UNIVERSITY OF
LATVIA

FACULTY OF SCIENCE
AND TECHNOLOGY

Insa Cremer

ORDERINGS AND STRUCTURE OF STRONG RICKART RINGS

DOCTORAL THESIS
Submitted for the Doctoral degree (Ph. D.)
of Mathematics
Subfield of Algebra and Mathematical Logic

Riga 2025

The doctoral thesis was carried out at the Department of Mathematics, Faculty of Science and Technology, University of Latvia from 2016 to 2024.

The thesis contains the abstract, table of contents, list of symbols, preface, introduction, eight chapters, conclusion, bibliography, list of conferences and list of the author's publications.

Form of the thesis: dissertation in the field of mathematics and subfield of algebra and mathematical logic.

Thesis supervisor:

Dr. math. **Jānis Cīrulis**, Associate Professor

Reviewers:

1. **Gregor Dolinar**, *Dr. math.*, Professor, University of Ljubljana, Slovenia
2. **Kalle Kaarli**, *Dr. math.*, Professor Emeritus, University of Tartu, Estonia
3. **Pēteris Daugulis**, *Dr. math.*, Leading researcher, University of Daugavpils, Latvia

The thesis will be defended at the public session of the Doctoral Commitee of Mathematics, University of Latvia, on 16th May 2025 at Jelgava street 3.

The thesis is available at the Library of the University of Latvia, Kalpaka blvd. 4, Riga.

Chairman of the Doctoral Committe *Dr. math.*, Prof. Uldis Strautiņš
Secretary of the Doctoral Commitee *Dr. math.*, Assoc. Prof. Jānis Bajārs

© University of Latvia, 2025
© Insa Cremer, 2025

Abstract

This thesis is concerned with several partial orders on certain classes of Rickart rings (possibly one-sided ones) satisfying that or other *strongness condition* – a condition which ensures that, without the need for an involution, the ring resembles a Rickart *-ring in certain aspects. This approach allows for the transfer of knowlegde from the field of Rickart *-rings to a wider class of rings not necessarily having involution, showing that the results obtained for Rickart *-rings can often be at least partly preserved.

In this spirit, we introduce a version of the strong right star order to so-called *right-strong Rickart rings*, obtaining a relatively orthocomplemented poset like for a Rickart *-ring.

Parts of the thesis focus on investigating conditions for the existence of meets and joins under several partial orders on Rickart rings satisfying a strongness condition. For example, we obtain a series of equivalent conditions under which two elements of a so-called *strong Rickart ring* have the join under the star order.

By a unified approach that involves weak BCK-algebras, we obtain, for most of the studied partial orders, conditions under which the respective poset is a meet semilattice.

Another focus of the thesis is on strong semilattice decompositions of certain reducts of reduced Rickart rings. Among other things, it is shown that, given a reduced Rickart ring $\langle R, +, \cdot, 1\rangle$ with the unary operation $^\circ$ mapping every element to its minimal idempotent duplicator, the algebra $\langle R, \cdot, ^\circ, 1\rangle$ is a strong semilattice of right-cancellative D-semigroups, and that this strong semilattice representation is essentially unique.

MSC: Primary: 06A06, 06F25, 16W99, 06F99, 16W10.
Secondary: 06C15, 03G25, 20M10, 20M25, 06A12, 16U99, 20M99

Keywords: Rickart ring, Rickart *-ring, strong Rickart ring, reduced Rickart ring, star order, one-sided star order, diamond order, sharp order, Abian order, join, meet, meet semilattice, relatively orthocomplemented poset, weak difference, weak BCK-algebra, strong semilattice decomposition

Acknowledgements

First, I would like to express my sincere gratitude to my supervisor Professor Jānis Cīrulis. He has steadily helped me grow as a mathematician from when I first met him during my bachelor studies until now. His friendly character and respectful demeanor created a fruitful and pleasant cooperation. He was always very patient and gave detailed feedback which improved both the thesis and my publications a lot. I consider myself lucky to have found such a great supervisor.

I am also very thankful to Professor Janko Marovt from University of Maribor. I really enjoyed my two visits to Slovenia which were made possible by our bilateral project. The second visit in particular was also a big boost in motivation to finish my thesis. It was also a pleasure to write an article together with Janko.

I would also like to thank Professor Valdis Laan and the other organizers of the AAA103 conference in Tartu, which gave me another motivational boost.

While I was a student at University of Latvia, I was attending the weekly algebra seminar of Professor Jānis Buls. I would like to thank the participants of the seminar, because it was a great opportunity to occasionally present my research in an early stage to a forgiving audience.

Of course, I am also very grateful to my family and friends for their support during the long years of studies.

My parents Ingrid and Ufke Cremer deserve the deepest gratitude. They always accepted my decisions even if they didn't like them, they always warmly welcomed me at home in Germany for long visits, and without their patient support this thesis would not exist.

I also wish to sincerely thank my beloved husband Jiří Janda. I often turned to him for advice and I am really happy to have somebody whom I can just ask some mathematical question at the kitchen table without fear of embarassment. Of course, his emotional support was very important for me, too.

There are several people among my wider family who inspired me (and/or fed me great food while I was working). I would like to thank my uncle Folkhard Cremer, my aunt Edelgard Lüttich, my mother-in-law Věra Jandová and also my deceased grandmother Charlotte Scheil.

Another source of emotional support and/or tasty food were my friends: Ieva Birzleja, Laura Bužinska, Shraddha Christian, Katrīna Kazačionaka, Ginta Kazuša, Judith Lampe, Janina Mutz, Liene Strīķe and Mira Wolters.

Finally I would like to thank the forests and waters of Lilaste for helping me keep my sanity (in particular during the pandemic) by giving me the opportunity for long walks and short swims whenever I was taking a break.

Contents

List of Symbols

S^1	the monoid obtained from the semigroup S by either $S^1 := S$, if S has an identity, or otherwise $S^1 := S \cup \{1\}$ with the multiplication on S^1 defined to be the same as on S for elements which are both in S, and $1 \cdot x = x \cdot 1 = x$ 15, 149, 151
$\in$	is element of 28
0	zero element (of a ring or semigroup) 28
$\mathrm{ann_l}(a)$	left annihilator of a 28
$\mathrm{ann_r}(a)$	right annihilator of a 28
$'$	focal operation (usually right focal operation) 28
$`$	left focal operation 28
1	multiplicative identity (of a ring or semigroup) 29
$a - b$	either additive inverse operation (when used as a unary operation) or subtraction in a ring or additive group, i.e., $a - b := a + (-b)$ (when used as a binary operation) 29
$''$	shorthand for double (right) focal operation, i.e., $a'' = (a')'$ 29, 150
$\ulcorner$	alternative symbol for (right) focal operation 29
ran	range (of an operation or a function), also column space of a matrix 29, 48, 153
$:=$	is defined to be equal to 29
$\urcorner$	alternative symbol for left focal operation 29
$\cup$	union (of sets) 29
$\mathrm{RI}(a)$	set of right idempotents of a 30
$\mathrm{LI}(a)$	set of left idempotents of a 30
$+$	addition (in a ring or additive group) 30
$^\perp$	Boolean complement operation 30
i	index from some arbitrary index set 31
$\mathbb{N}$	the set of all natural numbers 31
a^n	a raised to the power of n (where $n \in \mathbb{N}$) 31
$^\circ$	a unary operation characterizing a reduced Rickart ring (related to the focal operation by $a^\circ = a''$) 32, 109
$\mathrm{Mat}_n(\mathbb{R})$	the ring of $n \times n$ real matrices 32
$m \times n$	size of a matrix having m rows and n columns 32
$\mathcal{B}(\mathcal{H})$	the ring of linear bounded operators from $\mathcal{H}$ to $\mathcal{H}$ (where $\mathcal{H}$ is a Hilbert space) 32, 34, 138
$\mathbb{R}$	the set of all real numbers 32

$\mathbb{C}^n$	the n-dimensional complex vector space 32
$\mathbb{C}$	the set of all complex numbers 32
ker	kernel of a linear operator 32
im	range (a.k.a. image) of a linear operator 32
$^\perp$	orthogonal complement of the subspace 32, 138
$-a$	additive inverse of a (in a ring or group) 32, 35
$\cdot$	multiplication (on a ring or a (semi)group) – though usually omitted 33
$\overline{W}$	closure of the subspace W 34
*	involution (in a ring or a semigroup) 35
P	the set of all projection in a *-ring 35
$\mathbb{Z}$	the set of all integers 36
$\times$	Cartesian product 36
$\mathbb{Z}_3$	the ring of integers modulo 3 (i.e., the three-element field) 36
P_{r}	range of the right focal operation in a right-focal right Rickart ring 38, 52
P_{rl}	range of both focal operations in a left-right-focal Rickart ring with a matching pair of focal operations 38, 47
$\leq_E$	standard order of idempotents (in a ring or semigroup) 39, 137
E	the set of all idempotents (of a ring or a semigroup) 39
$\overset{*}{\leq}$	star order 43, 44
$\sqsubset$	space preorder 44
$\sqsubset^w$	weak space preorder 44
$\overset{*}{\vee}$	join under the star order 45
$[0,x]_{\overset{*}{\leq}}$	initial segment below x under the star order 47
$*\preceq$	(strong) left star order 48, 49
$\mathrm{Mat}_n(\mathbb{C})$	the ring of $n \times n$ complex matrices 48
A^*	Hermitian adjoint of the operator A (i.e., conjugate transpose in the matrix case) 48, 49, 50
$\preceq*$	(strong) right star order 48, 49, 60
$*\leq$	weak left star order 49, 50, 51
$\leq*$	weak right star order 49, 51
$a^\dagger$	Moore-Penrose generalized inverse of a 49, 153, 154
$[0,x]_{\leq*}$	initial segment below x under the weak right star order 52
$\vee*$	join under the weak right star order 52, 70
$\wedge*$	meet under the weak right star order 52, 70
$a_x^\perp$	sectional orthocomplementation of an element a in initial segment below x under the weak right star order 52
$\bigwedge A$	greatest lower bound of the set A 53
$\bigvee A$	least upper bound of the set A 53
$\overset{\leftarrow}{\wedge}$	skew meet on a right-strong right Rickart ring 53
$[0,x]_{\preceq*}$	initial segment below x under the strong right star order 54, 70
$a_x^{\underline{\wr}}$	sectional orthocomplement of a in initial segment below x under the strong right star order 54

$\overset{\diamond}{\leq}$	diamond order 55
$\overset{\sharp}{\leq}$	sharp order 55, 56
$a^\sharp$	group inverse of a 55, 155
$U \oplus V$	direct sum of subspaces U and V of a Hilbert space 56
$\mathcal{I}_R$	the subset of a unitary ring R consisting of all elements x such that there exists an idempotent $p \in R$ with the properties that $\mathrm{ann_l}(x) = \mathrm{ann_l}(p)$ and $\mathrm{ann_r}(x) = \mathrm{ann_r}(p)$ 56, 103
p_x	the unique idempotent p such that $\mathrm{ann_l}(p) = \mathrm{ann_l}(x)$ and $\mathrm{ann_r}(p) = \mathrm{ann_r}(x)$ 56, 103
$\leq_{\mathrm{Ab}}$	Abian order 56
$\sqsubset_r^w$	weak right space preorder 64
$\sqsubset_r$	right space preorder 64
$\sqsubset_r^s$	the strong right space preorder 64
$\preceq_*'$	the partial order defined by $a \preceq_*' b$ if and only if $a = ba''$ and $a \sqsubset_r b$ 65
$[0, x]_{\overset{\diamond}{\leq}}$	initial segment below x under the diamond order 68
$\curlywedge *$	meet under the strong right star order 70
$\curlyvee *$	join under the strong right star order 71
$\neq$	not equal 80
$x^{\perp}$	sectional orthocomplementation in the inital segment below x under the strong right star order 82
$\perp_x$	orthogonality relation in the initial segment below x under the strong right star order 87
$\perp$	orthogonality of the relatively orthocomplemented poset $\langle R, \preceq * \rangle$ 87
$\lesssim_r$	right preorder 90
$\sim_r$	right equivalence 90
$\widetilde{E}$	the family of all sets of left idempotents $\{\mathrm{LI}(e) \mid e \in E\}$ 91
$\ominus$	usually a difference or weak difference on a poset 94, 141
$\ominus'$	(D1)-restriction of a weak difference $\ominus$ 94
$\nleq$	negation of $\leq$ 96
$[0, x]_*$	initial segment below x under either the weak right star order or the strong right star order 102
${}_x^{\perp *}$	sectional orthocomplementation in the initial segment below x under either the weak right star order or the strong right star order 102
$\smallsetminus_*$	logical subtraction for either the weak right star order or the strong right star order 103
$\smallsetminus_\sharp$	logical subtraction for the sharp order 104
$\overset{\sharp}{\wedge}$	meet under the sharp order 104
$\smallsetminus_{\mathrm{Ab}}$	logical subtraction for the Abian order 105
$\wedge_{\mathrm{Ab}}$	meet under the Abian order 105
$\circ$	a unary operation characterizing an m-domain ring (see Definition 7.1.2) 108
M_e	the m-domain $\{x \in R \mid x^\circ = e\}$ 109

$\cdot_e$	restriction of the ring multiplication to the m-domain M_e 110
$\mathcal{M}$	the family of all m-domains in an m-domain ring 110
Φ	the family of homomorphisms ϕ_e^f 110
$\operatorname{sys} R$	the inverse system of m-domains of an m-domain ring R 110
$\mathfrak{S}\langle\mathcal{A},\mathcal{H}\rangle$	strong semilattice of semigroups obtained from the inverse system $\langle\mathcal{A},\mathcal{H}\rangle$ 111, 148
$\preceq$	the partial order on the family of semigroups of an inverse system of semigroups 111
$\curlywedge$	the meet under the partial order on the family of semigroups of an inverse system of semigroups 111
h_A^B	the homomorphism from B to A which is part of an inverse system 111
$\circ_A$	multiplication on a semigroup A in an inverse system $\mathcal{A}$ 111
$\mathcal{M}_1^\circ$	the family of all m-domains (seen as D-monoids) in an m-domain ring 116
$\operatorname{sys}_1^\circ R$	the inverse system of m-domains (seen as D-monoids) of an m-domain ring R 116
$\operatorname{sys}^\circ R$	the inverse system of m-domains (seen as D-semigroups) of an m-domain ring R 116
$\mathcal{M}^\circ$	the family of all m-domains (seen as D-semigroups) in an m-domain ring 116
$\bullet$	the unary operation on a strong semilattice of D-semigroups 116
$\top$	greatest element of a poset 116
$\mathbf{1}$	the identity on a strong semilattice of D-monoids over a semilattice with top element 116
$\mathfrak{S}_1^\circ\langle\mathcal{A}^\circ,\mathcal{H}\rangle$	strong semilattice of D-monoids obtained from inverse system $\langle\mathcal{A}^\circ,\mathcal{H}\rangle$ 116
$\mathcal{A}_1^\circ$	the family of D-monoids obtained from a family $\mathcal{A}^\circ$ of D-semigroups with identities by including the identities of the D-semigroups into their signatures 118
$\overleftarrow{\wedge}$	skew meet 119, 150
$\overrightarrow{\wedge}$	the left skew meet (on a reduced Rickart ring) 119
$\vartriangle$	usually denotes the band operation on a band-enriched monoid 120
$\overrightarrow{\vartriangle}$	the band operation on a left band-enriched monoid 120
$\overleftarrow{\wedge}_e$	restriction of skew meet to the m-domain M_e 121
$\operatorname{bem}\langle A,\cdot,^\circ,1\rangle$	the band-enriched monoid induced by the D-monoid $\langle A,\cdot,^\circ,1\rangle$ 122
$\operatorname{dmon}\langle A,\vartriangle,\circ,1\rangle$	the D-monoid induced by the band-enriched monoid $\langle A,\vartriangle,\circ,1\rangle$ 123
$\mathcal{M}_1^\vartriangle$	the family of all m-domains (seen as band-enriched monoids) of a reduced Rickart ring 124
$\operatorname{sys}_1^\vartriangle R$	the inverse system of band-enriched monoids of a reduced Rickart ring R 124
$\blacktriangle$	the band operation on a strong semilattice of band-enriched monoids 125

$\mathfrak{S}_1^{\triangle}$	strong semilattice of band-enriched monoids induced by inverse system of band-enriched monoids $\langle \mathcal{A}_1^{\triangle}, \mathcal{H}\rangle$ 125
$\blacktriangleleft$	the natural order of (the band reduct of) a strong semilattice of band-enriched monoids 125
$\leq$	some partial order 136
$^{\perp}$	complementation 136
$\wedge$	meet (i.e., greatest lower bound) operation 136
$\vee$	join (i.e., least upper bound) operation 136
$\perp$	orthogonality relation (on an orthoposet) 136
$\wedge_E$	meet under the standard order of idempotents in the set of idempotents 137
$\vee_E$	meet under the standard order of idempotents in the set of idempotents 137
$\wedge_C$	meet under the standard order of idempotents in the set of central idempotents 137
$\mathcal{S}(\mathcal{H})$	the set of all closed subspaces of a Hilbert space $\mathcal{H}$ 138
$^{\perp}$	g*- or m-complementation 139
$[0,x]_{\leq}$	initial segment below x (in a set ordered by partial order $\leq$) 140
$^{\perp}_{x}$	sectional g*-(or m-, b*-or ortho-) complementation 140
$\perp_x$	local orthogonality on initial segment $[0,x]_{\leq}$ of sectionally orthocomplemented poset 140
$\perp$	orthogonality of a relatively orthocomplemented poset 140
$\setminus$	usually the binary operation of a BCK-algebra or a weak BCK-algebra 142
fg	composition of the functions f and g, i.e., $(fg)(x) = f(g(x))$ 147
$\langle \mathcal{A}, \mathcal{H}\rangle$	inverse system of the algebras A_s and the homomorphisms h_s^t (over some poset S) 147
$\bigcup_{s\in S} A_s$	union of the family of sets $\{A_s \mid s \in S\}$ 148
$\cdot_i$	multiplication on (the semigroup or semigroup reduct of) A_i 148
$\bullet$	multiplication on a strong semilattice of semigroups 148
$\leq_B$	natural order on a band B 149
$\wedge_B$	meet under the natural order on a band B 149
$\vee_B$	join under the natural order on a band B 149
$\sqsubseteq_B$	natural preorder on a band B 149
$\mathcal{J}$	Green equivalence defined by $a\mathcal{J}b$ iff ${}^1aB^1 = B^1bB^1$ 149
$A/\mathcal{X}$	quotient of a semigroup or other algebra A by an equivalence relation $\mathcal{X}$ 149
$^{\circ}$	the unary operation on a D-semigroup 150
P_W	projection onto closed subspace W of a Hilbert space 153
a^-	some inner generalized inverse of a 154
$\operatorname{rank} A$	rank of the matrix A 156

Preface

Directions of research on partial orders on Rickart rings

Several partial orders on Rickart rings and Rickart *-rings have been studied for more than 40 years. They were also often studied on specific examples, like matrix rings or the ring of bounded linear operators on a Hilbert space, or on special cases of Rickart rings which have a simpler structure, like reduced Rickart rings.

Different directions of research include investigating the connections and similarities between the various partial order (see, for example, [Dolinar and Marovt, 2018]) or the links of the partial orders to ring theoretic notions, for example, generalized inverses (see [Marovt, 2015]). Moreover, the general order-theoretic properties of the partial orders are a field of interest, for example, existence of meets and joins, as well as complementations, and the structure of the so-called initial segments (down-sets under some element). Another direction is transferring results from the more specific setting of Rickart *-rings to the more general Rickart rings or even one-sided Rickart rings (in some cases, even semigroups, see [Cīrulis, 2019]).

This thesis is mostly part of the research directions mentioned in the last two sentences of the previous paragraph, but another aim is to prove stronger results which usually hold only in specific Rickart rings. A summary of the structure of the thesis follows in the end of the introduction.

Strong Rickart rings

Strong Rickart rings were introduced in [Cīrulis, 2016] as a "star-free" generalization of Rickart *-rings (i.e., a generalization that does not require an involution). They are equipped with two unary operations called left, respectively right, focal operation. Every Rickart *-ring, as well as every reduced Rickart ring, can be seen as a strong Rickart ring. A strong Rickart ring is by definition a (both-sided) Rickart ring, but there is also a one-sided version of this notion, which was introduced in [Cīrulis, 2015d] and which we call *right-strong right Rickart rings*. We also introduce the partly one-sided notion of *right-strong Rickart ring* – a both-sided Rickart ring which is equipped with a right focal operation such that it is a right-strong right Rickart ring.

In this thesis, we study star-free versions of several partial orders, mostly the one-sided star orders, but also the star order and the diamond order (we also touch the sharp order and the Abian order, which are not confined to Rickart rings). The star order and the weak right star order were generalized to a star-free setting in [Cīrulis, 2015d] and [Cīrulis, 2016], and for the diamond order, this was done in Cīrulis [2017]. A generalization of the strong right star order to so-called *right-strong Rickart rings* is provided by this thesis.

The aim of the thesis

The aim of the thesis is to investigate whether certain properties which are known for these partial orders in Rickart *-rings also hold in the corresponding star-free settings in some way, to adapt certain notions which are used in the study of the star order to the one-sided star orders and to find comparable one-sided versions of results which are known for the star order.

There are three main results: First, it is proved that a right-strong Rickart ring under the strong right star order is a relatively orthocomplemented poset. Second, given an Abelian group G equipped with a partial order $\leq$ that satisfies certain conditions which hold for many partial orders on suitable types of Rickart rings (in particular the star order, the one-sided star orders and the sharp order), we find a necessary and sufficient condition for $\langle G, \leq \rangle$ to be a meet semilattice. The third result deals with the special case of reduced Rickart rings. We unite a strong semilattice decomposition of the multiplicative semigroup of a reduced Rickart ring R, which is a special case of a similar construction on so-called *right PP-monoids* described in [Fountain, 1976], with the strong semilattice decomposition of R equipped with the unary operation which maps each element to its minimal idempotent duplicator in the sense of N.V. Subrahmanyam. We also provide an analogous strong semilattice decomposition for R equipped with the skew meet operation from [Cīrulis, 2015d].

Approbation of the results and contribution of the author

Results obtained as a part of this thesis have been presented in 11 international conferences and six domestic conferences (for details, see the list of attended conferences in the end of the thesis).

The main results of this thesis are contained in four scientific papers: Cīrulis and Cremer [2022], Cremer [2024], Cremer and Marovt [ND] and Cremer [2025] (see also the list of author's publications).

Results from Sections 4.3 and 4.5 in Chapter 4 were published in the first article Cīrulis and Cremer [2022], which was joint work with Professor Jānis Cīrulis. In that publication, the author proved Proposition 3.2, Theorem 5.3 and a part of Theorem 5.4. She also checked the article for mistakes. The second article Cremer [2024] consists of content from Sections 3.1, 4.2 and 5.1. The results which will be published in the third article Cremer [2025] can be found in Sections 7.1 and 7.2. Finally, the results from Chapter 6 will be published in the fourth article Cremer and Marovt [ND]. In this joint work with Professor Janko Marovt, the author proved all the results and did most of the writing and editing.

Introduction

Literature review

We first introduce the different types of Rickart rings which are widely studied, then we give a summary on partial orders on these rings. After that we focus on the star order and the one-sided star order, summarizing the known results on the topic which are relevant for this thesis. Finally, we describe the structure of the thesis.

Rickart rings and Rickart *-rings

Rickart rings and Rickart *-rings were first defined by Maeda in Maeda [1960] as an algebraic generalization of the ring of bounded linear operators on a Hilbert space. Examples of Rickart rings also include Boolean rings, domains and certain subdirect products of domains (these are examples of reduced Rickart rings).

A *left* (*right*) *Rickart ring* is a ring R (we assume that rings are associative by definition) in which the left (right) annihilator of every element is a principal left (right) ideal generated by an idempotent. A *Rickart ring* is a ring which is both left Rickart and right Rickart. If a Rickart ring is moreover equipped with an involution and the left (or right) annihilator of every element is generated by a self-adjoint idempotent, then it is called a *Rickart *-ring*.

In the ring of bounded linear operators over a Hilbert space, the involution is the operation assigning to each operator its adjoint operator, and in the ring of complex $n \times n$ square matrices, the involution assigns to every matrix its conjugate transposed matrix.

Note that one-sided Rickart rings do not necessarily have the multiplicative identity, but Rickart rings and Rickart *-rings do, and the rings which we deal with in this thesis are usually unitary.

Partial orders on Rickart rings and Rickart *-rings

The star order The *star order* was first introduced in a very general setting of involutive semigroups in [Drazin, 1978] by defining $a \overset{*}{\leq} b$ if and only if $a^*b = a^*a = b^*a$ and $ab^* = aa^* = ba^*$. It was extensively studied also on Rickart *-rings, in particular in the special cases of matrix rings and rings of bounded linear operators on a Hilbert space. In [Cīrulis, 2016], the so-called *strong Rickart rings* were introduced as a ”star-free” generalization of Rickart *-rings (i.e., a generalization that does not require an involution). In that article, also the star order was generalized to strong Rickart rings.

The minus order The *minus order* was introduced in [Hartwig, 1980] on the ring of bounded linear operators on a finite dimensional Hilbert space (in other words, the ring of $n \times n$ square matrices) by defining $A \overset{-}{\leq} B$ if and only if $\operatorname{rank}(B - A) = \operatorname{rank}(B) - \operatorname{rank}(A)$. It was also noted in [Hartwig, 1980] (see [Djordjevic et al., 2015]) that, on a matrix ring, $A \overset{-}{\leq} B$ iff $A^- A = A^- B$ and $AA^- = BA^-$, where $AA^- A = A$ (i.e., A^- is a generalized inner inverse of A). Recently, the minus order was generalized first to the ring of bounded linear operators on an arbitrary Hilbert space in [Šemrl, 2010] and then to Rickart rings in [Djordjevic et al., 2015].

The diamond order The *diamond order* was introduced in [Baksalary and Hauke, 1990] for rectangular complex matrices by defining $A \overset{\diamond}{\leq} B$ if and only if $A \sqsubset B$ and $AA^*A = AB^*A$, where $\sqsubset$ denotes the *space preordering*, i.e., $A \sqsubset B$ if and only if $\operatorname{ran} A \subseteq \operatorname{ran} B$ and $\operatorname{ran} A^* \subseteq \operatorname{ran} B^*$. In [Lebtahi et al., 2014] the diamond order was generalized to *-regular rings as $a \overset{\diamond}{\leq} b$ if and only if $aR \subseteq bR$, $Ra \subseteq Rb$, and $aa^*a = ab^*a$, and in Cīrulis [2017] it was studied on Rickart rings.

The core order The *core order* was introduced for matrices in [Baksalary and Trenkler, 2010]. For a complex square matrix A, a *core inverse* of A is a matrix $A^{\textcircled{\#}}$ with the properties that $AA^{\textcircled{\#}}$ is the orthogonal projection onto the column space $\operatorname{ran} A$ and $\operatorname{ran} A^{\textcircled{\#}} \subseteq \operatorname{ran} A$ (see [Baksalary and Trenkler, 2010]). The core order is then defined by $A \overset{\textcircled{\#}}{\leq} B$ if and only if $A^{\textcircled{\#}}A = A^{\textcircled{\#}}B$ and $AA^{\textcircled{\#}} = BA^{\textcircled{\#}}$. It was generalized to Rickart *-rings in [Rakic, 2015].

The sharp order For a complex square matrix A, the so-called *group inverse* of A is a matrix $A^\sharp$ such that $AA^\sharp A = A$, $A^\sharp AA^\sharp = A$ and $AA^\sharp = A^\sharp A$. The group inverse of a matrix A exists if and only if $\operatorname{rank} A = \operatorname{rank} A^2$, and in that case it is unique.

On the set of complex square matrices which have the group inverse, the *sharp partial order* was defined in [Mitra, 1987] by $A \overset{\sharp}{\leq} B$ if and only if $A^\sharp A = A^\sharp B$ and $AA^\sharp = BA^\sharp$. This partial order was extended to a certain subset of the algebra of bounded linear operators on a Banach space X (namely, the set $\{A \in \mathcal{B}X \mid \overline{\operatorname{im} A} \oplus \ker A = X\}$) in [Efimov, 2013] and to the set of elements which have the group inverse in a general (unitary) ring in [Marovt, 2015]. On an arbitrary unitary ring, the sharp order was extended in [Rakic, 2015] to the relation defined by $a \overset{\sharp}{\leq} b$ if and only if $a = eb = be$ for some idempotent e such that the right, respectively left, annihilator of e coincides with the right, respectively left, annihilator of a. On the set of elements for which such an idempotent exists, this relation is a partial order. However, it is in general not a partial order on the whole ring.

The one-sided star orders One-sided (left and right) generalizations of the star order were first defined for matrices in [Baksalary and Mitra, 1991] by weakening one of the two defining conditions. The one-sided star orders from [Baksalary and Mitra, 1991] were generalized to rings of bounded operators over Hilbert spaces in two different ways.

The first version, which was introduced by Deng and Wang in Deng and Wang [2012], and independently by Dolinar et al. in [Dolinar et al., 2014], just transferred the definition from [Baksalary and Mitra, 1991] to a more general structure.

The second version was introduced by Cīrulis in Cīrulis [2014a] and Cīrulis [2015c] (see also Cīrulis [2015a]). It coincides with the first version on matrix rings, but it is an open question whether they coincide in the general case (they do coincide for operators which have the Moore-Penrose generalized inverse). It is known that the second version is weaker than the first, and therefore we follow Dolinar et al. [2021] and call these weaker partial orders the *weak one-sided star orders*. The first version will accordingly be called the *strong one-sided star orders* (deviating from the usual terminology).

Since both the weak and the strong right star order can be expressed in purely algebraic terms, they were introduced also for Rickart *-rings (for the strong right star order from Deng and Wang [2012], this was done by Marovt et al. in Marovt et al. [2015]; for the weak version, see Cīrulis [2014a]).

We now give a more detailed summary of some known results on the star order and the one-sided star orders on Rickart *-rings and their generalizations to Rickart rings satisfying a suitable strongness condition. These results often served as a motivation for finding related comparable results which are presented in this thesis.

Properties of the star order

What follows is a summary of some known properties of the star order, and how the knowledge evolved from more specific rings to either strong Rickart rings or at least Rickart *-rings.

Lattice operations

In the 1980s, Janowitz and Hartwig started to investigate conditions for the existence of meets and joins of elements of certain (or even general) Rickart *-rings under the star order.

Meets of elements under an upper bound Janowitz proved in Janowitz [1983] that elements a, b of a Rickart *-ring R which are bounded from above (i.e., $a, b \overset{*}{\leq} x$ for some $x \in R$) have the meet. Cīrulis provided also an explicit equational description for this meet (as for the join) in Cīrulis [2015a]. This result was extended to strong Rickart rings in [Cīrulis, 2016])

Which rings are meet semilattices? It was proved in [Hartwig and Drazin, 1982] that the Rickart *-ring $\mathcal{B}(\mathbb{C}^n)$ ordered by the star order is a meet semilattice. Also other special cases of Rickart *-rings were shown in [Janowitz, 1983] to be meet semilattices under the star order – the so-called Baer *-rings, which include the ring of bounded linear operators on a Hilbert space (hence in particular $\mathcal{B}(\mathbb{C}^n)$), and reduced Rickart *-rings. Some necessary and sufficient conditions for an arbitrary Rickart *-ring to be a meet semilattice were given in Cīrulis [2015a].

Structure of initial segments

The structure of the initial segments of a Rickart *-ring was studied in [Janowitz, 1983] and later, in a star-free setting, in [Cīrulis, 2016].

Orthomodularity of initial segments It is known from [Janowitz, 1983] that the initial segments of a Rickart *-ring under the star order are orthomodular posets. It was proved in [Cīrulis, 2016] that a strong Rickart ring under the star order is even a quasi-orthomodular poset (with the star orthogonality). Since every quasi-orthomodular poset can be seen as a relatively orthocomplemented poset (see [Cīrulis, 2014b]), this implies that a strong Rickart ring is relatively orthocomplemented, as was also noted in [Cīrulis, 2016].

It was also noted in [Cīrulis, 2016] that the additive group of a strong Rickart ring R is an orthomodular group and that R is a generalized orthomodular poset.

Lattice isomorphisms For the case of a Rickart *-ring R, Janowitz proved in [Janowitz, 1983] that every initial segment $[0, x]_{\overset{*}{\leq}}$ under the star order is not only orthomodular, but also a lattice which is isomorphic to a sublattice of the projection lattice. Cīrulis extended and improved this result to strong Rickart rings in [Cīrulis, 2016], showing that the lattice operations in the initial segment are the restrictions of the (partial) lattice operations in R. Cīrulis found a similar isomorphism between initial segments of a strong Rickart ring and a sublattice of a cartesian product of two sublattices of the lattice of so-called closed idempotents of the ring.

Reduced Rickart rings

Janowitz showed that, for a Rickart *-ring, being reduced is equivalent to being a specific meet semilattice under the star order in [Janowitz, 1983]. Cīrulis and the author also studied reduced Rickart rings in [Cīrulis and Cremer, 2018], proving that reduced Rickart rings form a variety and proving that every reduced Rickart ring is isomorphic to a certain subdirect product of domains called *Sussman ring* (see also [Sussman, 1958]).

Properties of the one-sided star orders

The weak right star order has been generalized also to right-strong right Rickart rings (a one-sided version of strong Rickart rings) by Cīrulis in [Cīrulis, 2015d], although it is mentioned only in Remark 2 of that article that the partial order studied there is a generalization of the one-sided star order. The star-free generalization of the strong right star order is a subject of this thesis. Therefore, we summarize here some properties of the weak right star order on a right-strong right Rickart ring and the properties of the strong right star order on a Rickart *-ring.

The weak right star order The properties of the weak right star order on a right-strong right Rickart ring were investigated in [Cīrulis, 2015d]. For the special cases that the ring is a Rickart *-ring or even the ring $\mathcal{B}(\mathcal{H})$ of bounded linear operators over a Hilbert space, see also [Cīrulis, 2014a] and [Cīrulis, 2015c]. In [Dolinar et al., 2021], mappings on the ring $\mathcal{B}(\mathcal{H})$ which preserve the right (or left) star order in both directions were studied.

It was proved in [Cīrulis, 2015d] that several properies of the star order on a strong Rickart ring hold also for the weak right star order on a right-strong right Rickart ring: A right-strong right Rickart ring with the weak right star order has the upper bound property, and moreover, it is a relatively orthocomplemented poset in which the meet exists for any two elements having a common upper bound. An isomorphism between initial segments of the ring and initial segments of its lattice of so-called closed idempotents was also found in [Cīrulis, 2015d].

In [Cīrulis, 2015d], an particular binary operation $\overleftarrow{\wedge}$, called *skew meet*, was defined on a right-strong right Rickart ring R, and it was proved that $\langle R, \overleftarrow{\wedge}\rangle$ is a band. Moreover, it was proved that the natural order of this band is the weak right star order.

Even more, it was also proved in [Cīrulis, 2015d] that the band $\langle R, \overleftarrow{\wedge}\rangle$ is right normal, and therefore it is a so-called right normal *skew nearlattice* (a skew nearlattice is a band having the upper bound property with respect to its natural order).

It remains an open question whether the star order or the strong right star order also arises as the natural order of a similar skew nearlattice.

The strong right star order The strong right star order and the strong left star order were studied in abstract Rickart *-rings in [Marovt et al., 2015], [Marovt, 2015] and [Krēmere, 2016].

In [Dolinar et al., 2021], mappings on the ring $\mathcal{B}(\mathcal{H})$ of bounded linear operators over a Hilbert space which preserve the right (or left) star order in both directions were studied.

The one-sided star orders were also generalized to *-regular rings and it was proved that, on *-regular Rickart *-rings, this generalization coincides with the one for Rickart *-rings (see [Marovt et al., 2015, Definitions 12, 13 and Theorem 14])

In [Krēmere, 2016], the author proved that a Rickart *-ring with the left star order is a relatively orthocomplemented poset. Later in this thesis, we prove the result in star-free terms in the more general setting of right-strong Rickart rings, using the corresponding result from [Cīrulis, 2015d] for the weak right star order on right-strong right Rickart rings.

Structure of the thesis

The thesis is divided into two parts: Part I deals with the preliminaries and contains the first two chapters (Chapters 1 and 2), while Part II contains the original results (Chapters 3 to 7). There are also three appendices (Appendix A, Appendix B and Appendix C), which some prerequisites that were not included in Part I because they are not directly related to partial orders on Rickart rings.

First, in Chapter 1, we define Rickart rings (in Section 1.1), Rickart *-rings (in Section 1.2) and the recently introduced strong Rickart rings, right-strong right Rickart rings and right-strong Rickart rings (in Section 1.3), and illustrate all these notions with examples.

The second preliminary chapter is Chapter 2, which deals with specific partial orders on different types of Rickart rings. In Section 2.1 and Section 2.2, we recall the star order, the one-sided star orders and the weak one-sided star orders and summarize the relevant results from the literature on them. These partial orders are the main topic of this thesis, but we also mention some other partial orders on Rickart rings in Section 2.3.

Part II begins in Chapter 3, where we define the strong right star order on right-strong Rickart ring and prove that it is indeed a partial order. We also provide an alternative characterization of the weak right star order and introduce another partial order on a right-strong right Rickart ring which is weaker than the strong right star order, but stronger than the weak right star order, and thus can be regarded as a third generalization of the right star order.

In Chapter 4, we deal with conditions for the existence of joins and meets under different partial orders on Rickart rings satisfying some strongness condition. First we briefly describe minimal upper bounds and joins under the diamond order in Section 4.1. Then, in Section 4.2, we obtain a sufficient condition for the existence of the meet and a necessary condition for the existence of the join under the strong right star order for elements which have a common upper bound. In Section 4.3, we introduce star-free notions of different types of coherence and compare them to each other. We use these notions in Section 4.4 Section 4.5 in order to state conditions for the existence of specific meets and/or joins under the weak right star order and under the star order.

In Chapter 5, we investigate the order properties of a right-strong Rickart ring with the strong right star order. The main result is that this poset is relatively orthocomplemented. This is equivalent to saying that it is quasi-orthomodular (in particular, initial segments are orthomodular posets). This enables us to define an orthogonality relation on this poset, which we call *strong right star orthogonality*, and investigate its properties. Finally, we provide order embeddings of initial segments into certain ordered sets which are related to idempotents of the ring and we characterize the images of such isomorphisms.

In Chapter 6, we move to a much more general setting: We generalize the notion of a difference on a poset, introducing *weak differences*. We prove a necessary and sufficient condition for a poset with a certain weak difference to be a meet semilattice. This condition involves the existence of a certain binary operation which turns the poset into a weak BCK-algebra. As a special case, we examine an Abelian group equipped with a partial order satisfying some conditions. Examples of this structure are abundant and include strong Rickart rings with the star order, right-strong right Rickart rings with the weak right star order and right-strong Rickart rings with the strong right star order, as well as a certain subset of an arbitrary unitary rings with the sharp order. Note that these structures are not partially ordered groups (except in the trivial case of the one-element group).

In the last chapter, Chapter 7, we focus on the special case of reduced Rickart rings. The main result is a strong semilattice decomposition of the multiplicative semigroup of a reduced Rickart ring equipped with a unary operation that turns it into a D-semigroup. We also derive an analogous strong semilattice decomposition which uses the skew meet instead of the D-semigroup operation. That is, we construct a "double" strong semilattice decomposition of an algebra consisting of the underlying set of the ring equipped with two semigroup operations (we call such this algebra the *band-enriched monoid of the ring*). We define the *inverse system* of a reduced Rickart ring and prove that the band-enriched monoid of every reduced Rickart ring is a strong semilattice induced by its inverse system.

Part I

Preliminaries

Chapter 1

Rickart rings, Rickart *-rings and strongness conditions

We begin the thesis with this introductory chapter on different types of (one-sided or both-sided) Rickart rings. The term *ring* is used in the standard sense, i.e., all rings are assumed to be associative.

In Section 1.1, we introduce Rickart rings and their left and right versions with a particular focus on some special unary operations which can be defined on them and which are called *focal operations*. We also provide a collection of examples of Rickart rings in order to illustrate the definitions and show that they are quite common structures.

Section 1.2 deals with Rickart *-rings, a special subclass of Rickart rings which are equipped by an involution. They are of interest for this thesis because in later chapters, some results which are known for Rickart *-rings are generalized to some wider class of (sometimes even one-sided) Rickart rings. Again, we first give the necessary definitions and then provide examples of Rickart *-rings.

These wider classes are introduced in Section 1.3, where we define several types of (one-sided or both-sided) Rickart rings whose focal operation(s) satisfies (satisfy) certain conditions which we will refer to collectively as "strongness conditions". Some examples are also provided, including the class of Rickart *-rings. We are thus dealing with what can be considered to be a "star-free" generalization of Rickart *-rings, because these rings do not need to have an involution, but the strongness condition makes sure that they retain certain properties of Rickart *-rings.

1.1 Rickart rings

This section consists of two subsections. Section 1.1.1 recalls several standard notions which will appear throughout the thesis. First, we state the definition of a Rickart ring, as well as a right (or left) Rickart ring, and also recall the notion of *focal operations* – particular idempotent-valued unary operations which can be defined on a Rickart ring. Then we settle the terminology concerning Rickart rings seen as algebras whose signature includes the focal operations. We introduce a handy property called *normality*, which is not satisfied by every focal operation, but on a given Rickart ring, we can always define a focal operation that does satisfy it. Finally, we recall an alternative definition of a Rickart ring which is related to special sets of idempotents associated with an element of the ring (the so-called *left and right idempotents*).

Section 1.1.2 collects several well-known examples and particular subclasses of Rickart rings. First, we mention reduced Rickart rings, including Boolean rings, domains and direct products of domains (the definition of a reduced ring and some of its basic properties are also provided). We also state some of their properties, as well as their characterization in terms of existence of a particular unary operation. Then, we turn to matrix rings as a source of examples of Rickart rings, including an example of a ring which is right Rickart, but not left Rickart. Finally, we mention the standard motivating example of a Rickart ring: the ring of bounded linear operators on a Hilbert space.

1.1.1 Basic notions

Rickart rings and focal operations

We first remind the following standard notion, because it appears in the definition of a Rickart ring and also elsewhere in this thesis.

Definition 1.1.1. Let a be an element of a ring R. The set $\{x \in A \mid xa = 0\}$ is called the *left annihilator of* a and is denoted by $\mathrm{ann}_{\mathrm{l}}(a)$. Analogously, the *right annihilator of* a is the set $\{x \in A \mid ax = 0\}$ and is denoted by $\mathrm{ann}_{\mathrm{r}}(a)$.

Rickart rings were first defined by Maeda in [Maeda, 1960] as an algebraic generalization of the ring of bounded linear operators on a Hilbert space.

Definition 1.1.2. [Maeda, 1960] A ring is said to be a *right Rickart ring*, if the right annihilator of every element is the right ideal generated by some idempotent. It is said to be a *left Rickart ring* if the left annihilator of every element is the left ideal generated by some idempotent. A *Rickart ring* is a ring which is both right Rickart and left Rickart.

Every Rickart ring is necessarily unitary (i.e., has the multiplicative identity), but one-sided Rickart rings may not be unitary.

Paraphrasing Definition 1.1.2, a right (left) Rickart ring is a ring R in which for all $a \in R$ there exists an idempotent $e \in R$ such that, for all $x \in R$, $ax = 0$ if and only if $ex = x$ ($xa = 0$ if and only if $xe = x$). Obviously, this means that a ring R is a right Rickart ring if and only if there exists a unary operation $'$ such that, for every $a \in R$, a' is an idempotent with the property that, for every $x \in R$,

$$ax = 0 \text{ iff } a'x = x. \tag{1.1}$$

Such an operation is called *right focal operation* in [Cīrulis, 2016], adapting a term from [Foulis, 1960].

Similarly, a ring R is a left Rickart ring if and only if there exists a unary operation $^{`}$ such that, for every $a \in R$, $a^{`}$ is an idempotent with the property that, for every $x \in R$,

$$xa = 0 \text{ iff } xa^{`} = x. \tag{1.2}$$

Such an operation is called *left focal operation* in [Cīrulis, 2016]. By a *focal operation* we mean an operation which is either right focal or left focal.

A right focal operation on a given (right) Rickart ring is not necessarily unique, because, for an element a, there can be more than one idempotent a' satisfying Equation 1.1. The same goes for a left focal operation on a given left Rickart ring.

In cases when it is unique (reduced Rickart rings, as well as Rickart *-rings with the additional requirement that a' be self-adjoint), Equation (1.1) has been used as the defining property of a Rickart ring, see [Janowitz, 1976] and [Janowitz, 1983].

Rickart rings are also sometimes called *PP-rings*, and left (right) Rickart rings are called left-PP (right-PP) rings. This is because a ring is right (left) Rickart if and only if every right (left) principal ideal is projective as a right (left) module (*PP* is the abbreviation for *principal implies projective*, see [Lam, 1999, page 261]).

Right-focal right Rickart rings

In the sequel, it is important to distinguish between ”plain” (one-sided) Rickart rings and (one-sided) Rickart rings equipped with a particular focal operation. Therefore, we introduce the following term.

Definition 1.1.3. A Rickart ring (respectively, right Rickart ring) equipped with some right focal operation is called *right-focal* Rickart (respectively, *right-focal* right Rickart) ring.

Dually, a left focal operation gives rise to the notion of a *left-focal Rickart ring*. By a *left-right-focal Rickart ring* (or shorter, *focal Rickart ring*) we mean a Rickart ring equipped with both a left focal operation and a right focal operation.

Normality

Definition 1.1.4. [Cīrulis and Cremer, 2018] A focal operation on a unitary ring R is said to be *normal* if $a'' = 1 - a'$ for all $a \in R$ (where a'' is a shorthand for $(a')'$). A left or right Rickart ring whose focal operationis normal will sometimes also be called *normal* in this thesis.

On every unitary right Rickart ring R, we can define a normal right focal operation as follows: Suppose $^{\ulcorner}$ is some right focal operation on R. Let us denote its range by $\mathrm{ran}^{\ulcorner}$, and define $'$ as

$$a' := \begin{cases} a^{\ulcorner}, & \text{if } a \notin \mathrm{ran}^{\ulcorner} \\ 1 - a, & \text{if } a \in \mathrm{ran}^{\ulcorner}. \end{cases} \tag{1.3}$$

Obviously, the operation $'$ satisfies Equation (1.1). Moreover, a' is idempotent for every $a \in R$, because so is $a^{\ulcorner}$ and therefore also $1 - a^{\ulcorner}$. Hence, the operation $'$ is indeed a right focal operation.

Given a Rickart ring R, we define a *matching pair* of focal operations on R to be a pair $({}^{\backprime}, {}')$, where $^{\backprime}$ is a left focal operation and $'$ is a right focal operation, such that

$$a^{\backprime\prime} = a^{\backprime\backprime} \text{ and } a^{\prime\backprime} = a''. \tag{1.4}$$

Given a pair of normal focal operations $({}^{\urcorner}, {}^{\ulcorner})$ on a Rickart ring, we can always define a matching pair $({}^{\backprime}, {}')$ of normal focal operations by

$$a^{\backprime} := \begin{cases} a^{\urcorner}, & \text{if } a \notin \mathrm{ran}^{\urcorner} \cup \mathrm{ran}^{\ulcorner}, \\ 1 - a, & \text{if } a \in \mathrm{ran}^{\urcorner} \cup \mathrm{ran}^{\ulcorner}, \end{cases} \tag{1.5}$$

$$a' := \begin{cases} a^{\ulcorner}, & \text{if } a \notin \mathrm{ran}^{\urcorner} \cup \mathrm{ran}^{\ulcorner}, \\ 1 - a, & \text{if } a \in \mathrm{ran}^{\urcorner} \cup \mathrm{ran}^{\ulcorner}. \end{cases} \tag{1.6}$$

It was noted in [Cīrulis, 2016] that, if a left-right-focal Rickart ring has a matching pair of normal focal operations, then the ranges of these focal operations coincide.

Left and right idempotents

The notion of left and right idempotents, which also goes back to Maeda [Maeda, 1960], is particularly useful in unitary Rickart rings. However, left and right idempotents can be defined in arbitrary rings.

Definition 1.1.5. [Maeda, 1960] Let R be a ring and let $a \in R$. An idempotent $e \in R$ is called a *right idempotent of* a if, for all $x \in R$,

$$ax = 0 \text{ iff } ex = 0. \tag{1.7}$$

It is called *left idempotent of a* if, for all $x \in R$,

$$xa = 0 \text{ iff } xe = 0. \tag{1.8}$$

We denote the set of all right idempotents of an element a by $\mathrm{RI}(a)$ and the set of all left idempotents of a by $\mathrm{LI}(a)$.

Maeda chose the adjective *right* (*left*) because the right (left) idempotents of a are the idempotents whose right (left) annihilator coincides with the right (left) annihilator of a. Calling the idempotents satisfying Equation (1.7) *right* idempotents and the ones satisfying Equation (1.8) *left* idempotents also corresponds to the fact that their existence is equivalent to the ring being *right* or *left* Rickart, respectively (see Remark 1.1.7).

In [Cīrulis and Cremer, 2018], the right (left) idempotents are called "right (left) focal idempotents". This term in not used in the present thesis in order to avoid confusion with focal operations. The sets $\mathrm{RI}(a)$ and $\mathrm{LI}(a)$ are accordingly denoted $\mathrm{RFI}(a)$ and $\mathrm{LFI}(a)$ in that article.

In [Marovt et al., 2015] and [Krēmere, 2016], the corresponding notation is $\mathrm{RP}(a)$ for the set of right idempotents and $\mathrm{LP}(a)$ for the set of left idempotents of an element a. Both articles define these sets only in Rickart *-rings. Note that Berberian [Berberian, 1972] uses $\mathrm{LP}(a)$ and $\mathrm{RP}(a)$ in a different meaning. His $\mathrm{LP}(a)$ is our $a^{\backslash\backslash}$ and his $\mathrm{RP}(a)$ is our a'', where the right focal operations are the ones which map a to the unique self-adjoint among its left/right idempotents.

The following proposition states a trivial but useful property of the left/right idempotents of an element.

Proposition 1.1.6. *Let R be a unitary ring and let $e, a \in R$. If e is a left idempotent of a, then $ea = a$. If e is a right idempotent of a, then $ae = a$.*

Proof. Suppose e is a left idempotent of a. Since e is idempotent, we have $(1 - e)e = 0$. Therefore, Equation (1.8) yields $(1 - e)a = 0$, so $ea = a$. ☐

Remark 1.1.7. Note that an element e of a unitary ring is idempotent if and only if the element $1 - e$ is idempotent (because, if e is idempotent, then $(1 - e)^2 = 1 - e - e + e = 1 - e$). Moreover, given an idempotent e and an arbitrary element x of a ring, obviously $ex = x$ if and only if $(1 - e)x = 0$. So, if for some element a of a ring there exists an idempotent a' satisfying Equation (1.1), then the element $1 - a'$ is a right idempotent. Conversely, if an element a of a ring has a right idempotent f, then the element $1 - f$ is a possible choice for a'. Hence, a unitary ring R is a right Rickart ring if and only if, for every $a \in R$, there exists a right idempotent of a. Similarly, a unitary ring R is a left Rickart ring if and only if every element of R has a left idempotent. This characterization was given as an alternative definition of unitary Rickart rings by Maeda [Maeda, 1960]. ◁

Remark 1.1.8. Note that, since $1 - a'$ is always a right idempotent of a, normality of a right focal operation $'$ implies that $a'' \in \mathrm{RI}(a)$ for every a. ◁

1.1.2 Examples of Rickart rings

Reduced Rickart rings

The most simple examples of Rickart rings are Boolean rings and domains.

Example 1.1.9. Let R be a Boolean ring. With $^{\perp}$ denoting the Boolean complement operation, it is easy to check that the operation $'$ defined on R as $a' := 1 - a^{\perp}$ is both a right focal operation and a left focal operation. Hence, R is a Rickart ring. ◁

Example 1.1.10. Let R be a domain (i.e., a ring without zero divisors). The operation $'$ defined on R as

$$a' := \begin{cases} 0, \text{ if } a \neq 0, \\ 1, \text{ if } a = 0 \end{cases}$$

is both a right focal operation and a left focal operation. Hence, R is a Rickart ring. ◁

We can generalize Example 1.1.10 in the following way.

Example 1.1.11. Let R be a direct product of domains D_i (with indexes i from some arbitrary index set). It can be shown that the operation $'$ on R which maps a to a' with

$$(a')_i = \begin{cases} 0, \text{ if } a_i \neq 0, \\ 1, \text{ if } a_i = 0 \end{cases}$$

is both a right focal operation and a left focal operation. Hence, R is a Rickart ring. ◁

We introduce a well-known (see, for example, [Rotman, 2002]) notion.

Definition 1.1.12. A ring R is called *reduced* if it has no non-zero nilpotent elements, that is, for every $a \in R$ and for every $n \in \mathbb{N}$, $a^n = 0$ implies $a = 0$.

The rings from Examples 1.1.9 to 1.1.11 have in common that they are reduced.

The following proposition collects some well-known useful facts about reduced rings.

Proposition 1.1.13. (a) *A ring R is reduced if and only if $a^2 = 0$ implies $a = 0$ for every $a \in R$.*

(b) [Abian, 1975] *Every reduced ring is commutative at zero (that is, $ab = 0$ if and only if $ba = 0$).*

(c) [Chacron, 1971] *In a reduced ring, all idempotents are central (i.e., $ae = ea$ for every idempotent e and every arbitrary element a).*

It is known from [Endo, 1960] that a reduced ring is a right Rickart ring if and only if it is a left Rickart ring. Hence, every reduced one-sided Rickart ring is a Rickart ring. It was noted in [Janowitz, 1976] that a reduced Rickart ring has a unique right focal operation. Of course, the dual statement (a reduced Rickart ring has a unique left focal operation) also holds. In [Cīrulis and Cremer, 2018], it was noted that the right focal operation on a reduced Rickart ring coincides with the left focal operation (the converse also holds: if a ring admits an operation which is both right focal and left focal, then it is a reduced Rickart ring, because $a^2 = 0$ implies $a'a = a$, whence $0 = a`a = a'a = a$). Therefore, when dealing with a reduced Rickart ring R, we just refer to this unique both-sided focal operation as *the focal operation* of R.

The uniqueness of the focal operation on a reduced Rickart ring has the following consequence for idempotents.

Proposition 1.1.14. [Cīrulis and Cremer, 2018, page 381] *Let R be a reduced Rickart ring and let $'$ be its focal operation. If $e \in R$ is idempotent, then $e' = 1 - e$, and $e'' = e$.*

It is also known (see [Huh et al., 2002]) that the class of reduced Rickart rings includes the class of commutative Rickart rings.

In [Cīrulis and Cremer, 2018], it was proved that a ring is a reduced Rickart ring if and only if it is isomorphic to a particular subdirect product of domains called a *Sussman ring* in that paper (and called *associate ring* in Sussman's paper [Sussman, 1958]).

In [Cīrulis and Cremer, 2018, Theorem 6.5], reduced Rickart rings are characterised using another unary operation which is closely related to the focal operation. Unfortunately, there is a mistake in that theorem. The correct version of it (see [Cīrulis and Cremer, 2020]) is as follows:

Lemma 1.1.15. *A ring with unity is a reduced Rickart ring if and only if it admits a unary operation* $^\circ$ *such that*

(a) $xx^\circ = x = x^\circ x$,

(b) $(xy)^\circ = x^\circ y^\circ$,

(c) $0^\circ = 0$.

In this case, the focal operation of the ring is given by $x' := 1 - x^\circ$.

Note that x° is always an idempotent, because $x^\circ = 1 - x' = x''$ by Proposition 1.1.14.

Matrix rings

Another source of not too complicated examples of Rickart rings are matrix rings. Since matrices over a field can be regarded as linear mappings and vice versa, we will use matrix and linear mapping notation and language interchangeably, according to what is more convenient in the context. So in this thesis, we do not distinguish between, for example, the matrix ring $\mathrm{Mat}_n(\mathbb{R})$ of $n \times n$ real square matrices and the ring $\mathcal{B}(\mathbb{R}^n)$ of linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Example 1.1.16. Let $R = \mathcal{B}(\mathbb{C}^n)$ be the ring of linear mappings from $\mathbb{C}^n$ to $\mathbb{C}^n$ (with multiplication defined as $(AB)(v) = A(B(v))$ for $v \in \mathbb{C}^n$). The idempotents of this ring are the projections onto subspaces of $\mathbb{C}^n$.

Let $A : \mathbb{C}^n \to \mathbb{C}^n$ be an element of $\mathcal{B}(\mathbb{C}^n)$. The right idempotents of A are the projections which have the same kernel as A. The left idempotents of A are the projections onto the image of A. Since such projections obviously exist for every linear mapping A, the ring $\mathcal{B}(\mathbb{C}^n)$ is a Rickart ring (see Remark 1.1.7).

There are various possible right focal operations. The map A' is some projection onto $\ker A$ (the kernel of A). The map $A^{`}$ is some projection onto $(\mathrm{im} A)^\perp$ (the orthogonal complement of the image of $A$). Hence, $\mathcal{B}(\mathbb{C}^n)$ is a Rickart ring. Since there are many projections onto a given subspace, there are many possible choices for $A'$ and $A^{`}$. So there are many distinct left-right-focal Rickart rings whose ring reduct is the ring $\mathcal{B}(\mathbb{R}^n)$.

By assuming normality of the right focal operations, we ensure that the kernel of A'' coincides with the kernel of A (note that $\ker(1 - A') = \ker A$ for every linear mapping A). ◁

Of course, in Example 1.1.16 we could replace the field $\mathbb{C}$ by $\mathbb{R}$ or any other field. So we can give a simple (two-dimensional) concrete example which illustrates Example 1.1.16 with some real numbers.

Example 1.1.17. Consider the ring $\mathcal{B}(\mathbb{R}^2)$ and let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}. \tag{1.9}$$

The kernel of this linear mapping is the subspace generated by $\left(\begin{smallmatrix} -2 \\ 1 \end{smallmatrix}\right)$. Since the right idempotents of A are projections with the same kernel, we can compute that

$$\mathrm{RI}(A) = \left\{ \begin{pmatrix} 1 - 2\eta & 2 - 4\eta \\ \eta & 2\eta \end{pmatrix} \,\middle|\, \eta \in \mathbb{R} \right\}. \tag{1.10}$$

The possible choices for A' are the projections onto the subspace $\{\left(\begin{smallmatrix} -2k \\ k \end{smallmatrix}\right) \mid k \in \mathbb{R}\}$. They are given by

$$A' = \begin{pmatrix} -2\lambda & -4\lambda - 2 \\ \lambda & 2\lambda + 1 \end{pmatrix} \tag{1.11}$$

for any real number λ.

The kernel of A' is then the subspace generated by the vector $\left(\begin{smallmatrix} 2\lambda+1 \\ -\lambda \end{smallmatrix}\right)$. Since A'' must be a projection onto the kernel of A', we can compute that, if $\lambda \neq -\frac{1}{2}$,

$$A'' = \begin{pmatrix} 1+\lambda\mu & (2\lambda+1)\mu \\ -\frac{\lambda(1+\lambda\mu)}{2\lambda+1} & -\lambda\mu \end{pmatrix}, \tag{1.12}$$

where μ is some real number (if $\lambda = -\frac{1}{2}$, then $A'' = \left(\begin{smallmatrix} 0 & 0 \\ \lambda(1-\mu) & -\lambda\mu \end{smallmatrix}\right)$ for some real number μ). If the right focal operation is normal, then

$$A'' = \begin{pmatrix} 1+2\lambda & 4\lambda+2 \\ -\lambda & -2\lambda \end{pmatrix}. \tag{1.13}$$

◁

So far all examples we have seen were both-sided Rickart rings. The following example shows a ring which is only right Rickart, but not left Rickart.

Example 1.1.18. [Cīrulis and Cremer, 2018] Consider the subring of $\mathrm{Mat}_2(\mathbb{R})$

$$R = \left\{ \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} \,\middle|\, a, b \in \mathbb{Z} \right\}. \tag{1.14}$$

First, note that this ring has no right unity, because, for example,

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{1.15}$$

for all $a, b \in \mathbb{Z}$. It does have left unities, because, for all $x, y \in \mathbb{Z}$,

$$\begin{pmatrix} 0 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix}. \tag{1.16}$$

It is well-known and easy to check that the set of idempotents of the matrix ring $\mathrm{Mat}_2(\mathbb{R})$ is the set $\mathcal{E} = \left\{ \left(\begin{smallmatrix} a & b \\ c & 1-a \end{smallmatrix}\right) \,\middle|\, a^2 + bc = a \right\} \cup \{ \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \} \cup \{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \}$. Hence, the non-zero idempotents of the subring R are of the form

$$\begin{pmatrix} 0 & 0 \\ c & 1 \end{pmatrix}. \tag{1.17}$$

Thus, every non-zero idempotent is a left unity.

We can define a right focal operation on R in the following way.

First, for an element A of R of the form $A = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \in R$, we have $\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ for all $x, y \in \mathbb{Z}$. Hence, A' has to be an idempotent such that $A'X = X$ for all $X \in R$. In other words, A' has to be a left unity, and as we just saw, there are plenty of those. We can just choose any non-zero idempotent from R.

Second, for an element B of R of the form $B = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}$ with $b \neq 0$, we have $\begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ if and only if $x = y = 0$. Hence, B' has to be an idempotent such that $B'X = X$ if and only if $X = 0$ (where 0 denotes the zero matrix $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$). Now the only possible choice is $B' := 0$. Together with the first step, this shows that there exist right focal operations on R. Thus R is a right Rickart ring.

However, if there was a left focal operation on R, then $0'$ would have to be an idempotent such that $X \cdot 0^{\backprime} = X$ for all $X \in R$, since $X \cdot 0 = 0$ for all $X \in R$. In other words, $0^{\backprime}$ would have to be a right unity. But we noted in the beginning of this example that there are no right unities in the ring R. Hence, we can not define a left focal operation, and thus R is not left Rickart. ◁

The argument in the end of Example 1.1.18 works in every one-sided Rickart ring which does not have a unity: The element $0'$ would have to be a left unity and the element $0^{`}$ would have to be a right unity, but if the ring has both a left and a right unity, then it has a unity. Hence, a non-unitary ring can not be a both-sided Rickart ring. In other words, every both-sided Rickart ring is unitary.

The focus of this thesis is on unitary (one-sided or both-sided Rickart) rings, because it is very convenient to have a normal focal operation, which requires a unity.

The ring of bounded linear operators on a Hilbert space

The motivating example for studying Rickart rings (see [Maeda, 1960]) is a generalization of Example 1.1.16 to infinite dimensions. In a finite dimensional Hilbert space, every subspace is closed. In the infinite case this is generally not true. In particular, the image of a bounded linear operator A on a Hilbert space may be non-closed, but if A is a projection, then its image is closed. The kernel, however, is closed for every bounded linear operator. The orthogonal complement of a subspace is also always closed.

The following example can partly be found in [Berberian, 1972, Chapter 1, §3, Proposition 5]. Although terminology like "right idempotents" or "focal operations" is not used there, it is clear that the operations RP and LP are actually *-focal operations (see Definition 1.2.5), which is why in that book the elements $\mathrm{RP}(a)$ and $\mathrm{LP}(a)$ are uniquely determined for a given a, contrary to our focal operations.

Example 1.1.19. Let $\mathcal{B}(\mathcal{H})$ be the ring of bounded linear operators on a Hilbert space $\mathcal{H}$ (with multiplication defined as $(AB)(v) = A(B(v))$ for $v \in \mathcal{H}$).

The idempotents of this ring are the projections onto closed subspaces of $\mathcal{H}$.

Let $A : \mathcal{H} \to \mathcal{H}$ be an element of $\mathcal{B}(\mathcal{H})$. The right idempotents of A are the projections which have the same kernel as A. The left idempotents of A are the projections onto $\overline{\operatorname{im} A}$, the closure of the image of A.

As in Example 1.1.16, since there are many projections onto a given subspace, there are various possible focal operations. For $A \in \mathcal{B}(\mathcal{H})$, the map A' is some projection onto $\ker A$ (the kernel of A). The map $A^{`}$ is some projection onto $(\operatorname{im} A)^{\perp}$ (the orthogonal complement of the image of A). Hence, $\mathcal{B}(\mathcal{H})$ is a Rickart ring. ◁

1.2 Rickart *-rings

We now turn our attention to a class of Rickart rings equipped with an involution (i.e., a unary operation which interacts with the multiplication in a particular way). The section is divided into two subsections.

Section 1.2.1 recalls the basic notions. We revisit the basics of involutive semigroups and *-rings in general, which enables us to then state the definition of a Rickart *-ring. We also introduce the notion of **-focal operation*, which is a particular focal operation on a Rickart *-ring, and state some basic properties of Rickart *-rings and their *-focal operations.

In Section 1.2.2, we provide examples of Rickart *-rings, most of which are based on examples of Rickart rings discussed in Section 1.1.2.

1.2.1 Basic notions

*-rings, self-adjoints, projections and Rickart *-rings

We begin by reminding the following well-known notion from semigroups and its adaptation for rings.

Definition 1.2.1. A (*semigroup*) *involution* on a semigroup S is a bijection $^*: S \to S$ such that $(x^*)^* = x$ and $(xy)^* = y^*x^*$ for all $x, y \in S$.

An (*ring*) *involution* on a ring R is a unary operation which is a semigroup involution on the semigroup reduct of R and additionally satisfies $(x+y)^* = x^* + y^*$ for all $x, y \in R$. A ring with an involution is called **-ring* or *involutory ring*.

Remark 1.2.2. If a semigroup with involution has the zero element 0, then $0^* = 0$ (because $0^* = (0x)^* = x^*0^*$ and $0^* = (x \cdot 0)^* = 0^*x^*$, so 0^* is also a zero element, thus it is zero).

If a semigroup with involution has the identity element 1, then $1^* = 1$ (because, for arbitrary $a \in S$, there is $x \in S$ with $x^* = a$, and then $1^*a = 1^*x^* = (x \cdot 1)^* = x^* = a$, and similarly $a1^* = a$).

In a *-ring R, we have moreover $(-a)^* = -a^*$ for every element $a \in R$, because $a^* + (-a)^* = (a-a)^* = 0^* = 0$. ◁

On a Rickart *-ring, we are particularly interested in the idempotents which are mapped to themselves by the involution, because they play the same role which is played by plain idempotents in a "plain" Rickart ring. We therefore recall the corresponding terminology in the following definition.

Definition 1.2.3. [Berberian, 1972, Chapter 1, § 1, Definition 2] An element a of a *-ring is called *self-adjoint* if $a^* = a$. A *projection* is an element which is both self-adjoint and idempotent. The set of projections is usually denoted by P.

Unfortunately, we have to use the word *projection* with two different meanings depending on the context. In linear algebra and functional analysis, a projection is a mapping that coincides with the identity mapping on its image. We will see in Example 1.2.8 that this meaning does not correspond to Definition 1.2.3.

We can now define the structure which is the topic of this section.

Definition 1.2.4. [Janowitz, 1976] A *-ring is called *Rickart *-ring* if, for every $a \in R$, there exists a projection p such that, for all $x \in R$, $ax = 0$ if and only if $px = x$.

*-focal operations

By Definition 1.2.4, a Rickart *-ring is a right Rickart ring R which admits a right focal operation $'$ such that a' is self-adjoint for every $a \in R$.

Definition 1.2.5. We call a right focal operation $'$ on a *-ring R with the property that a' is self-adjoint for every $a \in R$ a *right *-focal operation*. A *left *-focal operation* is defined dually.

Let us see that a Rickart *-ring is automatically also left Rickart. By Definition 1.2.1, $xa = 0$ if and only if $a^*x^* = 0$ (recall that $0^* = 0$). Hence, $xa = 0$ if and only if $(a^*)'x^* = x^*$, where $'$ is the right *-focal operation. Using Definition 1.2.1 again, we obtain $xa = 0$ if and only if $x((a^*)')^* = x$. But $(a^*)'$ is self-adjoint. So $((a^*)')^* = (a^*)'$, which is obviously idempotent (even a projection).

In other words, a given right *-focal operation $'$ on a *-ring induces a left *-focal operation $`$ on the same ring by

$$a^{`} := (a^*)'. \tag{1.18}$$

The following propostion reformulates some useful well-known facts (see, for example, in [Berberian, 1972]) about Rickart *-rings.

Proposition 1.2.6. *The *-focal operations on a Rickart *-ring R are normal and unique (in the sense that, if $^{\backprime}$ and $^{\urcorner}$ are left *-focal operations on R, then they are the same operation, and similarly for the right *-focal operation).*

However, a Rickart *-ring may admit other focal operations which do not coincide with the *-focal operations (that is, they map at least one element on an idempotent which is not self-adjoint).

1.2.2 Examples of Rickart *-rings

Many of the examples of Rickart rings presented in Section 1.1.2 can be equipped with an involution that turns them into Rickart *-rings.

Commutative Rickart rings

Example 1.2.7. Let R be a commutative Rickart ring. It is easy to check that, by commutativity, $a^* := a$ for all $a \in R$ defines an involution on R (referred to as the *trivial involution*). With this involution every element is self-adjoint, and therefore every idempotent is a projection. Hence, we obtain a Rickart *-ring. The unique left-and-right focal operation of the ring R (recall that every commutative Rickart ring is reduced) is also a left-and-right *-focal operation.

In particular, every Boolean ring and every integral domain (i.e., commutative ring without zero divisors) is a Rickart *-ring with a trivial involution. ◁

If a commutative Rickart ring is equipped with a non-trivial involution, it is not necessarily a Rickart *-ring. For example, the ring $\mathbb{Z}_3 \times \mathbb{Z}_3$ (which is a Rickart ring because it is a direct product of domains, see Example 1.1.11) with the involution $(a, b)^* := (b, a)$ is not a Rickart *-ring (see [Patil and Waphare, 2017, Example 2.6]).

Matrix rings

Example 1.2.8. Recall Example 1.1.16 (the ring $\mathcal{B}(\mathbb{C}^n)$ of linear mappings from $\mathbb{C}^n$ to $\mathbb{C}^n$, aka the ring of $n \times n$ complex square matrirces). This ring admits an involution * which maps every matrix A to its conjugate transpose (or, in the language of linear mappings, every linear mapping A to its Hermitian adjoint).

As mentioned in Example 1.1.16, the idempotents of this ring are the projections (in the sense of linear algebra) onto subspaces of $\mathbb{C}^n$. This includes also *oblique* (i.e., non-orthogonal) projections. However, the projections in the sense of Definition 1.2.3 on this ring are the *orthogonal* projections onto subspaces of $\mathbb{C}^n$.

Among the many possible focal operations on $\mathcal{B}(\mathbb{C}^n)$, consider the right focal operation $'$ such that A' is the orthogonal projection onto $\ker A$ and the left focal operation $^{\backprime}$ such that $A^{\backprime}$ is the orthogonal projection onto $(\operatorname{im} A)^{\perp}$. Since A' and $A^{\backprime}$ are projections in the sense of Definition 1.2.3, these focal operations are *-focal operations. ◁

Example 1.2.9. The ring $\mathrm{Mat}_n(\mathbb{R})$ of $n \times n$ real matrices is also a Rickar *-ring in the same way as the ring $\mathcal{B}(\mathbb{C}^n)$. The involution in the real case is just taking the transpose matrix, and the self-adjoints are the symmetric matrices. The focal operations are the same as in Example 1.2.8 ◁

Example 1.2.10. Recall Example 1.1.17, where we calculated the possible choices for A' and A'' for

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}. \tag{1.19}$$

So instead of taking as A' any arbitrary projection onto the kernel of A (i.e., the subspace $\{\left(\begin{smallmatrix}-2k\\k\end{smallmatrix}\right) \mid k \in \mathbb{R}\}$), we chose the orthogonal projection onto it.

$$A' = \frac{1}{5}\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}. \tag{1.20}$$

The mapping A'' is then the orthogonal projection onto $(\ker A)^{\perp}$:

$$A'' = \frac{1}{5}\begin{pmatrix}4 & -2\\ -2 & 1\end{pmatrix}. \tag{1.21}$$

◁

The *-ring of bounded linear operators on a Hilbert space

Example 1.2.11. Recall Example 1.1.19. Just as in the finite dimensional case (see Example 1.2.8), we can define an involution by defining A^* to be the Hermitian adjoint operator of A. The projections in the sense of Definition 1.2.3 are the orthogonal projections onto closed subspaces of $\mathcal{H}$.

We choose the focal operations such that $A^{\backprime}$ is the orthogonal projection onto $(\operatorname{im} A)^{\perp}$ and A' is the orthogonal projections onto $\ker A$. These are *-focal operations, and thus $\mathcal{B}(\mathcal{H})$ is a Rickart *-ring. ◁

1.3 Strongness conditions for Rickart rings and one-sided Rickart rings

In this section, we finally introduce the types of Rickart rings which we will deal with in Part II.

The term *strong* was introduced by Cīrulis in [Cīrulis, 2015d] to refer to a right focal right Rickart ring satisfying some additional property. The same word is also used in [Cīrulis, 2016], but in the context of both-sided Rickart rings satisfying a similar property. We shall introduce more precise terms that allow to distinguish between these cases.

The motivation for introducing notions of strongness was to create a star-free generalization of Rickart *-rings. That is, a strongness condition should ensure that a (one-sided or both-sided) Rickart ring satisfying it behaves like a Rickart *-ring (but without requiring an involution).

Section 1.3.1 introduces the strongness conditions which give rise to different types of (possibly one-sided) Rickart rings. Section 1.3.2 confirms that indeed any Rickart *-ring satisfies a strongness condition, and also mentions reduced Rickart ring as another example of strong Rickart rings.

1.3.1 Basic notions

In this subsection, we first recall the star-free version of projections, which then enables us to introduce different strongness conditions: a one-sided one, which gives rise to the so-called *right-strong Rickart rings* and the admittedly clumsily named *right-strong right Rickart rings*, and a both-sided one, which gives rise to *strong Rickart rings.* Then, we review the properties of the focal operations in these rings.

Closed idempotents

In order to create a star-free generalization of Rickart *-rings, we start with a star-free generalization of projections (in the sense of Definition 1.2.3, i.e., self-adjoint idempotents). In a Rickart *-ring, the projections play an important role, but their definition is based on the involution. However, the set of projections in a Rickart *-ring is the range of the (left or right) *-focal operations. This inspires the following definition, which is based on a term used in [Foulis, 1960] and [Foulis, 1963] that has recently been revived in [Cīrulis, 2015d].

Recall that, for a matching pair of normal focal operations, the ranges of the focal operations coincide.

Definition 1.3.1. (a) Let R be a right-focal right Rickart ring (left-focal left Rickart ring). An idempotent $e \in R$ is called *closed* if it is in the range of the right focal operation (left focal operation) (i.e., $e = a'$ for some $a \in R$). We will denote the range of the right focal operation (left focal operation) by P_{r}.

(b) Let R be a left-right-focal Rickart ring with a pair of normal focal operations such that

$$a^{\backprime\prime} = a^{\backprime\backprime} \text{ and } a^{\prime\backprime} = a^{\prime\prime}. \tag{1.22}$$

An idempotent $e \in R$ is called *closed* if it is in the range of the focal operations. The range of the focal operations is denoted by P_{rl}.

When dealing with normal focal operations, we have the following alternative characterization of closed idempotents.

Remark 1.3.2. In a right-focal right Rickart ring R whose right focal operation is normal (and also in a left-right-focal Rickart ring with a matching pair of focal operations), an element $p \in R$ is a closed idempotent if and only if $p'' = p$ (see [Cīrulis and Cremer, 2022]). ◁

Right-strong right Rickart rings and right-strong Rickart rings

The following definition formulates the strongness condition which was introduced in [Cīrulis, 2015d].

Definition 1.3.3. A *right-strong right Rickart ring* is a right-focal right Rickart ring R such that

(a) the right focal operation is normal,

(b) for all $p, q \in P_{\mathrm{r}}$,

$$pq \in P_{\mathrm{r}} \text{ iff } pq = qp. \tag{1.23}$$

A *right-strong Rickart ring* is a right-focal Rickart ring (i.e., a both-sided Rickart ring equipped with a right focal operation) satisfying the same conditions.

We can define *left-strong left Rickart rings* and *left-strong Rickart rings* dually. Of course, every right-strong Rickart ring is also a right-strong right Rickart ring, because every right-focal Rickart ring is also a right-focal right Rickart ring. Similarly, every left-strong Rickart ring is also a left-strong left Rickart ring.

Strong Rickart rings

The right-strong right Rickart rings from Definition 1.3.3 were called "strong Rickart rings" in [Cīrulis, 2015d]. However, the same term was used with a different meaning in [Cīrulis, 2016]. In this thesis, we use the term exclusively in the sense of [Cīrulis, 2016].

Definition 1.3.4. [Cīrulis, 2016] A *strong Rickart ring* is a left-right-focal Rickart ring such that

(a) both focal operations are normal and they satisfy Equation (1.22) (i.e., $a^{\backprime\prime} = a^{\backprime\backprime}$ and $a^{\prime\backprime} = a^{\prime\prime}$),

(b) for all $p, q \in P_{\mathrm{rl}}$,

$$pq \in P_{\mathrm{rl}} \text{ iff } pq = qp. \tag{1.24}$$

Obviously, every strong Rickart ring can be seen as both a right-strong Rickart ring and a left-strong Rickart ring by ”forgetting” one of the focal operations.

Properties of the focal operations

The following proposition provides a list of properties of the right focal operation of a right-strong right Rickart ring, most of which can be found in [Cīrulis, 2015d]. Note that not all items require strongness: item (a) holds in every right-focal right Rickart ring, and items (b) and (e) hold in every right-focal right Rickart ring whose right focal operation is normal.

Proposition 1.3.5. *Let R be a right-strong right Rickart ring. Then for all $a, b \in R$,*
(a) $aa' = 0$,
(b) $aa'' = a$,
(c) $(ab)'' \leq_E b''$ *under the standard order of idempotents, i.e.,* $(ab)''b'' = (ab)'' = b''(ab)''$,
(d) $(ab)'' = (a''b)''$.
(e) $0'' = 0$.

Proof. Items (a) to (d) are proved in [Cīrulis, 2015d, Proposition 3.2].
By Equation (1.1), $0x = 0$ if and only if $0'x = x$. Since $0x = 0$ for every $x \in R$, this means that $0'$ is a left identity. Hence, $0' = 1$, because we assumed in the beginning of this section that our rings have the multiplicative identity. By normality, this yields $0'' = 0$. ☐

The dual statement for left-strong left Rickart rings is the following (as in Proposition 1.3.5, not all items require strongness: item (a) holds in every left-focal left Rickart ring, items (b) and (e) hold if the left focal operation is normal).

Proposition 1.3.6. *Let R be a left-strong left Rickart ring. Then for all $a, b \in R$,*
(a) $a^{\backprime}a = 0$,
(b) $a^{\backprime\backprime}a = a$,
(c) $(ab)^{\backprime\backprime} \leq_E a^{\backprime\backprime}$ *under the standard order of idempotents, i.e.,* $(ab)^{\backprime\backprime}a^{\backprime\backprime} = (ab)^{\backprime\backprime} = a^{\backprime\backprime}(ab)^{\backprime\backprime}$,
(d) $(ab)^{\backprime\backprime} = (ab^{\backprime\backprime})^{\backprime\backprime}$.
(e) $0^{\backprime\backprime} = 0$.

Proof. Analogous to the previous Proposition. ☐

On a reduced Rickart ring, we have a property which (by normality of the focal operation on a reduced Rickart ring) is stronger that Proposition 1.3.5(d) and Proposition 1.3.6(d).

Proposition 1.3.7. [Cīrulis and Cremer, 2018, Proposition 3.8(g)] *Let R be a reduced Rickart ring and let $'$ be its (unique, see Section 1.1.2) focal operation. Then $(ab)'' = a''b''$ for all $a, b \in R$.*

1.3.2 Examples of strong Rickart rings

In this subsection, we review two subclasses of strong Rickart rings: Reduced Rickart rings and Rickart *-rings.

Example 1.3.8. Let R be a reduced Rickart ring and let E be its set of idempotents.
We already mentioned that R has a unique (left and right) focal operation $'$. This focal operation must be normal, because otherwise there would be another focal operation distinct from $'$ (because we can always construct a normal focal operation from a given one, as in Equation (1.3)). Similarly, the pair $(', ')$ must satisfy Equation (1.22), because otherwise we could construct another pair of focal operations distinct from $'$ as in Equation (1.5). So Definition 1.3.4(a) is satisfied.

If $e \in R$ is idempotent, then $ex = 0$ if and only $(1-e)x = x$. This makes $1-e$ a possible choice for e' (see Equation (1.1)), and thus, by uniqueness of the focal operation, $e' = 1-e$. Obviously, $1-e$ is also idempotent, and $(1-e)' = e$. Hence, every idempotent is closed. So $P_{\mathrm{r}} = E$.

Since the idempotents of R are central (by Proposition 1.1.13), they commute with each other. Therefore, the product of idempotents e and f is also an idempotent (since $(ef) \cdot (ef) = eeff = ef$).

In other words, we have $ef \in P_{\mathrm{r}}$ for all $e, f \in P_{\mathrm{r}}$, and we also have $ef = fe$ for all $e, f \in P_{\mathrm{r}}$. So obviously Definition 1.3.4(b) is also satisfied. We conclude that every reduced Rickart ring equipped with (two copies of) its unique focal operation is a strong Rickart ring. ◁

Example 1.3.9. [Cīrulis, 2016] Let R be a Rickart *-ring and let $^{\backprime}$ and $'$ be its left and right *-focal operations, which are known to be normal by Proposition 1.2.6.

For every $a \in R$, we have $a^{\backprime} = (a^*)'$ (see Equation (1.18)) and thus also $a' = ((a^*)^*)' = (a^*)^{\backprime}$. Hence, $a^{\backprime\prime} = (a^*)''$, but also $a^{\backprime\backprime} = (((a^*)')^*)' = (a^*)''$, because $(a^*)'$ is self-adjoint (since it is a projection). So $a^{\backprime\prime} = a^{\backprime\backprime}$. Similarly, $a^{\prime\backprime} = ((a')^*)' = a''$, because a' is self-adjoint. So the *-focal operations satisfy Equation (1.22). Hence, Definition 1.3.4(a) is satisfied.

It is clear from Definition 1.2.5 that every closed idempotent is a projection. Conversely, if p is a projection, then $1-p$ is also a projection (it is obviously idempotent, and $(1-p)^* = 1^* - p^* = 1-p$ by Remark 1.2.2 and since p is self-adjoint). So $1-p$ is a possible choice for p', because $px = 0$ if and only if $(1-p)x = x$. Now uniqueness of the *-focal operations (Proposition 1.2.6) yields that $p = (1-p)'$. So every projection is a closed idempotent.

Now suppose p, q are closed idempotents, i.e., projections. If $pq = qp$, then pq is also a projection, because $(pq) \cdot (pq) = ppqq = pq$ and $(pq)^* = q^*p^* = qp = pq$. Conversely, if pq is a projection, then $qp = q^*p^* = (pq)^* = pq$. So pq is a closed idempotent if and only if p and q commute. So by Definition 1.3.4 we conclude that (the star-free reduct of) every Rickart *-ring equipped with its *-focal operations is a strong Rickart ring. ◁

Since every Rickart *-ring can be seen as a strong Rickart ring, Example 1.2.11 yields the following less abstract example.

Example 1.3.10. The ring $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$ with focal operations taken as in Example 1.2.11 is a strong Rickart ring. ◁

Chapter 2

Partial orders on Rickart rings

This is the last preliminary chapter before the exposure of research results which starts in Chapter 3. After introducing the different types of one-sided and both-sided Rickart rings satisfying a strongness condition in Chapter 1, we are now sufficiently equipped for getting acquainted with the partial orders which we shall investigate in Chapters 3 to 6: the star order, the one-sided star orders, the diamond order, the sharp order and the Abian order. These are all related to each other in some way and share certain properties, and with exception of the Abian order they all can be defined on a (sometimes even one-sided) Rickart ring satisfying a suitable strongness condition.

Among these partial orders, the one which in this chapter takes up more pages than any other is the star order – a very well-known partial order whose properties in different semigroup or ring settings, in particular on matrices, have been extensively investigated (see, for example, [Mitra et al., 2010, Chapter 5]). We shall focus on those of its properties which are in some way connected to the research topics which we will explore in Part II, providing not only the necessary preliminaries for studying the star order, but also some context for exploring other partial orders akin to it. A few of the known results stated in this chapter for the star order on Rickart *-rings will later be generalized to strong Rickart rings in Part II, others will be transferred to related partial orders. In other words, since the properties of the star order are already well-known, we often use them mostly as a motivation or inspiration for finding corresponding properties of the related partial orders, and more rarely engage in investigating the star order itself.

Therefore, there is comparably much less content on the star order in Part II than in this introductory chapter. The most prominent partial order in the chapters containing research results is probably a generalization of the right star order. This partial order has its origin in matrix rings (see [Baksalary and Mitra, 1991]). Recently, it has been generalized to the ring of bounded linear operators on a Hilbert space in two different ways, giving rise to what we shall call the strong right star order and the weak right star order[1] (see [Deng and Wang, 2012] and [Dolinar et al., 2014] for the strong right star order and [Cīrulis, 2014a] and [Cīrulis, 2015c] for the weak right star order). This has been accompanied by generalizations to abstract Rickart *-rings (in [Marovt et al., 2015] and [Cīrulis, 2014a], respectively), and in the case of the weak right star order, even to right-strong right Rickart rings (in [Cīrulis, 2015d]). The generalization of the strong right star order to right-strong Rickart rings and the description of the structure of the resulting poset is one of the main results of the research presented in Part II (see Sections 3.1 and 4.2, Chapter 5, as well as the relevant part of Section 6.4.1).

Note: Starting from this chapter, it will be assumed that the reader is familiar with some other parts of algebra, namely a few order theoretic topics and a few semigroup theoretic ones. For convenience, all relevant definitions and results from these fields are collected in the Appendix: See Appendix A for partially ordered sets with additional structure (orthomodular posets, posets with an orthogonality relation, generalizations of orthocomplemented posets, posets with difference and weak BCK-algebras) and Appendix B for some basic topics from semigroup theory (inverse systems, strong semilattices, bands and D-semigroups). Just as with the content of normal chapters, we shall often refer to the definitions and results contained in these two appendix chapters. This ensures that, whenever it is assumed that the reader is familiar with some result or definition which is not included in the main text, there is still a precise reference and thus the clarity of the argumentation does not suffer. Therefore, references to these two appendices occur quite frequently.

A third appendix chapter, Appendix C, contains the definitions and some basic properties of a few well-known generalized inverses. Although they are referred a few times in this chapter, they are not important for understanding Part II and are included mainly to add some context to the partial orders treated in this chapter.

The structure of the chapter is as follows: Section 2.1 deals with the star order from its origin in involutive semigroups to its recent generalization to strong Rickart rings, summarizing the necessary preliminaries concerning this partial order. In Section 2.1.1, we will first mention the original definition of the star order from involutive semigroups. Then we will give the recent generalization of the star order to strong Rickart rings. In general, a strong Rickart ring with this partial order is not a lattice, and not even a semilattice. However, there are many results about meets and joins of elements satisfying certain conditions. We summarize first what is known on meets and joins of elements which are bounded from above (from [Cīrulis, 2016]) in Section 2.1.3. Then we remind different types of coherence (coherence, precoherence and Djikić coherence, see the Definitions 2.1.8, 2.1.10 and 4.3.7), as well as some of the known facts about the meet and/or join of coherent, precoherent or Djikić-coherent elements (from [Cīrulis, 2015a], [Djikić, 2016b] and [Djikić and Djordjević, 2016]) in Section 2.1.4. After that, we present some conditions for a Rickart *-ring to be a meet semilattice under the star order (see [Cīrulis, 2015a]) in Section 2.1.5. Finally, in Section 2.1.6, we describe the structure of initial segments of a strong Rickart ring under the star order in terms of orthogonality and sectional orthocomplementations (see [Cīrulis, 2016]).

Section 2.2 is devoted to the two known generalizations of the right star order. It starts by stating the original definition on matrices and both of its generalizations to more general structures in Section 2.2.1. Section 2.2.2 deals with the properties of the weak right star order on a right-strong right Rickart ring, while Section 2.2.3 provides an overview over the properties of the strong right star order on a Rickart *-ring. As for the star order, we focus on the properties which are most relevant for the research presented in this thesis: Conditions for existence of meets and joins for the weak right star order, as well as the structure of initial segments under both right star orders, stressing in particular the sectional orthocomplementations and orthogonality relations corresponding to these partial orders. For the weak right star order, we also mention its connection to the so-called *skew meet* operation introduced in [Cīrulis, 2015d].

Section 2.3 collects the preliminaries concerning a few more partial orders: the diamond order and its generalization to strong Rickart rings (Section 2.3.1), the sharp order and its generalization to general unitary rings (Section 2.3.2), as well as the Abian order (Section 2.3.3), which is a bit out of place, because its natural habitat is a reduced ring and it admits no meaningful generalization to Rickart rings (on a reduced Rickart ring, it actually coincides with all the other partial orders treated in this chapter).

[1]This is slightly non-standard terminology – both of them are often referred to as just the "right star order", because most papers are focussed on only one of them. Sometimes the term "weak right star order" is used if it is necessary to distinguish. However, since both of them are equally legitimate generalizations of the right star order, it

2.1 The star order

Each of the orders treated in this Chapter admits some star-free generalization. For the star order, the setting for this generalization is a strong Rickart ring. However, some properties of the star order have not been proved in this general setting. Therefore, in this section, after introducing both the original definition and its generalization to strong Rickart rings, we have to jump a bit between different types of rings: The relation to the space preorder (Section 2.1.2), the meets and joins in initial segments (Section 2.1.3) and the structure of initial segments (Section 2.1.6) have been described for strong Rickart rings, but difference types of coherence (Section 2.1.4), as well as the meet semilattice condition (Section 2.1.5), were studied only on Rickart *-rings (the generalization of notions of coherence and to strong Rickart rings and their application for finding conditions for the existence of joins was part of the research conducted for this thesis and can therefore be found in Part II, see Sections 4.3 and 4.5, while the generalization of the meet semilattice condition to strong Rickart rings can be achieved in exactly the same manner as meet semilattice conditions are obtained for other partial orders in Section 6.4).

2.1.1 Origin and generalization to strong Rickart rings

Origin (involutive semigroups)

The star order was first introduced in the very general setting of involutive semigroups, but it was extensively studied for the special cases of matrix rings and rings of bounded linear operators on a Hilbert space.

Definition 2.1.1. [Drazin, 1978] Let S be a proper *-semigroup (see Definition C.1.2) with the involution denoted by *. The *star order* (sometimes called *Drazin's order*) is the relation $\overset{*}{\leq}$ defined on S by $a \overset{*}{\leq} b$ if and only if

$$a^*b = a^*a = b^*a \text{ and } ab^* = aa^* = ba^*. \tag{2.1}$$

The relation $\overset{*}{\leq}$ was proved to be a partial order on a proper *-semigroup in [Drazin, 1978]. Note that Equation (2.1) is equivalent to a slightly shorter condition

$$a^*b = a^*a \text{ and } ab^* = aa^*. \tag{2.2}$$

Indeed, if $a^*b = a^*a$, then also $b^*a = a^*b^* = a^*a^* = a^*a$, and similarly, if $ab^* = aa^*$, then also $ba^* = aa^*$.

It was noted in [Hartwig, 1979] that on a *-ring, Equation (C.1) is equivalent to the *star cancellation law*

$$\text{if } a^*a = 0, \text{ then } a = 0. \tag{2.3}$$

It was further noted in [Ara and Menal, 1984] that every Rickart *-ring satisfies this law.

The following proposition provides star-free versions of both parts of Equation (2.2).

Proposition 2.1.2. [Cīrulis, 2015a, Lemma 3.2(a,c)], [Marovt et al., 2015, Lemmas 2.4(i,iii) and 2.5(i,iii)] *Let R be a Rickart *-ring and $a, b \in R$. Then $aa^* = ba^*$ if and only if $a = ba''$, and $a^*a = a^*b$ if and only if $a = {}''ab$.*

seems awkward and possibly even confusing to designate one of them as either "weak" or "strong", thereby creating the implicit impression that the other one is the standard or usual generalization.

Generalization to strong Rickart rings

Proposition 2.1.2 provides a star-free characterization of the star order in a Rickart *-ring which makes sense in every left-right-focal Rickart ring. Therefore, the following was chosen as a definition of the star order on a suitable left-right-focal Rickart ring in [Cīrulis, 2016].

Definition 2.1.3. [Cīrulis, 2016] Let R be a left-right-focal Rickart ring with normal focal operations that satisfy Equation (1.22). We define a relation $\overset{*}{\leq}$ on R as follows: $a \overset{*}{\leq} b$ for $a, b \in R$ if and only if

$$a^{\backprime\backprime}b = a = ba''. \tag{2.4}$$

The relation $\overset{*}{\leq}$ thus defined on a focal Rickart ring with suitable focal operations was proved to be a partial order on a strong Rickart ring in [Cīrulis, 2016], and of course, it is called *star order*. In general, focal operations on a Rickart ring are not unique (and it is not known whether a given Rickart ring can have different pairs of focal operations which make it a strong Rickart ring). Therefore, there might be various star orders on a given Rickart ring, but once we chose the focal operations and work on a particular strong Rickart ring, this problem disappears.

Note that, although Definition 2.1.3 does not require R to be strong, we only deal with the star order on a strong Rickart ring, because that is the setting in which it actually is a partial order.

Basic properties

The following proposition collects a few basic properties of the star order, including how it is related to the standard order of idempotents.

Proposition 2.1.4. [Cīrulis, 2016, Lemma 2.3] *Let R be a focal Rickart ring with matching normal focal operations. Then*

(a) 0 *is the bottom element of* $\langle R, \overset{*}{\leq}\rangle$,

(b) *on the set of closed idempotents* P_{rl}*, the star order* $\overset{*}{\leq}$ *coincides with the standard order of idempotents* $\leq_E$,

(c) $P_{\mathrm{rl}} = [0, 1]_{\overset{*}{\leq}}$,

(d) *if the meet of closed idempotents under the star order exists in* P_{rl}*, then it is also their meet in* R.

2.1.2 Relation to the space preorder

The star order $\overset{*}{\leq}$ on a strong Rickart ring was proved in [Cīrulis, 2016] to be a subrelation of the so-called weak space preorder. On a regular strong Rickart ring, the latter coincides with the usual space preorder.

Definition 2.1.5. The *space preorder* $\sqsubset$ (see, for example, [Rakic, 2012]) is a preorder defined on an arbitrary ring R as

$$a \sqsubset b \text{ if and only if } aR \subseteq bR \text{ and } Ra \subseteq Rb. \tag{2.5}$$

A weaker version of this relation was introduced in [Cīrulis, 2016] on a focal Rickart ring.

Definition 2.1.6. [Cīrulis, 2016] Let R be a focal Rickart ring. The relation $\sqsubset^w$ is defined on R as follows: For $a, b \in R$, $a \sqsubset^w b$ if and only if $b^{\backprime\backprime}a^{\backprime\backprime} = a^{\backprime\backprime}$ and $a''b'' = a''$ (equivalently, $b^{\backprime\backprime}a = a$ and $ab'' = a$). The relation $\sqsubset^w$ is called the *weak space preorder*.

It was proved in [Cīrulis, 2016] that, if $a \sqsubset b$, then also $a \sqsubset^w b$ (which justifies calling the latter relation *weak*). Moreover, it was noted in [Cīrulis, 2017, Theorem 3.1] that these preoreders coincide on any strong Rickart ring which is regular (see Definition C.2.3). On the ring of bounded linear operators $\mathcal{B}(\mathcal{H})$ on a Hilbert space, they are known from [Cīrulis, 2017, Example 3.3] to coincide on the set of operators whose ranges are closed (the range of a bounded linear operator is known to be closed if and only if it has the Moore-Penrose generalized inverse). So in particular, they coincide for all linear operators on a finite-dimensional Hilbert space.

On a strong Rickart ring, the weak space preorder can be characterized by

$$a \sqsubset^w b \text{ if and only if } a^{\backprime\backprime} \leq_E b^{\backprime\backprime} \text{ and } a'' \leq_E b''. \tag{2.6}$$

It was also shown in [Cīrulis, 2016] that $a \overset{*}{\leq} b$ if and only if $b^{\backprime\backprime}a = a^{\backprime\backprime}b$, $ab'' = ba''$ and $a \sqsubset^w b$.

2.1.3 Joins and meets of elements of a strong Rickart ring which are bounded from above

A strong Rickart ring (and even a Rickart *-ring) with the star order is, in general, not a lattice and not even a semilattice. Therefore, conditions for existence of meets and joins have been investigated.

For meets and joins of matrices, see [Mitra et al., 2010, Chapter 12.3].

Joins (the upper bound property)

Recall that a *-ring R is called **-regular* if every element has the Moore-Penrose generalized inverse (Definition C.2.4). It was proved in [Hartwig, 1979] that a *-regular ring has the upper bound property (a poset with the upper bound property was called *pseudo upper semilattice* in that paper).

Xu et al. [Xu et al., 2010] proved that the upper bound property holds on the ring $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$. It was even already stated in [Janowitz, 1983] that the upper bound property holds in general on every Rickart *-rings with the star order. However, Cīrulis [Cīrulis, 2015a] noticed a mistake in the proof, but fortunately he also provided a correction, showing that the statement itself is true. Finally, it was proved in [Cīrulis, 2016] that the upper bound property holds even for a strong Rickart ring.

Theorem 2.1.7. [Cīrulis, 2016, Theorem 5.2(b)] *(see also* [Cīrulis, 2015a, Theorem 4.2(b)]*) Let R be a strong Rickart ring and $a, b, x \in R$. If $a, b \overset{*}{\leq} x$, then $a \overset{*}{\vee} b$ exists and $a \overset{*}{\vee} b = x(a'' \vee b'')$ (or equivalently, $a \overset{*}{\vee} b = (a^{\backprime\backprime} \vee b^{\backprime\backprime})x$), where $\overset{*}{\leq}$ denotes the star order, $\overset{*}{\vee}$ its partial join operation and $\vee$ denotes the join of closed idempotents (see Lemma A.2.7).*

Meets

Janowitz proved in [Janowitz, 1983] that elements a, b of a Rickart *-ring R which are bounded from above (i.e., $a, b \overset{*}{\leq} x$ for some $x \in R$) have the meet. Cīrulis provided also an explicit equational description for this meet (as for the join) in [Cīrulis, 2015a], and he extended this result to strong Rickart rings in [Cīrulis, 2016].

2.1.4 Different types of coherence on Rickart *-rings

Since meets and joins in general do not exists for arbitrary elements of a Rickart *-ring, it is interesting to characterize pairs of elements which do have the meet or join (possible satisfying some additional condition) under the star order. This led to different notions of coherence, which were introduced independently by different authors. In order to reduce confusion, we use different names for different notions of coherence. We will mostly use the terminology from [Cīrulis, 2015a] and [Cīrulis and Cremer, 2022], changing the names of some notions which were introduced in [Djikić, 2016b] and [Djikić and Djordjević, 2016].

Coherence

In [Cīrulis, 2015a], elements a, b of a Rickart *-ring R were called *coherent* if

$$a^*b = b^*a \text{ and } ab'' = ba''. \tag{2.7}$$

A star-free generalization of coherence, as well as one-sided versions of it, was introduced by Cīrulis for strong Rickart rings in [Cīrulis and Cremer, 2022]. We shall deviate from [Cīrulis and Cremer, 2022] by defining each of the versions of coherence for the most general Rickart ring possible, because we need to use one of them in Section 4.4 in a right-strong right Rickart ring.

Definition 2.1.8 (Cīrulis)**.** [Cīrulis and Cremer, 2022, Definition 3.5] Elements a, b of a left-strong left Rickart ring are called *left coherent* if $b^{\backprime\backprime}a = a^{\backprime\backprime}b$.

Elements a, b of a right-strong right Rickart ring are called *right coherent* if $ab'' = ba''$.

Elements a, b of a strong Rickart ring are called *coherent* if they are both left coherent and right coherent.

It was noted in [Cīrulis and Cremer, 2022] that, in a Rickart *-ring, elements a and b are left coherent if and only if a^* and b^* are right coherent.

Right Djikić coherence

The word "coherent" is used with another meaning in [Djikić, 2016b] and [Djikić and Djordjević, 2016] and other articles by those authors. For the sake of clarity and in order to facilitate comparison to coherence in the sense of Definition (2.1.8), we rename the notion from [Djikić, 2016b] in this thesis.

Definition 2.1.9. [Djikić and Djordjević, 2016] Let R be a Rickart *-ring. We call elements $a, b \in R$ *right Djikić-coherent* if there exists an element $x \in R$ such that

$$aa^* = xa^* \text{ and } bb^* = xb^*. \tag{2.8}$$

See Chapter 4 for the the star-free version of right Djikić coherence and for its dual notion *left Djikić coherence*.

It was noted in [Djikić and Djordjević, 2016, page 66] that right Djikić coherence is a more general notion than coherence on a Rickart *-ring.

Right precoherence

A notion of precoherent elements was first introduced in [Djikić and Djordjević, 2016]. The precoherence from [Djikić and Djordjević, 2016] was then called "*right precoherence*" in [Cīrulis and Cremer, 2022] (a corresponding notion of *left precoherence* was also introduced there). We choose the terminology from [Cīrulis and Cremer, 2022].

Definition 2.1.10. [Cīrulis and Cremer, 2022], see also [Djikić and Djordjević, 2016] Elements a, b of a Rickart *-ring R are called *right precoherent* if

$$a(a'' \wedge_P b'') = b(a'' \wedge_P b''). \tag{2.9}$$

They are called *left precoherent* if a^* and b^* are right precoherent, that is, if

$$(a^{\backslash\backslash} \wedge_P b^{\backslash\backslash})a = (a^{\backslash\backslash} \wedge_P b^{\backslash\backslash})b. \tag{2.10}$$

Elements $a, b \in R$ are called *precoherent* if they are both right precoherent and left precoherent.

Note that a and b are left precoherent if and only if a^* and b^* are right precoherent, because $a^*((a^*)'' \wedge_P (b^*)'') = a^*(a^{\backslash\backslash} \wedge_P b^{\backslash\backslash}) = ((a^{\backslash\backslash} \wedge_P b^{\backslash\backslash})a)^*$ and similarly, $b^*((a^*)'' \wedge_P (b^*)'') = ((a^{\backslash\backslash} \wedge_P b^{\backslash\backslash})b)^*$.

Both right precoherence and right Djikić coherence originated in the ring of bounded linear operators on a Hilbert space, although the term *precoherent* was introduced only in [Djikić and Djordjević, 2016], when the concept was already generalized to Rickart *-rings (see Example 3.1 there for the specal case of operators). Right Djikić coherence was introduced in [Djikić, 2016b]. There is a PhD thesis on the topic (mostly in the context of operators) [Djikić, 2016a].

2.1.5 Conditions for a Rickart *-ring to be a meet semilattice

We give some necessary and sufficient condition for an arbitrary Rickart *-ring to be a meet semilattice which is due to [Cīrulis, 2015a].

Reduced Rickart *-rings

The following theorem shows that, for a Rickart *-ring, being reduced is equivalent to being a specific meet semilattice under the star order.

Theorem 2.1.11. [Janowitz, 1983, Theorem 8] *Let R be a Rickart *-ring. The following conditions are equivalent:*

(a) *R is reduced;*

(b) *R is a meet semilattice under the star order in which $c(a \overset{*}{\wedge} b) = ac \overset{*}{\wedge} bc$ for all $a, b, c \in R$,*

(c) *R is a meet semilattice under the star order in which $(a \overset{*}{\wedge} b)c = ac \overset{*}{\wedge} bc$ for all $a, b, c \in R$,*

2.1.6 Posets with additional structure under the star order

Since a strong Rickart ring under the star order has the bottom element 0 (see Proposition 2.1.4(a)), it makes sense to investigate its initial segments.

We noted in Proposition 2.1.4 (items (b) and (c)) that, on the set of closed idempotents P_{rl} on a focal Rickart ring with matching normal focal operations, the star order coincides with the standard order of idempotents, and that $P_{\mathrm{rl}} = [0, 1]_{\overset{*}{\leq}}$ is an initial segment. It is also mentioned in Lemma A.2.7 that, in a strong Rickart ring, the set of closed idempotents P_{rl} with the standard order of idempotents forms an orthomodular lattice.

In this subsection, we describe initial segment of a strong Rickart ring R under the star order and how they are related to the orthomodular lattice of closed idempotents P_{rl} of the ring R.

For the case of a Rickart *-ring R, Janowitz proved in [Janowitz, 1983, Theorem 5] that every initial segment $[0, x]_{\overset{*}{\leq}}$ under the star order is not only orthomodular, but also a lattice which is isomorphic to a sublattice of the projection lattice. Cīrulis extended and improved this result to strong Rickart rings in the following way.

Theorem 2.1.12. [Cīrulis, 2016, Theorem 5.3] *Let R be a strong Rickart ring. Every initial segment $[0, x]_{\overset{*}{\leq}}$ under the star order is an orthomodular lattice. The meets and joins of elements in $[0, x]_{\overset{*}{\leq}}$ agree with those in R.*

2.2 The right star order and its generalizations

This section provides the necessary preliminaries concerning the so-called one-sided versions of the star order. Although the definitions will be given for both left star orders and right star orders in Section 2.2.1, for reviewing their properties in Sections 2.2.2 and 2.2.3 we focus mainly on the right ones, since the left ones are dual.

2.2.1 The right star order on matrix rings and two possible generalizations

Origin: the right star order on the ring of square matrices

One-sided (left and right) generalizations of the star order were first defined for matrices in [Baksalary and Mitra, 1991] by weakening one of the two defining conditions. A relation $*\preceq$ was defined on $\mathrm{Mat}_n(\mathbb{C})$ by setting, for $A, B \in \mathrm{Mat}_n(\mathbb{C})$, $A *\preceq B$ if and only if

$$A^*A = A^*B \text{ and } \operatorname{ran} A \subseteq \operatorname{ran} B, \tag{2.11}$$

where $\operatorname{ran} A$ and $\operatorname{ran} B$ denote the column space of A, respectively, B. The relation $*\preceq$ was called the *left star order on* $\mathrm{Mat}_n(\mathbb{C})$. Dually, we define a relation $\preceq*$ by setting, for $A, B \in \mathrm{Mat}_n(\mathbb{C})$, $A \preceq* B$ if and only if

$$AA^* = BA^* \text{ and } \operatorname{ran} A^* \subseteq \operatorname{ran} B^*. \tag{2.12}$$

We call the relation $\preceq*$ the *right star order on* $\mathrm{Mat}_n(\mathbb{C})$.

The original definition was actually for general (not necessarily square) complex matrices, but in this thesis we focus on $n \times n$ square matrices (i.e., linear mappings from the ring $\mathcal{B}(\mathbb{R}^n)$).

Two generalizations to bounded linear operators

The one-sided star orders from [Baksalary and Mitra, 1991] were generalized to rings of bounded operators over Hilbert spaces in two different ways.

The strong one-sided star orders The first version, which was introduced by Deng and Wang in [Deng and Wang, 2012], and independently by Dolinar et al. in [Dolinar et al., 2014], just transferred the definition from [Baksalary and Mitra, 1991] to a more general structure (the column space of a matrix is just the image of the corresponding linear mapping).

So according to [Deng and Wang, 2012] and [Baksalary and Mitra, 1991], the (*strong*) *left star order on* $\mathcal{B}(\mathcal{H})$ is a binary relation $*\preceq$ which is defined as follows: For operators $A, B \in \mathcal{B}(\mathcal{H})$, $A *\preceq B$ if and only if

$$A^*A = A^*B \text{ and } \operatorname{im} A \subseteq \operatorname{im} B, \tag{2.13}$$

where $\operatorname{im} A$ and $\operatorname{im} B$ are the images of the operators. Dually, the (*strong*) *right star order on* $\mathcal{B}(\mathcal{H})$ $\preceq*$ is defined by setting, for operators $A, B \in \mathcal{B}(\mathcal{H})$, $A \preceq* B$ if and only if

$$AA^* = BA^* \text{ and } \operatorname{im} A^* \subseteq \operatorname{im} B^*. \tag{2.14}$$

These relations were proved to be partial orders in [Deng and Wang, 2012].

The weak one-sided star orders For the second generalization, which was introduced by Cīrulis in [Cīrulis, 2014a] and [Cīrulis, 2015c] (see also [Cīrulis, 2015a]), recall that, in a Hilbert space of finite dimension, every subspace is closed. So for operators on $\mathbb{C}^n$ (a finite dimensional

Hilbert space), the second part of Equation (2.11) is equivalent to $\overline{\operatorname{ran} A} \subseteq \overline{\operatorname{ran} B}$. Dually, the second part of Equation (2.12) is equivalent to $\overline{\operatorname{ran} A^*} \subseteq \overline{\operatorname{ran} B^*}$.

According to the generalization from [Cīrulis, 2015c], the relation $*\leq$ is defined on $\mathcal{B}(\mathcal{H})$ as follows: for operators $A, B \in \mathcal{B}(\mathcal{H})$, $A *\leq B$ if and only if

$$A^*A = A^*B \text{ and } \overline{\operatorname{im} A} \subseteq \overline{\operatorname{im} B}, \tag{2.15}$$

where $\overline{\operatorname{im} A}$ denotes the closure of the subspace $\operatorname{im} A$. Dually, the relation $\leq*$ is defined on $\mathcal{B}(\mathcal{H})$ by setting, for $A, B \in \mathcal{B}(\mathcal{H})$, $A \leq* B$ if and only if

$$AA^* = BA^* \text{ and } \overline{\operatorname{im} A^*} \subseteq \overline{\operatorname{im} B^*}. \tag{2.16}$$

The relations $*\leq$ and $\leq*$ were proved to be partial orders in [Cīrulis, 2015c].

For operators over finite dimensional Hilbert spaces, the partial orders $*\preceq$ and $*\leq$ coincide, and so do $\preceq*$ and $\leq*$. On the ring $\mathcal{B}(\mathcal{H})$ for an arbitrary Hilbert space $\mathcal{H}$, it is obvious that $A *\preceq B$ implies $A *\leq B$ and that $A \preceq* B$ implies $A \leq* B$. Therefore, we follow [Dolinar et al., 2021] by calling the relation $*\leq$ the *weak left star order* and the relation $\leq*$ the *weak right star order on* $\mathcal{B}(\mathcal{H})$. (In [Cīrulis, 2015c], they are just called left/right star order, which is not so convenient if we want to talk about both $\preceq*$ and $\leq*$.)

The question whether the strong one-sided star orders coincide with their weak counterparts also on the ring $\mathcal{B}(\mathcal{H})$ for an arbitrary Hilbert space $\mathcal{H}$ remains open.

Note that the notation in this thesis differs from the one in [Cīrulis, 2015c] in the following way: In [Cīrulis, 2015c], the symbols $*\preceq$ and $\preceq*$ are used for the weak left and right star orders, while the strong left/right star order is denoted by $*\leq$ and $\leq*$, respectively.

Remark 2.2.1. It is known that the image of a bounded linear operator A on a Hilbert space is closed if and only if A has the Moore-Penrose generalized inverse $A^\dagger$ (an operator such that $A^\dagger AA^\dagger = A^\dagger$, $AA^\dagger A = A$ and $AA^\dagger$ and $A^\dagger A$ are self-adjoint). Hence, if A and B have the Moore-Penrose generalized inverse, then $A *\preceq B$ if and only if $A *\leq B$, and $A \preceq* B$ if and only if $A \leq* B$. ◁

From $\mathcal{B}(H)$ to Rickart *-rings

Since both the weak and the strong right star order can be expressed in purely algebraic terms, they were introduced also for Rickart *-rings (for the strong right star order from [Deng and Wang, 2012], this was done by Marovt et al. in [Marovt et al., 2015]; for the weak version, see [Cīrulis, 2014a]).

The strong one-sided star orders The strong one-sided star orders were generalized to unitary involutive rings as follows.

Definition 2.2.2. [Marovt et al., 2015, Definitions 10 and 11] Let R be a unitary *-ring. We define a binary relation $*\preceq$ on R as follows: For $a, b \in R$, $a *\preceq b$ whenever there exist a projection $p \in R$ and an idempotent $e \in R$ such that

(a) the left annihilator of a is $R(1-p)$,
(b) the right annihilator of a is $(1-e)R$,
(c) $pa = pb$,
(d) $ae = be$.

The relation $*\preceq$ is called the *strong left star order on* R. Dually, we define the relation $\preceq*$ by setting, for $a, b \in R$, $a \preceq* b$ if and only if there exist a projection $q \in R$ and an idempotent $f \in R$ such that

(a) the left annihilator of a is $R(1-f)$,
(b) the right annihilator of a is $(1-q)R$,
(c) $fa = fb$,
(d) $aq = bq$. The relation $\preceq\ast$ is called the *strong right star order on R.*

On a Rickart *-ring, the relations $\ast\preceq$ and $\preceq\ast$ were proved to be partial orders in [Marovt et al., 2015].

The definition of the weak one-sided star orders for bounded linear operators over a Hilbert space involves two conditions. The purely algebraic equalities $A^*A = A^*B$ for the weak left star and $AA^* = BA^*$ for the weak right star order order could be transferred to abstract Rickart *-rings directly. However, we will use equivalent formulations.

Proposition 2.2.3. [Cīrulis, 2015a, Lemma 3.2], [Marovt et al., 2015, Lemmas 2.4, 2.5] *Let R be a Rickart *-ring and $a, b \in R$. Then*

$$a^*a = a^*b \text{ iff } a = a^{\backprime\backprime}b \text{ iff } a = pb \text{ for some projection } p, \tag{2.17}$$

$$aa^* = ba^* \text{ iff } a = ba'' \text{ iff } a = bp \text{ for some projection } p. \tag{2.18}$$

For the second part of Equation (2.15), $\overline{\operatorname{im} A} \subseteq \overline{\operatorname{im} B}$, recall Example 1.2.11: For $A \in \mathcal{B}(\mathcal{H})$, the operator $A^{\backprime}$ is the orthogonal projection onto $(\operatorname{im} A)^{\perp}$ and A' is the orthogonal projection onto $\ker A$. Consequently, $A^{\backprime\backprime}$ is the orthogonal projection onto $\overline{\operatorname{im} A}$ and A'' is the orthogonal projection onto $(\ker A)^{\perp}$.

Recall also the one-to-one correspondence between closed subspaces and orthogonal projections in a Hilbert space (see, for example, [Dalla Chiara et al., 2004, pp. 28–29 and section 3.2]). This correspondence is even an order isomorphism between the lattice of closed subspaces of $\mathcal{H}$ (ordered by inclusion) and the lattice of orthogonal projections $\mathcal{P}_{\mathrm{rl}}$ (ordered by the standard order of idempotents): For Hilbert space orthogonal projections $P, Q \in \mathcal{P}_{\mathrm{rl}}$, we have $PQ = P$ if and only if $\operatorname{im} P \subseteq \operatorname{im} Q$ (since $\mathcal{B}(\mathcal{H})$ is a strong Rickart ring by Example 1.3.10, this is equivalent to the statement that $PQ = QP = P$ if and only if $\operatorname{im} P \subseteq \operatorname{im} Q$).

So $\overline{\operatorname{im} A} \subseteq \overline{\operatorname{im} B}$ can be "translated" into Rickart *-ring terms as $A^{\backprime\backprime} \leq_E B^{\backprime\backprime}$, because $A^{\backprime\backprime}$ and $B^{\backprime\backprime}$ are the orthogonal projections onto $\overline{\operatorname{im} A}$ and $\overline{\operatorname{im} B}$, respectively.

Concerning the second condition defining the weak right star order, $\overline{\operatorname{im} A^*} \subseteq \overline{\operatorname{im} B^*}$, first note that $\overline{\operatorname{im} A^*} = \left((\operatorname{im} A^*)^{\perp}\right)^{\perp} = (\ker A)^{\perp}$ by a standard functional analysis argument[2], and in the same way, $\overline{\operatorname{im} B^*} = (\ker B)^{\perp}$. So $\overline{\operatorname{im} A^*} \subseteq \overline{\operatorname{im} B^*}$ is equivalent to $(\ker A)^{\perp} \subseteq (\ker B)^{\perp}$. This can be "translated" into Rickart *-ring terms as $A'' \leq_E B''$.

Definition 2.2.4. [Cīrulis, 2014a] Let R be a Rickart *-ring. A relation $\ast\leq$ is defined on R as follows: For $a, b \in R$, $a \ast\leq b$ if and only if

$$a = a^{\backprime\backprime}b \text{ and } a^{\backprime\backprime} \leq_E b^{\backprime\backprime}. \tag{2.19}$$

[2]For the first equality, let $W = \operatorname{im} A^*$. Note that $\overline{W}^{\perp\perp} = \overline{W}$ by [Rudin, 1991, Corollary of Theorem 12.4], because $\overline{W}$ is closed. Since $W \subseteq \overline{W}$, we have $\overline{W}^{\perp} \subseteq W^{\perp}$ and thus $W^{\perp\perp} \subseteq \overline{W}$ by the properties of the orthogonal complement. It remains to show that also $\overline{W} \subseteq W^{\perp\perp}$. Let $w \in \overline{W}$. Then there exists a sequence $(w_n)_{n\in\mathbb{N}} \subseteq W$ which converges to w. Let x be an arbitrary element of $W^{\perp}$. Then $\langle w_n, x\rangle = 0$ for all w_n. So by continuity of the scalar product, also $\langle w, x\rangle = 0$ (recall that the scalar product is by definition linear in its first argument and, by the Schwartz inequality ([Rudin, 1991, Theorem 12.2]), it is also bounded, hence it must be continuous). Hence, w is orthogonal to all elements of $W^{\perp}$, or in other words, $w \in W^{\perp\perp}$. So indeed $\overline{W} \subseteq W^{\perp\perp}$.

The second equality is immediate from [Rudin, 1991, Theorem 12.10].

The relation $*\leq$ is called the *weak left star order* on R. Dually, the *weak right star order* $\leq*$ is defined on R by setting $a \leq* b$ if and only if

$$a = ba'' \text{ and } a'' \leq_E b''. \tag{2.20}$$

The relations $*\leq$ and $\leq*$ were proved to be partial orders on a Rickart *-ring in [Cīrulis, 2014a].

Remark 2.2.5. It was noted in [Cīrulis, 2014a, Theorem 3.1 and Definition 3.2] that in a von Neumann regular Rickart *-ring (see Definition C.2.3), the strong one-sided star orders coincide with their weak counterparts (this extends the observation concerning operators which was stated in Remark 2.2.1 to an abstract analogue). ◁

A Rickart *-ring with either of the partial orders $\preceq*$ and $\leq*$ in general does not have a top element (the exception are Boolean rings). For example, in $\langle \mathbb{Z}, \preceq* \rangle$, every non-zero element is maximal.

The star-free generalization of the weak right star order

The weak right star order has been generalized to right-strong right Rickart rings by Cīrulis in [Cīrulis, 2015d], although it is mentioned only in Remark 2 of that article that the partial order studied there is a generalization of the one-sided star order. (The star-free generalization of the strong right star order is a subject of this thesis and will be treated in Section 3.1.)

Definition 2.2.6. [Cīrulis, 2015d, Remark 2] Let R be a right-strong right Rickart ring. The relation $*\leq$ is defined on R as follows: For $a, b \in R$, $a *\leq b$ whenever

$$a = a^{\backprime\backprime}b \text{ and } a^{\backprime\backprime} \leq_E b^{\backprime\backprime}. \tag{2.21}$$

The relation is called the *weak left star order* on R. Dually, the *weak right star order* on R is defined by setting, for $a, b \in R$, $a \leq* b$ whenever

$$a = ba'' \text{ and } a'' \leq_E b''. \tag{2.22}$$

The choice of names for the relations $\leq*$ and $*\leq$ given in Definition 2.2.6 is justified by the following lemma:

Lemma 2.2.7. Cīrulis [2015d] *The weak left star order $*\leq$ and the weak right star order $\leq*$ as defined in Definition 2.2.6 on a right-strong right Rickart ring are partial orders.*

2.2.2 Properties of the weak right star order on a right-strong right Rickart ring

The properties of the weak right star order on a right-strong right Rickart ring were investigated in [Cīrulis, 2015d]. For the special cases that the ring is a Rickart *-ring or even the ring $\mathcal{B}(\mathcal{H})$ of bounded linear operators over a Hilbert space, see also [Cīrulis, 2014a] and [Cīrulis, 2015c]. In [Dolinar et al., 2021], mappings on the ring $\mathcal{B}(\mathcal{H})$ which preserve the right (or left) star order in both directions were studied.

Basic properties

The following lemma was proved in [Cīrulis, 2015d] (the first three items and the fifth item are from Lemma 4.2, the fourth item is stated on page 501, and for the last item, see the paragraph after the proof of Theorem 4.3).

Lemma 2.2.8. [Cīrulis, 2015d] *Let R be a right-strong right Rickart ring ordered by the weak right star order. Then*
(a) 0 *is the bottom element,*
(b) $P_{\mathrm{r}} = [0,1]_{\leq *}$,
(c) *every left invertible element is maximal,*
(d) *on the set of closed idempotents, the weak right star order agrees with the standard order of idempotents* $\leq_E$,
(e) *the meet* $p \wedge_P q$ *of closed idempotents* p, q *under the standard order of idempotents in the set* P_{r} *is also their meet in* R *under the weak right star order,*
(f) *the join* $p \vee_P q$ *of closed idempotents* p, q *under the standard order of idempotents in the set* P_{r} *is also their join in* R *under the weak right star order.*

Some of the properties of the star order with respect to meets and joins under upper bounds also hold for the weak right star order.

The upper bound property The following theorem shows that, just like a strong Rickart ring with the star order (Theorem 2.1.7), also a right-strong right Rickart ring with the weak right star order has the upper bound property.

Theorem 2.2.9. [Cīrulis, 2015d, Theorem 4.3] *Let R be a right-strong right Rickart ring. If elements $a, b \in R$ have an upper bound x, then they have the join under the weak right star order, and* $a \vee * \, b = x(a'' \vee_P b'')$.

Meets of elements under an upper bound Like in the case of the star order on a strong Rickart ring, also the meet exists for elements under an upper bound.

Theorem 2.2.10. [Cīrulis, 2015d, Equations (4.2), (4.3)] *Let R be a right-strong right Rickart ring. If $a, b \in R$ have an upper bound $x \in R$, then they have the meet under the weak right star order, and*

$$a \wedge * \, b = x(a'' \wedge_P b'') = a(a'' \wedge_P b'') = b(a'' \wedge_P b''). \tag{2.23}$$

Structure of initial segments

The structure of initial segments of a right-strong right Rickart ring under the weak right star order is similar to the structure of initial segments of a strong Rickart ring under the star order.

Orthocomplementations In a right-strong Rickart ring R, we can define on every initial segment $[0, x]_{\leq *}$ a unary operation $^{\perp}_{x}$ by

$$a^{\perp}_{x} := x - a = xa', \tag{2.24}$$

because, if $a \leq * \, x$, then also $x - a \leq * \, x$ (see [Cīrulis, 2015d] for the details).

Theorem 2.2.11. [Cīrulis, 2015d] *A right-strong Rickart ring with the weak right star order is a relatively orthocomplemented poset. The orthocomplementation on an initial segment* $[0,x]_{\leq *}$ *is the operation* $\frac{\perp}{x}$ *from* Equation (2.24)*. Moreover, for* $a,b \in R$*, if there exists* $x \in R$ *such that* $a,b \leq* x$*, then the join of* a *and* b *under the weak right star order exists and is given by*

$$a \vee * \, b = x(a'' \vee_P b''). \tag{2.25}$$

Theorem 2.2.11 implies that every initial segment is orthomodular by Remark A.4.6. By Theorems 2.2.9 and 2.2.10, every initial segment is an orthomodular lattice. For the ring $\mathcal{B}(\mathcal{H})$ of bounded linear operators over a Hilbert space, it was proved in [Cīrulis, 2015c, Proposition 4.1] that every initial segment $[0,x]_{\leq *}$ of $\mathcal{B}(\mathcal{H})$ is even a complete sublattice (in the sense that every subset A of the initial segment $[0,x]_{\leq *}$ has the meet $\bigwedge A$ and the join $\bigvee A$ in $\mathcal{B}(\mathcal{H})$, and both $\bigwedge A$ and $\bigvee A$ belong to $[0,x]_{\leq *}$.

Initial segment isomorphism

Theorem 2.2.12. [Cīrulis, 2015d, Corollary 5.2 and Theorem 4.4] *Let* R *be a right-strong right Rickart ring and let* $x \in R$*. The initial segment* $[0,x]_{\leq *}$ *of* R *under the weak right star order is an orthomodular lattice which is isomorphic to* $[0,x'']_{\leq}$ *via the isomorphism*

$$\begin{aligned} \phi_x^{\leq *} : \; [0,x]_{\leq *} &\to [0,x'']_{\leq} \\ a &\mapsto a'' \end{aligned} \tag{2.26}$$

Skew meets

In [Cīrulis, 2015d], an operation $\overleftarrow{\wedge}$ was defined on a right-strong right Rickart ring R by

$$a \overleftarrow{\wedge} b := b(a'' \wedge_P b''). \tag{2.27}$$

This operation was proved to be associative. Since also $a \overleftarrow{\wedge} a = a$, this shows that $\langle R, \overleftarrow{\wedge} \rangle$ is a band. Moreover, the following interesting observation was proved:

Proposition 2.2.13. *For a right-strong right Rickart ring* R*, the natural order of the band* $\langle R, \overleftarrow{\wedge} \rangle$ *is the weak right star order. Moreover, the band* $\langle R, \overleftarrow{\wedge} \rangle$ *is right normal (see* Definition B.2.2*).*

In view of Theorem 2.2.9 and Definition B.2.6, we have the following theorem.

Theorem 2.2.14. [Cīrulis, 2015d, Theorem 4.3] *Let* R *be a right-strong right Rickart ring and let* $\overleftarrow{\wedge}$ *be the operation defined in* Equation (2.27)*. Then* $\langle R, \overleftarrow{\wedge}, \vee * \rangle$ *is a right normal skew nearlattice whose natural order is the weak right star order.*

For elements a, b of a right-strong right Rickart ring that have a common upper bound x under the weak right star order, Equation (2.27) and Equation (2.23) yield that $a \overleftarrow{\wedge} b = a \wedge * \, b$.

2.2.3 Properties of the strong right star order on a Rickart *-ring

The strong right star order and the strong left star order were studied in abstract Rickart *-rings in [Marovt et al., 2015], [Marovt, 2015] and [Krēmere, 2016]. Mappings on the ring $\mathcal{B}(\mathcal{H})$ of bounded linear operators over a Hilbert space which preserve the right (or left) star order in both directions were investigated in [Dolinar et al., 2021].

The strong one-sided star orders were also generalized to *-regular rings (see Definition C.2.4) and it was proved that, on *-regular Rickart *-rings, this generalization coincides with the one for Rickart *-rings (see [Marovt et al., 2015, Definitions 12, 13 and Theorem 14])

Structure of initial segments

It was proved in [Cīrulis, 2015d] that a right-strong right Rickart ring with the weak right star order is a relatively orthocomplemented poset (see Theorem 2.2.12). This raised the question if this was also true for a right-strong Rickart ring with the strong right star order. In [Krēmere, 2016], the author proved that, in the special case of a Rickart *-ring, it is true (more precisely, the dual result for the strong left star order was proved there).

Later in this thesis, we prove the result in star-free terms in the more general setting of right-strong Rickart rings, using the corresponding result from [Cīrulis, 2015d] for the weak right star order on right-strong right Rickart rings.

Sectional orthocomplementations We have the following analogue of the first part of Theorem 2.2.11.

Theorem 2.2.15. Krēmere [2016] *A Rickart *-ring R under the strong left star order (or the strong right star order) is a relatively orthocomplemented poset, with orthocomplementations* $\overset{\wr}{_x}$ *on initial segments* $[0, x]_{\preceq *}$ *given by*

$$a_x^{\wr} = x - a \tag{2.28}$$

for every $a \in [0, x]_{\preceq *}$.

2.3 Other partial orders

We conclude this chapter with a collection of three more partial orders on rings which are part of the research in Part II: The diamond order (Section 2.3.1), the sharp order (Section 2.3.2) and the Abian order (Section 2.3.3).

Just as the right star orders, both the diamond order and the sharp order have their origin in matrices (with the sharp order being restricted to matrices which have the group inverse). We state the original definitions as well their recent generalizations to strong Rickart rings (in the case of the diamond order) and a certain subset of an arbitrary unitary ring (in the case of the sharp order). Both of these partial orders have quite a close relationship to generalized inverses, which will therefore also be mentioned, although it is not necessary for any of the results obtained in Part II.

The Abian order, on the other hand, does not completely fit into the company, because it has no connection to matrices at all, being defined on reduced rings (matrix rings are in general not reduced). This is the only of the partial orders treated in this chapter which is not generalized to any type of Rickart ring, because it would not make sense to attempt to do so. However, it is in many ways similar to the other partial orders mentioned.

2.3.1 The diamond order

We first state the original definition (for complex matrices) and the various generalizations to rings, in particular the recent generalization for strong Rickart rings. Then we collect a few results which relate the diamond order to the Moore-Penrose generalized inverse, as well as to the star order (on *-regular rings or Rickart *-rings).

Definition and origin

The following is the original definition of the diamond order for complex matrices, which can be found in [Baksalary and Hauke, 1990].

Definition 2.3.1. [Baksalary and Hauke, 1990] Let A, B be complex $n \times m$ matrices. We write $A \overset{\diamond}{\leq} B$ if and only if $A = AB^*A = AA^*A$ and $A \sqsubset B$ (where $\sqsubset$ denotes the space preorder, see Definition 2.1.5).

This definition was adapted to unitary *-regular rings (i.e., regular Rickart *-rings) in [Lebtahi et al., 2014] in a straightforward way, and it was proved in [Marovt et al., 2015, Theorem 16] that it defines a partial order not only on unitary *-regular rings, but even on any unitary ring equipped with a proper involution.

Definition 2.3.2. [Lebtahi et al., 2014], [Marovt et al., 2015] Let R be a unitary *-ring such that the involution is proper. The relation $\overset{\diamond}{\leq}$ is defined on R as follows: For elements $a, b \in R$, $a \overset{\diamond}{\leq} b$ if

$$aa^*a = ab^*a \text{ and } a \sqsubset b. \tag{2.29}$$

Star-free version for strong Rickart rings

For strong Rickart rings, a star-free version of the diamond order was recently introduced. It uses the weak space preorder (see Definition 2.1.6) instead of the space preorder (see Definition 2.1.5). (But recall that on regular strong Rickart rings, these preorders are the same, so in that case, the definition is equivalent to Definition 2.3.2. In particular, the definitions are equivalent on a regular Rickart *-ring (in other words, a *-regular *-ring, see Proposition C.2.6).).

Definition 2.3.3. [Cīrulis, 2017] In a strong Rickart ring, the *diamond order* is the relation $\overset{\diamond}{\leq}$ defined by $a \overset{\diamond}{\leq} b$ iff

$$a'' \leq_E b'', \ a^{\backprime\backprime} \leq_E b^{\backprime\backprime} \text{ and } a = a^{\backprime\backprime} b a''. \tag{2.30}$$

In other words, $a \overset{\diamond}{\leq} b$ if and only if $a \sqsubset^w b$ and $a = a^{\backprime\backprime} b a''$, where $\sqsubset^w$ is the weak space preorder (see Definition 2.1.6). The relation $\overset{\diamond}{\leq}$ was proved to be a partial order on a strong Rickart ring in [Cīrulis, 2017].

For square matrices, Definition 2.3.1 and Definition 2.3.3 are equivalent.

2.3.2 The sharp order

The sharp order differs from the other partial orders treated in this thesis by the fact that it is not a partial order on the whole ring, but just on a certain subset. It was first introduced for square matrices of index 1 (not necessarily complex ones – they could be over any field) in [Mitra, 1987].

Recall that the group inverse of a square matrix exists if and only if the index of the matrix is 1 (Theorem C.3.5).

Definition 2.3.4. [Mitra, 1987] Let $\mathcal{I}$ be the set of $n \times n$ square matrices which have index 1. We define a binary relation $\overset{\sharp}{\leq}$ on $\mathcal{I}$ as follows: For matrices $A, B \in \mathcal{I}$, $A \overset{\sharp}{\leq} B$ if and only if $A^\sharp A = A^\sharp B$ and $AA^\sharp = BA^\sharp$. The relation $\overset{\sharp}{\leq}$ is called the *sharp order*.

The sharp order was proved to be indeed a partial order in [Mitra, 1987].

It was extended to more general settings by several authors, specifically to rings of linear bounded operators on Banach spaces in [Efimov, 2013] and also to general unitary rings in [Marovt, 2015] and [Rakic, 2015]. In all of these cases, it is defined only on a certain subset of the respective ring. On the ring of bounded linear operators on a Banach space $\mathcal{B}(X)$, this subset is the set

$\{A \in \mathcal{B}(X) \mid \overline{\operatorname{im} A} \oplus \ker A = X\}$, i.e., all operators such that the Banach space is the direct sum of their kernel and the closure of their image (see [Efimov, 2013]). On a unitary ring R, in [Marovt, 2015], the set of completely regular elements is used. Another approach in a unitary ring, which can be found in [Rakic, 2015], uses the set of elements a of R for which there exists a (necessarily unique) idempotent $p_a \in R$ such that $\operatorname{ann}_l(a) = \operatorname{ann}_l(p_a)$ and $\operatorname{ann}_r(a) = \operatorname{ann}_r(p_a)$. We shall denote this set by $\mathcal{I}_R$, following [Rakic, 2015]. In general, the set $\mathcal{I}_R$ includes the set of completely regular elements, but not vice versa.

In this thesis, we shall use the most general version of the sharp order.

Definition 2.3.5. [Rakic, 2015] Let R be a unitary ring. Recall that $\operatorname{ann}_l(a)$ and $\operatorname{ann}_r(a)$ denote the left and right annihilators of an element a (see Definition 1.1.1). Let

$$\mathcal{I}_R := \{x \in R \mid (\exists p \in R)(p^2 = p,\ \operatorname{ann}_l(x) = \operatorname{ann}_l(p) \text{ and } \operatorname{ann}_r(x) = \operatorname{ann}_r(p))\}. \tag{2.31}$$

For a given element $x \in R$, the idempotent p from Equation (2.31) is unique if it exists. We may therefore denote it by p_x. Now the relation $\overset{\sharp}{\leq}$ is defined on the set $\mathcal{I}_R$ as follows: For $a, b \in \mathcal{I}_R$, $a \overset{\sharp}{\leq} b$ if and only if $a = bp_a = p_a b$. The relation $\overset{\sharp}{\leq}$ is called the *sharp order*.

It was proved in [Rakic, 2015] that the sharp order is indeed a partial order on the set $\mathcal{I}_R$.

2.3.3 The Abian order on reduced rings

We conclude this section with a brief description of a partial order which has nothing to do with matrices but, in spite of that, has certain similarities to the other partial orders introduced in this chapter.

Recall that a ring without non-zero nilpotent elements is called *reduced* (Definition 1.1.12). The *Abian partial order* on a reduced ring is defined as

$$x \leq_{\mathrm{Ab}} y \text{ if and only if } xy = xx. \tag{2.32}$$

Chacron proved in [Chacron, 1971, Lemmas 1 and 2] that this relation on an arbitrary ring is a partial order if and only if the ring is reduced. It is also known by many other names, including the *natural partial order* of a reduced ring, the *Conrad order* or the *reduced ring order*.

It is easy to see that on a reduced Rickart *-ring, the Abian order coincides with all the other orders which are investigated in this thesis, i.e., the star order, all one-sided star orders, the diamond order and the sharp order. On the set of idempotents of a reduced ring, the Abian order obviously coincides also with the natural order of idempotents.

The Abian order is also a partial order on certain semirings (see [Khatun et al., 2020]). It is noteworthy that the Abian order on the reduced semiring of self-adjoint bounded linear operators on a Hilbert space coincides with a restriction of the star order which (the restriction) is also called the *Gudder order* (for example, in [Dolinar and Molinár, 2007]) or the *logical order* (in [Gudder, 2006]) in this context. This connection was observed in [Pulmannová and Vinceková, 2007, page 2].

By the coincidence of the Abian order with the star order, it is immediate from Theorem 2.1.11 that a reduced Rickart *-ring with the Abian order is a meet semilattice. This is also true for Rickart rings without involution:

Theorem 2.3.6. [Burgess and Raphael, 2020, Examples 1.1(2)] *A reduced Rickart ring ordered by the Abian order is a meet semilattice.*

For other conditions under which the Abian order turns a reduced ring into a meet semilattice, see [Burgess and Raphael, 2018] and [Burgess and Raphael, 2020].

It is also known that a reduced ring is a join semilattice under the Abian order if and only if it is Boolean, see [Burgess and Raphael, 2018, Proposition 2.8].

Part II

Results

Chapter 3

Generalizations of the right star order to Rickart rings satisfying a strongness condition

This chapter deals with the possible generalizations of the right star order to one-sided or both-sided Rickart rings satisfying a suitable strongness contition and their characterizations.

As we saw in Section 2.2.1 (Definition 2.2.6), the weak right star order was recently generalized to right-strong right Rickart rings in [Cīrulis, 2015d]. We provide another characterization of this partial order involving right idempotents. As the main result of this chapter, we introduce the generalization of the strong right star order to right-strong Rickart rings. We also find a third generalization of the right star order to right-strong right Rickart rings (it remains an open question whether it actually coincides with one of the other two). Moreover, we introduce three one-sided versions of the space preorder which are related to the three right star orders in a way which is similar to how the space preorder relates to the star order (recall Section 2.1.2).

The structure of the chapter is as follows. In Section 3.1, after defining the strong right star order on a right-strong right Rickart ring, we first provide a shorter equivalent characterization of it which involves the set of left idempotents of the smaller element. Then we show that it is a subrelation of the weak right star orderand finally, we prove that on a right-strong Rickart ring, it is indeed a partial order (although the relation is defined on a one-sided Rickart ring, both-sided Rickartness is required for it to be a partial order).

Since the strong right star order can be characterized by a condition that involves left idempotents, the question arises whether an analogous characterization exists for the weak right star order. In Section 3.2, this question is answered affirmatively and the characterization involving right idempotents is given.

In Section 3.3, we introduce the three one-sided versions of the space preorder to (one-sided or both-sided) Rickart rings satisfying a suitable strongness condition and prove that they are indeed preorders. Then we see that two of them can be used to characterize the weak right star order and the strong right star order, respectively, by combining the condition $a = ab''$ with the requirement that a be below b in the respective preorder. By replacing the preorder in the requirement with the third one-sided version of the space preorder, we obtain yet another partial order. Finally, we establish the relations between the three one-sided space preorders, and deduce that the third right star order is stronger that the weak one, but weaker than the strong one (it is not known whether it coincides with any of them).

The results in Section 3.1 were published in [Cremer, 2024], while Sections 3.2 and 3.3 consist of unpublished results.

3.1 Introducing the strong right star order on right-strong Rickart rings

The strong right star order was generalized to Rickart *-rings in [Marovt et al., 2015, Definition 11] (see Definition 2.2.2 in Part I). We modify the definition from [Marovt et al., 2015] in the following way in order to make it meaningful also in a suitable star-free setting. For now, this setting can be a right-strong right Rickart ring, because the following definition does make sense in such a ring, but later, a one-sided Rickart ring will not be enough for ensuring that the relation defined in it is a partial order.

Definition 3.1.1. Let R be a right-strong right Rickart ring and let $a, b \in R$. We define a relation $\preceq*$ as follows: $a \preceq* b$ if and only if there exist an idempotent f and a closed idempotent p such that

($\preceq*$1) for all $x \in R$, $ax = 0$ if and only if $x \in (1-p)R$,

($\preceq*$2) for all $x \in R$, $xa = 0$ if and only if $x \in R(1-f)$,

($\preceq*$3) $ap = bp$,

($\preceq*$4) $fa = fb$.

If moreover the ring R is a both-sided Rickart ring (i.e., a right-strong Rickart ring), then we call the relation $\preceq*$ the *strong right star order* on R.

The justification for the chosen name of the relation follows later – see Lemma 3.1.4 and Theorem 3.1.7.

In order to shorten the proof of the next lemma, we state the following auxiliary proposition.

Proposition 3.1.2. [Cremer, 2024] *Let R be an arbitrary ring, let $a \in R$ and let $p, f \in R$ be idempotents. Then the following conditions ($\preceq*$1') and ($\preceq*$2') are equivalent to ($\preceq*$1) and ($\preceq*$2) from Definition 3.1.1, respectively.*

($\preceq*$1') *p is a right idempotent of a,*

($\preceq*$2') *f is a left idempotent of a.*

Proof. We prove only the equivalence of ($\preceq*$1) and ($\preceq*$1').

Let R be a ring. Note that, for an arbitrary element $x \in R$ and an arbitrary idempotent $e \in R$,

$$x \in eR \text{ if and only if } ex = x. \tag{3.1}$$

(The implication from right to left is clear. For the other direction, suppose $x \in eR$. Then $x = ey$ for some $y \in R$. So $ex = eey = ey = x$ by idempotency of e.)

Now let $a, p \in R$ such that p is an idempotent. Since $1-p$ is idempotent (see Remark 1.1.7), Equation (3.1) yields $x \in (1-p)R$ if and only if $(1-p)x = x$ for every $x \in R$. Since moreover $(1-p)x = x$ if and only if $px = 0$, we see that the elements a and p satisfy the condition ($\preceq*$1) if and only if

$$\text{for all } x \in R,\ ax = 0 \text{ if and only if } px = 0.$$

In other words, ($\preceq*$1) holds if and only if p is a right idempotent of a (see Definition 1.1.5).

The equivalence of the conditions ($\preceq*$2) and ($\preceq*$2') is proved in the left-right dual way. □

Proposition 3.1.2 implies that if the relation $\preceq*$ on a right-strong right Rickart ring is a partial order, then the ring must be also left Rickart, since reflexivity and ($\preceq*$2') imply that every element has a left idempotent (see Remark 1.1.7).

However, as we will see in the next lemma, even on a general right-strong right Rickart ring, we can characterize the relation $\preceq*$ in a shorter way, just as in a Rickart *-ring (see [Marovt et al., 2015, Theorem 9]). This characterization has the additional advantage that it bears some resemblance to the defininion of the weak right star order (Definition 2.2.6).

Lemma 3.1.3. [Cremer, 2024] *Let R be a right-strong right Rickart ring and let $a, b \in R$. Then $a \preceq* b$ if and only if*

$$a = ba'' \text{ and } a = eb \text{ for some } e \in \mathrm{LI}(a). \tag{3.2}$$

Proof. Let $a, b \in R$ for some right-strong right Rickart ring R.

"$\Leftarrow$"

Suppose Equation (3.2) holds. We need to prove that there exist an idempotent f and a closed idempotent p such that items ($\preceq*$1)-($\preceq*$4) from Definition 3.1.1 hold. Let $f = e$ and $p = a''$.

(a) By Remark 1.1.8, a'' is a right idempotent of a. In view of Proposition 3.1.2, this proves that ($\preceq*$1) holds.
(b) Since e is a left idempotent of a, Proposition 3.1.2 yields that ($\preceq*$2) holds.
(c) Since $aa'' = a$ by Proposition 1.3.5(b), it is immediate from the assumption $a = ba''$ (the first part of Equation (3.2)) that $aa'' = ba''$.
(d) Proposition 1.1.6 and the second part of Equation (3.2) immediately yield that $ea = eb$.

"$\Rightarrow$"

Suppose $a \preceq* b$. So there exist an idempotent f and a closed idempotent p such that items ($\preceq*$1)-($\preceq*$4) from Definition 3.1.1 hold. By Proposition 3.1.2, f is a left idempotent of a and p is a right idempotent of a. Hence, Proposition 1.1.6 yields that

$$fa = a, \tag{3.3}$$
$$ap = a. \tag{3.4}$$

For the first part of Equation (3.2), first note that Equation (3.4) in combination with ($\preceq*$3) from Definition 3.1.1 ($ap = bp$) yields that

$$a = bp. \tag{3.5}$$

Now by Proposition 1.3.5(c), we have $a'' = (bp)'' \leq_E p''$. Since p is closed and therefore $p'' = p$ by Remark 1.3.2, this implies $a'' \leq_E p$. That is, $pa'' = a''$. Hence, $ba'' = bpa'' = aa'' = a$ by Equation (3.5) and Proposition 1.3.5(b). So the first part of Equation (3.2) holds.

Finally, Equation (3.3) in combination with ($\preceq*$4) from Definition 3.1.1 ($fa = fb$) yields that $a = fb$, which is the second part of Equation (3.2). ☐

The following lemma shows that the relation $\preceq*$ from Definition 3.1.1 is stronger than the weak right star order $\leq*$ from Definition 2.2.6 in the sense that $\preceq*$ is a subrelation of $\leq*$. This will simplify many of the subsequent proofs. However, it remains an open question whether it is strictly stronger.

As we saw in Section 2.2, in some special cases the relation $\preceq*$ coincides with the weak right star order $\leq*$: By Remark 2.2.5, the converse implication to the one in Lemma 3.1.4 holds on every regular Rickart *-ring (hence, on any matrix ring, as we already know from Section 2.2.1). By Remark 2.2.1, it holds for any two linear bounded operators on a Hilbert space which have the Moore-Penrose generalized inverse (if if the ring $\mathcal{B}(\mathcal{H})$ itself is not regular).

Lemma 3.1.4. [Cremer, 2024] *Let R be a right-strong right Rickart ring and $a, b \in R$. If $a \preceq* b$, then $a \leq* b$.*

Proof. Let $a \preceq* b$ for elements a, b of a right-strong right Rickart ring R. Then, by Lemma 3.1.3, Equation (3.2) holds. According to Definition 2.2.6, we have to prove that Equation (2.22) holds, too. Note that Equations (2.22) and (3.2) differ only in their second parts. So we only need to prove that $a'' \leq_E b''$.

By Equation (3.2), we have $a = eb$ for some $e \in \mathrm{LI}(a)$. Now Proposition 1.3.5(c) yields that $a'' = (eb)'' \leq_E b''$. □

Remark 3.1.5. Obviously, for elements a and b of a right-strong right Rickart ring, we have

$$a \preceq* b \text{ if and only if } a \leq* b \text{ and } a = eb \tag{3.6}$$

for some left idempotent e of a. Now it becomes clear that, on the set of closed idempotents of a right-strong right Rickart ring, the relation $\preceq*$ coincides with the standard order of idempotents $\leq_E$ (Definition A.2.2). Indeed, if p and q are closed idempotents such that $p \leq_E q$, then $p \leq* q$ by Lemma 2.2.8(d), so we have $p \preceq* q$, since $p = pq$ and p is a left idempotent of itself. Conversely, if $p \preceq* q$ for closed idempotents p and q, then Lemma 3.1.4 and Lemma 2.2.8(d) yield that $p \leq_E q$. ◁

If $a \preceq* b$ for some elements a, b of a right-strong right Rickart ring, then there are a few things that we can say about left idempotents of a and b (if they exist). They are collected in the following Proposition.

Proposition 3.1.6. *Let R be a right-strong right Rickart ring and let $a, b \in R$ such that $a \preceq* b$. For every $e \in \mathrm{LI}(a)$ and every $f \in \mathrm{LI}(b)$, the following statements hold:*
- (a) *$fa = a$,*
- (b) *$fe = e$,*
- (c) *ef is idempotent,*
- (d) *if $a = eb$, then $ef \in \mathrm{LI}(a)$,*
- (e) *if $a = eb$, then $f - ef \in \mathrm{LI}(b - a)$.*

Proof. Let $a \preceq* b$, $e \in \mathrm{LI}(a)$ and $f \in \mathrm{LI}(b)$.
- (a) By Proposition 1.1.6, we have $fb = b$, because f is a left idempotent of b. Since $a = ba''$ by Lemma 3.1.3, this yields $fa = fba'' = ba'' = a$.
- (b) By item (a), we have $(1 - f)a = 0$. Hence $(1 - f)e = 0$, because e is a left idempotent of a. So $fe = e$.
- (c) By item (b) and idempotency of e, $(ef)^2 = e(fe)f = eef = ef$.
- (d) Suppose $a = eb$. The element ef is idempotent by item (c). Since f is a left idempotent of b, we have $xef = 0$ if and only if $xeb = 0$ for every $x \in R$. But $xeb = xa$, since we assumed that $a = eb$. Thus, ef is indeed a left idempotent of a.
- (e) Suppose $a = eb$. First, item (b), idempotency of f and item (c) yield that $(f - ef)^2 = f - fef - eff + (ef)^2 = f - ef - ef + ef = f - ef$. So the element $f - ef$ is idempotent. Second, we have $f - ef = (1 - e)f$. Since f is a left idempotent of b, this yields that, for every element x of the ring, $x(f - ef) = 0$ if and only if $x(1 - e)b = 0$. But $x(1 - e)b = x(b - eb) = x(b - a)$, since we assumed that $a = eb$. Therefore, we have $x(f - ef) = 0$ if and only if $x(b - a) = 0$. So the element $f - ef$ is indeed a left idempotent of the element $b - a$. □

Now we state the the main theorem of this section. Note that, while the propositions and lemmas up until here assumed just a right-strong right Rickart ring (i.e., a one-sided Rickart ring), this theorem wants more: It requires a right-strong Rickart ring, that is, a both-sided Rickart ring.

Theorem 3.1.7. [Cremer, 2024] *On a right-strong Rickart ring, the relation $\preceq*$ defined in Definition 3.1.1 is a partial order.*

Proof. Let R be a right-strong Rickart ring, and let $\preceq*$ be the relation defined in Definition 3.1.1.

Reflexivity Let $a \in R$. By Proposition 1.3.5(b), $aa'' = a$, and by Proposition 1.1.6, $a = ea$ for all $e \in \mathrm{LI}(a)$ (recall that by Remark 1.1.7, in $\mathrm{LI}(a) \neq \emptyset$, because R is also left Rickart). Thus, $a \preceq* a$.

Antisymmetry Let $a, b \in R$ such that $a \preceq* b$ and $b \preceq* a$. By Lemma 3.1.4, this implies that $a \leq* b$ and $b \leq* a$. Since the relation $\leq*$ is known to be a partial order (recall Lemma 2.2.7), this implies $a = b$.

Transitivity Let $a \preceq* b$ and $b \preceq* c$. By Lemma 3.1.4, this implies that $a \leq* b$ and $b \leq* c$. Hence, $a \leq* c$, since by Lemma 2.2.7 the relation $\leq*$ is known to be a partial order. So by Definition 2.2.6, we have in particular $a = ca''$.

So according to Lemma 3.1.3, in order to prove that $a \preceq* c$, it is sufficient to find some idempotent g such that $a = gc$ and $g \in \mathrm{LI}(a)$.

By Lemma 3.1.3,

$$a = eb \text{ for some } e \in \mathrm{LI}(a), \tag{3.7}$$

$$b = fc \text{ for some } f \in \mathrm{LI}(b). \tag{3.8}$$

We will prove that $g = ef$ is a suitable choice.

Obviously, $a = eb = efc$ by Equations (3.7) and (3.8). Moreover, by Proposition 3.1.6(d), the element ef is a left idempotent of a, (because $a \preceq* b$, $e \in \mathrm{LI}(a)$, $f \in \mathrm{LI}(b)$ and $a = eb$). So by Lemma 3.1.3, we conclude that $a \preceq* c$. □

In view of Theorem 3.1.7 and Lemma 3.1.4, it makes sense to call the relation $\preceq*$ on a right-strong Rickart ring the *strong right star order*.

3.2 Alternative characterization of the weak right star order

In Lemma 3.1.3, we characterized the strong right star order $\preceq*$ by a conjunction of the condition $a = ba''$ and a condition involving existence of a specific left idempotent e of a. The first condition is also part of the definition of the weak right star order $\leq*$ (Definition 2.2.6), but the second condition of Definition 2.2.6 ($a'' \leq_E b''$) is quite different from the left idempotent condition in Lemma 3.1.3. In this section, we see that we can replace the second condition of Definition 2.2.6 by a condition involving a right idempotent of b, obtaining an alternative characterization of the weak right star order which bears a bit more resemblance to the characterization of the strong right star order from Lemma 3.1.3.

We start with an auxiliary proposition which shows that for every $a \in R$ the set $\mathrm{RI}(a)$ of right idempotents of a contains a single closed idempotent.

Proposition 3.2.1. *Let a be an element of a right-strong right Rickart ring R. Then* $\mathrm{RI}(a) \cap P_{\mathrm{r}} = \{a''\}$.

Proof. It is clear that a'' is a closed idempotent (see Definition 1.3.1. We also know from Remark 1.1.8 that $a'' \in \mathrm{RI}(a)$.

Now assume that p is also a closed idempotent which is a right idempotent of a. Then, since both p and a'' are right idempotents of a, for every $x \in R$ we have that $px = 0$ if and only if $a''x = 0$. Since $p(1-p) = 0$, this yields $a''(1-p) = 0$, and thus

$$a'' = a''p. \tag{3.9}$$

Similarly, since $a''(1-a'') = 0$, we have

$$p = pa''. \tag{3.10}$$

But p is a closed idempotent, so we have $pa'' = a''p$, because R is right-strong(see Definition 1.3.3). Hence Equations (3.9) and (3.10) yield that $p = a''$. □

As a consequence of Proposition 3.2.1, we have the following.

Corollary 3.2.2. *Let a, b be elements of a right-strong right Rickart ring R. Then* $\mathrm{RI}(a) = \mathrm{RI}(b)$ *if and only if $a'' = b''$.*

Proof. If $\mathrm{RI}(a) = \mathrm{RI}(b)$, then by Proposition 3.2.1 $\{a''\} = \mathrm{RI}(a) \cap P_\mathrm{r} = \mathrm{RI}(b) \cap P_\mathrm{r} = \{b''\}$. Conversely, if $a'' = b''$, then $\mathrm{RI}(a) = \mathrm{RI}(a'') = \mathrm{RI}(b'') = \mathrm{RI}(b)$. □

Now we obtain the following alternative characterization of the strong right star order $\leq *$.

Proposition 3.2.3. *Let R be a right-strong right Rickart ring and $a, b \in R$. Then $a \leq * \, b$ if and only if*

$$a = ba'' \text{ and } a = af \text{ for some } f \in \mathrm{RI}(b). \tag{3.11}$$

Proof. Let $a \leq * \, b$. Then $a = ba''$ and $a'' \leq_E b''$. Hence, $ab'' = aa''b'' = aa'' = a$ by Proposition 1.3.5(b). Since $b'' \in \mathrm{RI}(b)$, this proves that Equation (3.11) holds.

Conversely, suppose that $a = ba''$ and $a = af$ for some $f \in \mathrm{RI}(b)$. Note that $f'' = b''$, because $\mathrm{RI}(f) = \mathrm{RI}(b)$. Hence, $a'' = (af)'' \leq_E f'' = b''$ by Proposition 1.3.5(c), which proves that $a \leq * \, b$. □

3.3 Right star orders and right space preorders

Both in the Definition 2.2.6 (the definition of the weak right star order) and in Lemma 3.1.3 (a characterization of the strong right star order) we encounter the equality $a = ba''$ as a part of the condition which characterizes the respective version of the right star order. In this section, we focus on the second part of these conditions, in which the right star orders differ from each other. For the weak right star order, the second part of the condition can be stated using a one-sided version of the weak space preorder. Also for the strong right star order, we will define a preorder $\sqsubset_r^s$ such that $a \preceq * \, b$ if and only if $a = ba''$ and $a \sqsubset_r^s b$. We also define a one-sided version of the space preorder which we denote by $\sqsubset_r$, and in turns out that the conjunction $a = ba''$ and $a \sqsubset_r b$ defines another partial order, which therefore can be regarded as a third right star order.

Definition 3.3.1. (a) Let R be a right-focal right Rickart ring. The *weak right space preorder* $\sqsubset_r^w$ is defined as follows: for $a, b \in R$, $a \sqsubset_r^w b$ if and only if $a''b'' = a''$.

(b) Let R be a ring. The *right space preorder* $\sqsubset_r$ is defined as follows: for $a, b \in R$, $a \sqsubset_r b$ if and only if $Ra \subseteq Rb$.

(c) Let R be a ring. The *strong right space preorder* $\sqsubset_r^s$ is defined as follows: for $a, b \in R$, $a \sqsubset_r^s b$ if and only if there exists $e \in \mathrm{LI}(a)$ such that $a = eb$ and $fe = e$ for all $f \in \mathrm{LI}(b)$.

Proposition 3.3.2. *The relations $\sqsubset_r^w$ and $\sqsubset_r$ as defined in Definition 3.3.1 are preorders. The relation $\sqsubset_r^s$ is a preorder if the ring R is a right-strong Rickart ring. In such a ring, we call the relation $\sqsubset_r^s$* strong right space preorder.

Proof. (a) Reflexivity is obvious. For transitivity, assume that $a \sqsubset_r^w b$ and $b \sqsubset_r^w c$, that is, $a''b'' = a''$ and $b''c'' = c''$. Then $a''c'' = a''b''c'' = a''b'' = a''$.

(b) Reflexivity is again obvious. For transitivity, first note that $a \sqsubset_r b$ if and only if there exists $x \in R$ such that $a = xb$. Now let $a \sqsubset_r b$ and $b \sqsubset_r c$. Then $a = xb$ and $b = yc$ for some $x, y \in R$. Hence, $a = xb = xyc$, i.e. $a \sqsubset_r b$.

(c) Let R be a right-strong Rickart ring. Since R is a left Rickart ring, every element a has some left idempotent e by Remark 1.1.7. So by Proposition 1.1.6, we have $ea = a$ with $e \in \mathrm{LI}(a)$. Moreover, for arbitrary $f \in \mathrm{LI}(a)$, $fe = e$ by Proposition 3.1.6 (since $a \preceq* a$). Hence, the relation $\sqsubset_r^s$ is reflexive.
For transitivity, let $a \sqsubset_r^s b$ and $b \sqsubset_r^s c$. Then there are $e \in \mathrm{LI}(a)$ and $f \in \mathrm{LI}(b)$ such that $a = eb$ and $b = fc$. Obviously, $a = eb = efc$.
We will prove that ef is a left idempotent of a. Since $f \in \mathrm{LI}(b)$, we have $fe = e$, and therefore ef is idempotent, since $(ef)^2 = efef = ef$. Moreover, $xef = 0$ if and only if $xeb = 0$ (since $f \in \mathrm{LI}(b)$). But $eb = a$ by assumption, so indeed $ef \in \mathrm{LI}(a)$.
Now let l be an arbitrary left idempotent of c. We know that, for all $g \in \mathrm{LI}(b)$, $ge = e$, and for all $h \in \mathrm{LI}(c)$, $hf = f$ (by Definition 3.3.1(c)). So in particular $fe = e$ and $lf = f$. Hence, $lef = lfef = fef = ef$. This proves that $a \sqsubset_r^s c$.

□

Remark 3.3.3. On a right-strong right Rickart ring, it is obvious from Remark A.2.6 that the weak right space preorder can be characterized by

$$a \sqsubset_r^w b \text{ if and only if } a'' \leq_E b''. \tag{3.12}$$

◁

Proposition 3.3.4. *1. In a right-strong right Rickart ring, $a \leq* b$ if and only if $a = ba''$ and $a \sqsubset_r^w b$.*

2. In a right-strong Rickart ring, $a \preceq* b$ if and only if $a = ba''$ and $a \sqsubset_r^s b$.

Proof. The first item is obvious from Definition 2.2.6 and Remark 3.3.3. For the second item, it is clear from Lemma 3.1.3 that $a = ba''$ and $a \sqsubset_r b$ implies $a \preceq* b$. Conversely, if $a \preceq* b$, then by Lemma 3.1.3, we have $a = ba''$ and $a = eb$ for some $e \in \mathrm{LI}(a)$. Moreover, by Proposition 3.1.6(b), we have $fe = e$ for every $f \in \mathrm{LI}(b)$. □

The question arises whether also the right space preorder in combination with $a = ba''$ defines some partial order. It does indeed.

Theorem 3.3.5. *Let R be a right-focal right Rickart ring and let $\preceq'_*$ be the relation defined on R as*

$$a \preceq'_* b \text{ if and only if } a = ba'' \text{ and } a \sqsubset_r b. \tag{3.13}$$

If R is a right-strong right Rickart ring, then $\preceq'_*$ is a partial order.

Proof. It is obvious that the relation $\preceq'_*$ is reflexive.

For antisymmetry, let $a \preceq'_* b$ and $b \preceq'_* a$. Then in particular $b = ya$ for some $y \in R$ (from Equation (3.13), since $b \preceq'_* a$). Hence, $b'' = (ya)'' \leq_E a''$ by Proposition 1.3.5(c). Now, noting that $a = ba''$ (since $a \preceq'_* b$) we obtain from Proposition 1.3.5(b) that $a = ba'' = bb''a'' = bb'' = b$.

For transitivity, let $a \preceq'_* b$ and $b \preceq'_* c$. Then $a \sqsubset_r b$ and $b \sqsubset_r c$, and thus also $a \sqsubset_r c$ by Proposition 3.3.2. It remains to show that $ca'' = a$. Since $a \sqsubset_r b$, we have $a = xb$ for some $x \in R$. So $ca'' = c(xb)'' = cb''(xb)''$ by Proposition 1.3.5(c). But $cb'' = b$, since $b \preceq'_* c$. Hence, $ca'' = b(xb)'' = ba'' = a$, since $a \preceq'_* b$.

□

The proof of the second item of the following proposition is essentially the same as the respective part of the proof of the corresponding assertion for the both-sided space preorder and weak space preorder (Theorem 2.1 in [Cīrulis, 2016]).

Proposition 3.3.6. (a) *For elements a, b of an arbitrary ring, if $a \sqsubset_r^s b$, then $a \sqsubset_r b$.*
(b) *For elements a, b of a normal right-focal right Rickart ring, if $a \sqsubset_r b$, then $a \sqsubset_r^w b$.*

Proof. The first item is obvious from Definition 3.3.1. For the second item, let R be a right-focal right Rickart ring and let $a, b \in R$ such that $a \sqsubset_r b$. Then there is some $x \in R$ such that $a = xb$. Hence, $ab' = xbb' = 0$ by Proposition 1.3.5(a), and therefore $a''b' = 0$ by Remark 1.1.8, since the right-focal operation is normal. So $a''b'' = a''$ (also by normality). Hence, $a \sqsubset_r^w b$ by Definition 3.3.1. □

Proposition 3.3.6, Proposition 3.3.4, and Theorem 3.3.5 yield the following consequence.

Theorem 3.3.7. *Let a, b be elements of a right-strong Rickart ring R.*
(a) *If $a \preceq* b$, then $a \preceq'_* b$.*
(b) *If $a \preceq'_* b$, then $a \leq* b$.*

In the remainder of the thesis, we shall deal only with two of the right star orders treated in this chapter. The weak right star order $\leq*$ is further studied in Section 4.4, while the strong right star order $\preceq*$ is the topic of Chapter 5. Both of them also are treated in Section 6.4.1. The partial order $\preceq'_*$ on the other hand is an opportunity for further research.

Chapter 4

Minimal upper bounds, joins and meets under suitable conditions

In this chapter, we find conditions for the existence of meets and/or joins under different partial orders on a one-sided or both-sided Rickart ring satisfying a strongness condition. We deal with the diamond order (Section 4.1), the strong right star order (Section 4.2), the weak right star order (Section 4.4) and the star order (Section 4.5). In order to find conditions for the latter two orders, the star-free generalization of different notions of coherence are useful tools, so they are discussed before (Section 4.3).

A condition for the existence of the meet of two elements of a strong Rickart ring ordered by the diamond order is already known from [Cīrulis, 2017, Lemma 5.2] . In Section 4.1, we find a condition for the existence of the join under the diamond order (existence of diamond joins was also investigated in [Cīrulis, 2017], but only on star order initial segments – see [Cīrulis, 2017, Proposition 5.6]).

Section 4.2 explores the interplay between the weak right star order and the strong right star order on a right-strong Rickart ring and uses the connections between these closely related partial orders in order to establish conditions for the existence of the meet and join of two elements of a right-strong Rickart ring under the strong right star order.

In Section 4.3, we discuss the star-free generalizations of precoherence, weak coherence, coherence and Djikić coherence and establish how they relate to each other. We also provide some examples to illustrate their relationships.

The notions of right precoherence and right coherence are further investigated in Section 4.4. We find equivalent characterizations of right precoherence and right coherence which involve the existence of meets and joins under the weak right star order satisfying some additional condition for elements bounded from above.

In a similar way, but without assuming the existence of an upper bound, we obtain a theorem which characterizes coherence of two elements in a strong Rickart ring as being equivalent to them having the join under the star order and satisfying some additional condition in Section 4.5. We also provide an example showing that an additional condition is indeed necessary, because elements can have the star join without being coherent.

The results in Sections 4.1 and 4.4 are unpublished. Section 4.2 consists of results which were published in [Cremer, 2024, Section 5]. Most of the results in Sections 4.3 and 4.5 were published in [Cīrulis and Cremer, 2022]. Some of them are due to Cīrulis, who is the first author of [Cīrulis and Cremer, 2022]. Those results are included in this thesis because they are necessary to prove the author's own results from the same paper, and they are placed into this chapter (instead of

Section 2.1.4 in Part I) because it would be hard to separate the author's results from those of Cīrulis.

Concerning the sections which contain results from collaboration (i.e., Sections 4.3 and 4.5), everything that belongs to Cīrulis has his name attached to it in parenthesis. If this is not the case, then the respective result is either the work of the author or joint work whose authorship is hard to disentangle. In the latter case, this will be mentioned in the text. There are a few minor pieces of content in Sections 4.3 and 4.5 which did not make it into the article [Cīrulis and Cremer, 2022], hence sometimes there is no citation.

4.1 Joins under the diamond order

Recall the diamond order $\overset{\diamond}{\leq}$ on a strong Rickart ring (see Definition 2.3.3). This section deals with the structure of initial segments under the diamond order on a strong Rickart ring. First we prove two propositions and then combine them into the main result of this section, providing several equivalent conditions under which two elements which are bounded from above have the least upper bound in their respective initial segment.

Given an upper bound x of two elements a and b, the first proposition of this section constructs a certain element which, if it is also an upper bound of a and b, is a minimal upper bound of a and b

Proposition 4.1.1. *Let R be a strong Rickart ring and $a, b, x \in R$ such that $a, b \overset{\diamond}{\leq} x$. Let $u = (a^{\backprime\backprime} \vee_P b^{\backprime\backprime})x(a'' \vee_P b'')$.*

(a) *If $a, b \overset{\diamond}{\leq} u$, then $u^{\backprime\backprime} = a^{\backprime\backprime} \vee_P b^{\backprime\backprime}$ and $u'' = a'' \vee_P b''$.*

(b) *If $u^{\backprime\backprime} = a^{\backprime\backprime} \vee_P b^{\backprime\backprime}$ and $u'' = a'' \vee_P b''$, then u is a minimal upper bound of a and b.*

Proof. (a) Let $a, b \overset{\diamond}{\leq} u$. Then by Definition 2.3.3 $a^{\backprime\backprime}, b^{\backprime\backprime} \leq_E u^{\backprime\backprime}$ and $a'', b'' \leq_E u''$. Hence, $a^{\backprime\backprime} \vee_P b^{\backprime\backprime} \leq_E u^{\backprime\backprime}$ and $a'' \vee_P b'' \leq_E u''$. But by Proposition 1.3.6 (c), we have $u^{\backprime\backprime} = ((a^{\backprime\backprime} \vee_P b^{\backprime\backprime})x(a'' \vee_P b''))^{\backprime\backprime} \leq_E (a^{\backprime\backprime} \vee_P b^{\backprime\backprime})^{\backprime\backprime} = a^{\backprime\backprime} \vee_P b^{\backprime\backprime}$, and similarly, by Proposition 1.3.5 (c), $u'' = ((a^{\backprime\backprime} \vee_P b^{\backprime\backprime})x(a'' \vee_P b''))'' \leq_E a'' \vee_P b''$. Thus, $u^{\backprime\backprime} = a^{\backprime\backprime} \vee_P b^{\backprime\backprime}$ and $u'' = a'' \vee_P b''$.

(b) Let $u^{\backprime\backprime} = a^{\backprime\backprime} \vee_P b^{\backprime\backprime}$ and $u'' = a'' \vee_P b''$. Then $a^{\backprime\backprime}, b^{\backprime\backprime} \leq_E u^{\backprime\backprime}$ and $a'', b'' \leq_E u''$. Further, $a^{\backprime\backprime}ua'' = a^{\backprime\backprime}(a^{\backprime\backprime} \vee_P b^{\backprime\backprime})x(a'' \vee_P b'')a'' = a^{\backprime\backprime}xa'' = a$, because $a^{\backprime\backprime} \leq_E a^{\backprime\backprime} \vee_P b^{\backprime\backprime}$ and $a'' \leq_E a'' \vee_P b''$ and $a \overset{\diamond}{\leq} x$ (see Equation (2.30)). Similarly, also $b^{\backprime\backprime}ub'' = b$. So $a, b \overset{\diamond}{\leq} u$ by Definition 2.3.3.

To see that the upper bound u is minimal, let $y \in R$ be such that $a, b \overset{\diamond}{\leq} y \overset{\diamond}{\leq} u$. Then $a^{\backprime\backprime}, b^{\backprime\backprime} \leq_E y^{\backprime\backprime}$ by Definition 2.3.3, and therefore also $a^{\backprime\backprime} \vee_P b^{\backprime\backprime} \leq_E y^{\backprime\backprime}$. Similarly, $a'' \vee_P b'' \leq_E y''$. So $y = y^{\backprime\backprime}uy'' = y^{\backprime\backprime}(a^{\backprime\backprime} \vee_P b^{\backprime\backprime})x(a'' \vee_P b'')y'' = (a^{\backprime\backprime} \vee_P b^{\backprime\backprime})x(a'' \vee_P b'') = u$.

□

Proposition 4.1.1 provides a condition under which the element u is a minimal upper bound. There might, however, be multiple minimal upper bounds which are incomparable to each other. Fortunately we can achieve uniqueness by restricting the focus on the initial segment below x. We obtain the following proposition.

Proposition 4.1.2. *Let R be a strong Rickart ring and let $a, b, x \in R$ such that $a, b \overset{\diamond}{\leq} x$. Let $u = (a^{\backprime\backprime} \vee_P b^{\backprime\backprime})x(a'' \vee_P b'')$. If $u'' = a'' \vee_P b''$ and $u^{\backprime\backprime} = a^{\backprime\backprime} \vee_P b^{\backprime\backprime}$, then the element u is the least upper bound of the elements a and b in the initial segment $[0, x]_{\overset{\diamond}{\leq}}$.*

Proof. Let a, b, x and u be elements of a strong Rickart ring as given in the assumption, and suppose that

$$u'' = a'' \vee_P b'' \text{ and } u^{\backprime\backprime} = a^{\backprime\backprime} \vee_P b^{\backprime\backprime}. \tag{4.1}$$

Let w be an upper bound of a and b in the initial segment $[0,x]_{\overset{\diamond}{\leq}}$, i.e., $a, b \overset{\diamond}{\leq} w \overset{\diamond}{\leq} x$. Then, since $a, b \overset{\diamond}{\leq} w$, we have

$$
\begin{aligned}
u^{\backprime\backprime} &= a^{\backprime\backprime} \vee_P b^{\backprime\backprime} \leq_E w^{\backprime\backprime}, \\
u'' &= a'' \vee_P b'' \leq_E w''
\end{aligned}
\tag{4.2}
$$

by Equation (4.1) and Definition 2.3.3. Further, since $w \overset{\diamond}{\leq} x$, Definition 2.3.3 and Equation (4.2) yield $u^{\backprime\backprime} w u'' = (a^{\backprime\backprime} \vee_P b^{\backprime\backprime}) w^{\backprime\backprime} x w'' (a'' \vee_P b'') = (a^{\backprime\backprime} \vee_P b^{\backprime\backprime}) x (a'' \vee_P b'') = u$. Together with Equation (4.2), this means that $u \overset{\diamond}{\leq} w$. Thus, the element u is indeed the least upper bound of a and b in the initial segment $[0,x]_{\overset{\diamond}{\leq}}$. □

By combining Propositions 4.1.1 and 4.1.2, we obtain the following series of equivalent conditions.

Theorem 4.1.3. *Let R be a strong Rickart ring and let $a, b, x \in R$ such that $a, b \overset{\diamond}{\leq} x$ and let $u = (a^{\backprime\backprime} \vee_P b^{\backprime\backprime}) x (a'' \vee_P b'')$. Then the following are equivalent.*

(a) *$a, b \overset{\diamond}{\leq} u$,*
(b) *$u^{\backprime\backprime} = a^{\backprime\backprime} \vee_P b^{\backprime\backprime}$ and $u'' = a'' \vee_P b''$.*
(c) *u is a minimal upper bound for a and b,*
(d) *u is the least upper bound for a and b in the initial segment $[0,x]_{\overset{\diamond}{\leq}}$.*

Proof. (a) ⇒ (b) follows from Proposition 4.1.1(a).
(b) ⇒ (c) follows from Proposition 4.1.1(b).
(b) ⇒ (d) follows from Proposition 4.1.2.
(c) ⇒ (a) and (d) ⇒ (a) are obvious.

□

4.2 Conditions for existence of meets and joins under the strong right star order $\preceq *$

In this section, we first prove a lemma which establishes an interesting connection between the weak right star order and the strong right star order which is useful for dealing with the latter, since more is known about the former. The lemma is then used to prove both theorems of this section, which deal with meets and joins of elements which are bounded from above.

The first of them deals with meets, establishing a sufficient condition under which the meet under the strong right star order exists and coincides with the meet under the weak right star order. The second theorem tells us that, whenever two elements which are bounded from above have the join under the strong right star order, then this join is also their join under the weak right star order.

All results in this section were published in [Cremer, 2024].

Lemma 4.2.1. [Cremer, 2024, Lemma 5.1] *Let R be a right-strong Rickart ring and let $a, b, c \in R$ such that $a \leq * \, b \leq * \, c$. If $a \preceq * \, c$, then also $a \preceq * \, b$.*

Proof. Let a, b, c be elements of a right-strong Rickart ring R such that $a \leq * \, b \leq * \, c$. So by Definition 2.2.6 we have, in particular,

$$a = ba'', \tag{4.3}$$

$$a'' \leq_E b'', \tag{4.4}$$

$$b = cb''. \tag{4.5}$$

Now suppose that $a \preceq* c$. Then, by Lemma 3.1.3,

$$a = ec \text{ for some } e \in \mathrm{LI}(a). \tag{4.6}$$

So

$$\begin{aligned} eb &= ecb'' \text{ (by Equation (4.5))} \\ &= ab'' \text{ (by Chapter 4)} \\ &= aa''b'' \text{ (by Proposition 1.3.5(b))} \\ &= aa'' \text{ (by Equation (4.4))} \\ &= a \text{ (by Proposition 1.3.5(b))}. \end{aligned}$$

Together with Equation (4.3), this yields that $a \preceq* b$ by Lemma 3.1.3.

□

Recall Theorems 2.2.9 and 2.2.10: if two elements a, b of a right-strong right Rickart ring have an upper bound x under the weak right star order, then that their meet and join (under the weak right star order) exist, and $a \wedge* b = x(a'' \vee_P b'')$ and $a \vee* b = x(a'' \vee_P b'')$. If $a, b \preceq* x$ for elements a, b, x of a right-strong Rickart ring R, then also $a, b \leq* x$ by Lemma 3.1.4. So their meet and join under the weak right star order $\leq*$ exist. The following two theorems provide a sufficient condition for the existence of the meet and a necessary condition for the existence of the join of elements of an initial segment under the strong right star order $\preceq*$.

Theorem 4.2.2. [Cremer, 2024, Theorem 5.2] *Let R be a right-strong Rickart ring and let a, b be elements of R such that $a, b \preceq* x$ for some $x \in R$.*
(a) *If $a \wedge* b \preceq* x$, then $a \curlywedge* b$ exists and $a \curlywedge* b = a \wedge* b$.*
(b) *If $a \curlywedge* b$ exists, then it is, under the weak right star order $\leq*$, the least upper bound in $[0, x]_{\preceq*}$.*

Proof. Let R be a right-strong Rickart ring and let $a, b, x \in R$ such that

$$a, b \preceq* x. \tag{4.7}$$

(a) Assume that $a \wedge* b \preceq* x$. Then $a \wedge* b \leq* a, b \leq* x$ and $a \wedge* b \preceq* x$. So Lemma 4.2.1 yields that $a \wedge* b \preceq* a, b$. Now suppose that $y \in R$ is also a lower bound of a and b, that is,

$$y \preceq* a, b. \tag{4.8}$$

Then, by Lemma 3.1.4, we also have $y \leq* a, b$, whence $y \leq* a \wedge* b$. But Equation (4.8) also yields that $y \preceq* x$. Together with $y \leq* a \wedge* b \leq* x$, this yields that $y \preceq* a \wedge* b$ by Lemma 3.1.4. So $a \wedge* b$ is indeed also the greatest lower bound of a and b under the strong right star order $\preceq*$.
(b) Assume that $a \curlywedge* b$ exists. Since $a \curlywedge* b \preceq* a, b$, we also have $a \curlywedge* b \leq* a, b$ by Lemma 3.1.4. Hence,

$$a \curlywedge* b \leq* a \wedge* b. \tag{4.9}$$

So we have $a \curlywedge* b \leq* a \wedge* b \leq* x$, which together with $a \curlywedge* b \preceq* x$ yields that $a \curlywedge* b \preceq* a \wedge* b$ (by Lemma 4.2.1).
Now let $y \leq* a, b$ for some $y \preceq* x$. Then $y \leq* a, b \leq* x$ and $y \preceq* x$ yield $y \preceq* a, b$. So $y \preceq* a \curlywedge* b$. By Lemma 3.1.4, we obtain $y \leq* a \curlywedge* b$, which proves that $a \curlywedge* b$ is the greatest lower bound in $[0, x]_{\preceq*}$ under $\leq*$.

□

Theorem 4.2.3. [Cremer, 2024, Theorem 5.3] *Let R be a right-strong Rickart ring and let a, b be elements of R such that $a, b \preceq\!* \, x$ for some $x \in R$.*

(a) *If $a \curlyvee\!* \, b$ exists, then $a \curlyvee\!* \, b = a \vee\!* \, b$.*

(b) *If $a \vee\!* \, b \preceq\!* \, x$, then $a \vee\!* \, b$ is, under the strong right star order $\preceq\!*$, the least upper bound of a and b in $[0, x]_{\leq *}$.*

Proof. Let R be a right-strong Rickart ring and let $a, b, x \in R$ such that $a, b \preceq\!* \, x$. First note that, since we have $a, b \leq\!* \, a \vee\!* \, b \leq\!* \, x$ and $a, b \preceq\!* \, x$, Lemma 4.2.1 yields that

$$a, b \preceq\!* \, a \vee\!* \, b. \tag{4.10}$$

Now assume that $a \curlyvee\!* \, b$ exists. By Equation (4.10), $a \curlyvee\!* \, b \preceq\!* \, a \vee\!* \, b$. So by Lemma 3.1.4,

$$a \curlyvee\!* \, b \leq\!* \, a \vee\!* \, b. \tag{4.11}$$

Moreover, since $a, b \preceq\!* \, a \curlyvee\!* b$, we also have $a, b \leq\!* \, a \curlyvee\!* b$ by Lemma 3.1.4. Hence, $a \vee\!* b \leq\!* \, a \curlyvee\!* b$. Together with Equation (4.11), this yields that $a \curlyvee\!* \, b = a \vee\!* \, b$.

Conversely, assume that $a \vee\!* \, b \preceq\!* \, x$. Let $y \in [0, x]_{\leq *}$ be such that $a, b \preceq\!* \, y$. Then also $a, b \leq\!* \, y$ by Lemma 3.1.4. So $a \vee\!* \, b \leq\!* \, y$. Now $a \vee\!* \, b \leq\!* \, y \leq\!* \, x$ together with the assumption $a \vee\!* \, b \preceq\!* \, x$ implies that $a \vee\!* \, b \preceq\!* \, y$ (by Lemma 4.2.1). □

4.3 Star-free notions of coherence

In this section, we generalize the known notions of coherence on Rickart *-rings, which were discussed in Section 2.1.4, to strong Rickart rings or even right-strong right Rickart rings (respectively, left-strong left Rickart rings). The first four subsections are devoted to four types of coherence – precoherence, weak coherence, coherence and Djikić coherence. In each of these subsections, we first state the star-free generalized definition of the respective version of coherence and then, if necessary, prove that on a Rickart *-ring, the star-free notion indeed coincides with the original one. All of these notions except weak coherence have also one-sided versions, which are also defined. In Section 4.3.5, we establish relations between some of these notions.

Most of the results presented in this section are joint work of the author and her supervisor Professor Jānis Cīrulis and are published in [Cīrulis and Cremer, 2022]. The notions of weak coherence and coherence on a strong Rickart ring were introduced by Cīrulis, while the generalization of Djikić coherence was done by the author (precoherence was already defined in a star-free way in its original form in [Djikić and Djordjević, 2016], so the definition did not have to be adapted). Since the results from the paper [Cīrulis and Cremer, 2022] are interconnected, this section also includes results which were proved solely by Cīrulis, but it is usually mentioned which results are due to whom.

4.3.1 Precoherence

Since the original definitions of precoherence and its one-sided versions (see Definition 2.1.10) are already star-free, it is obvious that they also make sense in strong Rickart rings (and their appropriate one-sided versions).

Definition 4.3.1. [Cīrulis and Cremer, 2022] Elements a and b of a right-strong right Rickart ring are called *right precoherent* if $a(a'' \wedge_P b'') = b(a'' \wedge_P b'')$.

Elements a, b of a left-strong left Rickart ring are called *left precoherent* $(a^{\backprime\backprime} \wedge_P b^{\backprime\backprime})a = (a^{\backprime\backprime} \wedge_P b^{\backprime\backprime})b$.

Elements a and b of a strong Rickart ring are called *precoherent* if they are both right precoherent and left precoherent.

4.3.2 Weak coherence

In this subsection, we introduce a notion which was named by Cīrulis in [Cīrulis and Cremer, 2022]. Contrary to the other star-free notions of coherence which are introduced in this section, it was never given its own name in a less general setting, but it is nevertheless a star-free generalization of an "anonymous" condition which has already been investigated on Rickart *-rings in [Djikić and Djordjević, 2016].

We shall first mention the original condition, which is given in the context of a Rickart *-ring, then we state the definition of weak coherence, which generalizes this condition to strong Rickart rings (i.e., in a star-free way), and finally we provide the proof that in a Rickart *-ring, the original condition from [Djikić and Djordjević, 2016] is indeed equivalent to the star-free version from [Cīrulis and Cremer, 2022].

Recall [Djikić and Djordjević, 2016, Lemma 3.4] . It provides the following necessary condition for two elements a, b of a Rickart *-ring R to be precoherent:

$$aa^*b = ab^*b \text{ and } ba^*a = bb^*a. \tag{4.12}$$

Cīrulis noted that the condition (4.12) can be reformulated in a star-free way and thereby generalized to strong Rickart rings. This observation lead to the following notion.

Definition 4.3.2 (Cīrulis)**.** [Cīrulis and Cremer, 2022, Definition 3.5] Let R be a strong Rickart ring. Elements $a, b \in R$ are said to be *weakly coherent* if

$$a^{\backprime\backprime}b = ab'' \text{ and } b^{\backprime\backprime}a = ba''. \tag{4.13}$$

Obviously, any two closed idempotents in a strong Rickart ring are weakly coherent.

The following proposition, which was proved by Cīrulis, justifies the claim that Definition 4.3.2 generalizes the condition (4.12).

Proposition 4.3.3 (Cīrulis)**.** [Cīrulis and Cremer, 2022, page 4] *On a Rickart *-ring, the condition* (4.13) *is equivalent to the condition* (4.12)*.*

4.3.3 Coherence

In [Cīrulis and Cremer, 2022], we mentioned that Definition 2.1.8 is adapted from [Cīrulis, 2015a], but we did not explain in detail why it is a valid generalization of the notion defined by Equation (2.7), because this was explained in [Cīrulis, 2015a] . For the sake of completeness, we present here a direct proof for the following proposition, which shows that, on a Rickart *-ring, coherence in the sense of [Cīrulis, 2015a] and coherence in the sense of [Cīrulis and Cremer, 2022] are indeed the same thing.

Proposition 4.3.4. *Let a and b be elements of a Rickart *-ring R. Then a and b satisfy Equation* (2.7) *if and only if they are both left coherent and right coherent.*

Proof. Let R be a Rickart *-ring and let $a, b \in R$.

Suppose that a and b are right coherent and left coherent. Of course, the second part of Equation (2.7) is immediate from right coherence. The first part of Equation (2.7) follows from left coherence: if a, b are left coherent (i.e., $b^{\backprime\backprime}a = a^{\backprime\backprime}b$), then by [Cīrulis and Cremer, 2022, Theorem 3.6(c)], $a^{\backprime\backprime}b^{\backprime\backprime} = b^{\backprime\backprime}a^{\backprime\backprime}$, and therefore $a^*b = a^*a^{\backprime\backprime}b^{\backprime\backprime}b = a^*b^{\backprime\backprime}a^{\backprime\backprime}b = (b^{\backprime\backprime}a)^*b^{\backprime\backprime}a = (a^{\backprime\backprime}b)^*b^{\backprime\backprime}a = b^*a^{\backprime\backprime}b^{\backprime\backprime}a = b^*b^{\backprime\backprime}a^{\backprime\backprime}a = b^*a$.

Conversely, suppose that a and b satisfy Equation (2.7). Obviously, they are then also right coherent by the second part of Equation (2.7). It remains to prove that

Since $a'' + a' = 1'$ (by normality of the focal operation) and $aa' = 0$ (by Proposition 1.3.5(a)), Equation (2.7) also yields $a^*b = a^*b(a'' + a') = a^*(ba'') + (a^*b)a' = a^*ab'' + b^*aa' = a^*ab''$. So we have $a^*(b - ab'') = 0$, whence $(a^*)''(b - ab'') = 0$ by Remark 1.1.8. Now Proposition 1.3.5(b) yields that $(a^*)''b = ab''$. But $(a^*)'' = a^{\backprime\backprime}$ by Equation (1.18). Hence, $a^{\backprime\backprime}b = ab''$.

Similarly, we have $b^{\backprime\backprime}a = ba''$ (later we shall call elements a, b with $a^{\backprime\backprime}b = ab''$ and $b^{\backprime\backprime}a = ba''$ *weakly coherent*). But now, using (the second part of) Equation (2.7) again, we obtain $a^{\backprime\backprime}b = ab'' = ba'' = b^{\backprime\backprime}a$. So we conclude that a and b are also left coherent.

□

The following lemma shows how the conditions of coherence and weak coherence behave on the set of closed idempotents P_{rl} of a strong Rickart ring.

Lemma 4.3.5. [Cīrulis and Cremer, 2022]
(a) *Any pair of closed idempotents is weakly coherent.*
(b) *Closed idempotents are coherent if and only if they commute.*

Proof. Let p, q be closed idempotents in some strong focal Rickart ring R. Then

$$p^{\backprime\backprime} = p = p'' \text{ and } q^{\backprime\backprime} = q = q'' \tag{4.14}$$

.

(a) By Equation (4.14), $p^{\backprime\backprime}b = pq''$ and $q^{\backprime\backprime}p = qp''$, that is, p and q satisfy Equation (4.13) and therefore are weakly coherent (see Definition 4.3.2).
(b) By Equation (4.14), we have $q^{\backprime\backprime}p = p^{\backprime\backprime}q \iff pq = qp$ and also $p''q = q''p \iff pq = qp$. So obviously, p and q satisfy the conditions for being coherent (see Definition 2.1.8) if and only if they commute. Hence, by Definition 2.1.8, they are coherent if and only if they commute.

□

4.3.4 Djikić coherence

A few remarks on Djikić coherence in a Rickart *-ring

Right Djikić coherence is not only more general than coherence(see [Djikić and Djordjević, 2016, page 66]), but even more general than right coherence: For right coherent elements a, b, an element x satisfying Equation (2.8) is $x = a + b - ab''$.

We introduce a dual notion to right Djikić coherence in the following way:

Definition 4.3.6. We call elements a, b of a Rickart *-ring R *left Djikić-coherent* if there exists $x \in R$ such that

$$a^*a = a^*x \text{ and } b^*b = b^*x. \tag{4.15}$$

If a and b are both right Djikić-coherent and left Djikić-coherent, then we call them *Djikić-coherent*.

Then elements a, b are right Djikić-coherent if and only if a^* and b^* are left Djikić-coherent. To see this, assume that a, b are right Djikić-coherent. So $aa^* = xa^*$ and $bb^* = xb^*$ for some x. Hence, $(a^*)^*a^* = aa^* = (aa^*)^* = (xa^*)^* = (a^*)^*x^*$, and similarly, $(b^*)^*b^* = (b^*)^*x^*$. So indeed a^* and b^* are left Djikić-coherent.

Similarly, a, b are left Djikić-coherent if and only if a^* and b^* are right Djikić-coherent.

Now right coherence implies right Djikić coherence, left coherence implies left Djikić coherence and coherence implies Djikić coherence.

Djikić coherence in a strong Rickart ring

We now introduce a star-free generalization of Djikić coherence to strong Rickart rings. Note that its one-sided versions make sense in even more general settings: right Djikić coherence can be defined in a right-strong right Rickart ring, while left Djikić coherence can be defined in a left-strong left Rickart ring.

Definition 4.3.7. Let R be a strong Rickart ring. Elements a, b of R are said to be
(a) right Djikić-coherent if $xa'' = a$ and $xb'' = b$,
(b) left Djikić-coherent if $a^{\backprime\backprime}x = a$ and $b^{\backprime\backprime}x = b$,
(c) Djikić-coherent if they are both right Djikić-coherent and left Djikić-coherent.

This is indeed a generalization of the original notions from Rickart *-rings (recall Definition 2.1.9 and Definition 4.3.6), as the following rather trivial proposition shows.

Proposition 4.3.8. [Cīrulis and Cremer, 2022, page 2] *Let R be a Rickart *-ring and $a, b \in R$. Then the following two statements are equivalent to each other:*
(a) $xa'' = a$ *and* $xb'' = b$*,*
(b) $aa^* = xa^*$ *and* $bb^* = xb^*$*.*
Analogously, the following two statements are equivalent to each other:
(a) $a^{\backprime\backprime}x = a$ *and* $b^{\backprime\backprime}x = b$*,*
(b) $a^*a = a^*x$ *and* $b^*b = b^*x$*.*

Proof. This is immediate from Proposition 2.1.2. □

4.3.5 Relations between the coherence, Djikić coherence, weak coherence and precoherence

This subsection deals with the relations between the different notions of coherence defined in the previous subsections. First we show that Djikić coherence implies precoherence. Then we mention a lemma proved by Cīrulis which relates the other notions to each other and provide some examples which show that at least two of the implications in the lemma are really just implications.

The following proposition relates Djikić coherence to precoherence.

Proposition 4.3.9. *Let R be a strong Rickart ring and $a, b \in R$. If the elements a and b are right Djikić-coherent, then they are right precoherent.*

Proof. Assume a and b are elements of a strong Rickart ring which are right Djikić-coherent, i.e., by Definition 4.3.7, $xa'' = a$ and $xb'' = b$ for some x.

Recall that, on the set of closed idempotents P_{rl}, the star order coincides with the standard order $\leq_E$ of idempotents (by Proposition 2.1.4(b). Moreover, by Proposition 2.1.4(d), any meet of closed idempotents in P_{rl} is also their meet under the star order the entire ring R. Since the set of closed idempotents is a lattice under the standard order $\leq_E$ by Lemma A.2.7, the meet of the closed idempotents a'' and b'' must exist.

So

$$a(a'' \wedge_P b'') = xa''(a'' \wedge_P b'') = x(a'' \wedge_P b'') = xb''(a'' \wedge_P b'') = b(a'' \wedge_P b''),$$

since $a'' \wedge_P b'' \leq_E a'', b''$. □

The following lemma, which was proved by Cīrulis, clarifies how coherence, weak coherence and precoherence are related to each other on a strong Rickart ring.

Lemma 4.3.10 (Cīrulis)**.** [Cīrulis and Cremer, 2022, Theorem 3.6] *Let R be a strong Rickart ring.*

(a) *If elements $a, b \in R$ are coherent, then they are weakly coherent.*
(b) *If elements $a, b \in R$ are weakly coherent, then they are precoherent.*
(c) *Elements $a, b \in R$ are left coherent if and only if they are left precoherent and $a^{\backprime\backprime}b^{\backprime\backprime} = b^{\backprime\backprime}a^{\backprime\backprime}$. They are right coherent if and only if they are right precoherent and $a''b'' = b''a''$.*

For Rickart *-rings, it was already known from [Djikić and Djordjević, 2016, Lemma 3.4] that every pair of weakly coherent elements is also precoherent. Hence, Lemma 4.3.10(b) is a generalization of that result to strong Rickart rings.

Examples: weakly coherent, but not coherent, and precoherent, but not weakly coherent

We shall provide two examples which illustrate that the implications in items (a) and (b) of Lemma 4.3.10 are not equivalences (these examples are the work of the author, but were not included in the article [Cīrulis and Cremer, 2022]).

In order to construct concrete examples, we will use the strong focal Rickart ring described in the following example.

Example 4.3.11 ($A^{\backprime\backprime}$ **and** A'' **in** $\mathcal{B}(\mathbb{R}^2)$)**.** Let $\mathcal{B}(\mathbb{R}^2)$ be the ring of linear operators on $\mathbb{R}^2$ (in other words, the matrix ring $\mathrm{Mat}_2(\mathbb{R})$). This is a Rickart *-ring (the involution maps every operator to its adjoint (i.e., the transposed matrix)) and hence also a strong focal Rickart ring. The left focal operation $^{\backprime}$ maps every operator A to the orthogonal projection onto the orthogonal complement of its image $(\operatorname{im} A)^{\perp}$, and the right focal operation $'$ maps every operator A to the orthogonal projection onto its kernel $\ker A$. Then $A^{\backprime\backprime}$ is the orthogonal projection onto $\operatorname{im} A$ and A'' is the orthogonal projection onto $(\ker A)^{\perp}$.

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a non-zero, non-invertible operator from $\mathcal{B}(\mathbb{R}^2)$. Moreover, assume that $a \neq 0$ (this simplifies the calculations).

Then $\ker A$ is the subspace generated by the vector $\begin{pmatrix} -b \\ a \end{pmatrix}$, and if $\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right) \neq \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$, then $\operatorname{im} A$ is the subspace generated by the vector $\begin{pmatrix} a \\ c \end{pmatrix}$ (otherwise it is the subspace generated by $\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right)$).

It is easy to check that the projections, that is, the idempotent matrices in $\mathrm{Mat}_2(\mathbb{R})$, are the matrices of the form

$$E = \begin{pmatrix} x & y \\ z & 1-x \end{pmatrix}$$

with $x^2 + y^2 = x$. The orthogonal projections, that is, the idempotent self-adjoint matrices, are thus the ones that additionally satisfy $z = y$.

This allows us to easily calculate the matrices $A^{\backprime\backprime}$ and A'' (this is where the assumption $a \neq 0$ is convenient). We obtain

$$A^{\backprime\backprime} = \frac{1}{a^2 + c^2} \begin{pmatrix} a^2 & ac \\ ac & c^2 \end{pmatrix}, \quad A'' = \frac{1}{a^2 + b^2} \begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix}. \tag{4.16}$$

◁

Example 4.3.12 (**Precoherent elements which are not weakly coherent**)**.** For a concrete example of elements A and B which are precoherent, but not weakly coherent, let

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & -1 \\ 3 & 3 \end{pmatrix}.$$

Since A and B are non-zero and non-invertible and their top-left elements are non-zero, we can use Equation (4.16) to calculate

$$A^{\backprime\backprime} = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, A'' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$B^{\backprime\backprime} = \frac{1}{10}\begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix}, B'' = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

An orthogonal projection is uniquely determined by its range, and every subspace V determines a unique orthogonal projection onto V. Moreover, the lattice of orthogonal projections ordered by the standard order $\leq_E$ is isomorphic to the lattice of subspaces ordered by inclusion. Since we are considering the two-dimensional case, the lattice of subspaces consists of the null space as a bottom element, the whole space $\mathbb{R}^2$ as a top element, and between them are just the one-dimensional subspaces, which are obviously incomparable to each other. So if P and Q are orthogonal projections on some one-dimensional subspaces, then either $P = Q$ or $P \wedge_P Q = 0$.

Since we chose non-zero, non-invertible A and B, the orthogonal projections $A^{\backprime\backprime}$ and $B^{\backprime\backprime}$ (onto their images) are projections onto one-dimensional subspaces. Since they are obviously not the same, we have $A^{\backprime\backprime} \wedge_P B^{\backprime\backprime} = 0$. Similarly, also $A'' \wedge_P B'' = 0$.

Therefore, $(A^{\backprime\backprime} \wedge_P B^{\backprime\backprime})A = 0 = (A^{\backprime\backprime} \wedge_P B^{\backprime\backprime})B$ and $A(A'' \wedge_P B'') = 0 = B(A'' \wedge_P B'')$. So A and B are precoherent.

They are, however, not weakly coherent, since

$$A^{\backprime\backprime}B = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \neq \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = AB''.$$

◁

Example 4.3.13 (Weakly coherent elements which are not coherent). By Lemma 4.3.5, any closed idempotents which do not commute are weakly coherent, but not coherent. For example, in the ring $\mathcal{B}(\mathbb{R}^2)$ of linear operators on $\mathbb{R}^2$ with the focal operations described above, the operators

$$P = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \text{ and } Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

are closed idempotents, since they are orthogonal projections (see the previous example). So by Lemma 4.3.5(a) they are weakly coherent. But since they do not commute ($PQ = \frac{1}{5}\left(\begin{smallmatrix} 1 & 0 \\ 2 & 0 \end{smallmatrix}\right)$, $PQ = \frac{1}{5}\left(\begin{smallmatrix} 1 & 2 \\ 0 & 0 \end{smallmatrix}\right)$), they are not coherent by Lemma 4.3.5(b).

◁

4.4 Meets and join under the weak right star order for right-coherent elements

This section deals with the weak right star order on a right-strong right Rickart ring. We investigate meets and joins of elements satisfying certain conditions. We prove that elements are right-pcecoherent if and only if they have the meet and this meet satisfies a particular condition, and we give a similar necessary and sufficient condition for right coherence. It is known from [Cīrulis, 2015d] that elements of a right-strong right Rickart ring which have an upper bound also have the join. The main result of this section is a characterization of this join of elements which are not only bounded from above but also right-coherent.

The results presented in this section are unpublished.

The following one-sided version of Corollary 4.5 from [Cīrulis and Cremer, 2022] connects right-precoherence to weak right star meets by a necessary and sufficient condition for elements to be right-precoherent.

Proposition 4.4.1. *Elements a, b of a right-strong right Rickart ring R are right-precoherent if and only they have the meet under the weak right star order and $(a \wedge\!\ast\, b)'' = a'' \wedge_P b''$.*

Proof. Let a, b be right-precoherent, that is, $a(a'' \wedge_P b'') = b(a'' \wedge_P b'')$. We have $(a(a'' \wedge_P b''))'' = (a''(a'' \wedge_P b''))'' = a'' \wedge_P b''$. Since $a'' \wedge_P b'' \leq_E a''$, this yields that $a(a'' \wedge_P b'') \leq\!\ast\, a$. Similarly, $b(a'' \wedge_P b'') \leq\!\ast\, b$. So $a(a'' \wedge_P b'')$ is a lower bound for a and b. To see that it is their meet, let $y \leq\!\ast\, a, b$. Then $y'' \leq_E a'', b''$, and therefore also $y'' \leq_E a'' \wedge_P b''$. Hence, $a(a'' \wedge_P b'')y'' = ay'' = y$, since $y \leq\!\ast\, a$. So $y \leq\!\ast\, a(a'' \wedge_P b'')$. Thus, $a(a'' \wedge_P b'') = a \wedge\!\ast\, b$.

Now assume that a and b have the meet under the weak right star order and $(a \wedge\!\ast\, b)'' = a'' \wedge_P b''$. Then $a(a'' \wedge_P b'') = a(a \wedge\!\ast b)'' = a \wedge\!\ast b = b(a \wedge\!\ast b)'' = b(a'' \wedge_P b'')$. So a and b are right-precoherent. □

The following proposition was proved for strong Rickart rings in [Cīrulis and Cremer, 2022, Theorem 3.6(c)], but the proof used only right strongness, so it can be directly transferred to right-strong right Rickart rings.

Proposition 4.4.2. *In a right-strong right Rickart ring, if elements a, b are right-coherent, then they are right-precoherent and $a''b'' = b''a''$.*

For elements which have an upper bound, right-coherence is even equivalent to the condition $a''b'' = b''a''$.

Proposition 4.4.3. *Let a, b, x be elements of a right-strong right Rickart ring such that $a, b \leq\!\ast\, x$. Then a and b are right-coherent if and only if $a''b'' = b''a''$.*

Proof. The first direction follows from Proposition 4.4.2. For the converse, assume that $a, b \leq\!\ast\, x$ and $a''b'' = b''a''$. Since $a, b \leq\!\ast\, x$ yields $a = xa''$ and $b = xb''$, we have $ab'' = xa''b'' = xb''a'' = ba''$, i.e., a and b are right-coherent. □

The following proposition relates right coherence to weak right star meets in a way similar to Proposition 4.4.1 relating right precoherence to them.

Lemma 4.4.4. *Elements a, b of a right-strong right Rickart ring are right-coherent if and only if they have the meet under the weak right star order and $a \wedge\!\ast\, b = ab'' = ba''$.*

Proof. Obviously, if $a \wedge\!\ast\, b = ab'' = ba''$ for some elements a, b, then they are right-coherent by Definition 2.1.8.

Conversely, let a, b be right-coherent, i.e., $ab'' = ba''$.

By Proposition 4.4.2, we have $a''b'' = b''a''$. Now it can be easily checked that $a'' \wedge_P b'' = a''b''$. So in particular $a''b'' \leq_E a''$. Moreover, $a''b'' = a''b''(a''b'')'' = a''b''(ab'')'' = a''(ab'')'' = a''(ba'')'' = (ba'')''$ by Proposition 1.3.5(b) and (c). So we have $(ba'')'' \leq_E a''$.

Proposition 1.3.5(b) also yields that $ba'' \cdot a'' = ab'' \cdot a'' = aa''b'' = ab''$. Thus, $ba'' \leq\!\ast\, a$. Similarly, we can prove that $ab'' \leq\!\ast\, b$.

Now suppose $y \leq\!\ast\, a, b$. So $y'' \leq_E a'', b''$. Since $(ba'')'' = a''b'' = a'' \wedge_P b''$, we obviously have $y'' \leq_E (ba'')''$. Further, $(ba'')y'' = by'' = y$ (because, first, $y'' \leq_E a''$, and second, $y \leq\!\ast\, b$). This finishes the proof that $ba'' = a \wedge\!\ast\, b = ab''$. □

In the case of elements which are bounded from above, we can use the upper bound property (Theorem 2.2.9) to obtain further charactizations of right-coherent elements. We obtain the following improvement of Lemma 4.6 from [Cīrulis, 2015d] (see the relevant paragraph in the conclusion for the details of how it improves the latter result):

Theorem 4.4.5. *Let a, b be elements of a right-strong right Rickart ring R which have an upper bound under the weak right star order $\leq*$. Then the following are equivalent.*

(a) $a''b'' = b''a''$,

(b) *a and b are right-coherent,*

(c) $ab'' = a \wedge * \, b = ba''$,

(d) $a + ba' = a \vee * \, b = b + ab'$,

(e) $a \vee * \, b = a + b - (a \wedge * \, b)$,

(f) $a \vee * \, b = ab' + (a \wedge * \, b) + ba'$ *and* $a''b'' = b''a''$.

Proof. The equivalence of items (b) and (c) is clear from Lemma 4.4.4. By Proposition 4.4.3, they are also equivalent to item (a). We will first prove that the items (b) and (c) imply the remaining three items, and then the converse directions.

Let R be a right-strong right Rickart ring and let $a, b \in R$. Let x be an upper bound of a and b under the weak right star order. By Theorem 2.2.9, a and b have the join and

$$a \vee * \, b = x(a'' \vee_P b''). \tag{4.17}$$

Note that

$$(a \wedge * \, b)a'' = (a \wedge * \, b)(a \wedge * \, b)''a'' = a \wedge * \, b, \tag{4.18}$$

because $a \wedge * \, b \leq * \, a$ (and using Proposition 1.3.5(b)).

Assume that item (c) holds.

(c) $\Rightarrow$ (e): Since $a, b \leq * \, x$, we have $a = xa''$ and $b = xb''$. So by Equation (4.17) and Proposition A.2.4 and item (c), we obtain that $a \vee * \, b = x(a'' \vee_P b'') = x(a'' + b'' - a''b'') = a + b - ab'' = a + b - (a \wedge * \, b)$. Thus, item (e) holds.

(c) $\Rightarrow$ (d): For item (d), normality of the focal operation and item (c) yield $a+ba' = a+b(1-a'') = a + b - ba'' = a + b - (a \wedge * \, b)$. Now we can use item (e), since we have just proved that it follows from item (c), which yields $a + ba' = a \vee * \, b$. Similarly, $b + ab' = a \vee * \, b$.

(c) $\Rightarrow$ (f): For item (f), by normality of the right focal operationand item (c), we have $ab' + (a \wedge * \, b) + ba' = a - ab'' + (a \wedge * \, b) + b - ba'' = a + b - (a \wedge * \, b)$. Now we can again apply item (e), which yields the first identity of item (f). The second identity follows from item (b) by Proposition 4.4.2.

It remains to prove that each of items (d) to (f) implies right coherence of a and b.

(d) $\Rightarrow$ (c): First assume that item (d) holds. By normality of the right focal operation, $a+ba' = a+b-ab''$. Similarly, $b+ab' = a+b-ba''$. So, since by item (d) $a+ba' = a \vee * \, b = b+ab'$, we have $ab'' = ba''$, i.e., a and b are right-coherent.

(e) $\Rightarrow$ (c): Now assume that item (e) holds. Since $a \leq * \, a \vee * \, b$, we have $a = (a \vee * \, b)a'' = (a + b - (a \wedge * \, b))a'' = a + ba'' - (a \wedge * \, b)$ by Proposition 1.3.5(b) and Equation (4.18). Hence, $ba'' = a \wedge * \, b$. It is proved similarly that also $ab'' = a \wedge * \, b$. So a and b are right coherent.

(f) $\Rightarrow$ (c): Finally, assume that item (f) holds. Since $a \leq * \, a \vee * \, b$, we have $a = (a \vee * \, b)a'' = (ab' + (a \wedge * \, b) + ba')a'' = ab'a'' + (a \wedge * \, b) + 0 = a - ab''a'' + (a \wedge * \, b)$ by Equation (4.18), normality of the right focal operationand Proposition 1.3.5(b). So $ab''a'' = a \wedge * \, b$. But now the second part of item (f) ($a''b'' = b''a''$) and Proposition 1.3.5(b) yield that $a \wedge * \, b = aa''b'' = ab''$. Similarly, it can be proved that $a \wedge * \, b = ba''$. So indeed item (c) holds. □

4.5 Existence of upper bounds and joins under the star order

This section provides conditions for the existence of an upper bound or even the join of two elements of a strong Rickart ring under the star order. The first main result is a theorem stating that having an upper bound is equivalent to being both weakly coherent and right Djikić-coherent. The second main result, which is joint work with Cīrulis, shows that all coherent elements have the join, which moreover satisfies a certain equality. However, being coherent is not a necessary condition for having the join and we provide an example of elements which have the join but are not coherent.

It is necessary to include also some results which were proved by Cīrulis, and the authorship of each result is marked in the same way as in Section 4.3.

The following theorem is a star-free counterpart of the essential part of [Djikić and Djordjević, 2016, Theorem 4.1]. By and large, its proof is similar to the original one, but does not involve involution.

Theorem 4.5.1. [Cīrulis and Cremer, 2022] *Let R be a strong Rickart ring. Elements a and b of R have an upper bound under the star order if and only if they are weakly coherent and are right Djikić-coherent.*

Proof. Assume that a and b have an upper bound x. By the definition of star order (2.4) (Definition 2.1.3), then $xa'' = a$ and $xb'' = b$; thus, a and b are right Djikić-coherent (see Definition 4.3.7). Moreover, then also $a^{\backprime\backprime}b = a^{\backprime\backprime} \cdot xb'' = a^{\backprime\backprime}x \cdot b'' = ab''$, and similarly $b^{\backprime\backprime}a = ba''$, i.e., a and b satisfy Equation (4.13) from Definition 4.3.2, i.e., they are weakly coherent.

Now assume that a and b are right Djikić-coherent, i.e., that $xa'' = a$ and $xb'' = b$ for some x. If they are also weakly coherent, then the element $c := x(a'' \vee_P b'')$ is an upper bound of a and b. Indeed, $ca'' = x(a'' \vee_P b'')a'' = xa'' = a$ and similarly $cb'' = b$, and it remains to prove that $a^{\backprime\backprime}c = a$ and $b^{\backprime\backprime}c = b$. But (using Proposition 1.3.5(b) and Proposition 1.3.6(b)) $a^{\backprime\backprime}(a - c)a'' = a^{\backprime\backprime}(aa'' - ca'') = 0$ and $a^{\backprime\backprime}(a - c)b'' = a^{\backprime\backprime}ab'' - a^{\backprime\backprime}cb'' = ab'' - a^{\backprime\backprime}b = 0$ (see (4.13)). By Proposition A.2.8, now $a^{\backprime\backprime}(a - c)(a'' \vee_P b'') = 0$, i.e., $a^{\backprime\backprime}a(a'' \vee_P b'') = a^{\backprime\backprime}c(a'' \vee_P b'')$. By the choice of c, Proposition 1.3.5(b) and Proposition 1.3.6(b)) yield $a^{\backprime\backprime}c = a^{\backprime\backprime} \cdot c(a'' \vee_P b'') = a^{\backprime\backprime}a(a'' \vee_P b'') = a \cdot a''(a'' \vee_P b'') = aa'' = a$. Thus $a \overset{*}{\leq} c$, and likewise $b \overset{*}{\leq} c$, i.e., a and b have a common upper bound. □

Example 4.1 in [Djikić, 2016b] shows that the condition of weak coherence in the theorem cannot be omitted. It is noted in [Djikić and Djordjević, 2016, Remark 4.1] that in the Rickart *-ring of square matrices weak coherence alone is sufficient for the existence of a corresponding upper bound; there are also other such examples. The question if it is so in general was put in the concluding paragaraph of [Djikić, 2016b, Section 3] and is still open.

We need to state a few results which were not proved by the author, because they are used in the proof of the last theorem of this section.

Recall Theorem 2.1.7, which states that two elements of a strong Rickart ring have a meet under the star order if they have a common upper bound.

It immediately follows with the help of Proposition 1.3.5(b) that

$$(a^{\backprime\backprime} \wedge_P b^{\backprime\backprime})a = (a^{\backprime\backprime} \wedge_P b^{\backprime\backprime})b = a \overset{*}{\wedge} b = a(a'' \wedge_P b'') = b(a'' \wedge_P b'') \tag{4.19}$$

whenever the elements a and b have an upper bound; In particular, such elements are precoherent. However, a meet of two elements may exist also when they are not bounded.

Proposition 4.5.2 (Cīrulis)**.** [Cīrulis and Cremer, 2022] *Suppose that elements a and b of a strong Rickart ring have an upper bound under the star order. Then they are left (right) coherent if and only if $b^{\backprime\backprime}a = a \overset{*}{\wedge} b = a^{\backprime\backprime}b$ ($ab'' = a \overset{*}{\wedge} b = ba''$).*

The following theorem is joint work with Cīrulis. Therefore, only the part of the proof which was found by the author is included.

Theorem 4.5.3. [Cīrulis and Cremer, 2022, Theorem 5.4] *The following conditions on elements a and b of a strong Rickart ring R are equivalent:*

(a) *a and b are coherent,*

(b) *a and b have an upper bound under the star order, and $ab'' \overset{*}{\leq} a, ba'' \overset{*}{\leq} b$,*

(c) *$a \overset{*}{\vee} b$ exists, and $a + ba' = a \overset{*}{\vee} b = b + ab'$,*

(d) *$a \overset{*}{\vee} b$ exists, and*

  (1) *a and b are right coherent,*

  (1) *$a \overset{*}{\wedge} b$ exists and $a \overset{*}{\vee} b = ab' + a \overset{*}{\wedge} b + ba'$,*

(e) *$a \overset{*}{\vee} b$ exists and $a \overset{*}{\vee} b = a + b - a \overset{*}{\wedge} b$.*

Proof. (d) $\Rightarrow$ (e). If $a \overset{*}{\vee} b$ exists, then (4.19) holds. If also item (d)(1) is satisfied, then $a \overset{*}{\wedge} b = ab''$ (Proposition 4.5.2). Therefore, $a = ab' + ab'' = ab' + a \overset{*}{\wedge} b$ and likewise $b = ba' + a \overset{*}{\wedge} b$, whence, using also item (d)(1), $a + b = ab' + a \overset{*}{\wedge} b + ba' + a \overset{*}{\wedge} b = a \overset{*}{\vee} b + a \overset{*}{\wedge} b$. Then (e) follows.

(e) $\Rightarrow$ (a). Assume that $a \overset{*}{\vee} b$ exists and satisfies the equality in item (e). As $a \overset{*}{\leq} a \overset{*}{\vee} b$, then $a = (a \overset{*}{\vee} b)a'' = aa'' + ba'' - (a \overset{*}{\wedge} b)(a \overset{*}{\wedge} b)'' \cdot a'' = a + ba'' - a \overset{*}{\wedge} b$; see Equation (2.4) in Definition 2.1.3 and Proposition 1.3.5(b). Hence, $ba'' = a \overset{*}{\wedge} b$. Likewise $ab'' = a \overset{*}{\wedge} b$; so $ba'' = ab''$, and dually $a^{\backprime\backprime} b = b^{\backprime\backprime} a$. Thus a and b are coherent (Definition 2.1.8).

The rest of the theorem was proved by Cīrulis, see [Cīrulis and Cremer, 2022]. $\square$

Of course, the duals of items (b) to (d) also are equivalent to item (e) and, hence, to items (b) to (d) themselves. We do not know if also item (d)(1) is self-dual in this sense.

The following example shows that two elements of a strong Rickart ring may have an upper bound (even the join) under the star order without being coherent.

Example 4.5.4. As an example of elements which have the star join, but are not coherent, we can take again the operators P and Q from the Example 4.3.13 (or any other pair of closed idempotents P, Q which do not commute). Then, since P and Q are not coherent, we have $P \overset{*}{\vee} Q \neq P + Q - P \overset{*}{\wedge} Q$.

We know from Proposition 2.1.4(c) that the set of closed idempotents P_{rl} on a strong Rickart ring is the initial segment $[0,1]_{\overset{*}{\leq}}$ under the star order. But by Theorem 2.1.12, the initial segment $[0,1]_{\overset{*}{\leq}}$ is a lattice and the join of two elements in $[0,1]_{\overset{*}{\leq}}$ is also their join in the whole ring. So in particular P and Q must have the join in $\mathcal{B}(\mathbb{R}^2)$.

◁

Chapter 5

Properties of the strong right star order

This chapter deals with the detailed structure of a right-strong right Rickart ring R ordered by the strong right star order $\preceq*$. In Section 5.1, we introduce orthocomplementations on initial segments of R and prove that $\langle R, \preceq* \rangle$ is a relatively orthocomplemented poset. Based on this, we then introduce the strong right star orthogonality in Section 5.2. Finally, in Section 5.3, we provide an order isomorphism which also preserves this orthogonality relation from each initial segment of the ring onto a corresponding initial segment of its set of closed idempotents P_{r}.

The results presented in Section 5.1 were published in Cremer [2024], while the other two sections contain unpublished results.

5.1 Sectional orthocomplementations for the strong right star order

In [Krēmere, 2016], the author proved that a Rickart *-ring with the left star order is a quasi-orthomodular (and thus relatively orthocomplemented) poset. In this section, we generalize this result (more precisely, the right dual result) to right-strong right Rickart rings.

The content of this section was published in Cremer [2024].

We start by characterizing meets and joins of closed idempotents whose product is also a closed idempotent. (This is a special case of the general characterization for arbitrary closed idempotents (which was given in Cīrulis [2015d]), see Lemma A.2.7.)

Proposition 5.1.1. [Cīrulis, 2015d, Lemma 3.3(c), Proposition 3.4] *Let R be a right-strong Rickart ring and let $p, q \in P_{\mathrm{r}}$ be closed idempotents. If pq is also a closed idempotent, then the join of p and q is $p \vee_P q = p + q - pq$.*

This yields in particular that, for closed idempotents p, q in a right-strong Rickart ring,

$$\text{if } pq = 0, \text{ then } p \vee_P q = p + q. \tag{5.1}$$

It was also proved in Cīrulis [2015d] that the meet $p \wedge_P q$ (join $p \vee_P q$) of arbitrary closed idempotents p, q in the lattice of closed idempotents P_{r} under the usual order of idempotents is also their meet (join) in R under the weak right star order (see Lemma 4.2(d) and the paragraph under Theorem 4.3 in Cīrulis [2015d]).

It is known that, in a Rickart *-ring, the element 0 is the bottom element under both the strong right star order and the weak right star order, and for the weak right star order this has already been generalized to right-strong right Rickart rings (see Theorem 2.2.15 and Lemma 2.2.8). The following simple observation is the star-free version of this result for the strong right star order.

Proposition 5.1.2. [Cremer, 2024] *In a right-strong Rickart ring R, the element 0 is the bottom element under the partial order $\preceq*$.*

Proof. By Proposition 1.3.5(e), we have $0'' = 0$. This yields $0 = a \cdot 0 = a \cdot 0''$ for every $a \in R$. Moreover, 0 is obviously a left idempotent of itself. Since also $a = 0 \cdot a$ for every $a \in R$, Lemma 3.1.3 yields that $0 \preceq* a$. □

A right-strong Rickart ring with either of the partial orders $\preceq*$ and $\leq*$ in general does not have a top element (the exception are Boolean rings). For example, in $\langle \mathbb{Z}, \preceq* \rangle$, every non-zero element is maximal.

Recall that, by Theorem 2.2.11, a right-strong right Rickart ring with the weak right star order is a relatively orthocomplemented poset with orthocomplementations on initial segments given by Equation (2.24).

We are going to prove an analogous result for the strong right star order. In order to do that, we first need to establish an operation on every initial segment $[0, x]_{\preceq*}$, which then could be proved to be an orthocomplementation.

Recall that, by Lemma 3.1.4, if $a \preceq* x$ for elements a and x of a right-strong Rickart ring, then $a \leq* x$. So obviously $[0, x]_{\preceq*} \subseteq [0, x]_{\leq*}$. The next proposition tells us that every initial segment $[0, x]_{\preceq*}$ under the strong right star order is closed under the operation $\frac{\perp}{x}$ defined in Equation (2.24).

Proposition 5.1.3. [Cremer, 2024] *In a right-strong Rickart ring R, for $a, x \in R$, if $a \preceq* x$, then $a^{\perp}_{x} \preceq* x$.*

Proof. Let R be a right-strong Rickart ring and $a, x \in R$ such that $a \preceq* x$. Since $a \in [0, x]_{\leq*}$ by Lemma 3.1.4, we know from Theorem 2.2.11 that $a^{\perp}_{x} \in [0, x]_{\leq*}$. So $a^{\perp}_{x} = x(a^{\perp}_{x})''$ and $(a^{\perp}_{x})'' \leq_E x''$ by Definition 2.2.6. To prove that $a^{\perp}_{x} \preceq* x$, we need to find an idempotent $g \in \mathrm{LI}(a^{\perp}_{x})$ such that $a^{\perp}_{x} = gx$ (by Lemma 3.1.3).

Since $a \preceq* x$, there is an idempotent $e \in \mathrm{LI}(a)$ such that

$$a = ex. \tag{5.2}$$

Let f be an arbitrary left idempotent of x. So by Proposition 3.1.6(e), we have $f - ef \in \mathrm{LI}(x - a)$ for every $f \in \mathrm{LI}(x)$. But $x - a = a^{\perp}_{x}$ by Equation (2.24) in Theorem 2.2.11. So $f - ef \in \mathrm{LI}(a^{\perp}_{x})$.

Moreover, Proposition 1.1.6, Equation (5.2) and Equation (2.24) yield $(f - ef)x = fx - efx = x - ex = x - a = a^{\perp}_{x}$. This finishes the proof that $a^{\perp}_{x} \preceq* x$. □

Proposition 5.1.3 tells us that the restriction of the operation $\frac{\perp}{x}$ from Equation (2.24) to the initial segment $[0, x]_{\preceq*}$ is a unary operation on $[0, x]_{\preceq*}$. In the next lemma, we see that this operation is an orthocomplementation. Note that the initial segment $[0, x]_{\preceq*}$ under the strong right star order is a subset of the initial segment $[0, x]_{\leq*}$ under the weak right star order, because the strong right star order is a subrelation of the weak right star order.

Lemma 5.1.4. [Cremer, 2024] *A right-strong Rickart ring R with the strong right star order $\preceq*$ is sectionally orthocomplemented. For every $x \in R$, the orthocomplementation $x^{\perp}$ in the intitial segment $[0, x]_{\preceq*}$ is the restriction of the operation $\frac{\perp}{x}$ from Equation (2.24) to $[0, x]_{\preceq*}$.*

Proof. Let R be a right-strong Rickart ring, let $x \in R$ and let $^{\wr}_{x}$ be the restriction of the operation $^{\perp}_{x}$ defined in Equation (2.24) to the initial segment $[0, x]_{\preceq *}$. We need to prove that $^{\wr}_{x}$ is an orthocomplementation on $[0, x]_{\preceq *}$, i.e., that the conditions ($^{\perp}1$), ($^{\perp}2$), ($^{\perp}3$) and ($^{\perp}4$) hold for all $a, b \in [0, x]_{\preceq *}$ (see Definition A.1.1). It is easy to see that conditions ($^{\perp}1$) and ($^{\perp}3$) hold:

($^{\perp}1$) Since by Theorem 2.2.11 $(a^{\perp}_{x})^{\perp}_{x} = a$ holds for the operation $^{\perp}_{x}$ for every $a \in [0, x]_{\leq *}$, it must also hold for its restriction to $[0, x]_{\preceq *}$.

($^{\perp}3$) By Theorem 2.2.11, this condition holds for the operation $^{\perp}_{x}$ on $[0, x]_{\leq *}$, i.e, $0^{\perp}_{x} = x$ (since the bottom element of $[0, x]_{\leq *}$ is 0 and the top element is x). Obviously, 0 and x are also the bottom and top elements of $[0, x]_{\preceq *}$. So the condition holds.

($^{\perp}2$) Let $a, b \in [0, x]_{\preceq *}$. So in particular $b \preceq * \, x$, and therefore, by Lemma 3.1.3,

$$b = ex \text{ for some } e \in \mathrm{LI}(b). \tag{5.3}$$

We need to prove that, for all $a, b \in [0, x]_{\preceq *}$, $a \preceq * \, b$ implies $b^{\wr}_{x} \preceq * \, a^{\wr}_{x}$.
So suppose that

$$a \preceq * \, b. \tag{5.4}$$

First, Lemma 3.1.4 yields $a \leq * \, b$. Since by Theorem 2.2.11, the operation $^{\perp}_{x}$ is an orthocomplementation an $[0, x]_{\leq *}$, we also have $b^{\wr}_{x} = b^{\perp}_{x} \leq * \, a^{\perp}_{x} = a^{\wr}_{x}$, that is (by Definition 2.2.6),

$$b^{\wr}_{x} = a^{\wr}_{x} (b^{\perp}_{x})'', \tag{5.5}$$

$$(b^{\wr}_{x})'' \leq_E (a^{\wr}_{x})''. \tag{5.6}$$

According to Lemma 3.1.3, to prove that $b^{\wr}_{x} \preceq * \, a^{\wr}_{x}$, we only need to find some idempotent $g \in \mathrm{LI}(b^{\wr}_{x})$ such that $b^{\wr}_{x} = g a^{\wr}_{x}$.
Let f be an arbitrary left idempotent of x. Then by Equation (5.3) and Proposition 3.1.6(e), $f - ef$ is a left idempotent of $x - b$. But $x - b = b^{\perp} = b^{\wr}_{x}$ by Equation (2.24). So $f - ef \in \mathrm{LI}(b^{\wr}_{x})$.

Note that, since $a^{\wr}_{x} \preceq * \, x$ by Proposition 5.1.3 and since f is a left idempotent of x, Proposition 3.1.6(a) yields that

$$f a^{\wr}_{x} = a^{\wr}_{x}. \tag{5.7}$$

Moreover, Proposition 3.1.6(a) yields that

$$a = ea. \tag{5.8}$$

Therefore, $(f - ef) a^{\wr}_{x} = (1-e) f a^{\wr}_{x} = (1-e) a^{\wr}_{x} = (1-e)(x-a) = x - ex - a + a = x - b = b^{\wr}_{x}$. by Equations (5.7), (2.24), (5.8), (5.3) and again (2.24) (recall that $^{\wr}_{x}$ is just the restriction of $^{\perp}_{x}$ to the initial segment $[0, x]_{\preceq *}$). Together with Equation (5.5), this proves that $b^{\wr}_{x} \preceq * \, a^{\wr}_{x}$. So the the operation $^{\wr}_{x}$ satisfies condition ($^{\perp}2$) for the strong right star order.

($^{\perp}4$) Let $a \in [0, x]_{\preceq*}$, that is, $0 \preceq* a \preceq* x$. Obviously, 0 is a lower bound for a and $a_x^{\underline{\wr}}$ under the strong right star order $\preceq*$.

Suppose b is also a lower bound, i.e., $b \preceq* a$ and $b \preceq* a_x^{\underline{\wr}}$. Then by Lemma 3.1.4, $b \leq* a$ and $b \leq* a_x^{\underline{\wr}}$, so b is also a lower bound of a and $a_x^{\underline{\wr}}$ under the weak right star order $\leq*$. Since $\langle [0, x]_{\leq*}, \leq*, 0, x \rangle$ is orthocomplemented, the elements a and $a_x^{\underline{\wr}}$ have the meet under $\leq*$, and $a \wedge* a_x^{\underline{\wr}} = 0$. So, given that b is a lower bound, we conclude that $b \leq* 0$. This means that $b = 0$, because 0 is the bottom element of $[0, x]_{\leq*}$.

Hence $0 = a \curlywedge* a_x^{\underline{\wr}}$, and this proves that $\langle [0, x]_{\preceq*}, \preceq*, 0, x \rangle$ is orthocomplemented.

Since every initial segment $[0, x]_{\preceq*}$ is orthocomplemented, the poset $\langle R, \preceq* \rangle$ is sectionally orthocomplemented. □

Before proving the main result of this section, we need to state two auxiliary propositions which will be used in the proof. Since a'' is a right idempotent of a, Proposition 5.1.5 is an immediate consequence of [Cīrulis, 2015d, Theorem 5.3]. (That theorem states that, if $a, b \leq* x$ for some elements a, b, x of a right-strong Rickart ring, then the condition $a \leq* b_x^{\perp}$ is equivalent to both item (a) and item (c) of Proposition 5.1.5. Since a'' is a right idempotent of a, items (a) and (b) are obviously also equivalent.)

Proposition 5.1.5. [Cremer, 2024] *Let a, b, x be elements of a right-strong Rickart ring R such that $a, b \leq* x$. If $a \leq* b_x^{\perp}$, then*
(a) $a''b'' = 0$,
(b) $ab'' = 0$.
(c) $a \vee* b$ *exists and* $a \vee* b = a + b$,

For dealing with the strong right star order, Proposition 5.1.5 is not enough. We need to say something also about left idempotents.

Proposition 5.1.6. [Cremer, 2024]
Let a, b, x be elements of a right-strong Rickart ring R such that $a, b \leq* x$. Let e be a left idempotent of a such that $a = ex$, and let f be a left idempotent of b. If $a''b'' = 0$, then
(a) $eb = 0$,
(b) $ef = 0$,
(c) $fa = 0$.

Proof. Let $a, b \in R$ such that $a, b \leq* x$, and suppose that

$$a''b'' = 0 \tag{5.9}$$

Let e be a left idempotent of a such that

$$a = ex, \tag{5.10}$$

and let f be a left idempotent of b.
(a) Since $b \leq* x$, we have in particular $b = xb''$ by Definition 2.2.6. In combination with Equation (5.10), Proposition 1.3.5(b) and Equation (5.9), this yields $eb = exb'' = ab'' = aa''b'' = 0$.
(b) Since f is a left idempotent of b, the previous item implies $ef = 0$.
(c) The proof is similar to the proof of item (a).

□

Now we can finally prove the analogous result to the first part of Theorem 2.2.11 for the strong right star order:

Theorem 5.1.7. [Cremer, 2024] *A right-strong Rickart ring R with the strong right star order and sectional orthocomplementations as in Lemma 5.1.4 is relatively orthocomplemented. Moreover, if $a, b \preceq* x$ and $a \preceq* b_x^{\perp}$ for some $a, b, x \in R$, then $a \curlyvee* b = a \vee* b$.*

Proof. Let R be a right-strong Rickart ring. By Lemma 5.1.4, it is sectionally orthocomplemented. According to Definition A.4.4, to show that it is even relatively orthocomplemented, we have to prove that the conditions (ro1) and (ro2) hold.

(ro1) Let $a, b, x \in R$ such that

$$a \preceq* b_x^{\perp}. \tag{5.11}$$

So in particular $a, b \in [0, x]_{\preceq*}$, and by Lemma 3.1.4 also

$$a, b \leq* x. \tag{5.12}$$

We know from Theorem 2.2.11 that the join of a and b under the weak right star order exists and

$$a \vee* b = x(a'' \vee_P b''), \tag{5.13}$$

where $a'' \vee_P b''$ is the join of the closed idempotents a'' and b'' in the set of closed idempotents P_{r} under the usual order of (closed) idempotents. We are going to prove that the element $a \vee* b$ is also the join of a and b under the strong right star order, thereby proving also the second part of the theorem.

In view of Equations (5.11) and (5.12), we can use Proposition 5.1.5. Item (a) yields that

$$a''b'' = 0 = b''a'', \tag{5.14}$$

(the second identity follows from Definition 1.3.3, since a'' and b'' are closed idempotents and their product 0 is also a closed idempotent ($0 = 0''$ by Proposition 1.3.5(e)). Moreover, by Proposition 5.1.5(c), we can express the join of a and b under the weak right star order as

$$a \vee* b = a + b. \tag{5.15}$$

Now let us prove that $a \preceq* a \vee* b$. First, since $a \leq* a \vee* b$, we have $a = (a \vee* b)a''$ by Definition 2.2.6.

Second, we know from Lemma 3.1.3 that $a = ex$ for some $e \in \mathrm{LI}(x)$ and $b = fx$ for some $f \in \mathrm{LI}(b)$, because $a \preceq* x$. Proposition 5.1.6(b),(c) yield that $eb = 0$ and $fa = 0$, because $a''b'' = 0$ (Equation (5.14)). So $e(a \vee* b) = e(a + b) = ea + eb = a$ by Equation (5.15) and Proposition 1.1.6, and in the same way, $f(a \vee* b) = f(a + b) = fa + fb = b$

So indeed $a \preceq* a \vee* b$ and $b \preceq* a \vee* b$.

Now, to prove that $a \vee* b = a \curlyvee* b$, suppose that

$$a, b \preceq* y \tag{5.16}$$

for some $y \in R$. Then by Lemma 3.1.4, we also have $a, b \leq* y$. Hence $a \vee* b \leq* y$. In particular, $a \vee* b = y(a \vee* b)''$ by Definition 2.2.6. So in view of Lemma 3.1.3, to show that $a \vee* b \preceq* y$, we only need to find an idempotent $i \in \mathrm{LI}(a \vee* b)$ such that $a \vee* b = iy$.

By Lemma 3.1.3, Equation (5.16) yields that

$$a = hy \text{ for some } h \in \mathrm{LI}(a), \tag{5.17}$$

$$b = gy \text{ for some } g \in \mathrm{LI}(b), \tag{5.18}$$

So by Equation (5.15), we have $a \vee\!* \, b = a + b = (h + g)y$.

Let us prove that $h + g$ is a left idempotent of $a \vee\!* \, b$.

First note that, since $a, b \leq\!* \, y$, $a''b'' = 0$ and $a = hy$ (Equation (5.17)), we have $hg = 0$ by Proposition 5.1.6(b). We can again apply Proposition 5.1.6(b) to obtain that $gh = 0$, because $b, a \leq\!* \, y$, $b''a'' = 0$ and $b = gy$ (Equation (5.18)). So $h + g$ is idempotent: $(h + g)^2 = h^2 + hg + gh + g^2 = h + g$.

Obviously, if $u(h + g) = 0$ for some $u \in R$, then $u(a + b) = u(h + g)y = 0$. Conversely, suppose $u(a + b) = 0$. Then $ua = -ub$. So multiplying with a'' from the right gives $uaa'' = -uba'' = ubb''a'' = 0$ by Proposition 1.3.5(b) and Equation (5.14). Hence $ua = 0$, and therefore also $ub = 0$. From these identities we obtain thot also $uh = 0$ and $ug = 0$, since h is a left idempotent of a and g is a left idempotent of b. So indeed $u(h + g) = 0$.

This proves that $a \vee\!* \, b \preceq\!* \, y$. So indeed $a \vee\!* \, b = a \curlyvee\!* \, b$, which proves also the second part of the theorem.

(ro2) Let a, x, y be elements of R such that $a \preceq\!* \, x$ and $x \preceq\!* \, y$. We will show that $a_x^\perp = a_y^\perp (a_x^\perp)''$ and that there exists an idempotent $g \in \mathrm{LI}(a_x^\perp)$ such that $a_x^\perp = g a_y^\perp$. By Lemma 3.1.3, this is enough to prove that $a_x^\perp \preceq\!* \, a_y^\perp$.

First, the assumptions $a \preceq\!* \, x$ and $x \preceq\!* \, y$ yield that also

$$a_x^\perp \preceq\!* \, x, \tag{5.19}$$

and therefore also

$$a_x^\perp \preceq\!* \, y. \tag{5.20}$$

Moreover, by Equation (2.24), we have $a_x^\perp = x - a$ and $a_y^\perp = y - a$. Hence, $a_y^\perp = y - x + x - a = y - x + a_x^\perp$. So

$$\begin{aligned} a_y^\perp (a_x^\perp)'' &= (y - x + a_x^\perp)(a_x^\perp)'' \\ &= y(a_x^\perp)'' - x(a_x^\perp)'' + a_x^\perp \text{ (by Proposition 1.3.5(b))} \\ &= y(a_x^\perp)'' - a_x^\perp + a_x^\perp \text{ (by Equation (5.19) and Lemma 3.1.3)} \\ &= a_x^\perp - a_x^\perp + a_x^\perp \text{ (by Equation (5.20) and Lemma 3.1.3)} \\ &= a_x^\perp. \end{aligned}$$

It remains to find a left idempotent g of $a_x^\perp$ such that $a_x^\perp = g a_y^\perp$. By Equation (5.19) and Lemma 3.1.3,

$$a_x^\perp = ex \text{ for some } e \in \mathrm{LI}(a_x^\perp). \tag{5.21}$$

In the same way, the assumption $x \preceq\!* \, y$ yields that

$$x = fy \text{ for some } f \in \mathrm{LI}(x). \tag{5.22}$$

Hence, by Proposition 3.1.6(d), the element ef is a left idempotent of $a_x^\perp$. Moreover,

$$\begin{aligned} efa_y^\perp &= efya' \text{ (by Equation (2.24))} \\ &= exa' \text{ (by Equation (5.22))} \\ &= ea_x^\perp \text{ (by Equation (2.24))} \\ &= a_x^\perp \text{ (by Proposition 1.1.6, since } e \in \mathrm{LI}(a_x^\perp)). \end{aligned}$$

So ef is a left idempotent of $a_x^\perp$ with the necessary property, and thus, $a_x^\perp \preceq* a_y^\perp$. This finishes the proof of (ro2).

We conclude that the right-strong Rickart ring R with the strong right star order and the restrictions of the sectional orthocomplementations from Equation (2.24) is relatively orthocomplemented (see Definition A.4.4). □

Theorem 5.1.7 immediately yields that the initial segments $[0, x]_{\preceq*}$ are orthomodular posets (see Remark A.4.6).

5.2 The strong right star orthogonality

In this section, we characterize the orthogonality of the relatively orthocomplemented poset $\langle R, \preceq* \rangle$. We shall also see that the poset $\langle R, \preceq* \rangle$ is quasi-orthomodular with this orthogonality.

The content of this section is unpublished. However, in the special case of Rickart *-rings, the author has introduced and characterized the dual notion (the left-star orthogonality) in Krēmere [2016]. Hence this section contains star-free generalizations of the duals of [Krēmere, 2016, Definition 6, Theorem 5.4].

Let $\langle R, \preceq* \rangle$ be a right-strong Rickart ring with the strong right star order. By Theorem 5.1.7, $\langle R, \preceq* \rangle$ is a relatively orthocomplemented poset. As noted in Section A.4.3 in the appendix, the orthogonality relations $\perp_x$ induced by the orthocomplementations $\overset{\perp}{x}$ in the initial segments $[0, x]_{\preceq*}$ are compatible in the sense that they agree on the intersection of their respective initial segments, giving rise to a global orthogonality of the relatively orthocomplemented poset $\perp$ defined by Equation (A.11). It is known from Theorem A.4.7 that $\langle R, \preceq*, \perp \rangle$ is a quasi-orthomodular poset.

In Krēmere [2016], the left-star orthogonality was introduced by first defining a relation and then proving that is is an orthogonality relation which gives rise to a quasi-orthomodular poset (the fact that a Rickart *-ring under the left star order is relatively orthocomplemented was then derived from that). Here, we took a different approach, starting by proving that a right-strong Rickart ring with the strong right star order is relatively orthocomplemented, from which we can now deduce that is is quasi-orthomodular with the orthogonality relation obtained from Equation (A.11). However, it would be desireable to have also an alternative characterization of this orthogonality which would be a star-free version of the dual of [Krēmere, 2016, Definition 6]. This is achieved by the following Theorem.

Theorem 5.2.1. *Let R be a right-strong Rickart ring and let $\perp$ be the orthogonality of the relatively orthocomplemented poset $\langle R, \preceq* \rangle$ (where $\preceq*$ denotes the strong right star order). Then $a \perp b$ if and only if*

$$a'' \perp b'' \text{ and } fa = 0 \text{ for some } f \in \mathrm{LI}(b), \tag{5.23}$$

where $\perp$ denotes the orthogonality of closed idempotents from Equation (A.9).

Proof. Let R be a right-strong Rickart ring and let a, b be elements of R.

"$\Rightarrow$"

Let $a \perp b$. So by Equation (A.11) we have $a \perp_x b$ for some $x \in R$. In other words,

$$a \preceq\!* b_x^{\perp}. \tag{5.24}$$

(see Definition A.1.2). By Lemma 5.1.4, the sectional orthocomplementation ${}_x^{\perp}$ of $\langle R, \preceq\!*\rangle$ in the initial segment $[0, x]_{\preceq *}$ is the restriction (to $[0, x]_{\preceq *}$) of the sectional orthocomplementation ${}_x^{\perp}$ corresponding to the weak right star order, which is defined by Equation (2.24) as $b_x^{\perp} := x - b$. Hence, we can rewrite Equation (5.24) as

$$a \preceq\!* x - b. \tag{5.25}$$

By Lemma 3.1.3, this means in particular that

$$a = e(x - b)'' \tag{5.26}$$

for some $e \in \mathrm{LI}(a)$. Moreover, by Lemma 3.1.4 and Definition 2.2.6, Equation (5.25) yields also

$$a'' \leq_E (x - b)''. \tag{5.27}$$

Since also $b \preceq\!* x$, we again use Lemma 3.1.3 and obtain that $b = fx$ for some $f \in \mathrm{LI}(b)$. Let g be a left idempotent of x. Then Proposition 3.1.6(b) yields that $g - fg \in \mathrm{LI}((x - b))$. Now Equation (5.25) and Proposition 3.1.6(a) yield that

$$a = (g - fg)a. \tag{5.28}$$

But also $a \preceq\!* x$, so applying Proposition 3.1.6(a) again, we obtain $a = ga - fga = a - fa$. Therefore, $fa = 0$.

It remains to prove that $a'' \perp b''$. Recall that, by Equation (2.24), we can rewrite Equation (5.24) as

$$a \preceq\!* xb'. \tag{5.29}$$

Hence, Lemma 3.1.4 and Definition 2.2.6 yield that $a'' \leq_E (xb')''$. But by Proposition 1.3.5(c) and normality of the focal operation, this implies $a'' \leq_E (b')'' = b'$. Now we obtain from the definition of the standard order of idempotents $\leq_E$ (Definition A.2.2) and normality of the focal operation that $a'' = a''b' = a''(1 - b'') = a'' - a''b''$. This yields $a''b'' = 0$, so by Equation (A.9), we have $a'' \perp b''$.

"$\Leftarrow$"

Conversely, assume that Equation (5.23) holds for the elements $a, b \in R$. In order to prove that $a \perp b$, we need to find an element $x \in R$ such that $a, b \preceq\!* x$ and $a \perp_x b$. By Lemma 5.1.4 and Equation (2.24), that means that we are searching for x such that $a, b \preceq\!* x$ and $a \preceq\!* x - b$. We shall prove that these requirements are satisfied by the choice of $x = a + b$. The second requirement is obvious: By reflexivity, $a \preceq\!* (a + b) - b$.

To see that $a, b \preceq\!* a + b$, first note that, since the element f from Equation (5.23) is a left idempotent of b, Proposition 1.1.6 yields that

$$fb = b \tag{5.30}$$

and therefore, in view of Equation (5.23), we have

$$f(a + b) = fa + fb = 0 + b = b. \tag{5.31}$$

Moreover, Proposition 1.3.5(b), the first part of Equation (5.23) and again Proposition 1.3.5(b) yield that $(a+b)b'' = (aa''+b)b'' = aa''b'' + bb'' = 0 + bb'' = b$. Together with Equation (5.31) this means that

$$b \preceq\!* \ a+b \tag{5.32}$$

(by Lemma 3.1.3).

It remains to prove that also $a \preceq\!* \ a+b$. First, note that $(a+b)a'' = (a+bb'')a'' = aa''+bb''a'' = a$ (by applying twice Proposition 1.3.5(b) and then the first part of Equation (5.23)). So now, by Lemma 3.1.3, we only need to find a left idempotent e of a such that $a = e(a+b)$.

Let $g \in \mathrm{LI}((a+b))$. Then by Proposition 3.1.6(e), Equations (5.32) and (5.31) yield that $g - fg \in \mathrm{LI}((a+b)-b) = \mathrm{LI}(a)$. Further we have

$$\begin{aligned}(g-fg)(a+b) &= (1-f)g(a+b)\\ &= (1-f)(a+b) \text{ (by Proposition 1.1.6, since } f \in \mathrm{LI}(a+b))\\ &= a+b-f(a+b)\\ &= a+b-b \text{ (by Equation (5.31))}\\ &= a.\end{aligned} \tag{5.33}$$

We conclude that indeed $a \preceq\!* \ a+b$, which finishes the proof. □

Remark 5.2.2. On the set of closed idempotents, the strong right star orthogonality $\perp$ obviously coincides with the orthogonality relation $\perp$ defined in Equation (A.9). ◁

Since a right-strong Rickart ring with the strong right star order is a relatively orthocomplemented poset (Theorem 5.1.7), the following corollary is immediate from Theorem A.4.7 and Theorem 5.2.1.

Corollary 5.2.3. *Let R be a right-strong Rickart ring, let $\preceq\!*$ denote the strong right star order on R and let $\perp$ be the relation characterized by Equation* (5.23) *(i.e., $a \perp b$ if and only if Equation* (5.23) *holds). Then $\langle R, \preceq\!*, \perp\rangle$ is a quasi-orthomodular poset.*

In particular, since $\perp$ is an orthogonality relation, the symmetry property (see Definition A.3.1) implies that we could alternatively characterize $\perp$ by the equivalence $a \perp b$ if and only if

$$a'' \perp b'' \text{ and } eb = 0 \text{ for some } e \in \mathrm{LI}(a). \tag{5.34}$$

5.3 Isomorphisms for initial segments

For the star order on a Rickart *-ring, an ortho-isomorphism between an initial segment $[0,x]_\leq$ and a certain subset of the initial segment $[0,x'']_\leq$ was given in Janowitz [1983]. In the more general case of a strong Rickart ring, it is known from [Cīrulis, 2016, Theorem 5.6] that the star order initial segments are isomorphic to initial segments of a certain sublattice of ${P_\mathrm{r}}^2$. Similarly, it is known from [Cīrulis, 2015d, Corollary 5.2 and Theorem 4.4] that the initial segments $[0,x]_{\leq *}$ under the weak right star order are order isomorphic to initial segments $[0,x'']_\leq$ of the lattice of closed idempotents (see also Theorem 2.2.12).

For the strong right star order, we present two order isomorphisms between initial segments and suitable isomorphic posets in this section. The first of them (see Section 5.3.1) maps the initial segment onto a subset of the corresponding initial segment of closed idempotents, and the second (see Section 5.3.2) is a less appealing poset consisting of equivalence classes in the Cartesian product of $E \times P_\mathrm{r}$.

The results presented in this section are unpublished.

5.3.1 Isomorphism onto subset of initial segment of P_{r}

Recall the isomorphism $\phi_x^{\leq *}$ defined in initial segments of a right-strong right Rickart ring under the weak right star order from Theorem 2.2.12

$$\phi_x^{\leq *} : [0, x]_{\leq *} \to [0, x'']_{\leq}$$
$$a \mapsto a''$$

Of course, by Lemma 4.2.1, the restriction of the order isomorphism $\phi_x^{\leq *}$ to the initial segement $[0, x]_{\preceq *}$ is an order embedding. We have the following theorem.

Theorem 5.3.1. *Let R be a right-strong Rickart ring and let $x \in R$. The map*

$$\phi_{\preceq *} : [0, x]_{\preceq *} \to \{p \in [0, x'']_{\leq} \mid (\exists e \in \mathrm{LI}(xp))ex = xp\}$$
$$a \mapsto a''$$

is an order isomorphism which preserves the orthogonality relation $\perp$.

Proof. Let $\phi_{\preceq *}$ be the restriction of the map $\phi_x^{\leq *}$ from Theorem 2.2.12. It is obviously an order embedding, since ϕ is an order isomorphism.

It remains to prove that $\mathrm{im}\, \phi_{\preceq *} = \{p \in [0, x]_{\leq} \mid (\exists e \in \mathrm{LI}(xp))ex = xp\}$. Obviously, if $a \preceq * \, x$, then $xa'' = a = ex$ for some $e \in \mathrm{LI}(a) = \mathrm{LI}(xa'')$. Since also $a'' \leq_E x''$, we see that a'' belongs to the set in question.

Conversely, suppose $p \in [0, x'']_{\leq}$ is such that $xp = ex$ for some $e \in \mathrm{LI}(xp)$. Then by Lemma 3.1.3, since $x(xp)'' = xp''(xp)'' = xp(xp)'' = xp$ (by Proposition 1.3.5(c), closedness of p and by Proposition 1.3.5(b)), we have that $xp \preceq * \, x$. But then, since $p \leq_E x''$, we have $\phi_{\preceq *}(xp) = (xp)'' = (x''p)'' = p'' = p$ by Proposition 1.3.5((d)) and closedness of p. So indeed $p \in \mathrm{im}\, \phi_{\preceq *}$. Hence, $\phi_{\preceq *}$ is an order isomorphism.

To see that the order isomorphism $\phi_{\preceq *}$ preserves orthogonality, recall that the orthogonality relation $\perp$ agrees with the orthogonality relation $\perp$ on the set of closed idempotents (Remark 5.2.2). Let $a \perp b$ for some $a, b \in [0, x]_{\preceq *}$. Then $a'' \perp b''$ by Theorem 5.2.1. Conversely, let a, b be elements of the initial segment $[0, x]_{\preceq *}$ such that $a'' \perp b''$. Then there exists $f \in \mathrm{LI}(b)$ such that $fx = b$. Hence, we have $fa = fxa'' = ba'' = bb''a'' = 0$, since $a \preceq * \, x$ and $a'' \perp b''$ (using Proposition 1.3.5(b)). This proves that indeed $a \perp b$ by Theorem 5.2.1. □

5.3.2 Isomorphism into subset of equivalence classes of $E \times P_{\mathrm{r}}$

The second isomorphism requires a few auxiliary propositions and definitions.

Definition 5.3.2. The *right preorder* on the set of idempotents E of a right-strong Rickart ring (or a semigroup) is defined by

$$e \lesssim_r f \text{ iff } fe = e. \tag{5.35}$$

It is obvious that the relation $\lesssim_r$ is reflexive. To see that it is also transitive, let $e \lesssim_r f$ and $f \lesssim_r g$. So $fe = e$ and $gf = f$. But then also $ge = gfe = fe = e$, that is, $e \lesssim_r g$.

We will also need the corresponding equivalence relation.

Definition 5.3.3. The *right equivalence* on the set of idempotents of a right-strong Rickart ring (or a semigroup) is the equivalence relation $\sim_r$ defined by

$$e \sim_r f \text{ iff } (ef = f \text{ and } fe = e) \tag{5.36}$$

Proposition 5.3.4. *For an idempotent e in a right-strong Rickart ring R, the equivalence class of e under the relation $\sim_r$ is the set* $\mathrm{LI}(e)$.

Proof. Let $e, f \in E$. We have to prove that $e \sim_r f$ if and only if $e \in \mathrm{LI}(f)$.

Suppose $e \sim_r f$ and let $x \in R$. We have $fe = e$ and $ef = f$. So, if $xe = 0$, then also $xf = xef = 0 \cdot f = 0$. Similarly, if $xf = 0$, then also $xe = xfe = 0$. So indeed e is a left idempotent of f (and vice versa).

Conversely, assume that $e \in \mathrm{LI}(f)$. So $xe = 0$ if and only if $xf = 0$. Choosing $x = 1 - e$, we obtain $(1-e)f = 0$, because $(1-e)e = 0$. Thus, $f = ef$. Similarly, choosing $x = 1 - f$ yields that $e = fe$. Hence, $e \sim_r f$.

□

Let

$$\widetilde{E} := \{\mathrm{LI}(e) \mid e \in E\}. \tag{5.37}$$

The partial order induced on $\widetilde{E}$ by the preorder $\lesssim_r$ will be denoted by $\leq$ (i.e., for idempotents e and f, $\mathrm{LI}(e) \leq \mathrm{LI}(f)$ if and only if $e \lesssim_r f$).

Proposition 5.3.5. *For idempotents e, f in a right-strong Rickart ring R, if $e \lesssim_r f$, then ef is an idempotent and $ef \sim_r e$.*

Proof. Let $e \lesssim_r f$. Then $fe = e$, and therefore, $(ef)^2 = efef = eef = ef$, so ef is idempotent. Since $efe = ee = e$ and $eef = ef$, we also have $ef \sim_r e$. □

Proposition 5.3.6. *If $a, b \preceq\!* x$, then $a \preceq\!* b$ if and only if $a'' \leq_E b''$.*

Proof. Let $a, b \preceq\!* x$. Obviously, if $a \preceq\!* b$, then by Lemma 3.1.4 and Definition 2.2.6, $a'' \leq_E b''$. Conversely, if $a'' \leq_E b''$, then, by the isomorphism theorem for the weak right star order (Theorem 2.2.12), we have $a \leq\!* b$. But then, by Lemma 4.2.1, we also have $a \preceq\!* b$. □

The next theorem provides an isomorphism between initial segments of R under the strong right star order and initial segments of a subset of $\widetilde{E} \times P_\mathrm{r}$.

Theorem 5.3.7. *Let R be a right-strong Rickart ring and let $x \in R$. Then the map*

$$\begin{aligned} \phi :\ [0, x]_{\preceq *} &\to \widetilde{E} \times P_\mathrm{r} \\ a &\mapsto (\mathrm{LI}(a), a'') \end{aligned}$$

is an order embedding and

$$\operatorname{im}\phi = \{(\mathrm{LI}(e), p) \in \widetilde{E} \times P_\mathrm{r} \mid \mathrm{LI}(e) \leq \mathrm{LI}(x) \text{ and } p \leq_E x'' \text{ and } ex = xp\}. \tag{5.38}$$

Proof. Let $a, b \preceq\!* x$.

Obviously, if $a \preceq\!* b$, then $\mathrm{LI}(a) \leq \mathrm{LI}(b)$ by Proposition 3.1.6(b). Also $a'' \leq_E b''$ by Lemma 3.1.4 and Definition 2.2.6. So $(\mathrm{LI}(a), a'') \leq (\mathrm{LI}(b), b'')$ in the Cartesian product $\widetilde{E} \times P_\mathrm{r}$.

Conversely, assume that $(\mathrm{LI}(a), a'') \leq (\mathrm{LI}(b), b'')$. So in particular $a'' \leq_E b''$, and thus Proposition 5.3.6 yields that $a \preceq\!* b$.

For the image of ϕ, let us denote

$$Q_x := \left\{(\mathrm{LI}(e), p) \in \widetilde{E} \times P_\mathrm{r} \,\middle|\, \mathrm{LI}(e) \leq \mathrm{LI}(x) \text{ and } p \leq_E x'' \text{ and } ex = xp\right\}. \tag{5.39}$$

It is easy to see that, for $a \preceq\!* x$, the image $\phi(a)$ is in Q_x: Since $a \preceq\!* x$, there exists $e \in \mathrm{LI}(a)$ such that $a = ex$. Then $fe = e$ for all $f \in \mathrm{LI}(x)$ (by Proposition 3.1.6), so $\mathrm{LI}(e) \leq \mathrm{LI}(f)$. By

Lemma 3.1.4 and Definition 2.2.6, we have $a'' \leq_E x''$. Since $ex = a = xa''$ and $a'' \in P_{\mathrm{r}}$, this shows that $\phi(a) \in Q_x$.

Conversely, let $(\mathrm{LI}(e), p) \in Q_x$, and let

$$a = ex. \tag{5.40}$$

We will prove that $a \preceq * \, x$ and $\phi(a) = (\mathrm{LI}(e), p)$.

Let f be a left idempotent of x. Then, since $\mathrm{LI}(f) = \mathrm{LI}(x)$, we have $\mathrm{LI}(e) \leq \mathrm{LI}(f)$ by Equation (5.39), and therefore $e \lesssim_r f$. So by Proposition 5.3.5, ef is an idempotent and $ef \sim_r e$. But ef is a left idempotent of a: for $t \in R$, $ta = 0$ if and only if $tex = 0$, and $tex = 0$ if and only if $tef = 0$. So also e is a left idempotent of a. So

$$\mathrm{LI}(e) = \mathrm{LI}(a). \tag{5.41}$$

In view of Equation (5.40), to prove that $a \preceq * \, x$ it remains to show that $a = xa''$. Obviously, $a = ex = efx$.

Moreover, since $a = ex = xp$ (by the definition of Q_x), we have $xa'' = x(xp)'' = xp''(xp)'' = xp(xp)'' = xp = a$ by Proposition 1.3.5(c) and (b) (recall that $p'' = p$, since p is closed, see Remark 1.3.2).

So by Lemma 3.1.3, $a \preceq * \, x$. Further, by Equation (5.39), we also have $p'' \leq_E x''$, and therefore, $a'' = (xp)'' = (x''p)'' = p'' = p$ by Proposition 1.3.5(d) and closedness of p. So $\phi(a) = (\mathrm{LI}(a), a'') = (\mathrm{LI}(e), p)$ (by Equation (5.41)).

□

Chapter 6

Weak differences, weak BCK-algebras and applications to certain partial orders on rings

This chapter introduces a weak difference on a poset as a generalization of a difference (see Definition A.5.1) on a poset and deals with connections between weak differences and weak BCK-algebras (see Definition A.6.3 for the latter). In particular, conditions under which a poset with a certain weak difference is a meet semilattice are provided. These conditions are then applied to some partial orders on rings, namely the weak right star order and the strong right star order on a right-strong Rickart ring, as well as the sharp order on a certain subset of an arbitrary unitary ring (recall Definitions 2.2.6 and 2.3.5 and Definition 3.1.1 of the respective orders).

One of the main results of this chapter, Theorem 6.2.1, provides some necessary and sufficient conditions (involving the existence of a particular weak BCK-algebra) under which a poset equipped with a weak difference and satisfying some additional properties is a meet semilattice. Theorem 6.2.1 serves as a unified approach by which we obtain analogous meet semilattice conditions for several partial orders on certain rings.

A meet semilattice condition which was given for the star order on a Rickart *-ring in [Cīrulis, 2015a, Lemma 5.4 and Theorem 5.5] involves the existence of a specific binary operation. The operation was called *star minus* there and it was noted that it is the binary operation of a weak BCK-algebra. This semilattice condition is actually a special case of our Theorem 6.2.1. In this chapter, we find analogous semilattice conditions also for the one-sided versions of the star order and the sharp order.

The structure of the chapter is as follows. Section 6.1 connects weak differences to weak BCK-algebras in a way which is similar to the connections between differences and BCK-algebras given in [Dvurečenskij and Kim, 1998]. First, Section 6.1.2 deals with obtaining a weak BCK-algebra from a semilattice equipped with a weak difference and the bottom 0. In Section 6.1.3, we construct a certain binary operation on a particular given weak BCK-algebra and investigate under what conditions it is a weak difference.

In Section 6.2, we deal with a poset with a bottom element equipped with a weak difference satisfying an additional condition. Section 6.2.1 contains the main result, which provides necessary and sufficient conditions for such a poset to be a meet semilattice. These conditions involve the existence of a weak BCK-algebra on the poset which is connected to the weak difference in a particular way. Therefore, in Section 6.2.2, we examine the properties of the corresponding weak BCK-algebra.

In Section 6.3, we explore the special case of a weak difference which is a restriction of the subtraction on an Abelian group, obtaining corollaries of the results of Section 6.2 for a subset of an Abelian group equipped with a particular partial order.

Then, in Section 6.4, we apply these corollaries to the weak and the strong right star orders on a particular Rickart ring (in Section 6.4.1), as well as to the sharp order on a certain subset of a unitary ring (in Section 6.4.2), obtaining meet semilattice conditions for all of these partial orders.

All the results proved in this chapter are part of an accepted article (Cremer and Marovt [ND]), except Corollary 6.4.5.

6.1 Weak differences and their connections to weak BCK-algebras

In this section, we introduce the new notion of weak difference and investigate its connections to weak BCK-algebras. First, we show that every meet semilattice equipped with a weak difference can be seen as a commutative weak BCK-algebra if and only if it is bounded from below (Theorem 6.1.2). After that we look at a partial operation on a weak BCK-algebra which for comparable elements coincides with the weak-BCK-subtraction and find out under which circumstances this partial operation is a weak difference (Theorem 6.1.5). We thereby obtain weak-BCK-versions of Theorems 4.1 and 5.5 in [Dvurečenskij and Kim, 1998].

6.1.1 Introducing weak differences

We introduce the following generalization of differences.

Definition 6.1.1. [Cremer and Marovt, ND] Let $\langle P, \leq \rangle$ be a poset. We call a partial operation $\ominus$ on P a *weak difference* if the following conditions hold:

($\ominus$1) If $a \leq b$, then $b \ominus a$ is defined,

($\ominus$2) if $a \leq b$, then $b \ominus a \leq b$,

($\ominus$3) if $a \leq b$, then $b \ominus (b \ominus a)$ is defined and $b \ominus (b \ominus a) = a$,

($\ominus$4) if $a \leq b \leq c$, then $c \ominus b \leq c \ominus a$.

Let us compare the Definitions A.5.1 and 6.1.1 to see exactly in what way the latter generalizes the former. The conditions ($\ominus$1) and ($\ominus$4) of Definition 6.1.1 are indeed weaker than conditions (D1) and (D3), respectively, while (D2) is just the conjunction of ($\ominus$3) and ($\ominus$2). The distinction between conditions ($\ominus$4) and (D3) is the more essential one, because for any partial operation $\ominus$ satisfying condition ($\ominus$1), there exists a restriction $\ominus'$ of $\ominus$ which satisfies the first difference axiom (D1). This restriction is obtained by letting $b \ominus' a$ be defined if and only if $a \leq b$, and in that case defining $b \ominus' a := b \ominus a$. We will call the partial operation $\ominus'$ the (D1)-*restriction* of $\ominus$.

6.1.2 Commutative weak BCK-algebra on a meet semilattice with weak difference

In this subsection, we show that a weak difference on a meet semilattice with a bottom element gives rise to a commutative weak BCK-algebra.

The following theorem is a weak-BCK-version of [Dvurečenskij and Kim, 1998, Theorem 5.5] (see also [Dvurecenskij and Pulmannová, 2013, Theorem 6.3.4]). However, in the BCK-version, the necessary and sufficient condition for being a commutative BCK-algebra is a condition which in general is not satisfied in a weak BCK-algebra (in fact, in combination which another condition it would turn a weak BCK-algebra into a BCK-algebra, see [Cīrulis, 2013, Theorem 2.4]), while the existence of a bottom element follows from the assumptions. For weak BCK-algebras and weak differences, the existence of zero does not follow from the assumptions, but instead it serves as a necessary and sufficient condition.

Theorem 6.1.2. [Cremer and Marovt, ND] *Let $\langle X, \leq \rangle$ be a meet semilattice equipped with a weak difference $\ominus$. Let*

$$b \setminus a := b \ominus (a \wedge b). \tag{6.1}$$

Then, for all $a, b \in X$ such that $a \leq b$, we have

$$b \setminus a = b \ominus a. \tag{6.2}$$

Moreover, given an element $0 \in X$, the algebra $\langle X, \setminus, 0 \rangle$ is a commutative weak BCK-algebra with the underlying order $\leq$ if and only if 0 is the bottom element of the semilattice $\langle X, \leq \rangle$.

Proof. If $a \leq b$, then $a \wedge b = a$, so (6.2) follows immediately from (6.1).

"$\Rightarrow$" This is obvious.

"$\Leftarrow$" First, note that, for all $x, y \in X$, $x \ominus (x \wedge y)$ is defined by ($\ominus$1), since $x \wedge y \leq x$. By ($\ominus$2), we have $x \ominus (x \wedge y) \leq x$. Hence, (6.1), (6.2) and ($\ominus$3) yield $x \setminus (x \setminus y) = x \setminus (x \ominus (x \wedge y)) = x \ominus (x \ominus (x \wedge y)) = x \wedge y$. So for all $x, y \in X$, we have

$$x \setminus (x \setminus y) = x \wedge y. \tag{6.3}$$

Now let $x, y, z \in X$, and let 0 be the bottom element of $\langle X, \leq \rangle$. We shall prove that $\langle X, \setminus, 0 \rangle$ is a commutative weak BCK-algebra by proving that, for all elements $x, y, z \in X$, the conditions (a), (b) and (c) from Proposition A.6.4, as well as (A.14) hold.

(a) This is immediate from (6.3).

(b) Let $x \leq y$. Then, for arbitrary $z \in X$, we have $x \wedge z \leq y \wedge z \leq z$. Hence, $z \ominus (y \wedge z) \leq z \ominus (x \wedge z)$ by ($\ominus$4). In other words, $z \setminus y \leq z \setminus x$ (by (6.1)).

(c) By ($\ominus$2) and ($\ominus$3), we have

$$x \ominus x \leq x \tag{6.4}$$

and

$$x \ominus (x \ominus x) = x \tag{6.5}$$

Since we assumed 0 to be the bottom element, we also have

$$0 \leq x \ominus x. \tag{6.6}$$

Now it follows by ($\ominus$4) from Equations (6.6) and (6.4) that $x \ominus (x \ominus x) \leq x \ominus 0$. By (6.5), this means that $x \leq x \ominus 0$. But $0 \leq x$ by assumption, so using ($\ominus$2), we obtain $x \ominus 0 \leq x$. Thus, $x = x \ominus 0$. By (6.2), this means that also $x \setminus 0 = x \ominus 0 = x$, since $0 \leq x$.

Commutativity (see (A.14)) follows by (6.3) from commutativity of the meet. So $\langle X, \setminus, 0 \rangle$ is indeed a commutative weak BCK-algebra with the underlying order $\leq$. ☐

6.1.3 Weak difference on a commutative weak BCK-algebra

In this subsection, we define a certain partial operation on a weak BCK-algebra and show that it is a weak difference if and only if the weak BCK-algebra is commutative. As a corollary, we also obtain a necessary and sufficient condition for this partial operation to be even a difference.

A poset with a weak difference does not necessarily have a bottom element. However, since weak BCK-algebras are by definition bounded from below and we want to define a weak difference on a weak BCK-algebra, we shall focus our interest on weak differences defined on a poset with a bottom element 0. In particular, we are interested in such weak differences for which the bottom element 0 is also a right identity. The following proposition is rather trivial, but useful.

Proposition 6.1.3. [Cremer and Marovt, ND] *If a partial operation $\ominus$ on a poset $\langle P, \leq \rangle$ with the bottom element* 0 *satisfies the conditions* ($\ominus$1), ($\ominus$4) *and, for all $a \in P$,*

($\ominus$5) *$a \ominus 0$ is defined and $a \ominus 0 = a$,*

then also the condition ($\ominus$2) *holds.*

Proof. Let $a \leq b$. By ($\ominus$1), $b \ominus a$ is defined. Since $0 \leq a \leq b$, we obtain $b \ominus a \leq b \ominus 0 = b$ from conditions ($\ominus$5) and ($\ominus$4). ☐

Remark 6.1.4. [Cremer and Marovt, ND] There is a very close connection between weak differences satisfying condition ($\ominus$5) and sectionally de Morgan complemented posets.

Let $\langle P, \leq \rangle$ be a poset with the bottom element 0 and let $\ominus$ be a partial operation on P satisfying condition ($\ominus$1). Further let $\{^{\perp}_{x}\}_{x \in P}$ be a family of unary operations such that every operation $^{\perp}_{x}$ is defined on the corresponding initial segment $[0, x]_{\leq}$, and let the unary operations be connected to the partial binary operation by

$$a^{\perp}_{x} = x \ominus a. \tag{6.7}$$

for $a \leq x$. It is easy to check that the partial operation $\ominus$ is a weak difference if and only if the unary operations $^{\perp}_{x}$ are de Morgan complementations.

Let $\langle P, \leq \rangle$ be a poset with the bottom element 0 equipped with a weak difference $\ominus$ satisfying ($\ominus$5). By properties ($\ominus$1) and ($\ominus$2), if $a \in [0, x]_{\leq}$, then $x \ominus a$ is defined and $x \ominus a \in [0, x]_{\leq}$ for all $a, x \in P$. Hence, (6.7) always defines a operation on the initial segment $[0, x]_{\leq}$. So every weak difference satisfying ($\ominus$5) gives rise to a family of sectional de Morgan complementations satisfying (6.7) (and this family is completely determined by (6.7)).

On the other hand, given a sectionally m-complemented poset, we can obviously define a partial operation $\ominus$ satisfying ($\ominus$1) by letting $b \ominus a$ be defined by (6.7) if and only if $a \leq b$ (and undefined otherwise). Hence, every sectionally de Morgan complemented poset admits a weak difference satisfying ($\ominus$5). Note that the partial operation $\ominus$ is not completely determined by (6.7), because there are no constraints on $b \ominus a$ for $a \nleq b$ – we can chose to leave it undefined (as it is done in the case of a difference), but we do not have to.

(There are a few reasons for defining a weak difference in a way that leaves open the possibility that $\ominus$ may not be a proper partial operation (proper in the sense of being undefined for some a, b). First, the aim of this chapter is the application of results on weak differences to certain rings, and we shall see that the usual ring minus works as a weak difference for several partial orders. Hence it is convenient that we do not have to create multiple restrictions of the ring minus depending on the partial order in question – we just take the ring minus and use it as a weak difference. In some occasions the notion of weak difference is too general, though, and therefore we sometimes need to use the (D1)-restriction of a weak difference. Since both notions are thus necessary, we could as well chose the (D1)-restriction as the main one and mention the weak difference only when necessary in the applications to rings. However, in order to avoid repetition and keep maximum generality, we chose the more general notion as the main one.) ◁

The following theorem was inspired by [Dvurečenskij and Kim, 1998, Theorem 4.1] (see also [Dvurecenskij and Pulmannová, 2013, Theorem 6.3.1]). It provides a necessary and sufficient condition under which a particular partial operation on a weak BCK-algebra is a weak difference.

Theorem 6.1.5. [Cremer and Marovt, ND] *Let $\langle X, \diagdown, 0\rangle$ be a weak BCK-algebra with the underlying partial order $\leq$, and let $\ominus$ be a partial operation on X satisfying the condition ($\ominus$1) with the property*

$$\text{if } x \leq y, \text{ then } y \ominus x = y \diagdown x. \tag{6.8}$$

Then the partial operation $\ominus$ satisfies also the conditions ($\ominus$2), ($\ominus$4), ($\ominus$5) and, for all $a, b \in X$,

($\bar{\ominus}$3) *if $a \leq b$, then $b \ominus (b \ominus a)$ is defined and $b \ominus (b \ominus a) \leq a$.*

Moreover, the partial operation $\ominus$ is a weak difference if and only if the weak BCK-algebra $\langle X, \diagdown, 0\rangle$ is commutative.

Proof. Let $\langle X, \diagdown, 0\rangle$ be a weak BCK-algebra, let $\leq$ be its underlying partial order, and let $\ominus$ be a partial operation on X such that, if $x \leq y$ for some elements $x, y \in X$, then $y \ominus x$ is defined and (6.8) holds.

We shall first prove that the conditions ($\ominus$5), ($\ominus$4), ($\bar{\ominus}$3) and ($\ominus$2) hold. Let $a, b, c \in X$.

($\ominus$4) Let $a \leq b \leq c$. By ($\ominus$1), both $c \ominus a$ and $c \ominus b$ are defined. Further, $c \ominus a = c \diagdown a$ and $c \ominus b = c \diagdown b$ by (6.8) (since $a, b \leq c$). But $c \diagdown b \leq c \diagdown a$ by Proposition A.6.4 (b). Hence condition ($\ominus$4) holds.

($\ominus$5) By ($\ominus$1), $a \ominus 0$ is defined, since $0 \leq a$. It follows immediately from Proposition A.6.4 (c) and (6.8) that $a \ominus 0 = a$.

($\bar{\ominus}$3) Let $a \leq b$. By condition ($\ominus$1) and (6.8), $b \ominus a$ is defined and $b \ominus a = b \diagdown a$. From Proposition A.6.6, we have $b \diagdown a \leq b$. Hence, also $b \ominus (b \ominus a)$ is defined (by ($\ominus$1)) and, by (6.8), we have $b \ominus (b \ominus a) = b \ominus (b \diagdown a) = b \diagdown (b \diagdown a)$. Now Proposition A.6.4(a) yields $b \ominus (b \ominus a) \leq a$.

($\ominus$2) This follows immediately from the properties ($\ominus$1), ($\ominus$4) and ($\ominus$5) by Proposition 6.1.3.

It remains to prove the second part of the theorem. First note that, by Proposition A.6.10, a weak BCK-algebra X is commutative if and only if, for all $x, y \in X$, if $x \leq y$, then $x \leq y \diagdown (y \diagdown x)$. But by Proposition A.6.4(a), we have $y \diagdown (y \diagdown x) \leq x$ for all $x, y \in X$. Hence, a weak BCK-algebra X is commutative if and only if, for all $x, y \in X$, if $x \leq y$, then $x = y \diagdown (y \diagdown x)$.

"$\Rightarrow$" Assume that the partial operation $\ominus$ is a weak difference. Let $x, y \in X$ such that $x \leq y$. Then, by property ($\ominus$2), we have $y \ominus x \leq y$. So after using condition ($\ominus$3), we can use (6.8) twice, obtaining $x = y \ominus (y \ominus x) = y \diagdown (y \ominus x) = y \diagdown (y \diagdown x)$. Hence, the weak BCK-algebra $\langle X, \diagdown, 0\rangle$ is commutative.

"$\Leftarrow$" Now assume that the weak BCK-algebra $\langle X, \diagdown, 0\rangle$ is commutative, and let $a \leq b$. Then $b \diagdown (b \diagdown a) = a$ by Proposition A.6.4(a) and Proposition A.6.10. Moreover, condition ($\bar{\ominus}$3) yields that $b \ominus (b \ominus a)$ is defined. By conditions ($\ominus$1) and ($\ominus$2), we have $b \ominus a \leq b$. Using (6.8) twice, we obtain that $b \ominus (b \ominus a) = b \diagdown (b \ominus a) = b \diagdown (b \diagdown a) = a$ So indeed condition ($\ominus$3) holds.

□

The following corollary follows immediately from Theorem 6.1.5 and the definition of a difference Definition A.5.1. It is a weak-BCK-version of [Dvurečenskij and Kim, 1998, Corollary 4.3] (see also [Dvurecenskij and Pulmannová, 2013, Corollary 6.3.2]).

Corollary 6.1.6. [Cremer and Marovt, ND] *Let $\langle X, \setminus, 0\rangle$ be a weak BCK-algebra and let $\ominus$ be a partial operation on X defined as in Theorem 6.1.5. Then the (D1)-restriction $\ominus'$ of the partial operation $\ominus$ is a difference if and only if $\langle X, \setminus, 0\rangle$ is a commutative weak BCK-algebra such that, for all $a, b, c \in X$,*

$$\text{if } a \leq b \leq c, \text{ then } (c \setminus a) \setminus (c \setminus b) = b \setminus a. \tag{6.9}$$

6.2 When is a poset with a weak difference satisfying ($\ominus$5) a meet semilattice?

In this section, we provide some necessary and sufficient conditions for a poset with a weak difference satisfying some additional properties to be a meet semilattice. The conditions involve the existence of a certain weak BCK-algebra on the poset, and we also investigate the properties of this weak BCK-algebra.

6.2.1 Semilattice conditions

We begin with a theorem that provides some semilattice conditions for a poset equipped with a bottom element and a weak difference satisfying condition ($\ominus$5).

Theorem 6.2.1. [Cremer and Marovt, ND]

Let $\langle P, \leq, 0\rangle$ be a poset with the bottom element 0 equipped with a weak difference satisfying condition ($\ominus$5). Then the following are equivalent:

(a) *$\langle P, \leq\rangle$ is a meet semilattice.*

(b) *There exists a binary operation $\setminus$ on P such that $\langle P, \setminus, 0\rangle$ is a commutative weak BCK-algebra with the underlying order $\leq$ and, for all $a, b \in P$,*

 ($\setminus$1) if $a \leq b$, then $b \setminus a = b \ominus a$.

(c) *There exists a binary operation $\setminus$ on P such that $\langle P, \setminus, 0\rangle$ is a weak BCK-algebra with the underlying order $\leq$ and condition ($\setminus$1) holds.*

(d) *There exists a binary operation $\setminus$ on P which satisfies condition ($\setminus$1) such that, for all $a, b, c \in P$,*

 ($\setminus$2) $a \ominus (a \setminus b)$ is defined and $a \ominus (a \setminus b) \leq b$,

 ($\setminus$3) if $a \leq b$, then $c \setminus b \leq c \setminus a$.

In the case when these conditions are fulfilled, the binary operation in each item ((b), (c) and (d)) is unique and they coincide, and the meet of elements x and y in P is given by:

$$x \wedge y = x \ominus (x \setminus y). \tag{6.10}$$

Proof. Let $\langle P, \leq, 0\rangle$ be a poset with the bottom element 0 and let $\ominus$ be a weak difference on P such that condition ($\ominus$5) holds.

(a)⇒(b) Assume that $\langle P, \leq\rangle$ is a meet semilattice. We define an operation $\setminus$ on P by

$$x \setminus y = x \ominus (x \wedge y). \tag{6.11}$$

Now it is immediate from Theorem 6.1.2 that $\langle P, \setminus, 0\rangle$ is a commutative weak BCK-algebra, since 0 is the bottom element of the semilattice $\langle P, \leq\rangle$.

(b)⇒(c) Obvious.

(c)⇒(d) Suppose that $\langle P, \smallsetminus, 0\rangle$ is a weak BCK-algebra such that the condition ($\smallsetminus 1$) is satisfied. We use the conditions from Proposition A.6.4, which hold for every weak BCK-algebra. It is obvious that ($\smallsetminus 3$) holds, since it is identical to item (b) of Proposition A.6.4. By Proposition A.6.4(b), ($\smallsetminus 1$) and ($\ominus 5$), we have $a \smallsetminus b \leq a \smallsetminus 0 = a \ominus 0 = a$, since $0 \leq b$ and $0 \leq a$ (because 0 is assumed to be the bottom element). Hence, $a \ominus (a \smallsetminus b)$ is defined, and ($\smallsetminus 1$) further yields that $a \ominus (a \smallsetminus b) = a \smallsetminus (a \smallsetminus b)$. Now item (a) of Proposition A.6.4 implies ($\smallsetminus 2$).

(d) ⇒ (a) Assume that P admits a binary operation $\smallsetminus$ satisfying the conditions ($\smallsetminus 2$), ($\smallsetminus 3$) and ($\smallsetminus 1$).

First, note that, since 0 is the bottom element of the poset, we have

$$x \smallsetminus 0 = x \tag{6.12}$$

for every $x \in P$, because the conditions ($\smallsetminus 1$) and ($\ominus 5$) yield $x \smallsetminus 0 = x \ominus 0 = x$.

Moreover, for arbitrary $x, y \in P$, we have $0 \leq y$ and therefore, $x \smallsetminus y \leq x \smallsetminus 0$ by condition ($\smallsetminus 3$). So (6.12) yields that, for all $x, y \in P$,

$$x \smallsetminus y \leq x. \tag{6.13}$$

Now let $a, b \in P$ and let

$$u := a \ominus (a \smallsetminus b). \tag{6.14}$$

We will prove that $u = a \wedge b$.

By ($\smallsetminus 2$), we have $u \leq b$. By (6.13), we have $0 \leq a \smallsetminus b \leq a$. This implies $a \ominus (a \smallsetminus b) \leq a \ominus 0$ (by condition ($\ominus 4$)). Using Equations (6.12) and (6.14), we obtain $u \leq a$. This proves that u is a lower bound of a and b.

Let $w \in P$ also be a lower bound of a and b. Then, since $w \leq b$, condition ($\smallsetminus 3$) yields $a \smallsetminus b \leq a \smallsetminus w$. Moreover, by (6.13), $a \smallsetminus w \leq a$. So by condition ($\ominus 4$), $a \ominus (a \smallsetminus w) \leq a \ominus (a \smallsetminus b)$. By (6.14), this means $a \ominus (a \smallsetminus w) \leq u$. But since we assumed that $w \leq a$, we obtain from conditions ($\smallsetminus 1$) and ($\ominus 3$) that $a \ominus (a \smallsetminus w) = a \ominus (a \ominus w) = w$. Hence, $w \leq u$.

So the element u is indeed the meet of a and b. We conclude that $\langle P, \leq\rangle$ is a meet semilattice. Moreover, by the definition of u in (6.14), we see that (6.11) holds.

We shall now prove first that (6.10) holds and then that the operations are unique and coincide.

For (6.10), assume that $\smallsetminus$ is a binary operation on P which satisfies the conditions given in either item (b), (c) or (d). Since in the proofs of (b)⇒(c) and of (c)⇒(d), we just take the operation from the respective first item and prove that it satisfies the conditions of the respective second item, in all cases we know that the binary operation satisfies the conditions in item (d). Therefore, (6.11) holds (see (d)⇒(a)).

Uniqueness of the operation $\smallsetminus$ follows from condition ($\ominus 3$) and (6.11): Since $a \smallsetminus b \leq a$ by Proposition A.6.6, we have $a \smallsetminus b = a \ominus (a \ominus (a \smallsetminus b)) = a \ominus (a \wedge b)$. Thus, the operation $\smallsetminus$ is determined by the partial operation $\ominus$ and the meet operation $\wedge$, and hence unique. □

Remark 6.2.2 (Alternative proof of (a)⇔ (b)⇔(c)). In view of Remark 6.1.4, the (a)⇔(b)⇔(c) part of Theorem 6.2.1 could be alternatively deduced from some known facts about sectionally m-complemented posets, because it is known that a sectionally m-complemented poset is a meet semilattice if and only if it is a weak BCK-algebra (see [Cīrulis, 2015b, Proposition 3.2, Proposition 3.3] and [Cīrulis, 2013, Proposition 6.1]). ◁

6.2.2 Properties of the weak BCK-algebra

Given a meet semilattice $\langle P, \leq, 0\rangle$ with bottom element 0 and equipped with a weak difference satisfying condition ($\ominus$5), we already know from Theorem 6.2.1 that the weak BCK-algebra arising from P is commutative. We proceed to investigate under which conditions it has any of the other two properties introduced in Definition A.6.8.

The following two lemmas provide conditions under which the weak BCK-algebra obtained from a meet semilattice with a weak difference satisfying ($\ominus$5) is a weak Henkin algebra or implicative.

Lemma 6.2.3. [Cremer and Marovt, ND] *Let $\langle P, \leq, 0\rangle$ be a meet semilattice with bottom element 0 equipped with a weak difference satisfying condition ($\ominus$5). Let $\setminus$ be the weak-BCK operation obtained on P by Theorem 6.2.1.*

Then the weak BCK-algebra $\langle P, \setminus, 0\rangle$ is a weak Henkin algebra if and only if, for all $a, b \in P$

$$\text{if } a \leq b \text{ and } b \ominus a \leq a, \text{ then } b = a. \tag{6.15}$$

Proof. $\langle P, \leq, 0\rangle$ be a meet semilattice with bottom element 0 equipped with a weak difference $\ominus$. Let $\setminus$ be the weak-BCK operation from Theorem 6.2.1.

Suppose (6.15) holds for all elements of P, and let x, y be elements of P such that $x \setminus y \leq y$. So by (6.11), we have $x \ominus (x \wedge y) \leq y$. But we also have $x \ominus (x \wedge y) \leq x$ by the property ($\ominus$2). Thus, $x \ominus (x \wedge y) \leq x \wedge y$. But now the assumption yields that $x = x \wedge y$. So $x \leq y$, and this proves (see Definition A.6.8) that the weak BCK-algebra $\langle G, \setminus, 0\rangle$ is indeed a weak Henkin algebra.

Conversely, if $\langle P, \setminus, 0\rangle$ is a weak Henkin algebra and $a, b \in P$ such that $a \leq b$ and $b \ominus a \leq a$, then, by ($\setminus$1), $b \setminus a \leq a$. So (A.15) yields that $b \leq a$. Thus, $a = b$. ☐

Lemma 6.2.4. [Cremer and Marovt, ND] *Let $\langle P, \leq, 0\rangle$ be a meet semilattice with bottom element 0 equipped with a weak difference satisfying condition ($\ominus$5). Let $\setminus$ be the weak-BCK operation obtained on P by Theorem 6.2.1, and let $\ominus'$ be the (D1)-restriction of the weak difference $\ominus$. Then the following conditions are equivalent:*

(a) *The weak BCK-algebra $\langle P, \setminus, 0\rangle$ is implicative.*

(b) *The weak BCK-algebra $\langle P, \setminus, 0\rangle$ is relatively orthocomplemented (with $a_x^{\perp} = x \setminus a$ for $a \leq x$).*

(c) *The partial operation $\ominus'$ is a difference and $\langle P, \ominus', 0, \leq\rangle$ is a regular poset with difference.*

Moreover, if the conditions hold, then the weak BCK-algebra $\langle P, \setminus, 0\rangle$ satisfies (6.9).

Proof. $\langle P, \leq, 0\rangle$ be a meet semilattice with bottom element 0 equipped with a weak difference $\ominus$. Let $\setminus$ be the weak-BCK operation from Theorem 6.2.1.

(a) $\Leftrightarrow$ (b) This is immediate from Proposition A.6.11, since $\langle P, \leq\rangle$ is by assumption a meet semilattice.

(b) $\Rightarrow$ (c) Let the weak BCK-algebra $\langle P, \setminus, 0\rangle$ be relatively orthocomplemented. Then by Lemma A.6.12, there is a difference $\bar{\ominus}$ on P such that $b \bar{\ominus} a = b \setminus a$ whenever $a \leq b$. This difference must be the restriction of $\ominus$ to pairs (a, b) with $a \leq b$, because by the definition of a difference, $b \bar{\ominus} a$ is defined if and only if $a \leq b$, and by ($\setminus$2), we have $b \ominus a = b \setminus a = b \bar{\ominus} a$ whenever $a \leq b$.

(b) $\Leftarrow$ (c) Conversely, let $\langle P, \ominus', 0\rangle$ be a regular poset with difference, where $\ominus'$ is the (D1)-restriction of $\ominus$. Then, by Lemma A.6.12, the weak BCK-algebra is relatively orthocomplemented.

It is easy to see that the weak BCK-algebra $\langle P, \setminus, 0\rangle$ satisfies (6.9) if the conditions hold: Since by condition (c) the operation $\bar{\ominus}$ is a difference, it follows from Corollary 6.1.6 that the weak BCK-algebra satisfies (6.9). ☐

6.3 Special case on an Abelian group

In this section, we obtain consequences of the results from the previous section for the special case when the weak difference is the subtraction on an Abelian group.

We will thus deal with two different binary operations that could be called subtraction or subtraction-like. To make the distinction clearer, a weak-BCK operation connected to a weak difference $\ominus$ by the condition ($\diagdown$1) will be called *logical subtraction* (note that a logical subtraction must be unique, because a weak BCK-algebra is completely determined by the structure of its initial segments, see Cīrulis [2015b]). The weak BCK-algebra corresponding to the logical subtraction will be called the weak BCK-algebra *induced by* $\ominus$.

Let us consider an algebra $\langle G, -, 0\rangle$ such that $-$ is a binary operation and $0 \in G$. Assume that the properties

(-1) $x - (x - y) = y$,

(-2) $x - 0 = x$

are satisfied for all $x, y \in G$. Let S be a subset of G equipped with a partial order $\leq$ such that $0 \in S$ and, for all $a, b, c \in S$, the following conditions hold

($\leq$1) if $a \leq b$, then $b - a \in S$,

($\leq$2) $0 \leq a$,

($\leq$3) if $a \leq b \leq c$, then $c - b \leq c - a$.

We can define a partial operation $\ominus$ on the set S by restricting the operation $-$ to S. That is, for $x, y \in S$, the expression $x \ominus y$ is defined if and only if $x - y \in S$, and in that case, $x \ominus y := x - y$.

The partial operation $\ominus$ on the poset $\langle S, \leq, 0\rangle$ obviously satisfies conditions ($\ominus$1), ($\ominus$4) and ($\ominus$5). Hence, by Proposition 6.1.3, condition ($\ominus$2) holds as well. To see that the partial operation $\ominus$ also satisfies condition ($\ominus$3), let a, b be elements of S such that $a \leq b$. Then $b \ominus a$ is defined by ($\ominus$1), and by ($\ominus$4), we have $b \ominus a \leq b \ominus 0$, because $0 \leq a \leq b$. But $b \ominus 0 = b$ by ($\ominus$5), so we have $b \ominus a \leq b$. Hence, by ($\ominus$1), also $b \ominus (b \ominus a)$ is defined. Now ($\leq$1) implies that $b \ominus (b \ominus a) = a$. Hence, condition ($\ominus$3) holds. So the partial operation $\ominus$ is a weak difference.

So by Theorem 6.2.1, the poset $\langle S, \leq\rangle$ is a meet semilattice if and only if it admits a logical subtraction, and in that case, the meet of elements $x, y \in S$ is

$$x \wedge y = x - (x \diagdown y). \tag{6.16}$$

For the applications which follow in Section 6.4, we use the special case where the algebra $\langle G, -, 0\rangle$ is a reduct of an Abelian group. The usual subtraction $-$ on an Abelian group $\langle G, +\rangle$ (defined by $a - b := a + (-b)$, where $-b$ is the inverse of b) obviously satisfies conditions (-1) and (-2). Hence, we obtain the following corollary of Theorem 6.2.1.

Corollary 6.3.1. [Cremer and Marovt, ND] *Let $\langle G, +, -, 0\rangle$ be an Abelian group. Let S be a subset of G equipped with a partial order $\leq$ such that $0 \in S$ and, for all $a, b, c \in S$, the conditions ($\leq$1), ($\leq$2) and ($\leq$3) hold. Then the following are equivalent.*

(a) *The poset $\langle S, \leq\rangle$ is a meet semilattice.*

(b) *The set S admits a binary operation $\diagdown$ satisfying conditions ($\diagdown$1), ($\diagdown$2) and ($\diagdown$3).*

(c) *The set S admits a binary operation $\diagdown$ such that $b - a = b \diagdown a$ whenever $a \leq b$ and $\langle G, \diagdown, 0\rangle$ is a weak BCK-algebra.*

If this is the case, then the weak BCK-algebra $\langle S, \diagdown, 0\rangle$ is commutative and the meet of elements $x, y \in S$ in S is given by (6.16).

Note that, if furthermore the set S is actually the whole group G, then the condition ($\leq$1) is satisfied automatically. Abelian groups with a partial order satisfying the conditions ($\leq$2) and ($\leq$3) were also studied by Hedlíková and Pulmannová in [Hedlíková and Pulmannová, 1996, Example 4.6] . These include as a special case the so-called *orthomodular groups* introduced by Chevalier in Chevalier [1993] .

We also obtain a corollary from Lemmas 6.2.3 and 6.2.4.

Corollary 6.3.2. [Cremer and Marovt, ND] *Let $\langle G, +, -, 0\rangle$ be an Abelian group with a partial order $\leq$ satisfying* ($\leq$2) *and* ($\leq$3)*, and let $\langle G, \leq\rangle$ be a meet semilattice. Then the induced weak BCK-algebra is a weak Henkin algebra if and only if, for all $a, b \in G$, if $a \leq b$ and $b - a \leq a$, then $b = a$. Moreover, the following conditions are equivalent:*

(a) *The induced weak BCK-algebra is implicative.*

(b) *The induced weak BCK-algebra is relatively orthocomplemented.*

(c) *For all elements $a, b \in G$, if $a \leq b$ and $a \leq b - a$, then $a = 0$.*

Proof. The weak Henkin part is obvious. For the implicativity part, note that the restriction $-'$ of the group subtraction $-$ to pairs (a, b) with $a \leq b$ is always a difference. Hence, regularity of the poset with difference $\langle G, -', 0\rangle$ is sufficient for implicativity of the weak BCK-algebra. Rewriting the regularity condition with $-$ instead of $-'$ (since $a \leq b$), we obtain the condition in the corollary. □

Note that an Abelian group with a partial order satisfying ($\leq$2) and ($\leq$3) generally is not a partially ordered group (a partially ordered group is a group G equipped with a partial order such that $a \leq b$ implies $a + c \leq b + c$ and $c + a \leq c + b$ for all $a, b, c \in G$). A group with a partial order satisfying ($\leq$2) can only be a partially ordered group in the trivial case of being the group which has no other elements than 0. (This can be seen by taking an arbitrary group element b and chosing $a = 0$ and $c = -b$ in the condition defining a partially ordered group. From ($\leq$2), we have $0 \leq b$, whence $0 - b \leq b - b$, i.e., $-b \leq 0$. But then $b = 0$.)

6.4 Application to some partial orders on rings

This section deals with applications of results from the previous section to some of the partial order discussed in this thesis: the one-sided star orders and the sharp order.

The application presented in this section would work in the same way also for the star order, but the corresponding result is already known, so we will not treat the star order here. The diamond order, on the other hand, does not satisfy the necessary properties for the approach in this section to work. Namely, it is known from [Baksalary and Hauke, 1990, page 166] that the property ($\leq$1) does not hold for it even on $\mathcal{B}(\mathbb{R}^2)$. In that paper, counterexample is provided by the matrices $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$: We have $A \overset{\diamond}{\leq} B$, but $B - A \overset{\diamond}{\leq} B$ is not true.

6.4.1 Meet semilattice conditions for the one-sided star orders

We derive a semilattice condition for this case and we see that a right-strong Rickart ring which is a meet semilattice under either of the right star orders gives rise to an implicative weak BCK-algebra.

It was proved in [Cīrulis, 2015d] that a right-strong Rickart ring with the weak right star order is relatively orthocomplemented (see Theorem 2.2.11), and it was proved by the author in [Cremer, 2024] that a right-strong Rickart ring with the strong right star order is also relatively orthocomplemented (see Theorem 5.1.7). Orthocomplementation on the initial segment $[0, x]_*$ under either of the right star orders is defined as $a_x^{\perp *} := x - a$. So by Remark 6.1.4, the ring subtraction $-$ is a weak difference satisfying ($\ominus$5).

Hence, the following results are immediate consequences of Theorem 6.2.1. (Alternatively, instead of using Theorem 6.2.1 itself combined with Remark 6.1.4 we could use its Corollary 6.3.1 for the special case where the partial operation $\ominus$ is a restriction of the subtraction of an Abelian group to a subset of the group – the set to which it is "restricted" being the whole group in this case.)

Corollary 6.4.1. [Cremer and Marovt, ND] *Let R be a right-strong Rickart ring, and let $\leq^*$ be either the weak or the strong right star order. Then the following are equivalent.*

(a) *The poset $\langle R, \leq^* \rangle$ is a meet semilattice.*
(b) *The poset R admits a binary operation $\setminus_*$ satisfying the conditions* ($\setminus$2), ($\setminus$3) *and* ($\setminus$1).
(c) *The poset R admits a binary operation $\setminus_*$ such that $\langle R, \setminus_*, 0\rangle$ is a weak BCK-algebra and $b \setminus_* a = b - a$ whenever $a \leq^* b$.*

In the case when these conditions are fulfilled, the binary operations in the latter two items are unique and they coincide, the weak BCK-algebra $\langle R, \setminus_*, 0\rangle$ is commutative and the meet of $a, b \in R$ under the respective version of the right star order is given by $a \wedge_* b = a - (a \setminus_* b) = (a \setminus_* b)_a^{\perp *}$.

Corollary 6.4.1 holds also for the star order on a Rickart *-ring, but most of that is already known: It is Lemma 5.4 and Theorem 5.5 in Cīrulis [2015a] . However, weak BCK-algebras are not mentioned in Cīrulis [2015a].

Note that, although for the sake of brevity we used the same symbol $\setminus_*$ for both the logical subtraction corresponding the weak right star order and the one corresponding to the strong right star order, the two logical subtraction do in general not coincide.

Since a right-strong Rickart ring with either the weak or the strong right star order is a relatively orthocomplemented poset, we have the following result as a consequence of Lemma 6.2.4.

Corollary 6.4.2. [Cremer and Marovt, ND] *Let R be a right-strong Rickart ring and let $\leq^*$ be either the weak or the strong right star order. If the poset $\langle R, \leq^* \rangle$ is a meet semilattice, then the weak BCK-algebra $\langle R, \setminus_*, 0\rangle$ is implicative (where $\setminus_*$ denotes the logical subtraction corresponding to the weak right star order or strong right star order, respectively).*

Moreover, let $\ominus'$ be the (D1)*-restriction of the ring subtraction $-$. Then $\ominus'$ is a difference and $\langle R, \ominus', 0, \leq^* \rangle$ is a regular poset with difference.*

6.4.2 Meet semilattice condition for the sharp order

Recall the definition of the sharp order and the set on which it is defined (Definition 2.3.5). After stating an auxiliary result, we prove that the set on which the sharp order is defined satisfies the properties necessary for applying Corollary 6.3.1. Then we derive a semilattice condition for the sharp order.

In order to use Corollary 6.3.1 on the subset $\mathcal{I}_R$ of the ring R, we need to prove that the conditions ($\leq$1), ($\leq$2) and ($\leq$3) hold on the poset $\langle \mathcal{I}_R, \overset{\sharp}{\leq} \rangle$. For that we first need the following auxiliary result.

Lemma 6.4.3. [Cremer and Marovt, ND] *Let R be a unitary ring and let $\mathcal{I}_R$ be the subset of R defined by Equation (2.31). Let $a, b \in \mathcal{I}_R$ such that $a \overset{\sharp}{\leq} b$. Let us denote $e = p_a$ and $f = p_b$. Then*

(a) $fa = a = af$,
(b) $fe = e = ef$,
(c) $\mathrm{ann_l}(f - e) = \mathrm{ann_l}(b - a)$ *and* $\mathrm{ann_r}(f - e) = \mathrm{ann_r}(b - a)$,
(d) $b - a \in \mathcal{I}_R$ *and* $p_{b-a} = f - e$,
(e) $b - a \overset{\sharp}{\leq} b$.

Proof. Let $a, b \in \mathcal{I}_R$, $e = p_a$, $f = p_b$ and $a \overset{\sharp}{\leq} b$. By the definition of the sharp order (Definition 2.3.5),

$$a = be = eb. \tag{6.17}$$

Note that, since $e(1-e) = (1-e)e = 0$, we have $a(1-e) = 0 = (1-a)a$, and thus, $ae = ea = a$. In the same way,

$$bf = fb = b. \tag{6.18}$$

(a) Using Equations (6.17), (6.18) and again (6.17), we obtain $fa = fbe = be = a$.
(b) Since $(1-f)a = 0$ by the previous item, we have $(1-f)e = 0$, i.e., $fe = e$. Similarly, $ef = e$.
(c) Let $x(b-a) = 0$. Then $x(f-e) = x(f-ef) = x(1-e)f$ by item (b). Hence, we have $x(f-e) = 0$ if and only if $x(1-e)b = 0$, since $\mathrm{ann_l}(b) = \mathrm{ann_l}(f)$. But $(1-e)b = b-a$ by (6.17). So indeed $x(f-e) = 0$ if and only if $x(b-a) = 0$. Similarly, we can prove that $(f-e)x = 0$ if and only if $(b-a)x = 0$. Hence, $\mathrm{ann_l}(f-e) = \mathrm{ann_l}(b-a)$ and $\mathrm{ann_r}(f-e) = \mathrm{ann_r}(b-a)$.
(d) It follows from item (b) that $f-e$ is idempotent, since $(f-e)^2 = f - fe - ef + e = f - e$. So by the previous item, $b - a \in \mathcal{I}_R$ with $f - e = p_{b-a}$.
(e) We have $b(f-e) = b-a$ by Equations (6.18) and (6.17), and in the same way, also $(f-e)b = b-a$. Thus, since $f - e = p_{b-a}$ by the previous item, we obtain $b - a \overset{\sharp}{\leq} b$ from the definition of $\overset{\sharp}{\leq}$. □

Now we can prove the following semilattice condition for the sharp order.

Corollary 6.4.4. [Cremer and Marovt, ND] *Let R be a unitary ring and let $\mathcal{I}_R$ be the subset of R as defined in (2.31). Then the following are equivalent.*

(a) *The poset $\langle \mathcal{I}_R, \overset{\sharp}{\leq} \rangle$ is a meet semilattice.*
(b) *The set $\mathcal{I}_R$ admits a binary operation $\smallsetminus_\sharp$ such that $b \smallsetminus_\sharp a = b - a$ whenever $a \overset{\sharp}{\leq} b$ and $\langle \mathcal{I}_R, \smallsetminus_\sharp, 0 \rangle$ is a weak BCK-algebra.*
(c) *The set $\mathcal{I}_R$ admits a binary operation $\smallsetminus_\sharp$ such that conditions ($\smallsetminus$1), ($\smallsetminus$2) and ($\smallsetminus$3) are satisfied.*

In the case when these conditions are fulfilled, the binary operations in **??** *and* **??** *are unique and they coincide, the weak BCK-algebra $\langle \mathcal{I}_R, \smallsetminus_\sharp, 0 \rangle$ is commutative and the meet of elements $x, y \in \mathcal{I}_R$ is given by $x \overset{\sharp}{\wedge} y = x - (x \smallsetminus_\sharp y)$.*

Proof. Let R be a unitary ring. We use Corollary 6.3.1 on the subset $\mathcal{I}_R$, on which the sharp partial order $\overset{\sharp}{\leq}$ is defined. First, note that $0 \in \mathcal{I}_R$, since 0 is itself idempotent and thus $p_0 = 0$. We need to prove that conditions ($\leq$1), ($\leq$2) and ($\leq$3) hold. Let $a, b, c \in \mathcal{I}_R$.

($\leq$1) This is immediate from Lemma 6.4.3(d).

($\leq$2) Obviously, $a \cdot 0 = 0 = 0 \cdot a$. Since $p_0 = 0$, this means that $0 \overset{\sharp}{\leq} a$.

($\leq$3) Let $a \overset{\sharp}{\leq} b \overset{\sharp}{\leq} c$. Then it particular $b = cp_b = p_b c$ (since $b \overset{\sharp}{\leq} c$), which together with Lemma 6.4.3(b) yields $p_a c = p_a p_b c = p_a b$. But since $a \overset{\sharp}{\leq} b$, we have $p_a b = a$. So we conclude that $p_a c = a$. Similarly, we can prove that also $cp_a = a$, whence $a \overset{\sharp}{\leq} c$.

Now the proposition follows from Corollary 6.3.1. □

6.4.3 Meet semilattice condition for the Abian order

Recall the Abian order on a reduced ring (Equation (2.32) in Section 2.3.3). We obtain the following corollary of Theorem 6.2.1.

Corollary 6.4.5. *Let R be a reduced ring and let $\leq_{\text{Ab}}$ denote the Abian order. Then the following conditions are equivalent.*

(a) *The poset $\langle R, \leq_{\text{Ab}} \rangle$ is a meet semilattice.*

(b) *The set R admits a binary operation $\setminus_{\text{Ab}}$ such that $b \setminus_{\text{Ab}} a = b - a$ whenever $a \leq_{\text{Ab}} b$ and $\langle R, \setminus_{\text{Ab}}, 0 \rangle$ is a weak BCK-algebra.*

(c) *The set R admits a binary operation $\setminus_{\text{Ab}}$ such that conditions ($\setminus$1), ($\setminus$2) and ($\setminus$3) are satisfied.*

In the case when these conditions are fulfilled, the binary operations in **??** *and* **??** *are unique and they coincide, the weak BCK-algebra $\langle R, \setminus_{\text{Ab}}, 0 \rangle$ is implicative and the meet of elements $x, y \in R$ is given by $x \wedge_{\text{Ab}} y = x - (x \setminus_{\text{Ab}} y)$.*

Proof. Let R be a reduced ring. We first use Corollary 6.3.1 on the whole ring R to prove the equivalence of the conditions and the formula for the meet. Then we use Corollary 6.3.2 to prove that, in case if the conditions hold, the resulting weak BCK-algebra is implicative.

In order to use Corollary 6.3.1, we have to prove that the conditions ($\leq$1), ($\leq$2) and ($\leq$3) hold. Let $a, b, c \in R$.

($\leq$1) Let $a \leq_{\text{Ab}} b$. Then $(b-a)^2 = b^2 - ab - ba + a^2 = b^2 - ab - a^2 + a^2 = b^2 - ab = (b-a)b$ by Equation (2.32), and similarly, also $b(b-a) = (b-a)^2$.

($\leq$2) Obviously, $0 \cdot a = a \cdot 0 = 0^2$.

($\leq$3) Let $a \leq_{\text{Ab}} b \leq_{\text{Ab}} c$. By applying Equation (2.32) several times we obtain $(c-b)(c-a) = c^2 - bc - ca + ba = c^2 - bc - a^2 + a^2 = c^2 - bc = c^2 - bc - cb + cb = c^2 - bc - cb + b^2 = (c-b)^2$, since $a \leq_{\text{Ab}} c$, $a \leq_{\text{Ab}} b$ and $b \leq_{\text{Ab}} c$. Similarly, also $(c-a)(c-b) = (c-b)^2$.

So by Corollary 6.3.1, the conditions (a), (b) and (c) are equivalent and if they hold, then $x \wedge_{\text{Ab}} y = x - (x - \setminus_{\text{Ab}} y)$ for all $x, y \in R$.

Now assume that the poset $\langle R, \leq_{\text{Ab}} \rangle$ is indeed a meet semilattice. We need to prove that the weak BCK-algebra $\langle R, \setminus_{\text{Ab}}, 0 \rangle$ is implicative. Let $a, b \in R$ such that $a \leq b$ and $a \leq b - a$. Then $ab = a^2$ and $a(b-a) = a^2$ by Equation (2.32). Hence, $ab = ab - a^2$, and thus $a^2 = 0$. Now reducedness (Definition 1.1.12) yields that $a = 0$, which by Corollary 6.3.2 proves that the weak BCK-algebra is implicative. □

Recall Theorem 2.3.6: Any reduced Rickart ring is known to be a meet semilattice under the Abian order. So on a reduced Rickart ring, Corollary 6.4.5 yields in particular that the operation $\setminus_{\text{Ab}}$ defined by $a \setminus_{\text{Ab}} b = a - (a \wedge_{\text{Ab}} b)$ is a weak BCK-subtraction.

Chapter 7

Special case: Strong semilattice decompositions of reduced Rickart rings

This chapter is devoted to strong semilattice decompositions of certain reducts of reduced Rickart rings. Given a reduced Rickart ring $\langle R, +, \cdot, 1\rangle$, we obtain various reducts by forgetting about addition (and possibly also the multiplicative identity) and adding another operation to the signature, namely either the unary operation $^\circ$ from Lemma 1.1.15 or the skew meet $\overset{\leftarrow}{\wedge}$ from Equation (2.27). It is known from Fountain [1976] that the semigroup reduct $\langle R, \cdot\rangle$ admits a decomposition as a strong semilattice of semigroups. In this chapter, we define strong semilattices of D-semigroups, D-monoids and also of structures which we call band-enriched monoids and study the strong semilattice decomposition of the D-semigroup reduct and the D-monoid reduct, as well as the band-enriched monoid reduct, of a reduced Rickart ring.

Section 7.1 is partly a preliminary section. The strong semlattice decomposition of the D-semigroup reduct can be obtained for a slightly more general class of rings called m-domain rings. We therefore recall the relevant preliminaries on m-domain rings in Section 7.1.1[1] and prove that reduced Rickart rings are a subclass of m-domain rings in Section 7.1.2. In Section 7.1.3, we describe an inverse system of subsemigroups of the semigroup reduct of an m-domain ring, and in Section 7.1.4 we show that the strong semilattice decomposition induced by this inverse system is the original semigroup reduct of the ring. These constructions are an elaboration on similar observations on right PP-monoids from Fountain [1976].

In Section 7.2, after introducing D-monoids in Section 7.2.1, we see in Section 7.2.2 that the inverse system from Section 7.1.3 can be modified to turn it into an inverse system of D-monoids by equipping its constituent semigroups with additional unary and nullary operations. Then we introduce the notions of strong semilattices of D-semigroups and D-monoids in Section 7.2.3. In Section 7.2.4, we show that, given an m-domain ring, the strong semilattice of D-semigroups (D-monoids) induced by the inverse system constructed in Section 7.2.2 equals the reduct $\langle R, \cdot, ^\circ\rangle$ (the reduct $\langle R, \cdot, ^\circ, 1\rangle$).

[1]This deviates from the general structure of the thesis, which has all the preliminaries in Part I, with Part II being reserved for results of the research. The reason is that m-domain rings are a bit of a detour from the topic of the thesis. They do not really fit into the general context of the topics of the three preliminary chapters. The reason to include them in the thesis at all is that it would be weird to present a result only on reduced Rickart rings if it could just as easily be obtained on m-domain rings, which are the more general class.

In Section 7.3, we first recall the skew meet operation on a right-strong right Rickart ring and describe its properties in the special case that the ring is reduced in Section 7.3.1. This serves as the motivation to introduce another structure which we shall call *band-enriched monoid* in Section 7.3.2. A band-enriched monoid is a monoid equipped with a second binary operation which is associative and idempotent and satisfies certain conditions relating it to the monoid multiplication. In Section 7.3.3, we study how band-enriched monoids are related to certain D-monoids.

Finally, Section 7.4 deals with the strong semilattice decomposition of the band-enriched monoid reduct of a reduced Rickart ring. First, in Section 7.4.1, we observe that, given a reduced Rickart ring, its inverse system of D-monoids from Section 7.2.2 can also be modified into an inverse system of band-enriched monoids. We also define strong semilattices of band-enriched monoids and show that they are band-enriched monoids themselves. Finally, in Section 7.4.2, we use the results from Section 7.3.3 to obtain a strong semilattice decomposition of the band-enriched monoid reduct of a reduced Rickart ring which follows from and is analogous to the strong semilattice decomposition of its D-monoid reduct found in Section 7.2.4.

The results proved in Sections 7.1 and 7.2 are contained in the article Cremer [2025], which has been accepted for publication. Sections 7.3 and 7.4 consist of unpublished results.

7.1 A wider class of rings containing reduced Rickart rings: m-domain rings

This section recalls the notion of *m-domain rings* and clarifies their relation to reduced Rickart rings, namely, that the latter are a subclass of the former. The reason for diverging to a class of non-Rickart rings is that most of the results in this Chapter do not require the ring to be reduced Rickart, and therefore it seems more appropriate to state them for the more general m-domain rings.

We start by recalling the definition of *m-domain rings* and some preliminary results which are known from the literature in Section 7.1.1. Then, in Section 7.1.2, we prove that an m-domain ring is a reduced Rickart ring if and only if it is unitary. The last two subsections deal with an inverse system formed by subsemigroups of the semigroup reduct of an m-domain ring (Section 7.1.3) and the strong semilattice of semigroups induced by it (Section 7.1.4).

7.1.1 M-domain rings

In this subsection, we recall the definition of *m-domain rings* and some preliminary results which are known from the literature.

Definition 7.1.1. [Howie, 1995, page 61] A *right-cancellative* semigroup is a semigroup A in which $xa = ya$ implies $x = y$ for all $x, y, a \in A$. A *left-cancellative* semigroup is a semigroup A in which $ax = ay$ implies $x = y$ for all $x, y, a \in A$. A *cancellative* semigroup is a semigroup which is both right- and left-cancellative.

In Sussman [1960], associate rings in the sense of that paper were decomposed into such semigroups, which were called *m-domains* in that paper. This decomposition was generalized in Subrahmanyam [1960] for the class of rings defined in the following definition, which was therefore named accordingly.

Definition 7.1.2. Subrahmanyam [1960] A ring R is called a *multiplicative domain ring*, or shorter, an *m-domain ring* if, for every $a \in R$, there exists a central idempotent a° such that, for all $a, b \in R$,

(a) $aa^\circ = a$,
(b) if $e \in R$ is idempotent and $ea = ae = a$, then $a^\circ e = a^\circ$,
(c) $(ab)^\circ = a^\circ b^\circ$.

It is easy to check that for an element a of an m-domain ring, the central idempotent a° is unique. Therefore we treat $^\circ$ as an operation on the m-domain ring. We use the same symbol $^\circ$ as for the operation in Lemma 1.1.15, because, as we will see in the next subsection, the operation in Lemma 1.1.15 is a special case of the operation $^\circ$ on an m-domain ring. The algebra $\langle R, +, \cdot, ^\circ\rangle$ will be called an *enriched m-domain ring* if $\langle R, +, \cdot\rangle$ is an m-domain ring and $^\circ$ is the operation from Definition 7.1.2. Analogously, given a reduced Rickart ring $\langle R, +, \cdot, 1\rangle$ with the operation $^\circ$ from Lemma 1.1.15, we will call the algebra $\langle R, +, \cdot, ^\circ, 1\rangle$ *enriched reduced Rickart ring*.

Remark 7.1.3. Let e be an idempotent in an m-domain ring. From Definition 7.1.2(b) it follows that $e^\circ e = e^\circ$ (taking $a = e$). Since e° is central, this yields $ee^\circ = e^\circ$. But then from Definition 7.1.2(a) we obtain $e^\circ = e$. So the operation $^\circ$ maps every idempotent on itself. In particular, every idempotent in an m-domain ring is central. ◁

Observe that

$$a^\circ = 0 \text{ iff } a = 0 \tag{7.1}$$

holds in every m-domain ring (one direction follows immediately from Definition 7.1.2(a), the other one is obtained by by chosing $a = e = 0$ in Definition 7.1.2(b)).

Lemma 7.1.4. [Subrahmanyam, 1960, Theorem XIV] *Let R be an m-domain ring and for an idempotent e, let M_e denote the set $\{x \in R \mid x^\circ = e\}$. Then the following statements hold.*
(a) *For every idempotent $e \in R$, M_e is a cancellative subsemigroup of the multiplicative semigroup of the ring, and e is the identity of this subsemigroup.*
(b) *The sets M_e are distinct and form a partition of R.*

Observe that the partition mentioned in Lemma 7.1.4(b) corresponds to the kernel equivalence of the operation $^\circ$. It is obvious from Equation (7.1) that $M_0 = \{0\}$.

Since according to Lemma 7.1.4(a), the semigroups M_e are multiplicative domains (recall that this is just another word for *cancellative semigroups*), we follow the terminology of Sussman [1960] by using an abbreviated version of this term for the sets M_e.

Definition 7.1.5. For an idempotent e in an m-domain ring R, the set M_e is called an *m-domain*.

We will use the term not only for the set M_e, but also for the semigroup $\langle M_e, \cdot\rangle$, the monoid $\langle M_e, \cdot, e\rangle$, etc.

Remark 7.1.6. Obviously, an m-domain M_e is always closed under the operation $^\circ$ from Definition 7.1.2, since $e \in M_e$ for every idempotent e and $x^\circ = e$ for every $x \in M_e$ by Lemma 7.1.4. ◁

7.1.2 Relations between reduced Rickart rings and m-domain rings

After introducing m-domain rings, we are now able to connect them to reduced Rickart rings by the following result.

Theorem 7.1.7. [Cremer, 2025] *A ring is a reduced Rickart ring if and only if it is a unitary m-domain ring. The unary operation from Definition 7.1.2 coincides with the operation from Lemma 1.1.15 (both are denoted $^\circ$), and they are connected to the focal operation $'$ by $x^\circ = x''$.*

Proof. Let R be a unitary m-domain ring and let $^\circ$ be the unary operation from Definition 7.1.2. By Lemma 1.1.15 and Definition 7.1.2, the ring R is a reduced Rickart ring, because the operation $^\circ$

satisfies the identities of Lemma 1.1.15 (the first condition of Lemma 1.1.15 follows from Remark 7.1.3; the third condition follows from Equation (7.1)). Lemma 1.1.15 also yields the identity $x^\circ = 1 - x'$. By Proposition 1.1.14, this implies $x^\circ = x''$, since x' is idempotent.

Now let R be a reduced Rickart ring and let $'$ be its focal operation. Recall that every Rickart ring is unitary. We consider the operation $^\circ$ given in Lemma 1.1.15. As $x^\circ = 1 - x' = x''$ by Lemma 1.1.15 and Proposition 1.1.14, the element x° is idempotent for all $x \in R$ (and hence also central by Proposition 1.1.13(c)). Let e be an idempotent such that $ea = ae = a$. Proposition 1.1.14 yields $e'' = e$. Now Proposition 1.3.7 yields $a'' = (ae)'' = a''e'' = a''e$, i.e., $a^\circ = a^\circ e$. Hence, R is an m-domain ring. □

Note that reducedness is necessary in Theorem 7.1.7, because it ensures that the idempotent x° is central for every x (Proposition 1.1.13(c)). In a strong Rickart ring which is not reduced, the idempotents a'' and $a^{\backslash\backslash}$ are in general not central.

Remark 7.1.8. It is obvious from Remark 7.1.6 and Theorem 7.1.7 that the m-domains M_e of a reduced Rickart ring are closed under the operation $^\circ$ from Lemma 1.1.15. ◁

7.1.3 An inverse system of semigroups formed by the m-domains

This subsection describes an inverse system of semigroups which we construct from the family of m-domains of an m-domain ring.

Example 7.1.9. Let R be an m-domain ring and let $\langle E, \leq_E \rangle$ be its semilattice of idempotents. We are going to demonstrate that the family of m-domains together with a suitably chosen family of maps is an inverse system of semigroups over E.

For every idempotent e, let $\cdot_e$ be the restriction of the ring multiplication to the . m-domain M_e (recall that by Lemma 7.1.4 the m-domains are closed with respect to multiplication). Let $\mathcal{M}$ be the family of all the m-domains $\langle M_e, \cdot_e \rangle$ (by Lemma 7.1.4(b), the family is disjoint).

For each pair $e, f \in E$ such that $e \leq_E f$, we define a map ϕ_e^f in the following way:

$$\begin{array}{rrcl} \phi_e^f: & M_f & \to & M_e \\ & x & \mapsto & xe. \end{array} \tag{7.2}$$

This map is well-defined, because, for $x \in M_f$ and $e \leq_E f$, we have $(xe)^\circ = x^\circ e^\circ = fe = e$ by Definition 7.1.2(c), so indeed $xe \in M_e$ by the definition of M_e in Lemma 7.1.4. Moreover, the maps ϕ_e^f are semigroup homomorphisms (by Example 7.1.9, centrality of idempotents and the definition of $\cdot_e$ and $\cdot_f$ as restrictions of the ring multiplication).

Now let Φ be the set of all the maps ϕ_e^f. To demonstrate that $\langle \mathcal{M}, \Phi \rangle$ is an inverse system of semigroups and semigroup homomorphisms, it remains to show that the two conditions on homomorphisms given in Definition B.1.1 are satisfied.

(a) If $x \in M_e$ for some $e \in E$, then $x^\circ = e$ (from Lemma 7.1.4), and therefore $\phi_e^e(x) = xe = xx^\circ = x$ by Definition 7.1.2(a). Thus, ϕ_e^e is the identity map.

(b) Suppose $e \leq_E f \leq_E g$. The definition of the homomorphisms in Example 7.1.9 and the assumption $e \leq_E f$ (i.e., $ef = e = fe$) yield $\phi_e^f(\phi_f^g(x)) = \phi_e^f(xf) = (xf)e = xe$.

Hence, $\langle \mathcal{M}, \Phi \rangle$ is indeed an inverse system of semigroups. We shall denote this inverse system by sys R.

Since every m-domain M_e is a right-cancellative monoid (see Lemma 7.1.4), the result of the construction described in this Example is very similar to the result of a construction on PP-monoids which can be found in Fountain [1976].

7.1.4 Decomposition of the multiplicative semigroup of an m-domain ring

In this section we investigate further the inverse system of semigroups of an m-domain ring described in Example 7.1.9. The results in this section which deal with inverse systems of right-cancellative semigroups hold also for inverse systems of left-cancellative semigroups.

The next theorem shows that the multiplicative semigroup of an m-domain ring is a strong semilattice of semigroups induced by the inverse system $\operatorname{sys} R$ from Example 7.1.9.

Theorem 7.1.10. [Cremer, 2025]

(a) *Let $\langle R, +, \cdot \rangle$ be an m-domain ring. Then $\langle R, \cdot \rangle = \mathfrak{S}\,(\operatorname{sys} R)$.*

(b) *Let $\langle \mathcal{A}, \mathcal{H} \rangle$ be an inverse system of disjoint right-cancellative semigroups with identities over a lower semilattice S. If $\mathfrak{S}\langle \mathcal{A}, \mathcal{H} \rangle = \langle R, \cdot \rangle$ for some m-domain ring R, then $\langle \mathcal{A}, \mathcal{H} \rangle = \operatorname{sys} R$.*

Proof. (a) As described in Example 7.1.9, $\operatorname{sys} R = \langle \mathcal{M}, \Phi \rangle$, where $\mathcal{M}$ is the set of m-domains indexed by idempotents and Φ is the set of homomorphisms defined in Example 7.1.9. We construct the strong semilattice of semigroups induced by the inverse system $\langle \mathcal{M}, \Phi \rangle$ using Equation (B.1). Since the union of all the m-domains is R by Lemma 7.1.4, this strong semilattice will have the form $\langle R, \bullet \rangle$. It remains to prove that the multiplication $\bullet$ coincides with the original ring multiplication.

Suppose that $x \in M_e$ and $y \in M_f$ for some elements $x, y \in R$ and idempotents $e, f \in P$. By Lemma 7.1.4, this means

$$x^\circ = e \text{ and } y^\circ = f \tag{7.3}$$

Now by Equations (A.5) and (B.1) and Example 7.1.9, centrality of idempotents, Equation (7.3)) and Lemma 1.1.15(a) we have

$$\begin{aligned} x \bullet y &= \phi^e_{e \wedge_E f}(x) \cdot_{e \wedge_E f} \phi^f_{e \wedge_E f}(y) \\ &= \phi^e_{ef}(x) \cdot_{ef} \phi^f_{ef}(y) \\ &= xef \cdot yef \\ &= xx^\circ yy^\circ \\ &= x \cdot y. \end{aligned}$$

Thus $\langle R, \cdot \rangle = \mathfrak{S}\,\langle \mathcal{M}, \Phi \rangle$.

(b) Let $\langle \mathcal{A}, \mathcal{H} \rangle$ be an inverse system of disjoint right-cancellative semigroups with identities over a lower semilattice S and suppose $\mathfrak{S}\,\langle \mathcal{A}, \mathcal{H} \rangle = \langle R, \cdot \rangle$ for some m-domain ring R.

In order to simplify the notation, we will work with the partial order induced on the family $\mathcal{A}$ by the semilattice S instead of referring to the semilattice S itself. We will denote this induced partial order on $\mathcal{A}$ by $\preceq$ (i.e., $A_s \preceq A_t$ iff $s \leq t$). The corresponding meet operation will be denoted by $\curlywedge$. We will index the homomorphisms from $\mathcal{H}$ by their domain and range, i.e., h^B_A denotes the semigroup homomorphism from B to A for semigroups $A, B \in \mathcal{A}$ with $A \preceq B$. The multiplication on a semigroup $A \in \mathcal{A}$ will be denoted by $\circ_A$.

We need to prove that $\mathcal{A}$ is the family of m-domains of R and that $\mathcal{H}$ is the family of all the homomorphisms ϕ^f_e defined in Example 7.1.9.

Let $\langle \mathcal{M}, \Phi \rangle = \operatorname{sys} R$, i.e., $\mathcal{M}$ is the set of m-domains $\langle M_e, \cdot \rangle$ indexed by idempotents, and $\Phi = \{\phi^f_e \mid e, f \in P \text{ and } e \leq_E f\}$, where P is the set of idempotents.

Since $\mathfrak{S}\,\langle \mathcal{A}, \mathcal{H} \rangle = \langle R, \cdot \rangle$, obviously $\mathcal{A}$ is a partition of R. Moreover, for $x \in A$ and $y \in B$, Equation (B.1) translates to

$$xy = h^A_{A \curlywedge B}(x) \circ_{A \curlywedge B} h^B_{A \curlywedge B}(y) \tag{7.4}$$

(in particular, $xy \in A \curlywedge B$).

Obviously, the identities of the semigroups from $\mathcal{A}$ are idempotents of the ring R, since the multiplication on a semigroup $A \in \mathcal{A}$ is a restriction of the ring multiplication to the set A.

We will prove that

(1) for every $a \in R$ and every $A \in \mathcal{A}$, we have $a \in A$ iff $a^\circ \in A$,

(2) every semigroup $A \in \mathcal{A}$ is the m-domain corresponding to its identity,

(3) $\mathcal{M} = \mathcal{A}$,

(4) $\mathcal{H} = \Phi$.

(1) For a semigroup $A \in \mathcal{A}$, let e be the identity of A and let $a \in A$ be an arbitrary element of A. Let $B \in \mathcal{A}$ be the semigroup containing the element a°.

Since $ae = ea = a$, we have $a^\circ e = a^\circ$ by Definition 7.1.2(b). Therefore $a^\circ e \in B$, because $a^\circ \in B$. But since $\langle R, \cdot \rangle = \mathfrak{S}\langle \mathcal{A}, \mathcal{H} \rangle$, Equation (7.4) yields that $a^\circ e \in B \curlywedge A$. So $B \preceq A$.

On the other hand, by Definition 7.1.2(a), we have $a^\circ a = a \in A$. But now Equation (7.4) yields $a^\circ a \in B \curlywedge A$. So $B \curlywedge A = A$, i.e., $A \preceq B$.

Hence, $A = B$. Since A was arbitrary and B was chosen to be the semigroup containing a° for an arbitrary element $a \in A$, we conclude that a and a° are always contained in the same semigroup. I.e., for $a \in R$ and $A \in \mathcal{A}$, $a \in A$ if and only if $a^\circ \in A$.

(2) If $a \in A$, then not only $a^\circ \in A$, but a° must be the identity e of A, because by right-cancellativity of A, from $a^\circ a = a = ea$ (see Definition 7.1.2(a)) follows $a^\circ = e$.

Conversely, let e be the identity of A and $a \in R$ and $a^\circ = e$. Then by the previous paragraph (1), we have $a \in A$.

This proves that $A = M_e$, because the m-domain is defined as $M_e = \{x \in R \mid x^\circ = e\}$ (see Lemma 7.1.4). Since the multiplication $\circ_A$ is a restriction of the ring multiplication, A and M_e are equal not only as sets, but also as semigroups.

(3) It follows from the previous step that $\mathcal{A} \subseteq \mathcal{M}$. Since both $\mathcal{A}$ and $\mathcal{M}$ are partitions of the ring R, this immediately implies $\mathcal{A} = \mathcal{M}$.

(4) Let $h_A^B \in \mathcal{H}$. In view of the previous items, the domain and range of h_A^B are m-domains. So $h_A^B : M_f \to M_e$ for some idempotents $e, f \in E$. We will prove that $e \leq_E f$ and $h_A^B = \phi_e^f$.

First, observe that e is the only idempotent in M_e: If $g \in M_e$ is idempotent, then $eg = geg$ by centrality of idempotents (see Remark 7.1.3). So by cancellativity, $e = ge$. But $ge = g$, because e is the identity of M_e. Hence, $e = g$.

Let $x \in M_f$; then

$$\begin{aligned} h_A^B(x) &= e \circ_A h_A^B(x) \text{ (since } e \text{ is the identity of } M_e) \\ &= h_A^A(e) \circ_A h_A^B(x) \text{ (because } h_A^A \text{ is the identity map)} \\ &= ex \text{ (by Equation (7.4)).} \end{aligned}$$

By the first part of this theorem, $ex \in M_{e \wedge_E f}$, since $e \in M_e$ and $x \in M_f$. But since the range of h_A^B is M_e, we also have $ex \in M_e$. This yields $M_e = M_{e \wedge_E f}$. So $e = e \wedge_E f$, i.e., $e \leq_E f$. Now obviously $h_A^B(x) = ex = \phi_e^f(x)$ by Example 7.1.9.

So indeed $\langle \mathcal{A}, \mathcal{H} \rangle = \operatorname{sys} R$. □

Fountain obtained a result on a strong semilattice decomposition of right PP-monoids with central idempotents, see [Fountain, 1976, Theorem 1]. His result is similar to Theorem 7.1.10(a) and in particular Corollary 7.1.11(a), and there are similar ideas in the proofs. However, the author preferred a direct and independent proof of Theorem 7.1.10(a). This also establishes some notation which will be necessary in the proof of Theorem 7.2.11.

It follows from Theorem 7.1.10 that the multiplicative semigroups of m-domain rings R and R' are equal if $\operatorname{sys} R = \operatorname{sys} R'$ (the converse is also true by Example 7.1.9).

As a special case of Theorem 7.1.10, we obtain the following.

Corollary 7.1.11. [Cremer, 2025]

(a) *Let $\langle R, +, \cdot, 1\rangle$ be a reduced Rickart ring. Then $\langle R, \cdot\rangle = \mathfrak{S}(\operatorname{sys} R)$.*

(b) *Let $\langle \mathcal{A}, \mathcal{H}\rangle$ be an inverse system of disjoint right-cancellative semigroups over lower semilattice S. If $\mathfrak{S}\langle \mathcal{A}, \mathcal{H}\rangle = \langle R, \cdot\rangle$ for some reduced Rickart ring R, then $\langle \mathcal{A}, \mathcal{H}\rangle = \operatorname{sys} R$.*

Proof. (a) Immediate from Theorem 7.1.10(a).

(b) Suppose that $\mathfrak{S}\langle \mathcal{A}, \mathcal{H}\rangle = \langle R, \cdot\rangle$ for some reduced Rickart ring R. Since every reduced Rickart ring is also an m-domain ring, it suffices to prove that the semigroups from the inverse system $\langle \mathcal{A}, \mathcal{H}\rangle$ have identities. We use the fact that every reduced Rickart ring is unitary.

Since $\mathcal{A}$ is a partition of R, the element 1 of the ring is contained in some semigroup from $\mathcal{A}$. Let T be this semigroup. For every element x of the ring, if $x \in A$, then $x \cdot 1 \in A \curlywedge T$ by Equation (7.4). But obviously $x \cdot 1 = x \in A$. Therefore, $A = A \curlywedge T$. Since this is the case for all $A \in \mathcal{A}$, the semigroup T must be the greatest element of $\mathcal{A}$.

Therefore, for every $A \in \mathcal{A}$, there is a homomorphism h_A^T from T to A. Now consider the element $h_A^T(1) \in A$. For every element $a \in A$, since h_A^A is the identity map (see Definition B.1.2), we have $h_A^T(1) \circ_A a = h_A^T(1) \circ_A h_A^A(a) = 1 \cdot a$ by Equation (7.4). In the same way, $a \circ_A h_A^T(1) = a$. So the element $h_A^T(1)$ is the identity of A. □

It is already known from Fountain's result in Fountain [1976] that the semigroup $\mathfrak{S}\langle \mathcal{A}, \mathcal{H}\rangle$ mentioned in Corollary 7.1.11(b) is a left PP monoid with central idempotents.

It follows from Corollary 7.1.11 that the multiplicative semigroups of reduced Rickart rings R and R' are equal if $\operatorname{sys} R = \operatorname{sys} R'$ and that $\mathfrak{S}(\operatorname{sys} R)$ is the unique representation of the semigroup $\langle R, \cdot\rangle$ as a strong semilattice of right-cancellative semigroups.

7.2 Strong semilattice decomposition using D-semigroups and their inverse systems

The question arises if Theorem 7.1.10 can be modified to include also the operation $^\circ$. I.e., we want to find out what happens if instead of the inverse system of m-domains (seen as semigroups $\langle M_e, \cdot\rangle$) we deal with the inverse system of "enriched" m-domains (seen as algebras of the kind $\langle M_e, \cdot, ^\circ\rangle$ or $\langle M_e, \cdot, ^\circ, e\rangle$). Answering this question is the goal of this section.

7.2.1 D-monoids

Recall the definition of D-semigroups (Definition B.3.1). In this thesis, we will deal also with D-semigroups with identity $\langle A, \cdot, ^\circ, 1\rangle$, i.e., D-monoids. In particular, we need the following special subclasses of D-monoids.

Definition 7.2.1. A D-semigroup or D-monoid is said to be *right-cancellative* if the underlying semigroup is right-cancellative.

We call a D-monoid $\langle A, \cdot, ^\circ, 1\rangle$ *$^\circ$-trivial* if $a^\circ = 1$ for all $a \in A$.

Every monoid can be turned into a $^\circ$-trivial D-monoid in the following very straightforward way:

Proposition 7.2.2. [Cremer, 2025] *Let $\langle A, \cdot, 1\rangle$ be a monoid, and let $a^\circ := 1$ for all $a \in A$. Then $\langle A, \cdot, ^\circ, 1\rangle$ is a D-monoid.*

Proof. Easy calculations show that all the conditions of Definition B.3.1 are satisfied: $a^\circ a = 1a = a$, $(a^\circ)^\circ = 1 = 1^\circ$ and $(ab)^\circ a^\circ = a^\circ (ab)^\circ = (ab)^\circ = 1$. ☐

Proposition 7.2.3. [Cremer, 2025] *Let $\langle A, \cdot, 1\rangle$ be a right-cancellative monoid with a unary operation $^\circ$. Then the following are equivalent.*

(a) *The algebra $\langle A, \cdot, ^\circ, 1\rangle$ is a D-monoid.*
(b) *The operation $^\circ$ satisfies the conditions ($^\circ$1) and ($^\circ$2) from Proposition B.3.2.*
(c) *$a^\circ = 1$ for all $a \in A$.*

Proof. (a) $\Longrightarrow$ (b) is obvious.
(b) $\Longrightarrow$ (c) follows by right-cancellativity from $a^\circ a = 1 \cdot a$.
(c) $\Longrightarrow$ (a) is immediate from Proposition 7.2.2. ☐

The next lemma establishes the relationships between right-cancellative, $^\circ$-trivial and D-abundant D-monoids.

Lemma 7.2.4. [Cremer, 2025]
A D-monoid $\langle A, \cdot, ^\circ, 1\rangle$ is right-cancellative if and only if it is both D-abundant and $^\circ$-trivial.

Proof. Let $\langle A, \cdot, ^\circ, 1\rangle$ be a D-monoid.
First, assume that $\langle A, \cdot, ^\circ, 1\rangle$ is both D-abundant and $^\circ$-trivial. Let $a, x, y \in A$ such that $xa = ya$. Then $xa^\circ = ya^\circ$, since $\langle A, \cdot, ^\circ, 1\rangle$ is D-abundant (see Definition B.3.4). But since it is also $^\circ$-trivial (see Definition 7.2.1), $a^\circ = 1$, so $x = y$. Hence, $\langle A, \cdot, ^\circ, 1\rangle$ is right-cancellative.
For the opposite direction, assume that $\langle A, \cdot, ^\circ, 1\rangle$ is right-cancellative. Then it is obviously D-abundant, because for all $x, y, a \in A$, $xa = ya$ implies $x = y$. Thus also $xa^\circ = ya^\circ$, and $^\circ$-triviality follows from Proposition 7.2.3. ☐

m-domain rings as D-abundant D-rings

The purpose of the following proposition is to clarify the relation between D-rings and enriched m-domain rings by showing that every enriched m-domain ring is a D-abundant D-ring in a unique way.

Proposition 7.2.5. [Cremer, 2025]
Let $\langle R, +, \cdot, ^\circ\rangle$ be an enriched m-domain ring.

(a) *Then $\langle R, +, \cdot, ^\circ\rangle$ is a D-semiadequate D-abundant D-ring. The set U from Definition B.3.1 which corresponds to the operation $^\circ$ is the set of all idempotents E.*
(b) *If an algebra $\langle R, +, \cdot, ^+\rangle$ (i.e., the same ring with another unary operation $^+$) is a D-abundant D-ring, too, then the operations $^\circ$ and $^+$ coincide.*

Proof. (a) To prove that $\langle R, +, \cdot, ^\circ\rangle$ is a D-ring, we will verify that $\langle R, \cdot, ^\circ\rangle$ satisfies the conditions for being a D-semigroup given in Proposition B.3.2. Obviously, the first condition of Proposition B.3.2 holds, because it is identical to the first item of Definition 7.1.2 (recall that idempotents are central by Remark 7.1.3). The third condition of Proposition B.3.2 follows immediately from the third item of Definition 7.1.2 and from the fact that a° is a central idempotent for every $a \in R$. For the second condition of Proposition B.3.2, let $a \in R$. By Lemma 7.1.4, there exists some m-domain M_e such that $a \in M_e$, and $a^\circ = e$. Since $e \in M_e$, too, Lemma 7.1.4 also yields that $e^\circ = e$. Therefore, $(a^\circ)^\circ = e^\circ = e = a^\circ$. So $\langle R, +, \cdot, ^\circ\rangle$ is a D-ring.

To prove that $\langle R, \cdot, ^\circ \rangle$ is also D-abundant, let $x, y, a \in R$ be such that $xa = ya$. Then $(x - y)a = 0$.
So by Definition 7.1.2(c) and Equation (7.1), $(x-y)^\circ a^\circ = ((x-y)a)^\circ = 0^\circ = 0$. Hence, by Definition 7.1.2(a), $(x-y)a^\circ = (x-y)(x-y)^\circ a^\circ = 0$. Therefore, $xa = ya$ implies $xa^\circ = ya^\circ$, so by Definition B.3.4, $\langle R, +, \cdot, ^\circ \rangle$ is D-abundant.
D-semiadequateness follows immedidately from centrality of idempotents a° and b° (for arbitrary a and b).
Obviously, the set U from Definition B.3.1 must be the set of all idempotents, because the operation $^\circ$ maps every idempotent on itself (see Remark 7.1.3).

(b) For uniqueness of the operation $^\circ$, suppose $^+$ is another unary operation defined on R such that $\langle R, +, \cdot, ^+ \rangle$ is a D-abundant D-ring. Then

$$a^+ a = a = a^\circ a \tag{7.5}$$

for every $a \in R$ by Proposition B.3.2($^\circ$1). This yields $a^+ = a^+ a^+ = a^\circ a^+$ by Definition B.3.4. In the same way, Equation (7.5) also yields $a^\circ = a^\circ a^\circ = a^+ a^\circ$. So $a^+ = a^\circ a^+ = a^+ a^\circ = a^\circ$ by centrality of a°. □

From here on, given an enriched m-domain ring $\langle R, +, \cdot, ^\circ \rangle$, the D-semigroup $\langle R, \cdot, ^\circ \rangle$ will be called the *D-semigroup reduct of R*, and for an enriched reduced Rickart ring $\langle R, +, \cdot, ^\circ, 1 \rangle$, the D-monoid $\langle R, \cdot, ^\circ, 1 \rangle$ will be called the *D-monoid reduct of R*.

7.2.2 The inverse system of D-semigroups formed by the m-domains

In Theorem 7.1.10 we saw that, first, the multiplicative semigroup $\langle R, \cdot \rangle$ of an m-domain ring is the strong semilattice induced by its inverse system of semigroups, and, second, if the strong semilattice induced by a given inverse system $\langle \mathcal{A}, \mathcal{H} \rangle$ of right-cancellative semigroups with identity happens to be the multiplicative semigroup of some m-domain ring, then $\langle \mathcal{A}, \mathcal{H} \rangle$ is the inverse system of semigroups of the ring.

We are going to prove similar results about the D-semigroup reduct of an enriched m-domain ring and about the D-monoid reduct of an enriched reduced Rickart ring in Section 7.2.4. Therefore in this section we turn the m-domains into $^\circ$-trivial D-monoids and prove that the semigroup homomorphisms defined in Example 7.1.9 between the m-domains are also D-monoid homomorphisms. Hence we obtain an inverse system of D-monoids in an enriched m-domain ring.

Proposition 7.2.6. [Cremer, 2025] *Let R be an enriched m-domain ring and let E be its semilattice of idempotents. For every $e \in E$, let $\langle M_e, \cdot_e, {}^\circ_e, e \rangle$ be the algebra of type* $(2, 1, 0)$ *defined by*

$$a \cdot_e b := a \cdot b \tag{7.6}$$

(see also Example 7.1.9*) and*

$$a^\circ_e := e. \tag{7.7}$$

Then $\langle M_e, \cdot_e, {}^\circ_e, e \rangle$ is a right-cancellative D-monoid.

Proof. By Proposition 7.2.5, $\langle R, +, \cdot, ^\circ \rangle$ is a D-abundant D-ring. The operation $\cdot_e$ is the restriction of the ring multiplication $\cdot$ to the m-domain M_e, and the operation ${}^\circ_e$ is the restriction of the operation $^\circ$ from Definition 7.1.2 (recall that M_e is closed under $\cdot$ by Lemma 7.1.4 and under $^\circ$ by Remark 7.1.6). So $\langle M_e, \cdot_e, {}^\circ_e \rangle$ is a sub-D-semigroup of $\langle R, \cdot, ^\circ \rangle$. Therefore, it must be D-abundant, too.

Moreover, since M_e has the identity e (by Lemma 7.1.4), it is a D-monoid. It is evident from Equation (7.7) that the D-monoid $\langle M_e, \cdot_e, {}^\circ_e, e \rangle$ is $^\circ$-trivial. Now Lemma 7.2.4 yields that it is right-cancellative. □

Proposition 7.2.7. [Cremer, 2025] *Let R be an enriched m-domain ring, let E be its semilattice of idempotents and for every pair of idempotents e, f with $e \leq_E f$, let $\phi_e^f : M_f \to M_e$ be the map given by Example 7.1.9 (i.e., $\phi_e^f(x) = xe$). Let*

$$\mathcal{M}_1^\circ = \{\langle M_e, \cdot_e, {}^\circ_e, e\rangle \,|\, e \in E\} \tag{7.8}$$

with $\langle M_e, \cdot_e, {}^\circ_e, e\rangle$ as in Proposition 7.2.6, and let

$$\Phi = \{\phi_e^f \,|\, e, f \in E \text{ and } e \leq_E f\} \tag{7.9}$$

as in Example 7.1.9. Then $\langle \mathcal{M}_1^\circ, \Phi\rangle$ is an inverse system of right-cancellative D-monoids.

Proof. First let us prove that ϕ_e^f is a D-monoid homomorphism between the D-monoids $\langle M_f, \cdot_f, {}^\circ_f, f\rangle$ and $\langle M_e, \cdot_e, {}^\circ_e, e\rangle$.

We have already proved (see Example 7.1.9) that the map ϕ_e^f is a semigroup homomorphism between $\langle M_f, \cdot_f\rangle$ and $\langle M_e, \cdot_e\rangle$. It also preserves the identity, because by Example 7.1.9, $\phi_e^f(f) = fe = e$, since $e \leq_E f$. Since it preserves the identity, it must also preserve the unary operation, because the D-monoids are $^\circ$-trivial: By Equation (7.7) $\phi_e^f(x_f^\circ) = \phi_e^f(f) = e = (\phi_e^f(x))_e^\circ$. Hence, ϕ_e^f is a D-monoid homomorphism.

In Example 7.1.9 we already saw that for every $e \in E$, the map ϕ_e^e is the identity map, and that $\phi_e^f \phi_f^g = \phi_e^g$ for all $e, f, g \in E$ with $e \leq_E f \leq_E g$. We conclude that $\langle E, \mathcal{M}, \Phi\rangle$ is an inverse system of D-monoids. ☐

Definition 7.2.8. [Cremer, 2025] The inverse system of D-monoids $\langle \mathcal{M}_1^\circ, \Phi\rangle$ from Proposition 7.2.7 will be called the *inverse system of D-monoids of the enriched m-domain ring R* and we will denote it by $\text{sys}_1^\circ R$. When we want to treat the m-domains just as D-semigroups instead of D-monoids, then we will speak of the *inverse system of D-semigroups of the enriched m-domain ring R* and write $\text{sys}^\circ R$ to refer to the inverse system $\langle \mathcal{M}^\circ, \Phi\rangle$, where $\mathcal{M}^\circ = \{\langle M_e, \cdot_e, {}^\circ_e\rangle \,|\, e \in E\}$.

7.2.3 Strong semilattices of D-monoids

In this subsection, we will settle the terminology concerning strong semilattices of D-semigroups and D-monoids.

The following definition is a variation of the left/right dual construction of the one in [Fountain, 1976, Theorem 1]. It is more general in that it does not assume the D-monoids to be right-cancellative, whereas in Fountain [1976], the monoids are required to be left-cancellative (so the dual would require them to be right-cancellative).

Definition 7.2.9. [Cremer, 2025]

(a) Let $\langle \mathcal{A}^\circ, \mathcal{H}\rangle$ be an inverse system of pairwise disjoint D-semigroups over a lower semilattice S. On the union of all the D-semigroups $A = \bigcup_{s\in S} A_s$, we define a binary operation $\bullet$ as in Equation (B.1) and a unary operation $^\bullet$ in the following way: For $x \in A_s$,

$$x^\bullet := x_s^\circ. \tag{7.10}$$

Then we write $A = \mathfrak{S}^\circ \langle \mathcal{A}^\circ, \mathcal{H}\rangle$ and call the algebra $\langle A, \bullet, {}^\bullet\rangle$ a *strong semilattice of D-semigroups* and say that it is *induced by the inverse system $\langle \mathcal{A}^\circ, \mathcal{H}\rangle$*.

(b) If $\langle \mathcal{A}^\circ, \mathcal{H}\rangle$ is even an inverse system of D-monoids over a semilattice S which has the greatest element $\top$, then we can define also a constant $\mathbf{1}$ by

$$\mathbf{1} := 1_\top. \tag{7.11}$$

We write $A = \mathfrak{S}_1^\circ [\langle \mathcal{A}^\circ, \mathcal{H}\rangle]$ and call the algebra $\langle A, \bullet, {}^\bullet, \mathbf{1}\rangle$ a *strong semilattice of D-monoids*.

Obviously, if $\langle A, \bullet, {}^\bullet \rangle$ is a strong semilattice of D-semigroups, then $\langle A, \bullet \rangle$ is a strong semilattice of semigroups.

The purpose of the next proposition is to justify the terminology by giving a positive answer to the naturally arising question whether a strong semilattice of D-semigroups (D-monoids) is itself also a D-semigroup (D-monoid).

Proposition 7.2.10. [Cremer, 2025]
(a) *Every strong semilattice of D-semigroups is a D-semigroup.*
(b) *Every strong semilattice of D-monoids is a D-monoid.*

Proof. (a) Let S, $\mathcal{A}^\circ$ and $\mathcal{H}$ be as in Definition 7.2.9, and let $\bullet$ and ${}^\bullet$ be the operations defined on the union $A = \bigcup_{s \in S} A_s$ as in Definition 7.2.9(a).
Since $\langle A, \bullet \rangle$ is a strong semilattice of semigroups, it is clear that it is a semigroup. So we only need to prove that the identities from Proposition B.3.2 are satsified.
Let $x \in A_s$, $y \in A_t$ and $z \in A_u$.
By Equation (B.1), Equation (7.10), Definition B.1.1 (recall that $\langle \mathcal{A}^\circ, \mathcal{H} \rangle$ is an inverse system) and Proposition B.3.2(°1):

$$x^\bullet \bullet x = h^s_s(x^\circ_s) \cdot_s h^s_s(x) = x^\circ_s \cdot_s x = x$$

By Equation (7.10) and Proposition B.3.2(°2):

$$(x^\bullet)^\bullet = (x^\circ_s)^\circ_s = x^\circ_s = x^\bullet.$$

The third identity follows similarly from Proposition B.3.2(°3):

$$\begin{aligned}
&(x \bullet y)^\bullet \bullet x^\bullet \\
&= (h^s_{s\wedge t}(x) \cdot_{s\wedge t} h^t_{s\wedge t}(y))^\bullet \bullet x^\circ_s && \text{(by Equations (B.1) and (7.10))} \\
&= (h^s_{s\wedge t}(x) \cdot_{s\wedge t} h^t_{s\wedge t}(y))^\circ_{s\wedge t} \cdot_{s\wedge t} h^s_{s\wedge t}(x^\circ_s) && \text{(by Equations (7.10) and (B.1))} \\
&= (h^s_{s\wedge t}(x) \cdot_{s\wedge t} h^t_{s\wedge t}(y))^\circ_{s\wedge t} \cdot_{s\wedge t} (h^s_{s\wedge t}(x))^\circ_s && (h^s_{s\wedge t} \text{ is a D-semigroup morphism}) \\
&= (h^s_{s\wedge t}(x) \cdot_{s\wedge t} h^t_{s\wedge t}(y))^\circ_{s\wedge t} && \text{(by Proposition B.3.2(°3))} \\
&= (x \bullet y)^\bullet && \text{(by Equations (B.1) and (7.10))},
\end{aligned}$$

and similarly for the multiplication from the other side.

(b) Suppose the semilattice S has a top element $\top$ and $\langle \mathcal{A}^\circ, \mathcal{H} \rangle$ is an inverse system of D-monoids over S. By the previous item, $\langle A, \bullet, {}^\bullet \rangle$ is a D-semigroup.
The element $\mathbf{1}$ is the neutral element with respect to the operation $\bullet$, because 1_s is the neutral element for $\cdot_s$:

$$x \bullet \mathbf{1} = h^s_{s\wedge\top}(x) \cdot_{s\wedge\top} h^\top_{s\wedge\top}(1_\top) = h^s_s(x) \cdot_s h^\top_s(1_\top) = h^s_s(x) \cdot_s 1_s = x,$$

by Equation (B.1) and (7.11), since $\top$ is the top element of S and $h^\top_s$ is a monoid homomorphism. (The identity $\mathbf{1} \bullet x = x$ is proved in the same way.) Hence, $\langle A, \bullet, \mathbf{1} \rangle$ is a monoid. □

Observe that the operation $^\circ$ is used neither in item (b) nor in the proof of Proposition 7.2.10(b). So given an inverse system of monoids (instead of D-monoids) over a lower semilattice which has the greatest element, we can also define a constant $\mathbf{1}$ by item (b), and $\mathbf{1}$ is the identity of the strong semilattice of semigroups $\langle A, \bullet \rangle$.

7.2.4 Decomposition of the D-semigroup reduct of an enriched m-domain ring

In this subsection, we look at the strong semilattice of D-semigroups (D-monoids) induced by the inverse system $\mathrm{sys}^\circ R$ ($\mathrm{sys}^\circ_1 R$) of a given enriched m-domain ring (reduced Rickart ring) R, and we see that it is the D-semigroup $\langle R, \cdot, ^\circ \rangle$ (and for a reduced Rickart ring, the D-monoid $\langle R, \cdot, ^\circ, 1\rangle$). So the D-semigroup reduct of an enriched m-domain ring is a strong semilattice of right-cancellative D-semigroups, and the D-monoid reduct of a reduced Rickart ring is a strong semilattice of right-cancellative D-monoids.

The next theorem provides a positive answer to the question formulated in the beginning of this section: It is the D-semigroup version of Theorem 7.1.10, and it differs from the original in that it includes the operation $^\circ$. The theorem proves that the D-semigroup reduct $\langle R, \cdot, ^\circ\rangle$ of an enriched m-domain ring is a strong semilattice of right-cancellative D-semigroups.

Theorem 7.2.11. [Cremer, 2025]

(a) *Let $\langle R, +, \cdot, ^\circ\rangle$ be an enriched m-domain ring. Then $\langle R, \cdot, ^\circ\rangle = \mathfrak{S}^\circ(\mathrm{sys}^\circ R)$.*

(b) *Let $\langle \mathcal{A}^\circ, \mathcal{H}\rangle$ be an inverse system of pairwise disjoint right-cancellative D-semigroups with identities over some lower semilattice. Suppose $\mathfrak{S}^\circ\langle \mathcal{A}^\circ, \mathcal{H}\rangle = \langle R, \cdot, ^\circ\rangle$ for some enriched m-domain ring $\langle R, +, \cdot, ^\circ\rangle$. Then $\langle \mathcal{A}^\circ, \mathcal{H}\rangle = \mathrm{sys}^\circ R$.*

Proof. Let R be an enriched m-domain ring and let $\langle \mathcal{M}^\circ, \Phi\rangle = \mathrm{sys}^\circ R$. So $\mathcal{M}^\circ = \{\langle M_e, \cdot_e, \overset{\circ}{e}\,\rangle \mid e \in P\rangle\}$ (see Definition 7.2.8) and Φ is the family defined in Equation (7.9), i.e., $\Phi = \{\phi_e^f \mid e, f \in P \text{ and } e \leq_E f\}$, where ϕ_e^f are the maps defined in Example 7.1.9. Let $\langle R, \bullet, ^\bullet\rangle = \mathfrak{S}^\circ\langle \mathcal{M}^\circ, \Phi\rangle$.

(a) We need to prove that $\langle R, \bullet, ^\bullet\rangle = \langle R, \cdot, ^\circ\rangle$. To prove that the binary operation $\bullet$ coincides with $\cdot$, let us forget about the other operations for a moment. Consider the family of semigroups $\mathcal{M}$ obtained from $\mathcal{M}^\circ$ by replacing each element $\langle M_e, \cdot_e, \overset{\circ}{e}\,\rangle$ by its reduct $\langle M_e, \cdot_e\rangle$. By Definition 7.2.8, $\langle \mathcal{M}, \Phi\rangle = \mathrm{sys}\, R$. So by Theorem 7.1.10, $\mathfrak{S}\langle \mathcal{M}, \Phi\rangle = \langle R, \cdot\rangle$. But $\mathfrak{S}\langle \mathcal{M}, \Phi\rangle = \langle R, \bullet\rangle$. So the operation $\bullet$ indeed coincides with the ring multiplication $\cdot$. The unary operations $^\circ$ and $^\bullet$ obviously coincide, too, because $x^\bullet = x_e^\circ = e = x^\circ$ by Equations (7.10), Equation (7.7) and Lemma 7.1.4.

(b) Now let $\langle \mathcal{A}^\circ, \mathcal{H}\rangle$ be an inverse system of right-cancellative D-semigroups with identities over a lower semilattice such that $\mathfrak{S}^\circ\langle \mathcal{A}^\circ, \mathcal{H}\rangle = \langle R, \cdot, ^\circ\rangle$. We have to prove that $\langle \mathcal{A}^\circ, \mathcal{H}\rangle = \mathrm{sys}^\circ R$, i.e., that $\langle \mathcal{A}^\circ, \mathcal{H}\rangle = \langle \mathcal{M}^\circ, \Phi\rangle$ (with $\langle \mathcal{M}^\circ, \Phi\rangle$ as defined in Definition 7.2.8).
Let $\mathcal{M}$ and $\mathcal{A}$ be the families of semigroups obtained from $\mathcal{M}^\circ$ and $\mathcal{A}^\circ$ by replacing all the elements by their semigroup reducts. So $\mathfrak{S}\langle \mathcal{A}, \mathcal{H}\rangle = \langle R, \cdot\rangle$, and since the semigroups from $\mathcal{A}$ have identities, we can use Theorem 7.1.10(b), obtaining that $\langle \mathcal{A}, \mathcal{H}\rangle = \mathrm{sys}\, R$, i.e., $\mathcal{M} = \mathcal{A}$ and $\Phi = \mathcal{H}$. In particular, the underlying sets of the members of $\mathcal{A}$ are the m-domains of the ring R, and the multiplications $\cdot_e$ on them are just restrictions of the ring multiplication.
It remains to prove that the unary operations on each m-domain M_e also coincide. Let $\mathcal{A}^\circ = \{\langle M_e, \cdot_e, \overset{+}{e}\,\rangle \mid e \in P\}$. By Proposition B.3.2(°1), for every element a of the m-domain M_e, we have $a_e^+ a = a$, because $\langle M_e, \cdot_e, \overset{+}{e}\,\rangle$ is a D-semigroup. Since by Proposition 7.2.6 the m-domain M_e with the usual unary operation $\overset{\circ}{e}$ is also a D-semigroup, we also have $a_e^\circ a = a$. So $a_e^+ a = a_e^\circ a$. But by right-cancellativity of the semigroup $\langle M_e, \cdot_e\rangle$, this implies $a_e^+ = a_e^\circ$. So $\langle \mathcal{A}^\circ, \mathcal{H}\rangle = \mathrm{sys}^\circ R$. □

For an inverse system $\langle \mathcal{A}^\circ, \mathcal{H}\rangle$ of D-semigroups with identities, let us denote by $\mathcal{A}_1^\circ$ the family obtained from $\mathcal{A}^\circ$ by including the identities of the D-semigroups into their signatures. Then $\langle \mathcal{A}_1^\circ, \mathcal{H}\rangle$ is even an inverse system of D-monoids.

The following corollary shows that in the case of an enriched reduced Rickart ring, Theorem 7.2.11 can be modified to include the multiplicative identities not only into the inverse system, but also into the strong semilattice construction. In particular, it says that the D-monoid reduct $\langle R, \cdot, ^\circ, 1\rangle$ of an enriched reduced Rickart ring is a strong semilattice of right-cancellative D-monoids.

Corollary 7.2.12. [Cremer, 2025]

(a) *If R is an enriched reduced Rickart ring, then* $\langle R, \cdot, ^\circ, 1\rangle = \mathfrak{S}_1^\circ(\operatorname{sys}_1^\circ R)$.

(b) *Let* $\langle \mathcal{A}^\circ, \mathcal{H}\rangle$ *be an inverse system of pairwise disjoint right-cancellative D-semigroups with identities over some lower semilattice, and let* $\mathcal{A}_1^\circ$ *denote the family of D-monoids obtained from the family of D-semigroups* $\mathcal{A}^\circ$ *by including the identities into the signatures. Suppose* $\mathfrak{S}^\circ\langle \mathcal{A}^\circ, \mathcal{H}\rangle = \langle R, \cdot, ^\circ, 1\rangle$ *for some enriched reduced Rickart ring* R. *Then* $\langle \mathcal{A}_1^\circ, \mathcal{H}\rangle = \operatorname{sys}_1^\circ R$.

Proof. (a) If R is not only an enriched m-domain ring, but even an enriched reduced Rickart ring, then its lattice of idempotents E has the greatest element 1. Hence, we can apply the second part of Definition 7.2.9 to obtain a strong semilattice of D-monoids $\mathfrak{S}_1^\circ\langle \mathcal{M}_1^\circ, \Phi\rangle$. Let $\langle R, \bullet, ^\bullet, \mathbf{1}\rangle = \mathfrak{S}_1^\circ\langle \mathcal{M}_1^\circ, \Phi\rangle$.
It is clear from Theorem 7.1.10(a) andTheorem 7.2.11(a) that the operation $\bullet$ is the ring multiplication and that $^\bullet$ is the operation from Lemma 1.1.15.
By Definition 7.2.9, $\mathbf{1}$ is the identity element with respect to the operation $\cdot_\top$ of the D-monoid $\langle M_\top, \cdot_\top, ^\circ_\top, \top\rangle$ corresponding to the top element $\top$ of the lattice E. Since $\top = 1$, obviously $\mathbf{1} = 1$.

(b) By Theorem 7.2.11(b), we already know that $\langle \mathcal{A}^\circ, \mathcal{H}\rangle = \operatorname{sys}^\circ R$. So $\langle \mathcal{A}^\circ, \mathcal{H}\rangle$ is an inverse system over a semilattice which actually is (isomorphic to) the lattice of idempotents P, and $\mathcal{A}^\circ = \mathcal{M}^\circ$. Therefore, the constant $\mathbf{1}$ obtained from $\langle \mathcal{A}^\circ, \mathcal{H}\rangle$ by item (b) is the identity of the m-domain M_1, that is, $\mathbf{1} = 1$. Hence, $\langle \mathcal{A}_1^\circ, \mathcal{H}\rangle = \operatorname{sys}_1^\circ R$. □

7.3 Band-enriched monoids

In this section, we introduce and explore a new structure that describes the role of the skew meet from Equation (2.27) in a reduced Rickart ring while also being related to certain D-monoids. In Section 7.4, we will use this structure instead of D-monoids in a strong semilattice decomposition of reduced Rickart rings akin to the one described in Section 7.2.

7.3.1 The skew meet in a reduced Rickart ring

In this subsection, we examine the role of the skew meet operation on a reduced Rickart ring.

Recall that the skew meet operation $\overleftarrow{\wedge}$ on a right-strong right Rickart ring is defined by Equation (2.27). For elements a, b of a reduced Rickart ring, this definition simplifies to

$$a \overleftarrow{\wedge} b = a''b. \tag{7.12}$$

Cornish defined a similar operation on C-rings in Cornish [1980b]. If the C-ring is a reduced Rickart ring, then his operation $\wedge$ is defined by $a \wedge b := ab''$. To be more consistent with our notation, it could be written $\overrightarrow{\wedge}$. (See Remark 7.3.2 below for the connection between the operations $\overrightarrow{\wedge}$ and $\overleftarrow{\wedge}$ in a reduced Rickart ring.)

By Proposition 2.2.13, a right-strong right Rickart ring with equipped with the skew meet $\overleftarrow{\wedge}$ is a right normal band. So in view of Lemma B.2.5, we obtain the following proposition.

Proposition 7.3.1. *Let R be a reduced Rickart ring and let $\overleftarrow{\wedge}$ be the skew meet defined on it. Then the semigroup $\langle R, \overleftarrow{\wedge}\rangle$ is a strong semilattice of right zero bands.*

Remark 7.3.2. The skew meet could also be called *right skew meet*, since we obtain a right normal band from it. We could as well define the *left skew meet* in a reduced Rickart ring as $a \overrightarrow{\wedge} b := ab''$ (this is the operation which Cornish defined on C-rings in Cornish [1980b]). However, since the

Rickart ring considered in this chapter is reduced, there is no need to consider both right and left skew meets for the following reason.

Centrality of idempotents in a reduced ring yields $a \overrightarrow{\wedge} b = b \overleftarrow{\wedge} a$. Therefore, the left skew meet gives rise to a left normal band in the same way as the right skew meet gives rise to a right normal band. Since this left normal band is obviously not much different from its right normal counterpart, it does not provide any further insights. So we will focus only on the right skew meet, and we will continue to call it just "skew meet" for simplicity. ◁

In Example 7.1.9 we saw that the multiplicative semigroup $\langle R, \cdot \rangle$ of an m-domain ring (in particular, a reduced Rickart ring) is a strong semilattice of its m-domains. By Proposition 7.3.1, for a reduced Rickart ring R with the skew meet $\overleftarrow{\wedge}$, the band $\langle R, \overleftarrow{\wedge} \rangle$ is a strong semilattice, too. This is the motivation for introducing particular algebras with two associative operations which have the properties of the ring multiplication and the skew meet of a reduced Rickart ring, in order to be able to define strong semilattices of these algebras later.

7.3.2 Introducing band-enriched monoids

In this subsection, we introduce an algebra (called *band-enriched monoid*) with two binary operations and a constant satisfying the three properties which are satisfied by the multiplication and the skew meet on a reduced Rickart ring, thus making a reduced Rickart ring an instance of such an algebra. In the end of the subsection, we see that also the m-domains of R are band-enriched monoids.

Definition 7.3.3. A *band-enriched monoid* is an algebra $\langle A, \triangle, \circ, 1 \rangle$ such that $\langle A, \circ, 1 \rangle$ is a monoid, $\langle A, \triangle \rangle$ is a band and the following identities hold for all $a, b, c \in A$.

(BEM1) $(a \circ b) \triangle c = a \triangle b \triangle c$,

(BEM2) $a \triangle (b \circ c) = b \circ (a \triangle c)$,

(BEM3) $a \triangle (b \circ c) = (a \triangle b) \circ c$.

The next proposition shows how the skew meet on a reduced Rickart ring is connected to the ring multiplication.

Proposition 7.3.4. *Let R be a reduced Rickart ring with the multiplication $\cdot$, the skew meet $\overleftarrow{\wedge}$ and the multiplicative neutral element 1. Then $\langle R, \overleftarrow{\wedge}, \cdot, 1 \rangle$ is a band-enriched monoid.*

Proof. Obviously, $\langle R, \cdot, 1 \rangle$ is a monoid. We also know from Proposition 2.2.13 that $\langle R, \overleftarrow{\wedge} \rangle$ is a band.

The identities required by Definition 7.3.3 follow from Equation (7.12), Proposition 1.3.7 and centrality of idempotents in the following way:

(BEM1) $(a \cdot b) \overleftarrow{\wedge} c = (ab)''c = a''b''c = a''(b \overleftarrow{\wedge} c) = a \overleftarrow{\wedge} (b \overleftarrow{\wedge} c)$,

(BEM2) $a \overleftarrow{\wedge} (b \cdot c) = a''bc = ba''c = b \cdot (a \overleftarrow{\wedge} c)$,

(BEM3) $a \overleftarrow{\wedge} (b \cdot c) = a''bc = (a \overleftarrow{\wedge} b) \cdot c$.

□

Remark 7.3.5. Recall the operation $\overrightarrow{\wedge}$ (Remark 7.3.2). The algebra $\langle R, \overrightarrow{\wedge}, \cdot, 1 \rangle$ is not a band-enriched monoid. However, we could call a band-enriched monoid in the sense of Definition 7.3.3 *right band-enriched monoid* and introduce also a left-dual version – a *left band-enriched monoid* $\langle A, \overrightarrow{\triangle}, \circ, 1 \rangle$. In this terminology, the definition of a left band-enriched monoid is obtained from

Definition 7.3.3 by reversing the order of the band multiplication in the defining properties, i.e., by replacing the properties (BEM1), (BEM2) and (BEM3) by the dual properties $c \overrightarrow{\triangle} (a \circ b) = c \overrightarrow{\triangle} b \overrightarrow{\triangle} a$, $(b \circ c) \overrightarrow{\triangle} a = b \circ (c \vartriangle a)$ and $(b \circ c) \overrightarrow{\triangle} a = (b \vartriangle a) \circ c$, respectively.

However, for the same reason why we do not deal with left skew meets, we shall also not deal with left band-enriched monoids, and therefore there is no need for noting the distinction in the terminology. ◁

Definition 7.3.6. Given a reduced Rickart ring R with the multiplication $\cdot$, the skew meet $\overleftarrow{\wedge}$ and the multiplicative neutral element 1, we call the algebra $\langle R, \overleftarrow{\wedge}, \cdot, 1\rangle$ the *band-enriched monoid reduct of R.*

The following proposition identifies a particularly simple type of band-enriched monoid.

Proposition 7.3.7. *Let A be a set and let $\circ$, $\vartriangle$ and 1 be operations such that $\langle A, \circ, 1\rangle$ is a monoid and $\langle A, \vartriangle\rangle$ is a right zero band. Then $\langle A, \vartriangle, \circ, 1\rangle$ is a band-enriched monoid.*

Proof. To see that the identities required by Definition 7.3.3 hold, let $a, b, c \in A$. Then, since $x \vartriangle y = y$ for all $x, y \in A$,

1. $(a \circ b) \vartriangle c = c = a \vartriangle b \vartriangle c$,

2. $(a \vartriangle (b \circ c) = b \circ c = b \circ (a \vartriangle c)$ and

3. $a \vartriangle (b \circ c) = b \circ c = (a \vartriangle b) \circ c$. □

We will call such a band-enriched monoid a *right-zero-band-enriched monoid*. Similarly, we call a band-enriched monoid whose band reduct is right-normal a *right-normal-band-enriched monoid*.

We conclude this subsubsection with a proposition showing that the m-domains of a reduced Rickart ring are right-zero-band-enriched monoids.

Proposition 7.3.8. *Let R be a reduced Rickart ring whose skew meet operation is denoted by $\overleftarrow{\wedge}$, let E be its lattice of idempotents. For every $e \in E$, let $\langle M_e, \overleftarrow{\wedge}_e, \cdot_e, e\rangle$ be the algebra defined by*

$$a \overleftarrow{\wedge}_e b := b \tag{7.13}$$

and

$$a \cdot_e b := a \cdot b. \tag{7.14}$$

Then $\langle M_e, \overleftarrow{\wedge}_e, \cdot_e, e\rangle$ is a right-zero-band-enriched monoid.

Proof. As we noted in Example 7.1.9, the m-domains M_e are closed with respect to the ring multiplication. Obviously, e is the identity of the semigroup $\langle M_e, \cdot\rangle$, since by Proposition 1.3.5(b) $ae = aa'' = a$, and by centrality of idempotents also $ea = a$. Hence, $\langle M_e, \cdot_e, e\rangle$ is a monoid.

Obviously $\langle M_e, \overleftarrow{\wedge}_e\rangle$ is a right zero band.

Hence, by Proposition 7.3.7, the algebra $\langle M_e, \overleftarrow{\wedge}_e, \cdot_e, e\rangle$ is a right-zero-band-enriched monoid. □

7.3.3 The connection between band-enriched monoids and certain D-monoids

In this subsection, we investigate the close connection between band-enriched monoids and a subclass of D-monoids.

Proposition 7.3.9. *Let $\langle A, \cdot, ^\circ, 1\rangle$ be a D-monoid such that all idempotents in the range of the operation $^\circ$ are central (hence, it is obviously also D-semiadequate). Let $\vartriangle$ be the operation defined by*

$$a \vartriangle b := a^\circ b. \tag{7.15}$$

Then the algebra $\langle A, \vartriangle, \cdot, 1\rangle$ is a band-enriched monoid.

Proof. To see that the operation $\vartriangle$ is associative, we have $(a \vartriangle b) \vartriangle c = (a^\circ b)^\circ c = (a^\circ)^\circ b^\circ c = a^\circ b^\circ c$ by Equation (7.15), Definition B.3.3 and Proposition B.3.2($^\circ$1). We also have $a \vartriangle (b \vartriangle c) = a^\circ(b^\circ c)$ by Equation (7.15), so associativity holds.

Idempotency of the operation $\vartriangle$ is immediate from Proposition B.3.2($^\circ$1) and Equation (7.15). Hence, the algebra $\langle A, \vartriangle\rangle$ is a band. It remains to prove that the conditions in Definition 7.3.3 hold.

Applying Equation (7.15), Definition B.3.3 and again (twice) Equation (7.15), we obtain $(a \cdot b) \vartriangle c = (a \cdot b)^\circ \cdot c = a^\circ b^\circ c = a^\circ(b \vartriangle c) = a \vartriangle (b \vartriangle c) = a \vartriangle b \vartriangle c$. So (BEM1) holds.

By Equation (7.15) the assumption that idempotents in the range of $^\circ$ are central yields that $b \cdot (a \vartriangle c) = ba^\circ c = a^\circ bc = a \vartriangle (b \cdot c)$, so (BEM2) holds, too.

Equation (7.15) also yields $a \vartriangle (b \cdot c) = a^\circ bc = (a \vartriangle b) \cdot c$, proving (BEM3).

Hence, the algebra $\langle A, \vartriangle, \cdot, 1\rangle$ is indeed a band-enriched monoid. □

Definition 7.3.10. Given a D-monoid $\langle A, \cdot, ^\circ, 1\rangle$ such that the range of the operation $^\circ$ contains only central idempotents, we call the band-enriched monoid $\langle A, \vartriangle, \cdot, 1\rangle$ obtained as in Proposition 7.3.9 the *band-enriched monoid induced by* $\langle A, \cdot, ^\circ, 1\rangle$ and denote it by $\mathrm{bem}\langle A, \cdot, ^\circ, 1\rangle$.

Proposition 7.3.11. *Let $\langle A, \vartriangle, \circ, 1\rangle$ be a band-enriched monoid, and let $^\circ$ be the unary operation defined by*

$$a^\circ := a \vartriangle 1. \tag{7.16}$$

Then the algebra $\langle A, \circ, ^\circ, 1\rangle$ is a D-monoid such that the idempotents in the range of the operation $^\circ$ are central.

Proof. By definition of a band-enriched monoid, the algebra $\langle A, \circ, 1\rangle$ is a monoid.

We have to prove that the idempotents a° are central and that the conditions ($^\circ$1),($^\circ$2) and ($^\circ$3) hold.

For the centrality of a°, we use Equation (7.16) and (BEM3) to obtain $a^\circ b = (a \vartriangle 1) \circ b = a \vartriangle (1 \circ b) = a \vartriangle b$. But by (BEM2), we also have $ba^\circ = b \circ (a \vartriangle 1) = a \vartriangle (b \circ 1) = a \vartriangle b$. Hence, indeed $a^\circ b = ba^\circ$ for all $a, b \in A$.

To prove ($^\circ$1), we use Equation (7.16), property (BEM3), the fact that 1 is the neutral element of the monoid $\langle A, \circ, 1\rangle$ and the idempotency of the band operation $\vartriangle$, obtaining that $a^\circ a = (a \vartriangle 1) \circ a = a \vartriangle (1 \circ a) = a \vartriangle a = a$.

For ($^\circ$2), we obtain $(a^\circ)^\circ = (a \vartriangle 1) \vartriangle 1 = a \vartriangle 1 = a^\circ$ by Equation (7.16) and associativity and idempotency of the band operation $\vartriangle$.

For ($^\circ$3), first note that $(ab)^\circ = (a \circ b) \vartriangle 1 = a \vartriangle b \vartriangle 1$ by Equation (7.16) and (BEM1). Moreover, by Equation (7.16), (BEM3) and the assumption that $\langle A, \circ, 1\rangle$ is a monoid, we have $a^\circ b^\circ = (a \vartriangle 1) \circ (b \vartriangle 1) = a \vartriangle (1 \circ (b \vartriangle 1)) = a \vartriangle b \vartriangle 1$. Hence, $(ab)^\circ = a^\circ b^\circ$, so in particular, $a^\circ(ab)^\circ = a^\circ a^\circ b^\circ = a^\circ b^\circ = (ab)^\circ$, and thus also $(ab)^\circ = (ab)^\circ a^\circ$ (by centrality of a°). This proves ($^\circ$3). □

The last paragraph of the proof of Proposition 7.3.11 contains an observation which we will use again later, therefore it needs to be stated as a corollary.

Corollary 7.3.12. *In a D-monoid $\langle A, \cdot, ^\circ, 1\rangle$ which is obtained from a band-enriched monoid by Equation (7.16), for all $a, b \in A$, we have*

$$(ab)^\circ = a^\circ b^\circ. \tag{7.17}$$

Definition 7.3.13. Given a band-enriched monoid $\langle A, \vartriangle, \circ, 1\rangle$ such that the range of the operation $^\circ$ contains only central idempotents, we call the D-monoid $\langle A, \circ, ^\circ, 1\rangle$ obtained as in Proposition 7.3.11 the *D-monoid induced by* $\langle A, \vartriangle, \circ, 1\rangle$ and denote it by $\mathrm{dmon}\langle A, \vartriangle, \circ, 1\rangle$.

Proposition 7.3.14. *For every band-enriched monoid* $\langle A, \vartriangle, \circ, 1\rangle$*, we have* $\mathrm{bem}(\mathrm{dmon}\langle A, \vartriangle, \circ, 1\rangle) = \langle A, \vartriangle, \circ, 1\rangle$. *For every D-monoid* $\langle A, \cdot, ^\circ, 1\rangle$ *such that* ran° *contains only central idempotents, we have* $\mathrm{dmon}(\mathrm{bem}\langle A, \cdot, ^\circ, 1\rangle) = \langle A, \cdot, ^\circ, 1\rangle$.

Proof. Let $\langle A, \vartriangle, \circ, 1\rangle$ be a band-enriched monoid, let $\langle A, \circ, ^\circ, 1\rangle = \mathrm{dmon}\langle A, \vartriangle, \circ, 1\rangle$, and let $\langle A, \vartriangle', \circ, 1\rangle = \mathrm{bem}\langle A, \circ, ^\circ, 1\rangle$. Then by Equation (7.15), Equation (7.16) and Definition 7.3.3(BEM3), we have $x \vartriangle' y = x^\circ \circ y = (x \vartriangle 1) \circ y = x \vartriangle (1 \circ y) = x \vartriangle y$.

Let $\langle A, \cdot, ^\circ, 1\rangle$ be a D-monoid such that ran° contains only central idempotents, let $\langle A, \vartriangle, \cdot, 1\rangle = \mathrm{bem}\langle A, \cdot, ^\circ, 1\rangle$ and let $\langle A, \cdot, ^+, 1\rangle = \mathrm{dmon}\langle A, \vartriangle, \cdot, 1\rangle$. Then $x^+ = x \vartriangle 1 = x^\circ \cdot 1 = x^\circ$. □

Lemma 7.3.15. *Let* $\langle A, \cdot, 1\rangle$ *be a monoid, let* $^\circ$ *be a unary operation on* A *and let* $\vartriangle$ *be a binary operation on* A *such that the operations* $^\circ$ *and* $\vartriangle$ *are connected to each other by Equations* (7.16) *and* (7.15). *Then the following equivalences hold:*

(a) *The algebra* $\langle A, \vartriangle, \cdot, 1\rangle$ *is a band-enriched monoid if and only if the algebra* $\langle A, \cdot, ^\circ, 1\rangle$ *is a D-monoid such that the idempotents in the range of* $^\circ$ *are central.*

(b) *The algebra* $\langle A, \vartriangle, \cdot, 1\rangle$ *is a right-zero-band-enriched monoid if and only if the algebra* $\langle A, \cdot, ^\circ, 1\rangle$ *is a* $^\circ$*-trivial D-monoid.*

Proof. The first part of the proposition is obvious (by Propositions 7.3.9 and 7.3.11).

To prove the second equivalence, first let $\langle A, \vartriangle, \cdot, 1\rangle$ be a right-zero-band-enriched monoid. By the previous item, the algebra $\langle A, \cdot, ^\circ, 1\rangle$ is a D-monoid. By Equation (7.16) and Definition B.2.4, we have $a^\circ = a \vartriangle 1 = 1$ for all $a \in A$.

Conversey, assume that $\langle A, \cdot, ^\circ, 1\rangle$ is a $^\circ$-trivial D-monoid. The only element in the range of the operation $^\circ$ is 1, which is obviously a central idempotent. So the previous item yields that $\langle A, \vartriangle, \cdot, 1\rangle$ is a band-enriched monoid. Moreover, $a \vartriangle b = a^\circ b = b$ for all $a, b \in A$. □

Definition 7.3.16. Given a band-enriched monoid $\langle A, \vartriangle, \cdot, 1\rangle$ and a D-monoid $\langle A, \cdot, ^\circ, 1\rangle$ as in Lemma 7.3.15, we will call the algebra $\langle A, \cdot, ^\circ, \vartriangle, 1\rangle$ a *band-enriched D-monoid.*

Proposition 7.3.17. *Given a reduced Rickart ring* R *with the skew meet* $\overleftarrow{\wedge}$*, the multiplication* $\cdot$ *and the operation* $^\circ$ *from Lemma* 1.1.15*, the algebra* $\langle R, \cdot, ^\circ, \overleftarrow{\wedge}, 1\rangle$ *is a band-enriched D-monoid.*

The following rather trivial proposition shows that, on band-enriched D-monoids, band-enriched monoid homomorphisms and D-monoid homomorphisms are the same thing.

Proposition 7.3.18. *Let* $\langle A, \cdot_A, ^\circ_A, \vartriangle_A, 1_A\rangle$ *and* $\langle B, \cdot_B, ^\circ_B, \vartriangle_B, 1_B\rangle$ *be band-enriched D-monoids, and let* $\phi : A \to B$ *be a function. Then* ϕ *is a band-enriched monoid homomorphism from* $\langle A, \vartriangle_A, \cdot_A, 1_A\rangle$ *to* $\langle B, \vartriangle_B, \cdot_B, 1_B\rangle$ *if and only if* ϕ *is a D-monoid homomorphism from* $\langle A, \cdot_A, ^\circ_A, 1_A\rangle$ *to* $\langle B, \cdot_B, ^\circ_B, 1_B\rangle$.

Proof. "⇒" Assume that ϕ is a band-enriched monoid homomorphism from $\langle A, \vartriangle_A, \cdot_A, 1_A\rangle$ to $\langle B, \vartriangle_B, \cdot_B, 1_B\rangle$. Then $\phi(x^\circ) = \phi(x \vartriangle_A 1_A) = \phi(x) \vartriangle_B \phi(1_A) = \phi(x) \vartriangle_B 1_B = x^\circ_B$ for every $x \in A$, hence ϕ is also a D-monoid homomorphism.

"⇐" Assume that ϕ is a D-monoid homomorphism. Then $\phi(x \vartriangle_A y) = \phi(x^\circ_A \cdot_A y) = \phi(x^\circ_A) \cdot_B \phi(y) = x^\circ_B \cdot_B \phi(y) = \phi(x) \vartriangle_B \phi(y)$ for arbitrary $x, y \in A$. Hence, ϕ is also a band-enriched monoid homomorphism. □

7.4 Strong semilattice decomposition using band-enriched monoids and their inverse systems

In this section we will work with strong semilattices of band-enriched monoids analogously to strong semilattices of semigroups, D-semigroups or D-monoids. In Section 7.4.1, we define them and prove that every strong semilattice of band-enriched monoids is itself a band-enriched monoid. We also investigate the properties of strong semilattices of right-zero-band-enriched monoids. In Section 7.4.2, we obtain a strong semilattice of band-enriched monoids from the m-domains on a reduced Rickart ring, showing that it is possible to decompose the band-enriched monoid reduct of a reduced Rickart ring R into an inverse system of right-zero-band-enriched monoids.

7.4.1 Inverse systems and strong semilattices of band-enriched monoids

Inverse systems of band-enriched monoids

Lemma 7.3.15 immediately yields the following proposition concerning reduced Rickart rings.

Proposition 7.4.1. *Let R be a reduced Rickart ring, let E be its lattice of idempotents and for every pair of idempotents e, f with $e \leq_E f$, let $\phi_e^f : M_f \to M_e$ be the map given by Example 7.1.9 (i.e., $\phi_e^f(x) = xe$). Then ϕ_e^f is a band-enriched monoid homomorphism.*

Since every band-enriched monoid is a D-monoid such that all idempotents the range of the operation $^\circ$ are central and vice versa, and moreover these structures have the same homomorphisms, it is obvious that any inverse system of D-monoids $\langle \mathcal{A}_1^\circ, \mathcal{H} \rangle$ can be seen as an inverse system of band-enriched monoids $\langle \mathcal{A}_1^\vartriangle, \mathcal{H} \rangle$ (and vice versa). This inverse system of band-enriched monoids is obtained by replacing the family of D-monoids $\mathcal{A}_1^\circ$ by the family of band-enriched monoids induced by those D-monoids, i.e., $\mathcal{A}_1^\vartriangle = \{\mathrm{bem}\langle A, \cdot, ^\circ, 1\rangle \mid \langle A, \cdot, ^\circ, 1\rangle \in \mathcal{A}_1^\circ\}$ (and similarly in the opposite direction). We abuse the notation a bit and denote the inverse system of band-enriched monoids obtained in this way from $\langle \mathcal{A}_1^\circ, \mathcal{H} \rangle$ by $\mathrm{bem}\langle \mathcal{A}_1^\circ, \mathcal{H} \rangle$. Similarly, given an inverse system of band-enriched monoids $\langle \mathcal{A}_1^\vartriangle, \mathcal{H} \rangle$, we denote the inverse system of D-monoids obtained from it by $\mathrm{dmon}\langle \mathcal{A}_1^\vartriangle, \mathcal{H} \rangle$. Note that, by Proposition 7.3.14, we have

$$\mathrm{dmon}(\mathrm{bem}\langle \mathcal{A}_1^\circ, \mathcal{H} \rangle) = \langle \mathcal{A}_1^\circ, \mathcal{H} \rangle \tag{7.18}$$

for every inverse system of D-monoids with range of $^\circ$ contained in the set of central idempotents and

$$\mathrm{bem}(\mathrm{dmon}\langle \mathcal{A}_1^\vartriangle, \mathcal{H} \rangle) = \langle \mathcal{A}_1^\vartriangle, \mathcal{H} \rangle \tag{7.19}$$

for every inverse system of band-enriched monoids.

In particular, given a reduced Rickart ring R with the set of idempotents E, let $\mathcal{M}_1^\vartriangle = \{\langle M_e, \overleftarrow{\wedge}_e, \cdot_e, e\rangle \mid e \in E\}$, and let Φ be the family of homomorphisms defined in Equation (7.9). Then $\langle \mathcal{M}_1^\vartriangle, \Phi \rangle$ is an inverse system of band-enriched monoids over E. Moreover,

$$\langle \mathcal{M}_1^\vartriangle, \Phi \rangle = \mathrm{bem}\,\mathrm{sys}_1^\circ R \text{ and } \mathrm{sys}_1^\circ R = \mathrm{dmon}\langle \mathcal{M}_1^\vartriangle, \Phi \rangle. \tag{7.20}$$

Hence, we can state the band-enriched monoid analogue of Definition 7.2.8.

Definition 7.4.2. The inverse system of band-enriched monoids $\langle \mathcal{M}_1^\vartriangle, \Phi \rangle$ will be called the *inverse system of band-enriched monoids of the reduced Rickart ring R* and we will denote it by $\mathrm{sys}_1^\vartriangle R$.

Remark 7.4.3. Recall the Abian order $\leq_{\mathrm{Ab}}$ (Equation (2.32)), which on a reduced Rickart ring coincides with all the other partial orders treated in this thesis. We can characterize the Abian order in terms of the inverse system of a reduced Rickart ring in the following way.

Let R be a reduced Rickart ring and let $\langle \mathcal{M}_1^\triangle, \Phi\rangle = \mathrm{sys}_1^\triangle R$. For elements $a, b \in R$, we have

$$a \leq_{\mathrm{Ab}} b \text{ iff there is some } \phi_e^f \in \Phi \text{ such that } a = \phi_e^f(b). \tag{7.21}$$

To see this, first observe that the homomorphisms ϕ_e^f map every element $x \in M_f$ to some element which is below it under the Abian order: $x \cdot \phi_e^f(x) = xex = xeex = (ex)^2 = (\phi_e^f(x))^2$ by Example 7.1.9, idempotency of e, centrality of idempotents and again Example 7.1.9; and similarly, $\phi_e^f(x) \cdot x = exx = (ex)^2 = (\phi_e^f(x))^2$. Conversely, assume that $a \leq_{\mathrm{Ab}} b$. Obviously, a is in the m-domain $M_{a''}$ and b is in the m-domain $M_{b''}$. Since the Abian order coincides with the weak right star order $\leq *$, we have $a = ba'' = \phi_{a''}^{b''}(b)$ by Definition 2.2.6 and Example 7.1.9. This proves Equation (7.21).

In other words, ϕ_e^f maps each element $x \in M_f$ on an element which lies not only in the m-domain M_e, but also in the initial segment $[0, x]_{\leq_{\mathrm{Ab}}}$. In fact, $\phi_e^f(x)$ is the only element contained in the intersection of the m-domain M_e with the initial segment $[0, x]_{\leq_{\mathrm{Ab}}}$. To see this, assume that $y \in M_e \cap [0, x]_{\leq_{\mathrm{Ab}}}$. Then $y'' = e$, since $y \in M_e$ (see Remark 7.1.6 and Equation (7.21), also $y \leq_{\mathrm{Ab}} x$.

The m-domains themselves are thus antichains under the Abian order: for $x, y \in M_e$, we have $\phi_e^e(x) = y$ if and only if $x = y$, since by Definition B.1.1, ϕ_e^e is the identity homomorphism (alternatively, we could use Remark B.2.7 and the fact that $\langle M_e, \overleftarrow{\wedge}\rangle$ is a right zero band whose natural order is the weak right star order – see Propositions 2.2.13 and 7.3.8).

◁

Strong semilattices of band-enriched monoids

We define a strong semilattice of band-enriched monoids in the obvious way:

Definition 7.4.4. Let S be a lower semilattice with the greatest element $\top$. Let $\mathcal{A}_1^\triangle = \{\langle A_s, \triangle_s, \circ_s, 1_s\rangle | s \in S\}$ be a family of disjoint band-enriched monoids and $\mathcal{H} = \{h_s^t | s, t \in S \text{ and } s \leq t\}$ a family of band-enriched monoid homomorphisms such that $\langle \mathcal{A}_1^\triangle, \mathcal{H}\rangle$ is an inverse system of band-enriched monoids. On the union $A = \bigcup_{s \in S} A_s$, we define an operation $\bullet$ as in Equation (B.1), a constant $\mathbf{1}$ as in item (b) and an operation $\blacktriangle$ as follows: If $x \in A_s$ and $y \in A_t$, then

$$x \blacktriangle y \quad := \quad h_{s\wedge t}^s(x) \triangle_{s \wedge t} h_{s \wedge t}^t(y), \tag{7.22}$$

Then we write $A = \mathfrak{S}_1^\triangle$ and call the algebra $\langle A, \blacktriangle, \bullet, \mathbf{1}\rangle$ a *strong semilattice of band-enriched monoids* and say that it is *induced by the inverse system* $\langle \mathcal{A}_1^\triangle, \mathcal{H}\rangle$.

Obviously, if $\langle A, \blacktriangle, \bullet, \mathbf{1}\rangle$ is a strong semilattice of band-enriched monoids, then both $\langle A, \blacktriangle\rangle$ and $\langle A, \bullet\rangle$ are strong semilattices of semigroups.

Proposition 7.4.5. *1. Every strong semilattice of band-enriched monoids is a band-enriched monoid. Moreover, given an inverse system of band-enriched monoids $\langle \mathcal{A}_1^\triangle, \mathcal{H}\rangle$, we have*

$$\mathrm{dmon}(\mathfrak{S}_1^\triangle \langle \mathcal{A}_1^\triangle, \mathcal{H}\rangle) = \mathfrak{S}_1^\circ(\mathrm{dmon}\langle \mathcal{A}_1^\triangle, \mathcal{H}\rangle) \tag{7.23}$$

2. Every strong semilattice of right-zero-band-enriched monoids is a right-normal-band-enriched monoid whose natural order $\blacktriangleleft$ satisfies

$$x \blacktriangleleft y \text{ if and only if } x \blacktriangle y = x. \tag{7.24}$$

Proof. Let S, $\mathcal{A}_1^{\vartriangle}$ and $\mathcal{H}$ be as in Definition 7.4.4, and let $\blacktriangle$, $\bullet$ and $\mathbf{1}$ be the operations defined on the union $A = \bigcup_{s\in S} A_s$ in Definition 7.4.4.

1. Let $\langle \mathcal{A}_1^{\circ}, \mathcal{H}\rangle = \mathrm{dmon}\langle \mathcal{A}_1^{\vartriangle}, \mathcal{H}\rangle$ (i.e., $\langle \mathcal{A}_1^{\circ}, \mathcal{H}\rangle$ is the inverse system of D-monoids from $\langle \mathcal{A}_1^{\vartriangle}, \mathcal{H}\rangle$ by replacing every band-enriched monoid $\langle A_s, \vartriangle_s, \circ_s, 1_s\rangle$ by its induced D-monoid.). By Lemma 7.3.15, the D-monoids in $\mathcal{A}_1^{\circ}$ have only central idempotents in the ranges of their unary operations.

   By Proposition 7.2.10, the strong semilattice of D-monoids $\mathfrak{S}_1^{\circ}\langle \mathcal{A}_1^{\circ}, \mathcal{H}\rangle$ is a D-monoid. Moreover, the range of its unary operation $^{\bullet}$ (defined by Equation (7.10)) also contains only central idempotents: For arbitrary $s, t \in S$ and $x \in A_s$ and $y \in A_t$, we have

$$\begin{aligned} x^{\bullet} \bullet y = x_s^{\circ} \bullet y = & \\ &= h^s_{s\wedge t}(x_s^{\circ}) \circ_{s\wedge t} h^t_{s\wedge t}(y) \\ &= h^t_{s\wedge t}(y) \circ_{s\wedge t} h^s_{s\wedge t}(x_s^{\circ}) \\ &= y \bullet x_s^{\circ} \\ &= y \bullet x^{\bullet}, \end{aligned} \tag{7.25}$$

   because $h^s_{s\wedge t}(x_s^{\circ}) = h^s_{s\wedge t}(x)^{\circ}_{s\wedge t}$ is a central idempotent in $A_{s\wedge t}$.

   Now we return to the strong semilattice of band-enriched monoids $\mathfrak{S}_1^{\vartriangle}\langle \mathcal{A}_1^{\vartriangle}, \mathcal{H}\rangle$. It turns out that the operations $\blacktriangle$ and $^{\bullet}$ satisfy Equation (7.15): For $x \in A_s$ and $y \in A_t$ (where s and t are arbitrary), we have

$$\begin{aligned} x \blacktriangle y =& h^s_{s\wedge t}(x) \vartriangle_{s\wedge t} h^t_{s\wedge t}(y) \\ =& (h^s_{s\wedge t}(x))^{\circ}_{s\wedge t} \circ_{s\wedge t} h^t_{s\wedge t}(y) \\ =& (h^s_{s\wedge t}(x_s^{\circ})) \circ_{s\wedge t} h^t_{s\wedge t}(y) \\ =& x_s^{\circ} \bullet y \\ =& x^{\bullet} \bullet y. \end{aligned} \tag{7.26}$$

   Hence, by Proposition 7.3.9, $\mathfrak{S}_1^{\vartriangle}\langle \mathcal{A}_1^{\vartriangle}, \mathcal{H}\rangle = \mathrm{bem}(\mathfrak{S}_1^{\circ}\langle \mathcal{A}_1^{\circ}, \mathcal{H}\rangle)$, which proves in particular, $\mathfrak{S}_1^{\vartriangle}\langle \mathcal{A}_1^{\vartriangle}, \mathcal{H}\rangle$ is a band-enriched monoid. For Equation (7.23)

2. Suppose the algebras A_s are right-zero-band-enriched monoids. By the previous item, $\langle A, \blacktriangle, \bullet, \mathbf{1}\rangle$ is a band-enriched monoid. Moreover, by Lemma B.2.5, the band $\langle A, \blacktriangle\rangle$, being a strong semilattice of right zero bands, is right normal.

   Equation (7.24) follows from right normality in the following way. First observe that one direction of the biconditional is trivial: Obviously, if $x \blacktriangleleft y$, then $x \blacktriangle y = x$. For the opposite direction, assume $x \blacktriangle y = x$ for some elements $x, y \in A$. By idempotency and right normality of the operation $\blacktriangle$, this yields $y \blacktriangle x = y \blacktriangle x \blacktriangle x = x \blacktriangle y \blacktriangle x = x \blacktriangle x = x$. ☐

7.4.2 Decomposition of the band-enriched monoid reduct of a reduced Rickart ring

In this subsubsection, we investigate the inverse system $\mathrm{sys}_1^{\vartriangle} R$ of band-enriched monoids on a reduced Rickart ring, obtaining that the strong semilattice which it induces is precisely the band-enriched monoid $\langle R, \overleftarrow{\wedge}, \cdot, 1\rangle R$.

Theorem 7.4.6. (a) *Let R be an enriched reduced Rickart ring. Then $\langle R, \overleftarrow{\wedge}, \cdot, 1\rangle = \mathfrak{S}_1^{\vartriangle}(\mathrm{sys}_1^{\vartriangle} R)$.*

(b) *Let* $\langle \mathcal{A}_1^\vartriangle, \mathcal{H}\rangle$ *be an inverse system of pairwise disjoint band-enriched monoids over some lower semilattice. Suppose* $\mathfrak{S}_1^\vartriangle\langle \mathcal{A}_1^\vartriangle, \mathcal{H}\rangle = \langle R, \overleftarrow{\wedge}, \cdot, 1\rangle$ *for some enriched reduced Rickart ring* R*. Then* $\langle \mathcal{A}_1^\vartriangle, \mathcal{H}\rangle = \mathrm{sys}_1^\vartriangle R$.

Proof. (a) Let $\langle \mathcal{M}_1^\vartriangle, \Phi\rangle = \mathrm{sys}_1^\vartriangle R$ be the inverse system of a reduced Rickart ring R and let $\langle R, \overleftarrow{\blacktriangle}, \bullet, \mathbf{1}\rangle$ be the strong semilattice $\mathfrak{S}_1^\vartriangle\langle M_e, \phi_e^f\rangle$ induced by it.

We need to prove that $\langle R, \overleftarrow{\wedge}, \cdot, 1\rangle = \langle R, \overleftarrow{\blacktriangle}, \bullet, \mathbf{1}\rangle$. By Definition 7.4.4, for $x \in M_e$ and $y \in M_f$, we have

$$x \overleftarrow{\blacktriangle} y \quad := \quad \phi^e_{e\wedge_E f}(x) \overleftarrow{\wedge}_{e\wedge_E f} \phi^f_{e\wedge_E f}(y), \tag{7.27}$$

$$x \bullet y \quad := \quad \phi^e_{e\wedge_E f}(x) \cdot_{e\wedge_E f} \phi^f_{e\wedge_E f}(y), \tag{7.28}$$

and $\mathbf{1}$ is the identity element with respect to the operation $\cdot_\top$ of the band-enriched monoid $\langle N_\top, \overleftarrow{\wedge}_\top, \cdot_\top, \top\rangle$ corresponding to the top element $\top$ of the lattice E. Since $\top = 1$, obviously $\mathbf{1} = 1$.

To prove that the binary operations $\overleftarrow{\blacktriangle}$ and $\bullet$ coincide with $\overleftarrow{\wedge}$ and $\cdot$, let $x \in M_e$ and $y \in M_f$ for some idempotents $e, f \in E$. Then

$$\begin{aligned} x \overleftarrow{\blacktriangle} y &= \phi^e_{e\wedge_E f}(x) \overleftarrow{\wedge}_{e\wedge_E f} \phi^f_{e\wedge_E f}(y) \\ &= \phi^f_{e\wedge_E f}(y) \text{ (by Equation (7.13))} \\ &= y \cdot (e \wedge_E f) \\ &= y \cdot e \cdot f \\ &= ye \\ &= yx'' \\ &= x \overleftarrow{\wedge} y \end{aligned}$$

and

$$\begin{aligned} x \bullet y &= \phi^e_{e\wedge_E f}(x) \cdot_{e\wedge_E f} \phi^f_{e\wedge_E f}(y) \\ &= \phi^e_{e\wedge_E f}(x) \cdot \phi^f_{e\wedge_E f}(y) \text{ (by Equation (7.14)} \\ &= (x \cdot e \cdot f) \cdot (y \cdot e \cdot f) \\ &= x \cdot e \cdot y \cdot f \\ &= x \cdot x'' \cdot y \cdot y'' \\ &= x \cdot y \end{aligned}$$

(b) Let $\langle \mathcal{A}_1^\vartriangle, \mathcal{H}\rangle$ be an inverse system of band-enriched monoids such that

$$\mathfrak{S}_1^\vartriangle\langle \mathcal{A}_1^\vartriangle, \mathcal{H}\rangle = \langle R, \overleftarrow{\wedge}, \cdot, 1\rangle \tag{7.29}$$

So we have $\mathrm{dmon}(\mathfrak{S}_1^\vartriangle\langle \mathcal{A}_1^\vartriangle, \mathcal{H}\rangle) = \mathrm{dmon}\langle R, \overleftarrow{\wedge}, \cdot, 1\rangle = \langle R, \cdot, ^\circ, 1\rangle$ by Proposition 7.3.17. But by Equation (7.23), this can be rewritten as $\mathfrak{S}_1^\vartriangle(\mathrm{dmon}\langle \mathcal{A}_1^\vartriangle, \mathcal{H}\rangle) = \langle R, \cdot, ^\circ, 1\rangle$. Now we can apply Corollary 7.2.12, obtaining that $\mathrm{dmon}\langle \mathcal{A}_1^\vartriangle, \mathcal{H}\rangle = \mathrm{sys}_1^\circ R$. Finally, Equation (7.19), Definition 7.4.2 and Equation (7.20) yield that $\langle \mathcal{A}_1^\vartriangle, \mathcal{H}\rangle = \mathrm{bem}(\mathrm{dmon}\langle \mathcal{A}_1^\vartriangle, \mathcal{H}\rangle) = \mathrm{bem}(\mathrm{sys}_1^\circ R) = \mathrm{sys}_1^\vartriangle R$. □

Conclusion

To sum up the overall contribution of the thesis, let us outline the results achieved by each of the chapters in Part II and recap how these findings advanced the previous research on the respective topics.

Chapter 3 The main result of this chapter is the generalization of the strong right star order to right-strong Rickart rings. This is an improvement from Marovt et al. [2015], where the strong right star order was introduced to Rickart *-rings. We provide star-free versions of [Marovt et al., 2015, Definition 11, Theorem 9(iii) and Theorem 10] in Definition 3.1.1, Lemma 3.1.3 and Theorem 3.1.7. The relation $\preceq *$ can be defined even on a right-strong right Rickart ring, but only on a right-strong Rickart ring, it is indeed a partial order. It is nice to know, although not very surprising, that the strong right star order on a right-strong Rickart ring is stronger than the weak right star order on the same ring (Lemma 3.1.4), just as it is the case in the ring of bounded linear operators on a Hilbert space (as observed in [Cīrulis, 2014a, page 2]) (this results also comes handy in Chapters 4 and 5).

The alternative characterization of the weak right star order given in Proposition 3.2.3 is interesting because it uses right idempotents in a way similar as to how the characterization of the strong right star order from Lemma 3.1.3 uses left idempotents, thereby pointing out what the two right star orders have in common and what separates them.

The introduction of different one-sided versions of the space order which correspond to one-sided version of the star order provides one-sided versions of the observation from [Cīrulis, 2016, page 3] relating the star order to the weak space preorder. This discussion also gives rise to possibly a third version of the right star order. However, it remains an open question for further research whether this is actually a new partial order or just an equivalent formulation of one of the known right star orders.

Chapter 4 Concerning lattice operations for partial orders on strong Rickart rings, right-strong Rickart rings and right-strong right Rickart rings, a collection of small results was obtained among which we can not single out any as the "main" one.

For the diamond order on a strong Rickart ring, we found a sufficient condition for the existence of the join of two elements in an initial segment, involving also an explicit description of this join (Theorem 4.1.3). It is not known whether or under what conditions this is also a global join – this remains a question for further research. This result continues the research carried out in Cīrulis [2017], where the diamond order was first introduced to strong Rickart rings and the corresponding order structure was investigated. In that article, there were several results on meets, but the only result on joins is for elements of a set which is an initial segment under the star order ([Cīrulis, 2017, Proposition 5.6]). Theorem 4.1.3 is rather in the spirit of the explicit formula for the meet of elements satisfying certain conditions stated in [Cīrulis, 2017, Lemma 5.2].

For the strong right star order on a right-strong Rickart ring, we used its close connection to the weak right star order to find some conditions concerning the existence of meets and joins in initial segments (Theorems 4.2.2 and 4.2.3). We proved that, for elements a, b of an initial segment under the strong right star order, if the strong right star join $a \curlyvee\!* \, b$ exists, then it is equal to the weak right star join $a \vee\!* \, b$. Moreover, if the weak right star meet $a \wedge\!* \, b$ is contained in the initial segment, then the strong right star meet exists and equals $a \wedge\!* \, b$. The results achieved are weaker than those known for the weak right star order (for which it was shown in [Cīrulis, 2015d, Theorem 4.3, Equations (4.2) and (4.3)] that any two elements bounded from above have both the meet and the join). However, the results presented in this thesis establish conditions under which the meets and joins under both right star orders are equal, which is also interesting.

Concerning the various notions of coherence introduced on Rickart *-rings in Djikić and Djordjević [2016] (right precoherence and right Djikić coherence, called just precoherence and coherence in that article) and Cīrulis [2015a] (coherence), we provided a star-free version of Djikić coherence and show that, on a strong Rickart ring, right Djikić coherence implies right precoherence just as it does in Rickart *-rings (see [Djikić and Djordjević, 2016, Lemma 3.1]).

For the weak right star order on a right-strong right Rickart ring, we found that right precoherence is a sufficient condition for two elements to have the meet (Proposition 4.4.1). This can be seen as an improvement from [Cīrulis, 2015d, Equation (4.2)], where the meet was given for elements which are bounded from above. Alternatively, we can say that it helps to characterize the notion of right precoherence, because Proposition 4.4.1 says that elements are right precoherent if and only if their meet exists and satisfies a certain condition. Lemma 4.4.4 is a similar characterization of right coherence.

Theorem 4.4.5 concerning both meets and joins of elements bounded from above under the weak right star order is an improvement from [Cīrulis, 2015d, Lemma 4.6]. Both results deal with elements a, b bounded from above, but Theorem 4.4.5 does not require any further conditions, while [Cīrulis, 2015d, Lemma 4.6] has the additional assumption that the elements *ϕ-commute* (that is, that $a''b'' = b''a''$). We see from Theorem 4.4.5 that the condition that a and b ϕ-commute actually follows from items (a), (b) and (c) of Lemma 4.6 in Cīrulis [2015d], and only the last item, (d), requires the assumption of ϕ-commuting. Moreover, Theorem 4.4.5 shows that those first three items of Lemma 4.6 in Cīrulis [2015d] are also equivalent to right precoherence of the elements, while [Cīrulis, 2015d, Lemma 4.6] does not mention right precoherence at all.

For the star order, Theorem 4.5.3 is both an improvement and a star-free version of [Cīrulis, 2015a, Theorem 4.5].

Chapter 5 One of the main results of this thesis is the description of the order structure of a right-strong Rickart ring R ordered by the strong right star order $\preceq\!*$. Namely, we proved that $\langle R, \preceq\!*\rangle$ is a relatively orthocomplemented poset (Theorem 5.1.7). As a corollary, we obtained that $\langle R, \preceq\!*\rangle$ is a quasi-orthomodular poset (Corollary 5.2.3), since quasi-orthomodular posets and relatively orthocomplemented posets are equivalent. We also described the corresponding orthogonality relation in Theorem 5.2.1.

So we have obtained a star-free version of previous research by the author which was limited to Rickart *-rings – it was shown in [Krēmere, 2016, Theorem 5.4] that a Rickart *-ring with the strong right star order is quasi-orthomodular. The strong right star orthogonality characterized in Theorem 5.2.1 is the star-free (and right dual) version of the orthogonality relation from [Krēmere, 2016, Definition 6].

Contrary to the orthogonality relation, the isomorphisms of initial segments described in Theorems 5.3.1 and 5.3.7 were not investigated at all on Rickart *-rings. They are, however, inspired by similar results on other partial orders.

All the results from Chapter 5 expand related research on structure of a star ordered strong Rickart ring, as well as a right-strong right Rickart ring equipped with the weak right star order. Compare Theorem 5.1.7 and Corollary 5.2.3, respectively, with [Cīrulis, 2015d, Theorem 5.1] (the corresponding result for the weak right star order) and [Cīrulis, 2016, page 7] (the corresponding result for the star order). Similarly, the isomorphism theorems are strong right star versions of [Cīrulis, 2015d, Corollary 5.2] and [Cīrulis, 2016, Theorem 4.5].

Theorem 5.1.7 also provides another sufficient condition for equality of joins under the right star orders: If elements are strong right star orthogonal, then their strong right star join equals their weak right star join.

Chapter 6 In this chapter, which contains another main result of the thesis, we obtained more general results which then were applied to some partial orders on rings, including on Rickart rings. Therefore, this part of the research sometimes left the realm of the theory of partial orders on Rickart rings, sneaking instead into the field of quantum structures. This had the advantage of providing some insight into properties that are shared by several of the partial orders studied in this thesis.

The quantum structure part of the research led to the generalization of the notion of difference from [Kôpka and Chovanec, 1994, Definition 1] by introducing weak differences in our Definition 6.1.1. We find results connecting weak differences to the weak BCK-algebras introduced in [Cīrulis, 2010b, page 728] in a similar way as difference were connected to BCK-algebras in Dvurečenskij and Kim [1998]: Theorem 6.1.2 ressembles [Dvurečenskij and Kim, 1998, Theorem 5.5], and Theorem 6.1.5 is the weak-BCK-version of [Dvurečenskij and Kim, 1998, Theorem 4.1].

In Theorem 6.2.1, a key finding of the chapter, we provide necessary and sufficient conditions for a poset with a certain weak difference to be a meet semilattice. By virtue of its generality, this result is quite fruitful, yielding semilattice conditions for the weak right star order, the strong right star order and the sharp order. These conditions are analogous to the corresponding semilattice condition for the star order given in [Cīrulis, 2015a, Lemma 5.4 and Theorem 5.5] (which could alternatively be obtained from Theorem 6.2.1, too). It is possible that there are other partial orders on rings, Abelian groups or other structures for which Theorem 6.2.1 might be useful – this is an opportunity for further research.

Chapter 7 In this chapter, for the third main result of this thesis, we used a semigroup theoretic approach to describe the structure of a particular subclass of strong Rickart rings – reduced Rickart rings. We studied both the D-monoid reduct and the band-enriched monoid reduct of a reduced Rickart ring, as well as the D-semigroup reduct of an m-domain ring.

This builds on a similar result on right PP monoids, see [Fountain, 1976, Theorem 1]. Contrary to that article, we obtained results also in a non-unitary setting (m-domain rings) and we included the additional operations – the unary operation $^\circ$ of an m-domain ring (Theorem 7.1.10) and the skew meet $\overleftarrow{\wedge}$ on a reduced Rickart ring (Theorem 7.4.6) – into the strong semilattice construction.

We also elaborated on the connection between the skew meet and the unary operation $^\circ$ on a reduced Rickart ring in a more abstract setting by investigating the relationship between D-monoids and band-enriched monoids in the results in Section 7.3 (for example Lemma 7.3.15). This connects the field of D-semigroups Stokes [2015] with the research from Cīrulis [2015d], where the skew meet operation was first defined.

The connection to the general spirit of this thesis, which mostly focussed on partial orders on Rickart rings, is maybe not completely obvious. It lies in the origin of the skew meet. The general abstract version of this operation emerged as the binary operation of a skew nearlattice in Cīrulis [2010a], i.e., it arose as a kind of non-commutative meet. In [Cīrulis, 2015d, Section 4], a concrete instance of this general order theoretic concept was introduced to a right-strong right Rickart ring – this is the operation $\overleftarrow{\wedge}$ which we encountered in this thesis. It was noted in that paper that the partial order for which it serves as a non-commutative analogue of the meet is the weak right star order.

On a reduced Rickart ring, the weak right star order coincides with all the other partial orders studied in this thesis (the star order, the strong right star order, the diamond order, the sharp order and the Abian order), and therefore, the skew meet on a reduced Rickart ring can be seen as a non-commutative analogue of a meet for all these partial orders. In particular, the natural order of the band $\langle R, \overleftarrow{\wedge} \rangle$ coincides with all of these orders. So the strong semilattice of band-enriched monoids from Theorem 7.4.6 decomposes a band whose natural order is any of the partial orders studied in this thesis.

Unfortunately, the results are not obtained for general strong Rickart rings. The only generalization is the one to m-domain rings, which are not Rickart rings in general. It could be a direction for further research to extend the results on strong semilattice decompositions to suitable reducts of wider classes of Rickart rings.

Appendix

Appendix A

Posets with certain additional structures

In this appendix chapter, we review some order theoretic preliminaries. The description of the order theoretic properties of Rickart rings (ordered by the partial orders defined in Chapter 2) was a key objective of the research and thus forms most of the content of Part II of the thesis.

The so-called *orthomodular lattices* were introduced in the middle of the 20th century. The axiom which is now known as the orthomodular law (see Definition A.1.3) was first mentioned in [Husimi, 1937], but not in a very prominent way and only in the quantum theoretic context of what was called "propositions" there, i.e., orthogonal projections on a Hilbert space. The first article to study specifically the orthomodular law on a lattice was [Loomis, 1955]. The axiom was however called "the weak modular identity" in that paper – according to [Kalmbach, 1983, page 12], the term "orthomodular" is due to Kaplansky.

An important feature of orthomodular lattices is that they describe the order structure of the set of closed subspaces of a Hilbert space ordered by inclusion (or equivalently, the set of orthogonal projections on a Hilbert space ordered by a partial order that we will encounter later in this section). Since the ring of bounded linear operators on a Hilbert space is the standard example of a Rickart ring, it should thus be clear that orthomodularity is a very relevant concept when studying partial orders on Rickart rings. In this section, we therefore recall the notions of orthomodular posets and lattices and different generalizations of these notions.

First, we provide basic information on orthocomplemented and orthomodular posets in Section A.1, including examples from the rings treated in Chapter 1. Then we continue by recalling the notion of orthogonality relation on a poset in Section A.3. After that, we collect a few generalizations of orthocomplemented posets in Section A.4. In Section A.5, we turn our attention to posets equipped with a particular partial operation called difference. Finally, BCK-algebras and their connection to posets with difference, as well as their generalization weak BCK-algebras, are addressed in Section A.6.

A.1 Orthomodular posets

In this subsection, we recall orthomodular posets and orthomodular lattices. The standard literature on this topic is Kalmbach's book [Kalmbach, 1983].

A.1.1 Orthocomplementations

On a lattice $\langle A, \leq \rangle$ with a bottom element 0 and a top element 1, a *complementation* is a unary operation $^\perp$ with the property that $a \wedge a^\perp = 0$ and $a \vee a^\perp = 1$ for all $a \in A$. We shall, however, use this term with a broader meaning – in Section A.4, we define so-called *dual Galois, DeMorgan* and *dual Brower* complementations, which are not complementations in the narrow sense just mentioned. The notion which the following definition recalls is an actual complementation in the narrow sense, though.

Definition A.1.1. [Kalmbach, 1983] A unary operation $^\perp$ on a poset $\langle A, \leq \rangle$ with both a top element 1 and a bottom element 0 is said to be an *orthocomplementation*, if, for all $a, b \in A$,

($^\perp$1) $(a^\perp)^\perp = a$,

($^\perp$2) if $a \leq b$, then $b^\perp \leq a^\perp$,

($^\perp$3) $0^\perp = 1$,

($^\perp$4) $0 = a \wedge a^\perp$ (equivalently, if $1 = a \vee a^\perp$ for all $a \in A$).

A poset equipped with an orthocomplementation is called *orthocomplemented poset* or *orthoposet*.

Definition A.1.2. On an orthocomplemented poset, the relation $\perp$ defined by

$$a \perp b \text{ if and only if } a \leq b^\perp \tag{A.1}$$

(or, equivalently, $a \perp b$ if and only if $b \leq a^\perp$) is called the *orthogonality relation induced by the orthocomplementation*.

A.1.2 Orthomodular posets

Definition A.1.3. [Kalmbach, 1983, page 27] An *orthomodular poset* is an orthocomplemented poset A satisfying

(OM1) if $a \perp b$, then $a \vee b$ exists in A,

(OM2) if $a \leq b$, then $b = a \vee (a^\perp \wedge b)$ (the *orthomodular law*).

An *orthomodular lattice* is just a lattice ordered orthomodular poset. Note that item (OM1) holds for every orthocomplemented lattice, so an orthomodular lattice is an orthocomplemented lattice satisfying the orthomodular law.

The orthomodular law is a weaker form of the modular law

$$\text{if } a \leq b, \text{ then } (a \vee x) \wedge b = a \vee (x \wedge b). \tag{A.2}$$

Hence, every orthocomplemented modular lattice is orthomodular (but not vice versa).

A.2 Examples from the world of Rickart rings

The following examples are meant to illustrate not only the definitions above, but also their close connection to certain subsets of idempotents in Rickart rings.

A.2.1 Boolean rings

As we saw, a Boolean ring is a very simple example of a Rickart ring. In the same way, a Boolean lattice is a very simple example of an orthomodular lattice.

Example A.2.1. Let $\langle A, \leq \rangle$ be a Boolean lattice. The Boolean complementation is an orthocomplementation. Since every Boolean lattice is modular, the modular law is satisfied, and thus also the orthomodular law. ◁

Since on every Boolean lattice, we can define operations that turn it into a Boolean ring ($ab := a \wedge b$ and $a + b := (a \vee b) \wedge (a \wedge b)^{\perp}$) and vice versa ($a \wedge b := ab$, $a \vee b := a + b + ab$ and $a^{\perp} := 1 - a$), we can easily reformulate Example A.2.1 in terms of Boolean rings. In particular, the orthocomplementation on a Boolean ring is defined by $a^{\perp} := 1 - a$.
This can be partly extended to the set of idempotents of an arbitrary unital ring.

A.2.2 Idempotents in a ring

Let us first recall the standard partial order on the set of idempotents of a ring or semigroup.

Definition A.2.2. By the *standard order of idempotents* on a semigroup or ring we mean the partial order $\leq_E$ defined on the set of idempotents of a semigroup (or ring) by

$$e \leq_E f \text{ if and only if } ef = e = fe. \tag{A.3}$$

We denote the meet and join of idempotents e and f under this order by $e \wedge_E f$ and $e \vee_E f$, respectively.

Example A.2.3. Let R be a unital ring and let E be its set of idempotents ordered by the standard order of idempotents. Then the additive identity 0 is the bottom element and the multiplicative identity 1 is the top element of E and

$$e^{\perp} = 1 - e \tag{A.4}$$

defines an orthocomplementation on E which turns E into an orthomodular poset. ◁

In general, the orthomodular poset of idempotent elements of a unital ring is not a lattice. However, under certain conditions meets and joins are known to exist.

Proposition A.2.4. *Let e, f be idempotents in a ring R. If $ef = fe$, then e and f have the meet and the join under the usual order of idempotents, and $e \wedge_E f = ef$ and $e \vee_E f = e + f - ef$.*

In the set of central idempotents of a semigroup, Equation (A.3) reduces to $e \leq_E f$ iff $ef = e$. This order turns the set of central idempotents of a semigroup into a lower semilattice with

$$e \wedge_C f = ef. \tag{A.5}$$

Recall that in a reduced ring R, all idempotents are central (see Proposition 1.1.13(c)). Hence it is obvious that Equation (A.5) holds on the set P of all idempotents in R. Moreover, it can be shown that not only meets but also joins exist for any pair of idempotents in P. Since by Example A.2.3 the poset $\langle P, \leq_E \rangle$ is an orthomodular poset, this means that $\langle P, ^{\perp}, \wedge_E, \vee_E \rangle$ is even a Boolean algebra whose lattice operations are given by

$$e \wedge_E f = ef \text{ and } e \vee_E f = e + f - ef. \tag{A.6}$$

A.2.3 Closed subspaces of a Hilbert space

The example which motivated the introduction of orthomodular lattices in the middle of the 20th century was the following. It can be found, for example, in [Kalmbach, 1983, Chapter 1§5, Proposition 1 (page 65)].

Example A.2.5. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{S}(\mathcal{H})$ be its set of closed subspaces. The set $\mathcal{S}(\mathcal{H})$ is ordered by inclusion, and it is a lattice (for closed subspaces U and V of $\mathcal{H}$, $U \wedge V = U \cap V$ and $U \vee V$ is the smallest closed subspace containing both U and V). By defining $U^{\perp}$ to be the orthogonal complement of the subspace U (which is always closed), we obtain an orthocomplementation on the lattice $\mathcal{S}(\mathcal{H})$. It can be proved that this orthocomplemented lattice also satisfies the orthomodular law, thus it is an orthomodular lattice. Moreover, it is even complete. ◁

Since in a Hilbert space, there is a one-to-one correspondence between closed subspaces and orthogonal projections, Example A.2.5 provides a structure of orthomodular lattice for the set of orthogonal projections on a Hilbert space. Recall that the orthogonal projections are the elements of $\mathcal{B}(\mathcal{H})$ which are projections in the Rickart *-ring sense (Definition 1.2.3). Or, if we prefer star-free notions, we can also characterize the orthogonal projections as the closed idempotents of the right-strong right Rickart ring $\mathcal{B}(\mathcal{H})$. Since Rickart *-rings were inspired by the ring $\mathcal{B}(\mathcal{H})$, this raises the question whether the set of projections is also an orthomodular lattice, and similarly for the set of closed idempotents of a right-strong right Rickart ring.

A.2.4 Projections and closed idempotents

Example A.2.3 shows that also the set of closed idempotents P_{r} of a right-strong right Rickart ring is orthocomplemented, since $0, 1 \in P_{\mathrm{r}}$ and also $p^{\perp} = p' \in P_{\mathrm{r}}$ for every $p \in P_{\mathrm{r}}$.

Remark A.2.6. In a Rickart *-ring, for projections p and q, if $pq = p$, then $p = p^* = (pq)^* = q^*p^* = qp$, which simplifise Equation (A.3). It was noted in [Cīrulis, 2015d] that also in the set closed idempotents of a right-strong right Rickart ring (thus also in the set P_{rl} on a strong Rickart ring), Equation (A.3) can be simplified to

$$p \leq_E q \text{ if and only if } pq = p \tag{A.7}$$

(or, equivalently, $p \leq_E q$ iff $qp = p$). ◁

The set of closed idempotents on a right-strong right Rickart ring was recently proved to be an orthomodular lattice, extending the analogous results for Rickart *-rings (see, for example, [Berberian, 1972, Chapter 1, §3, Proposition 7, page 14] and [Janowitz, 1983]) to a star-free setting.

Lemma A.2.7. [Cīrulis, 2015d, Proposition 3.4] *and* [Cīrulis, 2016, Proposition 3.1] *The set of closed idempotents P_{r} on a right-strong right Rickart ring (alternatively, the set of closed idempotents P_{rl} on a strong Rickart ring) is an orthomodular lattice under the standard order of idempotents $\leq_E$. The orthocomplementation is the right focal operation, and the lattice operations are given by*

$$p \wedge_P q = (p'q)'q = q - (p'q)'' \text{ and } p \vee_P q = ((pq')'p')' = q + (pq')''. \tag{A.8}$$

The following proposition shows that the intersection of the set of closed idempotents with an annihilator of an arbitrary element is a sublattice of the lattice P_{rl}.

Proposition A.2.8. [Cīrulis, 2015d, Proposition 3.5(a)] [Cīrulis, 2016, Proposition 3.2(g)] *Let P_{r} be set of closed idempotents on a right-strong right Rickart ring R (alternatively, the set of closed idempotents on a strong Rickart ring R). For every $a \in R$, the subsets $\{p \in P_{\mathrm{r}} \mid pa = 0\}$ and $\{p \in P_{\mathrm{r}} \mid ap = 0\}$ are sublattices of P_{r}.*

The orthogonality relation induced by the orthocomplementation on the set P_{r} is given by

$$p \perp q \text{ iff } pq = 0 \text{ iff } qp = 0. \tag{A.9}$$

A.3 Posets with an orthogonality relation

In this subsection, we recall some generalizations of orthomodular posets and orthomodular lattices which are based on the properties of the induced orthogonality of an orthoposet.

According to the standard book on orthomodular lattices [Kalmbach, 1983, page 257], there is no agreement on the precise meaning of the word "orthogonality". It always denotes a symmetric binary relation with some additional properties, though. Following Cīrulis [Cīrulis, 2014b], we adopt the following definition of orthogonality on a poset.

Definition A.3.1. [Cīrulis, 2014b] Let $\langle A, \leq \rangle$ be a poset with a bottom element 0. A binary relation $\perp$ is called an *orthogonality* if the following conditions hold for all $a, b, c \in A$:

($\perp$1) if $a \perp b$, then $b \perp a$,

($\perp$2) if $a \leq b$ and $b \perp c$, then $a \perp c$,

($\perp$3) $0 \perp a$.

A poset with orthogonality is said to have *orthojoins* if for all $a, b \in A$

($\perp$4) if $a \perp b$, then $a \vee b$ exists.

A *quasi-orthomodular poset* is a poset with orthogonality that has orthojoins and, for all $a, b, c \in A$, satisfies

($\perp$5) if $a \leq b$, then there exists $z \in A$ such that $b = a \vee z$ and $a \perp z$,

($\perp$6) if $a \perp b$, $a \perp c$ and $b \leq a \vee c$, then $b \leq c$.

The relation defined by Equation (A.1) on an orthoposet is an orthogonality in the sense of Definition A.3.1, which justifies calling it the induced orthogonality.

A.4 Generalizations of orthocomplemented posets

In this subsection, we present generalizations of the notion of orthocomplementation. One way of generalizing it is by weakening some of the conditions item ($^\perp$1), item ($^\perp$2), item ($^\perp$3) and item ($^\perp$4). Another approach, which yields a generalization that makes sense even in posets which do not necessarily have a top element, is to require not one "global" orthocomplementation, but a bunch of local (*sectional*) orthocomplementations, which are defined on initial segments of the poset. Of course, both ways can be combined.

A.4.1 Weaker complementations

Definition A.4.1. A unary operation $^\perp$ on a poset $\langle A, \leq \rangle$ with both a top element 1 and a bottom element 0 is said to be

(a) a *dual Galois complementation*, or shorter, a *g*-complementation* (see [Cīrulis, 2010b]), if it satisfies item ($^\perp$2), item ($^\perp$3) and, for all $a, b \in P$,

(g1) $a^{\perp\perp} \leq a$,

(b) a *DeMorgan complementation*, or shorter, an *m-complementation* (see [Cīrulis, 2010b]), if it satisfies item ($^\perp$1), item ($^\perp$2) and item ($^\perp$3).

Just as in the case of orthocomplementations, we call the binary relation defined by Equation (A.1) ($a \perp b$ if and only if $a \leq b^\perp$) on an m-complemented poset the *induced orthogonality* of the m-complemented poset. The induced orthogonality of an m-complemented (or orthocomplemented) poset is an orthogonality in the sense of Definition A.3.1.

A.4.2 Sectional complementations

On a poset $\langle A, \leq \rangle$ which has a bottom element 0 we call the down-set $[0, x]_{\leq} := \{a \in A \mid a \leq x\}$ the *initial segment below* x for every $x \in A$.

The following definition generalizes the notion of orthocomplementation (or m-, or g*-) to posets for which only the bottom element is assumed to exist.

Definition A.4.2. [Cīrulis, 2015d] A poset with a bottom element $\langle A, \leq, 0 \rangle$ is said to be *sectionally orthocomplemented* if for every $x \in A$, there exists an orthocomplementation $\frac{\perp}{x}$ on the initial segment $[0, x]_{\leq} = \{a \in A \mid a \leq x\}$.

Sectionally m- and g-complemented* posets are defined analogously (see [Cīrulis, 2010b]).

Remark A.4.3. This yields a relation $\perp_x$ for every initial segment $[0, x]_{\leq}$ of an m- (ortho-) complemented poset, which in the case of a sectionally orthocomplemented poset is an orthogonality relation on $[0, x]_{\leq}$. It was noted in [Cīrulis, 2015d] that, in the case of a sectionally orthocomplemented poset, the union of these local orthogonatities also is an orthogonality, called the *induced orthogonality on A*. ◁

A.4.3 Relatively orthocomplemented posets

When dealing with sectionally orthocomplemented posets, it is convenient if the sectional complementations are in some way compatible with each other.

Definition A.4.4. [Cīrulis, 2015d] A sectionally orthocomplemented poset $\langle A, \leq, 0 \rangle$ with sectional complementations denoted by $\frac{\perp}{x}$ on every initial segment $[0, x]_{\leq}$ is said to be *relatively orthocomplemented* if for all $a, b, x, y \in A$,

(ro1) if $a \leq b^{\perp}_{x}$, then the join $a \vee b$ in A exists (this property is called having *orthojoins*),

(ro2) if $a \leq x \leq y$, then $a^{\perp}_{x} \leq a^{\perp}_{y}$.

Note that the property item (ro1) is a weaker version of the following notion, which is widely used in the literature, for example, in [Cornish and Noor, 1982] and [Cīrulis, 2010b].

Definition A.4.5. A poset $\langle A, \leq \rangle$ is said to have the *upper bound property* (or *least upper bound property*) if every pair of elements $a, b \in A$ having a common upper bound also has the join. Such a poset is also called *nearsemilattice* (a term introduced in [Cīrulis, 1998]).

Hence, on a sectionally orthocomplemented poset which has the upper bound property, the property item (ro2) is sufficient for the poset to be relatively orthocomplemented.

Remark A.4.6. It is know that, in a relatively orthocomplemented poset, every initial segment is an orthomodular poset and, if $a \leq x \leq y$, then

$$a^{\perp}_{x} = a^{\perp}_{y} \wedge x \tag{A.10}$$

(see [Cīrulis, 2015d] for more details). ◁

Another convenient property of a relatively orthocomplemented poset $\langle A, \leq \rangle$ is the fact that, if $a \perp_x b$ in some initial segment, then $a \perp_y b$ in every initial segment $[0, y]_{\leq}$ with $a, b \in [0, y]_{\leq}$. Thus, the relative orthocomplementations induce a binary relation $\perp$ by

$$a \perp b \text{ if and only if } a \perp_x b \text{ for some } x \in A. \tag{A.11}$$

This relation is called the *orthogonality of the relatively orthocomplemented poset*, because it is an orthogonality on the poset $\langle A, \leq \rangle$. Even more, as the following theorem shows, the poset $\langle A, \leq \rangle$ equipped with the relation $\perp$ is even quasi-orthomodular.

Theorem A.4.7. Cīrulis [2014b] *Let $\langle A, \leq \rangle$ be a poset with a bottom element* 0 *and a binary relation $\perp$ defined on A. Then the following assertions are equivalent:*

(a) *$\langle A, \leq \rangle$ is relatively orthocomplemented and $\perp$ is its induced orthogonality.*

(b) *$\langle A, \leq \rangle$ is sectionally m-complemented, has orthojoins (see item ($\perp$4)) and every local orthogonality $\perp_x$ is the restriction of $\perp$ to $[0, x]_{\leq}$.*

(c) *$\langle A, \leq, \perp \rangle$ is a quasi-orthomodular poset.*

A.5 Posets with difference

In this subsection, we recall the notion of a difference on a poset, as well as related notions like D-posets, orthomodular groups and generalized difference posets. Differences and D-posets were introduced in [Kôpka and Chovanec, 1994], while generalized difference posets were first defined in [Hedlíková and Pulmannová, 1996]. Orthomodular groups are a special case of generalized difference posets that was studied before them in [Chevalier, 1993]. See also Chapter 1.1 in [Dvurecenskij and Pulmannová, 2013] for an introduction into all of these structures together.

A.5.1 D-posets and generalized difference posets

Definition A.5.1. A *difference* on a poset $\langle P, \leq \rangle$ was defined in [Kôpka and Chovanec, 1994] as a partial operation $\ominus$ on P such that, for all $a, b, c \in P$, the following conditions hold:

(D1) $b \ominus a$ is defined if and only if $a \leq b$,

(D2) if $a \leq b$, then $b \ominus a \leq b$ and $b \ominus (b \ominus a) = a$,

(D3) if $a \leq b \leq c$, then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

It was noted in [Cīrulis, 2015b] that every poset with difference is a particular sectionally m-complemented poset with $a_x^{\perp} = x \ominus a$ if and only if $a \leq x$. Following [Cīrulis, 2015b, page 489], we call a poset with difference $\langle P, \leq, \ominus \rangle$ *regular* if it has a bottom element 0 and, for all $a, b \in P$,

$$\text{if } a \leq b \text{ and } a \leq b \ominus a, \text{ then } a = 0 \tag{A.12}$$

(see also condition (DOA) in [Hedlíková and Pulmannová, 1996] for the same notion).

A.5.2 Relations to Abelian groups

Orthomodular groups also have examples in certain rings equipped with suitable partial orders.

A.6 BCK-algebras and weak BCK-algebras

In this subsection, we provide some preliminaries concerning the notion *BCK-algebras* and one of their recent generalizations called *weak BCK-algebras*. Both of them are partially ordered sets bounded from below which are equipped with a binary operation that shares some characteristic properties with the set theoretical difference.

A.6.1 BCK-algebras

BCK-algebras were introduced in the 1960s by Imai and Iséki. They are a widely studied topic nowadays, but their beginnings are a bit murky, because in a few cases there are inconsistencies between different citations of the same article. The preliminaries presented in this subsection are taken from [Dvurecenskij and Pulmannová, 2013, Chapter 5], which is a modern introduction to the topic. Another book, which is cited by [Dvurecenskij and Pulmannová, 2013] as a comprehensive resource on BCK-algebras, is [Meng and Jun, 1994].

Definition A.6.1. [Imai and Iséki, 1966], see also [Dvurecenskij and Pulmannová, 2013, page 293]. A *BCK-algebra* is a set X with a binary operation $\diagdown$ and with a constant 0 such that the following axioms are satisfied for all $a, b, c \in X$:
(a) $((a \diagdown b) \diagdown (a \diagdown c)) \diagdown (c \diagdown b) = 0$,
(b) $(a \diagdown (a \diagdown b)) \diagdown b = 0$,
(c) $a \diagdown a = 0$,
(d) if $a \diagdown b = 0$ and $b \diagdown a = 0$, then $a = b$,
(e) $0 \diagdown a = 0$.

On a BCK-algebra X, the relation $\leq$ defined by

$$a \leq b \text{ if and only if } a \diagdown b = 0 \tag{A.13}$$

is a partial order which is called the *BCK-order*.

Alternatively, a BCK-algebra and its BCK-order can be characterized in the following way.

Proposition A.6.2. [Dvurecenskij and Pulmannová, 2013, page 294] *Let $\langle X, \leq \rangle$ be a poset with the least element* 0 *equipped with a binary operation $\diagdown$. Then $\langle X, \diagdown, 0 \rangle$ is a BCK-algebra with these operation if and only if, for all $a, b, c \in X$,*
(a) $a \diagdown (a \diagdown b) \leq b$,
(b) $(a \diagdown b) \diagdown (a \diagdown c) \leq c \diagdown b$,
(c) $a \leq b$ *if and only if* $a \diagdown b = 0$.

Obviously, the partial order $\leq$ in Proposition A.6.2 is then the BCK-order of the BCK-algebra $\langle X, \diagdown, 0 \rangle$ (item (c) of Proposition A.6.2) is identical to Equation (A.13)).

Connections between BCK-algebras and D-posets were studied in [Dvurečenskij and Kim, 1998].

A.6.2 Weak BCK-algebras

Let us now shift the focus to a generalization of BCK-algebras which was introduced in [Cīrulis, 2010b]. We will also state its connections to some of the other structures which we described in this section, including a result from [Cīrulis, 2015b] which connects it to differences.

Definition A.6.3. [Cīrulis, 2010b] An algebra $\langle A, \diagdown, 0 \rangle$ equipped with a partial order $\leq$ such that 0 is the bottom element is said to be a *weak BCK-algebra* if, for all $x, y, z \in A$,

(wBCK1) $x \leq y$ if and only if $x \diagdown y = 0$,

(wBCK2) if $x \diagdown y \leq z$, then $x \diagdown z \leq y$.

It was shown in [Cīrulis, 2010b] that every weak BCK-algebra is completely determined by the structure of its initial segments (i.e., principal down-sets) via a family of sectional complementations (see Section A.4).

We shall use the following alternative characterization of a weak BCK-algebra.

Proposition A.6.4. [Cīrulis, 2010b, Proposition 2.2] *A poset $\langle A, \leq \rangle$ with a least element* 0 *equipped with a binary operation $\smallsetminus$ on P is a weak BCK-algebra if and only if, for all $x, y, z \in A$,*

(a) $x \smallsetminus (x \smallsetminus y) \leq y$.
(b) *if $x \leq y$, then $z \smallsetminus y \leq z \smallsetminus x$,*
(c) $x \smallsetminus 0 = x$,

Remark A.6.5. Note that Proposition A.6.4 immediately yields the following observation: A poset with a least element 0 equipped with a binary operation $\smallsetminus$ is a weak BCK-algebra if and only if items (a) and (b) of Proposition A.6.4 and the property item (wBCK1) hold.

(By Proposition A.6.4, the three properties evidently hold in every weak BCK-algebra. Conversely, assume that items (a) and (b) of Proposition A.6.4 and property item (wBCK1) hold on a poset $\langle P, \leq \rangle$ equipped with a binary operation $\smallsetminus$. Now if, for arbitrary $a, b, c \in P$, we have $a \smallsetminus b \leq c$, then $a \smallsetminus c \leq a \smallsetminus (a \smallsetminus b) \leq b$ by items (a) and (b), so item (wBCK2) holds. Together with item (wBCK1), this yields that the algebra $\langle P, \smallsetminus, 0 \rangle$ is a weak BCK-algebra.) ◁

Proposition A.6.4 and Remark A.6.5 make it more visible how weak BCK-algebras generalize BCK-algebras: Obviously, item (a) of Proposition A.6.2 and item (a) of Proposition A.6.4 are identical, so this is also one of the characterizing properties of BCK-algebras. Moreover, as we noted in Remark A.6.5, item (c) of Proposition A.6.4 can be replaced by the property item (wBCK1), which in turn is identical to item (c) of Proposition A.6.2. Hence, the main difference between BCK-algebras and their weak counterparts lies in Proposition A.6.4(b). It is easy to see that Proposition A.6.4(b) is an immediate consequence of items (b) and (c) of Proposition A.6.2: If the two items hold, then $x \leq y$, then $x \smallsetminus y = 0$ by item (c), whence by item (b) $(z \smallsetminus y) \smallsetminus (z \smallsetminus x) \leq 0$; so $(z \smallsetminus y) \smallsetminus (z \smallsetminus x) = 0$, since 0 is the smallest element, and therefore, item (c) yields that $z \smallsetminus y \leq z \smallsetminus x$. So every BCK-algebra is a weak BCK-algebra. This justifies the terminology, showing that a weak BCK-algebra is indeed a weaker structure than a BCK-algebra but at the same time a very closely related one .

Items (a) and (c) of Proposition A.6.4 yield the following basic property of weak BCK-algebras, which we also need later.

Proposition A.6.6. [Cīrulis, 2010b, Proposition 2.1(e)] *Let $\langle X, \smallsetminus, 0 \rangle$ be a weak BCK-algebra. Then $y \smallsetminus x \leq y$ for all $x, y \in X$.*

The following example provides a weak BCK-algebra which is not a BCK-algebra.

Example A.6.7. [Cīrulis, 2013, Example 1]

$\smallsetminus$	0	x	y	z	1
0	0	0	0	0	0
x	x	0	x	x	0
y	y	y	0	0	0
z	z	z	y	0	0
1	1	x	z	y	0

Table A.1: Table defining the operation $\smallsetminus$ in Example A.6.7.

Let $\mathbf{N_5}$ denote the five-element non-distributive lattice with two maximal chains $0 < a < 1$ and $0 < b < c < 1$. Table A.1 defines a binary operation $\smallsetminus$ on it which, as can be easily checked, turns the poset $\mathbf{N_5}$ into a weak BCK-algebra. Now, the values $a = x$, $b = z$ and $c = 1$ falsify the axiom in Proposition A.6.2(b); so $\mathbf{N_5}$ is not a BCK-algebra. ◁

The following definition collects a few important types of weak BCK-algebras which we will encounter.

Definition A.6.8. A weak BCK-algebra $\langle A, \diagdown, 0\rangle$ is said to be

(a) *commutative* (see [Cīrulis, 2013, 2014b, 2015b]) if for all $x, y \in A$,

$$x \diagdown (x \diagdown y) = y \diagdown (y \diagdown x), \tag{A.14}$$

(b) a *weak Henkin algebra* (see [Cīrulis, 2010b]) if for all $x, y \in A$,

$$\text{if } x \diagdown y \leq y, \text{ then } x \leq y, \tag{A.15}$$

(c) *implicative* (see [Cīrulis, 2014b]) if for all $x, p, q \in A$,

$$\text{if } x \leq p \leq q, \text{ then } p \diagdown (q \diagdown x) = x. \tag{A.16}$$

Remark A.6.9. The three particular types of weak BCK-algebras which are introduced by Definition A.6.8 are weak analogues of corresponding special types of BCK-algebras – *commutative BCK-algebras*, *positive implicative BCK-algebras* and *implicative BCK-algebras*.

A commutative BCK-algebra is defined as a BCK-algebra for which Equation (A.14) holds (the definition can be found, for example, in [Dvurecenskij and Pulmannová, 2013, page 295]), so it is obvious that commutative weak BCK-algebras are just a straightforward generalization.

Weak Henkin algebras are the weak analogue of positive implicative BCK-algebras. A positive implicative BCK-algebra is defined as a BCK-algebra such that $x \diagdown y = (x \diagdown y) \diagdown y$ for all elements x, y ([Dvurecenskij and Pulmannová, 2013, page 301]). This equality is called the *contraction law*. It was noted in [Cīrulis, 2010b] that on a weak BCK-algebra, the contraction law implies Equation (A.15) (by item (wBCK1)). As noted (in the dual form) in [Cīrulis, 2007, Section 3], for BCK-algebras, also the converse implication is true: the class of BCK-algebras satisfying Equation (A.15) coincides with that of positive implicative algebras.

An implicative BCK-algebra is a BCK-algebra which is both commutative and positive implicative ([Dvurecenskij and Pulmannová, 2013, page 301]). It is known from [Abbott, 1967] that every implicative BCK-algebra is a meet semilattice whose initial segments are Boolean lattices. Similarly, it was proved in [Cīrulis, 2015b] that every implicative weak BCK-algebra is a meet semilattice whose initial segments are orthomodular lattices. In this sense, implicative weak BCK-algebras are the weak anologues of implicative BCK-algebras. ◁

It is known from [Cīrulis, 2013] that every implicative weak BCK-algebra is both commutative and a weak Henkin algebra (however, the weak Henkin part is rather implicit there, and a weak Henkin algebra is called a weak BCK-algebra satisfying the *contraction rule* (the contraction rule being (A.15))).

We will also use the following alternative characterization of commutativity of a weak BCK-algebra.

Proposition A.6.10. [Cīrulis, 2013, Theorem 3.2] *A weak BCK-algebra $\langle X, \diagdown, 0\rangle$ is commutative if and only if, for all $x, y \in X$,*

$$\text{if } x \leq y, \text{ then } x \leq y \diagdown (y \diagdown x). \tag{A.17}$$

The following proposition establishes a connection between implicative weak BCK-algebras and relatively orthocomplemented posets.

Proposition A.6.11. [Cīrulis, 2014b, Theorem 6.8] *Suppose that $\langle A, \leq, 0\rangle$ is a poset with the least element and that $\diagdown$ is a binary operation on A. Then the following assertions are equivalent:*

(a) *A is a relatively orthocomplemented meet semilattice and*

$$x \diagdown y = (x \wedge y)^{\perp}_{x}, \tag{A.18}$$

where $^{\perp}_{x}$ denotes the orthocomplementation in the initial segment $[0, x]_{\leq}$,

(b) $\langle A, \setminus, 0\rangle$ *is an implicative weak BCK-algebra with the underlying partial order* $\leq$ *and, for* $a \leq x$, $a^{\perp}_{x} = x \setminus a$.

On an initial segment $[0, x]_{\leq}$ of a weak BCK-algebra $\langle A, \setminus, 0\rangle$, we can always define a unary operation $^{\perp}_{x}$ by $a^{\perp}_{x} := x \setminus a$. This is in general a g*-complementation. By a sectionally De Morgan complemented (or sectionally orthocomplemented, or relatively orthocomplemented) weak BCK-algebra we always mean a weak BCK-algebra for which the operations $^{\perp}_{x}$ satisfy the respective properties from Definition A.4.2 or Definition A.4.4.

Differences are related to weak BCK-algebras in the following way.

Lemma A.6.12. [Cīrulis, 2015b, Corollary 4.4] *A weak BCK-algebra* $\langle A, \setminus, 0\rangle$ *is relatively orthocomplemented if and only if it is a regular poset with difference (in the sense that there is a difference* $\ominus$ *on A which is related to the weak BCK-operation* $\setminus$ *by* $b \ominus a = b \setminus a$ *if and only if* $a \leq b$*).*

Appendix B

Semigroup theoretic preliminaries

This appendix chapter collects some basics from different semigroup theoretic topics. In Section B.1, we recall the notion of strong semilattices of semigroups. Then we provide some basic facts about bands (a.k.a. idempotent semigroups) in Section B.2, including the strong semigroup decomposition of right normal bands. Finally, Section B.3 deals with semigroups equipped with a particular unary operation called D-semigroups.

The preliminaries provided in this section are needed almost exclusively for Chapter 7.

B.1 Inverse systems and strong semilattices

The aim of this section is to recall strong semilattices of semigroups and to settle the corresponding terminology and notation. To state the definition of a strong semilattice of semigroups, it is useful first to define inverse systems of semigroups.

B.1.1 Inverse systems

Definition B.1.1. See [Rotman, 2002, page 499]. Let $\langle S, \leq\rangle$ be a poset and let $\mathcal{A} = \{A_s | s \in S\}$ be a family of algebras of the same type. Let $\mathcal{H} = \{h_s^t | s, t \in S \text{ and } s \leq t\}$ be a family of homomorphisms $h_s^t : A_t \to A_s$. Suppose that for all $r, s, t \in S$

(a) the homomorphism h_t^t is the identity map

(b) if $r \leq s \leq t$, then $h_r^s h_s^t = h_r^t$.

Then the pair $\langle \mathcal{A}, \mathcal{H}\rangle$ is called an *inverse system* of the algebras A_s and the homomorphisms h_s^t (over the carrier S).

Some authors require the poset $\langle S, \leq\rangle$ to be (upwards) directed (i.e., every two elements need to have an upper bound), but for our work this is not necessary. In the sequel, $\langle S, \leq\rangle$ will always be a lower semilattice (thus downwards directed). Later it will have to be a lower semilattice which has a greatest element (and thus it will be directed in both directions).

It is also common to define an inverse system as a triple $\langle S, \mathcal{A}, \mathcal{H}\rangle$. However, we do not include the carrier $\langle S, \leq\rangle$ into the signature, because it is determined by $\langle \mathcal{A}, \mathcal{H}\rangle$ up to order isomorphism.

When dealing with inverse systems of monoids, it is sometimes necessary to clarify whether the identities of the monoids are included into their signatures or not. In the first case, we will speak of *inverse systems of monoids*, while in the latter case, we will say *inverse system of semigroups with identities*. I.e., an inverse system of monoids consists of a family of monoids and a family of monoid homomorphisms, while an inverse system of semigroups with identities consists of a family of semigroups that happen to have identities and a family of semigroup homomorphisms.

B.1.2 Strong semilattices of semigroups

Given an inverse system of semigroups over a lower semilattice, the following definition provides a semigroup which contains all the semigroups from the inverse system as subsemigroups.

Definition B.1.2. [Grillet, 1995, page 75] Let $\langle S, \wedge \rangle$ be a lower semilattice and let $\langle \mathcal{A}, \mathcal{H} \rangle$ be an inverse system of pairwise disjoint semigroups over $\langle S, \wedge \rangle$.

On the union $A = \bigcup_{s \in S} A_s$ of all the semigroups, we define an operation $\bullet$ in the following way. If $x \in A_s$ and $y \in A_t$, and $\cdot_{s\wedge t}$ denotes the multiplication on the semigroup $A_{s\wedge t}$, then

$$x \bullet y := h^s_{s\wedge t}(x) \cdot_{s\wedge t} h^t_{s\wedge t}(y). \tag{B.1}$$

Then we write $A = \mathfrak{S}\langle \mathcal{A}, \mathcal{H} \rangle$ and call the semigroup $\langle A, \bullet \rangle$ (the operation $\bullet$ is known to be associative, see e.g. [Howie, 1995]) a *strong semilattice of semigroups*.

The construction which is dual to the strong semilattice, using a so-called *direct system* instead of an inverse system, is also known as a *Płonka sum* (see [Płonka, 1967]). That is, given an inverse system $\langle \mathcal{A}, \mathcal{H} \rangle$ over a meet semilattice S, the strong semilattice of semigroups $A = \mathfrak{S}\langle \mathcal{A}, \mathcal{H} \rangle$ can be equivalently obtained as a Płonka sum in the following way: Let $\bar{S}$ be the order dual of the meet semilattice S (hence $\bar{S}$ is a join semilattice). Then $\langle \bar{S}, \mathcal{A}, \mathcal{H} \rangle$ is a direct system of semigroups, and A is the Płonka sum of this direct system.

In order to keep the notation simple, we will write $\mathfrak{S}\langle \mathcal{A}, \mathcal{H} \rangle$ for both the set A and the semigroup $\langle A, \bullet \rangle$.

In general, the strong semilattice of semigroups obtained from an inverse system of monoids is itself not a monoid. However, under certain conditions it is, and its identity can be obtained from the inverse system of monoids. We deal with this situation in Chapter 7, where we define strong semilattices of so-called D-monoids in a way that ensures that a strong semilattice of D-monoids is itself a D-monoid(this construction is applied to the multiplicative semigroup of a reduced Rickart ring in Chapter 7).

Since every strong semilattice of semigroups is itself a semigroup, we will avoid the usual term *strong semilattice of monoids* for a strong semilattice of semigroups which happen to be monoids (even if the strong semilattice is obtained from an inverse system of monoids), because such a strong semilattice might not be a monoid itself. We will speak of strong semilattices of semigroups with identities instead.

B.2 Bands

This subsection collects some basic facts about bands, a.k.a. idempotent semigroups. In particular, right-normal bands and their decomposition as a strong semilattice of right-zero bands are of interest for the purpose of this thesis.

B.2.1 Right-normal bands

This subsubsection contains some basic semigroup theory which can be found in any textbook on the subject like, for example, [Howie, 1995].

Definition B.2.1. [Howie, 1995, page 8] A *band* is a semigroup in which every element is idempotent.

Recall that the set of idempotents E of a semigroup A is ordered by the standard order of idempotents $\leq_E$ (Definition A.2.2).

In the case if a semigroup $\langle B, \cdot\rangle$ is a band, the standard order of idempotents is called the *natural order* relation on B (see [Howie, 1995, page 134]). We shall denote it by $\leq_B$. The join and meet of elements a, b of a band under the natural order will be denoted by $a \wedge_B b$ and $a \vee_B b$, respectively.

Definition B.2.2. [Howie, 1995, page 133] A band $\langle B, \cdot\rangle$ is called *right normal* if, for all $a, b, c \in B$,

$$a \cdot b \cdot c = b \cdot a \cdot c. \tag{B.2}$$

(The corresponding equation for defining *left normality* is $x \cdot y \cdot z = x \cdot z \cdot y$.)

Right normal bands were also called *restrictive semigroups* in the past, see, for example, [Vagner, 1962].

Remark B.2.3. A right normal band is also *right-handed*, i.e., if $x \cdot y = x$, then also $y \cdot x = x$ (because $x \cdot y = x$ yields $x = x^2 = x \cdot y \cdot x = y \cdot x \cdot x = y \cdot x$). Hence, in a right normal (or, more generally, in a right-handed) band, $x \leq_B y$ if and only if $x \cdot y = x$. ◁

Another relation on a right normal band which is worth mentioning is the *natural preorder* $\sqsubseteq_B$, which is defined as $x \sqsubseteq_B y$ if and only if $y \cdot x = x$ (it is easy to see that it is indeed a preorder). Hence, $x \leq_B y$ if and only if both $x \sqsubseteq_B y$ and $x \cdot y = y \cdot x$.

The following definition recalls a particularly simple class of semigroups.

Definition B.2.4. [Howie, 1995, page 3] A semigroup $\langle A, \cdot\rangle$ is called *right zero semigroup* iff $x \cdot y = y$ for all $x, y \in A$. (It is called *left zero semigroup* iff $x \cdot y = x$ for all $x, y \in A$.)

Obviously, every right zero band is also right normal.

We conclude this subsubsection with a well-known result about right normal bands which provides some further insight into how right zero bands and right normal bands are related to each other. It can be found, for Example, in [Howie, 1995]. Originally it was published in [Yamada and Kimura, 1958].

Lemma B.2.5. *A band $\langle B, \cdot\rangle$ is right normal if and only if it is a strong semilattice of right zero bands $B = \mathfrak{S}\langle B_i, h_i^j\rangle$. The right zero bands B_i are the $\mathcal{J}$-classes of the band B (where $\mathcal{J}$ is the Green equivalence defined by $a\mathcal{J}b$ iff $B^1aB^1 = B^1bB^1$), the semilattice Y is the quotient $B/\mathcal{J}$, and for $\mathcal{J}$-classes i, j with $i \leq_{\mathcal{J}} j$ (where $\leq_{\mathcal{J}}$ is the order of the $\mathcal{J}$-classes), the homomorphism h_i^j is the map which maps every $a \in B_j$ to the unique $b \in B_i$ with $b \leq_B a$, where $\leq_B$ is the natural order of the band.*

B.2.2 Bands with the upper bound property: Skew nearlattices

Recall that poset which has the upper bound property is also called *nearsemilattice* (see Definition A.4.5) when we want to emphasize that we consider this structure as a generalization of lattices. Similarly, a meet semilattice which has the upper bound property is called a *nearlattice*. The latter notion is obviously closer to lattices and has been studied more in the literature (see, for example, [Noor, 1980], [Cornish, 1980a] and [Cornish and Noor, 1982]). This subsection deals with skew nearlattices, which were introduced as a non-commutative generalization of nearlattices in [Cīrulis, 2004] (see also [Cīrulis, 2010a] for the subclass of so-called right skew nearlattices).

Definition B.2.6. [Cīrulis, 2010a] A *skew nearlattice* is a band which has the upper bound property with respect to its natural order. We will call the multiplication on such a band *skew meet*.

We shall denote the skew meet by $\overleftarrow{\wedge}$. In the literature, the notation $\overleftarrow{\wedge}$ is usually reserved for the particular case of skew meet where the underlying band is right regular (i.e., $a \overleftarrow{\wedge} b \overleftarrow{\wedge} a = b \overleftarrow{\wedge} a$ for all elements a, b), while in the dual case, that is, for a left regular band, the notation is $\overrightarrow{\wedge}$. The skew nearlattices which we encounter in this thesis have right regular underlying bands, but this fact is not important for our purpose.

Remark B.2.7. Observe that a right zero band ordered by its natural order is obviously an antichain. As the order of an antichain always has the upper bound property, any right zero band $\langle B, \cdot \rangle$ can be seen as a right zero right skew nearlattice $\langle B, \vee_B, \cdot \rangle$ (where $\vee_B$ is the partial join operation corresponding to the natural order of the band $\langle B, \cdot \rangle$, i.e., $x \vee_B y$ is defined only if $x = y$, and in this case $x \vee_B x = x$). ◁

A *right normal skew nearlattice* (sometimes shortened to *rns-nearlattice*) is a skew nearlattice which is right normal as a band.

B.3 D-semigroups

In this subsection, we recall the notions of D-semigroups and D-rings. Such semigroups (respectively rings) admit a certain unary operation that has some properties which are closely related to the properties of the double focal operation $''$ on right-focal Rickart rings, which is why they are of interest for this thesis.

The original definition, however, does not involve any unary operation yet:

Definition B.3.1. [Stokes, 2015] A semigroup A is said to be a *D-semigroup* if there exists some subset U of its set of idempotents E such that, for all $a \in A$, there exists a smallest $e \in U$ with the property that $ea = a$ (smallest in the sense of the standard partial order of idempotents, see Definition A.2.2).

But D-semigroups can also be characterized in the following way which does involve a unary operation.

Proposition B.3.2. [Stokes, 2015] *A semigroup A is a D-semigroup if and only if it can be equipped with a unary operation $^\circ$ satisfying the following for all $a, b \in A$:*

($^\circ$1) $a^\circ a = a$,

($^\circ$2) $(a^\circ)^\circ = a^\circ$,

($^\circ$3) $(ab)^\circ a^\circ = a^\circ (ab)^\circ = (ab)^\circ$.

In a D-semigroup A, the set U from Definition B.3.1 is the range of the operation $^\circ$ from Proposition B.3.2. For each element $a \in A$, the element a° from Proposition B.3.2 is the smallest $e \in U$ with the property that $ea = a$. Hence the set U from Definition B.3.1 uniquely determines the operation $^\circ$ from Proposition B.3.2 and vice versa.

We will treat D-semigroups as algebras of the kind $\langle A, \cdot, ^\circ \rangle$ (where $\cdot$ denotes the multiplication on the semigroup).

Note that, since $a^\circ a^\circ = (a^\circ)^\circ a^\circ = a^\circ$ by Proposition B.3.2item ($^\circ$2) and Proposition B.3.2item ($^\circ$1), the element a° is idempotent for every $a \in A$.

Definition B.3.3. [Stokes, 2015] A D-semigroup $\langle A, \cdot, ^\circ \rangle$ is called D-semiadequate if $a^\circ b^\circ = b^\circ a^\circ$ for all $a, b \in A$.

The notion of *D-abundant* D-semigroup is also defined in [Stokes, 2015], and in the same article, the following was proved to be equivalent to the original definition of a D-abundant D-semigroup.

Definition B.3.4. A D-semigroup $\langle A, \cdot, ^\circ \rangle$ is said to be *D-abundant* if, for all $x, y \in A^1$ and all $a \in A$, $xa = ya$ implies $xa^\circ = ya^\circ$ (where A^1 denotes the monoid created from the semigroup A by adding a new element which acts like an identity).

A ring is said to be a *D-ring* if its multiplicative semigroup is a D-semigroup (of course, every ring which has the multiplicative identity can be turned into a $^\circ$-trivial D-ring by chosing $a^\circ = 1$ for all elements a). It is immediate from Definition B.3.1 and the definition of a C-ring (see [Cornish, 1975]) that the multiplicative semigroup of a C-ring is a D-semigroup such that the set U from Definition B.3.1 is the set of all central idempotents. Hence, C-rings are examples of D-rings.

We will treat D-rings as algebras of the kind $\langle R, +, \cdot, ^\circ, 1 \rangle$ or $\langle R, +, \cdot, ^\circ \rangle$. A D-ring is said to be *D-abundant* if its multiplicative semigroup is D-abundant.

In [Stokes, 2015, Theorem 4.4], a connection between left Rickart rings and D-abundant D-rings was mentioned. In the terminology of this thesis, we can reformulate it as follows.

Theorem B.3.5. *Let R be a unitary D-ring. Then R is D-abundant if and only if R is a left-focal Rickart ring in which $a^{\backprime} = 1 - a^\circ$ for all $a \in R$.*

Obviously, the left focal operation thus defined on a unitary D-abundant D-ring has to be normal.

It is easy to check that every unitary left-strong left Rickart ring can be seen as a D-ring with $a^\circ = a^{\backprime\backprime}$: The properties item ($^\circ$1) and item ($^\circ$3) are immediate from items (b) and (c) of Proposition 1.3.6, respectively, while the property item ($^\circ$2) follows from normality of the focal operation: $a^{\circ\circ} = a^{\circ\backprime\backprime} = 1 - (1 - a^\circ) = a^\circ$. Hence, by Theorem B.3.5, every left-strong Rickart ring is a D-abundant D-ring.

See also [Cīrulis, 2020, Corollary 3.10] for a similar connection between unitary right-focal Rickart rings and so-called unitary *right π-abundant* rings (those are rings equipped with a unary operation that does not necessarily make them D-rings, but does satisfies the defining condition of D-abundance).

Appendix C

Generalized inverses

In this appendix chapter, we recall the concept of *generalized inverses*, which is closely related to many of the partial orders on Rickart rings which we encounter in this thesis from Chapter 2 onwards. The theory of generalized inverses originates from the context of matrices. In linear algebra and its applications it is usually convenient if a matrix has an inverse. Since that is not the case for every matrix, several generalized inverses were introduced which retain some of the properties of inverses while being defined for a wider range of matrices.

The generalized inverses presented here are not necessary for any of the results of the research, but knowing about them provides some background and context.

C.1 Moore-Penrose generalized inverse

The first notion which we shall consider was first introduced for matrices in [Moore, 1920], where it was called *general reciprocal* (actually, it was even investigated for integral and differential operators before it was studied on matrices (for the first mentioning in this context, see [Fredholm, 1903], where it was called "transformation pseudo-inverse" in French). However, Moore's general reciprocal did not spark great interest and was almost forgotten (later authors, such as [Ben-Israel and Greville, 2003], attribute this fate to the rather incomprehensible notation of [Moore, 1920]).

The following is a translation of the original definition from [Moore, 1920] into more understandable language which can be found in [Ben-Israel, 2002] (see also [Ben-Israel and Greville, 2003, Appendix A, page 373]).

Theorem C.1.1. *For every complex matrix A there exists a unique matrix $X : \operatorname{ran} A \to \operatorname{ran} A^*$ such that $AX = P_{\operatorname{ran} A}$, $XA = P_{\operatorname{ran} A^*}$. The matrix X is called the* general reciprocal *of A and we denote it by $A^\dagger$.*

Much later, the same notion was defined in [Penrose, 1955] in a different, but equivalent way under the name of *the generalized inverse*. The equivalence was noted in [Rado, 1956]. The defining equations from [Penrose, 1955], called *Penrose equations*, lead to the development of a rich theory of generalized inverses which satisfy only a subset of these equations. Since the defining equations in [Penrose, 1955] make sense not only for matrices, the Moore-Penrose generalized inverse was also studied in other (more general or otherwise different) settings. For example, it was introduced to Baer *-semigroups in [Foulis, 1963] and to proper *-semigroups in [Drazin, 1978]. For the purpose of this dissertation, the context of proper *-semigroups seems more relevant, because the multiplicative reduct of every Rickart *-ring is a proper *-semigroup. Let us therefore recall the following definition.

Definition C.1.2. [Drazin, 1978] A *proper *-semigroup* is a semigroup S equipped with an involution (see Definition 1.2.1) satisfying the additional condition (called *properness*)

$$\text{if } a^*a = a^*b = b^*a = b^*b, \text{ then } a = b. \tag{C.1}$$

A *proper *-ring* is a ring whose *-semigroup reduct is proper.

Definition C.1.3. [Penrose, 1955; Drazin, 1978] Let S be a proper *-semigroup and let $a \in S$. An element $x \in S$ is called the *Moore-Penrose generalized inverse* if the following conditions (called *Penrose equations*) hold:

(PE1) $axa = a$,

(PE2) $xax = x$,

(PE3) $(ax)^* = a^*x^*$,

(PE4) $(xa)^* = x^*a^*$.

It is usually denoted by $a^\dagger$.

Remark C.1.4. Sometimes, the Moore-Penrose generalized inverse is just called *the generalized inverse* or *the pseudoinverse*, but this will be avoided in this dissertation since it may cause confusion. ◁

C.2 Weaker generalized inverses

In this subsubsection, we review a few more generalized inverses.

The following well-known definition can be found, for example, in [Hartwig, 1980] or in [Baksalary and Trenkler, 2021].

Definition C.2.1. Let S be a semigroup and let $a \in S$.

An element $x \in S$ is called an *inner generalized inverse* if it satisfies the Moore-Penrose equation item (PE1).

An element $x \in S$ is called an *outer generalized inverse* if it satisfies the Moore-Penrose equation item (PE2).

An element $x \in S$ is called *reflexive generalized inverse* if it is both an inner and an outer generalized inverse.

Remark C.2.2. Sometimes, an inner generalized inverses are just called *generalized inverses* (for example, in [Mitra et al., 2010, Definition 2.3.2]), but in this dissertation, we will stick to the longer term in order to avoid confusion. ◁

Definition C.2.3. [Green, 1951; von Neumann, 1936]) Let S be a semigroup or a ring. An element $a \in S$ which admits an inner generalized inverse is called (*von Neumann*) *regular*. A (*von Neumann*) *regular semigroup* (*or ring*) is a semigroup (respectively ring) S such that every element in S is regular.

Obviously, every regular ring is also a Rickart ring: For every a in a regular ring R, let a^- be some inner generalized inverse of a. Then we can define focal operations on R by $a' = 1 - a^-a$ and $a^{`} = 1 - aa^-$.

There are also corresponding terms with the the Moore-Penrose generalized inverse instead of an arbitrary inner generalized inverse:

Definition C.2.4. Let S be a proper *-semigroup or *-ring. An element $a \in S$ which has the Moore-Penrose generalized inverse is called **-regular* (see, for example, [Drazin, 1978]). If every element of a proper *-semigroup S (or a ring R) has the Moore-Penrose inverse, then the semigroup S (respectively, the ring R) is called **-regular* (see, for example, [Lebtahi et al., 2014] or [Foulis, 1963] for this definition of the term **-regular*).

Remark C.2.5. In other literature (for example, [Patrício and Araújo, 2010], [Ara and Menal, 1984] or [Kaplansky, 1955]), a *-regular ring is defined as a ring with a proper involution which is von Neumann regular. This definition is equivalent to the one used in this thesis (see, for example, [Marovt et al., 2015, page 17] and [Urquhart, 1968] for an explanation). ◁

Concerning Rickart rings, the following well-known Proposition is of interest (see, for example, [Berberian, 1972, Exercise 6A, page 18]).

Proposition C.2.6. *A unitary *-ring is *-regular if and only if it is both regular and a Rickart *-ring.*

C.3 The group inverse

This subsections provides the definition and some basic facts for another classical generalized inverse, which seems to have originated first in semigroup and ring theory. For the details of its origin, see [Clifford, 1941], where it was introduced under the name of *relative inverse*, as well as [Azumaya, 1954] and [Drazin, 1958], where it was mentioned as a special case of the (*Drazin*) *pseudo-inverse*. Later it also appeared (maybe independently) in matrix theory (see [Englefield, 1966]), and now it is a standard notion in all of these fields.

Definition C.3.1. [Ben-Israel and Greville, 2003, page 156], [Mary, 2014] Let S be a semigroup and $a \in S$. An element $x \in S$ is called the *group inverse* of a if it is an inner and outer inverse of a (i.e., satisfies the first two of the Penrose equations item (PE1) and item (PE2)) and the following identity holds:

(GI1) $xa = ax$.

The group inverse of an element a is denoted by $a^{\sharp}$.

It is also called *commuting reciprocal inverse* (in [Englefield, 1966]), *commutative 1-2 inverse* (in [Hartwig, 1976]), *Drazin 1-inverse* (in [Lajos, 1978]) or $\{1,2,5\}$-*inverse* (in [Ben-Israel and Greville, 2003]).

The group inverse shares the following important property with the Moore-Penrose inverse.

Proposition C.3.2. Drazin [1958] *Let S be a semigroup. If the group inverse of an element $a \in S$ exists, then it is unique.*

Although the group inverse is a special case of the *Drazin pseudo-inverse*, which was defined in [Drazin, 1958], we shall not treat the latter in this thesis, since we do not deal with any partial order closely related to it. Moreover, the Drazin pseudo-inverse does not satisfy the first Penrose equation item (PE1), so in the very strict sense we could even argue that it is actually not a generalized inverse at all (recall that sometimes the term "generalized inverse" is used as a synonym of "inner generalized inverse"). It is, however, covered by at least two books on generalized inverses ([Mitra et al., 2010] and [Ben-Israel and Greville, 2003]).

As with the Moore-Penrose inverse, there is a special term for a semigroup in which every element has the group inverse.

Definition C.3.3. A group invertible element of a semigroup is called *completely regular* (see, for example, [Mary, 2014]). A semigroup is called *completely regular* if all its elements are completely regular (cf. [Howie, 1995]).

Completely regular semigroups are somewhat less common than regular or *-regular ones. The existence of the group inverse is not even guaranteed in the case of square matrices. For matrices, a necessary and sufficient condition for existence involves the following notion.

Definition C.3.4. [Ben-Israel and Greville, 2003, page 153] The *index* of a square matrix A is the smallest positive integer k such that $\operatorname{rank} A^k = \operatorname{rank} A^{k+1}$ (where rank denotes the rank of the matrix).

Now we can formulate the condition for existence.

Theorem C.3.5. Ben-Israel and Greville [2003] *Let A be a square matrix of index k. Then A has the group inverse if and only if $k = 1$.*

Bibliography

Abbott, J. C. (1967). Semi-Boolean algebra. *Mat. Vesn., N. Ser.*, 4:177–198.

Abian, A. (1975). Rings without nilpotent elements. *Mat. Časopis*, 25(3):289–291.

Ara, P. and Menal, P. (1984). On regular rings with involution. *Arch. Math.*, 42(2):126–130.

Azumaya, G. (1954). Strongly π-reguar rings. *Hokkaido Mathematical Journal*, 13(1).

Baksalary, J. K. and Hauke, J. (1990). A further algebraic version of Cochran's theorem and matrix partial orderings. *Linear Algebra Its Appl.*, 127:157–169.

Baksalary, J. K. and Mitra, S. K. (1991). Left-star and right-star partial orderings. *Linear Algebra Its Appl.*, 149:73–89.

Baksalary, O. M. and Trenkler, G. (2010). Core inverse of matrices. *Linear Multilinear Algebra*, 58(6):681–697.

Baksalary, O. M. and Trenkler, G. (2021). The Moore–Penrose inverse: A hundred years on a frontline of physics research. *The European Physical Journal H*, 46(1):9.

Ben-Israel, A. (2002). The Moore of the Moore-Penrose inverse. *The Electronic Journal of Linear Algebra*, 9:150–157.

Ben-Israel, A. and Greville, T. N. E. (2003). *Generalized Inverses: Theory and Applications*. Number 15 in CMS Books in Mathematics. Springer, New York, 2. edition.

Berberian, S. K. (1972). *Baer*-Rings*, volume 195 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag Berlin. 2nd printing 2011, reprint of the 1972 edition with errata list and later developments indicated.

Burgess, W. D. and Raphael, R. (2018). The reduced ring order and lower semi-lattices. arXiv preprint arXiv180207227.

Burgess, W. D. and Raphael, R. (2020). The reduced ring order and lower semi-lattices II. ArXiv preprint 200613785.

Chacron, M. (1971). Direct product of division rings and a paper of Abian. *Proc. Am. Math. Soc.*, 29(2):259–262.

Chevalier, G. (1993). Order and orthogonality relations in rings and algebras. *Tatra Mt. Math Publ*, 3:31–46.

Cīrulis, J. (1998). Subtractive nearsemilattices. *Proc. Latvian Acad. Sci*, 52(2):228–233.

Cīrulis, J. (2004). Knowledge representation systems and skew nearlattices. In *Proceedings of the 64th Workshop on General Algebra "64. Arbeitstagung Allgemeine Algebra", Olomouc, Czech Republic, May 30–June 2, 2002 and of the 65th Workshop on General Algebra "65. Arbeitstagung Allgemeine Algebra", Potsdam, Germany, March 21–23, 2003*, pages 43–51. Klagenfurt: Verlag Johannes Heyn.

Cīrulis, J. (2007). On implications in sectionally pseudocomplemented posets. ArXiv preprint ArXiv07053803.

Cīrulis, J. (2010a). Skew nearlattices: Some structure and representation theorems. In *Proc. Olomouc Conf. 2010 AAA79 CYA25*, volume 19 of *Contributions to General Algebra*, pages 33–44, Klagenfurt. Verlag Johannes Heyn.

Cīrulis, J. (2010b). Subtraction-like operations in nearsemilattices. *Demonstr. Math.*, 43(4):725–738.

Cīrulis, J. (2013). On commutative weak BCK-algebras. ArXiv preprint ArXiv13040999.

Cīrulis, J. (2014a). On one-sided star partial orders on a Rickart *-ring. ArXiv preprint ArXiv14104693.

Cīrulis, J. (2014b). Quasi-orthomodular posets and weak BCK-algebras. *Order*, 31(3):403–419.

Cīrulis, J. (2015a). Lattice operations on Rickart *-rings under the star order. *Linear Multilinear Algebra*, 63(3):497–508.

Cīrulis, J. (2015b). On some classes of commutative weak BCK-algebras. *Studia Logica*, 103(3):479–490.

Cīrulis, J. (2015c). One–sided star partial orders for bounded linear operators. *Oper. Matrices*, 9(4):891–905.

Cīrulis, J. (2015d). Relatively orthocomplemented skew nearlattices in Rickart rings. *Demonstr. Math.*, 48(4):493–508.

Cīrulis, J. (2016). Extending the star order to Rickart rings. *Linear Multilinear Algebra*, 64(8):1498–1508.

Cīrulis, J. (2017). The diamond partial order for strong Rickart rings. *Linear Multilinear Algebra*, 65(1):192–203.

Cīrulis, J. (2019). Focal Baer semigroups and a restricted star order. *Acta Scientiarum Mathematicarum*, 85(1-2):97–117.

Cīrulis, J. (2020). Order structure of U-semiabundant semigroups and rings. Part I: Left Lawson's order. *Acta Scientiarum Mathematicarum*, 86(3-4):359–403.

Cīrulis, J. and Cremer, I. (2018). Notes on reduced Rickart rings, I. *Beitr. Zur Algebra Geom. Algebra Geom.*, 59(2):375–389.

Cīrulis, J. and Cremer, I. (2020). Correction to Notes on reduced Rickart rings, I. Representation and equational axiomatizations. *Beiträge zur Algebra und Geometrie / Contributions to Algebra and Geometry*, 61(3):579–580.

Cīrulis, J. and Cremer, I. (2022). On existence of joins and meets under the star order in strong Rickart rings. *Linear and Multilinear Algebra*, 70(22):7370–7383.

Clifford, A. H. (1941). Semigroups Admitting Relative Inverses. *The Annals of Mathematics*, 42(4):1037.

Cornish, W. H. (1975). Boolean orthogonalities and minimal prime ideals. *Communications in Algebra*, 3(10):859–900.

Cornish, W. H. (1980a). 3-permutability and quasicommutative BCK-algebras. *Math Jpn.*, 25:477–496.

Cornish, W. H. and Noor, ASA. (1982). Standard elements in a nearlattice. *Bull. Aust. Math. Soc.*, 26(2):185–213.

Cornish, WH. (1980b). Boolean skew algebras. *Acta Math. Hung.*, 36(3-4):281–291.

Cremer, I. (2024). Order structure of a right-strong Rickart ring under the right star order. *Communications in Algebra*, 52(5):2015–2032.

Cremer, I. (2025). Decompositions of multiplicative semigroups of m-domain rings and reduced Rickart rings. *Int. Electron. J. Algebra*, 37:273–296.

Cremer, I. and Marovt, J. (N.D.). Weak differences, weak BCK-algebras and applications to some partial orders on rings. *Math. Slovaca*. To appear.

Dalla Chiara, M., Giuntini, R., and Greechie, R. (2004). *Reasoning in Quantum Theory*, volume 22 of *Trends in Logic*. Springer Netherlands, Dordrecht.

Deng, C. and Wang, S. (2012). On some characterizations of the partial orderings for bounded operators. *Math. Inequal. Appl*, 12:619–630.

Djikić, M. S. (2016a). *Coherent and Precoherent Operators*. PhD thesis, University of Niš, Serbia.

Djikić, M. S. (2016b). Properties of the star supremum for arbitrary Hilbert space operators. *J. Math. Anal. Appl.*, 441(1):446–461.

Djikić, M. S. and Djordjević, D. S. (2016). Coherent and precoherent elements in Rickart *-rings. *Linear Algebra Its Appl.*, 509:64–81.

Djordjevic, D. S., Rakic, D. S., and Marovt, J. (2015). Minus partial order in Rickart rings. *Publ. Math. Debrecen*, 87(3-4):291–305.

Dolinar, G., Guterman, A., and Marovt, J. (2014). Monotone transformations on $B(H)$ with respect to the left-star and the right-star partial order. *Math. Inequal. Appl*, 17(2):573–589.

Dolinar, G., Kuzma, B., Marovt, J., and Poon, E. (2021). One-sided star partial order preservers on $B(H)$. *Oper. Matrices*, 15(3):1141–1159.

Dolinar, G. and Marovt, J. (2018). On a generalized concept of order relations on $B(H)$. *Mathematica Slovaca*, 68(1):33–40.

Dolinar, G. and Molinár, L. (2007). Maps on quantum observables preserving the Gudder order. *Reports on Mathematical Physics*, 60(1):159–166.

Drazin, M. P. (1958). Pseudo-Inverses in Associative Rings and Semigroups. *The American Mathematical Monthly*, 65(7):506–514.

Drazin, M. P. (1978). Natural structures on semigroups with involution. *Bull. Am. Math. Soc.*, 84(1):139–141.

Dvurečenskij, A. and Kim, H. S. (1998). Connections between BCK-algebras and difference posetse[1] *Studia Logica*, pages 421–439.

Dvurecenskij, A. and Pulmannová, S. (2013). *New Trends in Quantum Structures*, volume 516 of *Mathematics and Its Applications*. Springer Science+Business Media, Dordrecht. Originally published by Kluwer Academic Publishers in 2000.

Efimov, M. A. (2013). On the $\overset{\sharp}{\leq}$-order on the set of linear bounded operators in Banach space. *Math. Notes*, 93(5-6):784–788.

Endo, S. (1960). Note on P.P. Rings: (A Supplement to Hattori's Paper). *Nagoya Mathematical Journal*, 17:167–170.

Englefield, M. J. (1966). The commuting inverses of a square matrix. *Mathematical Proceedings of the Cambridge Philosophical Society*, 62(4):667–671.

Foulis, D. J. (1960). Baer *-semigroups. *Proc. Am. Math. Soc.*, 11(4):648–654.

Foulis, D. J. (1963). Relative inverses in Baer *-semigroups. *Michigan Mathematical Journal*, 10:65–84.

Fountain, J. (1976). Right PP monoids with central idempotents. *Semigroup Forum*, 13(1):229–237.

Fredholm, I. (1903). Sur une classe d'équations fonctionnelles. *Acta Math.*, 27:365–390.

Green, J. A. (1951). On the Structure of Semigroups. *The Annals of Mathematics*, 54(1):163.

Grillet, P. A. (1995). *Semigroups: an introduction to the structure theory.* Marcel Dekker, Inc., New York.

Gudder, S. (2006). An order for quantum observables. *Mathematica Slovaca*, 56(5):573–589.

Hartwig, R. E. (1976). Block generalized inverses. *Archive for Rational Mechanics and Analysis*, 61(3):197–251.

Hartwig, R. E. (1979). Pseudolattice properties of the star-orthogonal partial ordering for star-regular rings. *Proc. Am. Math. Soc.*, 77(3):299–303.

Hartwig, R. E. (1980). How to partially order regular elements. *Math Jpn.*, 25(1):1–13.

Hartwig, R. E. and Drazin, M. P. (1982). Lattice properties of the *-order for complex matrices. *J. Math. Anal. Appl.*, 86(2):359–378.

Hedlíková, J. and Pulmannová, S. (1996). Generalized difference posets and orthoalgebras. *Acta Math Univ Comen.*, 45:247–279.

Howie, J. M. (1995). *Fundamentals of Semigroup Theory.* Number 12 in London Mathematical Society Monographs New Series. Oxford University Press, Inc., New York. Reprinted 2003.

Huh, C., Kim, H. K., and Lee, Y. (2002). p.p. rings and generalized p.p. rings. *J. Pure Appl. Algebra*, 167(1):37–52.

[1] The published article includes this typo.

Husimi, K. (1937). Studies on the foundation of quantum mechanics. I. *Proceedings of the Physico-Mathematical Society of Japan. 3rd Series*, 19:766–789.

Imai, Y. and Iséki, K. (1966). On axiom systems of propositional calculi, XIV. *Proceedings of the Japan Academy, Series A, Mathematical Sciences*, 42(1):19–22.

Janowitz, M. F. (1976). A note on Rickart rings and semi-Boolean algebras. *Algebra universalis*, 6(1):9–12.

Janowitz, M. F. (1983). On the *-order for Rickart *-rings. *Algebra universalis*, 16(1):360–369.

Kalmbach, G. (1983). *Orthomodular Lattices.* Number 18 in L.M.S. Monographs. Academic Press Inc. (London) Ltd, London.

Kaplansky, I. (1955). Any orthocomplemented complete modular lattice is a continuous geometry. *Ann. Math.*, pages 524–541.

Khatun, S., Sircar, J., and Abu Nayeem, Sk. Md. (2020). Abian's relation on semirings. *Southeast Asian Bulletin of Mathematics*, 44(1):87–98.

Kôpka, F. and Chovanec, F. (1994). D -posets. *Math. Slovaca*, 44(1):21–34.

Krēmere, I. (2016). Left-star order structure of Rickart *-rings. *Linear Multilinear Algebra*, 64(3):341–352.

Lajos, S. (1978). Characterizations of completely regular elements in semigroups. *Acta Sci. Math*, 40:297–300.

Lam, T.-Y. (1999). *Lectures on Modules and Rings*, volume 189 of *Graduate Texts in Mathematics.* Springer-Verlag, Berlin.

Lebtahi, L., Patrício, P., and Thome, N. (2014). The diamond partial order in rings. *Linear Multilinear Algebra*, 62(3):386–395.

Loomis, L. H. (1955). The lattice theoretic background of the dimension theory of operator algebras. *Mem. Am. Math. Soc.*, (18).

Maeda, S. (1960). On a ring whose principal right ideals generated by idempotents form a lattice. *J. Sci. Hiroshima Univ. Ser. Math. Phys. Chem.*, 24(3):509–525.

Marovt, J. (2015). On partial orders in Rickart rings. *Linear Multilinear Algebra*, 63(9):1707–1723.

Marovt, J., Rakić, D. S., and Djordjević, D. S. (2015). Star, left-star, and right-star partial orders in Rickart *-rings. *Linear Multilinear Algebra*, 63(2):343–365.

Mary, X. (2014). Classes of semigroups modulo Green's relation $\mathcal{H}$. *Semigroup Forum*, 88(3):647–669.

Meng, J. and Jun, Y. B. (1994). *BCK-algebras.* Kyung Moon Sa Co., Seoul.

Mitra, S. K. (1987). On group inverses and the sharp order. *Linear Algebra Its Appl.*, 92:17–37.

Mitra, S. K., Bhimasankaram, P., and Malik, S. B. (2010). *Matrix Partial Orders, Shorted Operators and Applications*, volume 10 of *Series in Algebra.* World Scientific Publishing Co. Pte. Ltd., Singapore.

Moore, E. H. (1920). On the reciprocal of the general algebraic matrix. *Bulletin of the American Mathematical Society*, 26:294–295.

Noor, A. S. A. (1980). Isotopes of nearlattices. *Bull. Aust. Math. Soc.*, 22(3):472–474.

Patil, A. and Waphare, B. N. (2017). Zero-divisor graphs of reduced rickart *-rings. *Discuss. Math. Gen. Algebra Appl.*, 37(1).

Patrício, P. and Araújo, C. M. (2010). Moore–Penrose invertibility in involutory rings: The case $aa^\dagger = bb^\dagger$. *Linear and Multilinear Algebra*, 58(4):445–452.

Penrose, R. (1955). A generalized inverse for matrices. *Mathematical Proceedings of the Cambridge Philosophical Society*, 51(3):406–413.

Płonka, J. (1967). On a method of construction of abstract algebras. *Fundam. Math.*, 61(2):183–189.

Pulmannová, S. and Vinceková, E. (2007). Remarks on the order for quantum observables. *Math. Slovaca*, 57(6):589–600.

Rado, R. (1956). Note on generalized inverses of matrices. *Mathematical Proceedings of the Cambridge Philosophical Society*, 52(3):600–601.

Rakic, D. S. (2012). Decomposition of a ring induced by minus partial order. *Electron. J. Linear Algebra*, 23(1):72.

Rakic, D. S. (2015). Generalization of sharp and core partial order using annihilators. *Banach J. Math. Anal.*, 9(3):228–242.

Rotman, J. J. (2002). *Advanced Modern Algebra, 1st edition.* American Mathematical Society.

Rudin, W. (1991). Functional analysis, 2nd edition. *International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York.*

Šemrl, P. (2010). Automorphisms of $B(H)$ with respect to minus partial order. *J. Math. Anal. Appl.*, 369(1):205–213.

Stokes, T. (2015). Domain and range operations in semigroups and rings. *Commun. Algebra*, 43(9):3979–4007.

Subrahmanyam, N. V. (1960). Structure theory for a generealized Boolean ring. *Math. Ann.*, 141(4):297–310.

Sussman, I. (1958). A generalization of Boolean rings. *Math. Ann.*, 136(4):326–338.

Sussman, I. (1960). Ideal structure and semigroup domain decomposition of associate rings. *Math. Ann.*, 140(2):87–93.

Urquhart, N. S. (1968). Computation of generalized inverse matrices which satisfy specified conditions. *SIAM Review*, 10(2):216–218.

Vagner, V. V. (1962). Restrictive semigroups. *Izv. Vysshikh Uchebnykh Zaved. Mat.*, (6):19–27.

von Neumann, J. (1936). On regular rings. *Proceedings of the National Academy of Sciences of the United States of America*, 22(12):707–712.

Xu, X.-M., Du, H.-K., Fang, X., and Li, Y. (2010). The supremum of linear operators for the *-order. *Linear Algebra Its Appl.*, 433(11-12):2198–2207.

Yamada, M. and Kimura, N. (1958). Note on idempotent semigroups II. *Proc. Jpn. Acad.*, 34(2):110–112.

List of attended conferences

While conducting the research presented in this thesis, the author has attended six domestic and 11 international conferences. However, the conference talks **C6**, **C7** and **C9** were not directly connected to the topic of this thesis.

C1 Insa Cremer. Reduced Rickart rings and skew nearlattices. *Algebra and its applications.* Taevaskoja, Estonia, 2017.

C2 Insa Cremer. Reduced Rickart rings and skew nearlattices. *Topology, Algebra and Categories in Logic 2017.* Prague, Czech Republic, 2017.

C3 Insa Cremer. Reducēta Rikarta gredzena dekompozīcija pusgrupu stingra pusrežģī (Decomposition of a reduced Rickart ring as a strong semilattice of semigroups). *76th International Scientific Conference of the University of Latvia.* Rīga, Latvia, 2018.

C4 Insa Cremer. Strong semilattice decomposition of semigroup-bands. *AAA96 – 96. Arbeitstagung Allgemeine Algebra.* Darmstadt, Germany, 2018.

C5 Insa Cremer. Reducētie Rikarta gredzeni un greizi gandrīzrežģi (Reduced Rickart rings and skew nearlattices). *12th Conference of the Latvian Mathematical Society.* Ventspils, Latvia, 2018.

C6 Insa Cremer. The variety of reduced Rickart rings. *Summer School on Algebra and Ordered Sets 2018.* Špindlerův Mlýn, Czech Republic, 2018.

C7 Insa Cremer. Prime ideals in reduced Rickart rings. *International Conference on Algebra and its Applications.* Tartu, Estonia, 2018.

C8 Insa Cremer. Weak BCK-algebras in strong Rickart rings. *AAA98 – 98. Arbeitstagung Allgemeine Algebra.* Dresden, Germany, 2019.

C9 Insa Cremer. Reducēto Rikarta gredzenu varietāte (The variety of reduced Rickart rings). *77th International Scientific Conference of the University of Latvia.* Rīga, Latvia, 2019.

C10 Insa Cremer. Weak BCK-algebra in Rickart rings. *78th International Scientific Conference of the University of Latvia.* Rīga, Latvia, 2020.

C11 Insa Cremer. Strong semilattice decomposition of m-domain rings. *AAA100 – 100. Arbeitstagung Allgemeine Algebra.* Online (organized by Jagiellonian University in Kraków), 2021.

C12 Insa Cremer. Order structure of a strong Rickart ring under the strong right star order. *AAA101 – 101. Arbeitstagung Allgemeine Algebra.* Online (organized by University of Novi Sad), 2021.

C13 Insa Cremer. Stingro Rikarta gredzenu sakārtojumi un struktūra (Orderings and structure of strong Rickart rings). *79th International Scientific Conference of the University of Latvia.* Online (organized by University of Latvia), 2021.

C14 Insa Cremer. Stingra pusrežģa dekompozīcija īpašām D-abundantām D-pusgrupām (Strong semilattice decomposition for special D-abundant D-semigroups). *80th International Scientific Conference of the University of Latvia.* Online (organized by University of Latvia), 2022.

C15 Insa Cremer. A generalization of differences and its connection to weak BCK-algebras. *AAA103 – 103. Arbeitstagung Allgemeine Algebra.* Tartu, Estonia, 2023.

C16 Insa Cremer. A generalization of differences and its connection to weak BCK-algebras. *82nd International Scientific Conference of the University of Latvia.* Rīga, Latvia, 2024.

C17 Insa Cremer. Certain partial orders on strong Rickart rings. *Summer School on Algebra and Ordered Sets 2024.* Karolinka, Czech Republic, 2024.

Author’s publications

Krēmere, I. (2016). Left-star order structure of Rickart *-rings. *Linear Multilinear Algebra*, 64(3):341–352.

Cīrulis, J. and Cremer, I. (2018). Notes on reduced Rickart rings, I. *Beitr. Zur Algebra Geom. Algebra Geom.*, 59(2):375–389.

Cīrulis, J. and Cremer, I. (2022). On existence of joins and meets under the star order in strong Rickart rings. *Linear and Multilinear Algebra*, 70(22):7370–7383.

Cremer, I. (2024). Order structure of a right-strong Rickart ring under the right star order. *Communications in Algebra*, 52(5):2015–2032.

Cremer, I. (2025). Decompositions of multiplicative semigroups of m-domain rings and reduced Rickart rings. *Int. Electron. J. Algebra*, 37:273–296.

Cremer, I. and Marovt, J. (N.D.). Weak differences, weak BCK-algebras and applications to some partial orders on rings. *Math. Slovaca*. To appear.