproduct of objects is found.

Now let us turn to product and coproduct of morphisms.

5.3. Products and coproducts of morphisms

Suppose such four objects X_1, Y_1 and X_2, Y_2 from the category SET(L) are given that the sets $\mathtt{Mor}_{\,\mathrm{SET}(L)}(X_1, Y_1)$ and $\mathtt{Mor}_{\,\mathrm{SET}(L)}(X_2, Y_2)$ are not empty. Let us choose an arbitrary morphism $f \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X_1, Y_1)$ and an arbitrary morphism $g \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X_2, Y_2)$. Then the following diagram can be created.

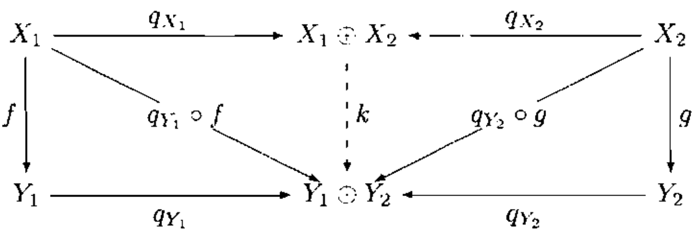


The definition of product of objects in the category SET(L) implies the existence of a unique morphism $h = f \times g$,

$$f \times g(x_1, x_2) = (f(x_1), g(x_2)), \ \forall (x_1, x_2) \in \widetilde{X \times Y}$$

which makes the diagram commutative and therefore is the product of morphisms f and g.

If we will use $X_1 \oplus X_2$ and $Y_1 \oplus Y_2$ instead of $X_1 \times X_2$ and $Y_1 \times Y_2$ then we will get the following diagram.



The definition of coproduct of objects in the category SET(L) implies the existence of a unique morphism $k = f \oplus g$,

$$f \oplus g(x) = \begin{cases} f(x), & x \in \tilde{X}_1 \\ g(x), & x \in \tilde{X}_2 \end{cases}$$

which makes the diagram commutative and therefore is the coproduct of morphisms f and g.

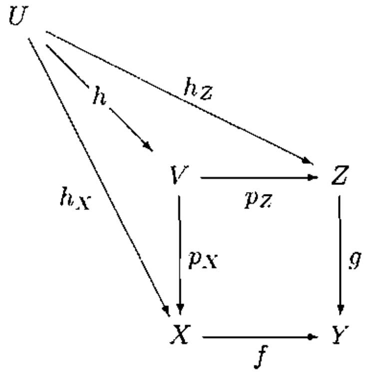
In the following section we will consider two last standard constructions pullback and pushout.

5.4. Pullbacks and pushouts

We will start by considering pullback in the category SET(L).

Suppose such three objects $X, Y, Z \in \mathtt{Obj}\,\mathrm{SET}(L)$ are given that the sets $\mathtt{Mor}_{\,\mathrm{SET}(L)}(X, Y)$ and $\mathtt{Mor}_{\,\mathrm{SET}(L)}(Z, Y)$ are not empty. Let us choose two arbitrary morphisms $f \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X, Y)$ and $g \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(Z, Y)$. We will try to find a pullback for these morphisms. According to the definition we have to find such an object $V \in \mathtt{Obj}\,\mathrm{SET}(L)$ and such two morphisms $p_X \in$

$\mathtt{Mor}_{\mathrm{SET}(L)}(V, X)$ and $p_Z \in \mathtt{Mor}_{\mathrm{SET}(L)}(V, Z)$ for which $g \circ p_Z = f \circ p_X$ that for each object $U \in \mathtt{Obj}\,\mathrm{SET}(L)$ and every two morphisms $h_X \in \mathtt{Mor}_{\mathrm{SET}(L)}(U, X)$ and $h_Z \in \mathtt{Mor}_{\mathrm{SET}(L)}(U, Z)$ for which $g \circ h_Z = f \circ h_X$ there exists a unique morphism $h \in \mathtt{Mor}_{\mathrm{SET}(L)}(U, V)$ such that the diagram



commutes.

Firstly we will construct the object V. Let $\tilde{X} \times \tilde{Z} \supset \tilde{V} = \{(x, z) \mid x \in \tilde{X},\ z \in \tilde{Z}$ and $f(x) = g(z)\}$. (Notice that in case of $\tilde{V}$ being an empty set the only object U which has the required morphisms h_X and h_Z is the object V itself, i.e., $\tilde{U} = \varnothing$, otherwise there exists some $u_0 \in \tilde{U}$ such that $f \circ h_X(u_0) = g \circ h_Z(u_0)$ and therefore $(h_X(u_0),\ h_Z(u_0)) \in \tilde{V}$. Obviously there exists a unique morphism $h \in \mathtt{Mor}_{\mathrm{SET}(L)}(U, V)$ which makes the above-mentioned diagram commute.) Let $V(x, z) = X(x) \wedge Z(z),\ \forall\,(x, z) \in \tilde{V}$. Then the necessary mappings p_X and p_Z will be as follows: $p_X(x, z) = x$ and $p_Z(x, z) = z,\ \forall\,(x, z) \in \tilde{V}$. Every two given morphisms h_X and h_Z will correspond to the mapping $h(u) = (h_X(u),\ h_Z(u)),\ \forall\,u \in \tilde{U}$.

Let us prove that $p_X \in \mathtt{Mor}_{\mathrm{SET}(L)}(V, X)$. For each point $v_0 = (x_0, z_0) \in \tilde{V}$ it follows that $X \circ p_X(v_0) = X \circ p_X(x_0, z_0) = X(x_0) \geqslant X(x_0) \wedge Z(z_0) = V(x_0, z_0) = V(v_0)$. The same way one can prove that $p_Z \in \mathtt{Mor}_{\mathrm{SET}(L)}(V, Z)$.

Now let us prove that $h \in \mathtt{Mor}_{\mathrm{SET}(L)}(U, V)$. For each point $u_0 \in \tilde{U}$ the point $h(u_0) = (h_X(u_0),\ h_Z(u_0))$ certainly belongs to $\tilde{V}$ because of $f \circ h_X(u_0) = g \circ h_Z(u_0)$. Then $V \circ h(u_0) = X \circ h_X(u_0) \wedge Z \circ h_z(u_0)$. The fact that h_X and h_Z are morphisms implies $X \circ h_X(u_0) \geqslant U(u_0)$ and $Z \circ h_Z(u_0) \geqslant U(u_0)$ and then $X \circ h_X(u_0) \wedge Z \circ h_z(u_0) \geqslant U(u_0)$ therefore h is indeed a morphism.

It is easy to verify that the above-mentioned diagram really commutes. The necessary pullback is found.

Now let us consider pushout in the category SET(L).

Suppose again we have such three objects $X, Y, Z \in \mathtt{Obj}\,\mathrm{SET}(L)$ that this time the sets $\mathtt{Mor}_{\mathrm{SET}(L)}(Y, X)$ and $\mathtt{Mor}_{\mathrm{SET}(L)}(Y, Z)$ are not empty. Let us choose two arbitrary morphisms $f \in \mathtt{Mor}_{\mathrm{SET}(L)}(Y, X)$ and $g \in \mathtt{Mor}_{\mathrm{SET}(L)}(Y, Z)$. We will try to find a pushout for these morphisms. According to the definition we have to find such an object $V \in \mathtt{Obj}\,\mathrm{SET}(L)$ and such two morphisms $q_X \in \mathtt{Mor}_{\mathrm{SET}(L)}(X, V)$ and $q_Z \in \mathtt{Mor}_{\mathrm{SET}(L)}(Z, V)$ for which $q_Z \circ g = q_X \circ f$

that for each $U \in \mathtt{Obj}\,\mathrm{SET}(L)$ and every two morphisms $h_X \in \mathtt{Mor}_{\mathrm{SET}(L)}(X, U)$ and $h_Z \in \mathtt{Mor}_{\mathrm{SET}(L)}(Z, U)$ for which $h_Z \circ g = h_X \circ f$ there exists a unique morphism $k \in \mathtt{Mor}_{\mathrm{SET}(L)}(V, U)$ such that the diagram dual to the diagram of pullback commutes.

For simplicity we will assume that f and g are injective.

Firstly we will construct the object V. Let $\tilde{V} = \tilde{V_1} \bigcup \tilde{V_2} \bigcup \tilde{V_3}$ where $\tilde{V_1} = \tilde{X} \backslash f(\tilde{Y})$, $\tilde{V_3} = \tilde{Z} \backslash g(\tilde{Y})$ and $\tilde{V_2} = \{v \,|\, v = (f(y), g(y)), y \in \tilde{Y}\}$, besides that $\tilde{V_i} \bigcap \tilde{V_j} = \varnothing$, $i \neq j$. The mapping V will be such that

$$V(v) = \begin{cases} X(v), & v \in \tilde{V_1} \\ Z(v), & v \in \tilde{V_3} \\ X(v_1) \vee Z(v_2), & v = (v_1, v_2) \in \tilde{V_2}. \end{cases}$$

Then the necessary mappings q_X and q_Z can be defined in the following way:

$$q_X(x) = \begin{cases} x, & x \in \tilde{X} \backslash f(\tilde{Y}) \\ v_x = (x, g \circ f^{-1}(x)), & x \in f(\tilde{Y}) \end{cases}$$

$$q_Z(z) = \begin{cases} z, & z \in \tilde{Z} \backslash g(\tilde{Y}) \\ v_z = (f \circ g^{-1}(z), z), & z \in g(\tilde{Y}). \end{cases}$$

Every two given morphisms h_X and h_Z will correspond to the mapping

$$k(v) = \begin{cases} h_X(v), & v \in \tilde{V_1} \\ h_Z(v), & v \in \tilde{V_3} \\ h_X(v_1) = h_Z(v_2), & v = (v_1, v_2) \in \tilde{V_2}. \end{cases}$$

Let us prove that $q_X \in \mathtt{Mor}_{\mathrm{SET}(L)}(X, V)$. For each point $x_0 \in \tilde{X}$ it follows that either $x_0 \in \tilde{X} \backslash f(\tilde{Y})$ or $x_0 \in f(\tilde{Y})$. If $x_0 \in \tilde{X} \backslash f(\tilde{Y})$ then $V \circ q_X(x_0) = V(x_0) = X(x_0)$ and if $x_0 \in f(\tilde{Y})$ then $V \circ q_X(x_0) = V(x_0, g \circ f^{-1}(x_0)) = X(x_0) \vee Y \circ g \circ f^{-1}(x_0) \geqslant X(x_0)$. Therefore q_X is indeed a morphism. The same way one can prove that $q_Z \in \mathtt{Mor}_{\mathrm{SET}(L)}(Z, V)$.

Now let us prove that $k \in \mathtt{Mor}_{\mathrm{SET}(L)}(V, U)$. For each point $v_0 \in \tilde{V}$ it follows that either $v_0 \in \tilde{V_1}$ or $v_0 \in \tilde{V_2}$ or $v_0 \in \tilde{V_3}$. If $v_0 \in \tilde{V_1}$ then $U \circ k(v_0) = U \circ h_X(v_0) \geqslant X(v_0) = V(v_0)$. In case of $v_0 \in \tilde{V_3}$ the same way we get $U \circ k(v_0) \geqslant V(v_0)$, whereas $v_0 = (v_{10}, v_{20}) \in \tilde{V_2}$ it follows that $U \circ k(v_0) = U \circ h_X(v_{10}) = U \circ h_Z(v_{20})$. From $U \circ h_X(v_{10}) \geqslant X(v_{10})$ and $U \circ h_Z(v_{20}) \geqslant Z(v_{20})$ we derive that $U \circ k(v_0) \geqslant X(v_{10}) \vee Z(v_{20}) = V(v_0)$. Therefore k is indeed a morphism.

It is easy to verify that the above-mentioned diagram really commutes. The necessary pushout is found.

At the beginning we have supposed that f and g are injective. If they are not then all the reasoning will be slightly different. For example the set $\tilde{V_2}$ will be $\tilde{V_2} = \{v \,|\, v = (V_X, V_Z), V_X \subset \tilde{X}, V_Z \subset \tilde{Z}\}$ and then for each $v_0 \in \tilde{V_2}$

$V(v_0) = (\vee_{v_{10}\in\widetilde{V_{X_0}}} X(v_{10})) \vee (\vee_{v_{20}\in\widetilde{V_{Z_0}}} Z(v_{20}))$ therefore the lattice L must be complete. The mappings q_X and q_Z will be the following:

$$q_X(x) = \begin{cases} x, & x \in \tilde{X}\backslash f(\tilde{Y}) \\ v = (\widetilde{V_X}, \widetilde{V_Z}), & x \in f(\tilde{Y}) \end{cases}$$

$$q_Z(z) = \begin{cases} z, & z \in \tilde{Z}\backslash g(\tilde{Y}) \\ v = (\widetilde{V_X}, \widetilde{V_Z}), & z \in g(\tilde{Y}). \end{cases}$$

And lastly if $v_0 \in \tilde{V_2}$ then $k(v_0) = h_X(v_{10})$ or $k(v_0) = h_Z(v_{20})$ where v_{10} and v_{20} are arbitrary points from accordingly $\widetilde{V_{X_0}}$ and $\widetilde{V_{Z_0}}$. All the rest is similar to the case when f and g are injective.

Acknowledgements

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REFERENCES

[1] Goguen, J. A., *L*-fuzzy sets. *J. Math. Anal. Appl.*, **18** (1967), 145–174.

[2] Goguen, J. A., The fuzzy Tychonoff theorem. *J. Math. Anal. Appl.*, **43** (1973), 737–742.

[3] Stout, L. N., A survey of fuzzy set and topos theory. *Fuzzy Sets Syst.*, **42** (1991), 3–14.

[4] Zadeh, L. A., Fuzzy sets. *Inf. Control*, **8** (1965), 338–365.

Dažas piezīmes par kategoriju SET(L)

Kopsavilkums

Darbā ir apskatītas dažas fiksēta režģa L L-vērtīgu kopu kategorijas SET(L) *īpašības. Ir atgādināta tās definīcija un aplūkoti daži speciāli objekti un morfismi, kā arī dažas universālas konstrukcijas (piem., objektu un morfismu reizinājums un koreizinājums) šajā kategorijā.*

On a category of L-valued L-topological spaces

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We introduce a topological-type structure on L-subsets of L-valued sets and consider the coresponding category L-TOP(L) whose objects are L-valued L-topological spaces and whose morphisms are naturally defined continuous mappings between such spaces. It is shown that L-TOP(L) is a topological category over the category L-SET(L) L-SET(L) of L-subsets of L-valued sets (or L-valued L-sets, for short) defined in our previous work.

Key words: L-set, L-topological space, L-valued set, GL-monoid

Mathematics Subject Classification (2000): primary 54A40, **secondary** 03E72

Introduction

The aim of this paper is to introduce the concept of an L-valued L-topological space, which is a certain synthesis of the concept of an L-(fuzzy) topological space in the sense of Chang-Goguen [1], [3], the concept of an L-subset of a set [6], [2] and the concept of a many-valued set in the sense of Höhle [4]. To do this we start with the concept of an L-subset of an L-valued set studied in our previous paper [5] which in its turn is a synthesis of the concept of an L-subset of a set in the sense of Zadeh-Goguen [6], [2] (that is a mapping $A : X \to L$) and the concept of a many-valued set in the sense of U. Höhle [4] (that is a usual set X equipped with the so called many-valued equality $E : X \times X \to L$). In the same paper [5] we introduced the category L-SET(L) of L-valued L-sets and investigated some properties of this category.

As it was said above, in this paper we introduce the concept of an L-valued L-topological space which is actually an L-subset of an L-valued set endowed with a naturally defined topological-type structure. Such spaces and naturally defined continuous mappings between them form a category L-TOP(L). We study some properties of this category. In particular, it is shown that L-TOP(L) is a topological category both over L-SET(L) and over the Goguen's category L-SET of L-sets with respect of the forgetfull functor.

To make the paper to some extent self-complete, we start with two preparatory sections: Section 1. prerequisities, where basic notions used in the paper are recalled, and Section 2 mainly containing basic definitions and results from our previous work [5]

1. Prerequisities

1.1. GL-monoids Let $(L, \leq, \wedge, \vee)$ be a complete infinitely distributive lattice, i.e. $(L, \leq)$ is a partially ordered set such that for every subset $A \subset L$ the join $\bigvee A$ and the meet $\bigwedge A$ are defined and $(\bigvee A) \wedge \alpha = \bigvee\{a \wedge \alpha \mid a \in A\}$ and $(\bigwedge A) \vee \alpha = \bigwedge\{a \vee \alpha \mid a \in A\}$ for every $\alpha \in L$. In particular, $\bigvee L =: 1$ and $\bigwedge L =: 0$ are respectively the universal upper and the universal lower bounds in L. We assume that $0 \neq 1$, i.e. L has at least two elements.

A $GL-$ monoid (see e.g. [4]) is a complete infinitely distributive lattice enriched with a monotone, commutative and associative binary operation $*$ such that

1. $\alpha * 1 = \alpha$ and $\alpha * 0 = 0$ for all $\alpha \in L$;
2. $\alpha * (\bigvee_j \beta_j) = \bigvee_j (\alpha * \beta_j) \quad \forall \alpha \in L, \quad \forall \{\beta_j : j \in J\} \subset L$;
3. If $\alpha \leq \beta$, then there exists $\gamma \in L$ such that $\alpha = \beta * \gamma$.

It is known that every $GL-$monoid is residuated, i.e. there exists a further binary operation " $\rightarrow$ " (implication) on L satisfying the following condition:

$$\alpha * \beta \leq \gamma \Longleftrightarrow \alpha \leq (\beta \rightarrow \gamma) \qquad \forall \alpha, \beta, \gamma \in L.$$

Explicitely implication is given by $\alpha \rightarrow \beta = \bigvee\{\lambda \in L \mid \alpha * \lambda \leq \beta\}$.

1.2. L-valued sets Following U. Höhle (cf. e.g. [4]) by an *L-valued set* we call a pair (X, E) where X is a set and E is an *L-valued equality*, i.e. a mapping $E : X \times X \rightarrow L$ such that

(1eq) $\quad E(x, x) = 1$;

(2eq) $\quad E(x, y) = E(y, x) \quad \forall x, y \in X$;

(3eq) $\quad E(x, y) * (E(y, y) \rightarrow E(y, z)) \leq E(x, z) \quad \forall x, y, z \in X$.

A mapping $f : (X, E_X) \rightarrow (Y, E_Y)$ is called *extensional* if

$$E_X(x, x') \leq E_Y(f(x), f(x'))\ \forall x, x' \in X.$$

Further, recall that an L-set, or more precisely, an L-subset of a set X is just a mapping $A : X \rightarrow L$. In case (X, E) is an L-valued set, its L-subset A is called *extensional* if

$$\bigvee_{x \in X} A(x) * E(x, x') \leq A(x') \quad \forall x' \in X.$$

2. On the category L-SET(L)

Let L be a fixed GL-monoid.

DEFINITION 2.1. [L-valued L-set] [5]. Let X be a set and A be its L-subset. An L-valued equality on A is a mapping $E : X \times X \to L$, such that

1. $E(x,x) = A(x)$;
2. $E(x,y) \le A(x) \wedge A(y)$ for all $x, y \in X$;
3. $E(x,y) = E(y,x)$;
4. $E(x,y) * (A(y) \to E(y,z)) \le E(x,z)$ for all $x,y,z \in X$.

The triple (X, A, E) is called *an L-valued L-set.*

REMARK 2.2. Notice, that if $E : X \times X \to L$ is an L-valued equality then defining $A : X \to L$ by $A(x) = E(x,x)$ for all $x \in X$ we obtain an L-valued L-set (X, A).

DEFINITION 2.3. [5] By a mapping f from an L-valued L-set (X, A, E_X) into an L-valued L-set (Y, B, E_Y) (in notation $f : (X, A, E_X) \to (Y, B, E_Y)$) we call a mapping $f : X \to Y$ such that $A(x) \le B(f(x))$ and $E_X(x,x') \le E_Y(f(x), f(x'))$.

If $f : (X, A, E_X) \to (Y, B, E_Y)$ and $g : (Y, B, E_Y) \to (Z, C, E_Z)$ are mappings of the corresponding L-valued L-sets, then obviously their set-theoretic composition is a mapping $g \circ f : (X, A, E_X)$ (Z, C, E_Z) of the corresponding L-valued L-sets. Besides, the identity mapping $id_X : X \to X$ can obviously be considered also as the identity mapping $id_{(X,A,E_X)} : (X, A, E_X) \to (X, A, E_X)$. From these observation we get

THEOREM 2.4. *L-valued L-sets and mappings between them form a category. This category will be denoted L-SET(L).*

THEOREM 2.5. *Given a family $\mathcal{E} = \{E_i \mid i \in \mathcal{I}\}$ of L-valued equalities on an L-set (X, A), let $E_0 : X \times X \to L$ be defined by $E_0(x,y) = \bigwedge\{E_i(x,y) \mid i \in \mathcal{I}\}$. Then E_0 is an L-valued equality on (X, A).*

Proof: The validity of the first three axioms of an L-valued equality on (X, A) for E_0 is clear. The validity of the fourth axiom follows from the next chain of (in)equalities:

$$E_0(x,y) * (A(y) \to E_0(y,z)) =$$

$$\bigwedge_{i \in \mathcal{I}} (E_{i\in\mathcal{I}}(x,y)) * (A(y) \to \bigwedge_{i\in\mathcal{I}} (E_{i\in\mathcal{I}}(y,z))) \le$$

$$\leq \bigwedge_{i\in\mathcal{I}}(E_{i\in\mathcal{I}}(x,y)) * (A(y) \to \bigvee_{i\in\mathcal{I}}(E_{i\in\mathcal{I}}(y,z)) =$$

$$= \bigwedge_{i\in\mathcal{I}}(E_{i\in\mathcal{I}}(x,y)) * \bigwedge_{i\in\mathcal{I}}((A(y) \to E_{i\in\mathcal{I}}(y,z)) \leq$$

$$\leq \bigwedge_{i\in\mathcal{I}}(E_{i\in\mathcal{I}}(x,y)) * (A(y) \to \bigwedge_{i\in\mathcal{I}}(E_{i\in\mathcal{I}}(y,z))) \leq$$

$$\leq \bigwedge_{i\in\mathcal{I}}(E_{i\in\mathcal{I}}(x,z)).$$

From the previous theorem we have

COROLLARY 2.6. *$\mathcal{E}(X,A)$ is a complete lattice. Its bootom element is E_* defined by $E_*(x,y) = 0$ if $x \neq y$ and $E_*(x,x) = A(x)$ for all $x,y \in X$, and its top element is E^* defined by $E^*(x,y) = A(x) \wedge A(y)$ for all $x,y \in X$.*

Now we can construct initial and final L-valued equalities on L-sets for a given family of mappings. Consider any family of mapings

$$\mathcal{F} = \{f_i : (X_i, A_i, E_i) \to (Y, B)\}.$$

We start with the case when the family $\mathcal{F}$ consists of a single mapping

$$f : (X, A, E) \to (Y, B).$$

Let $\mathcal{E}_f$ be the family of all such L-valued equalities E_j on (Y, B) for which $f : (X, A, E) \to (Y, B, E_j)$ is a morphism in L-SET(L). Then applying Theorem 2.5 we know that

$$f(E) := \bigwedge \mathcal{E}_f := \bigwedge\{E_j \mid E_j \in \mathcal{E}_f\}$$

is an L-valued equality on (Y, B). Besides it is easy to notice that $f : (X, A, E) \to (Y, B, E_0)$ is a morphism in L-SET(L) and that E_0 is the weakest of all L-valued equalities with this property.

Coming now to the general case of the family

$$\mathcal{F} = \{f_i : (X_i, A_i, E_i) \to (Y, B)\},$$

let $f_i(E_i)$ be defined as above and let

$$E_Y := \bigvee_{i\in\mathcal{I}} f_i(E_i)$$

be the supremum of this family in the lattice $\mathcal{E}$ of all L-valued equalities on (Y, B). Then, obviously E_Y is exactly the final L-valued equality on (Y, B) for the family $\mathcal{F}$.

We consider now the dual problem, namely the initial L-valued equality for a family of mappings. Explicitely, let $\mathcal{F} := \{f_i : (X, A) \to (Y_i, B_i, E_i) \mid i \in \mathcal{I}\}$ be a family of mappings. Our goal is to find the initial L-valued equality on

(X, A) for this family, that is the largest L-valued equality E_A on (X, A) for which all $f_i : (X, A, E_A) \to (Y_i, B_i, E_i)$ are morphisms in L-SET(L).

Again, we start with the case when there is only one mapping in the family $\mathcal{F}$:

$$f : (X, A) \to (Y, B, E).$$

Then we define $f^{-1}(E) := E_f$ by the equality:

$$E_f(x_1, x_2) = E(f(x_1), f(x_2)) \ \forall x_1, x_2 \in X.$$

It is easy to see that E_f thus defined is an L-valued equality on (Y, B) and besides it is the largest one for which the mapping $f : (X, A, E_f) \to (Y, B, E)$ is a morphism in L-SET(L).

Coming now to the general case of a family

$$\mathcal{F} := \{f_i : (X, A) \to (Y_i, B_i, E_i) \quad i \in \mathcal{I}\},$$

we define $E_{f_i} = f_i^{-1}(E_i)$ as above. From Theorem 2.5 it follows that

$$E_A := \bigwedge_{i \in \mathcal{I}} E_{f_i}$$

is the largest one of the L-valued equalities on (X, A) for which all f_i : $(X, A, E_A) \to (Y_i, B_i, E_i)$ are morphisms in L-SET(L). Thus E_A is the initial L-valued equality for this family. From here and, taking into account Corollary 2.6, we obtain the main result of this section:

THEOREM 2.7. *Category L-SET(L) is topological over the category L-SET.*

The existence of initial L-valued equalities guarantees that the operation of product is well defined in L-SET(L) while the existence of final equalities guarantees that the operation of co-product is well defined in this category. The details of definition of these operations are left to the reader.

3. Category L-TOP(L)

3.1. Basic definitions

DEFINITION 3.1. By an L-valued topology on an L-valued L-set (X, A, E) we call a family of L-subsets $\tau = \{U_i \in \tau : \forall i \in I\}$ of$X$, such that $U_i \leq A \ \forall i \in I$ and

1. $0 \in \tau; A \in \tau$;
2. if $\forall i, j \in I, \forall U_i, U_j \in \tau$, then $U_i \wedge U_j \in \tau$;
3. if $\forall i \in I, \forall U_i \in \tau$ then $\bigvee_{i \in i} U_i \in \tau$

4. if $\forall i \in I. \forall U_i \in \tau$: $U_i * E(x, x') \leq U_i(x')\ \forall x, x' \in X$.

The quadruple (X, A, E, τ) is called an L-valued L-topological space.

REMARK 3.2. The fourth axiom means that every L-subset of an L-set A from the family τ satisfies the extensionality type condition with respect to the L-valued equality E on (X, A).

DEFINITION 3.3. *By a continuous mapping from an L-valued L-topological space (X, A, E_A, τ_A) into an L-valued L-topological space (Y, B, E_B, τ_B) we mean a mapping $f : X \to Y$ such that*

1. $A(x) \leq B(f(x))\ \forall x \in X$;
2. $E_X(x, x') \leq E_Y(f(x), f(x'))\ \forall x, x' \in X$;
3. $\forall\ V \in \tau_B \to f^{-1}(V) \in \tau_A$.

Since composition of continuous mappings is obviously continuous and the identity mapping $f : (X, A, E_A, \tau_A) \to (X, A, E_A, \tau_A)$ is continuous, we obtain the following

THEOREM 3.4. *L-valued L-topological spaces as objects and continuous mappings between them as morphisms form a category. This category will be denoted L-TOP(L).*

3.2. The lattice of L-valued L-topologies. Final and initial structures in the categoty L-TOP(L) One can easily prove the following

THEOREM 3.5. *Let $\mathcal{T}$ be a family of L-valued L-topologies on an L-valued L-set (X, A, E). Then $\mathcal{T}$ is a complete lattice. Its top element is L-valued L-topology $\tau_1 = L_E^A$ consisting of all extensional L-valued L-subsets of A, and its bottom element is the L-valued L-topology $\tau_0 = \{A, 0\}$.*

Consider a family of L-valued L-topological spaces $\{(X_i, A_i, E_i, \tau_i)\ i \in I\}$, an L-valeud L-set (Y, B, E_B) and a family of mappings

$$\Phi := \{f_i : (X_i, A_i, E_i, \tau_i) \to (Y, B, E_B)\}.$$

Our aim is to find the final L-valued L-topology for this family. We start with the case when the family Φ consists of a single mapping

$$f : (X, A, E_A, \tau_A) \to (Y, B, E_B).$$

Let L_E^B be the family of all extentional L-subsets of (Y, B, E_B), and let

$$T_B = \{V \in L_E^B \mid f^{-1}(V) \in \tau_A\}.$$

Further, let an L-valued L-topology $\tau_B := f(\tau_A)$ be defined by T_B as a subbase. It is easy to notice that τ_B is the smallest L-valued L-topology on (Y, B, E_B), such that the mappig f is a morphism in L-TOP(L) and hence it is final for this mapping. Consider now to general case of

$$\Phi := \{f_i : (X_i, A_i, E_i, \tau_i) \to (Y, B, E_B)\}.$$

Let for each i $f(\tau_i)$ be defined as above, then the subbase of final L-valued L-topology τ_B on (Y, B, E_B) will be defined as a join of all L-topologies $T_B = \bigsqcup_{i\in I} f(\tau_i)$in the lattice of L-topoloies. Again, this is the smallest L-valued L-topology on (Y, B, E_B), such that all mappings f_i are morphism in L-TOP(L).

We consider now the dual problem, namely the initial L-valued L-topology for a family of mappings. Explisily, let

$$\Phi := \{f : (X, A, E_A) \to (Y_i, B_i, E_i, \tau_i)\}$$

be a family of mappings from an L-valued L-set (X, A, E_A) into L-valued L-topological space (Y_i, B_i, E_i, τ_i). Our goal is to find the initial L-valued topological space for this family. Again we start with the case when there is only one maping in the family Φ:

$$f : (X, A, E_A) \to (Y_i, B_i, E_i, \tau_i).$$

The initial L-valued topology $\tau_A := f^{-1}(\tau_B)$ on (X, A, E_A) is defined as $f^{-1}(\tau_B) = \{U = f^{-1}(V),\ \text{where } V \in \tau_B\}$

Consider now a family

$$\Phi := \{f : (X, A, E_A) \to (Y_i, B_i, E_i, \tau_i)\}.$$

Let $f^{-1}(\tau_{B_i})$ be defined as above, then τ_A will be construct from subbase

$$T_X = \bigcup_{i\in I} f^{-1}(\tau_{B_i}).$$

Thus we have established that both final and initial L-valued L-topologies for a family of mappings exist in L-TOP(L), and, moreover gave an explicite way how they can be constructed. From here it follows that both products and co-products and moreover, an explicite way of their construction is presented. Besides from here and, taking into account Theorem 3.5, we obtain the following fundamental result:

THEOREM 3.6. *The category L-TOP(L) is topological over the category L-SET(L) with respect to the forgetful functor*

$$\mathfrak{F} : L - TOP(L) \to L - SET(L).$$

Further, taking into account Theorem 2.7, it follows

COROLLARY 3.7. *The category L-TOP(L) is topological over the category L-SET with respect to the forgetful functor*

$$\mathfrak{G} : L - TOP(L) \to L - SET$$

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REFERENCES

[1] Chang, C. L., Fuzzy topological spaces. *J. Math. Anal. Appl.*, **24** (1968), 182–190.

[2] Goguen, J. A., L-fuzzy sets. *J. Math. Anal. Appl.*, **18** (1967), 145–167.

[3] Goguen, J. A., The fuzzy Tychonoff theorem. *J. Math. Anal. Appl.*, **43** (1973), 734–742.

[4] Höhle, U., M-valued sets and sheaves over integral commutative *cl*-monoids. In: Rodabaugh, S. E., Klement, E. P., Höhle, U., (eds.), *Applications of Category Theory to Fuzzy Subsets*. Kluwer Acad. Publ., Dodrecht, Boston, 1992, 33–72.

[5] Uljane, I., Šostaks, A., On a category of L-valued equalities on L-sets. *Journal of Electrical Engineering*, **55** (2004), 60–64.

[6] Zadeh, L. A., Fuzzy sets. *Inform. Control*, **8** (1965), 338–353.

Dažas piezīmes par L-vērtīgu L-topoloģisku telpu kategoriju

Kopsavilkums

Darbā ir definēts L-vērtīgas L-topoloģiskas telpas jēdziens, kas vispārina L-topoloģiskas telpas jēdzienu Čanga-Gogena nozīmē daudzvērtīgas kopas gadījumam. L-vērtīgas L-topoloģiskas telpas un dabiski definēti šo telpu nepārtraukti attēlojumi veido kategoriju L-TOP(L). Ir pierādīts, ka šī kategorija ir topoloģiska virs L-vērtīgu L-kopu kategorijas. L-SET(L), kuru mēs definējām iepriekšejā darbā

Multiple solutions of boundary value problems via the Schauder principle

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We consider solutions of the boundary value problem

$$x'' = f(t,x), \quad x(0) = x(1) = 0, \quad f \in C^1([0,1] \times R, R)$$

with respect to their types. The type of a solution ξ is defined via local oscillatory behavior of neighboring solutions and in the first approximation can be described in terms of the respective equation of variations $y'' = f_x(t,\xi(t))$. First we study quasi-linear equations with a linear part $x'' + k^2x = F(t,x)$ provided that F is bounded. We show that the boundary value problem for quasi-linear equation has a solution ξ such that the respective equation of variations oscillates like the linear one $x'' + k^2x = 0$. We rewrite the original equation in the form $x'' + k^2x = k^2x + f(t,x)$ for various k^2 and provide conditions for a priori boundedness of solutions of the modified problem. Multiplicity results are established and illustrative examples are analyzed.

Key words: Types of solutions, multiplicity results, nonlinear boundary value problems, Schauder principle, a priori bounds, quasi-linear equations

Mathematics Subject Classification (2000): 34B15

1. Introduction

In this paper we consider the boundary value problem

$$x'' = f(t,x), \quad t \in I = [0,1], \tag{1}$$

$$x(0) = 0, \quad x(1) = 0, \tag{2}$$

or such problem for an equation with a linear part

$$x'' + k^2x = f(t,x). \tag{3}$$

Function f is supposed to be continuous together with the partial derivative f_x in $(t,x) \in I \times R$. Then any extendable to the interval I solution $\xi(t)$ of equations (1) and (3) (and that of the related boundary value problem) can be described in terms of the variational equation

$$y'' = f_x(t,\xi(t))y \tag{4}$$

or, respectively,

$$y'' + k^2 y = f_x(t, \xi(t))y. \tag{5}$$

DEFINITION 1. We will say that a solution $\xi(t)$ of the BVP (1), (2) (respectively, (3), (2)) has index i, if a solution $y(t)$ to the Cauchy problem (4),

$$y(0) = 0, \quad y'(0) = 1 \tag{6}$$

(respectively, (5), (6)) either has exactly i zeros in the interval $(0, 1]$, or it has exactly i zeros in the interval $(0, 1)$ and $y(1) = 0$.

Consider the examples. Suppose that $\kappa(x)$ is continuously differentiable function such that $\kappa(x) = 1$ for $|x| \le 1$, $\kappa(x) = 0$ for $|x| \ge 2$ and $0 < \kappa(x) < 1$ for $1 < |x| < 2$. The nonlinear boundary value problem (with bounded right side)

$$x'' = -\left(\frac{\pi}{2}\right)^2 \kappa(x)x, \quad x(0) = 0, \quad x(1) = 0 \tag{7}$$

has only the trivial solution $\xi(t) = 0$ and the index of ξ is zero, since the Cauchy problem

$$y'' = -\left(\frac{\pi}{2}\right)^2 y, \quad y(0) = 0, \quad y'(0) = 1$$

has the solution $y(t) = \frac{2}{\pi}\sin\frac{\pi}{2}t$, which has no zeros in the interval $(0, 1]$.

The trivial solution of the quasi-linear problem

$$x'' + \left(\frac{3\pi}{2}\right)^2 x = \kappa(x)\left(\left(\frac{3\pi}{2}\right)^2 x - \left(\frac{5\pi}{2}\right)^2 x\right), \quad x(0) = 0, \quad x(1) = 0 \tag{8}$$

has index 2 since the Cauchy problem

$$y'' = -\left(\frac{5\pi}{2}\right)^2 y, \quad y(0) = 0, \quad y'(0) = 1$$

has the solution $y(t) = \frac{2}{5\pi}\sin\left(\frac{5\pi}{2}t\right)$, which vanishes twice in the interval $(0, 1)$ and is not zero at $t = 1$. It follows from the results below that this problem has also a solution of index one, induced by the linear part $x'' + \left(\frac{3\pi}{2}\right)^2 x$ of the equation.

REMARK 1.1. The above definition suits for the purposes of this paper. However, one might think of a solution $\xi(t)$ of the boundary value problem (1), (2) such that a respective $y(t)$ has its i-th zero at $t = 1$. Then either index i, or index $i + 1$ can be assigned to $\xi(t)$. Let us mention that we use the above definition in contexts which do not allow for ambiguities.

Our intent in this paper is to study quasi-linear boundary value problems with respect to solutions of different indices. We show that the linear part in (3) has influence on the index of possible solution. If different linear parts can be extracted from some equation of the form (1), then the boundary value problem (1), (2) admits multiple solutions.

We mention several papers on the relevant subject, namely, [5, 6, 7, 9]. It was shown in [5], for example, that under certain conditions the BVP (1), (2) possesses a solution $\xi(t)$ such that the equation (4) is disconjugate in I. Recall that the linear second order equation is called disconjugate in some open interval, if the only solution with more than one zero in this interval, is the trivial one. In the terminology of Definition 1 a solution ξ has index zero.

2. Quasilinear boundary value problems

Consider the problem

$$x'' + k^2 x = f(t, x), \tag{9}$$

$$x(0) = x(1) = 0, \tag{10}$$

where the following conditions are satisfied:

(A1) $f \in C(I \times R, R)$ and $f_x \in C(I \times R, R)$;

(A2) $\sup\{|f(t,x)| : 0 \le t \le 1,\ -\infty < x < +\infty\} = M < +\infty$;

(A3) the coefficient k belongs to one of the intervals

$$(0, \pi), \quad (\pi, 2\pi), \dots, (i\pi, (i+1)\pi), \dots \tag{11}$$

Since $k \neq \pi i$, the problem (9), (10) is non-resonant, that is, the linear homogeneous BVP

$$x'' + k^2 x = 0, \tag{12}$$

(10) has only the trivial solution and the respective Green function $G_k(t, s)$ exists.

LEMMA 2.1. *The Green function of the problem (12), (10) is given by*

$$G_k(t,s) = \begin{cases} \dfrac{\sin k(s-1) \cdot \sin kt}{k \sin k}, & 0 \le t \le s \le 1, \\[2ex] \dfrac{\sin k(t-1) \cdot \sin ks}{k \sin k}, & 0 \le s \le t \le 1 \end{cases} \tag{13}$$

and satisfies the estimate

$$|G_k(t,s)| \le \Gamma_k = \frac{1}{k \cdot |\sin k|} \tag{14}$$

LEMMA 2.2. *A set S of all solutions of the BVP (9), (10) is non-empty and compact in $C[0,1]$).*

Proof. Rewrite the problem (9), (10) in the integral form

$$x(t) = \int_0^1 G_k(t,s) f(s, x(s))\, ds,$$

where $G_k(t,s)$ is the Green function given by (13). Consider the mapping $T : C(I) \to C(I)$ given by

$$(Tx)(t) = \int_0^1 G_k(t,s) f(s, x(s))\, ds. \tag{15}$$

In order to show that T is completely continuous consider the ball $B_N = \{x \in C(I) : \|x\| \leq N\}$. One has, by virtue of the condition **(A2)** and (14), that

$$\|(Tx)(t)\|_{C(I)} \leq \frac{M}{k|\sin k|} = \Gamma_k \cdot M \tag{16}$$

for any $x \in C(I)$. Choose N so that

$$\Gamma_k \cdot M \leq N. \tag{17}$$

Then $T(B_N) \subset B_N$.

Let us show that the Arzela - Ascoli criterium is verified and $T(B_N)$ is therefore a compact set. Uniform boundedness of the set $T(B_N)$ follows from (16.) To prove equicontinuity take $\varepsilon > 0$. Let us find $\delta > 0$ such that

$$|(Tx)(t_2) - (Tx)(t_1)| < \varepsilon,$$

if $|t_2 - t_1| < \delta$, for any functions $(Tx)(t) \in T(B_\rho)$. One has that

$$\begin{aligned} |(Tx)(t_2) - (Tx)(t_1)| &\leq \left| \int_0^1 (G_k(t_2,s) - G_k(t_1,s)) f(s,x(s))\, ds \right| \\ &\leq M \cdot \left| \int_0^1 (G_k(t_2,s) - G_k(t_1,s))\, ds \right|. \end{aligned}$$

It follows from (13) that

$$\left| \int_0^1 (G_k(t_2,s) - G_k(t_1,s))\, ds \right| = \frac{4}{|k^2 \sin k} \cdot \sin\frac{k}{2} \cdot \sin k\frac{t_2 - t_1}{2} \cdot \sin\left(k\frac{t_1+t_2}{2} - \frac{k}{2}\right)|$$

$$\leq \frac{4}{k^2 \cdot |\sin k|} \cdot \frac{k \cdot |t_2 - t_1|}{2} = \frac{2}{k \cdot |\sin k|} \cdot |t_2 - t_1|,$$

whence

$$|(Tx)(t_2) - (Tx)(t_1)| \leq \frac{2M \cdot |t_2 - t_1|}{|k \cdot \sin k|}. \tag{18}$$

Thus the appropriate choice of δ for a given $\varepsilon > 0$ is

$$\delta = \frac{\varepsilon \cdot |k \cdot \sin k|}{2M}. \tag{19}$$

It follows from the arguments above that there exists a fixed point $x = x(t)$ of (15), which solves the boundary value problem (9), (10). Notice that a fixed point x satisfies the estimate

$$\|x_k(t)\|_{C(I)} \leq \Gamma_k M.$$

□

LEMMA 2.3. *A set S is compact also in $C^1(I)$.*

Proof. Consider the integral equation

$$x'(t) = \int_0^1 \frac{\partial G_k(t,s)}{\partial t} f(s, x(s))\, ds$$

and verify that the Arzela – Ascoli criterium is fulfilled for the set $S' = \{x'(t) : x \in S\}$. Consider the ball $B_N^* = \{x(t) \in C^1(I)\} : \|x(t)\|_{C^1(I)} \leq N\}$. We have from (13) that

$$\frac{\partial G_k(t,s)}{\partial t} = \begin{cases} \dfrac{\sin k(s-1) \cdot \cos kt}{\sin k}, & 0 \leq t \leq s \leq 1, \\ \dfrac{\cos k(t-1) \cdot \sin ks}{\sin k}, & 0 \leq s \leq t \leq 1, \end{cases} \tag{20}$$

and

$$\max_{[0,1]} |x'(t)| \leq$$

$$\leq \frac{M}{|\sin k|} \cdot \left(\max_{[0,1]} \left| \int_0^t \cos k(t-1) \cdot \sin ks\, ds \right| + \max_{[0,1]} \left| \int_t^1 \cos kt \cdot \sin k(s-1)\, ds \right| \right)$$

$$\leq \frac{M}{|\sin k|} \cdot \left(\left| \frac{-\cos kt + 1}{k} \right| + \left| \frac{-1 + \cos k(t-1)}{k} \right| \right) \leq \frac{4 \cdot M}{k \cdot |\sin k|}.$$

Then

$$\max_{[0,1]} |x'(t)| \leq \frac{4 \cdot M}{k \cdot |\sin k|}. \tag{21}$$

It follows from (16) and (21) that

$$\|(Tx)(t)\|_{C^1(I)} = \max_{[0,1]} |(Tx)(t)| + \max_{[0,1]} |x'(t)| \leq 5 \cdot \Gamma_k \cdot M.$$

Thus $T(B_N^*) \subset B_N^*$, if $N = 5 \cdot \Gamma_k \cdot M$. In order to show the compactness of the set $T(B_N^*)$ in $C^1(I)$ let us prove the equicontinuity of the set of derivatives of functions $y(t) = (Tx)(t)$.

In order to estimate the difference $|y'(t_2) - y'(t_1)|$, one has

$$|y'(t_2) - y'(t_1)| \leq$$

$$\leq \frac{M}{\sin k}\left|\int_0^{t_2} \cos k(t_2 - 1)\cdot \sin ks\, ds + \int_{t_2}^{1} \cos kt_2 \cdot \sin k(s-1)\, ds - \right.$$

$$\left. - \int_0^{t_1} \cos k(t_1 - 1)\cdot \sin ks\, ds - \int_{t_1}^{1} \cos kt_1 \cdot \sin k(s-1)\, ds\right| = \tag{22}$$

$$= \frac{4M}{|k\sin k|}\cdot\left|\sin\frac{k}{2}\right|\cdot\left|\sin\frac{k(t_2-t_1)}{2}\right|\cdot\left|\cos\frac{k(t_2+t_1-1)}{2}\right| \leq$$

$$\leq \frac{2M}{|\sin k|}\cdot|t_2 - t_1|.$$

Set

$$\delta_1 = \frac{\varepsilon\cdot|\sin k|}{2M}. \tag{23}$$

and choose the minimal of δ and δ_1 (defined by (19) and (23) respectively), that is

$$\delta^* = \min\{\frac{\varepsilon\cdot|k\sin k|}{2M}; \frac{\varepsilon\cdot|\sin k|}{2M}\}.$$

If $|t_2 - t_1| < \delta^*$, then both inequalities

$$|(Tx)(t_2) - (Tx)(t_1)| < \varepsilon, \qquad |y'(t_2) - y'(t_1)| < \varepsilon$$

hold for any function $y \in T(B_N^*)$ and this proves that $T(B_N^*) \subset C^1(I)$ is equicontinuous. □

LEMMA 2.4. *All solutions of (9) are extendable to the interval $[0,1]$ and are uniquely defined by the initial data.*

Proof. The first assertion follows from **(A2)**. Notice that since the continuous partial derivative f_x exists, equation (9) satisfies the Lipschitz condition in any compact subdomain of $[0,1] \times R$. Then solutions of (9) are uniquely defined by the initial data and continuously depend on the initial data. □

LEMMA 2.5. *There are elements $x^*(t)$ and $x_*(t)$ of S, which possess the properties:*

$$x^{*\prime}(0) = \max\{x'(0) : x \in S\}$$
$$x_*{}'(0) = \min\{x'(0) : x \in S\}.$$

Proof. If S consists of one element x then $x^* = x_* = x$. If S contains at most finite number of elements, then the assertion is obvious. Suppose that there are infinitely many elements in S. Since S is compact in $C^1(I)$, the set

$S_1 = \{x'(0) : x \in S\}$ is bounded. Then there exists $\sup S_1$. Consider a solution $x(t)$ of (9), which is defined by the initial data

$$x(0) = 0, \quad x'(0) = \sup S_1.$$

By definition of sup there exists a sequence $\{x_n\}$ of elements of S such that $x_n'(0) \to x'(0)$. One can prove, using the compactness of S, that $x \in S$. Thus $x^* = x$.

The proof for x_* is analogous. □

Denote by $x(t;\alpha)$ a solution of the Cauchy problem (1) (or (3)),

$$x(0) = 0, \quad x'(0) = \alpha. \tag{24}$$

Consider equation (9) with the initial conditions (24).

LEMMA 2.6. *Suppose that k in (9) satisfies*

$$i\pi < k < (i+1)\pi$$

for some $i = 0, 1, \ldots$. Let $\xi(t)$ be any element of S.

Then for $\alpha \to \pm\infty$ the difference $u(t;\alpha) = x(t;\alpha) - \xi(t)$ has exactly i zeros in the interval $(0,1)$ *and* $u(1;\alpha) \neq 0$.

Proof. Notice that both $x(t;\alpha)$ and $\xi(t)$ satisfy equation (9). Then $u(t;\alpha)$ is a solution of the initial value problem

$$u'' + k^2 u = f(t, x(t)) - f(t, \xi(t)),$$

$$u(0) = 0, \quad u'(0) = \alpha - \xi'(0).$$

Introduce a new variable v by $v := \dfrac{u}{\alpha - \xi'(0)}$. Then $v(t;\alpha)$ satisfies

$$v'' + k^2 v = \frac{f(t, x(t)) - f(t, \xi(t))}{\alpha - \xi'(0)}, \tag{25}$$

$$v(0) = 0, \quad v'(0) = 1.$$

In view of **(A2)** the right side in (25) tends to zero if $\alpha \to \pm\infty$. Then by continuous dependence on the right side $v(t;\alpha)$ tends to a solution of the homogeneous initial value problem $z'' - k^2 z = 0$, $z(0) = 0$, $z'(0) = 1$. Since $v(t;\alpha)$ and $u(t;\alpha)$ have the same zeros, the assertion of lemma follows. □

LEMMA 2.7. *Suppose the conditions* **(A1)** *and* **(A2)** *are satisfied. Let $\xi(t)$ be any element of S.*

Zeros $t_i(\alpha)$ of the function $u(t;\alpha) = x(t;\alpha) - \xi(t)$ are continuous functions of α in intervals of existence. If for some $\alpha_0 \neq \xi'(0)$ $u(t;\alpha_0) = 0$, then the respective $x(t;\alpha_0) \in S$.

Proof. The first assertion follows from continuous dependence of solutions of (9) on initial data and from the fact that $u(t;\alpha)$ cannot have double zeros. Indeed, if this were the case, then $x(t;\alpha) = \xi(t)$ and $x'(t;\alpha) = \xi'(t)$ at some point $t \in I$. Then $x \equiv \xi$, by the uniqueness of solutions of the Cauchy problem for (9).

Suppose that $u(t;\alpha_0) = 0$ for some $\alpha_0 \neq \xi'(0)$. Then $x(1;\alpha_0) = \xi(1) = 0$ and hence $x(t;\alpha_0)$ is a solution of the BVP (9), (10). □

3. Main results

THEOREM 3.1. *Suppose that the conditions* **(A1)**, **(A2)** *and* **(A3)** *are fulfilled. Suppose that k in (9) satisfies*

$$i\pi < k < (i+1)\pi$$

for some $i = 0, 1, \ldots$.

Then the both solutions $x^*(t)$ and $x_*(t)$, defined in Lemma 2.5, have index i.

Proof. Let $\xi := x^* \in S$. Consider the respective variational equation (5) together with the initial conditions (6). Let $y(t)$ stand for a solution of (5), (6). If the index of ξ is not equal to i, then

either: (1) $y(t)$ has less than i zeros in $(0, 1]$;

or (2) $y(t)$ has more than $i+1$ zeros in $(0, 1]$.

Consider the difference $u(t; \alpha)$, where $\alpha > \xi'(0)$. In case (1) $u(t; \alpha)$ has at most $i-1$ zeros in $(0, 1)$, if α is close to $\xi'(0)$. On the other hand, if $\alpha \to +\infty$, then $u(t; \alpha)$ has exactly i zeros in $(0, 1)$. An extra zero appears, if α varies from $\xi'(0)$ to $+\infty$, for some $\alpha_0 > \xi'(0)$. Then, by Lemma 2.7, $x(t; \alpha_0)$ is a solution of the BVP (9), (10). Since $x'(0; \alpha_0) = \alpha_0 > \xi'(0)$, this contradicts the definition of $\xi(t)$.

In case (2) $u(t; \alpha)$ has at least $i+1$ zeros in $(0, 1)$. If $\alpha \to +\infty$, then $u(t; \alpha)$ has exactly i zeros in $(0, 1)$. An extra zero appears, if α varies from $+\infty$ to $\xi'(0)$, for some $\alpha_0 > \xi'(0)$. Then the contradiction is obtained as above.

Proof for x_* is similar. □

COROLLARY 3.1. *If a solution x of the problem (9), (10), where $i\pi < k < (i+1)\pi$, is unique, then its index is i. If there are multiple solutions of the problem (9), (10), then at least two of them have indices i.*

THEOREM 3.2. *Suppose that f in (1) satisfies the condition* **(A1)** *and there exist k_j such that*

$$i_j\pi < k_j < (i_j + 1)\pi, \quad j = 1, \ldots, m,$$

the inequalities

$$\Gamma_{k_j} \cdot M_{k_j} < N_{k_j}$$

hold, where

$$\Gamma_{k_j} = \frac{1}{k_j|\sin k_j|}, \quad M_{k_j} = \max\{|k_j^2 x + f(t, x)| : 0 \le t \le 1, |x| \le N_{k_j}\}.$$

Then the problem (1), (2) has at least m solutions $\xi_1, \ldots, \xi_m$ such that the index of ξ_j is i_j.

Proof. To be definite, consider the case of $j = 1, 2$: $0 < k_1 < \pi$, $\pi < k_2 < 2\pi$. Consider the quasi-linear equations

$$x'' + k_1^2 x = \varphi_1(x)[k_1^2 x + f(t, x)], \tag{26}$$

$$x'' + k_2^2 x = \varphi_2(x)[k_2^2 x + f(t,x)], \tag{27}$$

where $\varphi_j(x)$ are $C^\infty(I)$ functions such that $\varphi_j(x) = 1$ for $|x| < N_{k_j}$, $\varphi_j(x) = 0$ for $|x| \geq N_{k_j} + \varepsilon_j$ and $0 < \varphi_j(x) < 1$ for remaining x (for construction of φ see Example 1 in the next section). The choice of ε_j will be discussed later. By Theorem 3.1 the problems (26), (10) and (27), (10) have solutions ξ_1 and ξ_2 respectively. The index of ξ_1 is zero and the index of ξ_2 is one.

Let us show that ξ_1 solves the original problem (1), (2). Denote the right side of (26) by F_1 and set

$$M_1 = \max\{|F_1(t,x)| : \ 0 \leq t \leq 1, \ -\infty < x < +\infty\}.$$

Evidently, $M_1 \to M_{k_1}$ as $\varepsilon_1 \to 0$ and the inequality $\Gamma_{k_1} \cdot M_1 < N_{k_1}$ holds for some $\varepsilon_1 > 0$. A solution ξ_1 satisfies the integral equation

$$\xi(t) = \int_0^1 G_{k_1}(t,s) F_1(s,\xi(s))\, ds$$

and therefore the estimate $\|\xi\|_{C(I)} \leq \Gamma_{k_1} \cdot M_1 < N_{k_1}$ holds. Since equation (26) reduces to (1) on the set $\Omega_1 := \{(t,x) : \ 0 \leq t \leq 1, \ |x| \leq N_{k_1}\}$, ξ_1 solves also the problem (1), (2). Its index with respect to equation (1) is zero, since the equations of variations with respect to ξ_1 for the nonlinear equations (26) and (1) on the set Ω_1 are identical.

A solution ξ_2 can be treated similarly. So both solutions ξ_1 and ξ_2 solve the problem (1), (2) and their indices are respectively zero and one.

We complete the proof by showing that ξ_1 and ξ_2 are different solutions of (1), (2). Consider the respective equations of variations

$$y'' + k_1^2 y = \frac{\partial F_1}{\partial x}(t,\xi_1(t))y,$$

$$y'' + k_1^2 y = \frac{\partial F_2}{\partial x}(t,\xi_2(t))y,$$

which reduce to

$$y'' = \frac{\partial f}{\partial x}(t,\xi_1(t))y, \tag{28}$$

$$y'' = \frac{\partial f}{\partial x}(t,\xi_2(t))y \tag{29}$$

on the sets Ω_1 and Ω_2. Denote by t_1 and τ_1 respectively the first zeros of solutions y_1 and y_2 in the interval $(0,1]$, where y_1 and y_2 are solutions of the Cauchy problems (28), (6) and (29), (6) respectively. It may happen that $t_1 = 1$ and $\tau_1 = 1$ also. Suppose that ξ_1 and ξ_2 are identical. Then $\xi_1'(0) = \xi_2'(0) =: \alpha_0$. Consider solutions $x(t;\alpha)$ of the initial value problem (1), (24). Suppose that $\alpha > \alpha_0$. We will show that $x(1;\alpha) > 0$ for α close enough to α_0. Indeed, if the difference $\alpha - \alpha_0 > 0$ is small enough, functions $x(t;\alpha)$ solve the equation (26) also. By Lemma 2.6, solutions $x(t;\alpha)$ of (26), (24) for large values of α are such that the difference $x(t;\alpha) - \xi_1(t)$ does not vanish in the interval $(0,1]$. If $x(1;\alpha) \leq 0$ for some $\alpha > \alpha_0$, then by the continuous dependence of solutions on initial data there exists $\alpha_1 > \alpha_0$ such that $x(1;\alpha_1) = 0$ and hence $x(t;\alpha_1)$ is a solution of the boundary value problem (26), (2), which contradicts the choice of $\xi_1(t)$ as a solution of the same problem with a maximal value of $x'(0)$.

On the other hand, one can show similarly, by analyzing the equation (26) in some vicinity of $\xi_2(t)$, that solutions $x(t;\alpha)$ of the initial value problem (1), (24) for $\alpha > \alpha_0$ and close to α_0 satisfy the relation $x(1;\alpha) > 0$. The contradiction obtained proves that ξ_1 and ξ_2 are different solutions of the problem (1), (2).

□

4. Examples and applications

EXAMPLE 1. Consider the problem

$$x'' = -\varphi(x)\pi^2 x, \quad x(0) = x(1) = 0, \tag{30}$$

where $\varphi(x)$ is a C^∞ function such that φ is equal to 1 for $|x| \le 1$, φ is zero for $|x| \ge 2$ and $0 < \varphi(x) < 1$ for $1 < |x| < 2$.

For instance,

$$\varphi(x) = \begin{cases} 1, & |x| \le 1, \\ 0, & |x| \ge 2, \\ \exp\frac{(x-1)^2}{(x-1)^2-1} & 1 < x < 2, \\ \exp\frac{(x+1)^2}{(x+1)^2-1} & -2 < x < -1. \end{cases}$$

The problem (30) has solutions of the form $A\sin \pi t$, where $|A| \le 1$. Consider solutions $x(t;\alpha)$ of the initial value problem

$$x'' = -\varphi(x)\pi^2 x, \quad x(0) = 0, \ x'(0) = \alpha.$$

Let $t_1(\alpha)$ be the first zero of $x(t;\alpha)$ in the interval $(0, +\infty)$. Obviously $t_1(\alpha) = 1$ for $|\alpha| \le 1$. For $|x| > 1$ the function $f(x) := \varphi(x)\pi^2 x$ grows slower than the linear one $l(x) := \pi^2 x$. Thus $t_1(\alpha) > 1$ for $|\alpha| > 1$. Since $f(x)$ is equal to zero for $|x| \ge 2$, solutions $x(t;\alpha)$ for $|\alpha|$ large enough do not vanish in the interval $(0, +\infty)$.

Evidently $x^*(t) = \sin \pi t$ and $x_*(t) = -\sin \pi t$. Since

$$f'(x) = \pi^2\varphi(x) + \pi^2 x\,\varphi'(x) = \pi^2 \quad \text{for} \quad |x| \le 1,$$

the equation of variations for $x^*(t)$ is $y'' + f_x(x^*(t))y = 0$ or simply

$$y'' + \pi^2 y = 0.$$

Therefore index 0 may be assigned to both solutions $x^*(t)$ and $x_*(t)$ of the problem (30).

EXAMPLE 2. Consider the boundary value problem

$$x'' = -\alpha^2 \cdot |x|^p \operatorname{sign} x, \quad x(0) = x(1) = 0, \tag{31}$$

where $\alpha > 0, \quad p > 0, \quad p \neq 1$.

The equivalent problem is

$$x'' + k^2 x = k^2 x - \alpha^2 \cdot |x|^p \operatorname{sign} x, \quad x(0) = x(1) = 0, \tag{32}$$

where k satisfies

$$i\pi < k < (i+1)\pi$$

for some i $(i = 0, 1, \ldots)$. Define

$$F_k(x) := k^2 x - \alpha^2 \cdot |x|^p \operatorname{sign} x.$$

This function is odd. Let us consider it for nonnegative values of x. Since

$$F_k'(x) = k^2 - \alpha^2 \cdot p \cdot x^{p-1},$$

a positive point of extremum (maximum point for $p > 1$ and minimum point for $p < 1$) x_0 is

$$x_0 = \left(\frac{k^2}{\alpha^2 p}\right)^{\frac{1}{p-1}}.$$

Set

$$M_k = |F_k(x_0)| = \left(\frac{k^2}{p}\right)^{\frac{p}{p-1}} \cdot |p-1| \cdot \alpha^{\frac{2}{1-p}}$$

and choose N_k so that

$$\forall (t, x): \quad t \in [0, 1], \quad |x| \leq N_k \quad \Rightarrow |F_k(x)| \leq M_k.$$

A constant N_k can be computed solving the equation

$$F_k(x) = -F_k(x_0),$$

or, equivalently,

$$k^2 x - \alpha^2 x^p = \left(\frac{k^2}{p}\right)^{\frac{p}{p-1}} \cdot (1-p) \cdot \alpha^{\frac{2}{1-p}}$$

with respect to x. Computation gives

$$N_k = \left(\frac{k^2}{\alpha^2}\right)^{\frac{1}{p-1}} \beta,$$

where a constant β is to be found from the equation

$$\beta^p = \beta + (p-1) \cdot p^{\frac{p}{1-p}}. \tag{33}$$

Equation (33) has a root $\beta > 1$ for any positive p.

In order to apply Theorem 3.2 one needs to verify the inequality

$$\Gamma_k \cdot M_k < N_k.$$

It can be written for the case under consideration as

$$\frac{1}{k \cdot |\sin k|} \cdot \left(\frac{k^2}{p}\right)^{\frac{p}{p-1}} \cdot |p-1| \cdot \alpha^{\frac{2}{1-p}} < \left(\frac{k^2}{\alpha^2}\right)^{\frac{1}{p-1}} \beta.$$

To simplify calculations, take $k = \dfrac{\pi}{2}(2n-1)$, where $n = 1, 2, \ldots$, then $|\sin k = 1.|$ The inequality above after simplification can be written as

$$k < \beta \cdot \frac{p}{|p-1|} \cdot p^{\frac{1}{p-1}}. \tag{34}$$

Since $\beta > 1$ and $\lim\limits_{p \to 1} \dfrac{p}{|p-1|} \cdot p^{\frac{1}{p-1}} = +\infty$, the right side in (34) tends to ∞ as $p \to 1$. The inequality (34) is satisfied therefore for arbitrarily large values of $k = \dfrac{\pi}{2}(2n-1)$.

Computations show that for $p \in [0.5, 1) \cup (1, 2]$ there exist at least two values of k

$$k_0 = \frac{\pi}{2}, \quad k_1 = \frac{3\pi}{2},$$

which satisfy (34).

For instance, consider the problem (31) for $p = 2$. The function $F_k(x) := k^2 x - \alpha^2 x^2$ has its positive maximum at the point $x_0 = \frac{k^2}{2\alpha^2}$, vanishes at $x_1 = \frac{k^2}{\alpha^2}$ and a solution to the equation $F_k(x_0) = -F_k(x)$ is $N_k = \frac{1+\sqrt{2}}{2} \cdot \frac{k^2}{\alpha^2}$. Define $M_k = F_k(x_0) = \frac{1}{4} \cdot \frac{k^4}{\alpha^2}$. Consider the quasi-linear equation

$$x'' - k^2 x = f_k(x),$$

where $f_k = F_k$ for $|x| \leq N_k$ and f_k is smooth. Evidently, f_k can be constructed so, that the estimate

$$\sup_R |f_k(x)| \leq M_k + \varepsilon$$

is valid for an arbitrarily small positive ε. The inequality $\Gamma_k \cdot (M_k + \varepsilon) < N_k$ writes as

$$\frac{1}{k} \cdot \left(\frac{1}{4}\frac{k^4}{\alpha^2} + \varepsilon\right) < \frac{1+\sqrt{2}}{2} \cdot \frac{k^2}{\alpha^2}.$$

It is satisfied for $k_0 = \frac{\pi}{2}$ and $k_1 = \frac{3\pi}{2}$, if ε is small, since

$$\frac{1}{4}k_0^3 \approx 0.969 < \frac{1+\sqrt{2}}{2}k_0^2 \approx 2.978$$

and

$$\frac{1}{4}k_1^3 \approx 26.162 < \frac{1+\sqrt{2}}{2}k_1^2 \approx 26.806.$$

Therefore at least two solutions ξ_0 and ξ_1 of the boundary value problem are expected, the first one of index zero, and the second one of index one.

The trivial solution $\xi_0 \equiv 0$ has index zero. A solution $\xi_1(t)$, which has no zeros in the interval $(0,1)$, attains its maximum at the point $t = \frac{1}{2}$ and $\xi_1(\frac{1}{2}) = 6 \cdot \frac{J_2^2}{\alpha^2}$, where $J_2 := \int_0^1 \frac{ds}{\sqrt{1-s^3}} \approx 1.402$. One has that

$$\max_{[0,1]} \xi_1(t) = \frac{6J_2^2}{\alpha^2} \approx \frac{11.794}{\alpha^2} < \Gamma_{\frac{3\pi}{2}} \cdot M_{\frac{3\pi}{2}} = \frac{1}{4\alpha^2}\left(\frac{3\pi}{2}\right)^3 \approx \frac{26.162}{\alpha^2}.$$

PROPOSITION 4.1. *For any $p \in [0.5, 2]$, $p \neq 1$ there exist at least two values of k of the form $k = \frac{\pi}{2}(2n-1)$, $n = 1, 2, \ldots$, which satisfy the inequality (34).*

Therefore there exist at least two solutions (of different types) of the problem (31), which satisfy the estimates $|x_k(t)| \leq N_k$.

PROPOSITION 4.2. *For any positive integer m there exists $\varepsilon > 0$ such that if $p \neq 1$ satisfies the inequalities*

$$1 - \varepsilon < p < 1 - \varepsilon,$$

then $k = \frac{\pi}{2}(2n-1)$, $n = 1, 2, \ldots, m$, satisfy the inequality (34). Therefore there exist at least m solutions (of different types) of the problem (31).

Represent p in the form $p = 1 + \dfrac{1}{s}$, if $p > 1$, or in the form $p = 1 - \dfrac{1}{s+1}$, if $0 < p < 1$ (in any case $s > 0$).

Calculation shows that the number of appropriate values of k for the problem (32) changes by unity if s changes by unity.

Indeed, if

1) $s \in [1; 2)$, then $p \in (\frac{3}{2}; 2]$ or $p \in [\frac{1}{2}; \frac{2}{3})$ and the appropriate values of k (those which satisfy the inequality (34)) are

$$k_0 = \frac{\pi}{2}, \quad k_1 = \frac{3\pi}{2};$$

2) $s \in [2; 3)$, then $p \in (\frac{4}{3}; \frac{3}{2}]$ or $p \in [\frac{2}{3}; \frac{3}{4})$ and the appropriate values of k are

$$k_0 = \frac{\pi}{2}, \quad k_1 = \frac{3\pi}{2}, \quad k_2 = \frac{5\pi}{2};$$

3) $s \in [3; 4)$, then $p \in (\frac{5}{4}; \frac{4}{3}]$ or $p \in [\frac{3}{4}; \frac{4}{5})$ and the appropriate values of k are

$$k_0 = \frac{\pi}{2}, \quad k_1 = \frac{3\pi}{2}, \quad k_2 = \frac{5\pi}{2}, \quad k_3 = \frac{5\pi}{2}$$

and so on.

PROPOSITION 4.3. *If $s \in [n; n-1)$, $n = 1, 2, \ldots$, then there exist exactly $n+1$ appropriate values of k of the form $k = \frac{\pi}{2}(2i+1)$, $i = 0, 1, \ldots, n$.*

Therefore the problem (31) has $n+1$ solutions $x_{k_i}(t)$ with indices $i = 0, 1, \ldots, n$, and these solutions satisfy the estimates $|x_{k_i}| \leq N_{k_i}$.

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REFERENCES

[1] Bernfeld, S., Lakshmikantham, V., *An Introduction to Nonlinear Boundary Value Problems*. Academic Press, New York, 1974, 386 pp.

[2] Hartman, P., *Ordinary Differential Equations*. Wiley, New York, 1964, 612 pp. Russian translation: Moscow, 1970.

[3] Coddington, E., Levinson, N., *Theory of Ordinary Differential Equations*. McGraw-Hill, New York, 1955, 429 pp.

[4] Krasnoselskii, M. A. et al., *Planar Vector Fields*. Acad. Press, New York, 1966.

[5] L.H. Erbe, L. H., Nonlinear boundary value problems for second order differential equations. *J. Diff. Equations*, **7** (1970), 459–472.

[6] Knobloch, H. W., Comparison theorems for nonlinear second order differential equations, *J. Diff. Equations*, **1** (1965), 1–26.

[7] Knobloch, H. W., Second order differential inequalities and a nonlinear boundary value problem, *J. Diff. Equations*, **5** (1969), 55–71.

[8] Jackson, L. K., Schrader, K. W., Comparison theorems for nonlinear differential equations. *J. Diff. Equations*, **5** (1969), 248–255.

[9] Reid, W., Comparison theorems for nonlinear vector differential equations. *J. Diff. Equations*, **5** (1969), 324–337.

Šaudera princips un vairāku nelineāro robežproblēmu atrisinājumu eksistence

Kopsavilkums

Tiek pētīta robežproblēma $x'' = f(t, x)$, $x(0) = x(1) = 0$. Atrisinājuma tips tiek definēts lineāra variaciju diferenciālvienādojuma terminos. Ir apskatīti ekvivalentie diferenciālvienādojumi ar izdalītu lineāru daļu. Pierādīts, ja modificētai robežproblēmai ir spēkā atrisinājumu apriorie novērtejumi, tad sākotnējai problēmai eksistē noteikta (atkarība no k izvēles) tipa atrisinajums.

Complete infinitely distributive lattices as topologies modulo an ideal

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Every lattice we consider is supposed to be a lattice of subsets of a set with the natural partial order (notice that every lattice is isomorphic to at least one lattice of that kind). It is well-known that every topology forms a complete infinitely distributive (cid) lattice, while the converse may not hold. We prove that every cid lattice is an i-topology (or a topology modulo an ideal), i.e., that every cid lattice is closed under arbitrary i-unions and finite i-intersections. Moreover, we provide an example of a complete completely distributive lattice (which is, then, a cid lattice) that is not closed under arbitrary i-intersections.

Key words: complete lattice, generalization, ideal, infinitely distributive lattice, pseudocomplemented lattice, topological space

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Preliminaries

Throughout this paper we use the common notations $\vee$ and $\wedge$ (or $\bigvee$ and $\bigwedge$ in case of a family of elements) for, respectively, the supremum and infimum in a complete lattice.

A collection $\mathcal{I}$ of subsets of a set X is called an ideal if it is closed under finite unions, contains all subsets of all its elements and does not contain X itself. An ideal $\mathcal{I}$ naturally generates a preorder on the family of all subsets of X. Indeed, defined as follows, the relation $\preceq$ is reflexive and transitive: $U \preceq V$ iff $U \setminus V \in \mathcal{I}$, for every $U, V \subseteq X$.

We refer a reader to [1] for the further standard definitions of Lattice Theory.

Z-emptiness. Equivalent forms of the infinite distributive law

Let L be a complete lattice, and $\leq$ be its partial order. With the respect to some fixed element z in L, we say that an element a of L is *z-empty* if $a \leq z$, and *z-nonempty* in any other case. If the converse is not stated, we always assume that some z in L is fixed. The following easy lemma is an immediate corollary from the completeness of the lattice L.

LEMMA 1. *The supremum of a family of z-empty elements of L is z-empty.*

The following proposition provides the equivalent form of the infinite distributive law for complete lattices to which we are going to refer in the proof of Lemma 5.

PROPOSITION 2. *Let L be a complete lattice, and $\leq$ be its partial order. The following are equivalent:*

(i) *The lattice L is infinitely distributive, that is, for every b in L and every subfamily $A \subseteq L$,*

$$b \wedge \bigvee A = \bigvee \{ b \wedge a \mid a \in A \} ;$$

(ii) *For every b and z from L and every subfamily $A \subseteq L$, if b is z-nonempty and $b \leq \bigvee A$ then there exists $a^* \in A$ such that $b \wedge a^*$ is z-nonempty.*

Proof. To show that (i) implies (ii), apply Lemma 1. Let us prove the converse implication. Assume (ii); fix b in L and a subfamily A of L. It holds that $b \wedge \bigvee A \leq \bigvee A$. Clearly, $(b \wedge \bigvee A) \wedge a$ is equal to $b \wedge a$ for every $a \in A$. Since for every $a \in A$ it holds that $b \wedge a \leq \bigvee \{ b \wedge a \mid a \in A \}$, by (ii) we conclude that $b \wedge \bigvee A \leq \bigvee \{ b \wedge a \mid a \in A \}$. The converse inequality holds in every complete lattice. The lemma is proved. □

We are going to call a complete lattice *infinitely trivially distributive* if it satisfies the condition (ii) of the previous proposition for $z = \varnothing$. Let us prove some equivalent forms of trivial distributivity.

PROPOSITION 3. *Let L be a complete lattice. The following are equivalent:*

(i) *The lattice L is infinitely trivially distributive;*

(ii) *For every b in L and every subfamily $A \subseteq L$, if $b \wedge a = 0$ holds for all a in A then $b \wedge \bigvee A = 0$;*

(iii) *The lattice L is pseudocomplemented.*

Proof. The proof of that (i) and (ii) are equivalent is the trivial case of the proof of the previous proposition. Let us prove the other implications. Let L be pseudocomplemented. Consider an element b of L and a subset $A \subseteq L$ with $b \wedge a = 0$ for all a in A. By the definition, $a \leq b^*$ holds for every a in A. Therefore, $\bigvee A \leq b^*$, and hence $b \wedge \bigvee A = 0$. Conversely, let L be infinitely trivially distributive. Consider an element a in L. Then $a^* = \bigvee \{ b \in L \mid b \wedge a = 0 \}$ is the pseudocomplement of a. Thus, we proved the equivalence of (ii) and (iii).

Main theorem

In what follows, we assume that a set X and a complete infinitely distributive lattice $(\mathfrak{L}, \subseteq)$ of its subsets, that contains the empty set and the whole X, are fixed. Unless specified, every subset or set will be supposed to be a subset of X. We use the notation $\mathfrak{L}$ for the lattice instead of the notation L, which we used above, and the upper case letters for the elements of the lattice instead of the lower case letters on purpose to emphasize that all the objects we consider now are sets.

A set A is going to be said to be *sup-* or *inf-generated* by the family $\mathfrak{U}$ of elements of $\mathfrak{L}$ if, respectively, $A = (\bigvee \mathfrak{U}) \setminus (\bigcup \mathfrak{U})$, or $A = (\bigcap \mathfrak{U}) \setminus (\bigwedge \mathfrak{U})$ for finite $\mathfrak{U}$. The family of all subsets of the set X that might be represented as such differences will be denoted $\mathcal{E}$. All the following results might not hold if we include in $\mathcal{E}$ the sets generated by the infinite infimums (see Example 9). We are going to make use of the ideal $\mathcal{I}$ generated by the family $\mathcal{E}$ and exploit the preorder $\preceq$ generated by this ideal. We will prove that this preorder happens to be a partial order on $\mathfrak{L}$.

LEMMA 4. *Let $\mathfrak{U} = \{ U_1, U_2, \ldots, U_k \}$ be a family of elements from $\mathfrak{L}$ and V be a Z-nonempty element from $\mathfrak{L}$ such that it is a subset of U_1. Then there exists a Z-nonempty W in $\mathfrak{L}$ that satisfies $W \subseteq V$ and one of the following conditions:*

(i) $W \subseteq \bigwedge \mathfrak{U}$;

(ii) *there is U_{j_*} in $\mathfrak{U}$ such that $W \wedge U_{j_*}$ is Z-empty.*

Proof. Let us define a sequence $\mathcal{V} = \{ V_1, V_2, \ldots, V_k \}$ as follows:

$$V_1 = V \text{ and } V_i = V_{i-1} \wedge U_i \text{ for } i \in \{ 1, 2, \ldots, k \}.$$

By the terms of the considered lattice, $\mathcal{V}$ is a decreasing chain and at least one its element is Z-nonempty, since $V_1 = V$. Put $W = V_{i_*}$, where V_{i_*} is the last Z-nonempty element of the chain. Clearly, it holds that $W \subseteq V$. There are two possibilities for i_*: either $i_* < k$ and then $W \wedge U_{i_*+1}$ is Z-empty, or $i_* = k$ and then $W \subseteq \bigwedge \mathfrak{U}$. The lemma is proved. ☐

LEMMA 5. *Let $Y \in \mathfrak{L}$ be Z-nonempty and satisfy $Y \setminus Z \subseteq \bigcup \mathcal{A}$ for some subfamily $\mathcal{A} = \{ A_1, A_2, \ldots, A_n \}$ of $\mathcal{E}$. Then there exist Z-nonempty W in $\mathfrak{L}$ and A_{i_*} lying in $\mathcal{A}$ that satisfy $W \subseteq Y$ and $W \cap A_{i_*} = \varnothing$.*

Proof. For every A_i lying in $\mathcal{A}$, we are going to denote by $\mathfrak{U}_i$ the corresponding family of elements from $\mathfrak{L}$ by that A_i is sup- or inf-generated. First, we define the common family $\mathfrak{U}_1$ as follows:

$$\mathfrak{U}_1 = \{ Z \} \cup \{ U \mid U \in \mathfrak{U}_i \text{ for some } i \in \{ 1, 2, \ldots, n \} \}.$$

It holds that $Y \subseteq \bigvee \mathfrak{U}_1$. By Proposition 2, we conclude that $V_1 = Y \wedge U$ is Z-nonempty for some $U \in \mathfrak{U}_1$. Clearly, it cannot be $V_1 \subseteq Z$. Thus, without loss of generality, we assume that $V_1 \subseteq U \in \mathcal{U}_1$. There are two possibilities for A_1: either A_1 is sup-generated and then $V_1 \cap A_1 = \varnothing$, we put $W = V_1$ and the proof is complete, or A_1 is inf-generated and then, applying Lemma 4 for $\mathcal{U}_1$ and V_1, we obtain the set $W_1 \subseteq V_1$. If $W_1 \subseteq \bigwedge \mathcal{U}_1$ then $W_1 \cap A_1 = \varnothing$: we put $W = W_1$ and the proof is complete. In the other case, we possess the set U_1 from $\mathcal{U}_1$ such that $W_1 \wedge U_1$ is Z-empty. We repeat the whole process from the beginning. Define the common family $\mathfrak{U}_2$ as follows:

$$\mathfrak{U}_2 = \{ Z \} \cup \{ U_1 \} \cup \{ U \mid U \in \mathcal{U}_i \text{ for some } i \in \{ 2, 3, \dots, n \} \},$$

It holds that $W_1 \subseteq \bigvee \mathfrak{U}_2$. Hence, $V_2 = W_1 \wedge U$ is Z-nonempty for some $U \in \mathfrak{U}_2$. We observe that neither $V_2 \subseteq Z$, since V_2 is Z-nonempty, nor $V_2 \subseteq U_1$, since $V_2 \subseteq W_1$ and $W_1 \wedge U_1$ is Z-empty, hold. Let us assume that $V_2 \subseteq U \in \mathcal{U}_2$. Again, there are two possibilities for A_2: either A_2 is sup-generated and then $W = V_2$ satisfies the conditions of the theorem, or A_2 is inf-generated and we apply Lemma 4 for $\mathcal{U}_2$ and V_2 and obtain the set $W_2 \subseteq V_2$. If it holds that $W_2 \subseteq \bigwedge \mathcal{U}_2$ then $W = W_2$ is the one we need. In the other case, we possess the set U_2 from $\mathcal{U}_2$ such that $W_2 \wedge U_2$ is Z-empty. We continue in the same way as above, defining the common family $\mathfrak{U}_3$.

Through the process, we obtain the decreasing chain of the Z-nonempty sets $Y \supseteq V_1 \supseteq W_1 \supseteq V_2 \supseteq \dots$. If the process stops at some $W = V_i$ or $W = W_i$, where $i \in \{ 1, 2, \dots, n - 1 \}$, it means that the proof is complete. Let us consider the other case. That is, we assume that we possess the chain $Y \supseteq W_1 \supseteq W_2 \supseteq \dots \supseteq W_{n-1}$. Then $W_i \wedge U_i$ is Z-empty for every $i \in \{ 1, 2, \dots, n - 1 \}$. As above, we define the common family $\mathfrak{U}_n$:

$$\mathfrak{U}_n = \{ Z \} \cup \{ U_1 \} \cup \{ U_2 \} \cup \dots \cup \{ U_{n-1} \} \cup \mathcal{U}_n.$$

It holds that $W_{n-1} \subseteq \bigvee \mathfrak{U}_n$. Hence, $V_n = W_{n-1} \wedge U$ is Z-nonempty for some $U \in \mathfrak{U}_n$. The only possibility for U is to be an element of the family $\mathcal{U}_n$. If A_n is sup-generated then we put $W = V_n$ and stop the proof. In the other case, applying Lemma 4 for $\mathcal{U}_n$ and V_n, we obtain the set $W_n \subseteq V_n$. If $W_n \subseteq \bigwedge \mathcal{U}_n$ then we take $W = W_n$ and the proof is complete. The other case is impossible. Indeed, if we assume that there exists U_n in $\mathcal{U}_n$ such that $W_n \wedge U_n$ is Z-empty then $W_n \subseteq \bigvee \mathfrak{U}_{n+1}$, where

$$\mathfrak{U}_{n+1} = \{ Z \} \cup \{ U_1 \} \cup \{ U_2 \} \cup \dots \cup \{ U_{n-1} \} \cup \{ U_n \}.$$

Hence, by Proposition 2 there exists U in $\mathfrak{U}_{n+1}$ such that $W_n \wedge U$ is Z-nonempty. But such U does not exist. The proof is complete. ❑

Recall that by $\preceq$ we denote the preorder on $\mathfrak{L}$ generated by the ideal $\mathcal{I}$. We are going to show now that $\preceq$ is equivalent to the partial order $\subseteq$ and, hence, is a partial order itself.

PROPOSITION 6. *For every Y, Z in $\mathfrak{L}$, it holds that $Y \preceq Z$ iff $Y \subseteq Z$.*

Proof. Fix two elements Y, Z in $\mathfrak{L}$. If $Y \subseteq Z$ then $Y \setminus Z = \varnothing \in \mathcal{I}$ and $Y \preceq Z$. Let us consider the other case when $Y \preceq Z$, i.e., when $Y \setminus Z \in \mathcal{I}$. Assume that $Y \setminus Z \neq \varnothing$. Then there exists a subfamily $\mathcal{A} = \{ A_1, A_2, \dots, A_n \} \subseteq \mathcal{E}$ such that $Y \setminus Z \subseteq \bigcup \mathcal{A}$. Then by Lemma 5 there exists a chain of Z-nonempty elements of the lattice $Y \supseteq W_1 \supseteq W_2 \supseteq \cdots \supseteq W_n$ satisfying $W_i \cap A_i = \varnothing$ for every $i \in \{ 1, 2, \dots, n \}$. Clearly, $W_n \cap (\bigcup \mathcal{A}) = \varnothing$. Since $W_n \subseteq Y$ and $Y \setminus Z \subseteq \bigcup \mathcal{A}$, we conclude that $W_n \subseteq Z$, that is, W_n is Z-empty. But a set cannot be both Z-nonempty and Z-empty! Therefore, we conclude that our assumption $Y \setminus Z \neq \varnothing$ is wrong and, hence, $Y \preceq Z$ implies $Y \setminus Z = \varnothing$, that is, $Y \subseteq Z$. The proof is complete.

□

THEOREM 7. *For every complete pseudocomplemented lattice* $(\mathfrak{L}, \subseteq)$ *of subsets of a set* X, *that contains the empty set and the whole* X, *there exists an ideal* $\mathcal{I}$ *on* X *that possesses the following properties:*

(1) *For all* $\mathfrak{U} \subseteq \mathfrak{L}$, *there exists* $A \in \mathcal{I}$ *such that*

$$\left(\bigcup \mathfrak{U}\right) \cup A \in \mathfrak{L};$$

(2) *For all* $\{ V_1, V_2, \dots, V_n \} \subseteq \mathfrak{L}$, *there exists* $B \in \mathcal{I}$ *such that*

$$(V_1 \cap V_2 \cap \dots \cap V_n) \setminus B \in \mathfrak{L};$$

(3) $\mathcal{I} \cap \mathfrak{L} = \{ \varnothing \}$.

If the lattice $(\mathfrak{L}, \subseteq)$ *is infinitely distributive then, in addition, it holds that*

(4) *The relation* $\preceq$ *generated by* $\mathcal{I}$ *is a partial order on* $\mathfrak{L}$;

(5) *The lattices* $(\mathfrak{L}, \subseteq)$ *and* $(\mathfrak{L}, \preceq)$ *are isomorphic.*

Proof. We generate an ideal $\mathcal{I}$ by the family of all sup- and inf-generated subsets of X. Then, clearly, (1) and (2) are satisfied. To prove (3), we need to apply Proposition 6 for $Y, Z \in \mathfrak{L}$ where $Z = \varnothing$. Indeed, if there lies $Y \in \mathfrak{L} \cap \mathcal{I}$ then it holds that $Y \preceq \varnothing$. By Proposition 6, it follows then that $Y = \varnothing$. The result of Proposition 6 for $Z = \varnothing$ follows from Lemma 5 for $Z = \varnothing$, but to prove Lemma 5 for $Z = \varnothing$ we only need the lattice $\mathfrak{L}$ to be trivially distributive. In assumption that the lattice $\mathfrak{L}$ is infinitely distributive, the statements (4) and (5) easily follow from Proposition 6. □

REMARK 1. The assumption that the lattice $\mathfrak{L}$ contains the empty set and the whole X is not a strong restriction. Indeed, let us consider a complete lattice $(\mathfrak{L}', \subseteq)$ that consists of sets. Since $\mathfrak{L}'$ is a complete lattice, there exist A_0 and A_1 in $\mathfrak{L}'$ such that $A_0 \subseteq A \subseteq A_1$ holds for every $A \in \mathfrak{L}'$. Then we put $\mathfrak{L} = \{ A \setminus A_0 \mid A \in \mathfrak{L}' \}$, that is, we just cut out the common part from each set in $\mathfrak{L}'$. The new family $\mathfrak{L}$ together with the partial order of inclusion is a complete lattice of subsets of a set $X = A_1 \setminus A_0$ and it contains the empty set and the whole X. Clearly, the lattices $(\mathfrak{L}', \subseteq)$ and $(\mathfrak{L}, \subseteq)$ are isomorphic.

REMARK 2. For distributive lattices we can prove the corresponding theorem similar to the previous one. That is, the results of the previous theorem hold if we substitute a complete lattice for a lattice; infinite distributivity for finite distributivity; a pseudocomplemented lattice, first, for a infinitely trivially distributive (see Proposition 3), and then for its finite analogue; arbitrary unions for finite unions.

Since the theorem is proved, we call every lattice $\mathfrak{L}$ that satisfies the conditions of the theorem an *i-topology* on X (or a *topology modulo an ideal*) and say that it is closed under arbitrary *i-unions* (supremums) and finite *i-intersections* (infimums).

We do not provide any examples since the objects we consider are complete infinitely distributive lattices that are widely presented in many sources. Though, we believe that the following example is worth to be mentioned here. Commonly, the lattice we are going to consider is checked to be a complete lattice and is not checked to be infinitely distributive. We are going to meet the lack.

PROPOSITION 8. *For every distributive lattice L, the family* $\mathrm{Id}(L)$ *of all its ideals together with the partial order of inclusion is a complete infinitely distributive lattice.*

Proof. Since L in an element of $\mathrm{Id}(L)$, and for every family $\{ I_s \mid s \in S \}$ of ideals of L the intersection $\bigcap \{ I_s \mid s \in S \}$ is an ideal, we conclude that $\mathrm{Id}(L)$ is a complete lattice with respect to $\subseteq$. Let us prove that it is infinitely distributive. Fix an ideal J of L, and a family $\{ I_s \mid s \in S \}$ of ideals of L; assume that $J \subseteq \bigvee \{ I_s \mid s \in S \}$. Then it holds that $\bigvee \{ J \wedge I_s \mid s \in S \} \subseteq J$. Let us verify the converse inclusion. If $a \in J$ then $a \in \bigvee \{ I_s \mid s \in S \}$, and hence there exists a finite subset $S_0 \subseteq S$ such that $a = \bigvee \{ a_s \mid a_s \in I_s \text{ and } s \in S_0 \}$. Since J is an ideal, and $a \in J$, we conclude that $a_s \in J$ holds for every $s \in S_0$. Hence, $a \in \bigvee \{ J \wedge I_s \mid s \in S \}$. Finally, it is clear that $J \wedge I_s \subseteq J$ holds for all $s \in S$. Therefore, by Proposition 2 the lattice $\mathrm{Id}(L)$ is infinitely distributive. ☐

Counterexample

The following example provides the set X and the complete completely distributive lattice of its subsets such that there exist two inf-generated sets the union of that is the whole X.

EXAMPLE 9. Let us consider the set X and the family $\mathcal{T}$ of its subsets that are defined as follows:

$$X = N_1(1) \cup N_2(1) \text{ and } \mathcal{T} = \{ N_1(k) \cup N_2(l) \mid k, l \in \mathbb{N} \} \cup \{ \varnothing \},$$

where $N_n(k) = \{ n \} \times \{ k, k+1, \ldots, \infty \}$ for $n \in \{ 1, 2 \}$ and $k \in \mathbb{N}$.

It is easy to check that $\mathcal{T}$ is a topology on X. Hence, $(\mathcal{T}, \subseteq)$ is a complete lattice satisfying the infinite distributive law. Moreover, this lattice is completely distributive. Let us show that. We have to prove that, for every family $\{ N_1(k_{i,j}) \cup N_2(l_{i,j}) \mid j \in J(i) \text{ and } i \in I \}$ of subsets of X, the following holds:

$$\bigwedge \Big\{ \bigvee \{ N_1(k_{i,j}) \cup N_2(l_{i,j}) \mid j \in J(i) \} \mid i \in I \Big\} =$$

$$= \bigvee \Big\{ \bigwedge \{ N_1\big(k_{i,\varphi(i)}\big) \cup N_2\big(l_{i,\varphi(i)}\big) \mid i \in I \} \mid \varphi \in \Phi \Big\},$$

where $\Phi = \prod \{ J(i) \mid i \in I \}$ is the family of choice functions. Let the left and right sides of the equivalence be denoted W_L and W_R, respectively. It is sufficient to show that $W_L \subseteq W_R$. Assume that W_L is nonempty. Then there exist natural numbers k_* and l_* such that $W_L = N_1(k_*) \cup N_2(l_*)$. We are going to consider two special functions φ and γ from Φ that satisfy $N_1(k_{i,j}) \subseteq N_1\big(k_{i,\varphi(i)}\big)$ and $N_2(l_{i,j}) \subseteq N_2\big(l_{i,\gamma(i)}\big)$ for all $j \in J(i)$ and $i \in I$. One can easily verify that such functions exist. Furthermore, $k_* = \max \{ k_{i,\varphi(i)} \mid i \in I \}$ and $l_* = \max \{ l_{i,\gamma(i)} \mid i \in I \}$. To complete the proof we consider two sets A and B that are defined as follows:

$$A = \bigwedge \{ N_1\big(k_{i,\varphi(i)}\big) \cup N_2\big(l_{i,\varphi(i)}\big) \mid i \in I \},$$

$$B = \bigwedge \{ N_1\big(k_{i,\gamma(i)}\big) \cup N_2\big(l_{i,\gamma(i)}\big) \mid i \in I \}.$$

Clearly, $W_L \subseteq A \cup B \subseteq W_R$. Thus, the proof is complete.

Now, we are going to show that there exist two inf-generated subsets of X the union of that is equal to the whole X. We construct two special subfamilies of $\mathcal{T}$ in the following way:

$$\mathcal{N}_1 = \{ N_1(1) \cup N_2(l) \mid l \in \mathbb{N} \} \text{ and } \mathcal{N}_2 = \{ N_1(k) \cup N_2(1) \mid k \in \mathbb{N} \}.$$

Then $\bigcap \mathcal{N}_1 = N_1(1) \cup \{ (2, \infty) \}$ and $\bigcap \mathcal{N}_2 = \{ (1, \infty) \} \cup N_2(1)$. It follows that both $\bigwedge \mathcal{N}_1$ and $\bigwedge \mathcal{N}_2$ are equal to the empty set. Hence, the union of two subsets of X that are inf-generated by the families $\mathcal{N}_1$ and $\mathcal{N}_2$ is equal to the whole X.

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REFERENCES

[1] Grätzer, G., *General Lattice Theory*. Birkhäuser, Basel, 2003

[2] Zvina, I., On i-topological spaces: generalization of the concept of a topological space via ideals. *Appl. Gen. Topol.* (to appear)

Pilnie bezgalīgi distributīvie režģi kā topoloģijas ar precizitāti līdz ideālam

Kopsavilkums

Mēs aplūkojam režģu klasi, kurā katrs režģis sastāv no kādas kopas apakškopām (dažādiem režģiem tās kopas var atšķirties) un veido režģi attiecībā pret dabisko iekļaušanas daļējo sakārtojumu (interesanti, ka katrs režģis ir izomorfs vismaz vienam tāda tipa režģim). Ir labi zināms, ka katra topoloģija kopā ar iekļaušanas daļējo sakārtojumu veido pilnu bezgalīgi distributīvu (pbd) režģi, tomēr minētajā klasē eksistē pbd režģi, kas nav topoloģijas. Mēs pierādām, ka katrs pbd režģis no dotas klases veido i-topoloģiju, t.i., ir slēgts attiecībā pret patvaļīgām i-apvienojuma un galīgām i-šķēluma operācijām. Pat vairāk: mēs konstruējam pilnu pilnīgi distributīvu režģi (tātad pbd režģi), kurš nav slēgts attiecībā pret patvaļīgiem i-šķēlumiem.

Reminiscences of ICM-2002 held in Beijing, August 20–28

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Impressions of a participant of the International Congress of Mathematicians 2002 held in Beijing, P.R. of China.

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Introduction. International Congresses of Mathematicians (ICM) have long history. The first one was held in 1897 in Zurich (which was the venue of three ICM). Since 1900 ICM took place every four years. At the Congress in Paris in 1900 D. Hilbert has formulated his famous problems, which considerably defined directions of mathematical research in XXth century. The venues of ICM were acknowledged centers of mathematical activity in Europe and North America. In 1990 for the first time ICM was organized in Asia (Kyoto, Japan). Then followed Zurich (1994) and Berlin (1998) and new meeting with Asia. The first ICM in new millennium and in XXIst century was held in the capital of People Republic of China the city of Beijing. Below follows some remarks and reminiscences of a participant of ICM-2002.

IMU. This abbreviation means International Mathematical Union. Among other objectives, one of the purposes of this organization is to help mathematicians from developing countries, Eastern Europe and other "less favorite regions" to participate in the work of International Congresses of Mathematicians. The objectives of IMU are "to promote international cooperation in mathematics", "support and assist the International Congresses of Mathematicians" etc. Countries can be adhered to IMU through adhering organizations, which may be its principal academies, mathematical societies, research institutions or government agencies. IMU provides requirements for ICM organizers, both financial and those with respect to infrastructure. According to these requirements, a potential host country may consider a budget of about 1.5 million US dollars, of which only about 0.6 million US dollars might be raised through registration fees. The host country is expected to lodge freely definite number of participants from developing countries, which get their trips paid by IMU through special funds. The officers of IMU preselected during the preceding ICM coordinate and supervise preparations to the next ICM. One can imagine what a huge work was done by members of a Program Committee, the members of which by tradition were kept secret. We know now that the Program Committee was a team of 11 prominent mathematicians with Y. Manin as a Chairman.

First impression. The McDonnell Douglas jet of Finnair brought me over Finnish Gulf, St-Petersburg, lake of Ladoga, darkness and golden spots of Russian cities to distant China. Early in the morning misty mountains emerged suddenly and landing in Beijing's International Airport followed soon. After the pass control has been finished we were brought by bus to downtown to the registration place. BICC (Beijing International Convention Centre) is a nice modern building facing the newly created architectural complex named Beichen (Northern Star). Nearby are Beijing Stadium and bulky Catic hotel. One have to cross the street where red lights, as explained local student volunteer, "recommend" vehicles to stop, and you arrive to the group of local "skyscrapers" where many of ICM participants had their lodging.

Opening Ceremony. The opening ceremony was held at the Great Hall of People in the very center of the city. The trip from BICC to Tian An Men (Way to Heaven) square took about forty minutes and one started to have impression of how this city is big. A lot of nice modern buildings are the evidences of new times and open policy of Chinese government. Great Hall of People was almost filled by participants of the congress. The President Jiang Zemin attended the ceremony. It was a surprising compliment to the mathematical community. Followed speeches of the President of IMU Jacob Palis, Vice Premier of the P.R. China Li Langing, President of the ICM-2002, Mayor of Beijing. Jacob Palis emphasized in his speech that this congress is the first in XXI century and it is taking place in the fastest growing country in the world. Great party followed the official ceremony. People were arranged by 12 and occupied seats at the round tables. The central part of the table was rotating and waiters changed dishes. It was the first meeting with Chinese cuisine for many of the guests.

Fields Medalists and the Rolf Nevanlinna prize winner. Laurent Lafforgue from I.H.E.S., Bures-sur-Yvette, France, has been awarded the Fields Medal for his proof of the Langlands correspondence for the full linear groups GLr over function fields. A global field is either a number field or a function field of characteristics $p > 0$ for some prime number p. The conjectural Langlands correspondence relates two fundamental objects, which are naturally attached to a global field. The Langlands correspondence embodies a large part of number theory, arithmetic algebraic geometry and representation theory of the groups. Small progress made towards the conjectural correspondence has already amazing consequences the most striking of them being the proof of Fermat's last theorem by Andrew Wiles.

Vladimir Voevodsky has studied at Moscow State University and now he is Professor at the Institute for advanced Study in Princeton. He was awarded the Fields Medal for his achievements in algebraic K-theory. Namely, he defined and developed motivic cohomology and the A1-homotopy theory of algebraic varieties; he proved the Milnor conjectures on the K-theory of fields.

The 2002 Rolf Nevanlinna's prize was awarded to Madhu Sudan, Associate Professor of Electrical Engineering and Computer Science, MIT, for his contribution to many areas of computer science theory including computational complexity theory, the design of efficient algorithms, algorithmic coding theory, and the theory of program checking and correcting. Complexity theory is concerned with how many resources are required to perform computational tasks. Examples of a computational task are finding a proof for a mathematical theorem and automatic verification of the correctness of a given mathematical proof. A large body of Sudan's work addresses the latter issue. Brief description of Madhu Sudan's work, especially of that on probabilistically checkable proof, was given by Shafi Goldwasser, Weizmann Institute of Science and MIT.

Plenary lectures. Each day the work has started with the plenary lectures. It was very traditional, just like as at the congress in Berlin. The scope of lectures was different however. Let me remind some of them.

Discrete Mathematics: Methods and Challenges by Noga Allon, Tel Aviv University. The lecture was devoted to combinatorics as a fundamental mathematical discipline. The author focused on the tight connection between discrete mathematics and theoretical computer science. A survey of two of the main general techniques of modern combinatorics was given: algebraic methods and probabilistic methods.

Differential Complexes and Numerical Stability by Douglas Arnold, University of Minnesota. The lecture was devoted to differential complexes, which have recently come to play an important role in analysis of numerical methods for partial differential equations. Consider a boundary value problem in partial differential equations as an operator $Lu = f$ in some space X. A numerical method discretizes this problem and defines an approximation solution uh by the equation $Lh\ uh = fh$. The well-posedness of a problem means that for a given right side f a unique solution u exists, and small changes in f induce small changes in u. The analogous question for the numerical method is the question of stability of the numerical method. The so called differential complexes associated with partial differential equations are very effective in establishing the stability of numerical methods.

Hyperbolic Systems of Conservation Laws in One Space Dimension by Alberto Bressan, S.I.S.S.A., Trieste. A specific system of the first order partial differential equations has been considered. Systems of this type express the balance equations of continuum physics, when small dissipation effects are neglected. A basic example is provided by the equations of non-viscous gases. The main focus was on the uniqueness and stability of entropy weak solutions and on the convergence of vanishing viscosity approximations.

Non Linear Elliptic Theory and the Monge-Ampere Equation by Luis A. Cafarelli, University of Texas at Austin. The main trust of the talk was to show "how the Monge-Ampere equation links in some way the ideas coming from the calculus of variations and those of fully non linear equations."

Emerging Applications of Geometric Multiscale Analysis by David L. Donoho, Stanford University. The lecturer has pointed out the unreasonable effectiveness of harmonic analysis, which was created in XIXth century as a tool for understanding of the equations of mathematical physics but has been applied in ways the inventors could not have anticipated. Further development of harmonic analysis has led to the classical multiscale analysis based on wavelets, which has a number of successful applications. The lecturer has mentioned that wavelets, however, are adapted to point singularities, and many problems in several variables exhibit different kind singularities, such as edges, filaments and sheets. The lecturer has discussed various attempts in order to replace the wavelet analysis in several applications by multiscale analysis adapted to intermediate-dimensional singularities.

Knotted Solitons by Ludwig D. Faddeev, St. Petersburg Department of Steklov Mathematical Institute. The author has traced the history of the notion of soliton in applied mathematics He pointed out that "the value of solitons for the particle physics consists in the possibility of going beyond the paradigm of the perturbation theory." He pointed out that soliton solutions are entirely nonlinear phenomena, and they disappear in their linearized form. Basic ideas about solitons was understood in the middle of 1970's and the developed methods were applicable only in 1 + 1 dimensional space-time. Further investigations showed that many features of 1 + 1 dimensional systems could not be generalized to 3 + 1 dimensions. The existence of "one-particle" soliton solutions was not excluded, however. In his talk the author described some 3 + 1 dimensional system, allowing solitons. First the model was introduced, then the description of the numerical treatment followed and, finally, several applications were discussed.

Mathematical Foundations of Modern Cryptography: Computational Complexity Perspective by Shafi Goldwasser, Weizmann Institute, Israel and MIT. This talk was devoted to the development of modern cryptography, the mathematics behind secret communications and protocols. The relations between cryptography and classical mathematics, as well as that between cryptography and information theory, were discussed first. The conventions and terminology were explained. A survey of the complexity theory underlying the cryptographic tasks of encryption followed. A constructive theory of pseudo randomness, including pseudo-randomness number generators and functions, was discussed. In the last section interactive protocols, interactive proofs, and zero knowledge interactive proofs were considered. The professional design of this presentation should be mentioned.

Singularities in String Theory by Edward Witten, Institute for Advanced Study, Princeton. The subject of this very interesting and energetic lecture was the string theory in a quantum theory. It "reproduces the results of general relativity at long distances but is completely different at short distances." Mathematical basics of string theory are of geometrical nature, and reduce to ordinary differential geometry when the curvature is asymptotically small. The author has described the results, which "were obtained about the behavior of string theory in spacetimes that develop singularities."

Undoubtedly, many interesting lectures are left beyond the scope of these notes.

Short communications. Short communications presentation was organized in parallel sessions. Each talk lasted about twenty minutes including discussion. Rooms in BICC were equipped not only with projectors but even with alarm clocks and a bottle of drinking water for each speaker.

Leisure hours. The city of Beijing is a splendid exotic recreation itself, especially for those who are new in Far East. One has to be cautious however. The first day I came to have an evening stroll I recognized cars turning to left on the red light. Next day local people taught me that red traffic lights in Beijing are recommendations for car drivers not commands to stop. One could easily travel in Beijing using the map where public transport lines with starting and final points were depicted. The only problem, as people mentioned, was that you know which line to use but you hardly could escape the transport mean at the right stop. Younger people in Beijing however often can communicate in English and generally one can get advice. The better transportation mean in Beijing probably is taxi. They are many, the price is very moderate and if you can explain were to go then you have no problems usually.

Footloose tours. Organizers took care of participants in the form of recruiting young people volunteers who accompanied visitors and showed them sightseeing absolutely free. One had the opportunity to visit nice places like Beijing university campus, parks and gardens, museums and pagodas, local computer companies and so on. It was a lovely place Beijing Botanic Garden where a piece of live exotic nature could be seen and not too many people were walking around. The only cat (the white one), so numerous in other places I have seen in Beijing lived just in Botanic Garden. It was a pity that I forgot to buy the collection of dried gigantic butterflies of bright colors that were sold by Chinese women at the entrance.

Excursions. A very special attraction of Beijing and China is the Great Wall. Some day buses took us and delivered after about forty minutes run to the so called pass. Imagine giant about 5000 miles wall crossing the country and several passages from South to North. The day was hot and rare person could climb the segment of the wall till the very top of the hill. Even from the upper point almost nothing was seen (almost nothing approx. 2 km) because of the intensive mist of unknown nature. The impression was deep but one could imagine also of the run along this road to unknown.

Welcome party. Imagine the lawn in front of the luxurious hotel overcrowded with people of different origins, speaking different languages, but mostly in English. By the perimeter of the lawn a chain of tables, some of them filled with food and drinks and some bearing lots of souvenirs, handicrafts, paintings in Chinese style, toys and adornments. In the hot atmosphere highlighted buildings forming this special region of Beijing stood around.

Beijing opera. Visit to renowned Chinese opera was scheduled to Sunday. I failed to book the ticket timely and decided to postpone this visit to some working day. It was surprising that evening performance at Wednesday was easily accessible and the hall was far from being filled. One hardly could follow the scenario of the tale but the show was so bright, dynamic and reach of effects that one scarcely needed to understand what happened on the scene.

Information and computer services. Every morning the newsletters were issued containing information on actualities, schedules of events and surprising facts. Look, for example, at the "ancient Chinese paradox." The person from the State Chu sold both shield and spear. He characterized the shield as "no spear can pierce my shield." He talked about his spear as "no object is my spear cannot pierce through." An audience reasonably asked: "What about using your spear to pierce your shield?" A hall on the ground floor was used for computer activity. It was permanently full. E-mail connection and Internet were Ok. In case of difficulties consultants were on duty and ready to help.

Bookstores and exhibitions. It is a tradition of ICM to arrange multiple bookstands of renowned publishing houses. This type of activity has attracted a lot of participants. Most of all people cluster at the stands of Springer, AMS and the local Higher Education Press stand. The latter could be explained by the fact that a lot of books originally printed by, say, Springer, were permitted to be republished (on the occasion of ICM-2002 in China) by the local publishing house. One could buy books at a relatively moderate price. For example, the "Variational Methods 2nd edition" by M. Struwe could be bought for approx. 7.5 US dollars.

Closing ceremony. The closing ceremony usually is a most exciting moment of ICM. Let me recall the same moment at ICM-1998 in Berlin where thousands of colleagues, sitting even on the floor in passages, some of them with little children in spinning wheels, in strained silence attended to speakers. It was so evident that despite geographical and menthol differences we all are one nation – the mathematicians. The representative of the Organizing Committee of ICM-2002 without any words have shown slides describing China and Beijing. But now it was Madrid turn. The day before booklets with transportation schemes in Madrid, maps and relevant tourist information were distributed, and the delegate of the Spanish IMU Committee Carles Casacuberta has red the words of invitation to the next ICM, to be held in Madrid in 2006. The newly elected president of IMU for 2003 2006 John Ball have promised that new Executive Committee and IMU Committees will work hard and that "we will have some progress to report on when we meet again in Spain in 2006."

Final impression. The last glimpse at Beijing from the board of the aircraft and one could see the thin line of the Great Wall running to the west towards high mountains. Green hills of China were replaced by yellowish Mongolian plains and after an 8 hour long flight the plane has landed in Helsinki where the stop was planned on the way to native Riga. It was calm and relaxing evening in the Finnish capital.

Conclusion. Some figures and exhaustible information about the ICM-2002 could be found at `http://www.icm2002.org.cn`. Proceedings of the ICM-2002 were published by Higher Education Press in 3 volumes. The first one contains plenary lectures and ceremonies. The CD is attached containing 324 MB mpg-file with description of the events during the Congress. An electronic version of the Proceedings of ICM-2002 will be available freely at the sites `http://front.math.ucdavis.edu/ICM2002/` and `http://www.cgtp.duke.edu/ICM2002/` as well as at a number of mirror sites worldwide.

The role, the ICM-2002 will play in the history of mathematics, will be clear in the distant future. One thing is evident just now, however. As Jacob Palis has said in his speech at the Opening ceremony, "this is in many ways very special Congress." He has remarked that mathematics has become more and more international and interaction among mathematicians both at a national and international level was the clear road for its development. The fact that the ICM was for the first time taking place in a developing country and in fact in the fastest growing country in the world, makes the ICM more inclusive and this meets the basic principle of IMU.

The Congress in China was special also in other ways. Two new prizes for mathematicians were founded. The first, in honor of Abel, will be awarded every year by the Norwegian Academy of Sciences. The second, called the Gauss Prize for Applications of Mathematics, is to be awarded jointly by IMU and the German Mathematical Society once every four years.

It is to the point to complete these notes with words from the announcement of the establishment of the Gauss prize: "Mathematics is an important and ancient discipline – no one doubts it. However, it seems that only the experts know that mathematics is a driving force behind many modern technologies. The Gauss prize has been created to help the rest of the world realize this fundamental fact."

REFERENCES

[1] *Proceedings of the International Congress of Mathematicians. Vol. 1: Plenary Lectures and Ceremonies.* Higher Education Press, Beijing, 2002, 657 pp.

[2] `http://www.icm2002.org.cn/`

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