University of Latvia

Sensitivity analysis for an optimal control problem of heat transfer

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Notations

$\mathbb{N}$	set of natural numbers.
$\mathbb{R}$	set of real numbers.
$\mathbb{R}_+$	set of strictly positive real numbers.
$c_1, c_2, \ldots$	constants of global scope.
$\hat{c}_1, \hat{c}_2, \ldots$	constants of a proof scope.
$\mathbb{R}^n$	set of real n-tuples.
$a \cdot b$	Euclidean inner product of $a, b \in \mathbb{R}^n$.
$\|a\|$	Euclidean norm of $a \in \mathbb{R}^n$.
$\overline{A}$	closure of a set $A \subset \mathbb{R}^n$ according to Euclidean norm.
$\mathrm{dist}(A, B)$	distance between sets $A, B \subset \mathbb{R}^n$.
$\mathrm{diam}(A)$	diameter of a set $A \subset \mathbb{R}^n$.
$\overline{ab}$	a closed line segment between $a, b \in \mathbb{R}^n$.
$B(a, \tau)$	an open ball with center $a \in \mathbb{R}^n$ and radius τ.
$S(a, \tau)$	a sphere with center $a \in \mathbb{R}^n$ and radius τ.
x, y, z	elements of $\mathbb{R}^2$, or $\mathbb{R}^3$.
$\partial\Omega$	boundary of a domain Ω.
ν	outward unit normal of a surface Γ.
ω_2	Lebesque measure over $\mathbb{R}^2$.
ω_3	Lebesque measure over $\mathbb{R}^3$.
λ_Γ	Lebesque surface measure over a surface Γ.
$f\|_A$	restriction (or trace) of a function $f : B \mapsto \mathbb{R}$ on the subset $A \subset B$.
f'	first order derivative of a function $f : \mathbb{R} \mapsto \mathbb{R}$.
f''	second order derivative of a function $f : \mathbb{R} \mapsto \mathbb{R}$.
$\partial_i f$	first order partial derivative along x_i-axis of a function $f : \Omega \mapsto \mathbb{R}$.
∇f	gradient of a function $f : \Omega \mapsto \mathbb{R}$.
$\nabla \cdot (\nabla f)$	Laplasian of a function $f : \Omega \mapsto \mathbb{R}$.
X^*	dual space for a normed space X.
$\\|v\\|_X$	norm of $v \in X$, where X is normed space.
$\langle \cdot, \cdot \rangle$	duality pairing for a normed space X and its dual space X^*.
$\mathcal{M}(A, \omega)$	vector space of ω-measurable functions $f : A \mapsto \mathbb{R}$.
$L_p(A)$	space of p-integrable ($p \in [1, +\infty]$) functions $f : A \mapsto \mathbb{R}$.
$\tilde{L}_p(\Omega)$	subspace of $L_p(\Omega)$ ($p \in [1, +\infty]$), which consists of functions $\{v \in L_p(\Omega) : \partial_3 v = 0\}$.
$H_1(\Omega)$	Sobolev space $\{v \in L_2(\Omega) : \partial_i v \in L_2(\Omega) \text{ for } i \in \{1, 2, 3\}\}$.
$H_1(\Omega, \Gamma)$	subspace of $H_1(\Omega)$, which consists of functions $\{v \in H_1(\Omega) : v\|_\Gamma = 0 \ \lambda_\Gamma\text{-a.e. on } \Gamma\}$, where $\Gamma \subset \partial\Omega$.
$V_p(\Omega, \Gamma)$	subspace of $H_1(\Omega)$, which consists of functions

	$\{v \in H_1(\Omega) : v\|_\Gamma \in L_p(\Gamma)\}$ and equipped with a norm $\\|\cdot\\|_{V_p(\Omega,\Gamma)} = \\|\cdot\\|_{H_1(\Omega)} + \\|\cdot\\|_{L_p(\Gamma)}$. Here we assume that $p \in [1, +\infty]$ and $\Gamma \subset \partial\Omega$.
$\dot{V}_p(\Omega, \Gamma_1, \Gamma_2)$	subspace of $H_1(\Omega, \Gamma_1)$, which consists of elements $\{v \in H_1(\Omega, \Gamma_1) : v\|_{\Gamma_2} \in L_p(\Gamma_2)\}$ and equipped with a norm $\\|\cdot\\|_{\dot{V}_p(\Omega,\Gamma_1,\Gamma_2)} = \\|\cdot\\|_{H_1(\Omega,\Gamma_1)} + \\|\cdot\\|_{L_p(\Gamma_2)}$. Here we assume that $p \in [1, +\infty]$ and $\Gamma_1, \Gamma_2 \subset \partial\Omega$.
$\mathrm{Lip}(A)$	space of Lipschitz continuous functions $f : A \mapsto \mathbb{R}$.
$C^1(A)$	space of functions $f : A \mapsto \mathbb{R}$ with continuous first order partial derivatives.
$\mathbf{a}, \mathbf{b}, \ldots$	lowercase bold Roman letters denote functionals over functional spaces.
$\mathbf{A}, \mathbf{B}, \ldots$	uppercase bold Roman letters denote operators between functional spaces.
$\mathbf{A} \circ \mathbf{B}$	composition of operators $\mathbf{A}$ and $\mathbf{B}$ (if $\mathbf{A}$, $\mathbf{B}$ are linear, then we simply write $\mathbf{AB}$).
$\mathbf{o}[v](w)$	denotes operators (functionals) of type $X \times Y \mapsto Z$ (X, Y, Z are normed spaces), such that $\\|\mathbf{o}[v](w)\\|_Z / \\|w\\|_Y \to 0$, as $\\|w\\|_Y \to 0$ (for every fixed $v \in X$).
$\mathbf{I}$	identity operator.
$\mathscr{L}(X, Y)$	space of linear bounded operators from a normed space X to a normed space Y.

Chapter 1

Introduction

This thesis was developed at Faculty of Physics and Mathematics of University of Latvia under supervision of professor Uldis Raitums. The research was supported by Europe Social Fund (contract 2004/0001/VPD1/ESF/PIAA/04/NP/3.2.3.1/0001/0001/0063) and Latvian Council of Science (grant 01.04 41).

This work is devoted to sensitivity analysis for an optimal control problem of conductive-radiative heat transfer arising in the glass fabric industry and primarily it is based on the author's earlier published papers ([Bi03], [Bi051], [Bi052]), where are covered various mathematical aspects of the optimal control model for treatment of glass fabric sheets in high temperature furnaces.

Mathematical models for treatment of simultaneous conductive, convective and radiative heat transfer were developed relatively recently. Here it is important to note the publications of Tiihonen and Laitinen (see [LT97], [LT98], [LT00]) at the end of 1990s and early 2000s. By mathematical formalization of simultaneous conductive-radiative heat transfer in grey materials they came to models with semilinear elliptic and parabolic boundary value problems, for which they established basic results on existence of solutions. We widely use this approach in this thesis, since there is only slight adaptation necessary for those models to serve our needs.

After the series of works of Tiihonen and Laitinen, there was a short break in research of the mathematical models of conductive-radiative heat transfer. However, in recent years some interesting papers have been published by Mayer, Philip and Tröltzsch (see [MPT04], [Ma05]). The research topic of these publications is optimal control of temperature gradient in high temperature furnaces of crystal growth. In order to deal with simultaneous conductive-radiative heat transfer, which play important role in this case, approach proposed by Tiihonen and Laitinen there is heavily exploited. The main problems considered by Mayer, Philip and Tröltzsch are necessary and sufficient optimality conditions, as well as existence of the optimal state.

Although mathematical models considered by Mayer, Philip, Tröltzsch and by us appear to be similar, they have some principal differences (we are using temperature on the surface of the heaters as a control, whereas in [MPT04], [Ma05] the control is volume heat source). As result, techniques that can be used for mathematical analysis of those models are different.

As it was already mentioned, the source for the problem considered by this thesis is glass fabric industry. Usually glass fiber is saturated with oil before weaving of fabric. After the fabric is produced, oil, which has been added before, must be removed. This can be achieved by oil burn out at high temperature in a special furnace. A sheet of glass fabric, after it has been weaved, is dragged with constant speed through this furnace, where it heats up and oil burns out. Unfortunately, quick glass fabric cooling after it has left the furnace leads to mechanical resistance degradation of produced product (see [BF99] for more details). Therefore, in order to maximize quality of produced fabric, a way to ensure uniform and slow cooling of the fabric sheet must be found via changing of physical characteristics of the furnace.

In this work we consider a simplified situation from [BF99], where oil burn out is neglected and process is steady in time. Here the furnace consists of heater system $\{\Omega_i \subset \mathbb{R}^3 : i \in \{1, \ldots, m\}\}$ (see Fig. 1.1). A thin fabric sheet $\Omega_f = (-l_1, l_1) \times (-l_2, l_2) \times (-\delta, \delta)$ of thickness 2δ is dragged through the furnace with a constant speed.

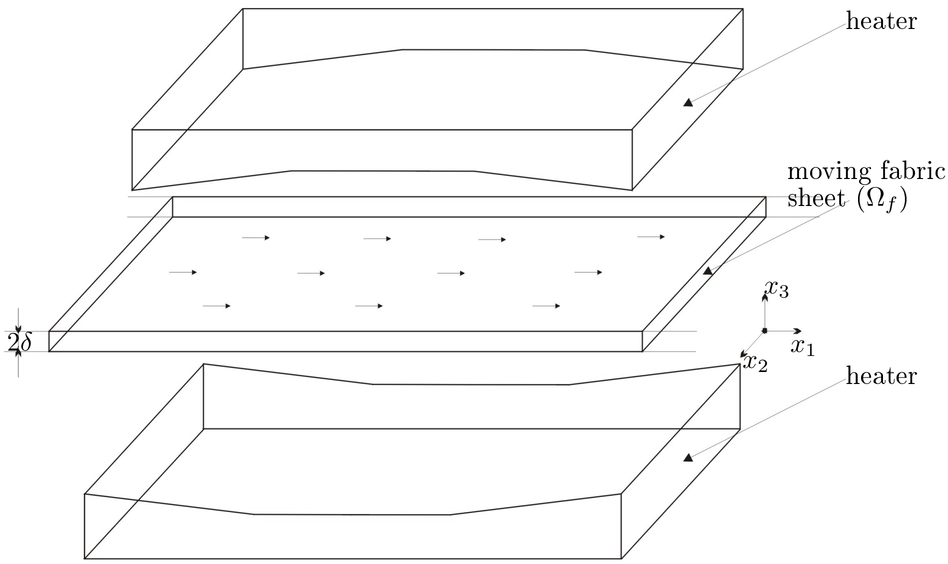

Figure 1.1: Construction of the furnace

Due to high temperature T_h on the heaters (1000-1100K) intensive heat transfer processes occur in the furnace (see Fig. 1.2). Since space between heaters and sheet is filled with transparent medium (air), but, on other hand, the fabric sheet and the heaters are opaque for heat radiation, then considerable heat flux exists on the surface of the sheet $\Sigma_1 = \bigcup_{i\in\{3,\dots,6\}} S_i$ (see Fig. 1.4 for more details about geometry of the problem) and on the heater-air interface $\Sigma_2 = \bigcup_{i\in\{1,\dots,m\}} \partial\Omega_i$ due to heat radiation emission, absorption and reflection. In addition, heat exchange with surrounding air also occurs on the interface $\Sigma = \Sigma_1 \cup \Sigma_2$ and this considerably influences overall heat flow in the furnace. On other hand, heat transfer inside the domain Ω_f occurs only via conduction and medium movement, since, as it was already mentioned, glass fabric is opaque for heat radiation.

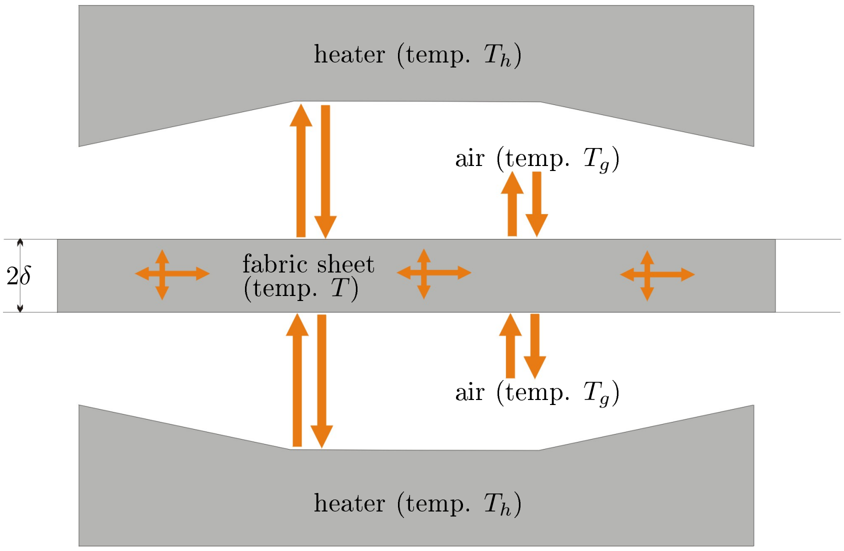

Figure 1.2: Heat exchange in the furnace

In mathematical terms the overall heat balance in Ω_f can be described by a boundary value problem:

$$\begin{cases} \nabla \cdot (k_1 \nabla T) - k_2(\partial_1 T) = 0 & \text{in } \Omega_f, \\ -\nu \cdot \nabla(k_1 T) = q & \text{on } \Sigma_1, \\ -\nu \cdot \nabla(k_1 T) = 0 & \text{on } S_2, \\ T = T_e & \text{on } S_1, \end{cases} \tag{1.1}$$

where T is temperature of the fabric in Ω_f, T_e is temperature of the fabric on the surface S_1 (see Fig. 1.3 for more details about geometry of the problem), q is heat flux through the interface Σ, ν is outer unit normal to $\partial\Omega_f$, k_1 is thermal conductivity of the fabric and k_2 is normalized velocity of sheet movement.

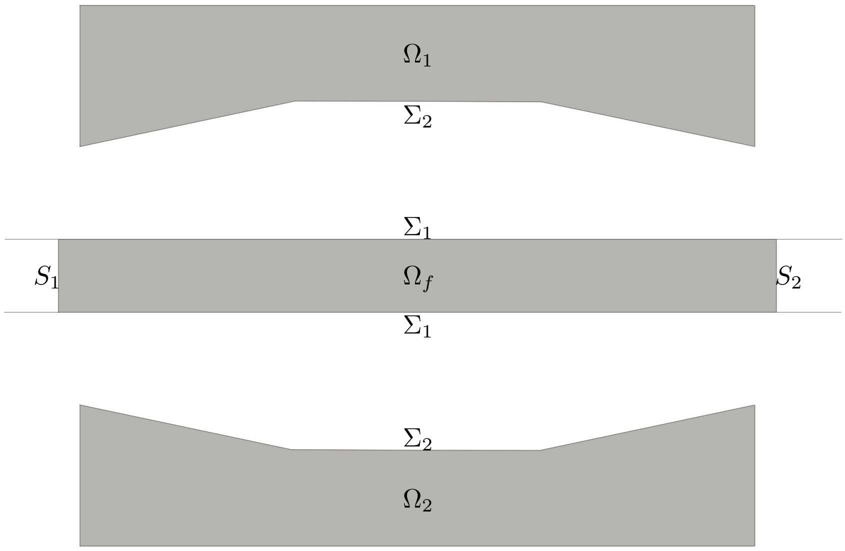

Figure 1.3: Geometry of the furnace

In order to optimize temperature field T and to achieve uniform and slow cooling effect of the fabric sheet after it has been heated up in the furnace, let us introduce an optimal control problem:

$$\begin{cases} \mathcal{J}(T) \mapsto \min, & \\ & \\ \nabla \cdot (k_1 \nabla T) - k_2(\partial_1 T) = 0 & \text{in } \Omega_f, \\ -\nu \cdot \nabla(k_1 T) = q & \text{on } \Sigma_1, \\ -\nu \cdot \nabla(k_1 T) = 0 & \text{on } S_2, \\ T = T_e & \text{on } S_1, \\ T \in V_5(\Omega_f,\, \Sigma_1), & \\ T_h \in \{v \in L_\infty(\Sigma_2) : 0 \le v(x) \le \mu\}, & \end{cases} \tag{1.2}$$

(for definition of the space $V_5(\Omega_f,\, \Sigma_1)$ see Notations), where T is chosen as state variable, T_h - as control variable, but the cost functional $\mathcal{J}(T)$ is defined as:

$$\mathcal{J}(T) := \int_{\Omega_f} (T - T_d)^2 \, d\omega_3.$$

Here T_d is a temperature field in Ω_f, which we would like to achieve by varying the control T_h. It should have high temperature near the entrance of the furnace to initialize oil burnout and it should also uniformly and slowly decrease at the exit of the furnace. Actually, the cost functional $\mathcal{J}(T)$ shows deviation of the real temperature field T from desired T_d (in L_2 sense).

Since radiative heat transfer is taken into account in our model, then the heat flux q has nonlocal and nonlinear dependence from temperature

$T_{sh} = (T|_{\Sigma_1}, T_h)$ on the interface Σ. In the explicit form it can be expressed by the following formula:

$$q = \mathbf{Q}(T_{sh}) = \mathbf{G}(|T_{sh}|^3 T_{sh}) + k_3(T_{sh} - T_g), \tag{1.3}$$

where T_g is temperature of air nearby Σ, k_3 is convective heat transfer coefficient (here it is assumed that k_3 is strictly positive constant, nevertheless the analysis can also be performed for the case, when $k_3 = 0$). Since Σ is diffuse grey surface in our case (see [BF99]), then $\mathbf{G} \in \mathscr{L}(L_{5/4}(\Sigma), L_{5/4}(\Sigma))$ and it will have the following representation formula:

$$\mathbf{G} = \sigma \mathbf{L}(\mathbf{I} - \mathbf{K})(\mathbf{I} - (\mathbf{I} - \mathbf{L})\mathbf{K})^{-1}\mathbf{L}$$

where σ is the Stefan-Boltzmann constant, $\mathbf{L} \in \mathscr{L}(L_{5/4}(\Sigma), L_{5/4}(\Sigma))$:

$$\mathbf{L}(v)(x) = \epsilon(x)v(x) \quad x \in \Sigma,$$

but $\mathbf{K} \in \mathscr{L}(L_{5/4}(\Sigma), L_{5/4}(\Sigma))$ represents irradiation on the interface Σ:

$$\begin{aligned}
\mathbf{K}(v)(x) &= \int_\Sigma k(x, y)\theta(x, y)v(y)\, d\lambda_\Sigma(y) && x \in \Sigma, \\
k(x, y) &= \frac{|\cos(\nu(x), (y - x))||\cos(\nu(y), (x - y))|}{\pi |x - y|^2} && x, y \in \Sigma, \\
\theta(x, y) &= \begin{cases} 1, & \text{if } x, y \text{ mutually see each other} \\ 0, & \text{otherwise} \end{cases} && x, y \in \Sigma.
\end{aligned}$$

The function ϵ characterizes emissivity on the interface Σ and for physical and mathematical reasons we assume that $c_1 \le \epsilon \le 1$ on Σ for some strictly positive constant c_1.

To overcome some mathematical problems that would rise to prove existence of T, in the formula (1.3) we write $|T_{sh}|^3 T_{sh}$ instead of physically more reasonable T_{sh}^4 (according to Stefan's law). As we will see later on, this replacement don't affect the final result in any way, because under appropriate mathematical conditions the function T_{sh} will be non-negative.

As we will see in Chapter 2, for every finite partition of Σ into λ_Σ-measurable and mutually disjoint subsets $\{A_i \subset \Sigma : i \in \{1, \dots, n\}, \Sigma = \bigcup_{i \in \{1, \dots, n\}} A_i\}$ the formula (1.3) can be rewritten in the matrix form:

$$\begin{pmatrix} q|_{A_1} \\ \vdots \\ q|_{A_n} \end{pmatrix} = \begin{pmatrix} \mathbf{Q}_{A_1 A_1} & \cdots & \mathbf{Q}_{A_n A_1} \\ \vdots & \ddots & \vdots \\ \mathbf{Q}_{A_1 A_n} & \cdots & \mathbf{Q}_{A_n A_n} \end{pmatrix} \begin{pmatrix} T_{sh}|_{A_1} \\ \vdots \\ T_{sh}|_{A_n} \end{pmatrix}, \tag{1.4}$$

where $\mathbf{Q}_{A_i A_j} : L_5(A_i) \mapsto L_{5/4}(A_j)$ $(i, j \in \{1, \dots, n\})$ are block components of the operator $\mathbf{Q}$.

Now, by putting $n = 2$, $A_1 = \Sigma_1$, $A_2 = \Sigma_2$ in the formula (1.4) and, hence, by expressing the flux $q|_{\Sigma_1}$ in an explicit form it becomes possible to rewrite the optimal control problem (1.2) in the new form:

$$\begin{cases} \mathcal{J}(T) \mapsto \min, & \\ & \\ \nabla \cdot (k_1 \nabla T) - k_2(\partial_1 T) = 0 & \text{in } \Omega_f, \\ -\nu \cdot \nabla(k_1 T) = \mathbf{Q}_{\Sigma_1\Sigma_1}(T) + \mathbf{Q}_{\Sigma_2\Sigma_1}(T_h) & \text{on } \Sigma_1, \\ -\nu \cdot \nabla(k_1 T) = 0 & \text{on } S_2, \\ T = T_e & \text{on } S_1, \\ T \in V_5(\Omega_f,\, \Sigma_1), & \\ T_h \in \{v \in L_\infty(\Sigma_2) : 0 \le v(x) \le \mu\}. & \end{cases} \tag{1.5}$$

As we will see later, every state T, satisfying the state equation of (1.5) for some admissible control T_h, can be represented in the form $T = u + T_*$, where $T_* \in V_5(\Omega_f,\, \Sigma_1)$ is extension of T_e into the domain Ω_f and u is solution of the variational equality:

$$\int_{\Omega_f} (k_1(\nabla(u + T_*) \cdot \nabla \eta) + k_2(\partial_1(u + T_*))\eta)\, d\omega_3 + \int_{\Sigma_1} \mathbf{Q}_{\Sigma_1\Sigma_1}(u + T_*)\eta\, d\lambda_{\Sigma_1} = -\int_{\Sigma_1} \mathbf{Q}_{\Sigma_2\Sigma_1}(T_h)\eta\, d\lambda_{\Sigma_1} \quad \forall \eta \in \dot{V}_5(\Omega_f,\, S_1,\, \Sigma_1) \tag{1.6}$$

(for definition of the spaces $V_5(\Omega_f,\, \Sigma_1)$, $\dot{V}_5(\Omega_f,\, S_1,\, \Sigma_1)$ see Notations).

Under certain mathematical assumptions the variational equality (1.6) has solutions in the function space $\dot{V}_5(\Omega_f,\, S_1,\, \Sigma_1)$. Existence of solutions for such type of variational equalities was already established by Tiihonen and Laitinen (see [LT00]). They proved coercivity and pseudomonotonity for an operator defined by a variational form and thereafter on basis of the Brezis' theorem came to a conclusion about existence of solutions. We solve this question in a slightly different manner. We prove that semilinear differential operator defined by left-hand side of (1.6) is coercive and can be represented as sum of a monotone and a weakly continuous operators. But then from general existence theorems (see [GGZ74]) it follows that there exists at least one solution u of (1.6).

As we will see later, natural constraints on T_g, T_e, T_h:

$$\begin{aligned} &0 \le T_g \le \mu && \text{on } \Sigma_1, \\ &0 \le T_e \le \mu && \text{on } S_1, \\ &0 \le T_h \le \mu && \text{on } \Sigma_2, \end{aligned}$$

and some specific properties of $\mathbf{Q}_{\Sigma_1\Sigma_1}$, $\mathbf{Q}_{\Sigma_2\Sigma_1}$ yield non-negativity and boundedness of $u + T_*$. Indeed, by using a technique introduced in [Bi051] and by bearing in mind that the variational equality (1.6) holds, it can be shown that:

$$0 \leq (u + T_*) \leq \mu \quad \text{a.e. in } \Omega_f. \tag{1.7}$$

The proof of (1.7) will be given in Chapter 3.

In continuation to analysis of (1.5) the next important issue to prove is uniqueness of the state T for every fixed control T_h and thus existence of a control-to-state operator $T = \mathbf{\Gamma}(T_h)$. Indeed, uniqueness of T can be established by putting special sample functions $\eta \in \dot{V}_5(\Omega_f,\, S_1,\, \Sigma_1)$ in the variational form (1.6) and, thereafter, by getting the following estimates for arbitrary state-control pairs $(T,\, T_h)$, $(\hat{T},\, \hat{T}_h)$:

$$\|T - \hat{T}\|_{L_\infty(\Omega_f)} \leq 4\mu^3 \|T_h - \hat{T}_h\|_{L_\infty(\Sigma_2)}, \tag{1.8}$$

$$\|T - \hat{T}\|_{L_\infty(\Sigma_1)} \leq 4\mu^3 \|T_h - \hat{T}_h\|_{L_\infty(\Sigma_2)}. \tag{1.9}$$

As it is easy to see, (1.8), (1.9) imply not only existence of $\mathbf{\Gamma}$, but also its Lipschitz continuity.

In conclusion of analysis of (1.5) we derive Fréchet differentiability of the cost functional $\mathcal{J}(\mathbf{\Gamma}(T_h))$:

$$\mathcal{J}(\mathbf{\Gamma}(\hat{T}_h)) - \mathcal{J}(\mathbf{\Gamma}(T_h)) = \ell[T_h](\hat{T}_h - T_h) + \mathbf{o}[T_h](\|\hat{T}_h - T_h\|_{L_\infty(\Sigma_2)}), \tag{1.10}$$

where $\ell[T_h] \in L^*_\infty(\Sigma_2)$, but $\hat{T}_h$, T_h are arbitrary admissible controls. This result is obtained by proving Fréchet differentiability of the operator $\mathbf{\Gamma}$ and, thereafter, by applying the chain rule on $\mathcal{J}(\mathbf{\Gamma}(T_h))$.

Nevertheless, although the problem (1.5) precisely models relationship between temperature T and T_h, its usage is inconvenient in practice. The main reason for this is the fact that geometry of Ω_f is strongly degenerated, since the fabric sheet is very thin. For example, sheet's thickness-width ratio is about 1/15000. Therefore, to handle this problem, some passage should be made from the original model (1.5) to a new model.

In Chapter 4 the asymptotic analysis is performed to investigate behaviour of T in the domain Ω_f and on the surface $\Sigma_0 = S_5 \cup S_6$, as $\delta \to 0$ (see Fig. 1.4 for more details about geometry of the problem). As it turns out, for small δ values T can be effectively approximated with some function $\tilde{T}$:

$$\frac{1}{\delta} \int_{\Omega_f} (T - \tilde{T})^2 \, d\omega_3 \to 0, \qquad \text{as } \delta \to 0, \tag{1.11}$$

$$\int_{\Sigma_0} (T - \tilde{T})^2 \, d\lambda_{\Sigma_0} \to 0, \qquad \text{as } \delta \to 0, \tag{1.12}$$

where $\tilde{T}$ is solution of the problem:

$$\begin{cases} \partial_3 \tilde{T} = 0 & \text{in } \Omega_f, \\ \mathbf{A}(\mathbf{Q}_{\Sigma_0\Sigma_0}(\tilde{T}) + \mathbf{Q}_{\Sigma_2\Sigma_0}(T_h)) = 0 & \text{on } \Sigma_0. \end{cases} \tag{1.13}$$

Here $\mathbf{A} \in \mathscr{L}(L_{5/4}(\Sigma_0),\, L_{5/4}(\Sigma_0))$ is defined as:

$$\mathbf{A}(v)(x_1,\, x_2,\, x_3) = \frac{v(x_1,\, x_2,\, \delta) + v(x_1,\, x_2,\, -\delta)}{2} \quad (x_1,\, x_2,\, x_3) \in \Sigma_0,$$

but $\mathbf{Q}_{\Sigma_0\Sigma_0} : L_5(\Sigma_0) \mapsto L_{5/4}(\Sigma_0)$, $\mathbf{Q}_{\Sigma_2\Sigma_0} : L_5(\Sigma_2) \mapsto L_{5/4}(\Sigma_0)$ are taken from (1.4) after putting $n = 3$, $A_1 = \Sigma_0$, $A_2 = \Sigma_1 \setminus \Sigma_0$, $A_3 = \Sigma_2$. Notice, that both T, $\tilde{T}$ are dependent from the parameter δ, but we will omit this dependence in our further notations.

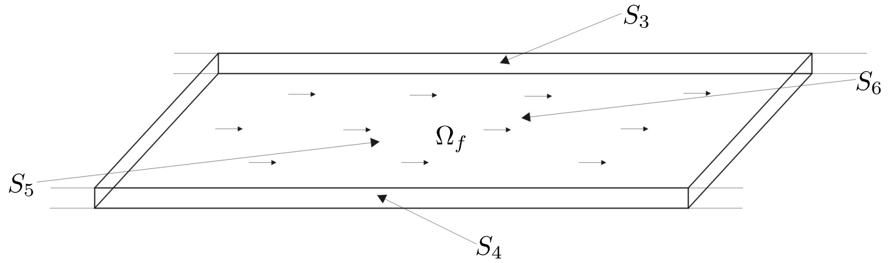

Figure 1.4: Glass fabric sheet

The previous result suggests that it would be reasonable to replace the problem (1.5) with the following one:

$$\begin{cases} \mathcal{J}(\tilde{T}) \mapsto \min, \\ \\ \mathbf{A}(\mathbf{Q}_{\Sigma_0\Sigma_0}(\tilde{T}) + \mathbf{Q}_{\Sigma_2\Sigma_0}(T_h)) = 0 \qquad \text{on } \Sigma_0, \\ \tilde{T} \in \{v \in L_5(\Omega_f) : \partial_3 v = 0\}, \\ T_h \in \{v \in L_\infty(\Sigma_2) : 0 \leq v(x) \leq \mu\}, \end{cases} \tag{1.14}$$

where $\tilde{T}$ is chosen as state function, but T_h - as control variable.

Similarly as it was in the case of (1.5), the analysis of the optimal control problem (1.14) we will start by proving existence the state $\tilde{T}$. Indeed, by using the contraction mapping theorem it can be shown that for every admissible control T_h the state equation of (1.14) can have one and only one solution $\tilde{T}$ in the functional space $\tilde{L}_\infty(\Omega_f)$ (for definition of the space $\tilde{L}_\infty(\Omega_f)$ see Notations), and, therefore, there exists an unique defined control-to-state operator $\tilde{T} = \tilde{\mathbf{\Gamma}}(T_h)$. As we will see in Chapter 4, $\tilde{\mathbf{\Gamma}}$ will be Lipschitz continuous and Fréchet differentiable as mapping between appropriate functional spaces. Furthermore, on basis of previously mentioned results we will be able to obtain an analogue of the formula (1.10):

$$\mathcal{J}(\tilde{\mathbf{\Gamma}}(\hat{T}_h)) - \mathcal{J}(\tilde{\mathbf{\Gamma}}(T_h)) = \tilde{\ell}[T_h](\hat{T}_h - T_h) + \mathbf{o}[T_h](\|\hat{T}_h - T_h\|_{L_\infty(\Sigma_2)}), \tag{1.15}$$

where $\tilde{\ell}[T_h] \in L^*_\infty(\Sigma_2)$, but $\hat{T}_h$, T_h are arbitrary admissible controls.

Chapter 2

Preliminaries

2.1 Geometry of the furnace

As it was set down in Chapter 1, let the furnace consists of heater system $\{\Omega_i \subset \mathbb{R}^3 : i \in \{1, \dots, m\}\}$ and let a thin fabric sheet $\Omega_f = (-l_1, l_1) \times (-l_2, l_2) \times (-\delta, \delta)$ is dragged through the furnace with a constant speed. Further the sides of Ω_f we will denote by

$$\begin{aligned} S_1 &:= \{x \in \partial\Omega_f : x_1 = l_1\}, & S_2 &:= \{x \in \partial\Omega_f : x_1 = -l_1\}, \\ S_3 &:= \{x \in \partial\Omega_f : x_2 = l_2\}, & S_4 &:= \{x \in \partial\Omega_f : x_2 = -l_2\}, \\ S_5 &:= \{x \in \partial\Omega_f : x_3 = \delta\}, & S_6 &:= \{x \in \partial\Omega_f : x_3 = -\delta\}. \end{aligned}$$

The surfaces Σ_0, Σ_1, Σ_2, Σ, mentioned in Chapter 1, we define as

$$\begin{aligned} \Sigma_0 &:= S_5 \cup S_6, \\ \Sigma_1 &:= S_3 \cup S_4 \cup S_5 \cup S_6, \\ \Sigma_2 &:= \bigcup_{i \in \{1, \dots, m\}} \partial\Omega_i, \\ \Sigma &:= \Sigma_1 \cup \Sigma_2, \end{aligned}$$

but overall volume in the furnace, filled with opaque medium, as

$$\Omega := \Omega_f \cup (\bigcup_{i \in \{1, \dots, m\}} \Omega_i).$$

Throughout this thesis we put the following assumptions on geometry of the furnace:

A1 $l_1 \in \mathbb{R}_+$, $l_2 \in \mathbb{R}_+$, $\delta \in \mathbb{R}_+$, $\delta \leq \min\{l_1, l_2\}$;

A2 For $\lambda_{S_1 \cup S_2}$-almost every $x \in S_1 \cup S_2$ and every $y \in \bigcup_{i \in \{1, \dots, m\}} \partial\Omega_i$ $\cos(\nu(x), (y - x)) \leq 0$.

Furthermore, we assume that there exist some $d \in \mathbb{R}_+$, $l \in \mathbb{R}_+$ such that:

A3 For every $i \in \{1, \ldots, m\}$ Ω_i is bounded Lipschitz domain with the constants d, l, i.e. for every $x_0 \in \partial\Omega_i$ there exist a transformation of coordinates $y = \Gamma[x_0](x - x_0)$ ($\Gamma[x_0]$ is orthogonal 3×3 matrix) and a Lipschitz continuous function $f[x_0] \in \mathrm{Lip}(\mathbb{R}^2)$ (with Lipschitz constant l) such that for transformed coordinates we have

$$\partial\Omega_i \cap C_{l,\,d} = \{(y_1,\, y_2,\, y_3) \in \mathbb{R}^3 : y_3 = f[x_0](y_1,\, y_2)\} \cap C_{l,\,d},$$
$$\Omega_i \cap C_{l,\,d} = \{(y_1,\, y_2,\, y_3) \in \mathbb{R}^3 : y_3 > f[x_0](y_1,\, y_2)\} \cap C_{l,\,d},$$

where

$$C_{l,\,d} = \{(y_1,\, y_2,\, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 \le d^2,\ -2ld \le y_3 \le 2ld\};$$

A4 For every $i \in \{1, \ldots, m\}$ $\mathrm{dist}(\Omega_f, \Omega_i) \ge d$;

A5 For every $i,\, j \in \{1, \ldots, m\}$, if $i \ne j$, then $\mathrm{dist}(\Omega_i, \Omega_j) \ge d$.

2.2 Further mathematical assumptions

We assume that the following conditions hold on quantities and functions occurring in (1.5), (1.6):

B1 $\mu \in [0, +\infty)$;

B2 $k_1 \in \mathbb{R}_+$, $k_2 \in [0, +\infty)$, $k_3 \in \mathbb{R}_+$;

B3 $T_e \in L_\infty(S_1)$, $0 \le T_e(x) \le \mu$ λ_{S_1}-a.e. on S_1;

B4 $T_h \in L_\infty(\Sigma_2)$, $0 \le T_h(x) \le \mu$ λ_{Σ_2}-a.e. on Σ_2;

B5 $T_g \in L_\infty(\Sigma)$, $0 \le T_g(x) \le \mu$ λ_{Σ_1}-a.e. on Σ_1;

B6 $T_* \in H_1(\Omega_f) \cap L_\infty(\Omega_f)$, $T_*(x) = T_e(x)$ λ_{S_1}-a.e. on S_1.

Notice, that the assumption B6 is easy to fulfill, if $T_e \in \mathrm{Lip}(S_1)$ and if we define the function T_* as

$$T_*(x_1,\, x_2,\, x_3) := T_e(x_2,\, x_3) \quad (x_1,\, x_2,\, x_3) \in \Omega_f.$$

Let

$$U = \{v \in L_\infty(\Sigma_2) : 0 \le v(x) \le \mu\ \lambda_{\Sigma_2}\text{-a.e. on } x \in \Sigma_2\}$$

be the set of admissible controls and let $T_{sh} = (T|_{\Sigma_1},\, T_h)$ denote temperature on the interface Σ.

2.3 Heat transfer on interface

As it was mentioned in Chapter 1 heat flux through the interface Σ exists due to radiative heat transfer and due to heat exchange with surrounding air.

In order to describe pure radiative heat transfer, we will adopt a mathematical model, proposed by Tiihonen and Laitinen in [LT00] for physical systems with entirely opaque and transparent components, separated by diffuse grey interfaces.

Let $\sigma \in \mathbb{R}_+$ be the Stefan-Boltzmann constant and let $\epsilon \in L_\infty(\Sigma)$ be the emissivity of the surface Σ. For mathematical and physical reasons we assume that

$$\exists c_1 \in (0,\, 1] : c_1 \leq \epsilon(x) \leq 1 \ \lambda_\Sigma\text{-a.e. on } x \in \Sigma. \tag{2.1}$$

In order to characterize intensity of radiative heat exchange between different interface points, let us introduce the function

$$k(x,\, y) := \begin{cases} \frac{|\cos(\nu(x),(y-x))||\cos(\nu(y),(x-y))|}{\pi|x-y|^2}, & \text{if } \exists\nu(x) \text{ and } \exists\nu(y) \\ 0, & \text{otherwise} \end{cases} \qquad x,\, y \in \Sigma$$

and the visibility factor

$$\theta(x,\, y) := \begin{cases} 1, & \text{if } \overline{xy} \cap \Omega = \emptyset \\ 0, & \text{otherwise} \end{cases} \qquad x,\, y \in \Sigma.$$

The visibility factor shows, if an arbitrary pair of interface points x, y mutually see each other and heat radiation can directly propagate between them. In proofs, what will follow, we will use the following properties of the visibility factor $\theta(x,\, y)$ and the function $k(x,\, y)\theta(x,\, y)$.

Lemma 2.3.1
If $x,\, y \in \Sigma$ and if there exists $\nu(x)$ and $\cos(\nu(x),\, (y-x)) < 0$, then $\theta(x,\, y) = 0$.

Lemma 2.3.2

i. $0 \leq \int_\Sigma k(x,\, y)\theta(x,\, y)\, d\lambda_\Sigma(y) \leq 1$ *for every* $x \in \Sigma_2$;

ii. $\exists c_2 \in [0,\, 1) : 0 \leq \int_\Sigma k(x,\, y)\theta(x,\, y)\, d\lambda_\Sigma(y) \leq c_2$ *for every* $x \in \Sigma_1$.

iii. $k(x,\, y)\theta(x,\, y) = k(y,\, x)\theta(x,\, y)$ *for all* $x \in \Sigma$, $y \in \Sigma$;

iv. $k(x,\, y)\theta(x,\, y) \geq 0$ *for all* $x \in \Sigma$, $y \in \Sigma$.

◀ *Proof of 2.3.2iii and 2.3.2iv* is straightforward.

Proof of 2.3.2i. Let us fix arbitrary $x \in \Sigma_2$. The inequality $0 \leq \int_\Sigma k(x,\, y)\theta(x,\, y)\, d\lambda_\Sigma(y)$ is direct consequence of 2.3.2iv, if only there exists an integral $\int_\Sigma k(x,\, y)\theta(x,\, y)\, d\lambda_\Sigma(y)$.

If $\nu(x)$ does not exist, then proof of the inequality $\int_\Sigma k(x,\, y)\theta(x,\, y)\, d\lambda_\Sigma(y) \leq 1$ is straightforward, since $k(x,\, y)\theta(x,\, y) = 0$ for every $y \in \Sigma$.

Let us prove the inequality $\int_\Sigma k(x, y)\, d\lambda_\Sigma(y) \leq 1$ for a nontrivial case, when $\nu(x)$ exists. For this reason we define sets:

$$\begin{aligned} &A_{x,\tau} := \Omega \setminus \overline{B(x,\, \tau)}, \\ &B_{x,\tau} := \{z \in \mathbb{R}^3 : z = x + t(y - x), y \in A_{x,\tau}, t \in [1,\, +\infty)\} \cap B(x,\, d_{max}), \end{aligned}$$

where $\tau \in (0,\, d_{min}]$, $d_{min} = \min\{d,\, 2ld\}$ and $d_{max} = \operatorname{diam}(\Omega)$. Notice, that $B_{x,\tau}$ is nonempty for every $\tau \in (0,\, d_{min}]$, therefore, we can define functions $f_i : B_{x,\tau} \mapsto \mathbb{R}$ ($i \in \{1,\, 2,\, 3\}$):

$$f_i(y) := \begin{cases} \frac{\cos(\nu(x),\, (y-x))}{\pi|y-x|^3}(y_i - x_i), & \text{if } \cos(\nu(x),\, (y - x)) \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Since $f_1 \in C^1(\overline{B}_{x,\tau})$, $f_2 \in C^1(\overline{B}_{x,\tau})$, $f_3 \in C^1(\overline{B}_{x,\tau})$ and $B_{x,\tau}$ can be represented as union of $m + 1$ Lipschitz domains, then the Gauss's formula holds:

$$\int_{B_{x,\tau}} \sum_{i=1}^{3} (\partial_i f_i(y))\, d\omega_3(y) = \int_{\partial B_{x,\tau}} \sum_{i=1}^{3} \nu_i(y) f_i(y)\, d\lambda_{\partial B_{x,\tau}}(y). \tag{2.2}$$

Now, let us define:

$$S_+(x,\, \tau) := \{z \in S(x,\, \tau) : \cos(\nu(x),\, (z - x)) \geq 0\},$$

where $\tau \in \mathbb{R}_+$. It is easy to verify that the following equalities hold:

$$\begin{aligned} &\sum_{i=1}^{3} \nu_i(y) f_i(y) = \frac{\cos(\nu(x),\, (x - y))}{\pi\tau^2} \quad \text{for } \lambda_{\partial B_{x,\tau}}\text{-a.e. } y \in \partial B_{x,\tau} \cap S_+(x,\, \tau), \\ &\sum_{i=1}^{3} \nu_i(y) f_i(y) = \frac{\cos(\nu(x),\, (y - x))}{\pi d_{max}^2} \quad \text{for } \lambda_{\partial B_{x,\tau}}\text{-a.e. } y \in \partial B_{x,\tau} \cap S_+(x,\, d_{max}), \\ &\sum_{i=1}^{3} \nu_i(y) f_i(y) = k(x,\, y)\theta(x,\, y) \quad \text{for } \lambda_{\partial B_{x,\tau}}\text{-a.e. } y \in \partial B_{x,\tau} \cap \Sigma, \\ &\sum_{i=1}^{3} \nu_i(y) f_i(y) = 0 \quad \text{for } \lambda_{\partial B_{x,\tau}}\text{-a.e. } y \in \partial B_{x,\tau} \setminus (S_+(x,\, \tau) \cup S_+(x,\, d_{max}) \cup \Sigma), \\ &\sum_{i=1}^{3} (\partial_i f_i(y)) = 0 \quad \text{for a.e. } y \in B_{x,\tau} \end{aligned}$$

and, therefore, (2.2) will give

$$
\begin{aligned}
\int_{\partial B_{x,\tau}\cap\Sigma} & k(x,\,y)\theta(x,\,y)\,d\lambda_{\partial B_{x,\tau}}(y) \\
&= \int_{\partial B_{x,\tau}\cap S_+(x,\,d_{max})} \frac{\cos(\nu(x),\,(y-x))}{\pi d_{max}^2}\,d\lambda_{\partial B_{x,\tau}}(y) \\
&\quad - \int_{\partial B_{x,\tau}\cap S_+(x,\,\tau)} \frac{\cos(\nu(x),\,(y-x))}{\pi\tau^2}\,d\lambda_{\partial B_{x,\tau}}(y) \\
&\quad \le \int_{S_+(x,\,d_{max})} \frac{\cos(\nu(x),\,(y-x))}{\pi d_{max}^2}\,d\lambda_{S(x,\,d_{max})}(y) = 1. \qquad (2.3)
\end{aligned}
$$

The family of sets $\{\partial B_{x,\tau}\cap\Sigma : \tau\in(0,\,d_{min}]\}$ is monotone and increasing, as the parameter τ decreases. Moreover, Lemma 2.3.1 guaranties that $\{y\in\Sigma : k(x,\,y)\theta(x,\,y)\neq 0\}\subset\bigcup_{\tau\in(0,\,d_{min}]}(\partial B_{x,\tau}\cap\Sigma)$, therefore, 2.3.2iv together with (2.3) will give:

$$
\int_\Sigma k(x,\,y)\theta(x,\,y)\,d\lambda_\Sigma(y) \le \int_{\bigcup_{\tau\in(0,\,d_{min}]}(\partial B_{x,\tau}\cap\Sigma)} k(x,\,y)\theta(x,\,y)\,d\lambda_\Sigma(y) \le 1.
$$

Proof of 2.3.2ii is almost identical with the proof of 2.3.2i. It slightly differs in case, when $x\in\Sigma_1$ and there exists $\nu(x)$. As we will see, in this case it is possible to get "better" estimate than (2.3). Indeed, if we denote

$$
C_x := \{z\in S_+(x,\,d_{max}) : (z_2-x_2)^2+(z_3-x_3)^2\ge d^2\},
$$

then due to A1, A2, A4 we will have $B_{x,\tau}\cap S_+(x,\,d_{max})\subset C_x$ for every $\tau\in(0,\,d_{min}]$. Therefore

$$
\begin{aligned}
\int_{\partial B_{x,\tau}\cap\Sigma} & k(x,\,y)\theta(x,\,y)\,d\lambda_{\partial B_{x,\tau}}(y) \\
&= \int_{\partial B_{x,\tau}\cap S_+(x,\,d_{max})} \frac{\cos(\nu(x),\,(y-x))}{\pi d_{max}^2}\,d\lambda_{\partial B_{x,\tau}}(y) \\
&\quad - \int_{\partial B_{x,\tau}\cap S_+(x,\,\tau)} \frac{\cos(\nu(x),\,(y-x))}{\pi\tau^2}\,d\lambda_{\partial B_{x,\tau}}(y) \\
&\qquad \le \int_{C_x} \frac{\cos(\nu(x),\,(y-x))}{\pi d_{max}^2}\,d\lambda_{S(x,\,d_{max})}(y) = c_2,
\end{aligned}
$$

where $c_2\in[0,\,1)$ does not depend on x and τ. The last estimate afterwards will yield

$$
\int_\Sigma k(x,\,y)\theta(x,\,y)\,d\lambda_\Sigma(y)\le c_2.
$$

►

According to proposed radiative heat transfer model (see [LT00] for more details), the function $k(x,\, y)\theta(x,\, y)$ can be used to calculate irradiation from the incoming radiation ρ:

$$\mathbf{K}(\rho)(x) := \int_{\Sigma} k(x,\, y)\theta(x,\, y)\rho(y)\, d\lambda_{\Sigma}(y) \quad x \in \Sigma.$$

Moreover, the ρ and T_{sh} on the interface Σ will be related in the following way:

$$(\mathbf{I} - (\mathbf{I} - \mathbf{L})\mathbf{K})(\rho) = \mathbf{L}(\sigma |T_{sh}|^3 T_{sh}), \tag{2.4}$$

where

$$\mathbf{L}(v)(x) := \epsilon(x)v(x) \quad x \in \Sigma.$$

This way, by taking into account radiative and convective heat transfer processes on the interface Σ, the overall heat flux q through Σ can be expressed by the following formula:

$$q = (\mathbf{I} - \mathbf{K})(\rho) + k_3(T_{sh} - T_g). \tag{2.5}$$

Now, let us enumerate some useful properties of the operators $\mathbf{K}$, $\mathbf{L}$.

Lemma 2.3.3
For every fixed $p \in [1,\, +\infty]$:

i. $\mathbf{K}$ *is linear and bounded from* $L_p(\Sigma)$ *to* $L_p(\Sigma)$ *and* $\|\mathbf{K}\|_{\mathscr{L}(L_p(\Sigma),\, L_p(\Sigma))} \leq 1$;

ii. $\langle \mathbf{K}(v),\, w\rangle = \langle v,\, \mathbf{K}(w)\rangle$ *for every* $v \in L_p(\Sigma)$, $w \in L_{p/(p-1)}(\Sigma)$;

iii. if $v \in L_p(\Sigma)$ *and* $v \geq 0$ λ_Σ-a.e. *on* Σ, *then* $\mathbf{K}(v) \geq 0$ λ_Σ-a.e. *on* Σ.

◀ The assertion 2.3.3i will follow due to estimate 2.3.2i (see [DS57] for more details). Similarly, the assertions 2.3.3ii, 2.3.3iii will follow from 2.3.2iii, 2.3.2iv, respectively.

▶

Lemma 2.3.4
For every fixed $p \in [1,\, +\infty]$:

i. $\mathbf{L}$ *is linear and bounded from* $L_p(\Sigma)$ *to* $L_p(\Sigma)$;

ii. $\|\mathbf{I} - \mathbf{L}\|_{\mathscr{L}(L_p(\Sigma),\, L_p(\Sigma))} = (1 - c_1) < 1$;

iii. if $v \in L_p(\Sigma)$ *and* $v \geq 0$ λ_Σ-a.e. *on* Σ, *then* $(\mathbf{I} - \mathbf{L})(v) \geq 0$ λ_Σ-a.e. *on* Σ.

Since 2.3.3i, 2.3.4ii, 2.3.3iii and 2.3.4iii hold, then there exists the inverse operator $(\mathbf{I} - (\mathbf{I} - \mathbf{L})\mathbf{K})^{-1}$.

Lemma 2.3.5
For every fixed $p \in [1, +\infty]$ there exists

$$(\mathbf{I} - (\mathbf{I} - \mathbf{L})\mathbf{K})^{-1} = \sum_{n=0}^{\infty} ((\mathbf{I} - \mathbf{L})\mathbf{K})^n$$

as linear and bounded operator from $L_p(\Sigma)$ to $L_p(\Sigma)$. Furthermore, if $v \in L_p(\Sigma)$ and $v \geq 0$ λ_Σ-a.e. on Σ, then $(\mathbf{I} - (\mathbf{I} - \mathbf{L})\mathbf{K})^{-1}(v) \geq 0$ λ_Σ-a.e. on Σ.

Now, by applying the operator $(\mathbf{I} - (\mathbf{I} - \mathbf{L})\mathbf{K})^{-1}$ on both sides of (2.4) we will get

$$\rho = (\mathbf{I} - (\mathbf{I} - \mathbf{L})\mathbf{K})^{-1}\mathbf{L}(\sigma|T_{sh}|^3 T_{sh}) = \sum_{n=0}^{\infty} ((\mathbf{I} - \mathbf{L})\mathbf{K})^n \mathbf{L}(\sigma|T_{sh}|^3 T_{sh}). \quad (2.6)$$

Next, for convenience let us define:

$$\mathbf{G} := \sigma\mathbf{L} - \sigma\mathbf{L}\mathbf{K}(\mathbf{I} - (\mathbf{I} - \mathbf{L})\mathbf{K})^{-1}\mathbf{L},$$
$$\mathbf{H} := \mathbf{I} - \mathbf{L} + \mathbf{L}\mathbf{K}(\mathbf{I} - (\mathbf{I} - \mathbf{L})\mathbf{K})^{-1}\mathbf{L}.$$

Basic properties of these operators are given by the following lemma.

Lemma 2.3.6
For every fixed $p \in [1, +\infty]$:

i. $\mathbf{G}$, $\mathbf{H}$ *are linear and bounded from $L_p(\Sigma)$ to $L_p(\Sigma)$;*

ii. $\mathbf{G} = \sigma(\mathbf{I} - \mathbf{H})$;

iii. if $v \in L_p(\Sigma)$ and $v \geq 0$ λ_Σ-a.e. on Σ, then $\mathbf{H}(v) \geq 0$ λ_Σ-a.e. on Σ.

iv. $\langle \mathbf{G}(v), w\rangle = \langle v, \mathbf{G}(w)\rangle$, $\langle \mathbf{H}(v), w\rangle = \langle v, \mathbf{H}(w)\rangle$ *for every $v \in L_p(\Sigma)$, $w \in L_{p/(p-1)}(\Sigma)$;*

v. $\mathbf{G}(1) \geq 0$ *λ_Σ-a.e. on Σ;*

vi. $\exists c_3 \in (0, 1] : \mathbf{G}(1) \geq \sigma c_3$ *λ_{Σ_1}-a.e. on Σ_1;*

vii. $\|\mathbf{H}\|_{\mathscr{L}(L_p(\Sigma), L_p(\Sigma))} \leq 1$.

◄ The assertions 2.3.6i-2.3.6v, 2.3.6vii can be proved by using results of Lemma 2.3.3, Lemma 2.3.4 and Lemma 2.3.5 (see [LT98], [LT00]). The assertion 2.3.6vi can be proved in a similar way as 2.3.6v, however, in this case the proof will be based on the estimate 2.3.2ii and 2.3.6iv.

►

Now, if we define

$$\mathbf{Q}(T_{sh}) := (\mathbf{I} - \mathbf{K})(\mathbf{I} - (\mathbf{I} - \mathbf{L})\mathbf{K})^{-1}\mathbf{L}(\sigma|T_{sh}|^3 T_{sh}) + k_3(T_{sh} - T_g),$$

then (2.5) together with (2.6) will yield the formula (1.3):

$$q = \mathbf{Q}(T_{sh}) = \mathbf{G}(|T_{sh}|^3 T_{sh}) + k_3(T_{sh} - T_g).$$

Next, in order to split the operator $\mathbf{Q}$ into "blocks" and to derive the formula (1.4), let us introduce some auxiliary operators. Let A is an arbitrary λ_Σ-measurable subset of Σ and let us define a restriction type operator on $\mathcal{M}(\Sigma,\, \lambda_\Sigma)$:

$$\mathbf{M}_A(v) := v|_A.$$

Let us also define an extension type operator on $\mathcal{M}(A,\, \lambda_A)$:

$$\mathbf{N}_A(v) := \begin{cases} v & \text{on } A \\ 0 & \text{on } \Sigma \setminus A. \end{cases}$$

Then for every $p \in [1,\, +\infty]$

$$\mathbf{M}_A \in \mathscr{L}(L_p(\Sigma),\, L_p(A)),\ \|\mathbf{M}_A\|_{\mathscr{L}(L_p(\Sigma),\, L_p(A))} = 1,$$
$$\mathbf{N}_A \in \mathscr{L}(L_p(A),\, L_p(\Sigma)),\ \|\mathbf{N}_A\|_{\mathscr{L}(L_p(A),\, L_p(\Sigma))} = 1.$$

Now, one can easily check that for every finite partition of Σ into λ_Σ-measurable and mutually disjoint subsets $\{A_i \subset \Sigma : i \in \{1, \dots, n\},\ \Sigma = \bigcup_{i\in\{1,\dots,n\}} A_i\}$ the formula (1.4) will hold

$$\begin{pmatrix} q|_{A_1} \\ \vdots \\ q|_{A_n} \end{pmatrix} = \begin{pmatrix} \mathbf{Q}_{A_1A_1} & \cdots & \mathbf{Q}_{A_nA_1} \\ \vdots & \ddots & \vdots \\ \mathbf{Q}_{A_1A_n} & \cdots & \mathbf{Q}_{A_nA_n} \end{pmatrix} \begin{pmatrix} T_{sh}|_{A_1} \\ \vdots \\ T_{sh}|_{A_n} \end{pmatrix},$$

if we will put

$$\mathbf{Q}_{A_iA_j} = \mathbf{M}_{A_j} \circ \mathbf{Q} \circ \mathbf{N}_{A_i} \quad i,\, j \in \{1, \dots, n\}.$$

Moreover, if we now define for arbitrary λ_Σ-measurable sets $A \subset \Sigma$, $B \subset \Sigma$ that

$$\mathbf{Q}_{AB} := \mathbf{M}_B \circ \mathbf{Q} \circ \mathbf{N}_A, \quad \mathbf{G}_{AB} := \mathbf{M}_B \mathbf{G} \mathbf{N}_A, \quad \mathbf{H}_{AB} := \mathbf{M}_B \mathbf{H} \mathbf{N}_A,$$

then these operators will have the following properties.

Lemma 2.3.7
For every fixed $p \in [1,\, +\infty]$ and arbitrary chosen λ_Σ-measurable sets $A \subset \Sigma$, $B \subset \Sigma$:

i. $\mathbf{Q}_{AB}$ maps $L_{4p}(A)$ to $L_p(B)$, but $\mathbf{G}_{AB}$, $\mathbf{H}_{AB}$ are linear and bounded from $L_p(A)$ to $L_p(B)$;

ii. if $v \in L_{4p}(A)$, then $\mathbf{Q}_{AA}(v) = \mathbf{G}_{AA}(|v|^3 v) + (v - T_g|_A)$;

iii. $\mathbf{G}_{AA} = \sigma(\mathbf{I} - \mathbf{H}_{AA})$;

iv. *if* $A \cap B = \emptyset$ *and* $v \in L_{4p}(A)$, *then* $\mathbf{Q}_{AB}(v) = \mathbf{G}_{AB}(|v|^3 v)$;

v. *if* $A \cap B = \emptyset$, *then* $\mathbf{G}_{AB} = -\sigma \mathbf{H}_{AB}$;

vi. *if* $v \in L_p(A)$ *and* $v \geq 0$ λ_A-a.e. *on* A, *then* $\mathbf{H}_{AB}(v) \geq 0$ λ_B-a.e. *on* B;

vii. $\langle \mathbf{G}_{AA}(v), w \rangle = \langle v, \mathbf{G}_{AA}(w) \rangle$, $\langle \mathbf{H}_{AA}(v), w \rangle = \langle v, \mathbf{H}_{AA}(w) \rangle$ *for every* $v \in L_p(A)$, $w \in L_{p/(p-1)}(A)$;

viii. *if* $A \cap B = \emptyset$ *and* $A \cup B = \Sigma$, *then* $\mathbf{G}_{AA}(1) + \mathbf{G}_{BA}(1) \geq 0$ λ_A-a.e. *on* A;

ix. $\|\mathbf{H}_{AA}\|_{\mathscr{L}(L_p(A),\, L_p(A))} \leq (1 - c_3) < 1$;

x. $\|\mathbf{H}_{AB}\|_{\mathscr{L}(L_p(A),\, L_p(B))} \leq 1$.

◀ The assertions 2.3.7i-2.3.7viii, 2.3.7x directly follow from Lemma 2.3.6, the formula (1.3) and properties of the operators $\mathbf{M}_B$, $\mathbf{N}_A$. The assertion 2.3.7ix can be proved in a similar way as 2.3.6vii. At the first step by using the estimate 2.3.6vi and 2.3.7vi, 2.3.7vii we can show that

$$\|\mathbf{H}_{AA}\|_{\mathscr{L}(L_1(A),\, L_1(A))} \leq (1 - c_3),$$
$$\|\mathbf{H}_{AA}\|_{\mathscr{L}(L_\infty(A),\, L_\infty(A))} \leq (1 - c_3).$$

Then from the Riesz interpolation theorem it will follow that for every $p \in (1, +\infty)$

$$\|\mathbf{H}_{AA}\|_{\mathscr{L}(L_p(A),\, L_p(A))} \leq (1 - c_3)$$

(see [LT98], [LT00]).

▶

2.4 Preliminaries for reduced problem

Let us define

$$Q_f := (-l_1, l_1) \times (-l_2, l_2).$$

In Chapter 4 we will investigate asymptotic behaviour of T, as $\delta \to 0$. Some proofs there will be based on technique, when functions of class L_∞ are made smooth and differentiable via applying some mollifying operator.

Therefore, for arbitrary $\tau \in \mathbb{R}_+$ and $v \in L_\infty(Q_f)$ let us define

$$\mathbf{T}_\tau(v)(x) := \int_{Q_f} \varsigma_\tau(x, y) v(y)\, d\omega_2 \quad x \in Q_f,$$

where

$$\varsigma_\tau(x, y) := \max\{\frac{3}{\pi \tau^2}(1 - \frac{|x - y|}{\tau}), 0\} \quad x, y \in Q_f.$$

As it turns out, $\mathbf{T}_\tau$ can be used as mollifier of functions of the class $L_\infty(Q_f)$. Indeed, it is easy to verify that images of this operator will have the following properties.

Lemma 2.4.1
If $v \in L_\infty(Q_f)$ and $0 \le v(x) \le \hat{c}_1$ a.e. in Q_f for some $\hat{c}_1 \in [0, +\infty)$, then for every $\tau \in \mathbb{R}_+$:

i. $\mathbf{T}_\tau(v) \in C^1(Q_f)$;

ii. $0 \le \mathbf{T}_\tau(v)(x) \le \hat{c}_1$ *a.e. in* Q_f;

iii. $|\nabla \mathbf{T}_\tau(v)(x)|^2 \le 18(\hat{c}_1/\tau)^2$ *a.e. in* Q_f.

Furthermore, $\mathbf{T}_\tau(v) \to v$ *in* $L_2(Q_f)$*, as* $\tau \to 0$.

Chapter 3

Analysis of original problem

3.1 Existence of the state T

Let us start sensitivity analysis of the optimal control problem (1.5) by rewriting the variational equality (1.6) in operator form. For this reason let us define functionals:

$$\mathbf{f}_1(v,\, \eta) := \int_{\Omega_f} (k_1(\nabla v \cdot \nabla \eta) + k_2(\partial_1 v)\eta)\, d\omega_3 + \int_{\Sigma_1} k_3 v\eta\, d\lambda_{\Sigma_1}$$
$$v,\, \eta \in \dot{V}_5(\Omega_f,\, S_1,\, \Sigma_1),$$

$$\mathbf{f}_2(v,\, \eta) := \int_{\Sigma_1} \mathbf{G}_{\Sigma_1\Sigma_1}(|v + T_*|^3(v + T_*))\eta\, d\lambda_{\Sigma_1}$$
$$v,\, \eta \in \dot{V}_5(\Omega_f,\, S_1,\, \Sigma_1),$$

and

$$\begin{aligned}\mathbf{f}_3(w,\, \eta) := &- \int_{\Sigma_1} \mathbf{G}_{\Sigma_2\Sigma_1}(|w|^3 w)\eta\, d\lambda_{\Sigma_1} + \int_{\Sigma_1} k_3 T_g \eta\, d\lambda_{\Sigma_1} \\ &- \int_{\Omega_f} (k_1(\nabla T_* \cdot \nabla \eta) + k_2(\partial_1 T_*)\eta)\, d\omega_3 - \int_{\Sigma_1} k_3 T_* \eta\, d\lambda_{\Sigma_1}\end{aligned}$$
$$w \in L_\infty(\Sigma_2),\, \eta \in \dot{V}_5(\Omega_f,\, S_1,\, \Sigma_1).$$

By using embedding theorems for $H_1(\Omega_f,\, S_1)$ and by taking into account properties of the functions T_g, T_h, T_* and the operators $\mathbf{G}_{\Sigma_1\Sigma_1}$, $\mathbf{G}_{\Sigma_2\Sigma_1}$, one can get that $\mathbf{f}_1(v,\, \cdot\,) \in \dot{V}_5^*(\Omega_f,\, S_1,\, \Sigma_1)$, $\mathbf{f}_2(v,\, \cdot\,) \in \dot{V}_5^*(\Omega_f,\, S_1,\, \Sigma_1)$, $\mathbf{f}_3(w,\, \cdot\,) \in \dot{V}_5^*(\Omega_f,\, S_1,\, \Sigma_1)$ for every fixed $v \in \dot{V}_5(\Omega_f,\, S_1,\, \Sigma_1)$, $w \in L_\infty(\Sigma_2)$. Therefore, if we define the following operators:

$$\mathbf{B}_1(v) := \mathbf{f}_1(v,\, \cdot), \quad \mathbf{B}_2(v) := \mathbf{f}_2(v,\, \cdot\,), \quad \mathbf{B}_3(w) := \mathbf{f}_3(w,\, \cdot\,),$$

then

$$\mathbf{B}_1 : \dot{V}_5(\Omega_f,\, S_1,\, \Sigma_1) \mapsto \dot{V}_5^*(\Omega_f,\, S_1,\, \Sigma_1),$$
$$\mathbf{B}_2 : \dot{V}_5(\Omega_f,\, S_1,\, \Sigma_1) \mapsto \dot{V}_5^*(\Omega_f,\, S_1,\, \Sigma_1),$$
$$\mathbf{B}_3 : L_\infty(\Sigma_2) \mapsto \dot{V}_5^*(\Omega_f,\, S_1,\, \Sigma_1).$$

Now, by using the introduced operators, we can rewrite the variational equality (1.6) in the following form:

$$\mathbf{B}_1(u) + \mathbf{B}_2(u) = \mathbf{B}_3(T_h). \tag{3.1}$$

Let us prove existence of solutions for this equation by using methodology from [GGZ74].

Lemma 3.1.1
The operator $\mathbf{B}_1$ is monotone, radial-continuous (see [GGZ74] for definitions) and there exists a constant $c_4 \in \mathbb{R}_+$ such that

$$c_4 \|v\|^2_{H_1(\Omega_f,\, S_1)} \le \mathbf{f}_1(v,\, v) \quad \forall v \in H_1(\Omega_f,\, S_1). \tag{3.2}$$

◀ Let us fix arbitrary $v \in H_1(\Omega_f,\, S_1)$. Then due to properties of the constants $k_1,\, k_2,\, k_3$:

$$\begin{aligned}\mathbf{f}_1(v,\, v) := \int_{\Omega_f} (k_1(\nabla v \cdot \nabla v) + k_2(\partial_1 v)v)\, d\omega_3 + \int_{\Sigma_1} k_3 v^2\, d\lambda_{\Sigma_1} \\ \ge \int_{\Omega_f} k_1(\nabla v \cdot \nabla v)\, d\omega_3 + \int_{\Omega_f} k_2(\partial_1 v) v\, d\omega_3 \\ = \int_{\Omega_f} k_1(\nabla v \cdot \nabla v)\, d\omega_3 + \int_{S_2} \frac{k_2 v^2}{2}\, d\lambda_{S_2} \\ \ge \int_{\Omega_f} k_1(\nabla v \cdot \nabla v)\, d\omega_3.\end{aligned} \tag{3.3}$$

As $v \in H_1(\Omega_f,\, S_1)$, then

$$\hat{c}_1 \|v\|^2_{L_2(\Omega_f)} \le \int_{\Omega_f} (\nabla v \cdot \nabla v)\, d\omega_3$$

for some constant $\hat{c}_1 \in \mathbb{R}_+$ such that it does not depend on chosen v (see [LU68]). Therefore, if we take $c_4 = k_1 \min\{1,\, \hat{c}_1\}/2$, then (3.3) will give

$$c_4 \|v\|^2_{H_1(\Omega_f,\, S_1)} \le \mathbf{f}_1(v,\, v).$$

Now, as $\dot{V}_5(\Omega_f,\, S_1,\, \Sigma_1) \subset H_1(\Omega_f,\, S_1)$ and as (3.2) holds, then $\mathbf{B}_1$ will be radial-continuous and monotone due to its linearity (see [GGZ74]).

▶

Lemma 3.1.2
The operator $\mathbf{B}_2$ *is weakly continuous (see [GGZ74] for definition).*

◀ Let us fix arbitrary sequence $\{v_n\}_{n\in\mathbb{N}} \subset \dot{V}_5(\Omega_f,\, S_1,\, \Sigma_1)$ such that there exists some $v_* \in \dot{V}_5(\Omega_f,\, S_1,\, \Sigma_1)$ that $v_n \rightharpoonup v_*$ in $\dot{V}_5(\Omega_f,\, S_1,\, \Sigma_1)$, as $n \to +\infty$.

As $\dot{V}_5(\Omega_f,\, S_1,\, \Sigma_1)$ is embedded both in $L_5(\Sigma_1)$ and in $H_1(\Omega_f,\, S_1)$, then $v_n \rightharpoonup v_*$ in $L_5(\Sigma_1)$ and also in $H_1(\Omega_f,\, S_1)$, as $n \to +\infty$. But then there must exist some constants $\hat{c}_1 \in \mathbb{R}_+$, $\hat{c}_2 \in \mathbb{R}_+$ such that:

$$\|v_n\|_{L_5(\Sigma_1)} \leq \hat{c}_1,\; \|v_n\|_{H_1(\Omega_f,\, S_1)} \leq \hat{c}_2 \quad \forall n \in \mathbb{N}. \tag{3.4}$$

Now, let us look at the sequence $\{|v_n+T_*|^3(v_n+T_*)\}_{n\in\mathbb{N}}$ and let us prove that $|v_n+T_*|^3(v_n+T_*) \rightharpoonup |v_*+T_*|^3(v_*+T_*)$ in $L_{5/4}(\Sigma_1)$, as $n \to +\infty$.

Since (3.4) holds, then there exists another constant $\hat{c}_3 \in \mathbb{R}_+$ such that

$$\||v_n+T_*|^3(v_n+T_*)\|_{L_{5/4}(\Sigma_1)} \leq \hat{c}_3 \quad \forall n \in \mathbb{N}.$$

Therefore, as $L_{5/4}(\Sigma_1)$ is reflexive, then there must exist at least one $L_{5/4}(\Sigma_1)$-weakly convergent subsequence of $\{|v_n+T_*|^3(v_n+T_*)\}_{n\in\mathbb{N}}$.

If we want to show that $|v_n+T_*|^3(v_n+T_*) \rightharpoonup |v_*+T_*|^3(v_*+T_*)$ in $L_{5/4}(\Sigma_1)$, as $n \to +\infty$, it is sufficientlly to prove that for every $L_{5/4}(\Sigma_1)$-weakly convergent subsequence $\{|v_{n_k}+T_*|^3(v_{n_k}+T_*)\}_{k\in\mathbb{N}}$ there will be $|v_{n_k}+T_*|^3(v_{n_k}+T_*) \rightharpoonup |v_*+T_*|^3(v_*+T_*)$ in $L_{5/4}(\Sigma_1)$, as $k \to +\infty$.

Let us fix arbitrary $L_{5/4}(\Sigma_1)$-weakly convergent subsequence $\{|v_{n_k}+T_*|^3(v_{n_k}+T_*)\}_{k\in\mathbb{N}}$, where $|v_{n_k}+T_*|^3(v_{n_k}+T_*) \rightharpoonup v_{**} \in L_{5/4}(\Sigma_1)$, as $k \to +\infty$. As embedding of $H_1(\Omega_f,\, S_1)$ into $L_2(\Sigma_1)$ is completely continuous, then $\|v_{n_k}-v_*\|_{L_2(\Sigma_1)} \to 0$, as $k \to +\infty$. But then without loss of generality it is reasonable to assume that $v_{n_k} \to v_*$ λ_{Σ_1}-a.e. on Σ_1 and thus $|v_{n_k}+T_*|^3(v_{n_k}+T_*) \to |v_*+T_*|^3(v_*+T_*)$ λ_{Σ_1}-a.e. on Σ_1, as $k \to +\infty$. Since simultaneously we have that $|v_{n_k}+T_*|^3(v_{n_k}+T_*) \rightharpoonup v_{**}$, as $k \to +\infty$, then there must be $|v_*+T_*|^3(v_*+T_*) = v_{**}$ (see [GGZ74], for instance).

Now, as both $\mathbf{G}_{\Sigma_1\Sigma_1} \in \mathscr{L}(L_{5/4}(\Sigma_1),\, L_{5/4}(\Sigma_1))$ and $|v_n+T_*|^3(v_n+T_*) \rightharpoonup |v_*+T_*|^3(v_*+T_*)$ in $L_{5/4}(\Sigma_1)$, as $n \to +\infty$, then also $\mathbf{G}_{\Sigma_1\Sigma_1}(|v_n+T_*|^3(v_n+T_*)) \rightharpoonup \mathbf{G}_{\Sigma_1\Sigma_1}(|v_*+T_*|^3(v_*+T_*))$ in $L_{5/4}(\Sigma_1)$, as $n \to +\infty$. Since $\dot{V}_5(\Omega_f,\, S_1,\, \Sigma_1)$ is embedded in $L_5(\Sigma_1)$, then this means that

$$\mathbf{f}_2(v_n,\, \eta) \to \mathbf{f}_2(v_*,\, \eta) \quad \forall \eta \in \dot{V}_5(\Omega_f,\, S_1,\, \Sigma_1),$$

as $n \to +\infty$.

▶

Lemma 3.1.3
The operator $\mathbf{B}_1 + \mathbf{B}_2$ *is coercive (see [GGZ74] for definition).*

◀ To prove this lemma, it should be sufficiently to show that there exist some constants $c_5 \in \mathbb{R}_+$, $c_6 \in [0, +\infty)$ such that:

$$c_5(\|v\|_{\dot{V}_5(\Omega_f, S_1, \Sigma_1)} - 1)^2 - c_6 \leq \mathbf{f}_1(v, v) + \mathbf{f}_2(v, v) \tag{3.5}$$

for every $v \in \dot{V}_5(\Omega_f, S_1, \Sigma_1)$, where $\|v\|_{\dot{V}_5(\Omega_f, S_1, \Sigma_1)} \geq 1$.

Let us fix arbitrary $v \in \dot{V}_5(\Omega_f, S_1, \Sigma_1)$. We can write

$$\begin{aligned}\mathbf{f}_2(v, v) = \int_{\Sigma_1} \mathbf{G}_{\Sigma_1\Sigma_1}(|v + T_*|^3(v + T_*))(v + T_*)\, d\lambda_{\Sigma_1} \\ - \int_{\Sigma_1} \mathbf{G}_{\Sigma_1\Sigma_1}(|v + T_*|^3(v + T_*))T_*\, d\lambda_{\Sigma_1}\end{aligned} \tag{3.6}$$

Due to 2.3.7i we have

$$\begin{aligned}&\int_{\Sigma_1} \mathbf{G}_{\Sigma_1\Sigma_1}(|v + T_*|^3(v + T_*))T_*\, d\lambda_{\Sigma_1} \\ &\qquad\leq \|\mathbf{G}_{\Sigma_1\Sigma_1}(|v + T_*|^3(v + T_*))\|_{L_{5/4}(\Sigma_1)}\|T_*\|_{L_5(\Sigma_1)} \\ &\qquad\leq \hat{c}_1\|v + T_*\|^4_{L_5(\Sigma_1)}\|T_*\|_{L_5(\Sigma_1)} \leq \hat{c}_2\|v + T_*\|^4_{L_5(\Sigma_1)} \\ &\qquad\qquad\leq \hat{c}_2(\|v\|_{L_5(\Sigma_1)} + \|T_*\|_{L_5(\Sigma_1)})^4.\end{aligned} \tag{3.7}$$

Due to 2.3.7i, 2.3.7iii, 2.3.7ix we also have

$$\begin{aligned}&\int_{\Sigma_1} \mathbf{G}_{\Sigma_1\Sigma_1}(|v + T_*|^3(v + T_*))(v + T_*)\, d\lambda_{\Sigma_1} \\ &= \sigma\int_{\Sigma_1} |v + T_*|^5\, d\lambda_{\Sigma_1} - \sigma\int_{\Sigma_1} \mathbf{H}_{\Sigma_1\Sigma_1}(|v + T_*|^3(v + T_*))(v + T_*)\, d\lambda_{\Sigma_1} \\ &\geq \sigma\|v + T_*\|^5_{L_5(\Sigma_1)} - \sigma\|\mathbf{H}_{\Sigma_1\Sigma_1}(|v + T_*|^3(v + T_*))\|_{L_{5/4}(\Sigma_1)}\|v + T_*\|_{L_5(\Sigma_1)} \\ &\qquad\geq \sigma c_3\|v + T_*\|^5_{L_5(\Sigma_1)} \geq \sigma c_3(\|v\|_{L_5(\Sigma_1)} - \|T_*\|_{L_5(\Sigma_1)})^5.\end{aligned} \tag{3.8}$$

Therefore, in view of (3.6), (3.7), (3.8), we will get

$$\mathbf{f}_2(v, v) \geq \sum_{i=0}^{5} \tilde{c}_i\|v\|^{5-i}_{L_5(\Sigma_1)}$$

and, since $\tilde{c}_0 = \sigma c_3 \in \mathbb{R}_+$, then there exist constants $\hat{c}_3 \in \mathbb{R}_+$, $\hat{c}_4 \in [0, +\infty)$ such that

$$\hat{c}_3\|v\|^5_{L_5(\Sigma_1)} - \hat{c}_4 \leq \mathbf{f}_2(v, v). \tag{3.9}$$

Notice, that both $\hat{c}_3$, $\hat{c}_4$ here do not depend on v.

Next, let us define for every $\tau \in [1, +\infty)$ the following set:

$$\Xi_\tau := \{\xi \in \dot{V}_5(\Omega_f, S_1, \Sigma_1) : \|\xi\|_{\dot{V}_5(\Omega_f, S_1, \Sigma_1)} = \tau\}.$$

Then for arbitrary chosen $\tau \in [1, +\infty)$, $v_\tau \in \Xi_\tau$ due to estimates (3.2), (3.9) we will have

$$\begin{aligned}\mathbf{f}_1(v_\tau, v_\tau) + \mathbf{f}_2(v_\tau, v_\tau) &\geq c_4(\tau - \|v_\tau\|_{L_5(\Sigma_1)})^2 + \hat{c}_3\|v_\tau\|^5_{L_5(\Sigma_1)} - \hat{c}_4 \\ &\geq \min_{\xi \in \Xi_\tau}\{c_4(\tau - \|\xi\|_{L_5(\Sigma_1)})^2 + \hat{c}_3\|\xi\|^5_{L_5(\Sigma_1)}\} - \hat{c}_4 \\ &\geq \frac{\min\{c_4, \hat{c}_3\}}{2}(\tau - 1)^2 - \hat{c}_4.\end{aligned}$$

Now, if we choose $c_5 = \min\{c_4, \hat{c}_3\}/2$, $c_6 = \hat{c}_4$, then we will get (3.5).

▶

Now, Lemma 3.1.1, Lemma 3.1.2, Lemma 3.1.3 together with the fact that the variational equality (1.6) can be rewritten as the equation (3.1) imply the following existence result (see [GGZ74]):

Theorem 3.1.4
For every fixed control $T_h \in U$ the variational equality (1.6) *has at least one solution $u \in \dot{V}_5(\Omega_f, S_1, \Sigma_1)$ and thus for every fixed control $T_h \in U$ there exists at least one feasible state T of the optimal control problem* (1.5).

3.2 Boundedness of the state T

In order to prove boundedness of the state T we will use technique introduced in [Bi051]. Let us start by proving that T is non-negative.

Theorem 3.2.1
If $u \in \dot{V}_5(\Omega_f, S_1, \Sigma_1)$ is a solution of (1.6) *for some fixed $T_h \in U$, then $0 \leq u(x) + T_*(x)$ a.e. in Ω_f. Therefore, if T is a feasible state of* (1.5) *for some fixed $T_h \in U$, then $0 \leq T(x)$ a.e. in Ω_f.*

◀ Let us suppose that inequality $0 \leq u(x) + T_*(x)$ does not hold a.e. in Ω_f.

Let us define the function

$$\eta(x) := \min\{u(x) + T_*(x), 0\} \quad x \in \Omega_f.$$

According to the previous assumption, $\eta \neq 0$. Moreover, since $T_*(x) \geq 0$ λ_{S_1}-a.e. on S_1 and $u \in \dot{V}_5(\Omega_f, S_1, \Sigma_1)$, then $\eta \in \dot{V}_5(\Omega_f, S_1, \Sigma_1)$.

As $\eta \neq 0$ and $\eta \in \dot{V}_5(\Omega_f, S_1, \Sigma_1)$, then

$$\int_{\Omega_f} k_1(\nabla(u + T_*) \cdot \nabla\eta)\, d\omega_3 = \int_{\Omega_f} k_1|\nabla\eta|^2\, d\omega_3 > 0. \tag{3.10}$$

In addition, the following inequality will hold:

$$\int_{\Omega_f} k_2(\partial_1(u+T_*))\eta\,d\omega_3 = \int_{\Omega_f} k_2(\partial_1\eta)\eta\,d\omega_3 = \int_{S_2} \frac{k_2\eta^2}{2}\,d\lambda_{S_2} \geq 0. \quad (3.11)$$

Next, since $T_h(x) \geq 0$ λ_{Σ_2}-a.e. on Σ_2 and $\eta(x) \leq 0$ λ_{Σ_1}-a.e. on Σ_1, then 2.3.7v, 2.3.7vi will yield

$$-\int_{\Sigma_1} \mathbf{G}_{\Sigma_2\Sigma_1}(|T_h|^3T_h)\eta\,d\lambda_{\Sigma_1} = \int_{\Sigma_1} \sigma\mathbf{H}_{\Sigma_2\Sigma_1}(|T_h|^3T_h)\eta\,d\lambda_{\Sigma_1} \leq 0. \quad (3.12)$$

Similarly, since $T_g(x) \geq 0$ λ_{Σ_1}-a.e. on Σ_1, then we will get

$$\int_{\Sigma_1} k_3T_g\eta\,d\lambda_{\Sigma_1} \leq 0. \quad (3.13)$$

In order to obtain an estimate for the integral

$$\int_{\Sigma_1} \mathbf{G}_{\Sigma_1\Sigma_1}(|u+T_*|^3(u+T_*))\eta\,d\lambda_{\Sigma_1}$$

we define the function

$$\gamma(x) := \max\{u(x)+T_*(x),\, 0\} \quad x \in \Omega_f.$$

Properties of η and γ yield:

$$|u(x)+T_*(x)|^3(u(x)+T_*(x)) = |\eta(x)|^3\eta(x) + |\gamma(x)|^3\gamma(x) \quad (3.14)$$

and

$$\eta(x)\gamma(x) = 0 \quad (3.15)$$

λ_{Σ_1}-a.e. on Σ_1.

Now, as the operator $\mathbf{G}_{\Sigma_1\Sigma_1}$ is linear, then the equality (3.14) implies

$$\begin{aligned}\int_{\Sigma_1} &\mathbf{G}_{\Sigma_1\Sigma_1}(|u+T_*|^3(u+T_*))\eta\,d\lambda_{\Sigma_1} \\ &= \int_{\Sigma_1} \mathbf{G}_{\Sigma_1\Sigma_1}(|\eta|^3\eta)\eta\,d\lambda_{\Sigma_1} + \int_{\Sigma_1} \mathbf{G}_{\Sigma_1\Sigma_1}(|\gamma|^3\gamma)\eta\,d\lambda_{\Sigma_1}. \quad (3.16)\end{aligned}$$

From 2.3.7i, 2.3.7iii, 2.3.7ix it follows that

$$\begin{aligned}\int_{\Sigma_1} \mathbf{G}_{\Sigma_1\Sigma_1}(|\eta|^3\eta)\eta\,d\lambda_{\Sigma_1} &= \sigma\int_{\Sigma_1} |\eta|^5\,d\lambda_{\Sigma_1} - \sigma\int_{\Sigma_1} \mathbf{H}_{\Sigma_1\Sigma_1}(|\eta|^3\eta)\eta\,d\lambda_{\Sigma_1} \\ &\geq \sigma\|\eta\|^5_{L_5(\Sigma_1)} - \sigma\|\mathbf{H}_{\Sigma_1\Sigma_1}(|\eta|^3\eta)\|_{L_{5/4}(\Sigma_1)}\|\eta\|_{L_5(\Sigma_1)} \\ &\geq \sigma c_3\|\eta\|^5_{L_5(\Sigma_1)} \geq 0. \quad (3.17)\end{aligned}$$

Since $\gamma(x) \geq 0$, $\eta(x) \leq 0$ λ_{Σ_1}-a.e. on Σ_1, then due to 2.3.7iii, 2.3.7vi, (3.15) we will get

$$\int_{\Sigma_1} \mathbf{G}_{\Sigma_1\Sigma_1}(|\gamma|^3\gamma)\eta \, d\lambda_{\Sigma_1} = \sigma \int_{\Sigma_1} |\gamma|^3\gamma\eta \, d\lambda_{\Sigma_1} - \sigma \int_{\Sigma_1} \mathbf{H}_{\Sigma_1\Sigma_1}(|\gamma|^3\gamma)\eta \, d\lambda_{\Sigma_1}$$
$$= -\sigma \int_{\Sigma_1} \mathbf{H}_{\Sigma_1\Sigma_1}(|\gamma|^3\gamma)\eta \, d\lambda_{\Sigma_1} \geq 0. \quad (3.18)$$

Therefore, the estimates (3.17), (3.18) together with the formula (3.16) will yield

$$\int_{\Sigma_1} \mathbf{G}_{\Sigma_1\Sigma_1}(|u+T_*|^3(u+T_*))\eta \, d\lambda_{\Sigma_1} \geq 0. \quad (3.19)$$

Next, we also have

$$\int_{\Sigma_1} k_3(u+T_*)\eta \, d\lambda_{\Sigma_1} = \int_{\Sigma_1} k_3\eta^2 \, d\lambda_{\Sigma_1} \geq 0. \quad (3.20)$$

Now, by putting the estimates (3.10), (3.11), (3.12), (3.13), (3.19), (3.20) together, we will get

$$\int_{\Omega_f} (k_1(\nabla(u+T_*) \cdot \nabla\eta + k_2(\partial_1(u+T_*))\eta) \, d\omega_3$$
$$+ \int_{\Sigma_1} \mathbf{G}_{\Sigma_1\Sigma_1}(|u+T_*|^3(u+T_*))\eta \, d\lambda_{\Sigma_1} + \int_{\Sigma_1} k_3(u+T_*)\eta \, d\lambda_{\Sigma_1}$$
$$+ \int_{\Sigma_1} \mathbf{G}_{\Sigma_2\Sigma_1}(|T_h|^3T_h)\eta \, d\lambda_{\Sigma_1} - \int_{\Sigma_1} k_3T_g\eta \, d\lambda_{\Sigma_1} > 0.$$

But this inequality contradicts with the equality (1.6) for the given η.

▶

As we see, the last theorem justifies using $|T_{sh}|^3T_{sh}$ instead of T_{sh}^4 to characterize intensity of emitted radiation by the interface Σ (for example, see (1.3), (2.4)), since now it turns out that T_{sh} is non-negative function for every given $T_h \in U$. If we would used T_{sh}^4 instead of $|T_{sh}|^3T_{sh}$ from the beginning, then it would be hard to prove coercivity of the operator $\mathbf{B}_1 + \mathbf{B}_2$ and, therefore, also existence of T.

After non-negativity of the state T is proved, we are ready to prove its boundedness from above.

Theorem 3.2.2
If $u \in \dot{V}_5(\Omega_f, S_1, \Sigma_1)$ is a solution of (1.6) for some fixed $T_h \in U$, then $u(x) + T_*(x) \leq \mu$ a.e. in Ω_f. Therefore, if T is a feasible state of (1.5) for some fixed $T_h \in U$, then $T(x) \leq \mu$ a.e. in Ω_f.

◀ As it was in the case of Theorem 3.2.1, here we will use similar proof technique. Therefore, let us suppose that inequality $u(x) + T_*(x) \leq \mu$ does not hold a.e. in Ω_f.

Let us define the functions:

$$\eta(x) := \max\{u(x) + T_*(x) - \mu,\, 0\} \quad x \in \Omega_f,$$
$$\gamma(x) := \min\{u(x) + T_*(x),\, \mu\} \quad x \in \Omega_f.$$

According to the previous assumption, $\eta \neq 0$. Moreover, since $T_*(x) \leq \mu$ λ_{S_1}-a.e. on S_1 and $u \in \dot{V}_5(\Omega_f,\, S_1,\, \Sigma_1)$, then $\eta \in \dot{V}_5(\Omega_f,\, S_1,\, \Sigma_1)$.

Next, the following estimates hold

$$\int_{\Omega_f} k_1(\nabla(u + T_*) \cdot \nabla\eta)\, d\omega_3 > 0, \tag{3.21}$$

$$\int_{\Omega_f} k_2(\partial_1(u + T_*))\eta\, d\omega_3 \geq 0 \tag{3.22}$$

by analogy with (3.10), (3.11).

In order to estimate the integral

$$\int_{\Sigma_1} (\mathbf{G}_{\Sigma_1\Sigma_1}(|u + T_*|^3(u + T_*)) + \mathbf{G}_{\Sigma_2\Sigma_1}(|T_h|^3 T_h))\eta\, d\lambda_{\Sigma_1}$$

let us introduce the sets:

$$A := \{x \in \Sigma_1 : \eta(x) > 0\},$$
$$B := \{x \in \Sigma_1 : \eta(x) \leq 0\}.$$

Properties of η and γ yield:

$$u(x) + T_*(x) = \eta(x) + \gamma(x) \;\; \lambda_{\Sigma_1}\text{-a.e. on } \Sigma_1.$$

Therefore, due to Theorem 3.2.1:

$$\begin{aligned} |u(x) + T_*(x)|^3(u(x) + T_*(x)) &= (u(x) + T_*(x))^4 \\ &= \eta(x)^4 + 4\eta(x)^3\gamma(x) + 6\eta(x)^2\gamma(x)^2 + 4\eta(x)\gamma(x)^3 \\ &\qquad + \gamma(x)^4 \;\; \lambda_{\Sigma_1}\text{-a.e. on } \Sigma_1. \end{aligned} \tag{3.23}$$

Furhermore, it is easy to check that the following estimates hold:

$$\gamma(x) = \mu \;\; \lambda_{\Sigma_1}\text{-a.e. on the } A, \tag{3.24}$$

$$\gamma(x) \leq \mu \;\; \lambda_{\Sigma_1}\text{-a.e. on the } B. \tag{3.25}$$

Now, as the operator $\mathbf{G}_{\Sigma_1\Sigma_1}$ is linear, then the equality (3.23) gives

$$\begin{aligned}\int_{\Sigma_1} &\mathbf{G}_{\Sigma_1\Sigma_1}(|u+T_*|^3(u+T_*))\eta\,d\lambda_{\Sigma_1}\\ &= \int_{\Sigma_1} \mathbf{G}_{\Sigma_1\Sigma_1}(\eta^4)\eta\,d\lambda_{\Sigma_1} + \int_{\Sigma_1} 4\mathbf{G}_{\Sigma_1\Sigma_1}(\eta^3\gamma)\eta\,d\lambda_{\Sigma_1}\\ &+ \int_{\Sigma_1} 6\mathbf{G}_{\Sigma_1\Sigma_1}(\eta^2\gamma^2)\eta\,d\lambda_{\Sigma_1} + \int_{\Sigma_1} 4\mathbf{G}_{\Sigma_1\Sigma_1}(\eta\gamma^2)\eta\,d\lambda_{\Sigma_1}\\ &+ \int_{\Sigma_1} \mathbf{G}_{\Sigma_1\Sigma_1}(\gamma^4)\eta\,d\lambda_{\Sigma_1}. \quad (3.26)\end{aligned}$$

Let us estimate the expression

$$\int_{\Sigma_1} \mathbf{G}_{\Sigma_1\Sigma_1}(\gamma^4)\eta\,d\lambda_{\Sigma_1} + \int_{\Sigma_1} \mathbf{G}_{\Sigma_2\Sigma_1}(|T_h|^3T_h)\eta\,d\lambda_{\Sigma_1}.$$

From (3.24), (3.25), 2.3.7vi it follows that $\mathbf{H}_{\Sigma_1\Sigma_1}(\gamma^4)(x) \le \mathbf{H}_{\Sigma_1\Sigma_1}(\mu^4)(x)$ λ_{Σ_1}-a.e. on Σ_1. Therefore, due to 2.3.7iii we will have:

$$\begin{aligned}\mathbf{G}_{\Sigma_1\Sigma_1}(\gamma^4)(x)\eta(x) &= \sigma(\gamma^4(x)\eta(x) - \mathbf{H}_{\Sigma_1\Sigma_1}(\gamma^4)(x)\eta(x))\\ &\ge \sigma(\mu^4\eta(x) - \mathbf{H}_{\Sigma_1\Sigma_1}(\mu^4)(x)\eta(x))\\ &= \mu^4\mathbf{G}_{\Sigma_1\Sigma_1}(1)(x)\eta(x)\ \lambda_{\Sigma_1}\text{-a.e. on } A. \quad (3.27)\end{aligned}$$

In addition, as we have that $\eta(x) = 0$ λ_{Σ_1}-a.e. on B, then:

$$\mathbf{G}_{\Sigma_1\Sigma_1}(\gamma^4)(x)\eta(x) = \mu^4\mathbf{G}_{\Sigma_1\Sigma_1}(1)(x)\eta(x) = 0\ \lambda_{\Sigma_1}\text{-a.e. on } B. \quad (3.28)$$

Next, due to B4 and 2.3.7vi we have $\mathbf{H}_{\Sigma_2\Sigma_1}(|T_h|^3T_h)(x) \le \mathbf{H}_{\Sigma_2\Sigma_1}(\mu^4)(x)$ λ_{Σ_1}-a.e. on Σ_1. Therefore, 2.3.7v will yield:

$$\begin{aligned}&\mathbf{G}_{\Sigma_2\Sigma_1}(|T_h|^3T_h)(x)\eta(x) = -\sigma\mathbf{H}_{\Sigma_2\Sigma_1}(|T_h|^3T_h)(x)\eta(x)\\ &\quad \ge -\sigma\mathbf{H}_{\Sigma_2\Sigma_1}(\mu^4)(x)\eta(x) = \mu^4\mathbf{G}_{\Sigma_2\Sigma_1}(1)(x)\eta(x)\ \lambda_{\Sigma_1}\text{-a.e. on } \Sigma_1. \quad (3.29)\end{aligned}$$

Now, due to (3.27), (3.28), (3.29), 2.3.7viii, we will have:

$$\begin{aligned}\int_{\Sigma_1} &\mathbf{G}_{\Sigma_1\Sigma_1}(\gamma^4)\eta\,d\lambda_{\Sigma_1} + \int_{\Sigma_1} \mathbf{G}_{\Sigma_2\Sigma_1}(|T_h|^3T_h)\eta\,d\lambda_{\Sigma_1}\\ &\ge \int_{\Sigma_1} \mu^4(\mathbf{G}_{\Sigma_1\Sigma_1}(1) + \mathbf{G}_{\Sigma_2\Sigma_1}(1))\eta\,d\lambda_{\Sigma_1} \ge 0. \quad (3.30)\end{aligned}$$

Next, let us estimate the integrals:

$$\int_{\Sigma_1} \mathbf{G}_{\Sigma_1\Sigma_1}(\eta^i\gamma^{4-i})\eta\,d\lambda_{\Sigma_1} \quad i \in \{1, 2, 3, 4\}.$$

From (3.24), (3.25), 2.3.7vi it follows that $\mathbf{H}_{\Sigma_1\Sigma_1}(\eta^i\gamma^{4-i})(x) \le \mathbf{H}_{\Sigma_1\Sigma_1}(\eta^i\mu^{4-i})(x)$ λ_{Σ_1}-a.e. on Σ_1. Therefore, 2.3.7iii will give

$$\begin{aligned}\mathbf{G}_{\Sigma_1\Sigma_1}(\eta^i\gamma^{4-i})(x)\eta(x) &= \sigma(\gamma^{4-i}(x)\eta^i(x) - \mathbf{H}_{\Sigma_1\Sigma_1}(\eta^i\gamma^{4-i})(x)\eta(x))\\ &= \sigma(\mu^{4-i}\eta^{i+1}(x) - \mathbf{H}_{\Sigma_1\Sigma_1}(\eta^i\gamma^{4-i})(x)\eta(x))\\ &\ge \sigma\mu^{4-i}(\eta^{i+1}(x) - \mathbf{H}_{\Sigma_1\Sigma_1}(\eta^i)(x)\eta(x))\ \lambda_{\Sigma_1}\text{-a.e. on } A. \end{aligned} \tag{3.31}$$

In addition, since $\eta(x) = 0$ λ_{Σ_1}-a.e. on B, then

$$\mathbf{G}_{\Sigma_1\Sigma_1}(\eta^i\gamma^{4-i})(x)\eta(x) = \sigma\mu^{4-i}(\eta^{i+1}(x) - \mathbf{H}_{\Sigma_1\Sigma_1}(\eta^i)(x)\eta(x)) = 0 \quad \lambda_{\Sigma_1}\text{-a.e. on } B. \tag{3.32}$$

Now, due to (3.31), (3.32), 2.3.7i, 2.3.7ix we will have

$$\begin{aligned}\int_{\Sigma_1} \mathbf{G}_{\Sigma_1\Sigma_1}(\eta^i\gamma^{4-i})\eta\, d\lambda_{\Sigma_1} &\ge \int_{\Sigma_1} \sigma\mu^{4-i}(\eta^{i+1} - \mathbf{H}_{\Sigma_1\Sigma_1}(\eta^i)\eta)\, d\lambda_{\Sigma_1}\\ &\ge \sigma\mu^{4-i}\|\eta\|^{i+1}_{L_{i+1}(\Sigma_1)} - \sigma\mu^{4-i}\|\mathbf{H}_{\Sigma_1\Sigma_1}(\eta^i)\|_{L_{(i+1)/i}(\Sigma_1)}\|\eta\|_{L_{i+1}(\Sigma_1)}\\ &\ge \sigma c_3\mu^{4-i}\|\eta\|^{i+1}_{L_{i+1}(\Sigma_1)} \ge 0. \end{aligned} \tag{3.33}$$

The estimates (3.30), (3.33) and the formula (3.26) yield:

$$\int_{\Sigma_1} (\mathbf{G}_{\Sigma_1\Sigma_1}(|u+T_*|^3(u+T_*)) + \mathbf{G}_{\Sigma_2\Sigma_1}(|T_h|^3T_h))\eta\, d\lambda_{\Sigma_1} \ge 0. \tag{3.34}$$

Next, since $u(x) + T_*(x) \ge \mu$ λ_{Σ_1}-a.e. on A and as $\eta(x) = 0$ λ_{Σ_1}-a.e. on B, then due to B5 we have:

$$(u(x) + T_*(x) - T_g(x))\eta(x) \ge 0 \ \lambda_{\Sigma_1}\text{-a.e. on } \Sigma_1.$$

Therefore, it follows that

$$\int_{\Sigma_1} k_3(u + T_* - T_g)\eta\, d\lambda_{\Sigma_1} \ge 0. \tag{3.35}$$

Now, by putting the estimates (3.21), (3.22), (3.34), (3.35) together, we will get

$$\begin{aligned}&\int_{\Omega_f} (k_1(\nabla(u+T_*)\cdot\nabla\eta + k_2(\partial_1(u+T_*))\eta)\, d\omega_3\\ &\quad + \int_{\Sigma_1} \mathbf{G}_{\Sigma_1\Sigma_1}(|u+T_*|^3(u+T_*))\eta\, d\lambda_{\Sigma_1} + \int_{\Sigma_1} k_3(u+T_*)\eta\, d\lambda_{\Sigma_1}\\ &\quad + \int_{\Sigma_1} \mathbf{G}_{\Sigma_2\Sigma_1}(|T_h|^3T_h)\eta\, d\lambda_{\Sigma_1} - \int_{\Sigma_1} k_3T_g\eta\, d\lambda_{\Sigma_1} > 0.\end{aligned}$$

But the last inequality contradicts with the equality (1.6) for the given η.

▶

3.3 Continuous dependence and uniqueness of solutions

In this section we will prove existence of the control-to-state operator $T = \mathbf{\Gamma}(T_h)$ and obtain its Lipschitz continuity in appropriate functional spaces. Let us start with the following lemma.

Lemma 3.3.1
For every two arbitrary chosen state-control pairs (T, T_h), $(\hat{T}, \hat{T}_h)$ the following inequalities hold:

$$\|T - \hat{T}\|_{L_\infty(\Omega_f)} \leq 4\mu^3 \|T_h - \hat{T}_h\|_{L_\infty(\Sigma_2)}, \tag{3.36}$$

$$\|T - \hat{T}\|_{L_\infty(\Sigma_1)} \leq 4\mu^3 \|T_h - \hat{T}_h\|_{L_\infty(\Sigma_2)}. \tag{3.37}$$

◀ At the first step of the proof let us prove that $T(x) - \hat{T}(x) \leq 4\mu^3 \|T_h - \hat{T}_h\|_{L_\infty(\Sigma_2)}$ a.e. in Ω_f.

Let us suppose that this inequality does not hold and let us define the constant

$$\hat{c}_1 := (4\mu^3 \|T_h - \hat{T}_h\|_{L_\infty(\Sigma_2)})^{1/4}$$

and the functions:

$$\eta(x) := \max\{T(x) - \hat{T}(x) - \hat{c}_1,\, 0\} \quad x \in \Omega_f,$$
$$\gamma(x) := \min\{T(x) - \hat{T}(x),\, \hat{c}_1\} \quad x \in \Omega_f.$$

According to our assumption, $\eta \neq 0$. Moreover, since $T(x) - \hat{T}(x) \leq \hat{c}_1$ λ_{S_1}-a.e. on S_1 and $(T - \hat{T}) \in \dot{V}_5(\Omega_f, S_1, \Sigma_1)$, then $\eta \in \dot{V}_5(\Omega_f, S_1, \Sigma_1)$.

Since $\eta \in \dot{V}_5(\Omega_f, S_1, \Sigma_1)$, then the following equalities must hold (see (1.6)):

$$\int_{\Omega_f} (k_1(\nabla T \cdot \nabla \eta) + k_2(\partial_1 T)\eta)\, d\omega_3 + \int_{\Sigma_1} \mathbf{Q}_{\Sigma_1\Sigma_1}(T)\eta\, d\lambda_{\Sigma_1} = -\int_{\Sigma_1} \mathbf{Q}_{\Sigma_2\Sigma_1}(T_h)\eta\, d\lambda_{\Sigma_1},$$

$$\int_{\Omega_f} (k_1(\nabla \hat{T} \cdot \nabla \eta) + k_2(\partial_1 \hat{T})\eta)\, d\omega_3 + \int_{\Sigma_1} \mathbf{Q}_{\Sigma_1\Sigma_1}(\hat{T})\eta\, d\lambda_{\Sigma_1} = -\int_{\Sigma_1} \mathbf{Q}_{\Sigma_2\Sigma_1}(T_h)\eta\, d\lambda_{\Sigma_1}.$$

If we now substract the second equality from the first one, then due to Theorem 3.2.1 we will get

$$\int_{\Omega_f} (k_1(\nabla(T-\hat{T})\cdot\nabla\eta) + k_2(\partial_1(T-\hat{T}))\eta)\,d\omega_3 + \int_{\Sigma_1} \mathbf{G}_{\Sigma_1\Sigma_1}(T^4-\hat{T}^4)\eta\,d\lambda_{\Sigma_1} + \int_{\Sigma_1} k_3(T-\hat{T})\eta\,d\lambda_{\Sigma_1} = -\int_{\Sigma_1} \mathbf{G}_{\Sigma_2\Sigma_1}(T_h^4-\hat{T}_h^4)\eta\,d\lambda_{\Sigma_1}. \quad (3.38)$$

Since $\eta \neq 0$ and $\eta \in \dot{V}_5(\Omega_f,\, S_1,\, \Sigma_1)$, then the following inequalities will hold:

$$\int_{\Omega_f} k_1(\nabla(T-\hat{T})\cdot\nabla\eta)\,d\omega_3 = \int_{\Omega_f} k_1|\nabla\eta|^2\,d\omega_3 > 0, \quad (3.39)$$

$$\int_{\Omega_f} k_2(\partial_1(T-\hat{T}))\eta\,d\omega_3 = \int_{\Omega_f} k_2(\partial_1\eta)\eta\,d\omega_3 = \int_{S_2} \frac{k_2\eta^2}{2}\,d\lambda_{S_2} \geq 0. \quad (3.40)$$

Now, one can easily check that

$$T(x) - \hat{T}(x) = \eta(x) + \gamma(x) \;\; \lambda_{\Sigma_1}\text{-a.e. on } \Sigma_1,$$

therefore, since the operator $\mathbf{G}_{\Sigma_1\Sigma_1}$ is linear, then

$$\int_{\Sigma_1} (\mathbf{G}_{\Sigma_1\Sigma_1}(T^4-\hat{T}^4) + \mathbf{G}_{\Sigma_2\Sigma_1}(T_h^4-\hat{T}_h^4))\eta\,d\lambda_{\Sigma_1} = \sum_{i=0}^{3}\sum_{j=0}^{4-i} \tilde{c}_{ij}\int_{\Sigma_1} \mathbf{G}_{\Sigma_1\Sigma_1}(\gamma^{4-i-j}\eta^j\hat{T}^i)\eta\,d\lambda_{\Sigma_1} + \int_{\Sigma_1} \mathbf{G}_{\Sigma_2\Sigma_1}(T_h^4-\hat{T}_h^4)\eta\,d\lambda_{\Sigma_1}, \quad (3.41)$$

where $\tilde{c}_{ij} \in \mathbb{R}_+$ for every $i \in \{0,\, 1,\, 2,\, 3\}$, $j \in \{0,\, \ldots,\, 4-i\}$. Furthermore, there will be $\tilde{c}_{00} = 1$.

Next, let us introduce the sets:

$$A := \{x \in \Sigma_1 : \eta(x) > 0\},$$
$$B := \{x \in \Sigma_1 : \eta(x) \leq 0\}.$$

Then properties of η and γ then will yield:

$$\gamma(x) = \hat{c}_1 \;\; \lambda_{\Sigma_1}\text{-a.e. on } A, \quad (3.42)$$
$$\gamma(x) \leq \hat{c}_1 \;\; \lambda_{\Sigma_1}\text{-a.e. on } B. \quad (3.43)$$

Now, let us estimate the expression

$$\int_{\Sigma_1} \mathbf{G}_{\Sigma_1\Sigma_1}(\gamma^4)\eta\,d\lambda_{\Sigma_1} + \int_{\Sigma_1} \mathbf{G}_{\Sigma_2\Sigma_1}(T_h^4-\hat{T}_h^4)\eta\,d\lambda_{\Sigma_1}.$$

From (3.24), (3.25), 2.3.7vi it follows that $\mathbf{H}_{\Sigma_1\Sigma_1}(\gamma^4)(x) \leq \mathbf{H}_{\Sigma_1\Sigma_1}(\hat{c}_1^4)(x)$ λ_{Σ_1}-a.e. on Σ_1. Therefore, 2.3.7iii will give

$$\begin{aligned}\mathbf{G}_{\Sigma_1\Sigma_1}(\gamma^4)(x)\eta(x) &= \sigma(\gamma^4(x)\eta(x) - \mathbf{H}_{\Sigma_1\Sigma_1}(\gamma^4)(x)\eta(x)) \\ &\geq \sigma(\hat{c}_1^4\eta(x) - \mathbf{H}_{\Sigma_1\Sigma_1}(\hat{c}_1^4)(x)\eta(x)) \\ &= \hat{c}_1^4\mathbf{G}_{\Sigma_1\Sigma_1}(1)(x)\eta(x)\ \lambda_{\Sigma_1}\text{-a.e. on } A. \end{aligned} \tag{3.44}$$

In addition, since we have $\eta(x) = 0$ λ_{Σ_1}-a.e. on B, then

$$\mathbf{G}_{\Sigma_1\Sigma_1}(\gamma^4)(x)\eta(x) = \hat{c}_1^4\mathbf{G}_{\Sigma_1\Sigma_1}(1)(x)\eta(x) = 0\ \lambda_{\Sigma_1}\text{-a.e. on } B. \tag{3.45}$$

Next, due to B4, 2.3.7vi we will have $\mathbf{H}_{\Sigma_2\Sigma_1}(T_h^4 - \hat{T}_h^4)(x) \leq \mathbf{H}_{\Sigma_2\Sigma_1}(4\mu^3\|T_h - \hat{T}_h\|_{L_\infty(\Sigma_2)})(x)$ λ_{Σ_1}-a.e. on Σ_1. Therefore, 2.3.7v will yield

$$\begin{aligned}\mathbf{G}_{\Sigma_2\Sigma_1}(T_h^4 - \hat{T}_h^4)(x)\eta(x) &= -\sigma\mathbf{H}_{\Sigma_2\Sigma_1}(T_h^4 - \hat{T}_h^4)(x)\eta(x) \\ &\geq -\sigma\mathbf{H}_{\Sigma_2\Sigma_1}(\hat{c}_1^4)(x)\eta(x) = \hat{c}_1^4\mathbf{G}_{\Sigma_2\Sigma_1}(1)(x)\eta(x)\ \lambda_{\Sigma_1}\text{-a.e. on } \Sigma_1. \end{aligned} \tag{3.46}$$

Now, due to (3.44), (3.45), (3.46), 2.3.7viii, we will have:

$$\begin{aligned}\int_{\Sigma_1} \mathbf{G}_{\Sigma_1\Sigma_1}(\gamma^4)\eta\, d\lambda_{\Sigma_1} &+ \int_{\Sigma_1} \mathbf{G}_{\Sigma_2\Sigma_1}(T_h^4 - \hat{T}_h^4)\eta\, d\lambda_{\Sigma_1} \\ &\geq \int_{\Sigma_1} \hat{c}_1^4(\mathbf{G}_{\Sigma_1\Sigma_1}(1) + \mathbf{G}_{\Sigma_2\Sigma_1}(1))\eta\, d\lambda_{\Sigma_1} \geq 0. \end{aligned} \tag{3.47}$$

Next, let us estimate the integrals

$$\int_{\Sigma_1} \mathbf{G}_{\Sigma_1\Sigma_1}(\gamma^{4-i-j}\eta^j\hat{T}^i)\eta\, d\lambda_{\Sigma_1},$$

where $i \in \{0, 1, 2, 3\}$, $j \in \{0, \ldots, 4-i\}$ and i, j are not simultaneously zeros. For this purpose let us define the functions $f_{ij} := \eta^j\hat{T}^i$. Then from (3.42), (3.43), 2.3.7vi it follows that $\mathbf{H}_{\Sigma_1\Sigma_1}(\gamma^{4-i-j}f_{ij})(x) \leq \mathbf{H}_{\Sigma_1\Sigma_1}(\hat{c}_1^{4-i-j}f_{ij})(x)$ λ_{Σ_1}-a.e. on Σ_1. Therefore, 2.3.7iii will give

$$\begin{aligned}&\mathbf{G}_{\Sigma_1\Sigma_1}(\gamma^{4-i-j}f_{ij})(x)\eta(x) \\ &\quad= \sigma(\gamma^{4-i-j}(x)f_{ij}(x) - \mathbf{H}_{\Sigma_1\Sigma_1}(\gamma^{4-i-j}f_{ij})(x))\eta(x) \\ &\quad= \sigma(\hat{c}_1^{4-i-j}f_{ij}(x) - \mathbf{H}_{\Sigma_1\Sigma_1}(\gamma^{4-i-j}f_{ij})(x))\eta(x) \\ &\quad\geq \sigma\hat{c}_1^{4-i-j}(f_{ij}(x) - \mathbf{H}_{\Sigma_1\Sigma_1}(f_{ij})(x))\eta(x)\ \lambda_{\Sigma_1}\text{-a.e. on } A. \end{aligned} \tag{3.48}$$

In addition, since $\eta(x) = 0$ λ_{Σ_1}-a.e. on B, then

$$\begin{aligned}\mathbf{G}_{\Sigma_1\Sigma_1}(\gamma^{4-i-j}f_{ij})(x)\eta(x) &= \sigma\hat{c}_1^{4-i-j}(f_{ij}(x) - \mathbf{H}_{\Sigma_1\Sigma_1}(f_{ij})(x))\eta(x) = 0 \\ &\lambda_{\Sigma_1}\text{-a.e. on } B. \end{aligned} \tag{3.49}$$

Now, due to (3.48), (3.49), 2.3.7i, 2.3.7ix we will have

$$\int_{\Sigma_1} \mathbf{G}_{\Sigma_1\Sigma_1}(\gamma^{4-i-j}\eta^j\hat{T}^i)\eta\, d\lambda_{\Sigma_1} \geq \int_{\Sigma_1} \sigma\hat{c}_1^{4-i-j}(f_{ij} - \mathbf{H}_{\Sigma_1\Sigma_1}(f_{ij}))\eta\, d\lambda_{\Sigma_1}$$
$$\geq \sigma\hat{c}_1^{4-i-j}\|\eta\|_{L_1(\Sigma_1)}(\|f_{ij}\|_{L_\infty(\Sigma_1)} - \|\mathbf{H}_{\Sigma_1\Sigma_1}(f_{ij})\|_{L_\infty(\Sigma_1)})$$
$$\geq \sigma c_3\hat{c}_1^{4-i-j}\|\eta\|_{L_1(\Sigma_1)}\|f_{ij}\|_{L_\infty(\Sigma_1)} \geq 0. \quad (3.50)$$

Now, the estimates (3.47), (3.50) together with the formula (3.41) imply that

$$\int_{\Sigma_1} \mathbf{G}_{\Sigma_1\Sigma_1}((T^4 - \hat{T}^4))\eta\, d\lambda_{\Sigma_1} + \int_{\Sigma_1} \mathbf{G}_{\Sigma_2\Sigma_1}(T_h^4 - \hat{T}_h^4)\eta\, d\lambda_{\Sigma_1} \geq 0. \quad (3.51)$$

Next, since $T(x) - \hat{T}(x) \geq \hat{c}_1$ λ_{Σ_1}-a.e. on A and $\eta(x) = 0$ λ_{Σ_1}-a.e. on B, then due to B5 we have

$$(T(x) - \hat{T}(x))\eta(x) \geq 0 \ \lambda_{\Sigma_1}\text{-a.e. on } \Sigma_1.$$

Therefore, it follows that

$$\int_{\Sigma_1} k_3(T - \hat{T})\eta\, d\lambda_{\Sigma_1} \geq 0. \quad (3.52)$$

Now, by putting the estimates (3.39), (3.40), (3.51), (3.52) together, we will get

$$\int_{\Omega_f} (k_1(\nabla(T - \hat{T}) \cdot \nabla\eta) + k_2(\partial_1(T - \hat{T}))\eta)\, d\omega_3$$
$$+ \int_{\Sigma_1} \mathbf{G}_{\Sigma_1\Sigma_1}(T^4 - \hat{T}^4)\eta\, d\lambda_{\Sigma_1} + \int_{\Sigma_1} k_3(T - \hat{T})\eta\, d\lambda_{\Sigma_1}$$
$$+ \int_{\Sigma_1} \mathbf{G}_{\Sigma_2\Sigma_1}(T_h^4 - \hat{T}_h^4)\eta\, d\lambda_{\Sigma_1} > 0.$$

But the last inequality contradicts with the equality (3.41), what shows that there must hold

$$T(x) - \hat{T}(x) \leq 4\mu^3\|T_h - \hat{T}_h\|_{L_\infty(\Sigma_2)} \text{ a.e. in } x \in \Omega_f. \quad (3.53)$$

Next, by performing similar analysis, we can prove that

$$\hat{T}(x) - T(x) \leq 4\mu^3\|T_h - \hat{T}_h\|_{L_\infty(\Sigma_2)} \text{ a.e. in } x \in \Omega_f. \quad (3.54)$$

But then by putting (3.53), (3.54) together we will get (3.36), (3.38).

►

Now, by using Lemma 3.3.1 it is easy to get the following result.

Theorem 3.3.2
There exists an unique defined control-to-state operator $T = \mathbf{\Gamma}(T_h)$ of the optimal control problem (1.5) as mapping from U to $H_1(\Omega_f)$. Furthermore, for every two arbitrary chosen controls $T_h \in U$, $\hat{T}_h \in U$ the following inequalities hold:

$$\|\mathbf{\Gamma}(T_h) - \mathbf{\Gamma}(\hat{T}_h)\|_{L_\infty(\Omega_f)} \leq 4\mu^3 \|T_h - \hat{T}_h\|_{L_\infty(\Sigma_2)},$$
$$\|\mathbf{\Gamma}(T_h) - \mathbf{\Gamma}(\hat{T}_h)\|_{L_\infty(\Sigma_1)} \leq 4\mu^3 \|T_h - \hat{T}_h\|_{L_\infty(\Sigma_2)}.$$

3.4 Linearized equation

In order to get the final result - Fréchet differentiability of the cost functional $\mathscr{J}(\mathbf{\Gamma}(T_h))$ (see the formula (1.10)), let us prove some results about a "linearized" operator of $\mathbf{B}_1 + \mathbf{B}_2$.

Let us define the functionals:

$$\mathbf{f}_4(v,\,\eta,\,\psi) := \int_{\Sigma_1} \mathbf{G}_{\Sigma_1\Sigma_1}(|\psi|v)\eta\, d\lambda_{\Sigma_1} \qquad v,\,\eta \in H_1(\Omega_f,\,S_1),\, \psi \in L_\infty(\Sigma_1),$$
$$\mathbf{f}_5(w,\,\eta,\,\vartheta) := \int_{\Sigma_1} \mathbf{G}_{\Sigma_2\Sigma_1}(|\vartheta|w)\eta\, d\lambda_{\Sigma_1} \qquad w,\,\vartheta \in L_\infty(\Sigma_2),\, \eta \in H_1(\Omega_f,\,S_1).$$

By using embedding theorems for $H_1(\Omega_f,\,S_1)$ and by taking into account properties of the operators $\mathbf{G}_{\Sigma_1\Sigma_1}$, $\mathbf{G}_{\Sigma_2\Sigma_1}$, one can get that $\mathbf{f}_4(v,\,\cdot,\,\psi) \in H_1^*(\Omega_f,\,S_1)$ for every fixed $v \in H_1(\Omega_f,\,S_1)$, $\psi \in L_\infty(\Sigma_1)$ and $\mathbf{f}_5(w,\,\cdot,\,\vartheta) \in H_1^*(\Omega_f,\,S_1)$ for every fixed $w,\,\vartheta \in L_\infty(\Sigma_2)$. Therefore, if we define the following operators:

$$\mathbf{B}_4[\psi](v) := \mathbf{f}_4(v,\,\cdot,\,\psi),$$
$$\mathbf{B}_5[\vartheta](w) := \mathbf{f}_5(w,\,\cdot,\,\vartheta),$$

then $\mathbf{B}_4[\psi] : H_1(\Omega_f,\,S_1) \mapsto H_1^*(\Omega_f,\,S_1)$ for every fixed $\psi \in L_\infty(\Sigma_1)$ and $\mathbf{B}_5[\vartheta] : L_\infty(\Sigma_2) \mapsto H_1^*(\Omega_f,\,S_1)$ for every fixed $\vartheta \in L_\infty(\Sigma_2)$. Furthermore, $\mathbf{B}_4[\psi] \in \mathscr{L}(H_1(\Omega_f,\,S_1),\,H_1^*(\Omega_f,\,S_1))$, $\mathbf{B}_5[\vartheta] \in \mathscr{L}(L_\infty(\Sigma_2),\,H_1^*(\Omega_f,\,S_1))$. In addition, it is easy to see that also $\mathbf{B}_1 \in \mathscr{L}(H_1(\Omega_f,\,S_1),\,H_1^*(\Omega_f,\,S_1))$.

In view of the previous considerations, it turns out that $\mathbf{B}_1 + \mathbf{B}_4[\psi] \in \mathscr{L}(H_1(\Omega_f,\,S_1),\,H_1^*(\Omega_f,\,S_1))$ for every fixed $\psi \in L_\infty(\Sigma_1)$. In order, to prove existence of an inverse operator $(\mathbf{B}_1 + \mathbf{B}_4[\psi])^{-1}$ for an arbitrary chosen $\psi \in L_\infty(\Sigma_1)$, let us obtain some temporary results.

Lemma 3.4.1
The equation

$$\mathbf{B}_1(\xi) + \mathbf{B}_4[\psi](\xi) = \mathbf{0} \tag{3.55}$$

can have only a trivial solution ξ for every fixed parameter $\psi \in L_\infty(\Sigma_1)$.

◄ Let us fix arbitrary $\psi \in L_\infty(\Sigma_1)$ and let us assume that the statement of this lemma does not hold and, therefore, there exists some nontrivial solution $\xi \in H_1(\Omega_f,\, S_1)$ of (3.55).

It is easy to prove that the equality $|\psi(x)|\xi(x) = 0$ can not hold λ_{Σ_1}-a.e. on Σ_1, because, if it would be so, then due to linearity of $\mathbf{G}_{\Sigma_1\Sigma_1}$ there will be

$$\int_{\Sigma_1} \mathbf{G}_{\Sigma_1\Sigma_1}(|\psi|\xi)\eta\, d\lambda_{\Sigma_1} = 0$$

for every $\eta \in H_1(\Omega_f,\, S_1)$, i.e. $\mathbf{B}_4[\psi](\xi) = \mathbf{0}$. Therefore, due to the equation (3.55) there will be $\mathbf{B}_1(\xi) = \mathbf{0}$. The estimate (3.2) in its turn (see Lemma 3.1.1) yields that there exists $\mathbf{B}_1^{-1} \in \mathscr{L}(H_1^*(\Omega_f,\, S_1),\, H_1(\Omega_f,\, S_1))$, and, therefore, it follows that $\xi(x) = 0$ a.e. in Ω_f. But this result contradicts with the assumption that ξ was a nontrivial solution of (3.55).

Let us define the functions:

$$\begin{aligned}
&\eta_\tau(x) := \max\{\min\{\xi(x)/\tau, 1\}, -1\} \quad x \in \Omega_f,\ \tau \in (0,\, 1]\\
&\eta(x) := \operatorname{sign}\{\xi(x)\} \quad x \in \Omega_f,\\
&\gamma_\tau(x) := \max\{\xi(x), \tau\} + \min\{\xi(x), -\tau\} \quad x \in \Omega_f,\ \tau \in (0,\, 1].
\end{aligned}$$

It is easy to see that $\eta_\tau \in H_1(\Omega_f,\, S_1)$, $\gamma_\tau \in H_1(\Omega_f,\, S_1)$ for every fixed $\tau \in (0,\, 1]$.

Now, we will have the following estimates:

$$\int_{\Omega_f} k_1(\nabla\xi \cdot \nabla\eta_\tau)\, d\omega_3 \geq \int_{\Omega_f} k_1\tau|\nabla\eta_\tau|^2\, d\omega_3 \geq 0 \tag{3.56}$$

and

$$\begin{aligned}
\int_{\Omega_f} k_2(\partial_1\xi)\eta_\tau\, d\omega_3 &= \int_{\Omega_f} k_2(\tau(\partial_1\eta_\tau)\eta_\tau + (\partial_1|\gamma_\tau|))\, d\omega_3\\
&= \int_{S_2} k_2(\frac{\tau\eta_\tau^2}{2} + |\gamma_\tau|)\, d\lambda_{S_2} \geq 0.
\end{aligned} \tag{3.57}$$

Next, 2.3.7i, 2.3.7iii, 2.3.7vii, 2.3.7ix give

$$\begin{aligned}
\int_{\Sigma_1} \mathbf{G}_{\Sigma_1\Sigma_1}(|\psi|\xi)\eta\, d\lambda_{\Sigma_1} &= \int_{\Sigma_1} |\psi|\xi\mathbf{G}_{\Sigma_1\Sigma_1}(\eta)\, d\lambda_{\Sigma_1}\\
&= \sigma(\int_{\Sigma_1} |\psi|\xi\eta\, d\lambda_{\Sigma_1} - \int_{\Sigma_1} |\psi|\xi\mathbf{H}_{\Sigma_1\Sigma_1}(\eta)\, d\lambda_{\Sigma_1})\\
&\geq \|\psi\xi\|_{L_1(\Sigma_1)} - \|\psi\xi\|_{L_1(\Sigma_1)}\|\mathbf{H}_{\Sigma_1\Sigma_1}(\eta)\|_{L_\infty(\Sigma_1)}\\
&\geq c_3\|\psi\xi\|_{L_1(\Sigma_1)} > 0.
\end{aligned} \tag{3.58}$$

Since $\|\eta_\tau - \eta\|_{L_1(\Sigma_1)} \to 0$, as $\tau \to 0$, then also

$$\int_{\Sigma_1} \mathbf{G}_{\Sigma_1\Sigma_1}(|\psi|\xi)\eta_\tau\, d\lambda_{\Sigma_1} \to \int_{\Sigma_1} \mathbf{G}_{\Sigma_1\Sigma_1}(|\psi|\xi)\eta\, d\lambda_{\Sigma_1},$$

as $\tau \to 0$. But then due to (3.58) there must exist some $\tau_0 \in (0,\, 1]$ such that the following estimate holds

$$\int_{\Sigma_1} \mathbf{G}_{\Sigma_1\Sigma_1}(|\psi|\xi)\eta_\tau \, d\lambda_{\Sigma_1} > 0 \tag{3.59}$$

for every $\tau \in (0,\, \tau_0]$.

In addition, we also have the inequality

$$\int_{\Sigma_1} k_3\xi\eta_\tau \, d\lambda_{\Sigma_1} \geq 0. \tag{3.60}$$

Now, by putting the estimates (3.56), (3.57), (3.59), (3.60) together, we will get

$$\int_{\Omega_f} (k_1(\nabla\xi \cdot \nabla\eta_\tau) + k_2(\partial_1\xi\eta_\tau)) \, d\omega_3 + \int_{\Sigma_1} k_3\xi\eta_\tau \, d\lambda_{\Sigma_1} + \int_{\Sigma_1} \mathbf{G}_{\Sigma_1\Sigma_1}(|\psi|\xi)\eta_\tau \, d\lambda_{\Sigma_1} > 0$$

for every $\tau \in (0,\, \tau_0]$. But the last inequality contradicts with the equation (3.55) and this shows that the assumption about non-triviality of ξ was wrong.

▶

Lemma 3.4.2
The operator $\mathbf{B}_4[\psi]$ is completely continuous for every fixed parameter $\psi \in L_\infty(\Sigma_1)$.

◀ Let us fix arbitrary $\psi \in L_\infty(\Sigma_1)$ and a bounded sequence $\{v_n\}_{n\in\mathbb{N}}$ in $H_1(\Omega_f,\, S_1)$. As embedding of $H_1(\Omega_f,\, S_1)$ in $L_2(\Sigma_1)$ is completely continuous, then there exists some subsequence $\{v_{n_k}\}_{k\in\mathbb{N}}$ and $v_* \in L_2(\Sigma_1)$ such that $v_{n_k} \to v_*$ in $L_2(\Sigma_1)$, as $k \to +\infty$. Next, due to 2.3.7i we will have that $\mathbf{G}_{\Sigma_1\Sigma_1}(|\psi|v_{n_k}) \to \mathbf{G}_{\Sigma_1\Sigma_1}(|\psi|v_*)$ in $L_2(\Sigma_1)$, as $k \to +\infty$. Finally, as $L_2(\Sigma_1)$ is embedded in $H_1^*(\Omega_f,\, S_1)$, then this will yield that $\mathbf{B}_4[\psi](v_{n_k}) \to \mathbf{B}_4[\psi](v_*)$ in $H_1^*(\Omega_f,\, S_1)$, as $k \to +\infty$.

▶

Now, on basis of the previous two lemmas it is possible to prove the following result.

Lemma 3.4.3
For every fixed parameter $\psi \in L_\infty(\Sigma_1)$ there exists the inverse operator $(\mathbf{B}_1 + \mathbf{B}_4[\psi])^{-1} \in \mathscr{L}(H_1^*(\Omega_f,\, S_1),\, H_1(\Omega_f,\, S_1))$.

◀ Let us fix arbitrary $\psi \in L_\infty(\Sigma_1)$. Due to the estimate (3.2) (see Lemma 3.1.1), there exists $\mathbf{B}_1^{-1} \in \mathscr{L}(H_1^*(\Omega_f,\, S_1),\, H_1(\Omega_f,\, S_1))$. Therefore, there will hold the following equalities:

$$\mathbf{B}_1 + \mathbf{B}_4[\psi] = \mathbf{B}_1(\mathbf{I} + \mathbf{B}_1^{-1}\mathbf{B}_4[\psi]), \tag{3.61}$$

$$\mathbf{I} + \mathbf{B}_1^{-1}\mathbf{B}_4[\psi] = \mathbf{B}_1^{-1}(\mathbf{B}_1 + \mathbf{B}_4[\psi]). \tag{3.62}$$

Since the opeator $\mathbf{B}_4[\psi]$ is completely continuous (see Lemma 3.4.2), then also the operator $\mathbf{B}_1^{-1}\mathbf{B}_4[\psi]$ will be completely continuous. Next, the equality (3.62) and Lemma 3.4.1 imply that an equation

$$\xi + \mathbf{B}_1^{-1}\mathbf{B}_4[\psi](\xi) = \mathbf{0}$$

can have only trivial solutions. But then, due to Fredholm alternative, there will exist the bounded inverse $(\mathbf{I} + \mathbf{B}_1^{-1}\mathbf{B}_4[\psi])^{-1}$.

Finally, since there exist $\mathbf{B}_1^{-1}$ and $(\mathbf{I} + \mathbf{B}_1^{-1}\mathbf{B}_4[\psi])^{-1}$, then the equality (3.61) implies existence of bounded inverse $(\mathbf{B}_1 + \mathbf{B}_4[\psi])^{-1}$.

▶

3.5 Fréchet differentiability

In this section we will finalize analysis of the optimal control problem (1.5). As a result, we will prove Fréchet differentiability of the control-to-state operator $T = \mathbf{\Gamma}(T_h)$ and the cost functional $\mathcal{J}(\mathbf{\Gamma}(T_h))$ (see the formula (1.10)).

Let us start with the Fréchet differentiability of the control-to-state operator $T = \mathbf{\Gamma}(T_h)$.

Theorem 3.5.1
For every two arbitrary chosen controls $T_h \in U$, $\hat{T}_h \in U$ the following formula holds:

$$\|\mathbf{\Gamma}(\hat{T}_h) - \mathbf{\Gamma}(T_h) - \mathbf{\Lambda}[T_h](\hat{T}_h - T_h)\|_{H_1(\Omega_f,\, S_1)} = \mathbf{o}[T_h](\|\hat{T}_h - T_h\|_{L_\infty(\Sigma_2)}), \tag{3.63}$$

where $\mathbf{\Lambda}[T_h] \in \mathscr{L}(L_\infty(\Sigma_2),\, H_1(\Omega_f,\, S_1))$.

◀ Let us fix arbitrary state-control pairs $(T,\, T_h)$, $(\hat{T},\, \hat{T}_h)$ and let us temporarily denote:

$$u = T - T_*, \quad \hat{u} = \hat{T} - T_*, \quad \psi = 4|T|^3, \quad \vartheta = 4|T_h|^3.$$

Lemma 3.4.3 and the equation (3.1) imply that

$$\begin{aligned}
\mathbf{\Gamma}(\hat{T}_h) - \mathbf{\Gamma}(T_h) - (\mathbf{B}_1 + \mathbf{B}_4[\psi])^{-1}(\mathbf{B}_3(\hat{T}_h) - \mathbf{B}_3(T_h)) \\
= (\hat{T} - T) - (\mathbf{B}_1 + \mathbf{B}_4[\psi])^{-1}(\mathbf{B}_3(\hat{T}_h) - \mathbf{B}_3(T_h)) \\
= (\mathbf{B}_1 + \mathbf{B}_4[\psi])^{-1}((\mathbf{B}_1 + \mathbf{B}_4[\psi])(\hat{T} - T) - (\mathbf{B}_1 + \mathbf{B}_2)(\hat{u}) + (\mathbf{B}_1 + \mathbf{B}_2)(u)) \\
= (\mathbf{B}_1 + \mathbf{B}_4[\psi])^{-1}(\mathbf{B}_4[\psi](\hat{T} - T) - \mathbf{B}_2(\hat{u}) + \mathbf{B}_2(u)).
\end{aligned}$$

By using the last equality, we can get that

$$
\begin{aligned}
\|\mathbf{\Gamma}(\hat{T}_h) - \mathbf{\Gamma}(T_h) - (\mathbf{B}_1 + \mathbf{B}_4[\psi])^{-1}(\mathbf{B}_3(\hat{T}_h) - \mathbf{B}_3(T_h))\|_{H_1(\Omega_f,\, S_1)} \\
\le \|(\mathbf{B}_1 + \mathbf{B}_4[\psi])^{-1}\|_{\mathscr{L}(H_1^*(\Omega_f,\, S_1),\, H_1(\Omega_f,\, S_1))} \\
\times \|\mathbf{B}_2(\hat{u}) - \mathbf{B}_2(u) - \mathbf{B}_4[\psi](\hat{T} - T)\|_{H_1^*(\Omega_f,\, S_1)}.
\end{aligned} \tag{3.64}
$$

Next, due to (3.64) and Lemma 3.3.1, we can write

$$
\begin{aligned}
&\frac{\|\mathbf{\Gamma}(\hat{T}_h) - \mathbf{\Gamma}(T_h) - (\mathbf{B}_1 + \mathbf{B}_4[\psi])^{-1}(\mathbf{B}_3(\hat{T}_h) - \mathbf{B}_3(T_h))\|_{H_1(\Omega_f,\, S_1)}}{\|\hat{T}_h - T_h\|_{L_\infty(\Sigma_2)}} \\
&\le \hat{c}_1 \frac{\|\mathbf{B}_2(\hat{u}) - \mathbf{B}_2(u) - \mathbf{B}_4[\psi](\hat{T} - T)\|_{H_1^*(\Omega_f,\, S_1)}}{\|\hat{T} - T\|_{L_\infty(\Sigma_1)}} \\
&\le \hat{c}_2 \frac{\|\mathbf{G}_{\Sigma_1\Sigma_1}(|\hat{T}|^3\hat{T} - |T|^3 T - 4|T|^3(\hat{T} - T))\|_{L_\infty(\Sigma_1)}}{\|\hat{T} - T\|_{L_\infty(\Sigma_1)}}.
\end{aligned} \tag{3.65}
$$

At the same time the following formula holds:

$$
\begin{aligned}
&|\hat{T}|^3\hat{T} - |T|^3 T - 4|T|^3(\hat{T} - T) \\
&= \int_0^1 12|T + t(\hat{T} - T)|(T + t(\hat{T} - T))(\hat{T} - T)^2(1 - t)\, dt.
\end{aligned}
$$

If we now put the last equality in the inequality (3.65), then due to Theorem 3.2.1, Theorem 3.2.2 and the assertion 2.3.7i we will get that

$$
\begin{aligned}
&\frac{\|\mathbf{\Gamma}(\hat{T}_h) - \mathbf{\Gamma}(T_h) - (\mathbf{B}_1 + \mathbf{B}_4[\psi])^{-1}(\mathbf{B}_3(\hat{T}_h) - \mathbf{B}_3(T_h))\|_{H_1(\Omega_f,\, S_1)}}{\|\hat{T}_h - T_h\|_{L_\infty(\Sigma_2)}} \\
&\le \hat{c}_3 \frac{\|\int_0^1 12|T + t(\hat{T} - T)|(T + t(\hat{T} - T))(\hat{T} - T)^2(1 - t)\, dt\|_{L_\infty(\Sigma_1)}}{\|\hat{T} - T\|_{L_\infty(\Sigma_1)}} \\
&\le \hat{c}_4 \|\hat{T} - T\|_{L_\infty(\Sigma_1)}.
\end{aligned} \tag{3.66}
$$

Next, let us temporarily define the functional

$$
\mathbf{f}(\eta) := \int_{\Sigma_1} \mathbf{G}_{\Sigma_2\Sigma_1}(\varpi(\hat{T}_h - T_h)^2)\eta\, d\lambda_{\Sigma_1} \quad \eta \in H_1(\Omega_f,\, S_1),
$$

where

$$
\varpi = \int_0^1 12|T_h + t(\hat{T}_h - T_h)|(T_h + t(\hat{T}_h - T_h))(1 - t)\, dt.
$$

By using embedding theorems for $H_1(\Omega_f,\, S_1)$ and by taking into account properties of the operator $\mathbf{G}_{\Sigma_2\Sigma_1}$ and the functions T_h, $\hat{T}_h$, one can get that $\mathbf{f} \in H_1^*(\Omega_f,\, S_1)$ and

$$
\|\mathbf{f}\|_{H_1^*(\Omega_f,\, S_1)} \le \hat{c}_5 \|\mathbf{G}_{\Sigma_2\Sigma_1}(\varpi(\hat{T}_h - T_h)^2)\|_{L_\infty(\Sigma_2)} \le \hat{c}_6 \|\hat{T}_h - T_h\|^2_{L_\infty(\Sigma_2)}. \tag{3.67}
$$

Since the following formula holds:

$$|\hat{T}_h|^3\hat{T}_h - |T_h|^3T_h - 4|T_h|^3(\hat{T}_h - T_h)$$
$$= \int_0^1 12|T_h + t(\hat{T}_h - T_h)|(T_h + t(\hat{T}_h - T_h))(\hat{T}_h - T_h)^2(1-t)\,dt,$$

then we can write:

$$\mathbf{B}_3(\hat{T}_h) - \mathbf{B}_3(T_h) = -\mathbf{B}_5[\vartheta](\hat{T}_h - T_h) - \mathbf{f}(\eta). \tag{3.68}$$

Now, from (3.66), (3.67), (3.68) it will follow that

$$\frac{\|\mathbf{\Gamma}(\hat{T}_h) - \mathbf{\Gamma}(T_h) - (\mathbf{B}_1 + \mathbf{B}_4[\psi])^{-1}(-\mathbf{B}_5[\vartheta](\hat{T}_h - T_h))\|_{H_1(\Omega_f,\, S_1)}}{\|\hat{T}_h - T_h\|_{L_\infty(\Sigma_2)}}$$
$$\leq \hat{c}_4\|\hat{T} - T\|_{L_\infty(\Sigma_1)} + \hat{c}_7\frac{\|\mathbf{f}\|_{H_1^*(\Omega_f,\, S_1)}}{\|\hat{T}_h - T_h\|_{L_\infty(\Sigma_2)}}$$
$$\leq \hat{c}_4\|\hat{T} - T\|_{L_\infty(\Sigma_1)} + \hat{c}_8\|\hat{T}_h - T_h\|_{L_\infty(\Sigma_2)}.$$

It is easy to see that the last inequality yields statement of this theorem, if one will choose

$$\mathbf{\Lambda}[T_h] = -(\mathbf{B}_1 + \mathbf{B}_4[\psi])^{-1}\mathbf{B}_5[\vartheta].$$

▶

Finally, one can easily check that the cost functional $\mathcal{J}(T)$ is Fréchet differentiable as mapping from $H_1(\Omega_f,\, S_1)$ to $\mathbb{R}$. If we now combine this result with the previous theorem, then by applying the chain rule for the functional $\mathcal{J}(\mathbf{\Gamma}(T_h))$ it is easy to get the following theorem.

Theorem 3.5.2
For every two arbitrary chosen controls $T_h \in U$, $\hat{T}_h \in U$ the following formula holds (see the formula (1.10)*):*

$$\mathcal{J}(\mathbf{\Gamma}(\hat{T}_h)) - \mathcal{J}(\mathbf{\Gamma}(T_h)) = \ell[T_h](\hat{T}_h - T_h) + \mathbf{o}[T_h](\|\hat{T}_h - T_h\|_{L_\infty(\Sigma_2)}),$$

where $\ell[T_h] \in L_\infty^*(\Sigma_2)$.

Chapter 4

Analysis of reduced problem

4.1 Boundedness of gradient

Before we start sensitivity analysis of the optimal control problem (1.14) and justify transition from the original problem (1.5) to reduced one, let us define the following functions:

$$
\begin{aligned}
h(x) &:= -\mathbf{G}_{\Sigma_2\Sigma_1}(T_h^4)(x) + k_3 T_g(x) + \sigma \mathbf{H}_{\Sigma_1\Sigma_1}(T^4)(x) && x \in \Sigma_1,\\
h_i(x) &:= -\mathbf{G}_{\Sigma_2 S_i}(T_h^4)(x) + k_3 T_g(x) + \sigma \mathbf{H}_{\Sigma_1 S_i}(T^4)(x) && x \in S_i,\\
g_i(x) &:= -\mathbf{G}_{\Sigma_2 S_i}(T_h^4)(x) + k_3 T_g(x) + \sigma \mathbf{H}_{\Sigma_0 S_i}(T^4)(x) && x \in S_i,\\
f_i(x) &:= -\mathbf{G}_{\Sigma_2 S_i}(T_h^4)(x) + k_3 T_g(x) && x \in S_i,
\end{aligned}
$$

where $i \in \{3,\, 4,\, 5,\, 6\}$. Due to Theorem 3.2.1 and Theorem 3.2.2 we have $0 \le T(x) \le \mu$ λ_{Σ_1}-a.e. on Σ_1. But this fact together with properties of the operators $\mathbf{G}_{\Sigma_2\Sigma_1}$, $\mathbf{G}_{\Sigma_2 S_i}$, $\mathbf{H}_{\Sigma_1\Sigma_1}$, $\mathbf{H}_{\Sigma_1 S_i}$, $\mathbf{H}_{\Sigma_0 S_i}$ (see Lemma 2.3.7) and B1, B2, B4, B5 will give boundedness of the functions h, h_i, g_i, f_i ($i \in \{3,\, 4,\, 5,\, 6\}$):

$$
\begin{aligned}
&h \in L_\infty(\Sigma_1),\ 0 \le h(x) \le 2\mu^4 + k_3\mu \ \lambda_{\Sigma_1}\text{-a.e. on } \Sigma_1,\\
&h_i \in L_\infty(S_i),\ 0 \le h_i(x) \le 2\mu^4 + k_3\mu \ \lambda_{S_i}\text{-a.e. on } S_i,\\
&g_i \in L_\infty(S_i),\ 0 \le g_i(x) \le 2\mu^4 + k_3\mu \ \lambda_{S_i}\text{-a.e. on } S_i,\\
&f_i \in L_\infty(S_i),\ 0 \le f_i(x) \le \mu^4 + k_3\mu \ \lambda_{S_i}\text{-a.e. on } S_i.
\end{aligned}
$$

Now, one can easily check that by replacing $u + T_*$ with T it is possible to rewrite the variational equality (1.6) as

$$
\begin{aligned}
\int_{\Omega_f} (k_1(\nabla T \cdot \nabla \eta) + k_2(\partial_1 T)\eta)\, d\omega_3 \\
+ \int_{\Sigma_1} \sigma T^4 \eta\, d\lambda_{\Sigma_1} + \int_{\Sigma_1} k_3 T \eta\, d\lambda_{\Sigma_1} \\
= \int_{\Sigma_1} h\eta\, d\lambda_{\Sigma_1} \quad \forall \eta \in \dot{V}_5(\Omega_f,\, S_1,\, \Sigma_1). \quad (4.1)
\end{aligned}
$$

Let us also introduce the following functions:

$$g_+(x_1,\,x_2,\,x_3) := \frac{g_5(x_1,\,x_2)+g_6(x_1,\,x_2)}{2} \qquad (x_1,\,x_2,\,x_3)\in\Omega_f,$$

$$g_-(x_1,\,x_2,\,x_3) := \frac{g_5(x_1,\,x_2)-g_6(x_1,\,x_2)}{2} \qquad (x_1,\,x_2,\,x_3)\in\Omega_f,$$

$$\dot{g}(x_1,\,x_2,\,x_3) := \frac{\mathbf{T}_{\delta^{1/4}}(g_5)(x_1,\,x_2)+\mathbf{T}_{\delta^{1/4}}(g_6)(x_1,\,x_2)}{2} \qquad (x_1,\,x_2,\,x_3)\in\Omega_f,$$

$$\xi(x_1,\,x_2,\,x_3) := \min\{\frac{x_1\min\{l_1^{1/2},\,l_2^{1/2}\}}{l_1\delta^{1/2}},\,1\} \qquad (x_1,\,x_2,\,x_3)\in\Omega_f,$$

$$\hat{g}(x_1,\,x_2) := \frac{g_5(x_1,\,x_2)+g_6(x_1,\,x_2)}{2} \qquad (x_1,\,x_2)\in Q_f,$$

$$\hat{f}(x_1,\,x_2) := \frac{f_5(x_1,\,x_2)+f_6(x_1,\,x_2)}{2} \qquad (x_1,\,x_2)\in Q_f.$$

Let us recall that $Q_f=(-l_1,l_1)\times(-l_2,l_2)$, but $\mathbf{T}_\tau$ is mollifier (see Chapter 2 for more details). After applying results of Lemma 2.4.1 it is possible to get that

$$\int_{\Omega_f}|\nabla\dot{g}|^2\,d\omega_3 \le 144c_8^2l_1l_2\delta^{1/2}, \tag{4.2}$$

$$\int_{\Omega_f}|\nabla\xi|^2\,d\omega_3 \le \frac{4\min\{l_1^{1/2},l_2^{1/2}\}l_2}{l_1}\delta^{1/2}. \tag{4.3}$$

Now, let us prove boundedness of $\||\nabla T|^2\xi^2\|_{L_1(\Omega_f)}$.

Lemma 4.1.1
There exists a constant $c_9\in[0,\,+\infty)$ such that for every state-control pair $(T,\,T_h)$ of (1.5) there is

$$\int_{\Omega_f}|\nabla T|^2\xi^2\,d\omega_3 \le c_9. \tag{4.4}$$

Furthermore, here c_9 does not depend on the parameter δ.

◄ Let us define the function

$$\eta := T\xi^2.$$

It is easy to see that $\eta\in\dot{V}_5(\Omega_f,\,S_1,\,\Sigma_1)$.

Since the functions T, h, η are uniformly bounded with respect to δ, then

$$|\int_{\Sigma_1}\sigma T^4\eta\,d\lambda_{\Sigma_1}|+|\int_{\Sigma_1}k_3T\eta\,d\lambda_{\Sigma_1}|+|\int_{\Sigma_1}h\eta\,d\lambda_{\Sigma_1}|\le\hat{c}_1. \tag{4.5}$$

Next, for arbitrary chosen $\tau \in \mathbb{R}_+$ we have

$$|\int_{\Omega_f} k_2(\partial_1 T)\eta \, d\omega_3| = |\int_{\Omega_f} k_2(\partial_1 T)T\xi^2 \, d\omega_3|$$
$$\leq k_2\tau \int_{\Omega_f} (\partial_1 T)^2\xi^2 \, d\omega_3 + \frac{k_2}{4\tau}\int_{\Omega_f} T^2\xi^2 \, d\omega_3$$

and, therefore, if we choose $\tau = k_1/4k_2$, then due to uniform boundedness of T with respect to δ we will have

$$|\int_{\Omega_f} k_2(\partial_1 T)\eta \, d\omega_3| \leq \frac{k_1}{4}\int_{\Omega_f} |\nabla T|^2\xi^2 \, d\omega_3 + \hat{c}_2. \tag{4.6}$$

For every arbitrary chosen $\tau \in \mathbb{R}_+$ we also have

$$\int_{\Omega_f} k_1(\nabla T \cdot \nabla\eta) \, d\omega_3 \geq k_1 \int_{\Omega_f} |\nabla T|^2\xi^2 \, d\omega_3$$
$$- k_1\tau \int_{\Omega_f} |\nabla T|^2\xi^2 \, d\omega_3 - \frac{k_1}{4\tau}\int_{\Omega_f} |\nabla\xi|^2 T^2 \, d\omega_3$$

and, if we choose $\tau = 1/4$ this time, then due to uniform boundedness of T with respect to δ and the estimate (4.3) we will have

$$\int_{\Omega_f} k_1(\nabla T \cdot \nabla\eta) \, d\omega_3 \geq \frac{3k_1}{4}\int_{\Omega_f} |\nabla T|^2\xi^2 \, d\omega_3 - \hat{c}_3, \tag{4.7}$$

Now, since $\eta \in \dot{V}_5(\Omega_f, S_1, \Sigma_1)$, then the integral equality (4.1) must hold for this η. If we will combine (4.1) with the estimates (4.5), (4.6), (4.7), then this will yield (4.4) for some $c_9 \in [0, +\infty)$. Here c_9 will be independent on the parameter δ ($\hat{c}_1$, $\hat{c}_2$, $\hat{c}_3$ are non-negative and independent on δ).

▶

4.2 Asymptotic behaviour of T in domain

In this section we investigate asymptotic behaviour of the state T in the degenerating domain Ω_f, as $\delta \to 0$.

Lemma 4.2.1
For every fixed $T_h \in U$ the corresponding state T of (1.5) has the following asymptotic behaviour:

$$\frac{1}{\delta}\int_{\Omega_f} (\sigma T^4 + k_3 T - g_+)^2 \, d\omega_3 \to 0,$$

as $\delta \to 0$.

◀ Let us define the function

$$\eta := (\sigma T^4 + k_3 T - \dot{g})\xi^2.$$

It is easy to see that $\eta \in \dot{V}_5(\Omega_f,\, S_1,\, \Sigma_1)$.

Since the estimates (4.2), (4.3), (4.4) hold and the functions T, $\dot{g}$ are uniformly bounded with respect to δ, then

$$\int_{\Omega_f} |\nabla \eta|^2 \, d\omega_3 \le \int_{\Omega_f} 48(\sigma T^3 + k_3)^2 \xi^4 |\nabla T|^2 \, d\omega_3 + \int_{\Omega_f} 12\xi^4 |\nabla \dot{g}|^2 \, d\omega_3 \\ + \int_{\Omega_f} 24(\sigma T^4 + k_3 T - \dot{g})^2 \xi^2 |\nabla \xi|^2 \, d\omega_3 \le \hat{c}_1 \quad (4.8)$$

Next, due to the same reasons we will have

$$\int_{\Omega_f} k_1 (\nabla T \cdot \nabla \eta) \, d\omega_3 \\ = k_1 \int_{\Omega_f} (4\sigma T^3 + k_3)\xi^2 |\nabla T|^2 \, d\omega_3 - k_1 \int_{\Omega_f} \xi^2 (\nabla T \cdot \nabla \dot{g}) \, d\omega_3 \\ + k_1 \int_{\Omega_f} 2(\sigma T^4 + k_3 T - \dot{g})\xi (\nabla T \cdot \nabla \xi) \, d\omega_3 \\ \ge -k_1 (\int_{\Omega_f} \xi^2 |\nabla \dot{g}|^2 \, d\omega_3)^{1/2} (\int_{\Omega_f} |\nabla T|^2 \xi^2 \, d\omega_3)^{1/2} \\ - k_1 (\int_{\Omega_f} (\sigma T^4 + k_3 T - \dot{g})^2 |\nabla \xi|^2 \, d\omega_3)^{1/2} (\int_{\Omega_f} |\nabla T|^2 \xi^2 \, d\omega_3)^{1/2} \\ \ge -\hat{c}_2 \delta^{1/4}, \quad (4.9)$$

$$\int_{\Omega_f} k_2 (\partial_1 T) \eta \, d\omega_3 = k_2 \int_{\Omega_f} (\partial_1 T)(\sigma T^4 + k_3 T - \dot{g})\xi^2 \, d\omega_3 \\ \ge -k_2 (\int_{\Omega_f} (\sigma T^4 + k_3 T - \dot{g})^2 \xi^2 \, d\omega_3)^{1/2} (\int_{\Omega_f} |\nabla T|^2 \xi^2 \, d\omega_3)^{1/2} \\ \ge -\hat{c}_3 \delta^{1/2}. \quad (4.10)$$

Now, let us get an upper estimate for the integral

$$-\int_{\Sigma_1} \sigma T^4 \eta \, d\lambda_{\Sigma_1}.$$

By using the Gauss's formula we can rewrite this integral as

$$-\int_{\Sigma_1} \sigma T^4 \eta \, d\lambda_{\Sigma_1} = -\int_{\Omega_f} \frac{\sigma}{\delta} (\partial_3 (T^4 \eta x_3)) \, d\omega_3 - \int_{S_3 \cup S_4} \sigma T^4 \eta \, d\lambda_{S_3 \cup S_4}. \quad (4.11)$$

We also have the following expansion:

$$\int_{\Omega_f} \frac{\sigma}{\delta}(\partial_3(T^4\eta x_3))\,d\omega_3 = \int_{\Omega_f} \frac{4\sigma}{\delta}T^3(\partial_3 T)\eta x_3\,d\omega_3 + \int_{\Omega_f} \frac{\sigma}{\delta}T^4(\partial_3\eta)x_3\,d\omega_3 + \frac{1}{\delta}\int_{\Omega_f} \sigma T^4\eta\,d\omega_3. \quad (4.12)$$

Since the estimates (4.4), (4.8) hold and the functions T, $\dot{g}$, η are uniformly bounded with respect to δ, then

$$\begin{aligned}\int_{\Omega_f} \frac{4\sigma}{\delta}T^3(\partial_3 T)\eta x_3\,d\omega_3 &\geq -[(\int_{\Omega_f} 16(\frac{\sigma}{\delta}x_3)^2T^6(\sigma T^4 + k_3T - \dot{g})^2\xi^2\,d\omega_3)^{1/2} \\ &\qquad \times (\int_{\Omega_f} |\nabla T|^2\xi^2\,d\omega_3)^{1/2}] \\ &\geq -\hat{c}_4(\int_{\Omega_f} (\frac{\sigma}{\delta}x_3)^2\,d\omega_3)^{1/2} \geq -\hat{c}_5\delta^{1/2}, \quad (4.13)\end{aligned}$$

$$\begin{aligned}\int_{\Omega_f} \frac{\sigma}{\delta}T^4(\partial_3\eta)x_3\,d\omega_3 &\geq -(\int_{\Omega_f} (\frac{\sigma}{\delta}x_3)^2T^8\,d\omega_3)^{1/2}(\int_{\Omega_f} |\nabla\eta|^2\,d\omega_3)^{1/2} \\ &\geq -\hat{c}_6(\int_{\Omega_f} (\frac{\sigma}{\delta}x_3)^2\,d\omega_3)^{1/2} \geq -\hat{c}_7\delta^{1/2} \quad (4.14)\end{aligned}$$

and

$$\int_{S_3\cup S_4} \sigma T^4\eta\,d\lambda_{S_3\cup S_4} \geq -\hat{c}_8\delta. \quad (4.15)$$

Now, if we put (4.11), (4.12), (4.13), (4.14), (4.15) together, then finally we will get

$$-\int_{\Sigma_1} \sigma T^4\eta\,d\lambda_{\Sigma_1} \leq -\frac{1}{\delta}\int_{\Omega_f} \sigma T^4\eta\,d\omega_3 + \hat{c}_9\delta^{1/2}. \quad (4.16)$$

Next, by using Gauss's formula and estimation techniques introduced before, one can get the following estimates:

$$\int_{\Sigma_1} h\eta\,d\lambda_{\Sigma_1} \leq \frac{1}{\delta}\int_{\Omega_f} g_+\eta\,d\omega_3 + \hat{c}_{10}\delta^{1/2}, \quad (4.17)$$

$$-\int_{\Sigma_1} k_3T\eta\,d\lambda_{\Sigma_1} \leq -\frac{1}{\delta}\int_{\Omega_f} k_3T\eta\,d\omega_3 + \hat{c}_{11}\delta^{1/2}. \quad (4.18)$$

Now, since $\eta \in \dot{V}_5(\Omega_f,\, S_1,\, \Sigma_1)$, then the integral equality (4.1) must hold for this η. If we will combine (4.1) with estimates (4.9), (4.10), (4.16), (4.17), (4.18), then this will yield the following inequality for some $\hat{c}_{12} \in [0,\, +\infty)$, which will be independent on the parameter δ ($\hat{c}_2$, $\hat{c}_3$, $\hat{c}_9$, $\hat{c}_{10}$, $\hat{c}_{11}$ are non-negative and independent on δ):

$$-\hat{c}_{12}\delta^{1/4} \le \frac{1}{\delta}\int_{\Omega_f} (g_+ - \sigma T^4 - k_3 T)\eta \, d\omega_3,$$

i.e.

$$\frac{1}{\delta}\int_{\Omega_f} (\sigma T^4 + k_3 T - g_+)\eta \, d\omega_3 \le \hat{c}_{12}\delta^{1/4}. \tag{4.19}$$

If we now consider the set $\Omega_f' := \{x \in \Omega_f : \xi(x) = 1\}$, then (4.19) will imply

$$\frac{1}{\delta}\int_{\Omega_f'} (\sigma T^4 + k_3 T - g_+)^2 \, d\omega_3 \le \hat{c}_{12}\delta^{1/4} + \frac{1}{\delta}\int_{\Omega_f'} (\sigma T^4 + k_3 T - g_+)|g_+ - \dot{g}| \, d\omega_3$$

and, therefore, for an arbitrary $\tau \in \mathbb{R}_+$ we will have

$$\frac{1}{\delta}\int_{\Omega_f'} (\sigma T^4 + k_3 T - g_+)^2 \, d\omega_3 \le \hat{c}_{12}\delta^{1/4} + \frac{\tau}{\delta}\int_{\Omega_f'} (\sigma T^4 + k_3 T - g_+)^2 \, d\omega_3 + \frac{1}{4\tau\delta}\int_{\Omega_f} (g_+ - \dot{g})^2 \, d\omega_3.$$

Now, if we put $\tau = 1/2$ in the last expression, then due to uniform boundedness of T with respect to δ we will get

$$\frac{1}{\delta}\int_{\Omega_f'} (\sigma T^4 + k_3 T - g_+)^2 \, d\omega_3 \le 2\hat{c}_{12}\delta^{1/4} + \frac{\hat{c}_{13}}{\delta}\int_{\Omega_f} (g_+ - \dot{g})^2 \, d\omega_3 \le 2\hat{c}_{12}\delta^{1/4} + \frac{\hat{c}_{13}}{2}\int_{S_5} (\mathbf{T}_{\delta^{1/4}}(g_5) - g_5)^2 \, d\lambda_{S_5} + \frac{\hat{c}_{13}}{2}\int_{S_6} (\mathbf{T}_{\delta^{1/4}}(g_6) - g_6)^2 \, d\lambda_{S_6}.$$

But the last inequality together with Lemma 2.4.1 imply that

$$\frac{1}{\delta}\int_{\Omega_f'} (\sigma T^4 + k_3 T - g_+)^2 \, d\omega_3 \to 0, \tag{4.20}$$

as $\delta \to 0$. At the same time $\Omega_f \backslash \Omega_f' = (0,\, \frac{l_1\delta^{1/2}}{\min\{l_1^{1/2}, l_2^{1/2}\}}) \times (-l_2,\, l_2) \times (-\delta,\, \delta)$, and, since the functions T, g_+ are uniformly bounded with respect to δ, then

$$\int_{\Omega_f \backslash \Omega_f'} (\sigma T^4 + k_3 T - g_+)^2 \, d\omega_3 \le \hat{c}_{14}\delta^{3/2},$$

and, therefore, we will have that

$$\frac{1}{\delta}\int_{\Omega_f\setminus\Omega_f'}(\sigma T^4+k_3T-g_+)^2\,d\omega_3\to 0, \tag{4.21}$$

as $\delta\to 0$.

If we now put the formulas (4.20), (4.21) together, then this will yield the statement of this lemma.

▶

4.3 Asymptotic behaviour of T on boundary

In this section we investigate asymptotic behaviour of the state T on the boundary of the degenerating domain Ω_f, as $\delta\to 0$.

Lemma 4.3.1
For every fixed $T_h\in U$ the corresponding state T of (1.5) *has the following asymptotic behaviour on the boundary:*

$$\int_{\Sigma_0}(\sigma T^4+k_3T-g_+)^2\,d\lambda_{\Sigma_0}\to 0,$$

as $\delta\to 0$.

◀ Proof of this lemma is similar to the proof of the previous lemma. Therefore, let us define the function

$$\eta:=(\sigma T^4+k_3T-\dot{g})\xi^2.$$

It is easy to see that $\eta\in\dot{V}_5(\Omega_f,\,S_1,\,\Sigma_1)$.

Next, the following estimates hold

$$\int_{\Omega_f}|\nabla\eta|^2\,d\omega_3\le\hat{c}_1, \tag{4.22}$$

$$\int_{\Omega_f}k_1(\nabla T\cdot\nabla\eta)\,d\omega_3\ge-\hat{c}_2\delta^{1/4}, \tag{4.23}$$

$$\int_{\Omega_f}k_2(\partial_1 T)\eta\,d\omega_3\ge-\hat{c}_3\delta^{1/2} \tag{4.24}$$

by analogy with (4.8), (4.9), (4.10).

Now, since the functions T, η are uniformly bounded with respect to δ on the surface Σ_0, then we get that

$$
\begin{aligned}
&-\int_{\Sigma_1} \sigma T^4 \eta \, d\lambda_{\Sigma_1} \\
&\qquad \le -\int_{S_3\cup S_4} \sigma T^4 \eta \, d\lambda_{S_3\cup S_4} - \int_{S_5\cup S_6} \sigma T^4 \eta \, d\lambda_{S_5\cup S_6} \\
&\qquad\qquad \le \hat{c}_4 \delta - \int_{S_5} \sigma T^4 \eta \, d\lambda_{S_5} - \int_{S_6} \sigma T^4 \eta \, d\lambda_{S_6}. \qquad (4.25)
\end{aligned}
$$

Similarly one can get

$$
\begin{aligned}
&\int_{\Sigma_1} (h - k_3 T)\eta \, d\lambda_{\Sigma_1} \le \hat{c}_5 \delta \\
&\qquad + \int_{S_5} (g_5 - k_3 T)\eta \, d\lambda_{S_5} + \int_{S_6} (g_6 - k_3 T)\eta \, d\lambda_{S_6}. \qquad (4.26)
\end{aligned}
$$

Since $\eta \in \dot{V}_5(\Omega_f,\, S_1,\, \Sigma_1)$, then the integral equality (4.1) must hold for this η. If we now combine it with the estimates (4.23), (4.24), (4.25), (4.26), then the following inequality will hold:

$$
\begin{aligned}
-\hat{c}_6 \delta^{1/4} &\le \int_{S_5} (g_5 - \sigma T^4 - k_3 T)\eta \, d\lambda_{S_5} \\
&\qquad + \int_{S_6} (g_6 - \sigma T^4 - k_3 T)\eta \, d\lambda_{S_6} \\
&\qquad = \int_{\Sigma_0} (g_+ - \sigma T^4 - k_3 T)\eta \, d\lambda_{\Sigma_0} \\
&\qquad\qquad - \int_{S_5} g_- \eta \, d\lambda_{S_5} + \int_{S_6} g_- \eta \, d\lambda_{S_6}. \qquad (4.27)
\end{aligned}
$$

Next, by using of Gauss's formula, we will have

$$
-\int_{S_5} g_- \eta \, d\lambda_{S_5} + \int_{S_6} g_- \eta \, d\lambda_{S_6} = \int_{\Omega_f} (\partial_3 (g_- \eta)) \, d\omega_3 = \int_{\Omega_f} g_- (\partial_3 \eta) \, d\omega_3. \qquad (4.28)
$$

Since the estimates (4.4), (4.22) hold and the function g_- is uniformly bounded with respect to δ, then

$$
\int_{\Omega_f} g_- (\partial_3 \eta) \, d\omega_3 \le \Big(\int_{\Omega_f} g_-^2 \, d\omega_3\Big)^{1/2} \Big(\int_{\Omega_f} |\nabla \eta|^2 \, d\omega_3\Big)^{1/2} \le \hat{c}_7 \delta^{1/2}. \qquad (4.29)
$$

Now, due to the equality (4.28) and the estimates (4.27), (4.29) the following inequality will hold for some $\hat{c}_9 \in [0, +\infty)$, which will be independent on δ (see (4.19)):

$$
\int_{\Sigma_0} (\sigma T^4 + k_3 T - g_+)\eta \, d\lambda_{\Sigma_0} \le \hat{c}_9 \delta^{1/4}. \qquad (4.30)
$$

Further proof of this lemma is similar with the proof of Lemma 4.2.1 (see after (4.19)). However, the proof in this case will be based on the estimate (4.30) instead of using of (4.19).

▶

4.4 Reduced problem

In this section we will perform sensitivity analysis of the optimal control problem (1.14). We will start by proving existence of feasible states for this problem. As it will turn out, there exists exactly one feasible state $\tilde{T}$ for every fixed admissible control T_h, therefore, this fact will automatically yield existence of the control-to-state operator $\tilde{T} = \tilde{\mathbf{\Gamma}}(T_h)$. Afterwards we will obtain Lipschitz continuity and Fréchet differentiability of the control-to-state operator $\tilde{T} = \tilde{\mathbf{\Gamma}}(T_h)$ in appropriate functional spaces. Finally, we will prove Fréchet differentiability of the cost functional $\mathcal{J}(\tilde{\mathbf{\Gamma}}(T_h))$ (see the formula (1.15)).

Let us start with some definitions. Let us define the function

$$a(\tau) := \sigma|\tau|^3\tau + k_3\tau \quad \tau \in \mathbb{R}.$$

Since

$$a'(\tau) = 4\sigma|\tau|^3 + k_3 > 0 \quad \forall \tau \in \mathbb{R},$$

then due to the implicit function theorem there exists a function $b : \mathbb{R} \mapsto \mathbb{R}$ such that

$$b(a(\tau)) = \tau \quad \forall \tau \in \mathbb{R}.$$

Next, let us also define the function

$$c(\tau) := \sigma|b(\tau)|^3 b(\tau) \quad \tau \in \mathbb{R}.$$

Lemma 4.4.1
There exist some constants $c_{10} \in [0, +\infty)$, $c_{11} \in [0, +\infty)$, $c_{12} \in [0, +\infty)$, $c_{13} \in [0, +\infty)$ *such that for all* $\tau \in \mathbb{R}$ *the following estimates hold:*

$$|b'(\tau)| \le c_{10}, \tag{4.31}$$

$$|c'(\tau)| \le c_{11}, \tag{4.32}$$

$$|b''(\tau)| \le c_{12}, \tag{4.33}$$

$$|c''(\tau)| \le c_{13}. \tag{4.34}$$

◀ Let us fix arbitrary $\tau \in \mathbb{R}$.

As b is inverse function of a, then we will have

$$|b'(\tau)| = \frac{1}{(4\sigma|b(\tau)|^3 + k_3)} \le \frac{1}{k_3}.$$

For the second order derivative the previous estimate will yield

$$|b''(\tau)| = \frac{24\sigma|b(\tau)|^2|b'(\tau)|}{(4\sigma|b(\tau)|^3 + k_3)^2} \leq \max\{\frac{24\sigma}{k_3}, \frac{24}{16k_3^5}\}.$$

Now, according to definition of the function c, we will get

$$|c'(\tau)| = 4\sigma|b(\tau)|^3|b'(\tau)| = \frac{4\sigma|b(\tau)|^3}{(4\sigma|b(\tau)|^3 + k_3)} \leq 1.$$

And again, for the second order derivative the estimate for $|b'(\tau)|$ yields

$$\begin{aligned}|c''(\tau)| &\leq \frac{12\sigma|b(\tau)|^2|b'(\tau)|}{(4\sigma|b(\tau)|^3 + k_3)} + \frac{48\sigma^2|b(\tau)|^5|b'(\tau)|}{(4\sigma|b(\tau)|^3 + k_3)^2} \\ &\leq \frac{12\sigma|b(\tau)|^2}{k_3(4\sigma|b(\tau)|^3 + k_3)} + \frac{48\sigma^2|b(\tau)|^5}{k_3(4\sigma|b(\tau)|^3 + k_3)^2} \\ &\leq \frac{2}{k_3}\max\{\frac{3}{k_3}, 12\sigma k_3, 48\sigma^2 k_3^3\}.\end{aligned}$$

Now, in order to finish the proof, let us take:

$$c_{10} := \frac{1}{k_3}, \qquad c_{11} := \max\{\frac{24\sigma}{k_3}, \frac{24}{16k_3^5}\},$$
$$c_{12} := 1, \qquad c_{13} := \frac{2}{k_3}\max\{\frac{3}{k_3}, 12\sigma k_3, 48\sigma^2 k_3^3\}.$$

►

Now, let us recall that $Q_f = (-l_1, l_1) \times (-l_2, l_2)$ and introduce the following operators:

$$\begin{aligned}
&\mathbf{W}_1(v)(x_1, x_2) := c(v(x_1, x_2)) && (x_1, x_2) \in Q_f, \\
&\mathbf{W}_2(v)(x_1, x_2) := b(v(x_1, x_2)) && (x_1, x_2) \in Q_f, \\
&\mathbf{A}_1(v)(x_1, x_2) := \frac{v(x_1, x_2, \delta) + v(x_1, x_2, -\delta)}{2} && (x_1, x_2) \in Q_f, \\
&\mathbf{A}_2(v)(x_1, x_2, x_3) := v(x_1, x_2) && (x_1, x_2, x_3) \in \Sigma_0, \\
&\mathbf{A}_3(v)(x_1, x_2, x_3) := v(x_1, x_2) && (x_1, x_2, x_3) \in \Omega_f
\end{aligned}$$

For convenience let us also define:

$$\mathbf{U}_1(v) := \mathbf{A}_1\mathbf{H}_{\Sigma_0\Sigma_0}\mathbf{A}_2(v), \quad \mathbf{U}_2(v) := -\mathbf{A}_1\mathbf{G}_{\Sigma_2\Sigma_0}(v^4) + k_3\mathbf{A}_1(T_g|_{\Sigma_0}).$$

Basic properties of these operators are formulated in the following lemma.

Lemma 4.4.2
For every fixed $p \in [1, +\infty]$:

i. $\mathbf{W}_1$ *is Lipschitz continuous from* $L_p(Q_f)$ *to* $L_p(Q_f)$ *with Lipschitz constant less or equal to* 1*;*

ii. $\mathbf{W}_2$ *is Lipschitz continuous from* $L_p(Q_f)$ *to* $L_p(Q_f)$ *with Lipschitz constant less or equal to* $1/k_3$*;*

iii. $(\mathbf{W}_2)^{-1}$ *maps* $L_{4p}(Q_f)$ *to* $L_p(Q_f)$*;*

iv. $\mathbf{A}_3$ *is linear and bounded from* $L_p(Q_f)$ *to* $\tilde{L}_p(\Omega_f) \subset L_p(\Omega_f)$ *(for definition of the spaces* $\tilde{L}_p(\Omega_f)$ *see Notations);*

v. $\mathbf{U}_1$ *is linear and bounded from* $L_p(Q_f)$ *to* $L_p(Q_f)$ *and* $\|\mathbf{U}_1\|_{\mathscr{L}(L_p(Q_f),\, L_p(Q_f))} \leq (1 - c_3) < 1$*;*

vi. $\mathbf{U}_2$ *maps* $L_{4p}(\Sigma_2)$ *to* $L_p(Q_f)$*;*

◀ Let us fix $p \in [1, +\infty]$ and some arbitrary functions $v \in L_p(Q_f)$, $\hat{v} \in L_p(Q_f)$. Since

$$b(0) = 0, \quad c(0) = 0,$$

then due to the estimates (4.31), (4.32) we will have:

$$\begin{aligned}
&|c(\hat{v}(x_1,\, x_2)) - c(v(x_1,\, x_2))| \leq |\hat{v}(x_1,\, x_2) - v(x_1,\, x_2)|, \\
&|b(\hat{v}(x_1,\, x_2)) - b(v(x_1,\, x_2))| \leq \frac{|\hat{v}(x_1,\, x_2) - v(x_1,\, x_2)|}{k_3}, \\
&|c(v(x_1,\, x_2))| \leq |v(x_1,\, x_2)|, \\
&|b(v(x_1,\, x_2))| \leq \frac{|v(x_1,\, x_2)|}{k_3}
\end{aligned}$$

for almost every $(x_1,\, x_2) \in Q_f$, and, therefore, this will yield 4.4.2i, 4.4.2ii.

Now, let us fix an arbitrary function $v \in L_{4p}(Q_f)$. Since

$$(\mathbf{W}_2)^{-1}(v) = \sigma|v|^3 v + k_3 v,$$

then 4.4.2iii follows automatically.

4.4.2iv also follows automatically. Since it can be proved that $\mathbf{A}_1 \in \mathscr{L}(L_p(\Sigma_0),\, L_p(Q_f))$, $\mathbf{A}_2 \in \mathscr{L}(L_p(Q_f),\, L_p(\Sigma_0))$ and $\|\mathbf{A}_1\|_{\mathscr{L}(L_p(\Sigma_0),\, L_p(Q_f))} = 1$, $\|\mathbf{A}_2\|_{\mathscr{L}(L_p(Q_f),\, L_p(\Sigma_0))} = 1$, therefore, due to 2.3.7i, 2.3.7ix, B5 and definitions of $\mathbf{U}_1$, $\mathbf{U}_2$ we will get 4.4.2v, 4.4.2vi.

▶

Now, since $\mathbf{A} = \mathbf{A}_2\mathbf{A}_1$ (for definition of $\mathbf{A}$ see Chapter 1), therefore we can rewrite the problem (1.13) as

$$\begin{cases} \partial_3 \tilde{T} = 0 & \text{in } \Omega_f, \\ \mathbf{A}_2\mathbf{A}_1(\mathbf{Q}_{\Sigma_0\Sigma_0}(\tilde{T}) + \mathbf{Q}_{\Sigma_2\Sigma_0}(T_h)) = 0 & \text{on } \Sigma_0, \end{cases}$$

where $\tilde{T} \in L_\infty(\Omega_f)$ is the unknown function. Furthermore, by using a transformation $\tilde{T} = \mathbf{A}_3(\dot{T})$ the last problem can be to reduced to a single equation

$$(\mathbf{I} - \mathbf{U}_1 \circ \mathbf{W}_1) \circ (\mathbf{W}_2)^{-1}(\dot{T}) = \mathbf{U}_2(T_h), \tag{4.35}$$

where now $\dot{T} \in L_\infty(Q_f)$ is the unknown function.

By using properties of the operators $\mathbf{W}_1$, $\mathbf{U}_1$ there can be proved the following result.

Lemma 4.4.3
For every fixed $p \in [1, +\infty]$ there exists a Lipschitz continuous inverse operator

$$(\mathbf{I} - \mathbf{U}_1 \circ \mathbf{W}_1)^{-1} = \sum_{n=0}^{\infty} (\mathbf{U}_1 \circ \mathbf{W}_1)^n, \tag{4.36}$$

which maps $L_p(Q_f)$ to $L_p(Q_f)$.

◀ Let us fix $p \in [1, +\infty]$, $w \in L_p(Q_f)$ and let us consider the equation

$$(\mathbf{I} - \mathbf{U}_1 \circ \mathbf{W}_1)(v) = w, \tag{4.37}$$

where v is the unknown variable.

There exists exactly one solution of (4.37) due to the contraction mapping theorem. Indeed, if we choose arbitrary functions $v \in L_p(Q_f)$, $\hat{v} \in L_p(Q_f)$, then 4.4.2i, 4.4.2v imply that

$$\begin{aligned}
&\|(\mathbf{U}_1 \circ \mathbf{W}_1)(\hat{v}) - (\mathbf{U}_1 \circ \mathbf{W}_1)(v)\|_{L_p(Q_f)} \\
&\qquad \le \|\mathbf{U}_1\|_{\mathscr{L}(L_p(Q_f),\, L_p(Q_f))} \|\mathbf{W}_1(\hat{v}) - \mathbf{W}_1(v)\|_{L_p(Q_f)} \\
&\qquad\qquad \le (1 - c_3)\|\hat{v} - v\|_{L_p(Q_f)}.
\end{aligned}$$

This shows that $\mathbf{U}_1 \circ \mathbf{W}_1$ is Lipschitz continuous as mapping from $L_p(Q_f)$ to $L_p(Q_f)$ and its Lipschitz constant is less than 1. Moreover, this property is sufficient for the assertion of the lemma.

▶

If we now apply the operator $\mathbf{W}_2 \circ (\mathbf{I} - \mathbf{U}_1 \circ \mathbf{W}_1)^{-1}$ to both sides of (4.35) then for every fixed $T_h \in U$ we will get

$$\dot{T} = \mathbf{W}_2 \circ (\mathbf{I} - \mathbf{U}_1 \circ \mathbf{W}_1)^{-1} \circ \mathbf{U}_2(T_h),$$

and, since $\tilde{T} = \mathbf{A}_3(\dot{T})$, then for every fixed $T_h \in U$ there will be

$$\tilde{T} = \mathbf{A}_3 \circ \mathbf{W}_2 \circ (\mathbf{I} - \mathbf{U}_1 \circ \mathbf{W}_1)^{-1} \circ \mathbf{U}_2(T_h). \tag{4.38}$$

Next, it can be shown that for arbitrary $T_h \in U$, $\hat{T}_h \in U$

$$\|\mathbf{U}_2(T_h) - \mathbf{U}_2(\hat{T}_h)\|_{L_\infty(Q_f)} \le 4\mu^3 \|T_h - \hat{T}_h\|_{L_\infty(\Sigma_2)}.$$

Therefore, since the operator $\mathbf{A}_3 \circ \mathbf{W}_2 \circ (\mathbf{I} - \mathbf{U}_1 \circ \mathbf{W}_1)^{-1}$ is Lipschitz continuous as mapping from $L_\infty(Q_f)$ to $L_\infty(\Omega_f)$ (see results of the previous two lemmas), then due to (4.38) we will have the following result.

Theorem 4.4.4
For every fixed control $T_h \in U$ there exists one and only one feasible state $\tilde{T}$ of the optimal control problem (1.14) *and, therefore, there exists an unique defined control-to-state operator $\tilde{T} = \tilde{\mathbf{\Gamma}}(T_h)$ of the optimal control problem* (1.14) *as mapping from U to $\tilde{L}_\infty(\Omega_f) \subset L_\infty(\Omega_f)$. Furthermore, there exits some constant $c_{14} \in [0, +\infty)$ such that for every two arbitrary chosen controls $T_h \in U$, $\hat{T}_h \in U$ the following inequality holds:*

$$\|\tilde{\mathbf{\Gamma}}(T_h) - \tilde{\mathbf{\Gamma}}(\hat{T}_h)\|_{L_\infty(\Omega_f)} \leq c_{14}\|T_h - \hat{T}_h\|_{L_\infty(\Sigma_2)}.$$

In order to get the final result - Fréchet differentiability of the cost functional $\mathcal{J}(\tilde{\mathbf{\Gamma}}(T_h))$ (see the formula (1.15)), let us prove Fréchet differentiability of the control-to-state operator $\tilde{T} = \tilde{\mathbf{\Gamma}}(T_h)$. Therefore, for every fixed $\psi \in L_\infty(Q_f)$ let us define:

$$\begin{aligned} &\mathbf{W}_3[\psi](v)(x_1, x_2) := c'(\psi(x_1, x_2))v(x_1, x_2) \qquad && (x_1, x_2) \in Q_f, \\ &\mathbf{W}_4[\psi](v)(x_1, x_2) := b'(\psi(x_1, x_2))v(x_1, x_2) \qquad && (x_1, x_2) \in Q_f. \end{aligned}$$

Lemma 4.4.5
For every fixed $p \in [1, +\infty]$ and $\psi \in L_\infty(Q_f)$ operators $\mathbf{W}_3[\psi]$, $\mathbf{W}_4[\psi]$ are linear and bounded from $L_p(Q_f)$ to $L_p(Q_f)$ and

$$\|\mathbf{W}_3[\psi]\|_{\mathscr{L}(L_p(Q_f),\, L_p(Q_f))} \leq 1, \ \|\mathbf{W}_4[\psi]\|_{\mathscr{L}(L_p(Q_f),\, L_p(Q_f))} \leq 1/k_3.$$

◀ Let us fix arbitrary $v \in L_p(Q_f)$, $\psi \in L_\infty(Q_f)$. Then due to (4.31), (4.32) we will have

$$\begin{aligned} |c'(\psi(x_1, x_2))v(x_1, x_2)| &\leq |v(x_1, x_2)|, \\ |b'(\psi(x_1, x_2))v(x_1, x_2)| &\leq \frac{|v(x_1, x_2)|}{k_3} \end{aligned}$$

for almost every $(x_1, x_2) \in Q_f$.

▶

Lemma 4.4.6
For every fixed $p \in [1, +\infty]$ and $\psi \in L_\infty(Q_f)$ there exists a linear and bounded inverse operator

$$(\mathbf{I} - \mathbf{U}_1\mathbf{W}_3[\psi])^{-1} = \sum_{n=0}^{\infty}(\mathbf{U}_1\mathbf{W}_3[\psi])^n, \tag{4.39}$$

which maps $L_p(Q_f)$ to $L_p(Q_f)$.

◀ This result can be easily proved by using the contraction mapping theorem. Indeed, due to 4.4.2v and Lemma 4.4.5 we have

$$\|\mathbf{U}_1\mathbf{W}_3[\psi]\|_{\mathscr{L}(L_p(Q_f),\,L_p(Q_f))} \le \|\mathbf{U}_1\|_{\mathscr{L}(L_p(Q_f),\,L_p(Q_f))}\|\mathbf{W}_3[\psi]\|_{\mathscr{L}(L_p(Q_f),\,L_p(Q_f))} < 1.$$

▶

Next, for convenience let us define

$$\mathbf{S}_1 := (\mathbf{I} - \mathbf{U}_1 \circ \mathbf{W}_1), \quad \mathbf{S}_2[\psi] := (\mathbf{I} - \mathbf{U}_1\mathbf{W}_3[\psi]),$$

where $\psi \in L_\infty(Q_f)$, and let us prove a temporary lemma.

Lemma 4.4.7
$\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{S}_1$, $\mathbf{S}_1^{-1}$ are Fréchet differentiable as mappings from $L_\infty(Q_f)$ to $L_\infty(Q_f)$, but $\mathbf{U}_2$ is Fréchet differentiable as mapping from U to $L_\infty(Q_f)$.

◀ Let us fix arbitrary $v \in L_\infty(Q_f)$, $\hat{v} \in L_\infty(Q_f)$. Since

$$\begin{aligned} c(\hat{v}(x_1,\,x_2)) &= c(v(x_1,\,x_2)) + c'(v(x_1,\,x_2))(\hat{v}(x_1,\,x_2) - v(x_1,\,x_2)) \\ &\quad + [\int_0^1 c''(v(x_1,\,x_2) + t(\hat{v}(x_1,\,x_2) - v(x_1,\,x_2)))(1-t)\,dt \\ &\qquad\qquad \times (\hat{v}(x_1,\,x_2) - v(x_1,\,x_2))^2] \end{aligned}$$

$$\begin{aligned} b(\hat{v}(x_1,\,x_2)) &= b(v(x_1,\,x_2)) + b'(v(x_1,\,x_2))(\hat{v}(x_1,\,x_2) - v(x_1,\,x_2)) \\ &\quad + [\int_0^1 b''(v(x_1,\,x_2) + t(\hat{v}(x_1,\,x_2) - v(x_1,\,x_2)))(1-t)\,dt \\ &\qquad\qquad \times (\hat{v}(x_1,\,x_2) - v(x_1,\,x_2))^2] \end{aligned}$$

for almost every $(x_1,\,x_2) \in Q_f$, then

$$\begin{aligned} &\frac{\|\mathbf{W}_1(\hat{v}) - \mathbf{W}_1(v) - \mathbf{W}_3[v](\hat{v}-v)\|_{L_\infty(Q_f)}}{\|\hat{v}-v\|_{L_\infty(Q)}} \\ &\qquad = \frac{\|\int_0^1 c''(v + t(\hat{v}-v))(\hat{v}-v)^2(1-t)\,dt\|_{L_\infty(Q_f)}}{\|\hat{v}-v\|_{L_\infty(Q_f)}} \\ &\qquad\qquad \le \hat{c}_1\|\hat{v}-v\|_{L_\infty(Q_f)}, \end{aligned}$$

$$\begin{aligned} &\frac{\|\mathbf{W}_2(\hat{v}) - \mathbf{W}_2(v) - \mathbf{W}_4[v](\hat{v}-v)\|_{L_\infty(Q_f)}}{\|\hat{v}-v\|_{L_\infty(Q)}} \\ &\qquad = \frac{\|\int_0^1 b''(v + t(\hat{v}-v))(\hat{v}-v)^2(1-t)\,dt\|_{L_\infty(Q_f)}}{\|\hat{v}-v\|_{L_\infty(Q_f)}} \\ &\qquad\qquad \le \hat{c}_2\|\hat{v}-v\|_{L_\infty(Q_f)}. \end{aligned}$$

But these estimates prove Fréchet differentiability of the operators $\mathbf{W}_1$, $\mathbf{W}_2$.

Next, since

$$\mathbf{S}_1 = (\mathbf{I} - \mathbf{U}_1 \circ \mathbf{W}_1),$$

then due to differentiability of $\mathbf{W}_1$ and 4.4.2v we will get differentiability of $\mathbf{S}_1$.

Now, let us prove differentiability of the operator $\mathbf{S}_1^{-1}$. Therefore, let us fix arbitrary $v \in L_\infty(Q_f)$, $\hat{v} \in L_\infty(Q_f)$ and let us denote

$$w := \mathbf{S}_1^{-1}(v), \quad \hat{w} := \mathbf{S}_1^{-1}(\hat{v}).$$

Since

$$\begin{aligned}
&\mathbf{S}_1^{-1}(\hat{v}) - \mathbf{S}_1^{-1}(v) - \mathbf{S}_2[\mathbf{S}_1^{-1}(v)]^{-1}(\hat{v} - v) \\
&\qquad = (\hat{w} - w) - \mathbf{S}_2[w]^{-1}(\mathbf{S}_1(\hat{w}) - \mathbf{S}_1(w)) \\
&\qquad\qquad = \mathbf{S}_2[w]^{-1}(\mathbf{S}_2[w](\hat{w} - w) - (\mathbf{S}_1(\hat{w}) - \mathbf{S}_1(w))),
\end{aligned}$$

then

$$\begin{aligned}
&\|\mathbf{S}_1^{-1}(\hat{v}) - \mathbf{S}_1^{-1}(v) - \mathbf{S}_2[\mathbf{S}_1^{-1}(v)]^{-1}(\hat{v} - v)\|_{L_\infty(Q_f)} \\
&\le \|\mathbf{S}_2[w]^{-1}\|_{\mathscr{L}(L_\infty(Q_f),\, L_\infty(Q_f))} \|\mathbf{S}_1(\hat{w}) - \mathbf{S}_1(w) - \mathbf{S}_2[w](\hat{w} - w)\|_{L_\infty(Q_f)}.
\end{aligned}$$

Next, due to Lipschitz continuity of $\mathbf{S}_1^{-1}$ (see Lemma 4.4.3) we will have

$$\begin{aligned}
&\frac{\|\mathbf{S}_1^{-1}(\hat{v}) - \mathbf{S}_1^{-1}(v) - \mathbf{S}_2[\mathbf{S}_1^{-1}(v)]^{-1}(\hat{v} - v)\|_{L_\infty(Q_f)}}{\|\hat{v} - v\|_{L_\infty(Q_f)}} \\
&\qquad \le \hat{c}_3 \frac{\|\mathbf{S}_1(\hat{w}) - \mathbf{S}_1(w) - \mathbf{S}_2[w](\hat{w} - w)\|_{L_\infty(Q_f)}}{\|\hat{v} - v\|_{L_\infty(Q_f)}} \\
&\qquad\qquad \le \hat{c}_4 \frac{\|\mathbf{S}_1(\hat{w}) - \mathbf{S}_1(w) - \mathbf{S}_2[w](\hat{w} - w)\|_{L_\infty(Q_f)}}{\|\hat{w} - w\|_{L_\infty(Q_f)}}.
\end{aligned}$$

Since $\mathbf{S}_1$ is differentiable, then the last inequality will yield Fréchet differentiability of the operator $\mathbf{S}_1^{-1}$.

Finally, in order to prove Fréchet differentiability of $\mathbf{U}_2$ it is sufficient to show differentiability of the operator $v \mapsto v^4$ as mapping from U to $L_\infty(\Sigma_2)$. But for arbitrary chosen $v \in U$, $\hat{v} \in U$ we have

$$\begin{aligned}
&\frac{\|\hat{v}^4 - v^4 - 4v^3(\hat{v} - v)\|_{L_\infty(\Sigma_2)}}{\|\hat{v} - v\|_{L_\infty(\Sigma_2)}} \\
&\qquad = \frac{\|\int_0^1 12(v + t(\hat{v} - v))^2(\hat{v} - v)^2(1 - t)\, dt\|_{L_\infty(\Sigma_2)}}{\|\hat{v} - v\|_{L_\infty(\Sigma_2)}} \\
&\qquad\qquad\qquad \le \hat{c}_5 \|\hat{v} - v\|_{L_\infty(\Sigma_2)}.
\end{aligned}$$

►

Now, on basis of the representation formula (4.38) and results of the previous lemma we will automatically obtain Fréchet differentiability of the control-to-state operator $\tilde{T} = \tilde{\mathbf{\Gamma}}(T_h)$.

Theorem 4.4.8
For every two arbitrary chosen controls $T_h \in U$, $\hat{T}_h \in U$ the following formula holds:

$$\|\tilde{\mathbf{\Gamma}}(\hat{T}_h) - \tilde{\mathbf{\Gamma}}(T_h) - \tilde{\mathbf{\Lambda}}[T_h](\hat{T}_h - T_h)\|_{L_\infty(\Omega_f)} = \mathbf{o}[T_h](\|\hat{T}_h - T_h\|_{L_\infty(\Sigma_2)}), \quad (4.40)$$

where $\tilde{\mathbf{\Lambda}}[T_h] \in \mathscr{L}(L_\infty(\Sigma_2),\, L_\infty(\Omega_f))$.

Finally, one can easily check that the cost functional $\mathcal{J}(\tilde{T})$ is Fréchet differentiable as mapping from $L_\infty(\Omega_f)$ to $\mathbb{R}$. If we now combine this result with the previous lemma, then by applying the chain rule for the functional $\mathcal{J}(\tilde{\mathbf{\Gamma}}(T_h))$ we will get the following theorem.

Theorem 4.4.9
For every two arbitrary chosen controls $T_h \in U$, $\hat{T}_h \in U$ the following formula holds (see the formula (1.15)):

$$\mathcal{J}(\tilde{\mathbf{\Gamma}}(\hat{T}_h)) - \mathcal{J}(\tilde{\mathbf{\Gamma}}(T_h)) = \tilde{\ell}[T_h](\hat{T}_h - T_h) + \mathbf{o}[T_h](\|\hat{T}_h - T_h\|_{L_\infty(\Sigma_2)}),$$

where $\tilde{\ell}[T_h] \in L^*_\infty(\Sigma_2)$.

4.5 Relationship between original and reduced problems

In this section we will establish relationship between the original problem (1.5) and the reduced problem (1.14). In other words, we will prove that for small values of δ the state function T can be effectively approximated by the state function $\tilde{T}$.

Theorem 4.5.1
For every fixed control $T_h \in U$ there exist the following limits:

$$\frac{1}{\delta}\int_{\Omega_f} (T - \tilde{T})^2 \, d\omega_3 \to 0,$$
$$\int_{\Sigma_0} (T - \tilde{T})^2 \, d\lambda_{\Sigma_0} \to 0,$$

as $\delta \to 0$.

◀ Let us define the functions

$$\begin{aligned}
\gamma_1(x_1, x_2) &:= T(x_1, x_2, -\delta) && (x_1, x_2) \in Q_f, \\
\gamma_2(x_1, x_2) &:= T(x_1, x_2, \delta) && (x_1, x_2) \in Q_f
\end{aligned}$$

and

$$\gamma_3 := \sigma\gamma_1^4 + k_3\gamma_1, \quad \gamma_4 := \sigma\gamma_2^4 + k_3\gamma_2.$$

Since

$$\|\gamma_3 - \gamma_4\|_{L_2(Q_f)} \le \|\gamma_3 - \hat{g}\|_{L_2(Q_f)} + \|\hat{g} - \gamma_4\|_{L_2(Q_f)}$$

(for definition of the function $\hat{g}$ see at the begining of this chapter), then, due to Lemma 4.3.1 we will get

$$\|\gamma_3 - \gamma_4\|_{L_2(Q_f)} \to 0,$$

as $\delta \to 0$. Furthermore, Lipschitz continuity of $\mathbf{W}_1$ (see Lemma 4.4.2) will imply that

$$\sigma\|\gamma_1^4 - \gamma_2^4\|_{L_2(Q_f)} = \|\mathbf{W}_1(\gamma_3) - \mathbf{W}_1(\gamma_4)\|_{L_2(Q_f)} \le \|\gamma_3 - \gamma_4\|_{L_2(Q_f)}.$$

Therefore, we will have

$$\|\gamma_1^4 - \gamma_2^4\|_{L_2(Q_f)} \to 0, \tag{4.41}$$

as $\delta \to 0$.

If we now define

$$\gamma_5(x_1, x_2, x_3) := \gamma_1(x_1, x_2) \quad (x_1, x_2, x_3) \in \Sigma_0,$$

then due to Lipschitz continuity of $\mathbf{S}_1$ (see Lemma 4.4.3) we will have

$$\begin{aligned}
\|\gamma_3 - \mathbf{S}_1^{-1}(\hat{f})\|_{L_2(Q_f)} &\le \hat{c}_1\|\mathbf{S}_1(\gamma_3) - \hat{f}\|_{L_2(Q_f)} \\
&= \hat{c}_1\|\gamma_3 - \sigma\mathbf{A}_1\mathbf{H}_{\Sigma_0\Sigma_0}(\gamma_5^4) - \hat{f}\|_{L_2(Q_f)} \qquad (4.42)
\end{aligned}$$

(for definition of the function $\hat{f}$ see at the begining of this chapter).

Since

$$\begin{aligned}
\gamma_5(x_1, x_2, x_3) &= \gamma_1(x_1, x_2) && (x_1, x_2, x_3) \in S_5, \\
T(x_1, x_2, x_3) &= \gamma_1(x_1, x_2) && (x_1, x_2, x_3) \in S_5, \\
\gamma_5(x_1, x_2, x_3) &= \gamma_1(x_1, x_2) && (x_1, x_2, x_3) \in S_6, \\
T(x_1, x_2, x_3) &= \gamma_2(x_1, x_2) && (x_1, x_2, x_3) \in S_6,
\end{aligned}$$

then (4.41) yields that

$$\|\gamma_5^4 - T^4\|_{L_2(\Sigma_0)} \to 0,$$

as $\delta \to 0$. Therefore, due to 2.3.7v and Lemma 4.3.1 we will get

$$\|\gamma_3 - \sigma \mathbf{A}_1 \mathbf{H}_{\Sigma_0\Sigma_0}(\gamma_5^4) - \hat{f}\|_{L_2(Q_f)} \to \|\gamma_3 - \sigma \mathbf{A}_1 \mathbf{H}_{\Sigma_0\Sigma_0}(T^4) - \hat{f}\|_{L_2(Q_f)} = \|\gamma_3 - \hat{g}\|_{L_2(Q_f)} \to 0,$$

as $\delta \to 0$. In addition, since the estimate (4.42) holds, then

$$\|\gamma_3 - \mathbf{S}_1^{-1}(\hat{f})\|_{L_2(Q_f)} \to 0 \tag{4.43}$$

as $\delta \to 0$.

Next, by using similar techniques, one can prove that

$$\|\gamma_4 - \mathbf{S}_1^{-1}(\hat{f})\|_{L_2(Q_f)} \to 0 \tag{4.44}$$

as $\delta \to 0$.

Finally, due to 4.4.2ii, 4.4.2iii we will have

$$\begin{aligned}
\int_{\Sigma_0} (T - \tilde{T})^2 \, d\lambda_{\Sigma_0}
&= \int_{Q_f} (\mathbf{W}_2 \circ \mathbf{W}_2^{-1}(\gamma_1) - \mathbf{W}_2 \circ \mathbf{S}_1^{-1}(\hat{f}))^2 \, d\omega_2 \\
&+ \int_{Q_f} (\mathbf{W}_2 \circ \mathbf{W}_2^{-1}(\gamma_2) - \mathbf{W}_2 \circ \mathbf{S}_1^{-1}(\hat{f}))^2 \, d\omega_2 \\
&\le \frac{1}{k_3^2} \int_{Q_f} (\gamma_3 - \mathbf{S}_1^{-1}(\hat{f}))^2 \, d\omega_2 + \frac{1}{k_3^2} \int_{Q_f} (\gamma_4 - \mathbf{S}_1^{-1}(\hat{f}))^2 \, d\omega_2,
\end{aligned}$$

and, therefore, (4.43), (4.44) will imply that

$$\int_{\Sigma_0} (T - \tilde{T})^2 \, d\lambda_{\Sigma_0} \to 0, \tag{4.45}$$

as $\delta \to 0$.

Now, let us define the function

$$\gamma := \sigma T^4 + k_3 T$$

and also

$$\begin{aligned}
T_\tau(x_1, x_2) &:= T(x_1, x_2, \tau) \qquad && (x_1, x_2) \in Q_f,\ \tau \in [-\delta, \delta], \\
\gamma_\tau(x_1, x_2) &:= \gamma(x_1, x_2, \tau) \qquad && (x_1, x_2) \in Q_f,\ \tau \in [-\delta, \delta].
\end{aligned}$$

Due to (4.43), Lemma 4.2.1 and Lemma 4.3.1 we will get

$$\begin{aligned}
&\frac{1}{\delta} \int_{-\delta}^{\delta} \int_{Q_f} (\gamma_\tau - \mathbf{S}_1^{-1}(\hat{f}))^2 \, d\omega_2 \, d\tau \\
&\quad \to \frac{1}{\delta} \int_{-\delta}^{\delta} \int_{Q_f} (\gamma_\tau - \gamma_3)^2 \, d\omega_2 \, d\tau \to \frac{1}{\delta} \int_{-\delta}^{\delta} \int_{Q_f} (\gamma_\tau - \hat{g})^2 \, d\omega_2 \, d\tau \\
&\quad = \frac{1}{\delta} \int_{\Omega_f} (\sigma T^4 + k_3 T - g_+)^2 \, d\omega_3 \to 0,
\end{aligned}$$

as $\delta \to 0$, and, therefore, 4.4.2ii, 4.4.2iii will imply that

$$\begin{aligned}
&\frac{1}{\delta}\int_{\Omega_f}(T-\tilde{T})^2\,d\omega_3 \\
&\qquad = \frac{1}{\delta}\int_{-\delta}^{\delta}\int_{Q_f}(\mathbf{W}_2\circ\mathbf{W}_2^{-1}(T_t)-\mathbf{W}_2\circ\mathbf{S}_1^{-1}(\hat{f}))^2\,d\omega_2\,d\tau \\
&\qquad\qquad \le \frac{1}{k_3^2\delta}\int_{-\delta}^{\delta}\int_{Q_f}(\gamma_\tau-\mathbf{S}_1^{-1}(\hat{f}))^2\,d\omega_2\,d\tau \to 0,
\end{aligned}$$

as $\delta \to 0$.

►

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