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В сборник включены результаты научных исследований, полученные сотрудниками и аспирантами математических кафедр Латвйского университета, института математики и информатики Латвйского университета, а также работы специалистов других вузов и учреждений, сотрудничающих с математиками Латвйского университета. Большинство из опубликованных в сборнике результатов получены за период 1992 - 1994 гг. Включенные в сборник рукописи, как правило, не рецензируются.

The proceedings contain the results of the investigations made in 1992 – 1994 by lecturers, research workers and post-graduate students of the Department of Mathematics and the Institute of Mathematics and Computer Sciences of the University of Latvia. Besides the proceedings reveal the results of the investigations made by some specialists from other universities and institutes collaborating with mathematicians of Latvian University. The works included in the proceedings generally are not reviewed by referees.

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Introducing Infinitary Lambda Calculus

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Zinovy Diskin *

Abstract. The paper introduces a notion of infinitary lambda calculus. While in the ordinary λ-calculus functions can be applied only to a single argument, our version allows multiple (particularly, infinite) applications and abstractions. So some λ-terms can involve infinitely many (or even all) variables, what makes syntactical machinery rather subtle. An algebraic semantics for the new calculus is constructed and the corresponding completeness and representation theorems are presented.

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0 Introduction

Untyped lambda calculus is a well known model of computability, and all effective procedures can be modeled by λ-terms. Such terms involve a finite number of variables, i.e. a modeled procedure can call a finite number of auxiliary subroutines. Perhaps, millions and millions calls. Once, their exact number (say, 10^7 or 10^7+5) turns into a tedious detail, and as the rich experience of mathematical analysis shows, that is just the moment to consider procedures with infinite number of subroutine calls and, respectively, λ-terms of infinitely many variables.

Though this and other interpretations may look intriguing, our research was initially motivated by the purely mathematical curiosity: what a λ-calculus with infinitary λ-terms does present from the algebraic point of view? It is interesting that in the universal-algebraic framework for λ-calculus metatheory, developed in [D91], [DB93], [PS93], infinitary versions of λ-calculus are described quite simply, while the feature of finitarity (each term depends just on a finite number of variables) cannot be expressed by the first order axioms. So we had a nice algebraic version -- of an unfamiliar theory, and the goals (first pointed in [D91] as an open problem) of the paper were just:

(1) to describe precisely syntax of some calculus allowing terms to be infinitary;
(2) to define axiomatically a class of algebras, intended to be an algebraic counterpart of the new-introduced calculus;
(3) to state adequateness of the corresponding algebraic semantics: the main result asserts that theories (1), and algebras (2) are different specifications of the same object.

There are two principal ways to introduce infinitary terms into calculus: one can either add infinitary applications and abstractions directly to syntax (thus obtaining term trees of finite depth and infinite breath,) or allow infinite iterations of unary applications and abstraction (in that case we have to extend the notion of reducibility to deal with such infinite-depth term trees). We chose the former approach, due to its closer connections to algebra. On the other hand, the latter shows similarity to graph rewriting technique (cf. [K92] and [KKSdV93], where generalizations of Church-Rosser property for infinitary systems are elaborated.) Generally, as soon as our calculus copes with infinitary composition, it appears to be a finite meta-theory for infinitary graph rewriting.

Certainly, an interesting question is how all this can be related to categorical versions of λ-calculus. Leaving it for future research, we note only that our machinery is expected to be interpreted in something like infinitary CCC, i.e. a category with infinite products and infinitary exponentials.

1 Calculus.

In the first section we describe our version of lambda calculus, where terms can be applied to infinitely many arguments and abstracted from infinitely many variables. Since a natural 'currying' property also has to be postulated for finite concatenations of (possibly infinite) arguments and indices, we found it relevant to define general operations on infinite tuples. Further we will introduce simple substitution-like operations on these terms, and equivalences compatible with these operations. These equivalences are infinitary counterparts of α and β conversions in the ordinary lambda calculus.

Our calculus will operate with disjoint sets Const of *constants*, Fvar of *Roman variables* (intended to be free) and Bvar of *Greek variables* (intended to be bound).

The number of constants is not restricted. Bvar and Fvar are countable and somehow enumerated by ω (the first infinite ordinal); normally we will use letters ξ,θ,ζ and x,y,z to refer to their elements. (The idea of separation between bound and free variables to avoid substitution collisions is well known; some formal machinery was developed by Keisler[Kei63] for the case of infinitary predicate calculus. Note, however, that algebraic behaviour of term-in-formula substitutions is quite different from that of term-in-term substitutions: while the latter constitute a monoid-like structure itself, the former amounts to a monoid homomorphism into another monoid arising from some closed set of logical operations).

On the other hand, a variable bound in some term may turn into free in its subterm. This forces us to introduce, at first, *general* terms where Greek variables can be free or bound (Roman variables are always free) and then to distinguish among them *regular* terms having all Greek variables bound. (Note that subterms of regular terms may not be regular). Just the regular terms considered up to α-equivalence will be our infinitary counterpart of ordinary λ-terms: while the latter make the set Λ(Const), we would like to convert the set of the former into a substitute for Λ_∞(Const), that is, to define over it infinitary substitutions with the proper behaviour. It proves to be indeed possible, however, in contrast with the ordinary Λ(Const) where substitutions can be defined almost trivially, the substitutions over Λ_∞(Const) are derived operations with hidden internal structure. Actually, converting Λ_∞(Const) into a 'good' substitution algebra turned out to be unexpectedly subtle and forced us to develop a monstrous machinery of handling Greek and Roman variables, several kinds of substitutions and many other frightening things demonstrated below. However, the result is good: we will see that infinitary substitution, application and abstraction over Λ_∞(Const) behave as desired and their monstrous origin is hidden.

1.1 Infinite tuples

What makes our calculus really infinitary, are infinitary operations of abstraction and application. Any functional can be abstracted from infinite tuple of variables, and be applied to infinite tuple of arguments. The ceiling for their dimensions is $\omega\omega$, that allows to concatenate several infinite tuples to a correct tuple again. Tuples of arguments and of lambda-indices are rather different, however we will use similar notation for them, what provides our considerations with more symmetric view.

1.1.1 Definition

An *argument tuple* is a function a: $|a| \longrightarrow$ Term, where $|a| < \omega\omega$ is the tuple's length. Concatenation of two tuples a,b is defined in an evident way:

$$|a+b| = |a|+|b|; \qquad (a+b)_i = \begin{cases} a_i \text{ if } i<|a| \\ b_j \text{ if } j<|b|,\ i=|a|+j \end{cases}$$

Unfortunately, we would meet difficulties trying to extend this obvious construction to indices. Namely, if we allowed variables to repeat in a lambda-index, we would not be able to equalize, say, $\lambda\eta\eta.\eta$ to $\lambda\zeta\xi.\xi$, or $\lambda\zeta_{i<\omega}.\vartheta$ to $\lambda(\eta\eta...).\vartheta$ without very complicated α-axioms. To avoid this, we impose a rigid restriction on variable usage in lambda operator: it must be indexed by irrepetitive string of Greek variables, and a special blank symbol □ is introduced for dummy variable. The precise definition is as follows:

1.1.2 Definition

A *lambda-index tuple* is a partial injection $a\colon |a| \hookrightarrow_p \mathrm{Bvar}$, where $|a|<\omega\omega$.
Concatenation is introduced as follows:

$$|a+b|=|a|+|b|, \qquad (a+b)_i=\begin{cases} a_i \text{ if } i<|a|,\ a_i \text{ defined, } a_i\notin \mathrm{Rg}\,b;\\ b_j \text{ if } j<|b|,\ b_j \text{ defined, } i=|a|+j;\\ \text{otherwise undefined.}\end{cases}$$

For example: $(\xi_1\xi_2\xi_3\xi_4)+(\xi_2\xi_4\xi_5) = \xi_1\square\xi_3\square\xi_2\xi_4\xi_5$

We use the symbol ε for empty tuples of both kinds (that is $|x|=0 \Leftrightarrow x=\varepsilon$). Assuming **Bvar** and **Term** fixed, we designate the set of argument tuples as **Args**, and index tuples as **Index**. It is easy to check that not only $(\mathbf{Args},+,\varepsilon)$, but also $(\mathbf{Index},+,\varepsilon)$ make monoids with left reduction (that is $a+\varepsilon=\varepsilon+a=\varepsilon$, $(a+b)+c=a+(b+c)$, and $a+b=a+c \Rightarrow b=c$ for any a,b and c, though $a+c=b+c \Rightarrow a=b$ is not necessarily true).

1.2 General and regular terms

1.2.1 Definition

The following rules introduce a set **Term** of *(general) terms* and corresponding functions $\mathrm{FV}\colon \mathbf{Term}\longrightarrow 2^{\mathrm{Fvar}\cup\mathrm{Bvar}}$ and $\mathrm{BV}\colon \mathbf{Term}\longrightarrow 2^{\mathrm{Bvar}}$.

	Context	Term	FV	BV
o	$c\in\mathrm{Const}$	c	$\varnothing$	$\varnothing$
i	$x\in\mathrm{Fvar}$	x	$\{x\}$	$\varnothing$
ii	$\xi\in\mathrm{Bvar}$	ξ	$\{\xi\}$	$\varnothing$

In the following two rules $\kappa<\omega\omega$ is a parameter ordinal.

	Context	Term	FV	BV
iii	$\xi_{i<\kappa}\in\mathrm{Index}$, $t\in\mathrm{Term}$, $\mathrm{Rg}\,\xi\subseteq\mathrm{FV}(t)$	$\lambda\xi_{i<\kappa}.t$	$\mathrm{FV}(t)\text{-}\mathrm{Rg}\,\xi$	$\mathrm{BV}(t)\cup\mathrm{Rg}\,\xi$
iv	$t\in\mathrm{Term}$, $s_{k<\kappa}\in\mathrm{Args}$	$t's_{k<\kappa}$	$\bigcup\mathrm{FV}(s_k)\cup\mathrm{FV}(t)$	$\bigcup\mathrm{BV}(s_k)\cup\mathrm{BV}(t)$

Since we have defined lambda terms inductively, we can use induction *by general term structure* to determine operations and to prove theorems concerning them. Later we will introduce another kind of induction, *by the alpha term structure*.

1.2.2 Definition. A regular term is a term t with $FV(t) \subseteq$ Fvar. The set of *regular terms* will be denoted by Regterm.

1.2.3 Examples. Regular terms: x, $\lambda\Box\xi_{k<\omega+7}.\xi_{k<\omega+7}$.

General, not regular, terms: ξ, $\lambda\xi.(\theta'\theta\xi x)$, $\lambda\Box\xi_{k<\omega+7}.\xi_{k<\omega+8}$

The following are not correct terms at all:

$$\lambda x.xy,\quad t'\xi_{k<\omega\omega},\quad \lambda\zeta.\lambda\zeta.\zeta,\quad \lambda\Box.\Box,\quad \lambda\Box\xi_{k<\omega+8}.\xi_{k<\omega+7}\ .$$

1.3 Operations over terms

We define three operations over terms -- variable renaming, term-in-term substitution, and lambda-quantification (*lambdaing*).

1.3.1 Definition. Given a map *Ren*: Bvar$\cup V \hookrightarrow$ Bvar, where $V \subseteq$ Fvar, we define *renaming* Ren:Term$\hookrightarrow$Term :

(o) $c \in$ Const $\Rightarrow$ Ren(c) = c

(i) $x \in$ Fvar $\Rightarrow$ Ren(x) = *Ren*(x) if $x \in$ Dom *Ren*, otherwise x

(ii) $\xi \in$ Bvar $\Rightarrow$ Ren(ξ) = *Ren*(ξ)

(iii) $\zeta \in$ Index, $t \in$ Term $\Rightarrow$ Ren($\lambda\zeta.t$) = λRen(ζ).Ren(t)

(iv) $t, s_{k<\kappa} \in$ Term $\Rightarrow$ Ren($t's_{k<\kappa}$) = Ren(t)'Ren($s_{k<\kappa}$)

Renaming involves all occurrences of variables, including lambda indices. If $V = \varnothing$, i.e. *Ren* is a permutation of Bvar, then Ren preserves regularity of terms.

1.3.2 Definition. Let Subst be a map Bvar$\cup$Fvar$\longrightarrow$Term. Then the operation of *substitution*, Subst*: Term$\longrightarrow$Term, is defined as follows:

(o) $c \in$ Const $\Rightarrow$ Subst*c=c

(i) $x \in$ Fvar $\Rightarrow$ Subst*x = Subst(x)

(ii) $\xi \in$ Bvar $\Rightarrow$ Subst*ξ = Subst(ξ)

(iii) $\zeta \in$ Index, $t \in$ Term $\Rightarrow$ Subst*$\lambda\zeta.t$ = $\lambda\zeta$.(Subst$|\overline{\text{Rg}\zeta}$)*$t$,

(where Subst$|\overline{\text{Rg}\zeta}$ coincides with Id on Rgζ and with Subst elsewhere.)

(iv) $t, s_{k<\kappa} \in$ Term $\Rightarrow$ Subst*$t's_{k<\kappa}$ = Subst*t'Subst*$s_{k<\kappa}$

Obviously, the above-introduced substitution also preserves regularity of terms. If Subst maps x_k to t_k, for all $k<\kappa$, where x:$\kappa \hookrightarrow$ Fvar$\cup$Bvar, t:$\kappa \hookrightarrow$ Term, $\kappa<\omega\omega$, and the other variables to themselves, we obtain a direct generalization of the ordinary substitution, denoted by [x:=t]. .

1.3.3 Examples. If $Ren=\begin{pmatrix}\eta & \zeta\\ \zeta & \eta\end{pmatrix}$, then $\mathrm{Ren}(x'\lambda\zeta.\zeta) = x'\lambda\eta.\eta$;

If $Ren=\begin{pmatrix}x & \eta_i\\ \eta_0 & \eta_{i+1}\end{pmatrix}_{i<\omega}$ then $\mathrm{Ren}(x'(\eta_2\eta_5)) = \eta_0'(\eta_3\eta_6)$;

$[\xi,y,z := \lambda\eta.\eta,\eta,\eta]\ (y'\xi'\lambda\xi.\xi) = \eta'\lambda\eta.\eta'\lambda\xi.\xi$;

$[\xi:=\theta]\ \xi'\lambda\xi.\lambda'\lambda\xi.\xi = \theta'\lambda\xi.\lambda'\lambda\xi.\xi$.

1.3.4 Definition. Given a κ-tuple $x{:}\kappa \longrightarrow \mathrm{Fvar}$, we define *lambdaing* λ^{x}: Regterm→Regterm, as

$$\lambda^{x}t = \lambda\mathrm{Shift}(x_{k<\kappa}).[\mathrm{Shift}]t,$$

where mapping $\mathrm{Shift}=\mathrm{Shift}^{x}_{t}$: $(\mathrm{Rgx}\cap\mathrm{FV}(t))\cup\mathrm{Bvar} \hookrightarrow \mathrm{Bvar}$ renames Roman variables, freely occurring in t, to Greek ones, (and the others, which occur only in x, to □'es.).

A concrete choice actually influences only names of bound variables, so we may take a restriction of an arbitrary mapping $\mathrm{Fvar}\cup\mathrm{Bvar}\hookrightarrow\mathrm{Bvar}$, for instance,

$$\mathrm{Fvar}_i \mapsto \mathrm{Bvar}_{2i}\ ,\ \mathrm{Bvar}_i \mapsto \mathrm{Fvar}_{2i+1}.$$

1.4 Alpha and beta-theories

1.4.1 Definition. *Alpha-theory* is determined by the following axioms and inference rules:

(α0) $a=b \vdash b=a$; $a=b, b=c \vdash a=c$;

(α1) for any Ren:Bvar$\hookrightarrow$Bvar, $a = \mathrm{Ren}(a)$;

(α2) for any $y{:}\omega\hookrightarrow\mathrm{Fvar}$, $a=b$, $v_{k<\omega}=u_{k<\omega} \vdash [y:=v_{k<\omega}]a = [y:=u_{k<\omega}]b$;

Alpha equivalence is the minimal α-theory.

Informally, two terms are alpha-equivalent iff they differ only in bound variables' naming. Regular term can be α-equivalent only to regular terms; moreover, α-equivalency is compatible with lambdaing and substitution.

Now we can turn to intended semantic, or logical, aspects of the calculus. In the previous section we did not require that $f'(ab)$ is the same as $(f'a)'b$ or that $\lambda\xi_{k<\omega+2}.a$ can stand for $\lambda\xi_0.\lambda(\xi_1..\xi_{\omega+1}).a$. We did not try at all to evaluate any terms. All this will be considered in the current section.

1.4.2 Concatenation axioms.

For any $\kappa,\iota<\omega\omega$, $t\in$**Term**, u,v:**Args**, $\vartheta,\xi\in$**Index**,

(c0) $$\lambda\varepsilon.a = a, \qquad a'\varepsilon = a,$$

(c1) $$\lambda\vartheta_{i<\iota}.\lambda\xi_{k<\kappa}.t = \lambda(\vartheta_{i<\iota}+\xi_{k<\kappa}).t\ ,$$

(c2) $$t'u_{i<\iota}'v_{k<\kappa} = t'(u_{i<\iota}+v_{k<\kappa}).$$

The reader can notice that our calculus does not support any limit construction, i.e., for example, we do not consider $t'u_{i<\omega}$ as a limit of $t'u_{i<n}$ when $n\to\omega$. So finite applications are separated from infinite ones, and cannot be converted one to the other. The same relates to lambda-operators. Introduction of infinitary concatenations would definitely break the $\omega\omega$ ceiling for tuple lengths.

1.4.3 Beta-axiom. For any $\kappa<\omega$, $\xi\in$**Index**$^\kappa$, $t\in$**Term**, $u:\kappa\hookrightarrow$**Regterm**,

(β) $$\lambda\xi.t'u_{k<\kappa} = [\xi:=u]t$$

1.4.4 Proposition. Neither β nor concatenation axioms break the regularity of terms.

1.4.5 Definition. A $\lambda\beta$-theory is a $\lambda\alpha$ theory closed under axioms $c0,c1,c2$, and β.

The most important for us are $\lambda\alpha$ and $\lambda\beta$-theories over regular terms, since the syntactical procedures of lambdaing and substitution behave on **Regterm**/α very similarly to λ-quantifier and substitution over ordinary λ-terms. All involved techniques become hidden. We refer to elements of **Regterm**/α as *alpha-terms*, and in fact they are the objects over which we will construct our algebra.

2 Algebras

Let A be a non-empty set of elements interpreted as alpha-terms, and the following families of operations be defined over it.

Name of operation	Parameter(s)	Symbol of operation	Arity	Notation example	Intended interpretation
variable	$i\in\omega$	x_i	0	x_i	x_i
application	$\kappa<\omega\omega$	'	$1+\kappa$	$u'z_{k<\kappa}$	$u'z_{k<\kappa}$
substitution	.	*	$\omega+1$	$w_{i<\omega}{}^*u$	$[x_i:=w_i]u$
λ-operator	$\kappa<\omega\omega$, $\mathbf{i}:\kappa\hookrightarrow\omega$	$\lambda\mathbf{i}$	1	$\lambda\mathbf{i}.u$	$\lambda^{\chi\mathbf{i}}u$

Figure 2.1.

(Symbols w_i,u,z_k in the 5th column mean elements of A, and in the 5th -- intended α-terms, where $x_{i<\omega}$ and $\chi_{i<\omega}$ are some enumerations of **Fvar** and **Bvar**.)

To extend the similarity with terms, we will use the abbreviations $[i:=a]b$ and $[\mathbf{i}:=\mathbf{a}]b$ with respect to algebras, ($i<\omega$, $\mathbf{i}<\kappa\hookrightarrow\omega$, $a,b\in A$, $\mathbf{a}\in A^{\kappa}$).

2.1 Definition. An infinitary $\lambda\alpha$ substitution algebra (or $i\lambda\alpha SA$) is an algebra of above-introduced signature subjected to the following identities:

Parameterization:

$$x_{i<\omega}{}^*a = a \qquad \text{(is1)}$$

$j<\omega$:
$$a_{i<\omega}{}^*x_j = a_j \qquad \text{(is2)}$$

$$a_{i<\omega}{}^*(b_{j<\omega}{}^*c) = (a_{j<\omega}{}^*b_j)_{j<\omega}{}^*c \qquad \text{(is3)}$$

$\kappa<\omega\omega$:
$$a_{i<\omega}{}^*(b'c_{k<\kappa}) = (a_{i<\omega}{}^*b)'(a_{i<\omega}{}^*c_k)_{k<\kappa} \qquad \text{(isA)}$$

$\iota<\omega\omega, \mathbf{j},\mathbf{k}:\iota\hookrightarrow\omega$:
$$([\mathbf{j}:=x_{\mathbf{k}}]a_i)_{i<\omega}{}^*\lambda\mathbf{j}.b = \lambda\mathbf{j}.([\mathbf{j}:=x_{\mathbf{k}}]a_i)_{i\notin \mathrm{Rg}(\mathbf{j})}{}^*b \qquad \text{(isL)}$$

$\iota<\omega\omega, \mathbf{i},\mathbf{j},\mathbf{k}:\iota\hookrightarrow\omega$, $\mathrm{Rg}(\mathbf{j})\cap\mathrm{Rg}(\mathbf{k})=\varnothing$:
$$\lambda\iota.[\mathbf{j}:=x_{\mathbf{k}}]a = \lambda\mathbf{j}.[\mathbf{i}:=x_{\mathbf{j}}][\mathbf{j}:=x_{\mathbf{k}}]a \qquad \text{(is}\alpha\text{)}$$

These identities have their origin in the *substitution algebras* [F82], *infinitary clones* [C90], λSAs [D91,DB93] and *λ-abstraction algebras* [PS93]. Moreover, they generalize λSA axioms written out in [DB93]. However they have some specifically infinitary fads: e.g. (isα) can be written out NOT for any tuple $\mathbf{j}$. If it ranges too wide (say, $\mathrm{Rg}\mathbf{j}=\omega\setminus\{1\}$), it may be impossible to allocate a tuple $\mathbf{k}$ of unused variable names.

To obtain $\lambda\beta$-algebras ($i\lambda\beta SA$s), we add three more identities (very similar to their calculus relatives):

$\iota<\omega\omega,\ \mathrm{j}:\iota\hookrightarrow\omega$	$(\lambda \mathrm{j}.\mathrm{a})'c_{i<\iota} = [\mathrm{j}:=c_{i<\iota}]a$	$(\mathrm{is}\beta)_{\iota \mathrm{j}}$
$\iota,\kappa<\omega\omega$	$(a'b_{i<\iota})'c_{k<\kappa} = a'(b_{i<\iota}+c_{k<\kappa})$	$(\mathrm{isAC})_{\iota}$
$\left.\begin{array}{r}\iota_0,\iota_1<\omega\omega,\mathrm{j}_0:\iota_0\hookrightarrow\omega\\ \mathrm{j}:\iota_0+\iota_1\hookrightarrow\omega\\ \mathrm{j}_1:\iota_1\hookrightarrow\omega,\mathrm{j}=\mathrm{j}_0+\mathrm{j}_1\end{array}\right\}$	$\lambda \mathrm{j}.a = \lambda \mathrm{j}_0.(\lambda \mathrm{j}_1.a)$	$(\mathrm{isLC})_{\mathrm{j}}$

2.2 Definition. *Dimension* is a mapping $\Delta:A\longrightarrow 2^{\omega}$, $\Delta a = \{i \mid [i:=x_{i+1}]a \neq a\}$. An element $a\in A$ is closed if $\Delta a=\varnothing$.

2.3 Proposition. The following are equivalent reformulations of $i\in\Delta a$:

$\forall j\neq i\ [i:=x_j]a\neq a$; $\quad [i:=x_{i+1}]a\neq a$; $\quad \exists t[i:=t]a\neq a$.

2.4 Proposition. (why the plain dimensions are sufficient)

If $\mathrm{a}\in A^{\kappa}, \mathrm{i}:\kappa\hookrightarrow\omega\setminus\Delta b$, then $[\mathrm{i}:=\mathrm{a}]b = b$.

2.5 Proposition. An arbitrary $\lambda\beta$-algebra is generated by its closed elements. It is worthwhile to note that this does not hold for $\lambda\alpha$-algebras, where unreachable elements can exist.

3 Results: Connections between Syntax and Algebras

If we, given infinitary λ-calculus, will interpret its constants as closed elements of infinitary $\lambda\beta$-algebras and syntactical operations over its Regterm/α as the corresponding algebraic operators from Fig.2.1, we can show that *iλSA*s make algebraic semantics for α-theories, that is, determine consequence relation $\vDash_{\lambda\alpha SA}$ and $\vDash_{\lambda\alpha\beta SA}$ · over the set of regular terms, Regterm/α.

The principal results here are as follows :

3.1 Theorem (Soundness) Let Γ be an $\alpha(\beta)$-theory, and $a,b\in$Regterm/α.

If $\Gamma\vdash_{\alpha(\beta)} a=b$, then $\Gamma\vDash_{\lambda\alpha(\beta)SA}\ a=b$;

3.2 Theorem. (Completeness) If $\Gamma\vDash_{\lambda\alpha(\beta)SA}\ a=b$, then $\Gamma\vdash_{\alpha(\beta)} a=b$;

3.3 Theorem. (Representation) For each $\lambda\beta$-algebra A (not $\lambda\alpha$! Cf. proposition 2.5.) there is some set of constants C with a theory Γ over it, such that $A\cong\Lambda(C)/\Gamma$.

The proofs can be found in [Be93].

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I. Beiļins, Z. Diskins. Infinitārie lambda rēķini

Anotācija. Rakstā tiek ieviests atšķirīgs no vispārpieņemtā infinitāro λ-rēķinu jēdziens. Atšķirība ir tā, ka funkcijas var tikt pielietotas ne vienam, bet bezgalīgi daudziem argumentiem. Tādējādi, λ-termi var būt atkarīgi no daudziem, pat visiem, mainīgajiem, kas būtiski sarežģī darbu ar tiem. Bez tam, uzkonstruētajiem rēķiniem tiek piekārtota algebriska semantika ar pilnības un reprezentācijas teorēmām, kas tipiskas tādos gadījumos.

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ABSTRACT ALGEBRAS OF FINITARY RELATIONS: SEVERAL NON-TRADITIONAL AXIOMATIZATIONS*

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Abstract. We show that several non-traditional classes of algebras related to first-order logic are definitionally equivalent to that of locally finite cylindric algebras.
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1 Introduction

E.M. Benjaminov has introduced and investigated in [B1], [B2] a class of algebras called by him *relational algebras*. The term comes from database theory, and it generally refers to algebras of a certain kind suggested by E.F. Codd [Co], [K]. Beniaminov described another version of the relational data model and also proposed a set of axioms to characterize abstractly his class of "concrete" relational algebras. However, he did not present any results concerning strength of the axiom system, neither did he compare his algebras with Codd algebras or other known algebras or relations.

It was shown early that every Codd algebra can be embedded in a cylindric set algebra [IL1], [V2]. In 1983, B.I. Plotkin put a question whether the concept of a(n abstract) relational algebra in the sense of Beniaminov is equipollent to that of a locally finite polyadic algebra (with equality). N.D. Volkov has given an affirmative answer to the question in the series of papers [V1], [V2], [V3], [V4] by showing that

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the categories of algebras of both kinds are equivalent. Unfortunately, the considerable total length of these papers (caused partly by unnecessary reproving of many results that could be find in literature on algebraic logic), the style of exposition and a lot of inaccuracies both in formulations and proofs make it difficult to follow them. The present author has made an attempt in [C1] to establish such an equivalence to cylindric algebras rather than polyadic ones. In that paper the axiom system of [B1] was considerably simplified and its incompleteness was noticed (see §3). Still there is an error in the proof of Lemma 3.2 and a gap in the proof of Lemma 6.2 in [C1].

Our aim in this paper is twofold. It is a revised version of [C1], and we present here a proof of the main result of [C1]—that Beniaminov relational algebras (with a minor modification) are indeed interdefinable with locally finite cylindric algebras of an appropriate dimension. To save space (and patience of the reader), we—just as in [C1]—try to involve several known facts and constructions; we therefore go through a number of non-traditional algebraic structures related to first-order logic. In this respect, the paper supplements the surveys in §5.6 of [HMT] and in §7 of [N].

The reader is supposed to be familiar with the notion of a category. Occasionally, some results are summed up in terms of isomorphism or equivalence of categories. As to algebraic logic, the paper includes all necessary definitions and formulations of results. However, proofs that are not original are usually omitted.

2 Boolean homomorphisms admitting conjugates

Let A and B be two Boolean algebras, and let $s\colon A \to B$, $t\colon B \to A$ be any mappings. We generalize the notion introduced for the case of one algebra (when $B = A$) in [JT] and call t a *conjugate* of s if, for all $a \in A$, $b \in B$,

$$sa \wedge b = 0 \Leftrightarrow a \wedge tb = 0. \qquad (1)$$

Clearly, if t is a conjugate of s, then s is a conjugate of t, and we may speak of that the mappings s and t are conjugate. As shown in [JT], s has at most one conjugate which we usually denote by s^* when it exists. So, $s^{**} = s$. Furthermore, a mapping that admits the conjugate is additive; in fact, it is even completely additive and preserves 0. Admitting $B = A$, we conclude that the identity map is self-conjugate. Finally, if C is one more Boolean algebra and mapings $s_1: A \to B, s_2: B \to C$ have conjugates, then the composition $s_2 s_1$ also has the conjugate and

$$(s_2 s_1)^* = s_1^* s_2^*. \tag{2}$$

If s has the conjugate and preserves complements, then it is a Boolean homomorphism. Moreover, in this case (1) is equivalent to

$$sa \geq b \Leftrightarrow a \geq tb \tag{3}$$

—the well-known condition characterizing the so called *residual pairs*, or (contravariant) *Galois connections*—see, eg. [GH]. Also, s is completely multiplicative and preserves 1. (3) is equivalent to the following collection of four inequalities:

$$sa_1 \leq s(a_1 \vee a_2), \qquad tb_1 \leq t(b_1 \vee b_2), \tag{4}$$

$$b \leq stb, \qquad tsa \leq a, \tag{5}$$

each of which can be rewritten in the form of equality (involving Boolean operations). We obtain as consequences some more relationships between conjugate mappings:

$$stsa = a, \qquad tstb = b, \tag{6}$$

$$t(sa \wedge b) = a \wedge tb. \tag{7}$$

$$s \text{ is injective } \Leftrightarrow t \text{ is surjective } \Leftrightarrow tsa = a \Leftrightarrow t1 = 1, \tag{8}$$

$$s^* = s^{-1} \text{ if } s \text{ is bijective.} \tag{9}$$

At last, in the case $A = B$ (3) and any of the conditions (8) imply that

$$\text{if mappings } s \text{ and } t \text{ are idempotent, then } s = \mathrm{id}_A = t. \tag{10}$$

For these and several other properties of Boolean homomorphisms admitting conjugates, see Lemmas 13–18 in [Cr]. (Only the case $A = B$ is considered there; this is of no significance, however.) See also [GH], Theorem 3.6 and Proposition 3.7.

Definition 2.1 *Let K be some category, and let $A_1(K)$ and $A_2(K)$ be classes of (heterogenous) algebras of kind*

$$(A_X, s_\alpha)_{X \in \mathrm{Ob}K, \alpha \in \mathrm{Mor}K},$$

respectively,

$$(A_X, \mathsf{s}_\alpha, \mathsf{t}_\alpha)_{X \in \mathrm{Ob}K, \alpha \in \mathrm{Mor}K},$$

where (a) *every* A_X *is a Boolean algebra,* (b) *for every morphism* $\alpha: X \to Y$, s_α *and* t_α *are operations of types* $A_X \to A_Y$ *and* $A_Y \to A_X$, *respectively. An algebra* $\mathbf{A} \in A_1(K)$ *is a* Boolean action algebra of K, *in symbols an* BAA(K), *or just a* Boolean K-act, *if each* s_α *is a Boolean homomorphism and the following axioms are satisfied for all* $\varepsilon, \alpha, \beta$:

a1: $\mathsf{s}_\varepsilon a = a$ *if* ε *is an identity morphism,*

a2: $\mathsf{s}_{\beta\alpha} = \mathsf{s}_\beta \mathsf{s}_\alpha a$ *if* α *and* β *are composable.*

If, moreover, every s_α *admits the conjugate, we call the algebra a* BAA(K) with conjugates, *or a* BAAC(K), *and consider it as an algebra from* $A_2(K)$.

We sometimes omit the adjective 'Boolean' in the above context and, following the practice of [HMT], use each of the abbreviations BAA(K) and BAAC(K) also as a denotation of the respective class of algebras. The following proposition is an easy conseqence of the general properties of conjugates.

Proposition 2.2 (a) *An algebra* $\mathbf{A} \in A_2(K)$ *is an* BAAC(K) *if and only if the following conditions are fulfilled:*

(i) *every* s_α *preserves complements,*

(ii) *all* s_α *and* t_α *are correlated by* (3) *or, equivalentially, by* (4) *and* (5),

(iii) *either axioms* **a1, a2** *or their duals*

a'1: $\mathsf{t}_\varepsilon a = a$ *if* ε *is an identity morphism,*

a'2: $\mathsf{t}_{\beta\alpha} = \mathsf{t}_\alpha \mathsf{t}_\beta a$ *if* α *and* β *are composable,*

hold for every $\varepsilon, \alpha, \beta$.

(b) *Both* **a1** *and* **a'1** *may be omitted in* (a) *if instead every pair* $(\mathsf{s}_\varepsilon, \mathsf{t}_\varepsilon)$ *obeys* (8).

(c) *In any* BAAC(K), *if* α *and* β *are* K-*morphisms with a common codomain, and if* $\alpha = \beta\gamma$, $\beta = \alpha\delta$ *for appropriate* γ *and* δ, *then* $\mathsf{s}_\alpha \mathsf{t}_\alpha = \mathsf{s}_\beta \mathsf{t}_\beta$.

Let us consider three concrete examples of the above situation.

Example 2.3 K may be a monoid, i.e. a one-object category. A trans-Boolean algebra in the sense of [Cr] is nothing else than a K-act with conjugates for such a K (satisfying a few additional axioms reproducing of which here is not necessery). It was proved in [Cr] that in the case K is a monoid of transformations of some set the concept of a trans-Boolean algebra is equipollent to that of an equality polyadic algebra. In [C2], we announced a result according to which the additional axioms of trans-Boolean algebras are superflous if the full transformation monoid and locally finite algebras are considered. We prove in §5 below a similar result for K a monoid of finite transformations (and relatively to cylindric algebras rather than polyadic ones).

Example 2.4 K may be a partially ordered set treated as a category. If it is directed, then any K-act is a direct family of Boolean algebras. A heterogeneous cylindric algebra in the sense of [Z] is a K-act, where K is the set of all finite subsets of some set. We shall consider algebras of this latter kind in §6 in more detail.

Example 2.5 Beniaminov algebras are also partially covered by the above scheme. Assume we are given a set of sorts Σ. A *sorted set* is a set each element of which is correlated with some sort. Where X and Y are two sorted sets, an *agreement* of X with Y is a sort preserving mapping $\varphi: X \to Y$ (see [B1]). Beniaminov begins with the category of all finite sorted sets and agreements which we denote by $\Sigma^!$, and the first four of his axioms describe, in fact, the class $\mathsf{BAAC}(\Sigma^!)$. The category $\Sigma^!$ is, however, too big: as the collection of all finite sets is a proper class, it is difficult to compare relational algebras in the original Beniaminov's sense with algebras of relations traditionally arising in algebraic logic. With this in mind, we shall restrict in the subsequent section the category $\Sigma^!$ to its small subcategory whose objects are subsets of some fixed set.

From the category theoretical viewpoint, a $\mathsf{BAA}(K)$ is determined by an action of a functor s from K to the category of Boolean algebras and may be identified with the functor. A $\mathsf{BAAC}(X)$ is then essentially a pair of functors (s,t), where t acts from K to the category of Boolean lattices (considered as posets). We do not take this position here; however, we notice that such an approach to Beniaminov algebras has been proposed in Chapter 8 of [Pl]. Moreover, there $\Sigma^!$ is replaced by a certain algebraic theory in the sense of Lawere. Of course, the category of Boolean algebras (lattices) also could be replaced e.g. by that of Heyting algebras (lattices).

3 Relational algebras

Thus, let V be any sorted set which is assumed to be fixed throughout the rest of the paper. Elements of V are usually called *variables* or *attributes*, and we assume that V contains infinitely many variables of all sorts.

A *relation type* is a finite subset of V. Let RT stands for the set of all such types. We shall base the concept of a Beniaminov algebra on the category $\Sigma^{I}V$ of all relational types and agreements over V rather than on Σ^{I} (see Example 2.5).

Subsets of V will also be used with a view to classify algebras of certain kinds according to their similarity types. We shall term a subset used this way a *dimension type* of the algebra, or just *type* of it if misunderstanding is unlikely.

Definition 3.1 *We call any* $\mathsf{BAAC}(\Sigma^{I}V)$ *a* weak relational algebra *of dimension type* V, *or briefly a* wRelA_V. *A* Beniaminov algebra *of type* V, *or briefly* BenA_V, *is a* wRelA_V *which satisfies the condition*

b: *for all* $X, Y, Z \in RT$ *such that* $Z \cap (X \cup Y) = \emptyset$, *and all agreements* $\alpha: X \to Y$,

$$\mathsf{t}_{\alpha'}(\mathsf{s}_{\iota_2} a \wedge \mathsf{s}_{\iota_3} b) = \mathsf{s}_{\iota_1} \mathsf{t}_{\alpha} a \wedge \mathsf{s}_{\iota_3} b,$$

where $\iota_1, \iota_2, \iota_3$ *are the embeddings* $X \to X \cup Z$, $Y \to Y \cup Z$ *and* $Z \to Y \cup Z$ *respectively, and* α' *is the agreement of* $X \cup Z$ *with* $Y \cup Z$ *that is an extension of* α *and acts as the identity map on* Z.

The axiom **b** was formulated in [B1] (cf. the last axiom there) in terms of direct sums of sets and mappings and was therefore even more involved. We shall see below that a simple particular case of **b** does the job. As noted in Introduction, we shall add one more axiom (see **r10** below). This way we come to a bit narrower class of algebras which we name *relational algebras*, the term being used by Beniaminov himself.

First we introduce a certain codification of morphisms in $\Sigma^{I}V$; this will ease our further analysis. The related notion of a transformation will be in use also in other sections.

By a *transformation* we mean any sort-preserving self-map of V. Let Tr_ω stand for the set of finite transformations, i.e. transformations α whose *effective domain* $\mathrm{ed}\alpha := \{x \in V: \alpha x \neq x\}$ is finite. Clearly, Tr_ω is a monoid: it contains the identity map ε and is closed under composition. Given two relation types X and Y, we denote by $Tr(X,Y)$ the set $\{\alpha \in Tr_\omega: \mathrm{ed}\alpha \subset X \text{ and } \alpha X \subset Y\}$. (Note that $Tr_X := Tr(X,X)$ is a submonoid of Tr_ω.) If $\alpha X \subset Y$, let $\binom{YX}{\alpha}$ stand for the agreement $\varphi: X \to Y$ such that $\varphi = \alpha|X$. In particular, if $X \subset Y$, then $\binom{YX}{\varepsilon}$ is the embedding of X into Y. Obviously, the transfer $\alpha \mapsto \alpha|X$ yields a one-to-one correspondence between $Tr(X,Y)$ and the set of all agreements of X with Y, and it is precisely this set the notation $\{\binom{YX}{\alpha}: \alpha \in Tr(X,Y)\}$ refers to. This way we obtain a "parametrization" of the set of all $\Sigma^{\sharp}V$-morphisms by finite transformations. Warning: although $\binom{ZY}{\beta}\binom{YX}{\alpha} = \binom{ZX}{\beta\alpha}$ for any $\alpha \in Tr(X,Y)$ and $\beta \in Tr(Y,Z)$, the composition $\beta\alpha$ itself belongs to $Tr(X,Z)$ iff $\mathrm{ed}\beta\alpha \subset X$ iff $\mathrm{ed}\beta \subset X$.

Now, given an algebra from $A_2(\Sigma^{\sharp}V)$, we shall write s_α^{YX} and t_α^{XY} for s_φ, resp. t_φ, if $\varphi = \binom{YX}{\alpha}$. So the algebra itself can be written as

$$(A_X, \mathsf{s}_\alpha^{YX}, \mathsf{t}_\alpha^{XY})_{X,Y \in RT,\ \alpha \in Tr(X,Y)},$$

where every s_α^{YX} is an operation of kind $A_X \to A_Y$ and t_α^{XY} is an operation $A_Y \to A_X$. In this notation, the wRelA_V axioms read as follows (see Proposition 2.2(a)):

r1: $\mathsf{s}_\alpha^{YX}(-a) = -\mathsf{s}_\alpha^{YX}a$,
r2: $\mathsf{s}_\alpha^{YX}a \leq \mathsf{s}_\alpha^{YX}(a \vee a')$,
r3: $\mathsf{t}_\alpha^{XY}b \leq \mathsf{t}_\alpha^{XY}(b \vee b')$,
r4: $b \leq \mathsf{s}_\alpha^{YX}\mathsf{t}_\alpha^{XY}b$,
r5: $\mathsf{t}_\alpha^{XY}\mathsf{s}_\alpha^{YX}a \leq a$,
r6: $\mathsf{s}_\varepsilon^{XX}a = a$,
r7: $\mathsf{s}_\beta^{ZY}\mathsf{s}_\alpha^{YX}a = \mathsf{s}_{\beta\alpha}^{ZX}a$,

while **b** takes the form

b': if $Z \cap (X \cup Y) = \emptyset$ and $\alpha \in Tr(X,Y)$, then

$$\mathsf{t}_\alpha^{(X\cup Z)(Y\cup Z)}(\mathsf{s}_\varepsilon^{(Y\cup Z)Y}a \wedge \mathsf{s}_\varepsilon^{(Y\cup Z)Z}b) = \mathsf{s}_\varepsilon^{(X\cup Z)X}\mathsf{t}_\alpha^{XY}a \wedge \mathsf{s}_\varepsilon^{(X\cup Z)Z}b.$$

Definition 3.2 *A relational algebra of type V, or a* RelA_V, *is a* wRelA_V *that satisfies the additional axiom*

r8: if $Z \cap (X \cup Y) = \emptyset$, $Y \cup Z \subset U$ and $\alpha \in Tr(X,Y)$, then

$$t_\alpha^{(X\cup Z)U} s_\varepsilon^{UY} a = s_\varepsilon^{(X\cup Z)X} t_\alpha^{XY} a.$$

We call attention to the following restriction of **r8** obtained by setting $U = Y \cup Z$:

r9: if $Z \cap (X \cup Y) = \emptyset$ and $\alpha \in Tr(X,Y)$, then

$$t_\alpha^{(X\cup Z)(Y\cup Z)} s_\varepsilon^{(Y\cup Z)Y} a = s_\varepsilon^{(X\cup Z)X} t_\alpha^{XY} a.$$

Remarkably that **r9** can be obtained also from **b'** by substituting 1 for b (recall that both $s_\varepsilon^{(Y\cup Z)Z}$ and $s_\varepsilon^{(X\cup Z)Z}$ are Boolean homomorphisms and thence preserve 1). The subsequent theorem shows the "distance" between BenA_V's and RelA_V's.

Theorem 3.3 (a) *A* wRelA_V *is a Beniaminov algebra iff it satisfies* **r9**.

(b) *A* BenA_V *is a relational algebra iff it satisfies the condition*

r10: $t_\varepsilon^{X(X\cup Y)} 1 = 1$, where Y does not contain variables of sorts presented in X.

Proof. (a) It remains to prove that **b'** holds in any wRelA_V satisfying **r9**. Assume that X, Y, Z and α are such as in **b'**. By **r7**, (7) and **r9**,

$$t_\alpha^{(X\cup Z)(Y\cup Z)}(s_\varepsilon^{(Y\cup Z)Y} a \wedge s_\varepsilon^{(Y\cup Z)Z} b) =$$

$$t_\alpha^{(X\cup Z)(Y\cup Z)}(s_\varepsilon^{(Y\cup Z)Y} a \wedge s_\alpha^{(Y\cup Z)(X\cup Z)} s_\varepsilon^{(X\cup Z)Z} b) =$$

$$t_\alpha^{(X\cup Z)(Y\cup Z)} s_\varepsilon^{(Y\cup Z)Y} a \wedge s_\varepsilon^{(X\cup Z)Z} b = s_\varepsilon^{(X\cup Z)X} t_\alpha^{XY} a \wedge s_\varepsilon^{(X\cup Z)Z} b.$$

(b) Assume axioms of BenA_V's. An application of **r10** yields

r11: every s_ε^{YX} is injective.

Indeed, let $X' = \{x \in Y\colon$ the sort of x is presented in $X\}$. Then $X \subset Y$, $\binom{YX}{\varepsilon} = \binom{YX'}{\varepsilon}\binom{X'X}{\varepsilon}$ and $s_\varepsilon^{YX} = s_\varepsilon^{YX'} s_\varepsilon^{X'X}$ (by **r7**). By **r10**, $t_\varepsilon^{X'Y} 1 = 1$, and by (8), $s_\varepsilon^{YX'}$ is injective. Choose $\alpha \in Tr(X', X)$ which agrees with ε on X; then $\binom{XX'}{\alpha}\binom{X'X}{\varepsilon} = \binom{XX}{\varepsilon}$. Again by **r7**, and **r6**, $s_\alpha^{XX'} s_\varepsilon^{X'X} = s_\varepsilon^{XX} = \mathrm{id}A_X$. So $s_\varepsilon^{X'X}$ also is injective, and **r11** follows.

By **r11** and (8),

r12: $t_\varepsilon^{XY} s_\varepsilon^{YX} a = a.$

Now we can derive **r8**: by **r7** and its dual, **r12** and **r9**,

$$t_\alpha^{(X\cup Z)U} s_\varepsilon^{UY} a = t_\alpha^{(X\cup Z)(Y\cup Z)} t_\varepsilon^{(Y\cup Z)U} s_\varepsilon^{U(Y\cup Z)} s_\varepsilon^{(Y\cup Z)Y} a =$$

$$t_\alpha^{(X\cup Z)(Y\cup Z)} s_\varepsilon^{(Y\cup Z)Y} a = s_\varepsilon^{(X\cup Z)X} t_\alpha^{XY} a.$$

Conversely, given a RelA_V, we set $Z = \emptyset$, $Y = X$ and $\alpha = \varepsilon$ in **r8**:

$$t_\varepsilon^{XU} s_\varepsilon^{UX} a = s_\varepsilon^{XX} t_\varepsilon^{XX} a.$$

By **r6** and its dual, the right-hand side equals to a. So $t_\varepsilon^{XU}1 = 1$ by (8), and **r10** follows. □

Remark 3.4 Therefore, $\mathsf{wRelA}_V \subset \mathsf{BenA}_V \subset \mathsf{RelA}_V$, and none of the inclusions is reversible. **r11** cannot be proved in full extent without using **r10** or some equivalent of it. The present author turned attention of N. Volkov to the fact that this was overlooked in Proposition 1 of his [V3]. (For relevant corrections see Section II of [V4].) E. Beniaminov communicated to the author in August, 1987, that he also had discovered independence of **r11** (of his axioms).

An inspection of the proof of **r12** shows that the identity is valid in any wRelA_V satisfying **r10**. This makes further splitting of **r8** possible.

Proposition 3.5 *The identities* **r1–r5**, **r7**, **r10** *and*

r13: $t_\varepsilon^{(X\cup Z)(Y\cup Z)} s_\varepsilon^{(Y\cup Z)Y} a = s_\varepsilon^{(X\cup Z)X} t_\varepsilon^{XY} a$ with $Z \cap Y = \emptyset$ (and $X \subset Y$),

r14: $s_\varepsilon^{YX} t_\alpha^{XX} b = t_\alpha^{YY} s_\varepsilon^{YX} b$ with $\alpha \in Tr_X$,

make up a complete axiom system for RelA_V*'s.*

Proof. Clearly, **r13** is contained in **r9**, while **r14** follows from **r9** by setting $Y = X$ and subsequent relettering of $Y \cup Z$. Hence, in view of Proposition 3.3 we must only show that **r6** and **r9** are derivable from the mentioned list of axioms. By **r10**, $t_\varepsilon^{XX}1 = 1$, and **r6** follows by Proposition **2.2**(b). Now assume that X, Y, Z and α are such as in **r9**, and let W stand for $X \cup Y$. Then by **r12**, the dual of **r7**, again the dual of **r7**, **r13**, **r14**, **r7** and its dual, again **r7** and its dual, **r12**,

$$s_\varepsilon^{(X\cup Z)X} t_\alpha^{XY} a = s_\varepsilon^{(X\cup Z)X} t_\alpha^{XY} t_\varepsilon^{YW} s_\varepsilon^{WY} a = s_\varepsilon^{(X\cup Z)X} t_\alpha^{XW} s_\varepsilon^{WY} a =$$

$$s_\varepsilon^{(X\cup Z)X} t_\varepsilon^{XW} t_\alpha^{WW} s_\varepsilon^{WY} a = t_\varepsilon^{(X\cup Z)(W\cup Z)} s_\varepsilon^{(W\cup Z)W} t_\alpha^{WW} s_\varepsilon^{WY} a =$$

$$\mathsf{t}_\varepsilon^{(X\cup Z)(W\cup Z)}\mathsf{t}_\alpha^{(W\cup Z)(W\cup Z)}\mathsf{s}_\varepsilon^{(W\cup Z)(W)}\mathsf{s}_\varepsilon^{(W)Y}a = \mathsf{t}_\alpha^{(X\cup Z)(W\cup Z)}\mathsf{s}_\varepsilon^{(W\cup Z)Y}a =$$
$$\mathsf{t}_\alpha^{(X\cup Z)(Y\cup Z)}\mathsf{t}_\varepsilon^{(Y\cup Z)(W\cup Z)}\mathsf{s}_\varepsilon^{(W\cup Z)(Y\cup Z)}\mathsf{s}_\varepsilon^{(Y\cup Z)Y}a = \mathsf{t}_\alpha^{(X\cup Z)(Y\cup Z)}\mathsf{s}_\varepsilon^{(Y\cup Z)Y}a,$$

and the proposition is proved. □

Now we leave relation algebras until §7, and turn to some other classes of algebras. Most of them, in that or other form, have already appeared in literature. We notice in this connection that, as a rule, dimension types of algebras usually dealt with in algebraic logic are unsorted sets. To compare the algebras considered above with these, we are forced either to extend the traditional definitions and results or to assume that in the rest the set of sorts, Σ, will be a singletone. We choose the latter alternative.

4 Cylindric algebras

In this section, we list some basic facts concerning cylindric algebras. The standard reference on cylindric algebras is [HMT]. For our purposes, it is more convenient to deviate from the tradition and to index the operations of a cylindric algebra by elements of an arbitrary set rather than by ordinals.

Definition 4.1 *By a* cylindric algebra of type X, *or briefly a* CA_X, *we mean an algebra* $\mathbf{A} := (A, \mathsf{c}_x, \mathsf{d}_{xy})_{x,y\in X}$, *where* $A := (A, \vee, \wedge, -, 0, 1)$ *is a Boolean algebra, every* c_x *is a unary operation on* A, *every* d_{xy} *is an element of* A, *and the following axioms are satisfied for all* $x, y, z \in X$:

c1: $\mathsf{c}_x 0 = 0$,

c2: $a \leq \mathsf{c}_x a$,

c3: $\mathsf{c}_x(a \wedge \mathsf{c}_x a') = \mathsf{c}_x a \wedge \mathsf{c}_x a'$,

c4: $\mathsf{c}_x \mathsf{c}_y a = \mathsf{c}_y \mathsf{c}_x a$,

c5: $\mathsf{d}_{xx} = 1$,

c6: $\mathsf{d}_{xz} = \mathsf{c}_y(\mathsf{d}_{xy} \wedge \mathsf{d}_{yz})$ if $y \neq x, z$,

c7: $\mathsf{c}_x(a \wedge \mathsf{d}_{xy}) \wedge \mathsf{c}_x(-a \wedge \mathsf{d}_{xy}) = 0$ if $x \neq y$.

We shall need the following additional properties of operations c_x in a cylindric algebra:

c8: $\mathsf{c}_x 1 = 1$,

c9: $\mathsf{c}_x a \leq \mathsf{c}_x(a \vee a')$, c10: $c_x c_x a = c_x a$,

c11: $c_x(-c_x a) = -c_x a$,

c12: $c_x d_{xy} = 1$,

c13: $c_x d_{yz} = d_{yz}$ if $x \neq y, z$.

Moreover, every c_x is a self-conjugate operation: $c_x^* = c_x$, so it is a completely additive closure operator.

Now suppose that **A** is a CA_X. For every $Z \subset X$, we define a generalized cylindrification, or *quantifier*, c_Z on **A** as follows (cf. [HMT, §1.7]):

$$\alpha 1: \quad c_Z a = \begin{cases} a & \text{if } Z = \emptyset, \\ c_{z_1} c_{z_2} \cdots c_{z_n} a & \text{otherwise,} \end{cases}$$

where $z_1, z_2, \ldots, z_n$ is a list of variables from Z in some fixed order. In fact, the order is irrelevant by **c4**. Obviously, every c_Z has properties analogous to **c1–c3**, **c8–c11**; moreover, also

c14: $c_\emptyset a = a$,

c15: $c_{Z_1} c_{Z_2} a = c_{Z_1 \cup Z_2} a$

hold. This motivates the following definition which we shall need later.

Definition 4.2 *Let* $RT(X)$ *stands for* $\overset{\circ}{RT} \cap X$. *We call a* quantifier algebra of type X, *or a* QA_X, *any algebra* $(A, c_Z)_{Z \in RT(X)}$, *where* A *is a Boolean algebra and all* c_X *are operations on* A *subject to* **c1–c4** *and* $(\alpha 1)$ *(or equivalentially,* **c1–3**, **c14** *and* **c15***).*

An element a of a CA_X **A** is said to be *independent of* x if $c_x a = a$. A *support* of a is a subset $Y \subset X$ such that a is independent of every $x \notin Y$. If all elements of A have finite supports, the algebra is called *locally finite.* In any case, the elements having the same support make up a subalgebra of the Boolean algebra A. Moreover, the subset $A|Y := \{a \in A: Y \text{ supports } a\}$, the *restriction* of A to Y, is closed under all operations c_x with $x \in Y$. By **c12** and **c13**, $A|Y$ contains also the elements d_{xy} for all $x, y \in Y$. So we come to the CA_Y $\mathbf{A}|Y := (A|Y, c_x, d_{xy})_{x,y \in Y}$, called a *neat Y-reduct* of **A** (cf. Definition 2.6.28 in [HMT]).

Two useful families of operations are defined in a cylindric algebra in the following way ([HMT, §1.5], [P, (3.6)]):

$$\beta1:\quad s^x_y a = \begin{cases} a & \text{if } x = y, \\ c_x(a \wedge d_{xy}) & \text{otherwise,} \end{cases}$$

$$\beta2:\quad t^x_y a = \begin{cases} a & \text{if } x = y, \\ c_x a \wedge d_{xy} & \text{otherwise,} \end{cases}$$

for all $a \in A$ and every $x, y \in X$. It was proved in [P] that the class CA_X with $|X| > 2$ can be characterized in terms of these operations. We shall consider this question in some detail.

Definition 4.3 *A substitution algebra of type X, or a SA_X, is an algebra $\mathbf{A} := (A, s^x_y)_{x,y\in X}$, where A is a Boolean algebra, every s^x_y is a unary operation on A, and the following axioms hold:*

s1: s^x_y is a Boolean endomorphism,

s2: $s^x_x a = a$,

s3: $s^y_z s^x_y a = s^y_z s^x_z a$,

s4: $s^x_z s^z_y a = s^x_y a \quad \text{if } x \neq y$,

s5: $s^u_v s^x_y a = s^x_y s^u_v a \quad \text{if } y \neq u \neq x \neq v$.

If, moreover, every operation s^x_y has the conjugate t^x_y, then $\mathbf{A}$ is called a SA *with conjugates, or SAC_X. In this case we consider it as an algebra of kind $(A, s^x_y, t^x_y)_{x,y\in X}$. We shall say that such an algebra is* commutative *if it satisfies one more axiom*

s6: $s^u_v t^x_y a = t^x_y s^u_v a \quad \text{if } y \neq u \neq x \neq v$.

According to (4) and (5) the class SAC_X is equationally definable. In a SAC_X $(A, s^x_y, t^x_y)_{x,y\in X}$ we set

$\gamma1$: $\quad c_x a = s^x_y t^x_y a$ with $y \neq x$,

$\gamma2$: $\quad d_{xy} = t^x_y 1$.

The definition $\gamma1$ is correct: $c_x a$ does not depend on the choise of y. Now we can restate the content of Theorem 2.7 of [P], in connection with the algebras we consider, as follows (see also the proposition (F) and the note after the proposition (G) on p. 176, Theorem 3.3 and the note at the bottom of the page 177 in [P]).

Proposition 4.4 *Assume that $|X| > 2$. For any two algebras*

$$(A, c_x, d_{xy})_{x,y\in X} \quad \textit{and} \quad (A, s^x_y, t^x_y)_{x,y\in X},$$

the following statements are equivalent:

a) $(A, c_x, d_{xy})_{x,y\in X}$ *is a* CA_X *and* $\beta1$, $\beta2$ *hold,*

b) $(A, s^x_y, t^x_y)_{x,y\in X}$ *is a* SAC_X *and* **s6**, $\gamma1$, $\gamma2$ *hold.*

In particular, $t^x_y = (s^x_y)^*$ in $\beta 1$, $\beta 2$. Via $\gamma 1$, the notion of a support applies also to arbitrary SAC's. The following immediate consequence of Theorem 2.12 and Lemma 2.13 of [P] is crucial (here, and below, Lf stands for 'locally finite').

Proposition 4.5 *Every* LfSAC_V *is commutative.*

Now it follows that the classes LfCA_V and LfSAC_V are definitionally equivalent and may be considered as indistinguishable; the more so as any cylindric homomorphism is an SAC homomorphism and vice versa. See also Theorem 5.4.

5 Transformation algebras

From the viewpoint of their structure, cylindric and relational algebras are too far from each other to be handily compared immediately. To reduce this difficulty, we now proceed from CA's to transformation algebras. Transformation algebras were introduced by the way by Halmos, and studied by Leblanc in [L]. Roughly, a transformation algebra is a K-act with K the transformation monoid of some set. For our purposes, it is more convenient to deal only with finite transformations. Following the tradition, we should use the term 'quasi-transformation algebra' in this case. However, we do not, partly because the difference vanishes as far as locally finite algebras (of infinite dimension type) are considered.

We first introduce some additional conventions concerning transformations. The notation $(y_1/x_1, \ldots, y_n/x_n)$ stands for a finite transformation α such that $\mathrm{ed}\alpha \subset \{x_1, \ldots, x_n\}$ and $\alpha x_i = y_i$ for every i. We call a *replacement* any transformation of kind (y/x), and a *transposition* any transformation of kind $(x, y) := (y/x, x/y)$. Every finite transformation can be produced as a composition of a finite number of replacements and transpositions (see [Cr], p.10); we shall refer to this fact as to the *decomposition property* (DP).

Definition 5.1 *Suppose that* $X \subset V$. *By a* transformation algebra of dimension X, *or* TA_X, *we shall mean any* Tr_X*–act. A* TA_X with conjugates, *or a* TAC_X, *is defined in accordance with Definition* 2.1.

In a TAC_X, the Proposition 2.2(c) can be concretized as follows:

$$\mathsf{s}_\alpha \mathsf{t}_\alpha a = \mathsf{s}_\beta \mathsf{t}_\beta a \quad \text{whenever } \alpha X = \beta X. \tag{11}$$

Indeed, assume that α and β satisfy the condition, and choose $\varphi \in Tr(X)$ such that $\varphi y \in \beta^{-1}y$ when $y \in \alpha X$. Then $\beta\varphi\alpha = \alpha$. Likewise, $\beta = \alpha\psi\beta$ for some $\psi \in Tr(X)$. Now the equality follows from Proposition 2.2(c).

Clearly, the reduct of a TAC_X obtained by omitting all operations s_α and t_α but those with α a replacement is an SAC_X. In particular, the notion of a support can be transferred to TA's and TAC's. Also, quantifiers can be defined in TAC's according to (γ1) and (α1); moreover, Propositions 4.5 and 4.4 imply that a TAC_V can be converted into a cylindric algebra (hence, a quantifier one as well).

Now we are going to show that any locally finite SAC_V can be expanded to a TAC_V.

By (9), any operation $\mathsf{s}_{(y,x)}$ of a TA_V has the conjugate, for the transposition is inverse to itself. Moreover, $(\mathsf{s}_{(y,x)})^* = \mathsf{s}_{(y,x)}$. Now the following proposition holds on the strength of the DP, .2 and (2).

Proposition 5.2 *If every operation* $\mathsf{s}_{(y/x)}$ *of a* TA_V $\mathbf{A}$ *has the conjugate, then* $\mathbf{A}$ *is a* TAC_V.

Let $\alpha := (y_1/x_1, \ldots y_n/x_n)$ be a finite transformation. For any element a of a locally finite substitution algebra $\mathbf{A}$, we set

$$\delta 1: \quad \mathsf{s}_\alpha a = \begin{cases} a & \text{if } \alpha = \varepsilon, \\ \mathsf{s}^{z_n}_{y_n} \cdots \mathsf{s}^{z_1}_{y_1} \mathsf{s}^{x_n}_{z_n} \cdots \mathsf{s}^{x_1}_{z_1} a & \text{otherwise;} \end{cases}$$

here the variables $z_1, \ldots, z_n$ are supposed to be distinct from each other and from $x_1, \ldots, x_n, y_1, \ldots, y_n$, and such that a does not depend on them. In [G, §4], it is shown that $\mathsf{s}_\alpha a$ does not depend on the choise of $z_1, \ldots, z_n$. Therefore, looking over all elements of A, we can define an operation s_α on A. Actually the first statement of the following proposition is implicit in [G]; see also [HMT, 1.11.9, 1.11.11, 1.11.12]. The other one follows from Proposition 5.2.

Proposition 5.3 *An algebra* $(A, \mathsf{s}_\alpha)_{\alpha \in Tr_\omega}$ *is a locally finite* TA_V *iff its reduct* $(A, \mathsf{s}_{(y/x)})_{x,y \in V}$ *is a locally finite* SA_V *and* $\delta 1$ *holds for all* a *and appropriate* $z_1, \ldots, z_n$. *Moreover, if one of the algebras admits conjugates, then so does the other.*

Now we can state in what sense the concept of a TAC is equipollent to those of the preceeding section. We call two categories L and M *indistinguishable* if there is an isomorphism $F: L \to M$ such that $F\alpha = \alpha$ for every $\alpha \in \mathrm{Mor}L$. A typical example is provided by the categories of Boolean algebras and Boolean rings with unit.

Theorem 5.4 *The classes* LfCA_V, LfSAC_V *and* LfTAC_V, *considered as categories, are indistinguishable.*

Proof. As to the first two of the classes, the result was essentially established at the end of §4. It is clear from the above discussion that the transfer from a TAC_V $(A, \mathsf{s}_\alpha, \mathsf{t}_\alpha)_{\alpha \in Tr_\omega}$ to its reduct $(A, \mathsf{s}_{(y/x)}, \mathsf{t}_{(y/x)})_{x,y \in V}$ yields a one-to-one correspondence between classes LfTAC_V and LfSAC_V. Of course, every TAC-homomorphism is an SAC-homomorphism between the corresponding SAC's. By $\delta 1$, any SAC-homomorphism h between locally finite algebras preserves all operations s_α. By the DP, **a2** and (2), every operation t_α of an LfTAC_V can be decomposed into a product of operations of kind $\mathsf{t}_{(y/x)}$ and $\mathsf{t}_{(x,y)}$; since $\mathsf{t}_{(x,y)} = \mathsf{s}_{(x,y)}$, h preserves t_α as well. Thus it is a TAC-homomorphism, and we have proved that LfSAC_V and LfTAC_V are indistinguishable. □

We still derive some more properties of TAC's which will be used in further sections.

Lemma 5.5 *Assume that* $\mathbf{A}$ *is a* TAC_V, $\alpha \in Tr_\omega$, $Z \in RT$, *and that* c_Z *is the operation defined on* A *by* $\alpha 1$ *and* $\gamma 1$. *Then*

(a) $\mathsf{c}_Z a = \mathsf{s}_\alpha \mathsf{t}_\alpha a$ if $\mathrm{ran}\alpha = \bar{Z}$,
(b) $\mathsf{s}_\alpha \mathsf{c}_Z a = \mathsf{c}_Z$ if $\mathrm{ed}\alpha \subset Z$,
(c) $\mathsf{c}_Z \mathsf{s}_\alpha a = \mathsf{s}_\alpha a$ if $Z \cap \mathrm{ran}\alpha = \emptyset$,
(d) $\mathsf{c}_Z \mathsf{s}_\alpha a = \mathsf{s}_\alpha \mathsf{c}_Z a$ if $Z \cap (\mathrm{ed}\alpha \cup \mathrm{er}\alpha) = \emptyset$.

Proof. (a) Assume that Z and α satisfy the condition. By 11, we may consider α to be idempotent. Then $Z = \mathrm{ed}\alpha$. Now if $Z = \{z_1, z_2, \ldots, z_n\}$ and $\alpha = (y_1/z_1, y_2/z_2, \ldots, y_n/z_n)$, then $z_i \neq y_j$ for all i and j, and $\alpha = (y_n/z_n) \cdots (y_1/z_1)$. Taking ($\alpha$1), ($\gamma$1), **s5**, Propositon 4.5 and **s6**, **a2** and **a2'** into account, we infer that

$$\mathsf{c}_Z a = \mathsf{c}_{z_1} \cdots \mathsf{c}_{z_n} a = \mathsf{s}^{z_1}_{y_1} \mathsf{t}^{z_1}_{y_1} \cdots \mathsf{s}^{z_n}_{y_n} \mathsf{t}^{z_n}_{y_n} a = \mathsf{s}^{z_n}_{y_n} \cdots \mathsf{s}^{z_1}_{y_1} \mathsf{t}^{z_1}_{y_1} \cdots \mathsf{t}^{z_n}_{y_n} a = \mathsf{s}_\alpha \mathsf{t}_\alpha a.$$

(b), (c), (d) are proved similarly, or they can be derived as particular cases from Theorem 1.11.12(vi) of [HMT]. □

Given a TAC_V $\mathbf{A}$, it is easy to see that every subset (in fact, a Boolean algebra) $A|X := \{a \in A : X \text{ supports } a\}$ is closed under those operations s_α with $\alpha \in Tr_X$: if $a \in A|X$, then $\mathsf{c}_y a = a$ for $y \notin X$, and we have that $\mathsf{c}_y \mathsf{s}_\alpha a = \mathsf{s}_\alpha \mathsf{c}_y a = \mathsf{s}_\alpha$ by (d). Using the dual of (d), we likewise obtain that $A|X$ is closed also under all operations t_α with $\alpha \in Tr_X$. We call the algebra $\mathbf{A}_X := (A|X, \mathsf{s}_\alpha, \mathsf{t}_\alpha)_{\alpha \in Tr_X}$ the *neat* X-*reduct* of $\mathbf{A}$.

6 Heterogeneous cylindric algebras

This section is "optional" in the sense that heterogeneous cylindric algebras are not concerned to the subsequent section directly. However, several constructions and methods considered here will be used afterwards in a less familiar context.

Definition 6.1 *A heterogeneous* CA_V, *or* HCA_V, *is an algebra*

$$(\mathbf{A}_X, f^{YX}, g^{XY})_{X \subset Y \in RT}, \tag{12}$$

where each $\mathbf{A}_X := (A_X, c_z^{(X)}, d_{zy}^{(X)})_{z,y \in X}$ *is a* CA_X *and, for all* X, Y, Z,

h1: f^{YX} *is a homomorphism from* $\mathbf{A}_X$ *to the* X*-reduct of* $\mathbf{A}_Y$,
h2: $f^{ZY} f^{YX} a = f^{ZX} a$,
h3: $g^{XY} f^{YX} a = a$,
h4: $f^{YX} g^{XY} b = c_{Y \setminus X}^{(Y)} b$,

$c_{Y\setminus X}^{(Y)}$ *being the operation defined in* $\mathbf{A}_Y$ *according to* $\alpha 1$.

Observe that each algebra $(A_X, c_Z^{(X)})_{Z \subset X}$ in a HCA_V is a QA_X.

Heterogenous CA's first appeared in [Z]. Zlatoš' original axiom system included also

h5: $f^{XX} a = a$,
h6: $c_z^{(Y)} f^{YX} a = f^{YX} a \quad \text{if } z \in Y \setminus X$.

However, **h5** follows from the above axioms by Propositions 6.6 (see below) and 2.2(b), while **h6** was shown superfluous in [C4, item (3.6)]: by **h4, h2, h3, h2,**

$$c_z^{(Y)} f^{YX} a = f^{Y(Y\setminus\{z\})} g^{(Y\setminus\{z\})Y} f^{YX} a =$$

$$f^{Y(Y\setminus\{z\})} g^{(Y\setminus\{z\})Y} f^{Y(Y\setminus\{z\})} f^{(Y\setminus\{z\})X} a = f^{Y(Y\setminus\{z\})} f^{(Y\setminus\{z\})X} a = f^{YX} a.$$

It was proved in [Z, Theorem 1] that the concepts of a locally fini e CA and a heterogeneous CA are equipollent in the following strict sense.

Theorem 6.2 *The categories* LfCA_V *and* HCA_V *are equivalent.*

We sketch a somewhat different proof of this theorem, several details of which will be referred to in the last section.

First, a *homomorphism* from a HCA_V $\mathbf{A}$ to another HCA_V $\mathbf{A}'$ is, as usual for heterogeneous algebras, a family $\Phi := (\varphi_X : X \in RT)$ of CA-homomorphisms $\varphi_X : \mathbf{A}_X \to \mathbf{A}'_X$ such that for all X, Y with $X \subset Y$

$$\varphi_Y f^{YX} a = f'^{YX} \varphi_X a, \quad \varphi_X g^{XY} a = g'^{XY} \varphi_Y a.$$

Φ is an identity homomorphism if all $\varphi_X : \mathbf{A}_X \to \mathbf{A}'_X$ are identity homomorphisms, and the composition of two HCA_V-homomorphisms is also defined componentwise. Φ is said to be an *isomorphism* if all φ_X are bijective.

We call a HCA_V *flat* if $A_X \subset A_Y$ whenever $X \subset Y$ and every monomorphism f^{YX} is the embedding that realizes this inclusion. In this case the axioms **h1**, **h2** become trivial while **h3** and **h4** reduce to equalities

$$g^{XY} a = a \;\text{ if } a \in A_X \quad \text{and} \quad g^{XY} b = \mathsf{c}^{(Y)}_{Y \setminus X} b \;\text{ if } b \in A_Y, \tag{13}$$

respectively. Moreover, then

$$\mathsf{d}^{(X)}_{xy} = \mathsf{d}^{(Y)}_{xy} \;\text{ and }\; \mathsf{c}^{(X)}_x a = \mathsf{c}^{(Y)}_x a \tag{14}$$

for $x, y \in X \subset Y$ and $a \in A_X$. The proof of the following proposition is straightforward; we only note that this is where **h5** is needed.

Proposition 6.3 *Every algebra from* HCA_V *is isomorphic to a flat algebra. Moreover, the category* HCA_V *is equivalent to its full subcategory determined by flat algebras.*

Now we associate with every flat HCA_V $\mathbf{A}$ an algebra $\mathbf{A}^{Lf} := (A, \mathsf{c}_x, \mathsf{d}_{xy})_{x,y \in V}$ by setting

$\varepsilon 0$: $A = \bigcup (A_X : X \in RT)$,

$\varepsilon 1$: $\mathsf{c}_x a = \mathsf{c}^{(X)}_x a$ for some $X \in RT$ such that $a \in A_X$ and $x \in X$,

$\varepsilon 2$: $\mathsf{d}_{xy} = \mathsf{d}^{(X)}_{xy}$ with $x, y \in X$.

By (14), definitions ($\varepsilon 1$) and ($\varepsilon 2$) are unambigous: any X which satisfies the conditions may be used. Conversely, given a locally finite CA_V $\mathbf{A}$, we construct an algebra $\mathbf{A}^F := (\mathbf{A}_X, f^{YX}, g^{XY})_{X \subset Y \in RT}$ as follows:

$\zeta 0$: $\mathbf{A}_X := \mathbf{A}|X$ is the neat X-reduct of $\mathbf{A}$,

$\zeta 1$: f^{YX} is the embedding of A_X into A_Y,

$\zeta 2$: g^{XY} is the restriction of $c_{Y\setminus X}$ to A_Y.

Proposition 6.4 (a) *If* $\mathbf{A}$ *is a flat* HCA_V $\mathbf{A}$, *then* $\mathbf{A}^{Lf}$ *is a* LfCA_V *and* $(\mathbf{A}^{Lf})^F = \mathbf{A}$.

(b) *If* $\mathbf{A}$ *is a* LfCA_V $\mathbf{A}$, *then* $\mathbf{A}^F$ *is a flat* HCA_V *and* $(\mathbf{A}^F)^{Lf} = \mathbf{A}$.

Proof. (a) Assume that $\mathbf{A}$ is a flat HCA_V. Clearly, $\mathbf{A}^{Lf} \in \mathsf{CA}_V$, for any finite number of elements of A belongs to some A_X. To verify that $\mathbf{A}^{Lf}$ is locally finite, it sufficies to show that X supports a whenever $a \in A_X$. Let $a \in A_X$, $x \notin X$ and $X \cup \{x\} \subset Y$. Then in $\mathbf{A}$, $c_x^{(Y)} f^{YX} a = f^{YX} a$ by **h6**, i.e. $c_x a = a$ in $\mathbf{A}^{Lf}$.

We have just seen that $A_X \subset A|X$. Conversely, if $b \in A|X$, then $c_y b = b$ whenever $y \notin X$. There is $Y \in RT$ such that $b \in A_Y$ and $X \subset Y$; so we have $b = c_{Y\setminus X} b = c^{(Y)}_{Y\setminus X} b$. By **h4**, then $b = f^{YX} a$ for some $a \in A_X$; since $\mathbf{A}$ is flat, we conclude that $b \in A_X$. Therefore, $A|X = A_X$. Now it is easily seen that $\mathbf{A}_X$ is the neat X-reduct of $\mathbf{A}^{Lf}$, indeed, and that ($\zeta 1$), ($\zeta 2$) hold as well.

(b) Straightforward. □

This is a routine job to check that if $\{\varphi_X : X \in RT\}$ is a HCA_V-homomorphism from $\mathbf{A}$ to $\mathbf{A}'$, then the mapping $\varphi := \bigcup(\varphi_X : X \in RT)$ is a homomorphism between the respective CA_V's. Conversely, if φ is a CA-homomorphism from $\mathbf{A}$ to $\mathbf{A}'$, then the family $(\varphi|X : X \in RT)$ is a homomorphism between the respective HCA_V's. These transformations are mutually inverse; therefore, we come to

Proposition 6.5 *The category* LfCA_V *is isomorphic to that of flat* HCA_V*'s.*

Now Theorem 6.2 follows from Propositions 6.3 and 6.5.

Returning to the structure of a HCA_V, we note that the following result was proved in [C4].

Proposition 6.6 *Assume that* $\mathbf{A}$ *is an algebra of kind* (12) *and that each* $\mathbf{A}_X$ *is of the form* $(A_X, c_z^{(X)})_{z\subset X}$. *Then the following conditions are equivalent:*

(a) *every* $\mathbf{A}_X$ *is a* QA_X *and* **h1–h4** *hold,*

(b) $(A_X, f^{YX}, g^{XY})_{X\subset Y\in RT}$ *is a* $\mathsf{BAAC}(RT)$, *and* **h3**, **h4** *as well as*

h7: $f^{Y(X\cap Y)}g^{(X\cap Y)X}a = g^{YZ}f^{ZX}a$

hold.

The conditions listed in (b) correspond to items (2.1), (4.2) and (2.3) in [C4]. **h3** can even be omitted: an instance of **h7** (with $X = Y$) gives $f^{XX}g^{XX}a = g^{XZ}f^{ZX}a$, and **h3** follows by **a1** and **a1'**. On the other hand, a weakened form

h8: $f^{Y(X\cap Y)}g^{(X\cap Y)X}a = g^{Y(X\cup Y)}f^{(X\cup Y)X}a$

of **h7** would suffice:

$$g^{YZ}f^{ZX}a = g^{Y(X\cup Y)}g^{(X\cup Y)Z}f^{Z(X\cup Y)}f^{(X\cup Y)X}a =$$

$$g^{Y(X\cup Y)}f^{(X\cup Y)X}a = f^{Y(X\cap Y)}g^{(X\cap Y)X}a$$

by **a2** and **a2'**, **h3**, **h8**.

Remark 6.7 Thus, a BAAC(RT) that satisfies **h7** may be treated as a "quantifier-free" heterogeneous quantifier algebra, **h4** being merely a definition of quantifiers. If endowed with diagonal elements in all $\mathbf{A}_X$'s (subject to appropriate axioms involving only operations f and g), such an RT-act becames essentially a heterogeneous cylindric algebra. For a class of algebras defined along these lines, see [C3].

7 Equivalence of RelA$_V$'s and LfCA$_V$'s

Now we are almost ready for proving the following result which shows, in particular, that the concept of a relational algebra is properly defined.

Theorem 7.1 *The category* RelA$_V$ *is equivalent to* LfCA$_V$.

We still need only one lacking link—heterogeneous TAC's! We shall imitate Definition 12; however, there is a trouble with quantifiers in the axiom **h4**. If we define $c_x^{(Y)}$ according to (γ1), the case $Y = \{x\}$ makes no sense while quantifiers $c_Y^{(Y)}$ prove to be unrelated with operations of the algebras $\mathbf{A}_Y$. For this reason, we use a different definition, restrict **h4** and add one more axiom instead.

Definition 7.2 *A heterogeneous* TAC_V, *or* HTAC_V, *is an algebra*

$$(\mathbf{A}_X, f^{YX}, g^{XY})_{X\subset Y\in RT},$$

where each $\mathbf{A}_X := (A_X, s_\alpha^{(X)}, t_\alpha^{(X)})_{\alpha\in Tr_X}$ *is a* TAC_X *and, for all* X, Y, Z, *the axioms* **h1**–**h3**,

$$\mathbf{h4}_1\colon\ f^{YX}g^{XY}a = c_{Y\setminus X}^{(Y)} \quad \text{for } X \neq \emptyset,$$

and

$$\mathbf{h9}\colon\ f^{Y\emptyset}g^{\emptyset X}a = g^{Y(X\cup Y)}f^{(X\cup Y)X} \quad \text{if } X\cap Y = \emptyset$$

hold, the operations $c_{Y\setminus X}^{(Y)}$ *in* $\mathbf{h4}_1$ *being defined by*

$$\eta 1\colon\quad c_{Y\setminus X}^{(Y)} = s_\alpha^{(Y)}t_\alpha^{(Y)}, \quad \text{where } \alpha Y = X.$$

Note that **h9** is included in **h8**. By Lemma 5.5(a), the right-hand side of ($\eta 1$) does not depend on the choise of α.

Now let **A** be any HTAC_V. We set

$$\eta 2\colon\quad c_Y^{(Y)}a = g^{YZ}c_Y^{(Z)}f^{ZY}a, \quad \text{where } Z \text{ is a proper superset of } Y.$$

Then

$$c_Y^{(Y)} = g^{YZ}c_Y^{(Z)}f^{ZY}a = g^{YZ}f^{Z(Z\setminus Y)}g^{(Z\setminus Y)Z}f^{ZY}a = f^{Y\emptyset}g^{\emptyset(Z\setminus Y)}f^{(Z\setminus Y)\emptyset}g^{\emptyset Y}a = f^{Y\emptyset}g^{\emptyset Y}a$$

by ($\eta 2$), $\mathbf{h4}_1$, **h9**, **h3**. Therefore, the operations $c_Y^{(Y)}$ are also well-defined. Moreower, we conclude that **h4** holds in **A** in full extent and that every f^{YX} is a homomorphism from $(A_X, c_Z^{(X)})_{Z\subset X}$ to $(A_Y, c_Z^{(Y)})_{Z\subset X}$. At last, the proof of **h6** remains valid, and **h5** again is a consequence of the following lemma.

Lemma 7.3 *The Boolean part* $(A_X, f^{YX}, g^{XY})_{X\subset Y\in RT}$ *of the algebra* **A** *is a* $\mathsf{BAAC}(RT)$.

Proof. For X nonempty, $\mathbf{h4}_1$ implies that every g^{XY} is isotone together with $c_{Y\setminus X}^{(Y)}$ (see (4)):

$$b_1 \le b_2 \Rightarrow c_{Y\setminus X}^{(Y)}b_1 \le c_{Y\setminus X}^{(Y)}b_2 \Rightarrow f^{YX}g^{XY}b_1 \le f^{YX}g^{XY}b_2 \Rightarrow g^{XY}b_1 \le g^{XY}b_2.$$

It immediately follows from ($\eta 2$) that also every $c_Y^{(Y)}$ is isotone, and even extensive: by ($\eta 2$), (5), **h3**:

$$c_Y^{(Y)}a = g^{YZ}c_Y^{(Z)}f^{ZY}a \ge g^{YZ}f^{ZY}a = a.$$

Then all mappings $g^{\emptyset Y}$ are isotone as well in a similar way, and we apply Proposition 2.2. □

The following theorem is the counterpart of Theorem 6.2 and is proved after the same fashion.

Theorem 7.4 *The categories* LfTAC$_V$ *and* HTAC$_V$ *are equivalent.*

The definition of a homomorphism between HTAC$_V$'s can be modelled after that of the previous section. Instead of $\varepsilon 1$ and $\varepsilon 2$ we now use the definitions

$\varepsilon 1'$: $\mathsf{s}_\alpha a = \mathsf{s}_\alpha^{(Y)} a$ for some $X \in RT$ such that $a \in A_X$ and $\alpha \in TR_X$.

$\varepsilon 2'$: $\mathsf{t}_\alpha a = \mathsf{t}_\alpha^{(Y)} a$ for some $X \in RT$ such that $a \in A_X$ and $\alpha \in TR_X$.

Of course, $(\eta 1)$ implies that $(\varepsilon 1)$ follows from $(\varepsilon 1')$ and $(\varepsilon 2')$.

Furthermore, when proving the TAC-analogue of Proposition 6.4, we must pay more attention to the inclusion $A|X \subset A_X$. The point is that the equality $b = \mathsf{c}_{Y\setminus X} b = \mathsf{c}_{Y\setminus X}^{(Y)} b$ presupposes that the operations $\mathsf{c}_{Y\setminus X}$ and $\mathsf{c}_{Y\setminus X}^{(Y)}$ satisfy $(\alpha 1)$. But this is the case: $\mathbf{A}^L f$ is a QA$_V$ (see the note just after the proof of (11)), and therefore every $(A_Y, \mathsf{c}_Z^{(Y)})_{Z \subset Y}$ is a QA$_Y$.

Now we move to relationships between HTAC$_V$'s and RelA$_V$'s.

Let us correlate an algebra $\mathbf{A}^H :\overset{\circ}{=} (\mathbf{A}_X, f^{YX}, g^{XY})_{X \subset Y \in RT}$ with every RelA$_V$ $\mathbf{A} := (A_X, \mathsf{s}_\alpha^{YX}, \mathsf{t}_\alpha^{XY})_{X,Y \in RT,\ \alpha \in Tr(X,Y)}$ by setting

$\theta 0$: $\mathbf{A}_X = (A_X, \mathsf{s}_\alpha^{(X)}, \mathsf{t}_\alpha^{(X)})_{\alpha \in Tr_X}$, where

$\theta 1$: $\mathsf{s}_\alpha^{(X)} = \mathsf{s}_\alpha^{XX}$,

$\theta 2$: $\mathsf{t}_\alpha^{(X)} = \mathsf{t}_\alpha^{XX}$,

as well as

$\theta 3$: $f^{YX} = \mathsf{s}_\varepsilon^{YX}$,

$\theta 4$: $g^{XY} = \mathsf{t}_\varepsilon^{XY}$.

Therefore, $\mathbf{A}^H$ is merely a reduct of $\mathbf{A}$. Also, let us correlate an algebra $\mathbf{A}^R := (A_X, \mathsf{s}_\alpha^{YX}, \mathsf{t}_\alpha^{XY})_{X,Y \in RT,\ \alpha \in Tr(X,Y)}$ with every HTAC$_V$ $\mathbf{A} := (\mathbf{A}_X, f^{YX}, g^{XY})_{X \subset Y \in RT}$ by setting

$\iota 0$: A_X is the Boolean algebra underlying $\mathbf{A}_X$,

$\iota 1$: $\mathsf{s}_\alpha^{YX} = g^{YU} \mathsf{s}_\alpha^{(U)} f^{UX}$,

$\iota 2$: $\mathsf{t}_\alpha^{XY} = g^{XU} \mathsf{t}_\alpha^{(U)} f^{UY}$, where U in (ι1), (ι2) is selected so that $X \cup Y \subset U$. The two definitions are correct: s_α^{YX} and t_α^{XY} do not depend on U. For example, since $\alpha \in Tr_{(X\cup Y)}$, we have by **h2** and its dual, **h1**, **h3**:

$$g^{YU}s_\alpha^{(U)}f^{UX}a = g^{Y(X\cup Y)}g^{(X\cup Y)U}s_\alpha^{(U)}f^{U(X\cup Y)}f^{(X\cup Y)X}a =$$

$$g^{Y(X\cup Y)}g^{(X\cup Y)U}f^{U(X\cup Y)}s_\alpha^{(X\cup Y)}f^{(X\cup Y)X}a = g^{Y(X\cup Y)}s_\alpha^{(X\cup Y)}f^{(X\cup Y)X},$$

and likewise for (ι2). (Actually, (ι2) is the dual of (ι1).) In particular, $s_\varepsilon^{YX} = g^{YY}s_\varepsilon^{(Y)}f^{YX} = f^{YX}$ by (ι1), **h5** and its dual, and **a'**. Therefore, we have

ι1': $s_\varepsilon^{YX} = f^{YX}$,

and, similarly,

ι2': $t_\varepsilon^{XY} = g^{XY}$.

Theorem 7.5 (a) *If* **A** *is a* RelA$_V$, *then* $\mathbf{A}^H$ *is a* HTAC$_V$ *and* $(\mathbf{A}^H)^R = \mathbf{A}$.

(b) *If* **A** *is a* HTAC$_V$, *then* $\mathbf{A}^R$ *is a* RelA$_V$ *and* $(\mathbf{A}^R)^H = \mathbf{A}$.

Proof. (a) Assume that **A** is a RelA$_V$. First of all, then every $\mathbf{A}_X$ in $\mathbf{A}^H$ is a TAC$_X$ (obvious) and every Boolean homomorphism f^{YX} preserves also the operations $s_\alpha^{(X)}$ and $t_\alpha^{(X)}$: by (θ1)–(θ4) and **r7**, we have

$$f^{YX}s_\alpha^{(X)}a = s_\varepsilon^{YX}s_\alpha^{XX}a = s_\alpha^{YX}a = s_\alpha^{YY}s_\varepsilon^{YX} = s_\alpha^{(X)}f^{YX}a,$$

and the other identity $f^{YX}t_\alpha^{(X)}b = t_\alpha^{(Y)}f^{YX}b$ is **r14**. So **h1** is valid in $\mathbf{A}^H$. Furthermore, **h2** is included in **r7**, and **h3** is **r12**. Finally, we obtain **h4**$_1$ by Proposition 2.2(c): oviously, $\binom{YY}{\alpha} = \binom{YX}{\varepsilon}\binom{YX}{\alpha}$ and $\binom{YX}{\varepsilon} = \binom{YY}{\alpha}\binom{YX}{\beta}$ for some β such that $\beta x \in \alpha^{-1}x$ whenever $x \in X$. By (θ3) and (θ4), **h9** is included in **r13**.

So $\mathbf{A}^H$ is a HTAC$_V$. Furthermore, by (θ1)–(θ4), **r7**, **r7**, **r12**,

$$g^{YU}s_\alpha^{(U)}f^{UX}a = t_\varepsilon^{YU}s_\alpha^{UU}s_\varepsilon^{UX}a = t_\varepsilon^{YU}s_\alpha^{UX}a = t_\varepsilon^{YU}s_\varepsilon^{UY}s_\alpha^{YX}a = s_\alpha^{YX}a,$$

i.e. (ι1) holds, and likewise (ι2) can be checked. We have proved (a).

To prove (b), assume that $\mathbf{A} \in$ HTAC$_V$ and apply Proposition 3.5. It follows immediately from definitions (ι1), (ι2) and (2) that the operations s_α^{YX} and t_α^{XY} of $\mathbf{A}^R$ are conjugate. Furthermore, **r10** follows by (8) from **h3** while **r13** and **r14** are included in **h8** and **h1**, respectively (by (ι1') and (ι2')). Finally, as to **r1** and **r7**, it is handily to assume that the algebra **A** is flat and then transfer the problem to the corresponding TAC_V.

If the initial HTAC_V is flat, the equations ($\iota 1$), ($\iota 2$), when transferred to the corresponding TAC_V $(A, \mathsf{s}_\alpha, \mathsf{t}_\alpha)_{\alpha \in T_{V_0}}$, read there as follows:

$$\mathsf{s}_\alpha^{YX} a = \mathsf{c}_{U \setminus Y} \mathsf{s}_\alpha a, \quad \mathsf{t}_\alpha^{XY} b = \mathsf{c}_{U \setminus X} \mathsf{t}_\alpha b \tag{15}$$

for $a \in A|X$, $b \in A|Y$ and $U \supset X \cup Y$ (see (13)). We assume that $U = X \cup Y$. Now, by (15), **r1**, Lemma 5.5(c), **c11**, (15)

$$\mathsf{s}_\alpha^{YX}(-a) = \mathsf{c}_{U \setminus Y} \mathsf{s}_\alpha(-a) = \mathsf{c}_{U \setminus Y}(-\mathsf{s}_\alpha a) =$$

$$\mathsf{c}_{U \setminus Y}(-\mathsf{c}_{U \setminus Y} \mathsf{s}_\alpha a) = -\mathsf{c}_{U \setminus Y} \mathsf{s}_\alpha a = -\mathsf{s}_\alpha^{YX} a.$$

By (15), Lemma 5.5(c), **a2**, 15 , if $X, Y, Z \subset U$,

$$\mathsf{s}_\beta^{ZY} \mathsf{s}_\alpha^{YX} a = \mathsf{c}_{U \setminus Z} \mathsf{s}_\beta \mathsf{c}_{U \setminus Z} \mathsf{s}_\alpha a = \mathsf{c}_{U \setminus Z} \mathsf{s}_\beta \mathsf{s}_\alpha a = \mathsf{c}_{U \setminus Z} \mathsf{s}_{\beta\alpha} a = \mathsf{s}_{\beta\alpha}^{ZX} a.$$

So $\mathbf{A}^R$ is a RelA_V. Furthermore, $\mathsf{s}_\varepsilon^{YX} a = g^{YY} \mathsf{s}_\varepsilon^{(Y)} f^{YX} a = f^{YX}$ by ($\iota 1$), the dual of **h5**, **a1**, i.e. ($\theta 3$) holds, and likewise ($\theta 4$) can be checked. □

Remark 7.6 In Proposition 3.5, **r13** could be replaced by its particular case

r15: $\mathsf{s}_\varepsilon^{Y\emptyset} \mathsf{t}_\varepsilon^{\emptyset X} a = \mathsf{t}_\varepsilon^{YZ} \mathsf{s}_\varepsilon^{ZX} a \quad$ if X and Y are disjoint

obtained by setting $X = \emptyset$ and appropriate relettering of types. Indeed, in the proof of (a) only **r14**, **r13** and **r12** (i.e. **r10**) were used along with wRelA_V axioms. Moreover, **r13** was only needed to justify **h9**. Therefore, axioms of HTAC_V are derivable from **r1**–**r5**, **r7**, **r10**, **r14**, **r15**, and we already have proved in (b) that **r13** holds in any HTAC_V. Note that, in fact, **r15** is, essentially, the same **h9**.

Obviously, every homomorphism between two RelA_V's is also a homomorphism between the respective HTAC_V's, and vice versa. Therefore, we have

Theorem 7.7 *The categories* HTAC_V *and* RelA_V *are indistinguishable.*

Together with Theorems 5.4 and 7.4, this leads to Theorem 7.1.

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J. Cīrulis. Abstraktas finitāru relāciju algebras: dažas netradicionālas aksiomatizācijas.

Anotācija. Darbā parādīts, ka vairākas netradicionālas algebru klases, kas saistītas ar pirmās pakāpes loģiku, ir definicionāli ekvivalentas lokāli galīgo cilindrisko algebru klasei.

Я. Цирулис. Абстрактные алгебры финитарных отношений: некоторые нетрадиционные аксиоматизации.

Аннотация. Показано, что несколько нетрадиционных классов алгебр, связанных с логикой первого порядка, дефинициально эквивалентны классу локально конечных цилиндрических алгебр.

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CORRECTIONS TO MY PAPER "AN ALGEBRAIZATION OF THE FIRST ORDER LOGIC WITH TERMS"

Jānis Cīrulis

Abstract. We present an improved list of axioms for term systems and correct a number of misprints in [1].

AMS 1991 Subject Classification 03G15.

We assume that the reader is familiar with [1] and has a copy of that paper before him/her.

1. In §1 of [1], we proposed a formalization of substitutions in term algebras. It turned out later that a couple of inaccuracies are admitted in subsequent considerations and that the axiom system T1–T4 is, in fact, insufficient for substantiating a technical device used in §2. We owe Z. Diskin for indication that there was something wrong.

The problems are concerned with the notation $[w_1/x_1, \ldots, w_m/x_m]w$ introduced on p. 132 just after the proof of Lemma 2.2. First, we have overlooked that it requires a special justification in the case $m = 1$, for the notation $[v/x]w$ was already used on the previous pages on its own rights as a shortening for $s_x(v, w)$ (see p. 128). What is needed here is the conditional identity suggested by (2.1)

T': $[v/y][y/x]w = [v/x]w$ if w ind y and $y \neq x$.

T' can be modified and given the form of a pure identity (see below).

Furthermore, to justify the notation $\bar{\sigma}w$, also introduced on p.132, we must be aware that, for example, $[w_1/x_1]w = [w_1/x_1, w_2/x_2]w$ when $w_2 = x_2$. Difficulties of this kind are eliminated by means of the identity

$[x/y][y/x]w = w$ if w ind y and $y \neq x$.

By T', it may be reduced to

T": $[x/x]w = w$.

The following is the corrected axiom list of term systems in the original notation of [1].

T1: $s_\kappa(v, x_\kappa) = v$,

T2: $s_\lambda(v, x_\kappa) = x_\kappa$ if $\lambda \neq \kappa$,

T3: $s_\kappa(x_\kappa, v) = v$,

T4: $s_\lambda(v, s_\kappa(x_\lambda, s_\lambda(x_\mu, w))) = s_\kappa(v, s_\lambda(x_\mu, w))$ if $\lambda \neq \kappa, \mu$ and $\kappa \neq \mu$,

T5: $s_\kappa(v, s_\kappa(x_\lambda, w)) = s_\kappa(x_\lambda, w)$ if $\lambda \neq \kappa$,

T6: $s_\lambda(s_\kappa(x_\mu, u), s_\kappa(v, w)) =$

$$s_\kappa(s_\lambda(s_\kappa(x_\mu, u), v), s_\lambda(s_\kappa(x_\mu, u), w)) \text{ if } \lambda \neq \kappa, \mu \text{ and } \kappa \neq \mu.$$

Here, T3 = T", T4 is the equational version of T', and T1, T2, T5, T6 are respectively the axioms T1, T2, T3, T4 of [1].

Meantime we have learned from [5] that a similar axiom system has been studied by N. Feldman already in [4]. This system (proposed by C. Pinter in 1972 for essentially the same purposes: to characterize substitutions in term algebras) is not properly equational; however the two non-equational axioms A4 and A6 of [4] can be given a form of an equation in the same manner as T' above (then A6 becames our T6). It is easy to show that the two axiom systems are equivalent under the assumption of local finiteness—see the paper by A. Silionova in this volume.

2. While investigating relationships between two axiomatizations of cylindric algebras with terms [1],[5], A. Silionova noticed that there is a trouble with Lemma 3.12(i) in [1]. Really, the axiom D!2 should read like the axiom (c) in [3], §2:

D!2: $d_{vw} = c_z(d_{vz} \wedge d_{zw})$ if v, w ind z.

This axiom, as well as D!3 and D!4, is not a pure identity. However, all of them can be given a form of an equation in the same way as T' was. E.g., D!2 is equivalent to

$d_{([y/z]v)([y/z]w)} = c_z(d_{z([y/z]v)} \wedge d_{z([y/z]w)})$, where $y \neq z$.

We take the opportunity to note that originally the following single generalization of D2 did the job of D!2 and D!4 (see [2]):

$c_z(d_{uv} \wedge d_{zw}) = d_{([u/z]u)([w/z]v)}$ if w ind z.

It is somewhat weaker than the axiom (e) in [5].

3. We also correct the most unpleasant misprints in [1].

Proof of Lemma 2.2: in the displayed formulas, read '$[w_m/y_m]\cdots$' for '$[w_m/y_m\cdots$'.

Proof of Lemma 2.3: in the displayed equality, omit the last ']' at the end of the first line and the first '[' at the beginning of the second line.

Read 'TS' for 'ST' and for 'ST' at the bottom of p. 133.

Read 'subalgebra' for 'superalgebra' in Definition 3.11.

Omit the first equality sign in 140_{11}.

Replace '1981' by '1986' in ref. [C1].

Read 'State' for 'Scientific' in refs [C4] and [C6].

Read 'Ukrainian' for 'Ukrarian' and replace '1980' by '1988' in ref. [MP].

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J. Cīrulis. Labojumi manam rakstam "An algebraisation of first order logic with terms".

Anotācija. Tiek koriģēts termu sistēmu aksiomu saraksts un izlabotas dažas iespiedkļūdas darbā [1].

Я. Цирулис. Исправления к моей статье "An algebraization of first order logic with terms".

Аннотация. Корригируется список аксиом термовых систем из [1] и исправляются некоторые опечатки.

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SUBSTITUTIONS IN TERM ALGEBRAS: EQUIVALENCE OF TWO AXIOMATIZATIONS

A. Silionova

Abstract. We prove that the axiom system of term systems [1,2] and that of substitution algebras [3] are equivalent in the case of locally finiteness.
AMS 1991 Subject Classification: primary 08A40, secondary 03G15.

We adduce the definition of term system from [1] (with corrections from [2]) and that of substitution algebra from [3]. We borrow the notation of [1].

Let α be a fixed ordinal.

Definition 1. A *term system* of dimension α is an algebra $T := (W, s_{\varkappa}, x_{\varkappa})_{\varkappa<\alpha}$ such that each $s_{\varkappa}$ is a binary operation on W, each $x_{\varkappa}$ is an element of W and the following conditions are fulfilled for all $\mu, \varkappa, \lambda < \alpha$:

T1: ${}_{\varkappa}(\upsilon, x_{\varkappa}) = \upsilon$,

T2: $s_{\lambda}(\upsilon, x_{\varkappa}) = x_{\varkappa}$, where $\lambda \neq \varkappa$,

T3: $s_{\varkappa}(x_{\varkappa}, \upsilon) = \upsilon$,

T4: $s_{\lambda}(\upsilon, s_{\varkappa}(x_{\lambda}, s_{\lambda}(x_{\mu}, w))) = s_{\varkappa}(\upsilon, s_{\lambda}(x_{\mu}, w))$, where $\lambda \neq \varkappa, \mu$ and $\varkappa \neq \mu$,

T5: $s_{\varkappa}(\upsilon, s_{\varkappa}(x_{\lambda}, w)) = s_{\varkappa}(x_{\lambda}, w)$, where $\lambda \neq \varkappa$,

T6: $s_{\lambda}(s_{\varkappa}(x_{\mu}, u), s_{\varkappa}(\upsilon, w)) =$

$$= s_{\varkappa}(s_{\lambda}(s_{\varkappa}(x_{\mu}, u), \upsilon), s_{\lambda}(s_{\varkappa}(x_{\mu}, u), w)),$$

where $\lambda \neq \varkappa, \mu$ and $\varkappa \neq \mu$. □

Definition 2. A *substitution algebra* of dimension α is an algebra $S := (W, s_{\varkappa}, x_{\varkappa})_{\varkappa<\alpha}$ such that each $s_{\varkappa}$ is a binary operation on W, each $x_{\varkappa}$ is an element of W and the following conditions are fulfilled for all $\mu, \varkappa, \lambda < \alpha$:

A1: $s_{\varkappa}(\upsilon, x_{\varkappa}) = \upsilon$,

A2: $s_{\lambda}(\upsilon, x_{\varkappa}) = x_{\varkappa}$, where $\lambda \neq \varkappa$, A3: $s_\varkappa(x_\varkappa, v) = v$,

A4: if $s_\lambda(u,v) = v$ for all $u \in W$, then

$s_\lambda(v, s_\varkappa(x_\lambda, w)) = s_\lambda(v, s_\varkappa(v,w))$,

A5: $s_\varkappa(s_\varkappa(v,u),w) = s_\varkappa(v, s_\varkappa(u,w))$,

A6: if $s_\varkappa(v,u) = u$ for all $v \in W$, then

$s_\lambda(u, s_\varkappa(v,w)) = s_\varkappa(s_\lambda(u,v), s_\lambda(u,w))$, where $\lambda \neq \varkappa$. □

It will be convenient to denote arbitrary variables by letters x, y, z, and to write $[v/x]w$ for $s_\varkappa(v,w)$ where $x = x_\varkappa$. We refer to the elements of W as to *terms* and to those of the set $X := (x_\varkappa : \varkappa < \alpha)$ - as to *variables*, the term $s_\varkappa(v,w)$ is said to be the result of substitution of v for $x_\varkappa$ in w.

Definition 3. We say that a term w is *independent* of the variable x in a term system T (in a substitution algebra S), w ind x, in short, if $[v/x]w = w$ for any $v \in W$. The term system T (the substitution algebra S) itself is said to be *locally finite*, if every term depends only on a finite number of variables. □

Note that, both in T and in S, if $\alpha > 1$, then

$$w \text{ ind } x \Leftrightarrow [y/x]w = w \text{ for some } y \neq x \qquad (*)$$

(see Theorem 2.1 in [3] and the observation after (1.2) in [1]).

Now T1--T6 and A1--A6 may be rewritten as follows (T4' and T6' are really not just other wordings of T4 and T6 but are equivalent to them by (*)):

T1' = A1': $[v/x]x = v$,

T2' = A2': $[v/y]x = x$, where $y \neq x$,

T3' = A3': $[x/x]v = v$,

T4': $[v/y][y/x]w = [v/x]w$, where $y \neq x$ and w ind y

A4': $[v/y][y/x]w = [v/y][v/x]w$, where w ind y,

T5': $[v/x][y/x]w = [v/x]w$, where $x \neq y$,

A5': $[v/x][w/x]w = [[v/x]u/x]w$,

T6' = A6': $[u/y][v/x]w = [[u/y]v/x][u/y]w$, where $x \neq y$ and u ind x.

If also v ind y in the last equality, then

$$[u/y][v/x]w = [v/x][u/y]w. \qquad (**)$$

and, provided that $\alpha > 2$,

$$u,v \text{ ind } y \Rightarrow [v/x]u \text{ ind } y \text{ if } x \neq y, \qquad (***)$$

(again, both in T and in S).

In the rest we consider that $\alpha \geq \omega$.

Theorem 4. If T is locally finite, then A1--A6 hold.

Proof. The assertions A1',A2',A3',A6' coincide with the axioms T1',T2',T3',T6', respectively. Now we prove that A5' holds in T. Let y be a variable distinct from x and such that u,v ind y. Then by T4',T3',T5',T4',

$$[v/x][u/x]w = [v/x][u/y][y/x]w = [[v/x]u/y][v/x][y/x]w =$$
$$= [[v/x]u/y][y/x]w = [[v/x]u/x]w.$$

Also, A4' holds in T. Let $z \neq x,y$, such that v,w ind z. If v ind y, Then by T4',T6' T1',(**),T6',T1',T4'

$$[v/y][y/x]w = [v/z][z/y][y/x]w = [v/z][[z/y]y/x][z/y]w =$$
$$= [v/z][z/y][z/x]w = [[v/z]z/y][v/z][z/x]w =$$
$$= [v/y][v/z][z/x]w = [v/y][v/x]w. \quad \square$$

Theorem 5. If S is locally finite, then T1--T6 hold.

Proof. The assertions T1',T2',T3',T6' coincide with the axioms A1',A2',A3',A6', respectively. Now we prove that T5' holds in S. By A5' and A2'

$$[v/x][y/x]w = [[v/x]y/x]w = [y/x]w.$$

Also, A4' holds in T. Let $z \neq x,y$ such that w,v ind z. If w ind y, then by (***),A4',A4',(***),A4',(***),

$$[v/y][y/x]w = [v/z][v/y][y/x]w =$$
$$= [v/z][z/y][y/x]w = [v/z][z/y][z/x]w =$$
$$= [v/z][z/x]w = [v/z][v/x]w = [v/x]w. \quad \square$$

Therefore, in the case $\alpha \geq \omega$ and the algebras under consideration are locally finite, every term system is a substitution algebra, and vice versa.

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A. Siļionova. Substitūcijas termu algebrās: divu aksiomatizējumu salīdzinājums

Anotācija. Darbā noskaidrots, ka termu sistēmu [1,2] aksiomu kopa un substitūciju algebru [3] aksiomu kopa ir ekvivalentas.

А. Силионова. Подстановки в алгебрах термов: сравнение двух аксиоматизаций.

Аннотация: В работе установлено, что система аксиом для термовых систем [1,2] и система аксиом для субституционных алгебр [3] эквивалентны.

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WHEN IS A SEMANTICALLY DEFINED LOGIC ALGEBRAIZABLE?

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Abstract. The basic paradigm of algebraic logic (particularly, categorical logic) consists in replacing theories by algebras, and models—by homomorphisms into similar algebras extracted from semantics.

The objective of the paper is to suggest a framework for thorough study of this paradigm in the general setting, in particular, for classification and comparison of various kinds of algebraiziations in the above sense and, at last, to clarify when a logic can be somehow algebraized.

As a formal substitute for a logic we take institutions by Goguen and Burstall (this notion is well-known in the area of algebraic specification languages). With each institution *I* there is correlated a specification system, *spec(I)*, so that a certain kind of *I*-logic's algebraization amounts to a presentation of *spec(I)* by means of another specification system arising from an algebraic (categorical) doctrine of the corresponding kind.

However, while in the algebraic logic standard such a presentation is an *a priori* assumption and the starting point of precise considerations, in the framework developed in this paper just the very possibility of the presentation is a fact which must be proved. The principal idea we will elaborate is to construct an algebraization of *spec(I)* out from some *algebraization of the very institution I*—the chief notion to be defined in this paper. The main theorem of the paper states that if an instituion is algebraizable then the associated specification system is algebraizable too.

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In order to study logical theories (specifications) and their models, it is often useful to be free from details of their representation determined by a concrete choice of the signature and the set of axioms. Algebraic logic, in particular categorical logic, supports such an intention by means of replacing theories by algebras, and models - by homomorphisms into similar algebras extracted from semantics, what enables one to use powerful machinery of algebraic manipulations.

The idea goes back to Tarski and Lindenbaum; at the beginning of sixties it got a new sound owing to deep Lawvere's ideas of replacing theories by categories with additional (algebraic!) structure while models - by this-structure-preserving functors into similar semantic categories.

In practice we often have different algebraizations of the same logic. For example, first order predicate logic (FOL) can be algebraized by means of polyadic or cylindric algebras in a universal algebra fashion (Halmos[Ha62], Henkin, Monk and Tarski[HMT_I,II]), or, alternatively, by hyperdoctrines in a indexed category fashion (Lawvere [Law70], see also Seely[See83]), or, else, by means of logical categories (logoses, pretoposes etc., see, eg, Makkai and Reyes [MR77]) which are, in fact, hyperdoctrines constructed internally.

Computer Science brought to life a plenty of logical systems for writing specifications; a majority of them can be (and often really done) algebraized in one or another way. So, algebraization of logic has become a paradigm whose study in a unified general setting looks atrractive and useful.

The objective of the present paper is to suggest a framework for such a study, and to present some results justifying the approach. In particular, we will give some sufficient condition when a certain algebraization of a logic is possible. In addition, fulfilment of these conditions in a majority of real logics is easily checked, thus, our results explain, in a sense, why a logic can be algebraized in a certain way.

Briefly, the approach is as follows.

As a formal substitute for a logic we take the familiar notion of institution by Goguen and Burstall ([GB84], see also [GB92]). However, since in the present paper our goal consists only in outlining ideas, to simplify things and to avoid 2-categorical machinery, we will deal with the so called *discrete* institutions when morphisms between models as well as between sentences (ie, proofs) are not considered. That is, by *an institution* we will mean a quadruple $\mathcal{I}$=(Sign,*Sen*,*Mod*,$\models$) where Sign is a category of *sigantures*, *Sen* and *Mod* are functors Sign --> Set and $\mathbf{Sign}^{op}$ --> SET assigning a small set *Sen*(Σ) of *sentenses* and a class *Mod*(Σ) of *models* resp. to each signature $\Sigma \in$ObSign, finally, $\models$ is a function assigning a binary *satisfaction* relation $\models_\Sigma \subset Mod(\Sigma)\times Sen(\Sigma)$ for each siganture Σ s.t. for each siganture morphism σ: Σ --> Σ' in Sign and any $m' \in Mod(\Sigma')$, $\varphi \in Sen(\Sigma)$ one has:

$$m' \vdash_{\Sigma'} \sigma(\varphi) \iff \sigma^*(m') \vdash_{\Sigma} \varphi ,$$

where *Sen(σ)* is again denoted by σ while σ^* denotes *Mod(σ)*.

Given an institution $\mathcal{I}$, *a specification* (or *a theory*, or *a presentation*) is defined to be a pair (Σ,Φ) with Σ a siganture and Φ a subset of $Sen(\Sigma)$. By defining a notion of *a specification morphism* in a suitable way, one gets the corresponding category of *specifications over* $\mathcal{I}$, Spec($\mathcal{I}$), equipped with a model functor Mod^*: spec($\mathcal{I}$) --> SET^{op}. by $Mod^*(\Sigma,\Phi) := \{m \in Mod(\Sigma): m \vdash \varphi$ for all $\varphi \in \Phi\}$.

Further, under *a specification system* we will mean a pair $\mathcal{S}$=(Spec,*Mdl*) with Spec a category and *Mdl* a functor as above.

To be able to speak about algebraization, we need a suitable general notion of algebra. For this end we take the notion of a generalized algebraic theory introduced by Cartmell [Ca86]. It is a direct generalization of the usual notion of a many-sorted equational theory: its extra generality is achieved by introduction of sort structures more general than those usually considered, in that sorts may denote sets as is usual, or they may denote families of sets, families of families of sets or the like. (A basic example is the generalized algebraic theory of categories, in which Ob appears as a sort to be interpreted as a set while Hom appears as a sort to be interpreted as a family of sets indexed by Ob×Ob). In more detail, a generalized algebraic theory T appears as an adjunction ($F\backslash U$): FamT $\overset{*}{\rightleftarrows}$ AlgT where AlgT is the (generalized) variety of T-algebras. FamT is the category of their carrier set structures, U is the underlying functor and F is the freely-generated-algebra functor. (For example, if T is the theory of categories, then an object of FamT is nothing but a family of sets, $(H_{ij} \in \mathrm{Set}: i,j \in I)$, indexed by the Cartesian squire of some set I, while a morphism between two such families, (say, from H_{ij} into H'_{ij}), is a pair (f,g) consisting of a function f mapping I into I' and a family of functions $g=(g_{ij}, i,j \in I)$ with g_{ij} mapping H_{ij} into $H'_{fi,fj}$). In fact, for any T, FamT is a full subcategory of the large category Fam of sets, families of sets, families of families of sets etc. described by Cartmell.

Now, *an algebraization* of a specification system $\mathcal{S}$ is defined to be a list of the following data.

A generalized algebraic theory, T, two classes of T algebras, $\mathcal{T}h, \mathcal{M}od \subset$ AlgT, an adjunction, $\alpha_s = (F\backslash G)$: Spec $\underset{F}{\overset{G}{\rightleftarrows}}$ $\mathcal{T}h$, and an isomorphic natural transformation α_m: *Mdl* -> F;Hom(-,$\mathcal{M}od$) (cf. the definition of categorical logic by Meseguer [Me87]).

* Further in the paper, this category is also denoted by Pres $\mathcal{I}$

By adopting the terminology of categorical logic, we will call a triple $(\mathsf{T},\mathcal{T}h,\mathcal{M}od)$ as above *a doctrine*. Each doctrine $\mathcal{D}$ determines a specification system **spec**($\mathcal{D}$) where **Spec** is $\mathcal{T}h$ and, given a specification T and a specification morphism $\tau: T \dashrightarrow T'$, $Mdl(T)$ is $\bigcup\{\mathrm{Hom}(T,M): M \in mod\}$ and $Mdl(\tau)$ is defined by composition. Now we can say that *an algebraization* of a specification system $\mathcal{S}$ is a pair $(\mathcal{D},\alpha)$ with $\mathcal{D}$ a doctrine and α *an algebraizing presentation* $\mathcal{S} \dashrightarrow \mathcal{D}$, that is, a kind of specification morphism from $\mathcal{S}$ into **spec**($\mathcal{D}$).

Note, in categorical logic, given a logic (institution) $\mathcal{I}$, identifying **spec**($\mathcal{I}$) with a certain **spec**($\mathcal{D}$) for some doctrine $\mathcal{D}$ is the starting point of precise considerations, while in the framework being developed in this paper just such an identification is a fact which must be proved. The principal idea we will be elaborating is to construct an algebraization of **spec**($\mathcal{I}$) out from *an algebraization of the very institution* $\mathcal{I}$ - the chief notion to be defined in this paper.

In comparison to algebarization of a specification system, to define an institution algebraization one needs a much more involved construction which we call *a predoctrine*, intended to model semantically defined logics in a algebraic manner. Roughly speaking, a predoctrine is again a generalized algebraic theory T coupled with two classes of T-algebras, but now the algebraic theory is assumed to be endowed with two set-valued functors, $\mathcal{P}$: **Fam**T --> **Set** and $\mathcal{D}$: **Alg**T --> **Set**, $\mathcal{D}A \subseteq \mathcal{P}UA$, intended to present the following.

If a T-algebra A is being thought of as the algebra of expressions (eg, terms) generated by some signature Σ, $A = \alpha_{sig}(\Sigma)$, then $\mathcal{P}UA$ is to be thought of as the set of propositions over Σ (eg, equations between Σ-terms), while if A is being thought as semantically generated by some Σ-model M, $A = \alpha_{mod}(M)$, then $\mathcal{D}A \subset \mathcal{P}UA$ is to be thought of as the truth set connected with M (eg, the diagonal of A), ie, $M \models \varphi$ for a proposition φ iff $(\mathcal{P}U\mathrm{h})\varphi \in \mathcal{D}(A)$ where $\mathrm{h} = \alpha_{hom}(M)$ is the homomorphism $\alpha_{sig}(\Sigma) \to \alpha_{mod}(M)$ connected with the model M.

Actually, by forgetting some additional (algebraic) information, each predoctrine $\mathcal{A}$ determines a (discrete) institution, **inst**($\mathcal{A}$), so that there is a forgetful functor from the category of predoctrines into the category of institutions.

Now, let $\mathcal{I}$ be an object institution. Briefly speaking, under algebraization of $\mathcal{I}$ (in a universal algebra fashion) we mean encoding the components of $\mathcal{I}$ by facilities offered by some predoctrine $\mathcal{A}$, in other words, to algebraize $\mathcal{I}$ we design a kind of presentation of $\mathcal{I}$ in $\mathcal{A}$. In fact, such a presentation proves to be a special kind of institution morphisms $\alpha{:}\mathcal{I} \dashrightarrow$ **inst**($\mathcal{A}$). Thus, *algebraization* of an institution $\mathcal{I}$ is defined to be a pair $(\mathcal{A},\alpha)$ with $\mathcal{A}$ a predoctrine and α *an algebraizing presentation*, in such a case we write $\alpha{:}\mathcal{I} \dashrightarrow \mathcal{A}$ and call the institution *algebraizable* (by means of α).

The main result of the paper can be formulated as follows: every algebraization $\alpha: \mathcal{I} \dashrightarrow \mathcal{A}$ of an institution $\mathcal{I}$ gives rise to an algebraizing presentation of the associated specification system, $\alpha^*: spec(\mathcal{I}) \dashrightarrow \mathcal{D}[\alpha]$, where the latter list is determined by $\mathcal{A}$ and α.

Now, some general words on the crucial in the paper notion, the notion of algebraizability of an institution, will be relevant.

Institutions introduced by Goguen and Burstall provide a general algebraic framework for describing specifications in various logical systems. However, the pure institutions themselves are rather poor algebraically in the sense that their algebraic facilities are exhausted by very general functoriality assumptions. At the same time, as is well known in software methodology, endowing a complex structure with algebraic machinery provides, as a rule, a much more handy and efficient way of using or operating the structure. Apparently, the most natural and evident way of applying this general principle for institutions is to build into the institution framework the old idea of classical algebraic logic in the style of Halmos, Tarski and Henkin: to relate a class C of algebras to the logic L in question in such a way that L-models could be considered as homomorphisms from syntactically generated C-algebras into C-algebras arising from L-semantics. This is just the way we adopt in the paper. An attempt to go along this line was also made by Goguen and Burstall who suggested in [GB85] a construction of *chartering* institution. However, in this construction, treatment of the semantic part of an institution is closely connected with Lawvere's idea of functorial semantics while the basic paradigm of algebraic logic presumes, besides the model-as-homomorphism idea, that the L-satisfaction relation should be determined by assigning a set of designated elements to every semantic C-algebra (that is, by converting semantic C-algebras into so called *logical matrices* - in the terminology of Polish school of algebraic logic going back to Łukasiewicz and Tarski [LT30]).

As rich experience of algebraic logic has shown, such an approach has an immediate consequence that L-theories prove to be in a precise correspondence with kernels of the above-described homomorphisms and so are connecting with the corresponding quotient algebras (usually called Lindenbaum-Tarski algebras). Thus, theory - congruence - quotient-algebra correspondence is taken in algebraic logic very seriously from the very beginning and, in fact, turns algebraic logic into a special part of universal algebra (see [ANS92],[Ne90], [Di92] for an explicit demonstration of this statement).

This paper is the first in a series of works wit a general intention to incorporate the above-outlined methodology into the institution framework and thus to "inject" rich intuition (based on a large body of results and experience) of algebraizing logics accumulated in classical algebraic logic. (For example, as far as I know, there are no powerful results induced by exploiting the idea of chartering institution, and, on the whole, the introducing work of Goguen and Burstall had taken no development. I can suppose that this is the case just because algebraic logic is not very popular in the Institutional Community and so the latter is denied very useful guidelines. Indeed, algebraic logic methodology immediately gives a bundle of schemes of definitions, constructions and presupposed theorems - some of them will be demonstrated in this paper).

The paper is organized as follows.

In section 1 we present an algebraization of first order logic in a universal algebraic fashion, in fact, very closely to classical algebraization via polyadic Boolean algebras due to Halmos [Ha62]. It is hoped that this section provides a general motivation for the forthcoming abstract considerations.

In section 2 the notion of a predoctrine is introduced and an adjunction between the category of its theories and a certain class of algebras is stated. In proofs of this section there is used standard but often non-evident universal algebraic machinery connected with congruence lattices.

In section 3 we deal immediately with algebraizing institutions and prove the above described theorem.

By the lack of space many intermediate results and proofs are only outlined or described informally.

0 Notation

Throughout the paper we designate categories by bold letters or abbreviations while subclasses of their objects by script letters, these classes can be also considered as full subcategories. We shall always assume they are abstract, i.e., closed under isomorphisms. **SET** is the category of large sets while **Set** - of small ones.

Let **K** be a category and f∈Mor**K**. Then we denote the domain of f by □f and the codomain by f□. If f□=□g, the composition is denoted by f;g. If $\mathcal{M}\cup\{A\}\in\mathcal{K}$=Ob**K** then the object class of the slice category $A/\mathcal{M}$ will be denoted by Hom($A,\mathcal{M}$). The corresponding functor **K**⟶**SET**^{op} will be denoted by Hom(-,$\mathcal{M}$).

If ≈ is an equivalence on morphisms of **K** compatible with composition then the corresponding quotient category will be denoted by **K**/≈; note, **K**/≈ and **K** have the same objert set and in the paper we will deal only with such quotients. Given an adjunction F:K⟶L, U:L⟶K between categories K and L with F the left adjoint and U the right one, we will also write it as $(F\backslash U)$:K $\overset{\longleftarrow}{\underset{*}{\longrightarrow}}$ L and call also F and U the lower and the upper adjoints resp.

We regard the power-set construction as a functor $\mathfrak{P}$: Set⟶Pos into the category of posets. If $f:X\longrightarrow Y$ is a (many-valued) mapping, the associated image and preimage mappings will be denoted by $f^{+}:\mathfrak{P}X\longrightarrow\mathfrak{P}Y$ and by f^{-1} or $f^{\cdot}$: $\mathfrak{P}Y\longrightarrow\mathfrak{P}X$ resp. Note, actually they make an adjunction with $f^{\cdot}$ the upper (right) adjoint and f^{+} the lower (left) adjoint.

D-Set and ⊢-Set denote the concrete categories whose objects are pairs (X,R) and morphisms $(X,R)\longrightarrow(X',R')$ are the set mapnings $f:X\longrightarrow X'$ s.t. $f(R)\subset R'$ where X denotes a set and R denotes a designated subset $D\subset X$ for the former category and a consequence relation $\vdash\ \subset \mathfrak{P}X\times X$ for the latter.

Forgetful functors from concrete over Set categories will be uniformly denoted by $|\_|$, this ambiguity will hopefully not be confusing.

$\mathbb{S}$ and $\mathbb{P}$ are standard universal algebra operations on classes of algebras - closures under subalgebras and products resp.

Following to traditions of universal algebra, as a rule we will designate the carrier set of an algebra and the very algebra by the same letter - a capital italic one.

1 Motivating considerations: an algebraization of first order logic

Though this section contains some technical results, the presentation is rather informal and incomplete, the point is that the true goal of the section is to exhibit only a certain style of algebraizing logics - not details - and, what is very important, to develop a definite intuition. So, technically involved formulations may be simply skipped without serious lack of understanding.

1.1 Under first order logic (f.o.l.) we mean the following institution.

A signature is a pair Σ=(Op,Pred) consisting of a set Op of one-sorted operation symbols (with their arities) and a set Pred of one-sorted predicate symbols (with their arities), all arities are finite.

Given Σ and a countable set of variables Var=$(x_i, i<\omega)$, the set of terms over Op, Term(Σ), and the set of ordinary f.o.l. formulas over Σ, Form(Σ), are built in the standard way. Now, our crucial step for algebraization is to identify formulas mutually convertible by renaming bound variables (a kind of α-conversion, it will be denoted by $\sim$) and then to turn the collection of all syntactical expressions (modulo ~) into a two-sorted algebra $A=F(\Sigma)=(T^A,\Phi^A,\mathrm{Si}^A)$ (so, the sort set is $\{T,\Phi\}$ and the signature is denoted by Si) with the carrier sets $T^A=\mathrm{Term}(\Sigma)$, $\Phi^A=\mathrm{Form}(\Sigma)/\sim$ and the following operation signature (i runs over ω):

$$x_i : \varnothing \longrightarrow T, \quad x_i^A = x_i,$$
$$Sbt_i: T\times T\longrightarrow T, \quad Sbt_i^A(u,v) = v[x_i/u],$$
$$\wedge : \Phi\times\Phi\longrightarrow\Phi, \quad \wedge^A(\varphi/\alpha,\psi/\alpha) = (\varphi\&\psi)/\alpha,$$
$$not: \Phi\longrightarrow\Phi, \quad not^A(\varphi/\alpha) = (\mathrm{not}\varphi)/\alpha,$$
$$\exists_i: \Phi\longrightarrow\Phi, \quad \exists_i(\varphi/\alpha) = (\exists x_i\varphi)/\alpha,$$
$$Sbf_i: T\times\Phi\longrightarrow\Phi, \quad Sbf_i^A(u,\varphi/\alpha) = (\varphi[x_i/u])/\alpha.$$

(Below we shall omit the superscript A near symbols of sorts and operations). It is easy to see that these definitions do not depend on choices of representatives in the ~-equivalence classes (and below we shall also omit the symbol /~ of factorization).

In addition, these operations meet the following equations for all $i,j,k<\omega$, $k\neq i\neq j$, and all $u,v,w\in T$, $\varphi,\psi,\chi\in\Phi$ (for better readability of the equations below we will always use italic and Greek letters above for elements of T and Φ resp., instead of $Sbt_i(v,u)$ and $Sbf_i(v,\varphi)$ we will write more suggestive $[v/i]u$ and $[v/i]\varphi$, and, finally, use the abbreviations u_{ki} and ϕ_{ki} for the expressions $[x_k/i]u$ and $[x_k/i]\varphi$ resp.):

$(\mathrm{Sb1})_i$ $\quad [x_i/i]u = u; \quad [x_i/i]\varphi = \varphi;$

$(\mathrm{Sb2})_i$ $\quad [u/i]x_i = u;$

$(\mathrm{Sb3})_{ij}$ $\quad [u/i]x_j = x_j;$

$(\mathrm{Sb4})_i$ $\quad [u/i][v/i]w = [[u/i]v/i]w; \quad [u/i][v/i]\varphi = [[u/i]v/i]\varphi;$

$(\mathrm{Sb5})_{ijk}$ $\quad [u_{ki}/j][v/i]w = [[u_{ki}/j]v/i][u_{ki}/j]w; \quad [u_{ki}/j][v/i]\varphi =$
$[[u_{ki}/j]v/i][u_{ki}/j]\varphi;$

$(\mathrm{SbP})_i$ $\quad [v/i](u1\wedge u2) = [v/i]u1\wedge[v/i]u2; \; [v/i]not\varphi = not[v/i]\varphi$

$(\mathrm{SbQ1})_i$ $\quad [v/i]\ \exists_i\mu = \exists_i\mu;$

$(\mathrm{SbQ2})_{ijk}$ $\quad [u_{ki}/j]\ \exists_i\varphi = \exists_i[u_{ki}/j]\varphi;$

$(\alpha)_{ijk}$ $\quad \exists_i\phi_{ki} = \exists_j[v_j/j]\phi_{ki}.$

These equations (being considered as identities with variables $u,v,w\in T$, $\varphi,\psi,\chi\in\Phi$) determine a variety of two-sorted algebras which in the terminology tradition of algebraic logic could be called *(finitary* or *quasi)polyadic substitution algebras* (abbreviated to PSA in the singular and PSAs in the plural). In fact, the list of identities is an amalgamation of identities for the variety SA of substitution algebras of Feldman [Fe82] and a modified version of a part of identities for polyadic Boolean algebras of Halmos [Ha62]. Note, these identities do not concern any logical laws as such, they only explicitly describe syntactical rules of substituting and interacting substitutions with quantifiers.

Note also that, owing to (Sb2) and (Sb3), $i \neq j$ implies $x_i \neq x_j$ if only $|T| > 1$, and we shall always assume this condition is fulfilled. So the mapping $i \longmapsto x_i$ states an isomorphism of ω onto the set $Var(A) := \{x_i: i<\omega\}$ of *A-variables* and we shall often identify i and x_i.

1.2 Definition. Given a PSA A, for any elements $u \in T, \varphi \in \Phi$ we introduce the so called *dimension sets*:

$$\Delta^t u := \{i<\omega: [v/i]u \neq u \text{ for some } v \in T\},$$

$$\Delta^f \varphi := \{i<\omega: [v/i]\varphi \neq \varphi \text{ for some } v \in T\},$$

and say that u or φ is *independent on i (the i-th variable)* if $i \notin \Delta^t u$, $i \notin \Delta^f \varphi$ (below we will often omit superscripts t,f if they can be reconstructed from the context).

From $(Sb4)_i$ and $(Sb3)_{ij}$, $i \neq j$, it follows that, given $w \in T$, $[u/i][x_j/i]w = [x_j/i]w$ for all $u \in T$, i.e., for any w, $i \notin \Delta[x_j/i]w$ if $i \neq j$. Therefore, in the list of identities above, $\mathbf{u}_{ki}$ (ϕ_{ki}) denotes an arbitrary element of T (of Φ) independent on i. Also, $i \notin \Delta \exists_i \varphi$ by $(SbQ1)_i$.

1.3 Definition. Given a PSA A, we call an element $a \in A = T \cup \Phi$ *finitary* if Δa is finite; a PSA A itself is said to be *locally finite* (l.f.) if Δa is finite for all $a \in A$.

Given a PSA A, for any $Y \subseteq \omega$ we introduce the set $A[Y] := \{a \in A: \Delta a \subseteq Y\}$ and call elements of $A[\varnothing] = T[\varnothing] \cup \Phi[\varnothing]$ *closed*: those of $T[\varnothing]$ - *closed* terms while those of $\Phi[\varnothing]$ - *closed* formulas.

For a given PSA A, we shall call the set $A_{fin} := \{a \in A: \Delta a \text{ is finite}\} = T_{fin} \cup \Phi_{fin}$ *the locally finite part of A*.

A more detailed examination of the structure of PSAs gives the following results (see [Fe82],[Ci88] for proofs).

1.4 Proposition. If A is a PSA, then for all $i<\omega$, $a \in A, u \in T, \varphi \in \Phi$:

(o) $i \notin \Delta a$ iff $[x_j/i]a = a$ for some $j \neq i$,

(i) $\Delta x_i = \{i\}$,

(ii) $\Delta([v/i]a) \subseteq (\Delta a - \{i\}) \cup \Delta v$,

(iii) $\Delta(\varphi 1 \& \varphi 2) \subseteq \Delta\varphi 1 \cup \Delta\varphi 2$, $\Delta(not\varphi) \subseteq \Delta\varphi$,

(iv) $\Delta(\exists_i \varphi) \subseteq \Delta\varphi - \{i\}$,

1.5 Proposition. For any homomorphism $h: A \longrightarrow B$ of PSAs and any $a \in A$ one has $\Delta ha \subseteq \Delta a$.

1.6 It is easy to see that in the above-described syntactical algebra $A = F(\Sigma)$, for all elements $u \in T, \varphi \in \Phi$ their dimension sets consist of exactly those variables which syntactically free occur in them, hence, A is a l.f. PSA owing to finite arities of Σ-symbols. Actually, with each $o \in$Op and each $\pi \in$Pred there are correlated certain sets of defining relations (in the sense of universal algebra), to wit:

$R_o = \{[x_j/i]o=o: i>\text{arity}(o), j<\omega\}$, $R_\pi = \{[x_j/i]\pi=\pi: i>\text{arity}(\pi), j<\omega\}$.

In fact, the algebra $F(\Sigma)$ is generated by the two sorted set Σ with defining relations $R_\Sigma = \{R_o: o\in\text{Op}\}\cup\{R_\pi:\pi\in\text{Pred}\}$. Thus, in the variety PSA there is a distinguished subclass (of l.f. algebras):

$$\mathcal{E}xpr = \{F(\Sigma): \Sigma \text{ is a f.o.l.signature}\}.$$

Note, this class is a proper subclass of all l.f. PSAs because by taking quotients of $\mathcal{E}xpr$-algebras we obtain a lot of l.f.PSA which are not freely generated by signatures, hence, do not belong to $\mathcal{E}xpr$.

Conversely, with each l.f. PSA A there is correlated a f.o.l. signature $\Sigma=G(A)$ with

Op $= \{u\in T^A: \Delta u=\{1...n\}$ for some $n<\omega\}$, Pred $= \{\varphi\in\Phi^A: \Delta\varphi=\{1...n\}$ for some $n<\omega\}$.

It is easy to see that actually we have an adjunction

$$(F\backslash G)\text{:}\mathbf{Sign} \underset{\longrightarrow}{\overset{\longleftarrow}{*}} \mathcal{E}xpr$$

where **Sign** is the category of the f.o.l. signatures.

1.7 Note also that if a PSA is freely generated by a two-sorted set X then X can be considered as a f.o.l. signature with countable arities of all its symbols; conversely, for each such a signature Σ_∞, the freely generated PSA, $F(\Sigma_\infty)$, is nothing but the algebra of f.o.l. expressions (modulo α-conversion) over Σ_∞. Moreover, as soon as we admit infinitary signatures there is a forgetful functor G_∞:PSA$\longrightarrow$Sign$_\infty$ and an adjunction $F_\infty\backslash G_\infty$.

1.8 Remark. It is easy to see that the class of one-sorted $\{x_i, Sbt_i, i<\omega\}$-reducts of l.f.PSA proves an algebraic counterpart of one-sorted equational logic. Moreover, by enriching the latter signature with a countable family of unary operations of λ-quantification and a binary operation of application, one can construct an algebraic version of type-free λ-calculus and algebraize its meta-theory (see [DB93] and [PS93] for these results).

1.9 Up to now we were concerning on f.o.l. without equality. Note, however, that an equality formula is nothing but a pair of terms, so we can capture equality in our framework by adding into the signature Si of item 1.1 the set of constants $\{d_{ij}$: i,j run over $\omega\}$ of the sort Φ subjected to the following identities (to be add to the list of identities in 1.1):

$(\text{Eq}1)_{ijk}$ $[x_k/i]d_{ij} = d_{kj}$;

$(\text{Eq}2)_i$ $d_{ii} = \mathbf{1}$;

$(\text{Eq}3)_{ikl}$ $[x_k/i]\varphi \wedge d_{kl} \le [x_l/i]\varphi$

where $\varphi\le\psi$ abbreviates $\varphi\wedge\psi=\varphi$ and $\mathbf{1}$ abbreviates $\neg(\varphi\wedge\neg\varphi)$.

The resulting variety will be denoted by PESA.

Thus, in the selected way of algebraizing f.o.l. we have some variety of algebras, PESA, with the forgetful functor U:PESA⟶Set×Set and a set-valued functor $\mathcal{P}$:Set×Set⟶Set, $\mathcal{P}(X,Y) = Y$, producing propositions; we will call it the *proposition functor*. So, given a signature Σ, the collection of expressions over Σ is constituted by the (two-sorted) carrier set (T,Φ) of the expression algebra $F(\Sigma)$, while the collection of all (open) propositions over Σ is constituted by the set $\#F(\Sigma)$ where # denotes the functor $U;\mathcal{P}$.

1.10. To algebraize semantics we proceed as follows

Let B be a non-empty (base) set. By OpB and Rel$B$ we designate respect. the set of all ω-ary operations on B (i.e. maps from B^ω into B) and the set of all ω-ary relations on B (i.e. subsets of B^ω). In fact, OpB and Rel$B$ contain also all finitary operations and relations which (being considered as their elements) depend actually on finite number of arguments only, while other arguments are dummy. Tuples from B^ω will be denoted by x,y,z, the very operations – by u,v,w and relations - by R,Q etc. Given $i<\omega$ and an operation2 u, there is defined a map $[u]_i:B^\omega \dashrightarrow B^\omega$ which sends a tuple x into the tuple y coinciding with x for all $j<\omega$ different from i, $y_j=x_j$, while $y_i=u(x)$.

The following standard operations are defined on the two-sorted set $M = M(B) =$ <OpB,Rel$B$>:

projections, $\pi_i \in$ OpB, $\pi_i x := xi$, $i<\omega$;

binary compositions, $Sbt_i(u,w)x := w[u]_i x$, $i<\omega$;

Boolean operations on Rel$B$; o

cylindrifications, $\exists_i$:Rel$B$⟶Rel$B$, $\exists_i R := \{x\in B^\omega\colon x|_{\omega-i}=y|_{\omega-i}$ for some $y\in R\}$;

substitutions, $Sbf_i(u,R) := [u]_i^{-1} R$, $i<\omega$.

It is easily checked that these operations convert M into a PESA. However, in contrast to yntactical PESAs, algebras arising from semantics satisfy additional equations and conditional equations reflecting concrete logical structures of OpB's and Rel$B$'s, for example, $R\cup Q=Q\cup R$, $R\cup notR=Q\cup notQ$, $\exists\exists R=\exists R$, $R\subset\exists R$, $R\subset Q \Rightarrow \exists R\subset\exists Q$ etc. (see, e.g., [HMT_I,II] for a complete list).

Thus, in the variety PESA besides the distinguished class $\mathcal{E}xpr$ there is another distinguished subclass, namely, the class

$$\mathcal{M}od = \{M(B)\colon B \text{ is a non-empty set}\}.$$

Now, let Σ=(Op,Pred) be a f.o.l signature and $\mathfrak{M}$ be a Σ-model, that is, $\mathfrak{M} = (B,<o^{\mathfrak{M}}>_{o\in \mathrm{Op}},<\pi^{\mathfrak{M}}>_{\pi\in \mathrm{Pred}})$ with $o^{\mathfrak{M}}\in$ OpB and $\pi^{\mathfrak{M}}\in$ Rel$B$ for some non-empty set B; in addition, if arity(o)=n then $\Delta(o^{\mathfrak{M}}) \subseteq n$ for all $o\in$Op and similarly for all $\pi\in$Rel. It is easy to see that any such a model is nothing but a signature morphism $(-)^{\mathfrak{M}}$: Σ⟶$G_\infty M(B)$. The latter uniquely determines an algebra homomorphism $h^{\mathfrak{M}}$: $F\Sigma$⟶M. Conversely, any homomorphism h: E⟶M from a PESA $E\in\mathcal{E}xpr$ into a PESA belonging to $\mathcal{M}od$ could be considered as a model of the signature GE because of proposition 1.5.

In fact, there is a canonical isomorphism between ModΣ and Hom($F\Sigma$,$\mathcal{M}od$).

Well, let $\mathfrak{M}$ be a Σ-model and $\phi\in\#F\Sigma$ is a Σ-proposition. What does it mean $\mathfrak{M}\models\varphi$ algebraically?

As we have seen, with $\mathfrak{M}$ there is correlated a homomorphism h: $E\longrightarrow M$ and the map #h: $\#E\longrightarrow\#M$ assigning to each proposition $\varphi\in\#E$ its semantic meaning $\|\varphi\| = (\#h)\varphi\in OpB\times OpB \cup RelB$. In addition,

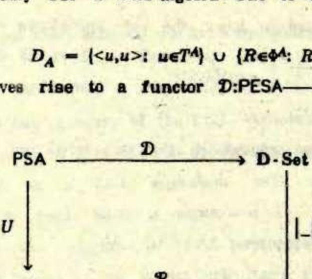
$$\mathfrak{M}\models\varphi \iff (\#h)\varphi\in D_M = Diag_{OpB}\cup \{1_{RelB}\}$$

where $Diag_{OpB}=\{<u,u>: u\in OpB\}$, $1_{RelB}=B^{\omega}$.

Thus, with any Σ-model $\mathfrak{M}$ there are actually correlated a PESA , M, together with a set of designated M-propositions, $D_M\subset\#M$, in such a way that for any Σ-proposition $\varphi\in\#F\Sigma$ we have $\mathfrak{M}\models\varphi \iff (\#h^{\mathfrak{M}})\varphi\in D_M$. (Just such a machinery is called *matrix semantics* in Polish tradition). In fact, a definite set of designated elements may be assigned not only for a $\mathcal{M}od$-algebra but to an arbitrary PESA algebra A as follows:

$$D_A = \{<u,u>: u\in T^A\} \cup \{R\in\Phi^A: R\vee notR = R\};$$

moreover, this gives rise to a functor $\mathcal{D}$:PESA$\longrightarrow$D-Set s.t. the following diagram commutes:



1.11 The above-described definition of satisfaction leads in the ordinary way to a family of consequence relations indexed by the class of expression algebras, $(\vdash_E, E\in\mathcal{E}xpr)$. In the terminology of the paper [HST89], this is a logic of validity type - all propositions are implicitly universally closed. To capture logics of truth-type into our framework we can proceed as follows.

The key observation is that propositions of a truth-type logic are rather sequences $\Psi\to\varphi$ than formulas themselves. So, we define a new proposition functor $\mathcal{P}^*$:Set×Set$\longrightarrow$Set by setting $\mathcal{P}^*(T,\Phi) = \mathfrak{P}_{\omega}\mathcal{P}(T,\Phi) \times \mathcal{P}(T,\Phi)$, in fact, $\mathcal{P}^*=\mathcal{P};<\mathfrak{P}_{\omega},Id>$, and then correspondingly define a new functor $\mathcal{D}^*$ by setting for all $A\in$PESA, $D^*_A = \{<\Psi,\varphi>\in\mathcal{P}^*UA$: if $\Psi\subset D_A$ then $\varphi\in D_A\}$, where D_A is the "old" set of designated elements. One can see that with such definitions of $\mathcal{P}^*$ and $\mathcal{D}^*$ we have

$$\Psi \vdash^t_{\Sigma} \varphi \iff (\#^*h)(\Psi,\varphi)\in D^*_M,\ \#^*=U;\mathcal{P}^* \text{ for all h: } F\Sigma\longrightarrow M\in\mathcal{M}od,$$

where on the left we have the ordinary f.o.l. consequence of truth-type. Thus, we see that the matrix semantics framework provides sufficient flexibility to describe various logics in a unifying way.

1.12 Up to now, we were dealing with one-sorted f.o.l. Our framework can be immediately generalized for the case of n-sorted f.o.l with arbitrary but fixed number of sorts, n, through the evident construction of *n*-sorted PESA over the underlying category $\mathbf{Set}^n\times\mathbf{Set}$ instead of $\mathbf{Set}\times\mathbf{Set}$. However, the situation becomes much more difficult when one has to deal with many-sorted logic with different sort sets, and hence algebras of the kind $F(\Sigma)$ may have different number of carrier sets for different Σ's.

A natural algebraic framework for working in such situations is the notion of generalized algebraic theory introduced by Cartmell [Ca86]. It is a direct generalization of the usual notion of a many-sorted equational theory: its extra generality is achieved by introduction of sort structures more general than those usually considered, in that sorts may denote sets as is usual, or they may denote families of sets, families of families of sets or the like. (A basic example is the generalized algebraic theory of categories, in which Ob appears as a sort to be interpreted as a set while Hom appears as a sort to be interpreted as a family of sets indexed by Ob×Ob). In more detail, a generalized algebraic theory T appears as an adjunction $(F\backslash U)$: $\mathbf{Fam}\mathsf{T} \underset{\longrightarrow}{\overset{\longleftarrow}{\ast}} \mathbf{Alg}\mathsf{T}$ where $\mathbf{Alg}\mathsf{T}$ is the (generalized) variety of T-algebras, $\mathbf{Fam}\mathsf{T}$ is the category of their carrier set structures, U is the underlying functor and F is the freely-generated-algebra functor. (For example, if T is the theory of categories, then an object of $\mathbf{Fam}\mathsf{T}$ is nothing but a family of sets, $(\mathrm{H}_{ij}\in\mathbf{Set}\colon i,j\in I)$, indexed by the Cartesian squire of some set I, while a morphism between two such families, (say, from H_{ij} into H'_{ij}), is a pair (f,g) consisting of a function f mapping I into I' and a family of functions $g=(g_{ij}, i,j\in I)$ with g_{ij} mapping H_{ij} into $\mathrm{H}'_{fi,fj}$). In fact, for any T, $\mathbf{Fam}\mathsf{T}$ is a full subcategory of the very large category $\mathbf{Fam}$ of sets, families of sets, families of families of sets etc. described by Cartmell.

As an example of algebraizing logic in a Cartmell's style, let us consider the following algebraization of the many-sorted equational logic.

Symbol	*Introductory rule*	*Notation*	
		...is denoted by....	
Sorts	$\vdash$ *Sorts* is a set		
Trm	$s\in Sorts \vdash Trm(s)$ is a set	$Trm(s)$	Trm_s
x_i	$s\in Sorts \vdash x_i(s)\in Trm_s$	$x_i(s)$	$x_i{:}s$
$Sb_{i:}$	$s,t\in Sorts, u\in Trm_s, v\in Trm_t \vdash Sb_{i:s}(u,v)\in Trm_t$	$Sb_{i:s}(u,v)$	$[u/i{:}s]v$

Axioms

$s,t\in Sorts, u\in Trm_s \vdash [x_i{:}s/i{:}s]u = u$

$s,t\in Sorts, u\in Trm_s \vdash [u/i{:}s]x_i{:}s = u$

$s,t\in Sorts, u\in Trm_s \vdash [u/j{:}s]x_i{:}s = x_i{:}s$

and the others counterparts of equations (Sb1)...(Sb5)

(By the way, similarly we can algebraize also typed λ-calculus what gives its algebraic version alternative to the categorical one via Cartesian closed categories).

So, meta-theory of a logic containing a *n*-sorted term system with arbitrary but fixed number of sorts, *n*, can be algebraized by means of a certain many-sorted algebraic theory while algebraization of logics with many-sorted term systems requires to use generalized algebraic theories of Cartmell.

Thus, generally speaking, a proper unified algebraic universe for algebraizing meta-theories of different logics must be not a variety but a generalized variety of algebras.

2 Metatheory of a logic via universal algebra4

In this section, under an algebraic theory we mean a generalized algebraic theory in the sense of Cartmell. With each such a theory T there is correlated an adjunction $(F\backslash U)$: **Fam**T $\underset{*}{\overset{\longleftarrow}{\longrightarrow}}$ **Alg**T where **Fam**T is a full subcategory of **Fam**, **Alg**T is the variety of T-algebras, U is the underlying functor and F is the freely-generated-algebra functor.

2.1 Definition. *A logic metatheory in algebraic form* consists of the following constructs.

(i) *A language* is defined to be a triple $\mathfrak{l}=(\mathsf{T},\mathcal{P},\mathcal{D})$ with T an algebraic **theory**, $\mathcal{P}$: **Fam**T $\longrightarrow$ **Set** and $\mathcal{D}$: **Alg**T $\longrightarrow$ **D-Set** functors s.t. the following diagram commutes:

$$\begin{array}{ccc}
\mathbf{Alg}\mathsf{T} & \xrightarrow{\ \mathcal{D}\ } & \mathbf{D\text{-}Set} \\
{\scriptstyle U}\downarrow & & \downarrow{\scriptstyle |_|} \\
\mathbf{Fam}\mathsf{T} & \xrightarrow{\ \mathcal{P}\ } & \mathbf{Set}
\end{array}$$

Given a language $\mathfrak{l}$, we will designate **Alg**T and **Fam**T as **Alg**$_{\mathfrak{l}}$ and **Fam**$_{\mathfrak{l}}$ and often omit the subindex. The functor $U;\mathcal{P}$ will be denoted by #. If for an algebra A $\mathcal{D}(A)=(X,D)$ then the set D will be denoted by D_A.

(ii) *A logic* is defined to be a triple $\mathcal{L}=(\mathfrak{l},\mathcal{E}xpr,\vdash)$ with $\mathcal{E}xpr\subset$**Alg**$_{\mathfrak{l}}$ a class of *expression algebras* and $\vdash:\mathcal{E}xpr \longrightarrow\!\!\!\!\!+$ **Set** a functor s.t. the following diagram commutes:

$$\begin{array}{ccc} \mathcal{E}xpr & \xrightarrow{\ \vdash\ } & \vdash\text{-Set} \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle |\_|} \\ \mathrm{Fam}_{\mathbb{I}} & \xrightarrow{\ \mathcal{P}\ } & \mathrm{Set} \end{array}$$

We shall also write a logic over a given language $\mathbb{I}$ as $\mathcal{L}=(\vdash_E, E\in\mathcal{E}xpr)$ and call it an $\mathbb{I}$-logic.

(iii) ***An algebraic semantics*** **is defined to be a pair $\mathcal{A}s=(\mathbb{I},\mathcal{M}od)$ with $\mathcal{M}od\subset\mathrm{Alg}_{\mathbb{I}}$ a class of** ***model algebras.***

2.2 Construction. Given an algebraic semantics $\mathcal{A}s$, each homomorphism $h:E\longrightarrow M\in\mathcal{M}od$ can be considered as a model of the expression algebra E (or some signature generating E). With this intuition in mind, elements of the set $\#E$ will be referred to as *propositions* and elements of $\#M$ - as *predicates (or relations)*; the elements of the set D_M can be thought of as *universally true predicates* in the sense that for any proposition $\varphi\in\#E$ we put

$$h \vDash_E \varphi :\Leftrightarrow (\#h)\varphi \in D_M.$$

This construction determines in the ordinary way a consequence relation $\vdash_E^{(\mathcal{M}od)}\subset\mathfrak{P}(\#E)\times(\#E)$ and, hence, the theory lattice $\mathrm{Th}^{(\mathcal{M}od)}E \subset \mathfrak{P}\#E$ and the consequence operator $\mathrm{Cn}_E^{(\mathcal{M}od)}: \mathfrak{P}\#E\longrightarrow\mathfrak{P}\#E$.

2.3 Definition. *An institution in algebraic form* or *a predoctrine* is defined to be a triple $\mathcal{A}=(\mathbb{I},\mathcal{E}xpr,\mathcal{M}od)$ with $\mathbb{I}$ a language and $\mathcal{E}xpr,\mathcal{M}od$ two subclasses of $\mathrm{Alg}_{\mathbb{I}}$ with members called *expression algebras* and *model algebras* resp.

By construction 2.2, with each predoctrine there is correlated an $\mathbb{I}$-logic $\mathcal{L}(\mathcal{A})=(\vdash_E^{(\mathcal{M}od)},E\in\mathcal{E}xpr)$, correspondingly, there are defined the family of theory lattices, $(\mathrm{Th}_E^{(\mathcal{M}od)},E\in\mathcal{E}xpr)$, and the family of closure operators $(\mathrm{Cn}_E^{(\mathcal{M}od)},E\in\mathcal{E}xpr)$. As a rule, below we will often write the superscript $(\mathcal{A})$ instead of $(\mathcal{M}od)$.

2.4 Basic Assumption Our next goal is to state some results about the introduced constructs necessary for proving our main theorem described in the introduction. Proofs of these results require rather involved universal algebraic machinery, moreover, at the present moment they are stated completely only for the special case of ordinary one-sorted algebraic theories when $\mathrm{Fam}\mathbb{T}$ is Set. Generalization for the case of many-sorted algebraic theories, $\mathrm{Fam}\mathbb{T}=\mathrm{Set}^n$ for some fixed number n, is immediate and tedious while the general situation of Cartmell's theories is much more difficult. The point is that our machinery is based on exploiting a certain universal algebraic technique of relating congruence lattices and quasi-varieties (see [Di92], some aspects were demonstrated also in the book [BP89]), however, for generalized universal algebra, the problem of stating such a relation is much more involved. (Indeed, as Cartmell explained in [Ca86], generalized algebraic theories are equal in descriptive power to the essentially algebraic theories of P.Freyd; but it is well known that constructions of a congruence and a quotient are rather capricious in the case of Freyd's algebras, at any rate, cannot be generalized in a straightforward way from ordinary algebras - this was clearly demonstrated, e.g., in the book of Reichel [Re87]).

Therefore, for the sake of transparency of the main ideas, on the one hand, and by the lack of a general proof, on the other, below we shall deal with the special situation of ordinary one-sorted algebraic theories when **Fam**Υ=**Set**. It is hoped, however, that transition to the general situation will require modifying proofs but not results.

Thus, our goal is a theorem on connection between theories and algebras in a given predoctrine, concretely, theorem 2.12

First of all, we need a refinement of the notion of a language.

2.5 Definition. A language $l=(\Upsilon,\mathcal{P},\mathcal{D})$ is said to be *regular* iff it meets the following two conditions:

(i) the functor $\mathcal{P}$, hence $\#=U;\mathcal{P}$, preserves inclusions, surjections and products,

(ii) the functor $\mathcal{D}$ preserves subobjects and products where (X,D) is a subobject of (Y,E) if $X\subseteq Y$ and $D=E\cap X$.

Remark. On one hand, these are quite natural algebraic conditions, on the other hand, it can be checked that a majority of algebraization languages appearing in practice are regular, thus, this constraint is not restrictive for practical using.

Now we need a portion of universal algebra.

2.6 Construction. Let V be a variety of algebras - the universe of our considerations. Given an algebra A, the lattice of its congruences will be denoted by **Con**A and if $\theta\in$**Con**A then the corresponding canonical epimorphism will be denoted by $\varepsilon_\theta:A\twoheadrightarrow A/\theta$. For any relation $\rho\subset A\times A$, the least congruence containing ρ will be denoted by $Cg_A\rho$. Further, if $h:A\longrightarrow B$ is a homomorphism and $\theta\in$**Con**B then $h^{-1}\theta$ = $(h\times h)^{-1}\theta$ is also a congruence on A. Hence, **Con** turns out to be a functor $V\longrightarrow$**Pos**^{op}, and since **Pos** can be regarded as a category, **Con** is an indexed category.

Now, for any class $\mathcal{K}$ of algebras, let $\mathrm{Con}^{(\mathcal{K})}A$ denotes the collection $\{\theta\in\mathrm{Con}A: A/\theta\in\mathcal{K}\}$. It can be shown that if $\mathcal{K}$ is a quasi-variety, $\mathcal{K}=\mathbb{SP}\mathcal{K}$, then $\mathrm{Con}^{(\mathcal{K})}A$ is closed under intersections for all $A\in V$ (this is well known) and, moreover, for any homomorphism $h:A\longrightarrow B$, if $\vartheta\in\mathrm{Con}^{(\mathcal{K})}B$ then $h^{-1}\vartheta\in\mathrm{Con}^{(\mathcal{K})}A$ (it seems that this simple fact as well as the converse statement that the above features of the family $(\mathrm{Con}^{(\mathcal{K})}A,\ A\in\mathcal{V})$ imply that $\mathcal{K}$ is a quasi-variety (see [Di1] for proofs) are not known even for the universal algebra community - so, relations between logic and algebra are fruitful in both directions!). Thus, if $\mathcal{K}$ is a quasivariety then the family $(\mathrm{Con}^{(\mathcal{K})}A, A\in\mathcal{V})$ can be also regarded as an indexed category $\mathrm{Con}^{(\mathcal{K})}:\mathcal{V}\longrightarrow\mathrm{Pos}^{op}$. As we will see immediately below this simple machinery turns out extremely useful in algebraic logic (see also [Di2]) and in algebraizing institutions.

2.7 Construction. Let $\mathcal{A}\mathfrak{s}=(\mathrm{I},\mathcal{M}od)$ be an algebraic semantics over a regular language and Q denotes the quasi-variety generated by $\mathcal{M}od$, i.e., $Q=\mathbb{SP}\mathcal{M}od$. We note that owing to preservation properties of $\mathcal{D}$ and $\mathcal{P}$, for any homomorphism $\mathrm{h}:A\longrightarrow B\in Q$ and any $\varphi\in\#A$, $(\#\mathrm{h})\varphi\in D_B$ iff $\varphi\in(\#\varepsilon_\vartheta)^{-1}D_{A/\vartheta}$ where ϑ denotes $ker\mathrm{h}$, in addition, $A/\vartheta\in Q$ since it is isomorphic to a subalgebra of B. So, if for each $A\in\mathrm{Alg}_\mathrm{I}$ we introduce an operator $\mathrm{H}_A:\mathrm{Con}^{(Q)}A\longrightarrow\mathfrak{P}\#A$ by setting $\mathrm{H}_A\vartheta\ :=\ (\#\varepsilon_\vartheta)^{-1}D_{A/\vartheta}$, then $\mathrm{h}\models\varphi$ iff $\varphi\in\mathrm{H}_A(ker\mathrm{h})$. Now one can see that for any $\vartheta\in\mathrm{Con}^{(Q)}A$, $\mathrm{H}_A\vartheta$ is a theory with respect to $\models_A^{(\mathcal{M}od)}$, and, moreover, each such a theory can be obtained in this way. Actually, it can be shown by using standard universal algebraic machinery that owing to preservation properties of $\mathcal{P}$ and $\mathcal{D}$, each H_A preserves meets (intersections), hence, its image is closed under intersections and this closure system is just $\mathrm{Th}^{(\mathcal{M}od)}A$. Since H_A preserves meets (and $\mathfrak{P}\#A$ is a complete lattice), H_A has the left (lower) adjoint Ω_A: $\mathfrak{P}\#A\longrightarrow\mathrm{Con}^{(\mathcal{M}od)}A$, in particular, there is a Galois insertion (see [MSS86] for a theory of adjoint situation for posets):

$$\mathrm{Con}^{(Q)}A\ \underset{\Omega_A}{\overset{\mathrm{H}_A}{\rightleftarrows}}\ \mathrm{Th}^{(\mathcal{M})}A\ ,$$

here and below, in diagrams $\mathcal{M}$ denotes the class $\mathcal{M}od$.

Finally, again owing to preservation properties of $\mathcal{P}$ and $\mathcal{D}$ (namely, their compatibility with inclusions and subobjects), the family $(\mathrm{H}_A,\ A\in\mathrm{Alg})$ is compatible with homomorphisms in the sense that for any homomorphism $\mathrm{h}:A\longrightarrow B$ one has $\mathrm{H}_A\mathrm{h}^{-1}\vartheta=(\#\mathrm{h})^{-1}\mathrm{H}_B\vartheta$ for all $\vartheta\in\mathrm{Con}B$.

$$\begin{array}{ccccc}
A & & \mathrm{Con}^{(Q)}A & \underset{\Omega_A}{\overset{\mathrm{H}_A}{\rightleftarrows}} & \mathrm{Th}^{(\mathcal{M})}A \\
\mathrm{h}\big\downarrow & & \mathrm{h}^{-1}\uparrow\downarrow\ \mathrm{h}^{+};\mathrm{Cg}_B^{(Q)} & & (\#\mathrm{h})^{+};\mathrm{Cn}_B^{(\mathcal{M})}\downarrow\uparrow\ (\#\mathrm{h})^{-1} \\
B & & \mathrm{Con}^{(Q)}B & \underset{\Omega_B}{\overset{\mathrm{H}_B}{\rightleftarrows}} & \mathrm{Th}^{(\mathcal{M})}B
\end{array}$$

In particular, this implies that if $T\in \mathrm{Th}_B^{(Mod)}$ then $(\#h)^{-1}T\in \mathrm{Th}_A^{(Mod)}$ and all the diagrams on the figure above are commutative.

In algebraic logic the latter condition is called *structurality condition* and a logic meeting it is called *structural.* Actually, structurality of a logic $\mathcal{L}=(\vdash_E, E\in\mathcal{E}xpr)$ is equivalent to the implication $\Psi\vdash\varphi \Rightarrow h\Psi\vdash h\varphi$ for any homomorphism h of $\mathcal{E}xpr$-algebras (see [Di92] for details and more precise formulations).

These considerations immediately provide the following result (from now on we begin to use the terminology adopted in the paper [TBG91]).

Let $\mathcal{AS}=(\mathrm{I}, \mathcal{M}od)$ be an algebraic semantics over a regular language I, Q denotes $SP\mathcal{M}od$ and $\mathcal{E}$ be a subclass of Alg_I.

2.8 Proposition. The families $(\mathrm{Con}^{(Q)}E, E\in\mathcal{E})$, $(\mathrm{Th}^{(Mod)}E, E\in\mathcal{E})$ prove to be indexed categories $\mathrm{Con}^{(Q)}:\mathcal{E}\longrightarrow \mathbf{Pos}^{op}$, $\mathrm{Th}^{(Mod)}:\mathcal{E}\longrightarrow \mathbf{Pos}^{op}\hookrightarrow \mathbf{Cat}^{op}$ while the family $(\mathrm{H}_E, E\in\mathcal{E}$ can be regarded as an indexed functor (natural transformation) $\mathrm{Con}^{(Q)}\longrightarrow \mathrm{Th}^{(Mod)}$. Moreover, this indexed functor is locally reversible.

□

Now, let $\mathbf{Theor}^{(Mod)}\mathcal{E}$ denotes the category whose objects are pairs (E,T) with $E\in\mathcal{E}$, $T\in \mathrm{Th}^{(Mod)}E$ and morphisms $(E,T)\longrightarrow(E',T')$ are the homomorphisms $h:E\longrightarrow E'$ s.t. $(\#h)T\subset T'$; and let $\mathbf{Congr}^{(Q)}\mathcal{E}$ be the similar category whose objects are pairs (E,θ) with $\theta\in \mathrm{Con}^{(Q)}E$. Since $\mathbf{Flat}(\mathrm{Con}^{(Q)})=\mathbf{Congr}^{(Q)}\mathcal{E}$ and $\mathbf{Flat}(\mathrm{Th}^{(Mod)})=\mathbf{Theor}^{(Mod)}\mathcal{E}$ by the very definition of flattening, theorem 3 of [TBG91] gives immediately an adjunction (which is, in fact, an embedding due to fact that the local adjunctions are Galois insertions):

$$\mathbf{Congr}^{(Q)}\mathcal{E} \underset{\Omega}{\overset{\mathbf{H}}{\rightleftarrows}} \mathbf{Theor}^{(Mod)}\mathcal{E} \qquad \square$$

To get our goal in searching an adjunction between theories and algebras we need an adjunction between $\mathbf{Congr}^{(Q)}\mathcal{E}$ and Q, however, there are some delicate universal algebraic points here. A natural functor from $\mathbf{Congr}^{(Q)}\mathcal{E}$ to Q is evident while to construct a reverse functor we need two refinements.

2.9 Definition. Let $h1,h2:(E,\theta)\longrightarrow(E',\theta')$ be two morphisms in $\mathbf{Congr}^{(Q)}\mathcal{E}$. They are said to be *equivalent*, $h1\approx h2$, if $\langle h1(e),h2(e)\rangle\in\theta'$ for all $e\in E$. The category of $\approx$-equivalence classes is a quotient category and will be denoted by $\mathbf{Congr}^{(Q)}\mathcal{E}/\approx$.

2.10 Definition. Let V be a variety of algebras. A class $\mathcal{E}\subset V$ is said to *possess the projectivity property* (*PP*) (or *PP holds* for $\mathcal{E}$) iff for any diagram $A\xrightarrow{f}C\twoheadleftarrow^{g}B$ with $A,B\in\mathcal{E}$ and g epi there is some $h\in \mathrm{Hom}(A,B)$ s.t. $h;g=f$; in other words, each algebra A of $\mathcal{E}$ is projective with respect to epis from $\mathcal{E}$-algebras (see e.g. Mac Lane [CWM71] for the standard definition of a projective object in a category). □

Natural universal algebraic considerations state the following

2.11 Lemma. Let V be a variety, $Q\subset V$ a quasi-variety and $\mathcal{E}\subset V$ a class with PP. Then the categories $\mathrm{Congr}^{(Q)}\mathcal{E}/\approx$ and $(Q\cap H\mathcal{E})$ are equivalent. □

Thus, we see, to get our goal we must modify the notion of a theory morphism and deal with the quotient category $\mathrm{Theor}^{(\mathcal{M}od)}\mathcal{E}/\approx$ where two $\mathrm{Theor}^{(\mathcal{M}od)}\mathcal{E}$-morphisms h1,h2: $(E,T)\longrightarrow(E',T')$ are *equivalent*, h1≈h2, if $<(\#h1)\varphi,(\#h2)\varphi>\in\#[\Omega_{E'},T']$ for all $\varphi\in\#E$; we will write a more suggestive $\mathrm{Theor}^{(\mathcal{M}od)}\mathcal{E}/\Omega$ instead of $\mathrm{Theor}^{(\mathcal{M}od)}\mathcal{E}/\approx$. Finally, given a predoctrine $\mathcal{A}$=(I,$\mathcal{E}xpr$,$\mathcal{M}od$), we will write Theor$\mathcal{A}$ and Congr$\mathcal{A}$ for $\mathrm{Theor}^{(\mathcal{M}od)}\mathcal{E}xpr$ and $\mathrm{Congr}^{(Q)}\mathcal{E}xpr$ resp.

Now, our chief result in this section is almost immediate

2.12 Theorem. Let $\mathcal{A}$=(I,$\mathcal{E}xpr$,$\mathcal{M}od$) be a predoctrine over a regular language and s.t. $\mathcal{E}xpr$ has projectivity property. Then, for some class of I-algebras $\mathcal{T}h$, there is an adjunctive embedding

$$\mathcal{T}h \underset{\Omega}{\overset{H}{\rightleftarrows}} \mathrm{Theor}\mathcal{A}/\Omega.$$

Algebras from that class may be called *Lindenbaum-Tarski algebras*.

2.13 Remark. The PP-requirement for $\mathcal{E}xpr$ may seem to be rather unnatural. However, there is well-known in algebraic logic that definite algebraic properties of classes of algebras correlated with logics are closely connected with properties of that logics. For example, amalgamation property (AP) for Q is a universal algebraic counterpart of Craig's interpolation property, all-epis-are-surjective property (ESP) is a counterpart of Beth's definability property and the like (see,e.g.,[HMT85). Thus, PP as well as its "colleagues" can be considered as rather respectable from the theoretical view point of algebraic logic. On the other hand, the expression algebra classes of a majority of logics in use has PP - this point provides a justification from the practical view point.

2.14 Definition. Theorem 2.12 makes it reasonable to introduce a special name for predoctrines over regular languages with PP-classes of expression algebras. We will call such predoctrines *regular*.

3 Algebraizing institutions

Let $\mathcal{I}$=(Sign,*Sen*,*Mod*,⊧) be an institution.

3.1 Definition. *An algebraization* of an institution $\mathcal{I}$ is defined to be a pair $(\alpha,\mathcal{A})$ with $\mathcal{A}$=(I,$\mathcal{E}xpr$,$\mathcal{M}od$) a regular predoctrine and α *a representation of* $\mathcal{I}$ *in* $\mathcal{A}$

consisting of the following data:

(A1) $\alpha_{sign}=(F_{sig}\backslash U_{sig})$: Sign $\rightleftarrows$ $\mathcal{E}\alpha p\imath$ an adjunction ,

(A2) α_{sen}: $Sen \longrightarrow F_{sig};\#;\mathfrak{P}$ a natural transformation,

(A3) α_{mod}: $Mod \longrightarrow F_{sig};\mathrm{Hom}(-,\mathcal{M}od)$ an isomorphic natural transformation,

such that the following condition is fulfilled:

$(A4)_\Sigma$ $\mathfrak{M} \models^{(\mathcal{I})}_{\Sigma} \varphi \iff (\#h)(\alpha_{sen_\Sigma}\varphi) \subset D_{h\Box}$ where h denotes $\alpha_{mod_\Sigma}\mathfrak{M}$, .

for all $\Sigma \in |\mathrm{Sign}|$, $\varphi \in Sen\Sigma$ and $\mathfrak{M} \in Mod(\Sigma)$.

In such a case we will say that the institution can be *algebraized via* $\alpha:\mathcal{I} \longrightarrow \mathcal{A}$.

3.2 Remark. With construction 2.2 condition (A4) means that for any signature Σ we have:

$(A4)'_\Sigma$ $\mathfrak{M} \models^{(\mathcal{I})}_{\Sigma} \varphi \iff \alpha_{mod_\Sigma}\mathfrak{M} \models^{(\mathcal{A})}_{F\Sigma} \alpha_{sen_\Sigma}\varphi$.

Naturalness of the definition is justified by the following

3.3 Fact. Institutions describing logics from the following list

- many-sorted equational logic, untyped and typed λ-calculi, polymorphic λ-calculi and their conditional versions;
- Horn f.o.l., universal f.o.l., full f.o.l with and without terms...

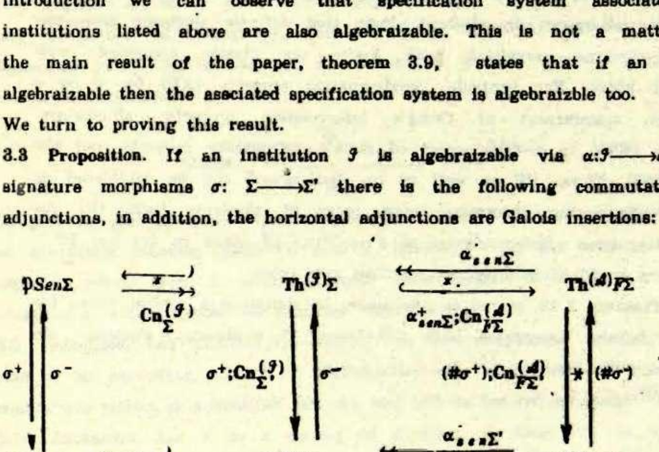
are algebraizable.

(It is hoped that considerations of section 1 give a notion of proving this fact).

Recalling the definition of specification system algebraizability described in the introduction we can observe that specification system associated with the institutions listed above are also algebraizable. This is not a matter of chance: the main result of the paper, theorem 3.9, states that if an institution is algebraizable then the asciated specification system is algebraizble too.

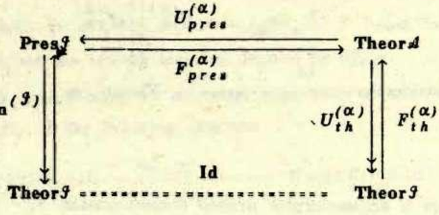
We turn to proving this result.

3.3 Proposition. If an institution $\mathcal{I}$ is algebraizable via $\alpha:\mathcal{I} \longrightarrow \mathcal{A}$ then for all signature morphisms σ: $\Sigma \longrightarrow \Sigma'$ there is the following commutative diagram of adjunctions, in addition, the horizontal adjunctions are Galois insertions:



Proof (sketch). The left square commutes by virtue of the institution satisfaction axiom, the right one commutes due to (A2) and structurality of the logic $\mathcal{L}(\mathcal{A})$. Adjunctions on the left square are checked straightforward, those on the right one are due to (A3) and (A4).

3.4 Corollary. If an institution $\mathcal{I}$ is algebraizble via $\alpha:\mathcal{I}\longrightarrow\mathcal{A}$ then there is the following commutative diagram of adjunctions where all functors F's are lower (left) adjoints and all the functors U's are upper (right) adjoints:



Proof (sketch). Owing to commutativity of the proposition 3.3 squares above, the families of upper arrows can be considered as indexed functors and we can apply theorem 8 from [TBG91] and the lemma below.

Lemma (a modification of theorem 8 of [TBG91]). Let $\mathbf{C}:\mathbf{I}^{op}\longrightarrow\mathbf{Cat}$, $\mathbf{D}:\mathbf{K}^{op}\longrightarrow\mathbf{Cat}$ be indexed categories, $\mathbf{f}:\mathbf{I}\longrightarrow\mathbf{K}$ a functor and $\mathbf{F}:\mathbf{C}\longrightarrow\mathbf{f};\mathbf{D}$ an indexed functor. Then $\mathbf{Flat(f,F)}$: $\mathbf{Flat(C)}\longrightarrow\mathbf{Flat(D)}$ has a right adjoint as soon as $\mathbf{f}$ has a right adjoint and $\mathbf{F}$ has a right adjoint locally. Here $\mathbf{Flat(f,F)}$ denotes the following functor Φ: $\Phi(i,a)=(fi,\mathbf{F}_i a)$ and $\Phi(\sigma,f)=(f\sigma,\mathbf{F}_i f)$: $(fi,\mathbf{F}_i a)$ --> $(fj,\mathbf{F}_j a)$
for all $(i,a)\in|\mathbf{Flat(C)}|$, $(\sigma,f):(i,a)\longrightarrow(j,b)$ in $\mathbf{Flat(C)}$.

We would like to state an adjunction between $\mathbf{Pres}_L\mathcal{I}$ and some class of $\mathcal{A}ld_I$-algebras. Corollary 3.4 and theorem 2.12 would be sufficient for this purpose if in theorem 2.12 we would have the category **Theor**$\mathcal{A}$ on the right. However, we have there the quotient category **Theor**$\mathcal{A}/\Omega$, hence, we must modify corollary 3.4 in order to capture Ω-factorization of **Theor**$\mathcal{A}$-morphisms. It is obvious that to this end we must introduce something similar Ω-equivalence on the set of **Theor**$\mathcal{I}$-morphisms.

3.5 Definition. Let $\mathcal{I}$ be an institution and $\sigma1,\sigma2:(\Sigma,\Psi)\longrightarrow(\Sigma',\Psi')$ be two $\mathbf{Pres}_L\mathcal{I}$-morphisms. They are said to be *semantically equivalent*, $\sigma1 \doteqdot \sigma2$, if, for any sentence φ, $\mathfrak{M}\models_{\Sigma'}(\sigma1)\varphi$ iff $\mathfrak{M}\models_{\Sigma'}(\sigma2)\varphi$ for all models $\mathfrak{M}$ of (Σ',Ψ'). It is checked immediately that $\doteqdot$ is compatible with composition and the corresponding quotient category will be denoted by **Theor**$\mathcal{I}/\doteqdot$.

With corollary 3.4, this gives

3.6 Proposition. If an institution $\mathcal{I}$ is algebraizble via $\alpha:\mathcal{I}\longrightarrow\mathcal{A}$, then $\sigma1\doteqdot\sigma2$ iff $F^{(\alpha)}_{pres}\sigma1 \approx F^{(\alpha)}_{pres}\sigma2$ for any presentation morphisms $\sigma1,\sigma2$

With corollary 3.4 this immediately gives

3.7 Theorem. If an institution $\mathcal{I}$ is algebraizable via $\alpha:\mathcal{I}\longrightarrow\mathcal{A}$ then there is the following commutative diagram of adjunctions where all functors F's are lower (left) adjoints and all the functors U's are upper (right) adjoints:

$$
\begin{array}{ccc}
\mathbf{Pres}\mathcal{I}_{\models}/\dashv\models & \overset{U^{(\alpha)}_{pres}}{\underset{F^{(\alpha)}_{pres}}{\rightleftarrows}} & \mathbf{Theor}\mathcal{A}/\Omega \\
\mathrm{Cn}^{(\mathcal{I})}\downarrow\uparrow & & U^{(\alpha)}_{th}\downarrow\uparrow F^{(\alpha)}_{th} \\
\mathbf{Theor}\mathcal{I}/\dashv\models & \overset{\mathrm{Id}}{\cdots\cdots\cdots} & \mathbf{Theor}\mathcal{I}/\dashv\models
\end{array}
\qquad \square
$$

3.8 Theorem. There is an isomorphic natural transformation

$$\alpha^{*}_{mod}\colon Mod^{*}\longrightarrow F^{(\alpha)}_{pres};\mathrm{Hom}(-,\mathcal{M}od)\ .$$

Proof (sketch). Firstly, for each presentation $\pi=(\Sigma,\Psi)\in|\mathbf{Pres}_{\models}\mathcal{I}|$ we define a map $\beta_{\Sigma}:\mathrm{Hom}(F^{(\alpha)}_{pres}\pi,\mathcal{M}od)\longrightarrow Mod^{*}\pi$ as follows.

Let $(E,T)=F^{(\alpha)}_{pres}\pi$, that is, $E=F_{alg}\Sigma$, $T=\mathrm{Cn}^{(\mathcal{A})}\alpha_{sen\sigma\Sigma}\Psi$ and let ϑ denotes $\Omega_E T$. For each $\mathrm{h}:E/\vartheta\longrightarrow M\in\mathcal{M}od$ we have

$(\# \mathrm{h})D_{E/\vartheta}\subset D_M$ since $\mathcal{D}\mathrm{h}$ is a D-Set (matrix) morphism,

$\Rightarrow$ $(\#\varepsilon_{\vartheta})^{-1}D_{E/\vartheta}\subset(\#\varepsilon_{\vartheta})^{-1}(\#\mathrm{h})^{-1}D_M$ by $f^{+}\backslash f^{-1}$ for any mapping f

$\Rightarrow$ $(\#\mathrm{h})(\#\varepsilon_{\vartheta})\mathrm{H}_E\Omega_E T\subset D_M$ by definition of H_E

$\Rightarrow$ $(\#\bar{\mathrm{h}})\mathrm{H}_E\Omega_E T\subset D_M$ (where $\bar{\mathrm{h}}=\varepsilon_{\vartheta};\mathrm{h}$) by functoriality of $\mathcal{P}$

$\Rightarrow$ $(\#\bar{\mathrm{h}})T\subset D_M$ since $T\subset\mathrm{H}_E\Omega_E T$ by $\Omega_E\backslash\mathrm{H}_E$

$\Rightarrow$ $(\#\bar{\mathrm{h}})\alpha_{sen\Sigma}\Psi\subset D_{M}$ □ since $T := \mathrm{Cn}^{(\mathcal{A})}_{F\Sigma}\alpha_{sen\Sigma}\Psi$

$\Rightarrow$ $\alpha^{-1}_{mod\Sigma}\bar{\mathrm{h}}\models^{(\mathcal{I})}_{\Sigma}\Psi$ owing to isomorphity of $\alpha_{mod\Sigma}$ and the direction ($\Leftarrow$) of $(\mathrm{A4})_{\Sigma}$.

Thus, with each $\mathrm{h}\in\mathrm{Hom}(F_{pres}\pi,\mathcal{M}od)$ there is correlated a Σ-model $\mathfrak{M}=\alpha^{-1}_{mod\Sigma}\bar{\mathrm{h}}\in Mod^{*}\pi$ and this defines a mapping

$$\beta_{\Sigma}\colon\ \mathrm{Hom}(F_{pres}\pi,\mathcal{M}od)\dashrightarrow Mod^{*}\pi.$$

Moreover, β_{Σ} is injective since

$\mathrm{h1}\neq\mathrm{h2}\Rightarrow\bar{\mathrm{h}}1\neq\bar{\mathrm{h}}2\Rightarrow\mathfrak{M}1\neq\mathfrak{M}2$ as ε_{ϑ} is epic and $\alpha^{-1}_{mod\Sigma}$ is injective by (A3).

Surjectivity of β_{Σ} is proved by the following arguments.

For each $\mathfrak{M}\in Mod^{*}\pi$ we have

$\mathfrak{M} \vdash_{\Sigma}^{(\mathcal{I})} \Psi$ by the definition of Mod^*

$\Rightarrow$ $\#h\alpha_{sen\Sigma}\Psi \subset D_{h\Box}$ (where h denotes $\alpha_{mod\Sigma}\mathfrak{M}$) by (A4)($\Rightarrow$)

$\Rightarrow$ $(\#h)Cn_E^{(\mathcal{A})}\alpha_{sen\Sigma}\Psi \subset D_{h\Box}$ by definition of $Cn^{(\mathcal{A})}$

i.e. $(\#h)T \le D_{h\Box}$ by our notation

$\Rightarrow$ $T \subset (\#h)^{-1}D_{E/\eta}$ (where $\eta = ker h$) by the first statement of constr. 2.7

i.e. $T \subset H_E\eta$ by definition of H_E

$\Rightarrow$ $\vartheta = \Omega_E T \subset \Omega_E H_E \eta \le \eta = ker h$ by $\Omega_E \backslash H_E$.

Hence, there is a homomorphism $\tilde{h}: E/\vartheta \longrightarrow (\alpha_{mod\Sigma}\mathfrak{M})\Box$ s.t. $\varepsilon_\vartheta;\tilde{h} = \alpha_{mod\Sigma}\mathfrak{M}$.

In addition, it is easy to see that $\beta_\Sigma\tilde{h} = \alpha^{-1}_{mod\Sigma}(\varepsilon_\vartheta;\tilde{h}) = \alpha^{-1}_{mod\Sigma}\cdot\alpha_{mod\Sigma}\mathfrak{M} = \mathfrak{M}$, that is, β_Σ is surjective and has the inverse mapping denoted by $\alpha^*_{mod\Sigma}$.

It remains to check compatibility of mappings α^*_{mod} with morphisms, in other words, to state commutativity of the following diagram:

$$\begin{array}{lccc}
(\Sigma,\Psi) & Mod^*(\Sigma,\Psi) & \xrightarrow{\ \alpha^*_{mod\Sigma}\ } & Hom(E/\vartheta, \mathcal{M}od) \\
\sigma \downarrow & \uparrow Mod^*\sigma/\dashv\vdash & & \uparrow (F\sigma/\approx;-) \\
(\Sigma',\Psi') & Mod^*(\Sigma',\Psi') & \xrightarrow{\ \alpha^*_{mod\Sigma'}\ } & Hom(E'/\vartheta', \mathcal{M}od)
\end{array}$$

where $(\#F\sigma)\vartheta \subset \vartheta'$ hence $(\#F\sigma)/\approx: E/\vartheta \longrightarrow E'/\vartheta'$ is followed from the very definition of the functor F_{pres} by flattening. Now, commutativity can be checked directly from definitions by using (A3). $\Box$

Together with the definition of specification system algebraizability described in the introduction, theorems 2.12, 3.7 and 3.8 immediately imply

3.9 Theorem. Any algebraization

$$\mathcal{I} = (Sign, Sen, Mod, \vdash) \xrightarrow{\ \alpha = (F_{sig}\backslash U_{sig}, \alpha_{sen}, \alpha_{mod})\ } \mathcal{A} = (I, \mathcal{E}qn, \mathcal{M}od)$$

of an institution $\mathcal{I}$ gives rise to the following algebraization

$$spec(\mathcal{I}) = (Spec(\mathcal{I}), Mod^*) \xrightarrow{\ \alpha^* = (F^{(\alpha)}_{pres}\backslash U^{(\alpha)}_{pres}, \alpha^*_{mod})\ } \mathcal{D} = (Alg_I, \mathcal{T}h[\mathcal{A}], \mathcal{M}od)$$

of $\mathcal{I}$'s specification system, where α^* is as above and $\mathcal{T}h[\mathcal{A}] = SP\mathcal{M}od \cap H\mathcal{E}qn$

4 Conclusion: towards generalizations

It seems that the principal contribution of the paper consists in definition 2.1 on the ground of which it is suggested to develop algebraic logic in a very general setting. Actually, the framework is a generalization of the familiar approach to algebraizing logics developed by Polish school (see, eg, [Wo88]). According to the latter, a logic is a consequence relation on a (countably generated) free algebra of some signature. Definition 2.1(ii) generalizes this construction in the following directions:

- there is considered a family (not a single one) of consequence relations (via functor $\vdash: \mathcal{E}xpn \dashrightarrow \vdash\!\mathbf{Set}$);
- there are considered algebras over arbitrary carrier structures, not only over sets (via machinery of monads);
- given a logic, there are distinguished its algebras of expressions from its sets of propositions (via functor $\mathcal{P}$).

The functor $\mathcal{D}$ provides capturing in our framework a crucial for the Polish approach notion of matrix semantics.

Thus, while the Polish approach is suitable for algebraizing only propositional logics, the theory developed in the paper hopefully provides a unified framework for studying algebraizations of a whole diversity of logics both semantically and axiomatically defined.

Indeed, as for the former, one nontrivial result, theorem 3.9, is presented in the paper, while others concerning, eg, investigations of compactness, can be hopefully got along the same lines (for example, for the case of logical languages with $\mathcal{P}$=Id and $\mathbb{T}$ being an ordinary algebraic theory, ie, Fam$\mathbb{T}$=Set, a bunch of results on compactness of semantically defined logics was obtained in [Di92]).

As for axiomatically defined logics, the following can be said.

A thorough classification of these logics was developed in a series of works by Avron (see, eg, [Av91],[Av92]). His framework can be easily captured in our setting as follows. The main feature of Avron's considerations is to deal with consequence relations (CR) over various kinds of sequent-carrier set structures: single as well as multiply-conclusioned CRs, CRs over sets, over multisets, over sequences etc. In our setting this is provided by the functor $\mathcal{P}$. For example, if $\mathbb{T}$ is an algebraic theory over Set, then the ordinary single-conclusioned sequents corresponds to the case when $\mathcal{P}=\mathfrak{P}_{Horn}:=\mathfrak{P}_{\omega}\times \mathrm{Id}$, sequents over multisets - $\mathcal{P}=\mathfrak{P}_{Avron}:=\mathfrak{P}^{\bullet}_{\omega}\times\mathfrak{P}^{\bullet}_{\omega}$, sequents over sequences - $\mathfrak{P}=\mathfrak{P}_{Gentz}:=\mathfrak{P}^{*}_{\omega}\times\mathfrak{P}^{*}_{\omega}$ where $\mathfrak{P}^{\bullet}_{\omega}X$ and $\mathfrak{P}^{*}_{\omega}X$ are the sets of all finite multisets and all finite sequences over a set X, other cases are now obvious. At the same time, our framework makes it possible to handle substitutions in a very natural and easy way via composition of morphisms in the Kleisly category of the theory $\mathbb{T}$.

Moreover, we conjecture that $\mathfrak{P}_{Horn}$, $\mathfrak{P}_{Avron}$, $\mathfrak{P}_{Gentz}$ and similar functors give rise

to monads over Fam$\mathsf{T}$, thus providing one more direction of algebraizing logics (eg, transition from equational logic to Horn equational logic can be described as free generation of a $\mathfrak{P}_{Horn}$-algebra, or, more generally, transition, say, from the Hilbert-style version of a logic to its Gentzen-style version can be described by a suitable monad $\mathfrak{P}_{Gentz}$).

As for formal inferential systems for generating CRs, in our framework they are naturally modeled by corresponding formal sequents (inference rules) over free algebras in the variety Alg$\mathsf{T}$. In more detail, given a language $I=(\mathsf{T},\mathcal{P})$, an inference rule is a pair (Γ,φ) with $\Gamma\cup\{\varphi\}\subset\mathfrak{P}_{\omega}[\mathcal{P}T(\mathrm{Var})]$ where Var is a set of meta-variables, T(Var) is the carrier set structure of the T-algebra freely generated by Var. A Kleisly morphism from Var into a T-algebra A is nothing but a substitution of formulas from A for meta-variables from Var - this enables us to generate CR via inference rules.

Some results about the construction for the case of ordinary T and $\mathcal{P}$=Id were obtained in [Di92]. Proofs of their counterparts as well as counterparts of the above-mentioned compactness results in the general setting developed in the paper are open problems.

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Zinovijs Diskins. Kad semantiski definēta loģika ir algebrizējama ?

Anotācija. Algebriskās loģikas pamatā ir teoriju aizstāšana ar algebrām un modeļu aizstāšana ar homomorfismiem algebrās, kas izdalītas no semantikas. Raksta mērķis ir piedāvāt bāzi šīs paradigmas sistemātiskai pētīšanai vispārīgā gadījumā un īpaši dažādu algebrizāciju klasifikācijai un salīdzināšanai, kā arī lai iegūtu teorēmu, kas izskaidro, kad algebrizācija iepriekš minētajā nozīmē vispār ir iespējama.

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Abstract queries, schema transformations and algebraic theories: an application of categorical algebra to database theory

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Report on work in progress

Summary. The general intention of the paper is to demonstrate the naturality of using category theory language and machinery for building specifications in database theory. Concretely speaking, notion of a query over a given database schema *S* is discussed from the viewpoint of transformations of both the schema *S* and the query schema. For the sake of independence on data model, a formal definition of an abstract data model (a.d.m.) is introduced and all considerations are put into the abstract framework provided by this notion. There is shown naturality and handiness of formalizing intuitive notion of derived data through setting a closure operator (monad) on the category of all schemas pertaining to the data model in question. In particular, query schemas over a schema *S* turn out to be nothing but subschemas of the closure *Der(S)* of the schema *S* while *S*-instances can be treated as homomorphisms from *Der(S)* into schemas arising from semantics, which are closed 'from the very beginning', i.e. appear as algebras; then extensions of queries are nothing but images of the corresponding subschemas under these homomorhisms. All this constitutes the essence of algebraizabiity of an a.d.m. and it is shown that relational data models are algebraizable.

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The general intention of the paper is to demonstrate naturality of using category theory language for the specification part of database theory. Given this purpose, the paper is oriented more on definitions than theorems. Concretely speaking, the notion of a query over a given database schema S is discussed from the viewpoint of transformations of both the schema S and the query schema. For the sake of independence on data model, a formal definition of an abstract data model (a.d.m.) is introduced and all considerations are put into the abstract framework provided by this notion. This constitutes a more wide (than just discussing queries) context of the paper -- to suggest a certain mathematical framework for *abstract data modeling* under which we mean a unified way of reasoning on data modeling constructs (schema, instance, query etc.) being independent on (but applicable for) any (reasonable) concrete data model: relational, higher-order relational, extended ER etc. In fact, our notion of an a.d.m. explicates it as an institution-like structure -- a kind of categorical constructs introduced in computer science by Goguen and Burstall in order to support specification formalisms completely independent on underlying logics (see [GB92] for a survey and further references).

The intended audience of the paper is assumed to be an amalgamation of the specification-oriented part of the database theory community and the institution theory community.
From the viewpoint of database theory, the main technical novelty of the paper consists in introducing transformations of database schemas into the notion of a data model. Such an action enables us to organize the collection of all schemas accepted by a given data model into a category, and then use the language, methodology and machinery of category theory -- it is hoped this provides far-reaching conceptual consequences for database theory. We believe that similarly to that institutions provide a powerful unifying framework for handling algebraic specifications and specification languages, the categorical framework will be extremely useful in specification aspects of data modeling.

From the viewpoint of institution theory, the paper introduces two developments over the basic institution standard. The first one is not very essential and consists in defining and justifying a special kind of generalized institutions.

The second development over the basic standard is principal. It is commonly accepted to think that the institution framework is an immediate formal categorical refinement of the framework of abstract model theory. We assert, however, that this is not the case by virtue of the lack in the former of any concreteness facilities, i.e., facilities enabling us to speak about elements of models and relations. Indeed, ordinary models considered in model theory or database instances considered in database theory are not abstract algebraic structures but structures over domains or lists of domains, which enables us to speak about model's (instance's) elements (values). So, each such a model (instance) structure must be considered together with its list of underlying domains, in fact, there is a forgetful functor into the category of many-sorted sets, **SSet**, and, thus, in categorical terms, categories of models are actually concrete over **SSet** categories. Moreover, in database theory there is well recognized the necessity of involving domain independence into discussing queries, so, categories of models must be endowed with another underlying functor which assigns to a model its active domain, i.e., the set of elements (values) actually appearing in structural components of this model (for example, in relations the model consists of). So, just a notion of an institution endowed with such facilities can pretend to be a formal refinement of the intuition behind abstract model theory while ordinary institutions provide a framework rather for abstract algebraic semantics considerations.

At last but not least, the paper presents a notion of algebraizability of an a.d.m. which seems to be new for both database and institution theories (this notion is related to, but seriously differs from, the notion of chartering institutions introduced by Goguen and Burstall in [GB86]). The essence of algebraizability of a.d.m. amounts to the following.

An a.d.m., say, $\mathcal{M}$, is algebraizable if, given any $\mathcal{M}$-database schema S, database instances over a set of domains $d=(d_i, i=1,\dots,n)$ can be treated as morphisms from S into another schema, S_d, built from **d** by definite semantic tools determined by $\mathcal{M}$. In addition, all such schemas arising from semantics are, in fact, algebras (e.g., of relations) and instances can be extended to homomorphisms of algebras freely generated by schemas into these semantic algebras.

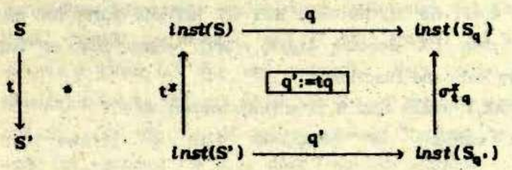
Thus, algebraizability supposes, first of all, availability of a closure operator over the collection of schemas which (freely) closes a schema up to an algebra and, secondly, a class of algebras, that is, closed schemas, arising from semantics and therefore endowed with some domain structure. (All this can be formulated in precise categorical terms of setting an algebraic theory (monad, triple) over the category of schemas). A formal definition capturing the idea of algebraizability is presented in section 4 under the name of *(algebraic) base*. Further, it is proved that any base $\mathcal{B}$ gives rise to an a.d.m., $\mathcal{M}(\mathcal{B})$, and an a.d.m. $\mathcal{M}$ is called *algebraizable* if $\mathcal{M}$ is isomorphic to $\mathcal{M}(\mathcal{B})$ for some base $\mathcal{B}$. We will show that relational data model is algebraizable and, hopefully, it will be easy to see how to extend this result and its proof for various higher-order relational data models. Moreover, we assert that various extended ER data models are also algebraizable (this point will be addressed in a forthcoming paper).

We do not use any advanced category theory tools, a very modest basis, for example, presented in the preliminary section 0 of [LS87] will be sufficient. Relevant to our purposes versions of some known, though not quite standard, notions are described in Appendix 2.

1 Motivating considerations

1.1 It is well known that for proper formulating one or another notion of a query (operation) a kind of genericity condition is required (see, e.g., [ABGG87]). This condition states that queries treat data values as uninterpreted objects and hence commute with permutations of values. However, it is natural to consider also "genericity with respect to schemas", that is, to require commutativity of queries with schema transformations. Indeed, given, for example, a relational schema S, query expressions over S in some query language are actually parameterized by relation-schemas so that transition from S to another schema S' forces the corresponding transformation of queries over S into queries over S'. Thus, query expressions are rather patterns for querying parameterized by schemas and respectively their semantic extensions are rather families of queries over different schemas than isolated query operations; certainly, these families are correlated between themselves via schema transformations in a definite way.

A bit more formally, let us fix a certain data model $\mathcal{M}$. This means, first of all, that there are defined notions of a *schema* (for representing data) and *an instance* (of a data structure or a database) over a schema. So, given $\mathcal{M}$, we have a collection of schemas, *Schema*, and, for each schema $S \in Schema$, a set of instances over S, *inst*(S). Furthermore, we assert that $\mathcal{M}$ must necessarily suppose the notion of a *mapping between schemas* called a *transformation* or *an interpretation* of schemas. In addition, if $t: S \longrightarrow S'$ is such a schema transformation then instances over S' can be also considered as instances over S, that is, there is a mapping t^*: *inst*(S') $\longrightarrow$ *inst*(S). In particular, if $t: S \longrightarrow S'$ is an inclusion, ie, S is a subschema of S', then, given $\iota' \in inst(S')$, $t^*(\iota')$,can be thought of as the restriction of ι' to S and the designation of $t^*(\iota')$ as $\iota'|_S$ will be relevant in such a case. Now, if q: *inst*(S) $\longrightarrow$ *inst*(S_q) is a query with the schema S_q,. and $t: S \longrightarrow S'$ is a schema transformation, then t translates q into a query q'=tq: *inst*(S') $\longrightarrow$ *inst*($S_{q'}$). In addition, S_q and $S_{q'}$ are related by means of a transformation σ_{tq}: $S_q \longrightarrow S_{q'}$ and, finally, commutativity of the following diagram expresses invariance of querying under change of notation:



1.2 Now we address to the problem of capturing the idea that queries only extract a part of derived information already implicitly contained in a database without any side effects. Leaving a full discussion till section 4, here we note only that if q, q_1 are queries over the same schema S and S_{q_1} is included into S_q, $S_{q_1} \hookrightarrow S_q$, then, for any $\iota \in inst(S)$, $(q\iota)|_{S_{q_1}} = q_1\iota$. For the special case when S_q=S and q is the identity mapping over *inst*(S), we obtain just the similar condition of [AK89].

Thus, we see, transformations of schemas indeed form a necessary facility of abstract data models, hence, any such a model should be based on not just a collection of (database) schemas but rather on a category of schemas, that is, a collection of schemas organized into a single whole by means of schema transformations. (Definition A.1 of Appendix 1 will help to clarify our intention).

2 Notation and terminology

Throughout the paper categories and functors are denoted by bold letters or abbreviations. Given a category **K**, we write $A \in K$, or, else, A ::**K**, and h: $A \longrightarrow B$ ::**K** to denote that A is an object of **K** and h is an arrow of **K** with the source A and the target B. We will also designate the source and the target of h as □h and h□ resp. If h□=□g, their composition is denoted by h;g.

SSet denotes the category of sorted sets: its objects are pairs (I,X) with I a set of sorts or indexes and X a family $(X_i, i \in I)$ of sets, its arrows are pairs (f,h) with f: $I \longrightarrow I'$ an ordinary function and $h=(h_i: X_i \longrightarrow X'_{f(i)} \mid i \in I)$ a family of functions. We write $\mathbf{X}=(I,X) \preceq \mathbf{Y} =(J,Y)$ iff $I \subseteq J$ and $X_i=Y_i$ for all $i \in I$; and $\mathbf{X} \subseteq \mathbf{Y}$ iff $I \subseteq J$ and $X_i \subseteq Y_i$ for all $i \in I$. If **X** ::**SSet** then $\bigcup \mathbf{X}$ denotes $\bigcup_{i \in I} X_i$ ::**Set**, where **Set** is the category of ordinary sets and functions.

Given sets X, Y, $X \subset_f Y$ means that X is a finite subset of Y.

$\mathbf{Cat}_{iso}$ is the category of categories with no arrows but isomorphisms.

All categories we consider in the paper are categories with inclusions and the corresponding image-factorization system, *if-categories*, in short (the precise definition appears in Appendix 2, however, an intuitive notion of inclusion of one object into another and the image of a subobject under a mapping will be sufficient for understanding the text). The fact that $A \longrightarrow B$ is an inclusion will be denoted by $A \hookrightarrow B$ since there is no more than one (if any) inclusion between objects; the very arrow will be also denoted by A. If **K** has an image-factorization system and h: $A \longrightarrow B$ is an arrow, then the image of h will be denoted by $Img(h) \hookrightarrow B$.

Sub$A$ denotes the set of all possible inclusions $X \hookrightarrow A$::**K**.

EpiK, **IsoK**, **IncK** denotes resp. classes of all epimorphisms, all isomorphisms and all inclusions of **K**.

$F \backslash G: K \overset{<---}{\underset{--->}{\bullet}} L$ denotes an adjunction F: K-->L, G: L-->K between categories **K** and **L** with F the left (lower) adjoint and G the right (upper) one.

3 Definition of an abstract data model

3.1 Definition. *A frame for data structuring* or simply *a frame* is defined to be a collection of the following components:

- An ii-category **Schema** of (database) *schemas* and *schema transformations*.
- A functor **inst**: $\mathbf{Schema} \longrightarrow \mathbf{Cat}^{op}_{iso}$ assigning to each schema S a category of *(database) instances over* S and their isomorphisms (to be thought of as those ones which are generated by permutations of underlying domains). Pairs (S,ι) with S∈*Schema* and ι∈*inst*(S) will be also called *instances* and the set of all instances in the latter sense will be denoted by *Inst*.
- *A domain structure over* **Inst**, **dmn=(dom,val)**, where **dom=(dom**$_S$,S∈*Schema*) and **val=(val**$_S$,S∈*Schema*), are families of functors, $\mathbf{dom}_S$: **inst(S)** --> **SSet** and $\mathbf{val}_S$: **inst(S)** --> **Set** resp., s.t. $\mathbf{val}_S(\iota) \subseteq \bigcup \mathbf{dom}_S(\iota)$ for any ι∈*inst*(S). Here, given an instance (S,ι), $\mathbf{dom}_S(\iota)$ must be thought of as the list of domains underlying ι, whereas $\mathbf{val}_S(\iota)$ - as the set of values actually involved by ι. We will often omit the subindex s if it is clear from the context.

Moreover, if t: S $\longrightarrow$ S' ::**Schema** then the following condition holds for all ι'∈*inst*(S'):

(ext) $\mathbf{dom}(t^*\iota') \leq \mathbf{dom}(\iota')$, $\mathbf{val}(t^*\iota') = \mathbf{val}(\iota') \cap \bigcup \mathbf{dom}(t^*\iota')$.

This condition explicates the intuitive idea that t^* produces only another structural view on that part of data contained in ι' which is captured by t without affecting data themselves, in other words, t^* changes the structure of that part but not the very values.

Now we turn to presentation of the notion of a query through our formalism.

Let $\mathcal{F}$=(**Schema**,**Inst**,**dmn**) be a frame. By adapting and generalizing to our context the definition of a deterministic query from [AK89] we come to the items (i) and (ii) of the definition following below. Item (iii) is intended to formalize the intuitive idea that queries are rather patterns for quering than isolated operations.

3.3 Definition. Let S be a schema, S∈*Schema*.

(i) *A (deterministic) query operation* or simply a *query over* S is defined to be a pair $q=(S_q,f_q)$ with S_q a *schema of* the query and f_q a functor $\mathbf{Inst}(S) \longrightarrow \mathbf{Inst}(S_q)$ s.t. the following condition holds for all $\iota \in inst(S)$:

(qr1) $\mathbf{dom}(q\iota) \le \mathbf{dom}(\iota)$, $\mathbf{val}(q\iota) \subseteq \mathbf{val}(\iota) \cap \bigcup\mathbf{dom}(q\iota)$.

here and furtheron q also denotes the very function f_q.

(ii) *A query system over* S is defined to be a set *qr*(S) of queries over S satisfying the following two conditions for any q1,q2∈*qr*:

(qr2) $id_{inst(S)} \in qr(S)$,

(qr3) if $S_{q1} \hookrightarrow S_{q2}$ then $q2(\iota)|_{S_{q1}} = q1(\iota)$ for all $\iota \in inst(S)$.

(We recall that if t: S ⟶ S' ::**Schema** is an inclusion and $\iota \in inst(S')$ then $t^*\iota$ is denoted by $_S\iota|$).

(iii) *A query hypersystem (over a frame* 𝔉*)* is defined to be a functor **qr**: **Schema** ⟶ **Set** assigning to each schema S a query system over S, in addition, with any schema transformation t: S1 --> S2 and q∈*qr*(S1) there is correlated a schema transformation σ_{tq}: $S_q \longrightarrow S_{tq}$ (here and furtheron tq denotes (**qr**t)q) s.t. the following diagram commutes for any q,q' ∈ *qr*(S1):

(qr4)
$$\begin{array}{ccccc} S1 & & S_q & \hookrightarrow & S_{q'} \; ::qr(S1) \\ \downarrow t & & \downarrow \sigma_{tq} & & \downarrow \sigma_{tq'} \\ S2 & & S_{tq} & \hookrightarrow & S_{tq'} \; ::qr(S2) \end{array}$$

Moreover, the following diagram also commutes for any transformation t:S1⟶S2 and query q ∈ *qr*(S1):

(qr5)
$$\begin{array}{ccccc} S1 & & inst(S1) & \xrightarrow{\;q\;} & inst(S_q) \\ \downarrow t & & \uparrow t^* & & \uparrow \sigma^*_{tq} \\ S2 & & inst(S2) & \xrightarrow{\;tq\;} & inst(S_{tq}) \end{array}$$

3.4 Remarks.

(i). Functoriality in the item (i) above means commutativity with isomorphisms of instances, that is, the well-known genericity condition.

(ii). Note, condition (qr5) together with condition (ext) of definition 3.1 imply that **val** $q(t^*\iota)$ = **val** $(tq)\iota$ for all $\iota \in inst(S2)$, that is, the extension of a given query does not depend on notation.

3.5 Definition. *An abstract (static) data model* is defined to be a couple of a frame $\mathfrak{F}$ and a query hypersystem (qr,σ) over $\mathfrak{F}$.

4 Algebraizing data models

In the definition of an a.d.m. instances were treated quite abstractly as some entities functorially connected with schemas but without specifying their nature. However, in all data models using in practice, instances (for example, of a relational schema S) are mappings which assign semantic meanings (relations) to the structural components of the schema (names of relations) in correspondence with their structural characteristics (relation-schemes). This suggests the idea of treating an instance ι of a schema S as a schema transformation of S into the corresponding schema arising from semantics, $\iota: S \longrightarrow S_d$, where **d** refers to the list of domains underlying the schema S_d. (Construction A-1.2(i) from Appendix 1 will help to clarify this idea). The image of S under the transformation can be considered as data being kept in the database.

Furthermore, from these data new derived data can be extracted, in addition, derived data appear as instances of other (query) schemas somehow related to S. Reflecting on this point suggests the idea to supply the category **Schema** with an operator **Der** which assigns to each schema S a very large schema **DerS** being thought of as the schema specifying the total combination of all possible derived data which can be extracted from ι, in fact, from $Img_{\iota}(S)$. That is, ι is assumed to be naturally extendible to $\bar{\iota}$: **DerS** $\rightarrow$ S_d so that just the image $Img_{\bar{\iota}}$(**DerS**) presents the totality of derived data. (Again, construction A-1.2(ii) will clarify these considerations). Now, a query Q appears as a subschema S_Q of **DerS**, $S_Q \hookrightarrow$**DerS**, and the answer is nothing else than $Img_{\bar{\iota}}(S_Q)$.

In precise categorical terms, these considerations mean that **Der** is a regular algebraic theory over **Schema** and S_d is a **Der**-algebra, $S_d \in Alg^{\mathbf{Der}}$. In the following definition we treat algebraic theories over a category in a equivalent way through adjoint functors.

4.1 Definition. *An algebraic base of a data model* or simply a *base* is defined to be a collection of the following data.

- A regular algebraic theory T, that is, an adjunction, F\G: $\mathbf{Schema}_{\infty}$ $\overset{\longleftarrow}{\underset{\longrightarrow}{\bullet}}$ **AlgT**), where elements of the ii-category $\mathbf{Schema}_{\infty}$ are to be thought of as *possible schemas* and their *transformations* and those of **AlgT** - as (*accepted*) *algebras* and their *homomorphisms*. The composition **Der** = F;G will be thought of as a *derived data operator* on $\mathbf{Schema}_{\infty}$.
- A full subcategory of $\mathbf{Schema}_{\infty}$ consisting of *accepted* schemas, **Schema**, which is closed under subobjects.
- A subcategory **Sem** of **AlgT** consisting of *semantic* algebras. The objects of **Sem** are to be thought of as algebras arising from semantics and arrows - as algebra isomorphisms generated by permutations of underlying domains. (Warning: as it was demonstrated by Shelah [She88], there are relational algebra isomorphisms between non-isomorphic data structures, that is, generally speaking, categories **Sem** can be non-fulll).
- *A domain structure over* **Sem**, **dmn**=(**dom**,**val**), where **dom**: **Sem**$\rightarrow$ **SSet** and **val**: **Sem** $\longrightarrow$ **Set** are functors s.t. **val**(A) $\subseteq$ $\bigcup$**dom**(A) for any $A \in$**Sem**. In addition, for any two $A,B \in$**Sem** s.t. G$A \subset$GB ::**Schema**, we have **dom**$A \subset$**dom**B and **val**$A \subset$**val**B. Here, given an algebra A, **dom**(A) must be thought of as the list of domains carrying A whereas **val**(A) - as the set of values

actually involved by A. ∎

Considerations before the definition suggest an evident construction which provides the following

4.2 Theorem. Each base $\mathcal{B}$ gives rise to an abstract data model $(\mathcal{F}(\mathcal{B}), qr(\mathcal{B}))$. ∎

Finally, considerations of Appendix 1 shows that any relational data model is generated by the corresponding algebraic base, that is, is algebraizable. Hopefully, it is easy to see that this is the case also for various higher-order relational schemas, thus,

4.3 Theorem. All kinds of relational data models are algebraizable, i.e. can be presented as $(\mathcal{F}[\mathcal{B}], qr[\mathcal{B}])$ for some algebraic base $\mathcal{B}$.

Appendix 1. Category of relational schemas. Relational algebras

A-1.1 Definition Let $\mathcal{U}$ be an arbitrary but fixed countable set of *attribute names.*

(i). *A (finitary) relational schema (over $\mathcal{U}$)* is defined to be a triple $S=(\mathcal{DOM}, dom, rel)$ with $\mathcal{DOM}$ a finite set of *domain names*, *dom* a function $\mathcal{U} \longrightarrow \mathcal{DOM}$ s.t. $dom^{-1}(D)$ is countable for any $D \in \mathcal{DOM}$ and *rel* a function taking each finite subset X of $\mathcal{U}$ to a finite set (possibly, empty!) of *relation names in scheme X*, $rel(X)$. (Well, in the standard definition of a relation scheme $rel(X)$ is either empty or a singleton). In addition, the set $\{X \subset_f \mathcal{U}: rel(X) \neq \emptyset\}$ is finite too.

(ii). *A transformation* of relational schemas, $S \longrightarrow S'$, is defined to be a triple $t=(t_d, t_a, t_r)$ consisting of a mapping t_d: $\mathcal{DOM} \longrightarrow \mathcal{DOM}'$, a bijection t_a: $\mathcal{U} \to \mathcal{U}$ s.t. $dom t_a(A) = t_d(dom A)$ for all $A \in \mathcal{U}$, and a family of mappings t_{rX}: $rel X \longrightarrow rel' t_a(X)$ indexed by finite subsets X of $\mathcal{U}$.

(iii). Composition of transformations and identity transformation are defined in a evident way, this constitutes *the category of (finitary) flat relational schemas and transformations*, $\mathbf{Rel}_f$.

A-1.2 Construction. (i). Given a finite set of domains, $d=(d_1,\ldots,d_n)$, we fix arbitrary partition of $\mathcal{U}$ by a mapping $\#:\mathcal{U} \longrightarrow d$ s.t. $\#^{-1}(d_i)$ is countable for all $i=1...n$ and then build the following (infinitary!) relational schema $S[d,\#]=(\mathcal{DOM}\#, dom\#, rel\#)$: $\mathcal{DOM}\#=d$, $dom\#=\#$, $rel\#(X) = rel_X d := Power_{fin}\{f: X \longrightarrow \cup d \mid f(A) \in \#A$ for all $A \in X\}$, $X \subset_f \mathcal{U}$. Such a schema satisfies the definition above except for requirements of finiteness of $rel(X)$ and finiteness of the set $\{X\subset_f \mathcal{U}: rel(X)\neq\emptyset\}$. If we remove these two requirements from the definition of $\mathbf{Rel}_f$, we get the definition of the category of *infinitary relational schemas and transformations*, $\mathbf{Rel}_\infty$, s.t. $\mathbf{Rel}_f$ is a full subcategory of $\mathbf{Rel}_\infty$. Now it is easy to see that a relational database instance ι over a schema $S=(\mathcal{DOM}, dom, rel)$ is nothing but the schema transformation $t=(t_d,t_a,t_r)$: $S \longrightarrow S[d,\#]$ with $\#A = \iota(A)$, the ι-domain of A, for all $A\in\mathcal{U}$, $t_d(D) = \iota(A)$ for any, hence all, $A\in\mathcal{U}$ s.t. $domA=D$, t_a the identity mapping of $\mathcal{U}$, and $t_{rX}(R) = \iota(R)$ for all $R\in rel(X)$, $X\subset_f \mathcal{U}$. Conversely, any such a transformation is an instance and, thus, we have an isomorphism $inst(S)\cong\bigcup\{\mathbf{Rel}_\infty(S,S')\text{: } S'\in Sem\}$ where *Sem* denotes the class of schemas of the form $S[d,\#]$ for various d and #.

(ii). Actually there are several relational data models depending on what a collection of operations over relations is accepted. On the other hand, fixing such a collection T determines the corresponding class of many-sorted algebras, we will say T-*algebras*, and any schema from the class *Sem* can be considered as a T-algebra. In addition, accepted in the data model in question instances of a schema S can be considered as homomorphisms of T-algebras of the kind $\bar{S} \longrightarrow S[d,\#]$ for various d and #, where $\bar{S}$ denotes the T-algebra freely generated by S. (The latter means that $\bar{S}$ is the labeled tree (whose nodes are relation names together with their schemes) generated from relation names occuring in S by means of symbols of T-operations, e.g. from leaves $R1\in rel(X)$ and $R2\in rel(Y)$ by means of the symbol $\bullet$ of join-operation we come to the node $R1\bullet R2$ with the scheme $X\cup Y$). Note, $\bar{S}$ is always an infinitary schema even though S is finitary.

(iii). It can be shown that a similar machinery is valid for complex relational data models and various kinds of graph-oriented, in particular, ER and extended ER data models. Thus, as it was already said, the majority of ex citing data models suppose availability of a closure operator over the category of schemas. To capture this construct in a formal way the notion of an algebraic theory (triple, monad) is just suitable.

Appendix 2. Image factorization systems for categories with inclusions

A-2.1 Definition. *A category with inclusions* is defined to be a pair (K,Inc) with K a category and Inc a class of K-monomorphisms called *inclusions* s.t. the following holds (Inc(*A*,*B*) denotes the set {i∈K(*A*,*B*): i∈Inc}):

(inc1) Inc(*A*,*B*) is either empty or a singleton, in the latter case we write i:*A*↪*B* ::K or simply *A*↪*B*;

(inc2) *A*=*B* as soon as *A*↪*B* and *B*↪*A*;

(inc3) Inc is closed under identity arrows, arrow composition and pullbacks along arbitrary maps (in other words, Inc is a *dominion* (see, eg, Moggi [Mo89]) over K).

(inc4) for any object *A*, the class Sub*A*={*X*↪*A*: *X*∈K} is a set (consisting of the *subobjects* of *A*).

We will often ambiguously designate the source of an inclusion morphism and the very morphism by one and the same letter, namely, the letter denoting the source.

A category with inclusions is said to possess *canonical image factorizations* if the closure of Inc under isomorphisms, Inc*, is the mono-component of a factorization system (Cov,Inc*) over K (about the latter see, eg, Barr [Ba91]); arrows from Cov will be called *covers*. In particular, this means that each h: *A* --> *B* ::K has a unique factorization h=c;i with c∈CovK⊆EpiK, i∈IncK⊆MonoK, thusly, with each morphism h: *A* --> *B* there are correlated its *image*, the inclusion *Img*(h)↪*B*, and its *kernel*, the cover c=*ker*h: *A* --->> *Img*(h).

An ii-category is a category with inclusions and corresponding canonical image factorizations. In any such a category there is a mapping Img_h: Sub*A* ⟶ Sub*B* defined by setting $Img_h X = Img(X;h)$.

A regular algebraic theory consists of an underlying ii-category K and a monad T=(*T*,μ,η) over K (see, e.g., [LS87] or [BW85]) which preserves image factorizations, that is, if c∈CovK, i∈IncK then *T*c∈CovK, *T*i∈IncK too.

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Zinovijs Diskins, Boriss Kadišs. **Abstraktie pieprasījumi, shēmu pārveidojumi un algebriskas teorijas: Kategoriju algebras pielietojums datubāzu teorijā.**

Anotācija. Rakstā parādīts kategoriju teorijas valodas lietošanas dabiskums datubāzu teorijas specifikāciju daļai. Pieprasījumi par doto datubāzu shēmu *S* tiek aplūkoti no shēmas *S* un pieprasījumu shēmas viedokļa. Lai nodrošinātu datu modeļa neatkarību, tiek ievesta *abstrakta datu modeļa (a.d.m.)* definīcija un turpmākie spriedumi balstās uz šo jēdzienu. Tiek parādīts intuitīvā *atvasināto datu* jēdziena formalizēšanas dabiskums un ērtums, ievedot slēguma operatoru (monādi) to shēmu kategorijā, kuras attiecas uz aplūkojamo datu modeli.

Piemēram, pieprasījumi pār shēmu *S*, izrādās, ir tikai shēmas *S* slēguma *Der(S)* apakšshēmas, bet *S*-instances var aplūkot kā homomorfismus no *Der(S)* uz shēmām, kas rodas semantikā. Tad pieprasījumu paplašinājumi nav nekas cits kā atbilstošie apakšshēmu attēli. Tas veido a.d.m. algebrizējamības būtību un tiek parādīts, ka relāciju datu modeļi ir algebrizējami.

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ON SEMIGROUPS OF OPEN TRANSFORMATIONS OF SYSTEMS OF SUBSETS.

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__Abstract__. The problem of determinability (up to homeomorphism) of a system of subsets by means of its semigroup of open transformations within different classes Γ is discussed. In the capacity of Γ are considered classes of chains, partitions, uniform quasiretract systems and generalized T_1- spaces.

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Many specialists investigated the relations between certain mathematical structures and algebraic objects associated with these structures. One of the main problems in this direction is the problem of determinability of an initial object by means of its derivative algebraic object. We mention the well-known result of the type - the fundamental theorem of Gelfand-Kolmogorov, according to which two compact topological spaces X and Y are homemorphic iff the rings of continuous functions of these spaces are isomorphic.

During the last decades transformation semigroups have been considered actively as derivative objects of different mathematical objects. In particular, the problem of determinability of a topological spaces X by semigroups of its continuous (and of other types) transformations was repeteadely studied (see e.g. [3, 5, 8], e.a.). The problem studied in this paper is to reveal the possibilities of a similar algebraic determinability of some systems of subsets by means of the semigroups of their open transformations. From this point of view we consider some classes of systems of subsets (such as chains, partitions, uniform quasiretract as well as generalizations of topological spaces). If our proof is a straightforward modification of the known proof of the similar result for topological spaces, then we state the result only. Such a situation is for the considered generalizations of the topological spaces. However, as a rule, the considered semigroups don't contain constant transformations in other cases and therefore a technique of proofs essentially differs from that successfully used for the topological spaces in the related investigations.

By a system of subsets we call a pair (X,τ), where X is a set, and τ is a subset of the $\mathcal{P}_X$, where $\mathcal{P}_X$ is the power set of X. (In the survey [8] Vechtomov calls such pairs by generalized topological spaces).

It is known that various mathematical structures on a set X are given by specifying of a subset τ of $\mathcal{P}_X$, where τ satisfies some axioms. Such is topologies. Other examples are filters, matroids, subset algebras, closure systems, e.c.).

Let SO be the category the objects of which are systems of subsets and the morphisms are the mapping $\varphi: X_1 \to X_2$ such that $\varphi(M) \in \tau_2$ for each $M \in \tau_1$. Morphisms (isomorphisms) in SO we call by open mappings (homeomorphisms) of systems of subsets. Two systems of subsets are called homeomorphic if they are isomorphic objects of the category SO. In the literature other terminology is used as well: homeomorphic systems call by equivalent or isomorphic (e.g. [6,9]).

Suppose that $T_c(T_0)$ is the complete subcategory of SO, elements of which are pairs (X,τ) where τ is a closed (resp. open) topology on X. It is evident, that the category $T_c(T_0)$ is in fact the category of topological spaces with closed (open) mappings.

For a system of subsets (X,τ) let $O(X,\tau)$ denote the semigroup of all its open transformations under the operation of multiplication of mappings.

Let Γ be a class of systems of subsets. We say that a system $(X,\tau) \in \Gamma$ is determined (up to homeomorphism) by the semigroup $O(X,\tau)$ within the class Γ if any isomorphism of semigroups $O(X,\tau)$ and $O(X_1,\tau_1)$ implies a homeomorphism of the systems (X,τ) and (X_1,τ_1) for every $(X_1,\tau_1) \in \Gamma$.

For any subset τ of $\mathcal{P}_X$, let $\tau^0 = \tau \cup \{\varnothing\}$, if τ does not contain empty subset, and let $\tau^{\circ}=\tau$, if $\emptyset\in\tau$. It is clear, that semigroups $O(X,\tau)$ and $O(X,\tau^{\circ})$ are equal. Therefore the following definition will turn out to be quite convenient.

Let Γ be a class of systems of subsets. We say that a system $(X,\tau)\in\Gamma$ is O-determined by the semigroup $O(X,\tau)$ within the class Γ, if for every $(X_1,\tau_1)\in\Gamma$ any isomorphism of the semigroups $O(X,\tau)$ and $O(X_1,\tau_1)$ implies a homeomorphism of the systems (X,τ°) and (X_1,τ_1°).

The difference between the determinability and the O-determinability is not very essential, as it is seen from the definitions. In particular, if Γ is a class of systems with $\emptyset\in\tau$ for all $(X,\tau)\in\Gamma$, then the notions of O-determinability and determinability are equivalent within the class Γ.

Throughout the remainder of the paper, the only considered systems of subsets are (X,τ) with $X\in\tau$. We use this remark without explicit mention.

By analogy with the topological terminology, a subset $M\subset X$ is called a quasiretract (resp. retract) of a system (X,τ), if $M=\varphi(X)$ for some $\varphi\in O(X,\tau)$ (resp. $\varphi\in O(X,\tau)$ and $\varphi^2=\varphi$). A system of subsets (X,τ) we call by quasiretract (resp. retract) if every nonempty subset $M\in\tau$ is a quasiretract (resp. retract) of the system (X,τ). For example, if $\overset{\circ}{\tau}$ is a closed T_1- topology on a set X, then the topological space (X,τ) is a retract system of subsets.

We say that a system of subsets (X,τ) is uniform if $(M-\{\alpha\})\cup\beta\in\tau$ for any $\emptyset\neq M\in\tau$, $\alpha\in M$, $\beta\in X$. One may easily verify, that if τ is a free filter (i.e. a filter with empty kernel) on a set X then the system of subsets (X,τ) is a uniform quasiretract system.

Theorem 1. Let Γ be a class of the uniform quasiretract systems of subsets. Every system $(X,\tau)\in\Gamma$ is O-determined by the semigroup $O(X,\tau)$ within the class Γ.

Proof. Let (X,τ) be an uniform, quasiretract system of subsets.

For distinct elements $\alpha,\beta\in X$ we define the transformation $\varphi_{\beta\alpha}$ as follows:

$$\varphi_{\beta\alpha}(\alpha)=\beta,\ \varphi_{\beta\alpha}(\gamma)=\gamma \text{ for } \gamma\in X,\gamma\neq\alpha.$$

Let F_X denotes the set of all transformations $\varphi_{\beta\alpha}$, where $\alpha,\beta\in X$. Each $\varphi_{\beta\alpha}\in F_X$ is an open transformation of the system of subsets (X,τ). Really, for every $M\in\tau$ we have

$$\varphi_{\beta\alpha}(M) = \begin{cases} (M-\{\alpha\})\cup\{\beta\} \text{ if } \alpha\in M \\ M \text{ otherwise.} \end{cases}$$

Since (X,τ) is a uniform system, it follows that $\varphi_{\beta\alpha}(M)\in\tau$ and hence $\varphi_{\beta\alpha}\in O(X,\tau)$. Besides , $\varphi_{\beta\alpha}$ is an idempotent of the semigroup $O(X,\tau)$.

The following remark follows from lemma 2.8 [4]. Two elements $\varphi_{\beta_1\alpha_1}$, $\varphi_{\beta_2\alpha_2}\in F_X$ are $\mathcal{R}$-equivalent elements of the semigroup $O(X,\tau)$ if and only if $\alpha_1=\alpha_2$. ($\mathcal{R}$ is the Green's equivalence). For every element $\alpha\in X$ we denote

$$K_\alpha = \{\varphi_{\beta\alpha} | \beta\in X\}.$$

The preceding observation yields that K_α is a class of $\mathcal{R}$- equivalent elements belonging to the set F_X. Moreover,

$$F_X/\rho = \{K_\alpha | \alpha\in X\}$$

where F_X/ρ denote the quotient of the set F_X by the equivalence relation induced in F_X by the relation $\mathcal{R}$.

The subset F_X of the semigroup $O(X,\tau)$ may be characterized by means of a first order formula in the language of semigroups.

Namely, a transformation $f\in O(X,\tau)$ belongs to the set F_X if and only if f satisfies the following two conditions:

(1) f is an idempotent and f is not the identity element of the semigroup $O(X,\tau)$

(2) f has no two-sided identities in $O(X,\tau)$ besides f and the identity of $O(X,\tau)$.

Necessity follows from 2.5 [ ]. Conversely, assume that $f\in O(X,\tau)$ has the properties (1), (2). Then $f(\alpha)=\beta\neq\alpha$ for some $\alpha\in X$. Since f is an idempotent it follows that $f(\beta)=\beta$, $\alpha\notin f$.Then it is easy to show that $f\varphi_{\beta\alpha}=\varphi_{\beta\alpha}f=f$, i.e. $\varphi_{\beta\alpha}$ is an identity of f. Besides, $\varphi_{\beta\alpha}\in O(X,\tau)$. Hence, by the condition (2), $f=\varphi_{\beta\alpha}\in F_X$.

Assume now that (X_1,τ_1) and (X_2,τ_2) are uniform quasiretract systems of subsets with the isomorphic semigroups $O(X_1,\tau_1)$ and $O(X_2,\tau_2)$ and let $\chi:O(X_1,\tau_1)\to O(X_2,\tau_2)$ be the corresponding isomorphism. Since the subset F_X may be defined within the semigroup $O(X\ ,\tau\ )$ by the formula of first order predicate calculus, it follows that $\chi(F_{X_1})=F_{X_2}$.

It is evident, that $f_1\mathscr{R}f_2$ in the semigroup $O(X_1,\tau_1)$ if and only if $\chi(f_1)\mathscr{R}\,\chi(f_2)$ in the $O(X_2,\tau_2)$. The foregoing allows us to define a mapping $\bar{\chi}$ of $F_{X_1}/\mathscr{R}$ in $F_{X_2}/\mathscr{R}$ by letting $\bar{\chi}(K_{\alpha_1})=$ $=K_{\alpha_2}$ $(\alpha_1\in X_1,\ \alpha_2\in X_2)$ if and only if the image $\chi(K_{\alpha_1})$ coincides with K_{α_2}. It is easy to notice that $\bar{\chi}$ is a bijection of $F_{X_1}/\mathscr{R}$ upon $F_{X_2}/\mathscr{R}$

Let $f_i\colon X_i\to F_{X_i}/\mathscr{R}$, $i=1,2$ be the natural bijection defined by the equality $f_i(x_i)=K_{x_i}$ $(x_i\in X_i, i=1,2)$.

Then the mapping $\psi=f_2^{-1}\,\bar{\chi}\,f_1$ is a bijection of X_1 upon X_2. In the sequel we write $\psi(x)=\bar{x}$ for $x\in X_1$ and $\psi(M)=\bar{M}$ for $M\subset X_1$. We remark that $\chi(K_x)=K_{\bar{x}}$ for every $K_x\subset F_{X_1}$. Really, for any $K_x\in F_{X_1}/\mathscr{R}$.

$$\bar{\chi}(K_x)=f_2\psi f_1^{-1}(K_x)=f_2\psi(x)=f_2(\bar{x})=K_{\bar{x}}$$

and then, by the definition of $\bar{\chi}$, it follows that $\chi(K_x)=K_{\bar{x}}$ for $K_x\subset F_{X_1}$.

Let M be a nonempty element of τ_1. Since (X_1,τ_1) is a quasiretract system there exists $\varphi\in O(X_1,\tau_1)$ with $\varphi(X_1)=M$. It is easy to verify that $x\in X_1$ belongs to $\varphi(X_1)$ if and only if $K_x\varphi\neq\varphi$.

Assume now that x belongs to the set M. Then $K_x\varphi\neq\varphi$ and hence

$$\chi(\varphi)\neq\chi(K_x\varphi)=\chi(K_x)\chi(\varphi)=K_{\bar{x}}\chi(\varphi).$$

This implies $\bar{x}\in\chi(\varphi)(X_2)$, i.e. $\bar{M}\subset\chi(\varphi)(X_2)$. Similarly we may verify the converse inclusion. Thus,

$$\psi(M)=\chi(\varphi)(X_2)\in\tau_2.$$

Analogously can be proved that $\psi^{-1}(N)\in\tau_1$ for every nonempty $N\in\tau_2$.

We conclude that ψ is the homeomorphism of the systems of subsets (X_1,τ_1^0) and (X_2,τ_2^0), as needed.

<u>Corollary</u>. Let Γ be a class of the uniform quasiretract systems of subsets (X,τ) with $\emptyset\notin\tau$. Every system $(X,\tau)\in\Gamma$ is determined by the semigroup $O(X,\tau)$ within Γ.

In order to prove theorems 2,3 below, we need some additional notations and some properties of retract systems.

A subset τ of a power set $\mathscr{P}_X$ one may consider as a partially ordered set with respect to inclusion. Let $(\tau,\subset)$ denote the corresponding partially ordered set.

Lemma 1. [7] Let (X_1,τ) and (X_2,τ_2) be retract systems of subsets. If the semigroups $O(X_1,\tau_1)$ and $O(X_2,\tau_2)$ are isomorphic then the partially ordered sets $(\tau_1,\subset)$ and $(\tau_2,\subset)$ are isomorphic.

For any semigroup S, let I be some set of idempotents of S together with the binary relation σ defined by

$$(\forall a,b\in I)(a\sigma b \leftrightarrow ba=a).$$

We put $\rho=\sigma\cap\sigma^{-1}$. It is easy to see that the relation σ is a quasiorder and ρ is an equivalence relation in I. The quasiorder σ naturally induces a partial order relation in the set of all ρ-classes I/ρ. This partial order relation we denote by $\bar{\sigma}$. The set I/ρ endowed with the relation $\bar{\sigma}$ we denote by $(I/\rho,\bar{\sigma})$.

Let (X,τ) be a system of subsets and $Id(X,\tau)$ be the set of all idempotents of the semigroup $O(X,\tau)$. For every $M\in\tau$ we set

$$I_M=\{\varphi\in Id(X,\tau)\,|\,\varphi(X)=M\} \qquad (1)$$

It is easy to prove that a nonempty I_M is a class of ρ- equivalent elements of the set $Id(X,\tau)$. Besides, we have

Lemma 2. Let (X,τ) be a retract system of subsets. Then

$$Id(X,\tau)/\rho=\{I_M\,|\,M\in\tau,\ M\neq\varnothing\}.$$

The following statement was proved essentially in [7], but was not stated explicitely.

Lemma 3.Let (X,τ) be a retract system of subsets. Then the mapping $\pi:\tau\longrightarrow Id(X,\tau)/\rho$ defined by $\pi(M)=I_M$ for each $M\in\tau$ is an isomorphism of the ordered sets $(\tau,\subset)$ and $(Id(X,\tau)/\rho,\bar{\sigma})$.

Theorem 2. Let Γ be a class of systems of subsets (X,τ) where $\tau\setminus\{X\}$ is a partition of the set X. Every system $(X,\tau)\in\Gamma$ such that all components of the partition $\tau\setminus\{X\}$ are of equal cardinality is determined by the semigroup $O(X,\tau)$ within the class Γ.

Proof. Assume that (X,τ) is a system of subsets where $\tau\setminus\{X\}$ is a partition of X. Let A be an element of the set τ of minimal cardinality. For the subset A, construct a transformation $\varphi:X\to X$ as follovs: $\varphi(M)=A$ for all $M\in\tau$, and the restriction φ to A is the identity. Obviously, such transformation φ exists and $\varphi \in Id(X,\tau)$. Hence A is a retract of the system (X,τ). Besides, if $M \in \tau$ and $|A| < |M|$ then M is not even a quasiretract of (X,τ).

Let us denote by $\mathfrak{A}_X$ the set of all transformations $f \in O(X,\tau)$ such that $fg=g$ for all non-identity idempotents $g \in Id(X,\tau)$. It is evident that $\mathfrak{A}_X$ is just the set of all open transformations of (X,τ), whose restrictions to all retracts $M \neq X$ of (X,τ) are the identities.

Note that $|\mathfrak{A}_X|=1$ if and only if the system (X,τ) is such that all elements of $\tau\setminus\{X\}$ are of equal cardinality. Really, sufficiency follows immediately from the descriptions of the retracts of (X,τ) and the set $\mathfrak{A}_X$. Conversely, assume that there exist $B,C \in \tau\setminus\{X\}$ with $|B|<|C|$. Then C is not retract. It is clear that a transformation $f: X \to X$, for which $f(\alpha)=\alpha (\alpha \in X\setminus C)$ and the restriction f to C is a bijection, belongs to $\mathfrak{A}_X$. Hence $|\mathfrak{A}_X|>1$.

Assume now that systems of subsets (X_1,τ_1) and (X_2,τ_2) belong to the class Γ mentioned in theorem 2. Let the semigroups $O(X_1,\tau_1)$ and $O(X_2,\tau_2)$ be isomorphic and let $\chi: O(X_1,\tau_1) \to O(X_2,\tau_2)$ be the corresponding isomorphism.

Besides, assume that all elements of $\tau_1\setminus\{X_1\}$ are of equal cardinality. Then $|\mathfrak{A}_{X_1}|=1$. On the other hand, the subset $\mathfrak{A}_X$ of $O(X,\tau)$ is defined (essentially) by a first order formula in the language of semigroups. It follows that $\chi(\mathfrak{A}_{X_1})= \mathfrak{A}_{X_2}$ Hence $|\mathfrak{A}_{X_2}|=1.$ This implies, from what has been shown, that all elements of $\tau_2\setminus\{X_2\}$ are of equal cardinality. For such a system of subsets every element is a retract.

Since the systems (X_1,τ_1) and (X_2,τ_2) are retract systems, it follows from lemma 1 that $|\tau_1|=|\tau_2|$.

Using lemma 2 and by the definition of ρ, since χ is an isomorphism, we obtain that for any $M_1 \in \tau_1\setminus\{X_1\}$ there exists $M_2 \in \tau_2\setminus\{X_2\}$ such that

$$\chi(Id(X_1,\tau_1)\setminus I_{M_1}) = Id(X_2,\tau)\setminus I_{M_2}. \qquad (2)$$

For each $M \in \tau\setminus\{X\}$ with $I_M \neq \emptyset$, let G_M be the set of all invertibe elements g of the semigroup $O(X,\tau)$ such that $gf= f$ for all $f \in Id(X,\tau)\setminus I_M$. It is easy to notice that a transformation $g:X \to X$ belongs to G_M if and only if g is the identical mapping when restricted to $X\setminus M$ and it is a bijection when restricted to M. Hence G_M is of equal cardinality with the symmetric group on M.

Assume that $M_1 \in \tau_1$ and $M_2 \in \tau_2$ are selected as in (2). According to (2) and the definition of G_M we obtain $\chi(G_{M_1})=G_{M_2}$. Hence $|M_1|=|M_2|$.

Taking into account the proved above, we may conclude now that the systems of subsets (X_1,τ_1) and (X_2,τ_2) are homemorphic.

Theorem 3. Let Γ be a class of systems of subsets (X,τ) where the partially ordered set $(\tau,\subset)$ is a chain. Every system of subsets $(X,\tau) \in \Gamma$, with τ finite, is O-determined by the semigroup $O(X,\tau)$ within the class Γ.

Proof. Let (X,τ) be a system of subsets where $(\tau,\subset)$ is a chain. For any nonempty subset $M \in \tau$ define a transformation $f:X \to X$ by setting $f(x)=x$ $(x \in M)$, $f(x) \in M$ $(x \in X\setminus M)$. Since τ is a chain, it follows that f is an open transformation of (X,τ). Besides, f is an idempotent, and hence M is a retract. Thus, (X,τ) is a retract system.

Let us take into addition that τ is finite, and $\tau=\{A_1,A_2,\ldots,A_n\}$ where $\emptyset\neq A_1\subset A_1\subset\ldots\subset A_n=X$. For any $A_i \in \tau$ let $\mathfrak{A}_i=\mathfrak{A}_{A_i}$ $(i=1,\ldots,n)$ denote the set of all invertible elements g of the semigroup $O(X,\tau)$ such that $gf=f$ for all $f\in I_{A_i}$ (cf.(1)). For the sake of uniformity we set $A_0=\emptyset$ and let $\mathfrak{A}_0=\mathfrak{A}_{A_0}$ be the group of all invertible elements of $O(X,\tau)$. Obviously, $\mathfrak{A}_i$ is a subgroup of the semigroup $O(X,\tau)$. Taking into account lemma 2, one may conclude that the subset $\mathfrak{A}_i$ is first-order definable in $O(X,\tau)$.

On the other hand, it is easy to notice that $\mathfrak{A}_i$ is just the set of all homeomorphisms of (X,τ) upon itself, for which the restriction to A_i is the identical mapping. We remark that, given any invertible element g of $O(X,\tau)$, we have $g(A_i)=A_i$ for each $A_i\in \tau$. Conversely, every bijection $g:X \to X$ such that $g(A_i)= A_i$ $(i=1,\ldots,n)$ is the invertible element of $O(X,\tau)$. In particular, any bijection on a set $A_i\setminus A_{i-1}$ coincides with the restriction of some invertible element of $O(X,\tau)$ to the set $A_i\setminus A_{i-1}(i=1,\dots,n)$.

Let us denote by $S_i = S_{A_{i+1}\setminus A_i}$ $(i=0,1,\dots,n-1)$ the symmetric group on the set $A_{i+1}\setminus A_i$ (specifically, $S_0=S_{A_1}$). We define a mapping $\nu_i:\mathfrak{H}_i\to S_i$ by letting

$$\nu_i(g)=g|_{A_{i+1}\setminus A_i} \quad \text{for each } g\in\mathfrak{H}_i.$$

From the foregoing, it follows that the mapping ν_i is a homomorphism of the group $\mathfrak{H}_i$ upon S_i. Obviously, the kernel of the homomorphism ν_i is $\mathfrak{H}_{i+1}$. Hence the factor group $\mathfrak{H}_i/\mathfrak{H}_{i+1}$ and the group S_i are isomorphic. In particular,

$$|\mathfrak{H}_i/\mathfrak{H}_{i+1}|=|S_i| \ (i=0,1,\dots,n-1). \tag{3}$$

Let us assume now that systems of subsets (X_1,τ_1) and (X_2,τ_2) belong to the class Γ, τ_1 is finite, the semigroups $O(X_1,\tau_1)$ and $O(X_2,\tau_2)$ are isomorphic and $\chi:O(X_1,\tau_1)\to O(X_2,\tau_2)$ is the corresponding isomorphism. Since the considered systems are retract systems, it follows by lemma 1 that $|\tau_1|=|\tau_2|$. Without loss of generality, one may assume that $\emptyset\notin\tau_1$ and $\emptyset\notin\tau_2$.

From the definition of ρ and from lemma 2, it follows that, by letting $\bar\chi\,(I_M)=I_N$ $(M\in{}_o\tau_1, N\in\tau_2)$ if and only if $\chi(I_M)=I_N$, a bijection

$$\bar\chi:Id\,(X_1,\tau_1)/\rho\to Id(X_2,\tau_2)/\rho$$

is defined. It is not difficult to verify that $\bar\chi$ is an isomorphism of the partially ordered sets $(Id\,(X_1,\tau_1)/\rho,\bar\sigma)$ and $(Id(X_2,\tau_2)/\rho,\bar\sigma)$.

Let $\pi_i:\tau_i\to Id(X_i,\tau_i)/\rho$, $i=1,2$ be a bijection defined by analogy with the mapping π from lemma 3. Then π_i is an isomorphism of the corresponding partially ordered sets. Hence the bijection $\nu=\pi_2^{-1}\bar\chi\,\pi_1:\tau_1\to\tau_2$ is an isomorphism of the partially ordered sets $(\tau_1,\subset)$ and $(\tau_2,\subset)$. In the sequel for $M\in\tau_1$, we write $\nu(M)=\bar M$. It is easy to see that

$$\chi(I_M)=I_{\bar M} \ (M\in\tau_1) \tag{4}$$

Elements of τ_1 we denote by $M_1, M_2,\dots,M_k$ where $M_1\subset M_2\subset\dots\subset M_k=X$. The foregoing shows that $\tau_2=\{\bar M_1,\bar M_2,\dots,\bar M_k\}$ where $\bar M_1\subset\bar M_2\subset\dots\subset\bar M_k=X_2$. Since χ is an isomorphism, it follows from (4) and the definition of $\mathfrak{H}_M$ that $\chi(\mathfrak{H}_M)=\mathfrak{H}_{\bar M}$ $(M\in\tau_1)$.

Hence the factor groups $\mathfrak{H}_{M_i}/\mathfrak{H}_{M_{i+1}}$ and $\mathfrak{H}_{\bar{M}_i}/\mathfrak{H}_{\bar{M}_{i+1}}$ are isomorphic. By (3) this implies

$$|M_1|=|\bar{M}_1|,\ |M_{i+1}\setminus M_i|=|\bar{M}_{i+1}\setminus\bar{M}_i|\ (i=1,\dots,n-1).$$

It is evident now that the systems (X_1,τ_1) and (X_2,τ_2) are homeomorphic. The proof is completed.

Next we shall state a few results concerning the determinability by means of $O(X,\tau)$ within some classes Γ, containing topological T_1-spaces (with closed topology). For these classes semigroups $O(X,\tau)$ contain all constant transformations of X. Constant transformations of X are just left zeroes in $O(X,\tau)$. The following statements can be easy proved in the same way as the similar results for T_1-spaces: any isomorphism χ of the semigroups $O(X_1,\tau_1)$ and $O(X_2,\tau_2)$ induces a bijection $\bar{\chi}$ between the sets of left zeroes of these semigroups, and in turn, $\bar{\chi}$ naturally gives rise a bijection of the corresponding sets X_1 and X_2 etc. Therefore the proofs are omitted.

In the descriptive set theory [1] a set $\tau\subset\mathscr{P}_X$ is called by T_1-separating if for any distinct $x,y\in X$ there exist subsets $H,L\in\tau$ such that $H\cap\{x,y\}=\{x\}$ and $L\cap\{x,y\}=\{y\}$. Following it we call a system of subsets (X,τ) by strong T_1-seperable if $\{x\}\in\tau$ for each $x\in X$.

<u>Theorem 4</u>. Let Γ be a class of strong T_1-separable quasiretract systems of subsets. Every system $(X,\tau)\in\Gamma$ is O-determined by the semigroup $O(X,\tau)$ within the class Γ.

This result strenghten the determinability theorem [2] by means of the semigroup of closed transformations for T_1-spaces.

<u>Corollary</u>. Let Γ be a class of strong T_1-separable systems of subsets (X,τ) such that τ is closed under finite intersections and unions of the type: $M\cup\{a\}\in\tau$ for all $M\in\tau, a\in X$. Every system $(X,\tau)\in\Gamma$ is determined by $O(X,\tau)$ within Γ.

To verify this statement it suffices to prove that (X,τ) is a quasiretact system under the hypotheses of the corollary. Really, for any $M\in\tau$ define a transformation $f:X\to X$ as follows: $f(x)=x\,(x\in M)$, $f(x)\in M\,(x\in X\setminus M)$. Then $f\in O(X,\tau)$ and $f(X)=M$. Hence M is a retract.

<u>Theorem 4'</u>. Let Γ be a class of strong T_1-separable

closure systems (X,τ) such that τ have a quasiretract multiplicative basis $\tau'\subset\tau$. Every $(X,\tau)\in\Gamma$ is determined by the semigroup $O(X,\tau)$ within Γ.

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V.Šteinbuks. Par apakškopu sistēmu valējo transformāciju pusgrupām.

Anotācija. Tiek pētīts šāds jautājums: cik pilnīgi apakškopu sistēmas atklātu transformāciju pusgrupa nosaka atbilstošo apakškopu sistēmu dažādās klasēs Г. Klases Г lomā tiek aplūkotas ķēdes, sadalījumi, homogēnas kvaziretraktu sistēmas un vispārinātas T_1- telpas.

В.Штейнбук. О полугруппах открытых преобразований систем подмножеств

Аннотация. Изучается вопрос об определяемости (с точностью до гомеоморфизма) систем подмножеств полугруппами их открытых преобразований в различных классах Г. В качестве Г рассматриваются классы цепей, разбиений, однородных квазиретрактных систем и обобщенных T_1- пространств.

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ON EVEN FUZZY TOPOLOGIES

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ABSTRACT

A special kind of fuzzy topologies in the sense of the second author, the so called even fuzzy topologies is introduced. Some properties of even fuzzy topologies and their role in the theory of (general) fuzzy topologies are considered. Besides the proximity counterpart of even fuzzy topologies - the so called even fuzzy proximities - is introduced and briefly discussed.

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Introduction

Soon after the inception of the concept of a fuzzy set by L.A. Zadeh [Za], C.L. Chang [Ch] in 1968 made the first attempt to extend fundamental notions of general topology for the case of fuzzy sets. Namely, according to Chang, a fuzzy topology on a set X is just a usual (i.e. crisp) subset τ of the fuzzy powerset I^X of X satisfying axioms which are natural analogs of the standard axioms of topology (see Definition 1.3 for the precise formulation). Chang's pioneering paper was followed by many others the authors of which either investigated different aspects of Chang fuzzy topological spaces or proposed alternative viewpoints on the subject of fuzzy topology and were developing corresponding theories. Among the last ones was [Šo$_1$] in which fuzzy topology on a set X was realized as a mapping $\mathcal{T}: I^X \longrightarrow I$ satisfying certain axioms, i.e. a fuzzy topology in accordance with this viewpoint is essentially a fuzzy subset of the fuzzy powerset of the given set (see Definition 1.1 for the precise formulation as well as the subsequent Comments 1.2.) In what follows the term "a fuzzy topology" without specification will be always understood in this sense.

It is well known that any fuzzy topology has a property which can be viewed upon as its "lower semicontinuity"; this property is described in 1.6. On the other hand fuzzy topologies "often" do not have the dual property of "upper semicontinuity" described in 1.7.

It is the principal aim of this paper to investigate fuzzy topologies which are uppersemicontinuous as well. They could naturally be called "continuous": however we prefer to use the term "even fuzzy topologies" since the term "continuous" when discussing topological concepts is quite ambiguous and may lead to wrong associations. The class of even fuzzy topologies is broad enough; in particular it contains all Chang fuzzy topologies.

We study basic properties of even fuzzy topologies and their relations with general fuzzy topologies. In the last Section we discuss a proximal counterpart of even fuzzy topologies: the so called even fuzzy proximities.

In what follows, I will denote the unit interval $[0,1]$. The two point set $\{0,1\}$ will be denoted by $\mathbf{2}$. If X is a set, then, as usually, I^X denotes the family of all fuzzy subsets of X and $\mathbf{2}^X$ denotes the ordinary powerset of X, i.e. the family of all crisp subsets of X. Given a fuzzy set $M \in I^X$, $M^c := 1-M$ denotes its complement. We do not distinguish in notation between crisp subsets of X and corresponding characteristic functions.

1. Even fuzzy topologies

Definition 1.1. [Šo₁] Let X be a set. By a ***fuzzy topology*** on X we call a function $\mathcal{T}: I^X \longrightarrow I$ such that

(FT1) $\mathcal{T}(0) = \mathcal{T}(1) = 1$;

(FT2) if $M, N \in I^X$, then $\mathcal{T}(M \wedge N) \geq \mathcal{T}(M) \wedge \mathcal{T}(N)$;

(FT3) if $M_\lambda \in I^X$ for all $\lambda \in \Lambda$, then $\mathcal{T}\left(\bigvee_\lambda M_\lambda\right) \geq \bigwedge_\lambda \mathcal{T}(M_\lambda)$.

A pair $(X, \mathcal{T})$ is called a fuzzy topological space.

In case a fuzzy topology $\mathcal{T}: I^X \longrightarrow I$ satisfies the following stronger version of the first axiom:

(FT1′) $\mathcal{T}(c) = 1$ for each constant $c \in I$

it is called laminated.

Comments 1.2. The idea of such an aproach to the subject of fuzzy topology first appeared in Höhle's paper [Hö]. However, there a fuzzy topology was realized as a fuzzy subset of the usual powerset of X, i.e. as a mapping $\mathcal{T}: 2^X \longrightarrow I$. In the present form the concept of a fuzzy topology was introduced in 1985 in [Šo₁]. In the middle eighties similar ideas were discussed by some other authors, see [Ha], [Ku], [Di], [Lo] and [Ge].

Definition 1.3. [Ch] Let X be a set. By a Chang fuzzy topology on X we mean a subset $\mathcal{T} \subseteq I^X$ such that

(CFT1) $0, 1 \in \mathcal{T}$;

(CFT2) if $M, N \in \mathcal{T}$ then $M \wedge N \in \mathcal{T}$;

(CFT3) if $M_\lambda \in \mathcal{T}$ for all $\lambda \in \Lambda$, then $\bigvee_\lambda M_\lambda \in \mathcal{T}$.

Remark 1.4. Chang fuzzy topologies can be interpreted as a special case of fuzzy topologies in the sense of 1.1. Namely, a fuzzy topology $\mathcal{T}: I^X \longrightarrow I$ is Chang iff it satisfies the following additional axiom:

(FTC) $\mathcal{T}(I^X) \subseteq 2$.

Definition 1.5. [Šo₁] A mapping $f: X \longrightarrow Y$ where $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are fuzzy topological spaces is called fuzzy continuous if $\mathcal{T}_X(f^{-1}(N)) \geq \mathcal{T}_Y(N)$ for each $N \in I^Y$.

Fuzzy topological spaces and continuous mappings of such spaces form a category which will be denoted **FT**.

Discussion 1.6. [Šo₃, Šo₄] Given a fuzzy topological space $(X, \mathcal{T})$ and $\alpha \in (0, 1]$ let $\mathcal{T}_\alpha = \{M \in I^X : \mathcal{T}(M) \geq \alpha\}$. Obviously $\mathcal{T}_\alpha$ is a Chang fuzzy topology on X and the family $\{\mathcal{T}_\alpha : \alpha \in (0, 1]\}$ is non increasing. $\mathcal{T}_\alpha$ will be referred to as the α-level Chang fuzzy topology of the given fuzzy topology $\mathcal{T}$ and the construction $\mathcal{T} \longrightarrow \{\mathcal{T}_\alpha : \alpha \in I\}$ will be referred to as the descomposition of $\mathcal{T}$ into the system of its α-level Chang fuzzy topologies.
It is easy to see that for any fuzzy topology $\mathcal{T}$ and any $\alpha \in (0, 1]$ it holds $\mathcal{T}_\alpha = \bigcap_{\alpha' < \alpha} \mathcal{T}_{\alpha'}$. Thus the system $\{\mathcal{T}_\alpha : \alpha \in I\}$ is in a certain sense "lower semicontinuous", or, as we prefer to say, *lower semieven*.
On the other hand given a non increasing family of Chang fuzzy topologies $\{\tau_\alpha : \alpha \in I\}$ on a set X one can define a fuzzy topology $\mathcal{T}: I^X \longrightarrow I$ by setting $\mathcal{T}(M) = \vee\{\tau_\alpha(M) \wedge \alpha : \alpha \in I\}$, $M \in I^X$. Besides τ_α is exactly the α-level Chang fuzzy topology $\mathcal{T}_\alpha$ of $\mathcal{T}$ iff the family $\{\tau_\alpha : \alpha \in I\}$ is lower semieven in α, i.e., if $\tau_\alpha = \bigcap_{\alpha' < \alpha} \tau_{\alpha'}$.

Discussion 1.7. Given a fuzzy topological space $(X, \mathcal{T})$ and $\alpha \in (0,1]$ let $\langle \bigcup_{\alpha'>\alpha} \mathcal{T}_{\alpha'} \rangle$ denote the Chang fuzzy topology generated by $\bigcup_{\alpha'>\alpha} \mathcal{T}_{\alpha'}$ as a base. Obviously, $\mathcal{T}_\alpha \supseteq \langle \bigcup_{\alpha'>\alpha} \mathcal{T}_{\alpha'} \rangle$ but in general the equality does not hold. It seems reasonable to consider a special kind of fuzzy topologies for which the equality is true for all $\alpha \in (0,1)$. We call such fuzzy topologies even.

Definition 1.8. Given a fuzzy topology $\mathcal{T}$ on a set X we say that it is *even* if $\mathcal{T}_\alpha = \langle \bigcup_{\alpha'>\alpha} \mathcal{T}_{\alpha'} \rangle$ for each $\alpha \in (0,1)$.
If besides $\mathcal{T}_\alpha = \langle \bigcup_{\alpha'>\alpha} \mathcal{T}_{\alpha'} \rangle$ also for $\alpha = 0$ then $\mathcal{T}$ is called *strictly even*.
It is easy to see that $\mathcal{T}$ is even (strictly even) if and only if for all $M \in I^X$ with $0 < \mathcal{T}(M) < 1$, (respectively $0 \leq \mathcal{T}(M) < 1$) there exists a collection $\{M_\epsilon\}_{\epsilon>0}$ such that $M_\epsilon \leq M_{\epsilon'}$ for $\epsilon \geq \epsilon'$, $\sup_{\epsilon>0} M_\epsilon = M$ and $\mathcal{T}(M_\epsilon) \geq \mathcal{T}(M) + \epsilon$ for all $\epsilon > 0$.

Even fuzzy topological spaces and continuous mappings of such spaces form a full subcategory of **FT** which will be denoted **EFT**.

Remark 1.9. Each Chang fuzzy topology is obviously even. On the other hand a Chang fuzzy topology is strictly even only in case it is discrete.

Given a fuzzy topology $\mathcal{T}$ on a set X we can define a collection of fuzzy closure operators $\{\mathrm{Cl}_\alpha\}_{\alpha\in(0,1]}$ where $\mathrm{Cl}_\alpha : I^X \longrightarrow I^X$, in the following way:
$\mathrm{Cl}_\alpha(M) = \wedge\{N \in I^X : N \geq M \text{ and } N^c \in \mathcal{T}_\alpha\}$.
It is easy to see that $\mathrm{Cl}_\alpha(M) \leq \bigwedge_{\alpha'>\alpha} \mathrm{Cl}_{\alpha'}(M)$.
Now we can give a characterization of even fuzzy topologies in terms of such families of closure operators.

Theorem 1.10. A fuzzy topology $\mathcal{T}$ on a set X is even if and only if for any $\alpha \in (0,1)$ and $M \in I^X$ we have that $\mathrm{Cl}_\alpha(M) = \bigwedge_{\alpha'>\alpha} \mathrm{Cl}_{\alpha'}(M)$.

Proof: Sufficiency. We have only to prove that $\mathcal{T}_\alpha \subseteq \langle \bigcup_{\alpha'>\alpha} \mathcal{T}_{\alpha'} \rangle$ for each $\alpha \in (0,1)$.
Take some $M \in \mathcal{T}_\alpha$, then $\mathrm{Cl}_\alpha(M^c) = M^c$ and therefore $M^c = \bigwedge_{\alpha'>\alpha} \mathrm{Cl}_{\alpha'}(M^c)$. So $M = \bigvee_{\alpha'>\alpha} \mathrm{Cl}_{\alpha'}(M^c)^c$ and $\mathrm{Cl}_{\alpha'}(M^c)^c \in \mathcal{T}_{\alpha'}$ for every $\alpha' > \alpha$. From here we can conclude that $M \in \langle \bigcup_{\alpha'>\alpha} \mathcal{T}_{\alpha'} \rangle$.

Necessity. Let $M \in I^X$, then $\mathrm{Cl}_\alpha(M)^c \in \mathcal{T}_\alpha = \langle \bigcup_{\alpha'>\alpha} \mathcal{T}_{\alpha'} \rangle$ and hence there exists a collection $\{N_{\alpha'}\}_{\alpha'>\alpha}$ such that $N_{\alpha'} \in \mathcal{T}_{\alpha'}$ for every $\alpha' > \alpha$ and

$\mathrm{Cl}_\alpha(M)^c = \bigvee_{\alpha'>\alpha} N_{\alpha'}$.
Thus, $\mathrm{Cl}_\alpha(M) = \bigwedge_{\alpha'>\alpha} N_{\alpha'}{}^c$.
Since, obviously, $M \le \mathrm{Cl}_\alpha(M) \le \mathrm{Cl}_{\alpha'}(M) \le N_{\alpha'}{}^c$ for every $\alpha' > \alpha$, it follows that $\mathrm{Cl}_\alpha(M) \le \bigwedge_{\alpha'>\alpha} \mathrm{Cl}_{\alpha'}(M) \le \bigwedge_{\alpha'>\alpha} (N_{\alpha'})^c = \mathrm{Cl}_\alpha(M)$, that is, $\mathrm{Cl}_\alpha(M) = \bigwedge_{\alpha'>\alpha} \mathrm{Cl}_{\alpha'}(M)$. ∎

In a similar way one can prove:

Theorem 1.10′. A fuzzy topology $\mathcal{T}$ on a set X is strictly even if and only if for any $\alpha \in [0,1)$ and $M \in I^X$ we have that $\mathrm{Cl}_\alpha(M) = \bigwedge_{\alpha'>\alpha} \mathrm{Cl}_{\alpha'}(M)$.

As shown by the next two theorems, the family of even fuzzy topologies on a given set X is in a certain sense "dense" in the family of all fuzzy topologies on this set.

Theorem 1.11. Any laminated fuzzy topology $\mathcal{T}$ on a set X is a supremum of some family of even fuzzy topologies.

Proof: Given a fuzzy topology $\mathcal{T}$ let $\{\mathcal{T}_\alpha\}_{\alpha\in I}$ be the family of its α-level topologies. Define the family of fuzzy topologies $\{\mathcal{T}^{\alpha,\epsilon}\}_{\alpha\in(0,1],\epsilon\in(0,\alpha)}$ in the following way:

For each $M \in I^X$ let $\mathcal{T}^{\alpha,\epsilon}(M) = \begin{cases} 1, & \text{if } M \in \{0,1\}; \\ \alpha - \epsilon \sup M, & \text{if } M \in \mathcal{T}_\alpha - \{0,1\}; \\ 0, & \text{if } M \notin \mathcal{T}_\alpha. \end{cases}$

This fuzzy topology is even for each $\alpha \in (0,1]$ and $\epsilon \in (0,\alpha)$. To show this notice that its level Chang fuzzy topologies:

$$\mathcal{T}^{\alpha,\epsilon}_\beta = \begin{cases} \{0,1\}, & \text{if } \beta \ge \alpha; \\ \{1\} \cup \{M \in \mathcal{T}_\alpha : M \le \frac{\alpha-\beta}{\epsilon}\}, & \text{if } \beta \in [\alpha-\epsilon,\alpha); \\ \mathcal{T}_\alpha, & \text{if } \beta \in (0,\alpha-\epsilon). \\ I^X, & \text{if } \beta = 0. \end{cases}$$

and hence $\mathcal{T}^{\alpha,\epsilon}_\beta = \langle \bigcup_{\beta'>\beta} \mathcal{T}^{\alpha,\epsilon}_{\beta'} \rangle$ for each $\beta \in (0,1]$

Indeed in case $\beta \ge \alpha$ and $\beta \in (0,\alpha-\epsilon)$ it is obvious; if $\beta \in [\alpha-\epsilon,\alpha)$ then $M \in \mathcal{T}^{\alpha,\epsilon}_\beta$ $(M \ne 1)$ if and only if $M \in \mathcal{T}_\alpha$ and $M \le \frac{\alpha-\beta}{\epsilon}$ and hence, noticing that $M = \sup_{\beta'\in(\beta,\alpha)} M \wedge \dfrac{\alpha-\beta'}{\epsilon}$ we conclude that $M \in \langle \bigcup_{\beta'>\beta} \mathcal{T}^{\alpha,\epsilon}_{\beta'} \rangle$.
To complete the proof we have to show only that $\mathcal{T} = \sup_{\alpha\in(0,1]} \sup_{\epsilon\in(0,\alpha)} \mathcal{T}^{\alpha,\epsilon}$.
The inequality "$\ge$" is obvious because $\mathcal{T} \ge \mathcal{T}^{\alpha,\epsilon}$ for each $\alpha \in (0,1]$ and $\epsilon \in (0,\alpha)$.
To prove the converse, suppose that there exist $M \in I^X - \{0,1\}$ and $\gamma \in I$ such

that $\mathcal{T}(M) \geq \gamma > \sup_{\alpha\in(0,1]} \sup_{\epsilon\in(0,\alpha)} \mathcal{T}^{\alpha,\epsilon}(M)$.
Since $M \in \mathcal{T}_\gamma$ it is clear that $\mathcal{T}^{\gamma,\epsilon}(M) = \gamma - \epsilon \sup M$ for all $\epsilon \in (0,\gamma)$ and therefore $\gamma = \sup_{\epsilon\in(0,\gamma)} \mathcal{T}^{\gamma,\epsilon}(M) \leq \sup_{\alpha\in(0,1]} \sup_{\epsilon\in(0,\alpha)} \mathcal{T}^{\alpha,\epsilon}(M)$. The obtained contradiction completes the proof. ∎

Theorem 1.12. Any fuzzy topology $\mathcal{T}$ on a set X is an infimum of some family of strictly even fuzzy topologies.

Proof: Given a fuzzy topology $\mathcal{T}$ let $\{\mathcal{T}_\alpha\}_{\alpha\in I}$ be de family of its α-level topologies. Define the family of fuzzy topologies $\{\mathcal{T}^{\alpha,\epsilon}\}_{\alpha\in[0,1),\epsilon\in(0,1-\alpha)}$ in the following way: $\mathcal{T}^{\alpha,\epsilon} = \sup_{\beta\in(0,1]} \mathcal{T}^{\alpha,\epsilon}_\beta \wedge \beta$ where

$$\mathcal{T}^{\alpha,\epsilon}_\beta = \begin{cases} \{0,1\}, & \text{if } \beta \geq 1-\epsilon; \\ \langle \mathcal{T}_\alpha \cup \{M : M \leq \frac{1-\epsilon-\beta}{1-\epsilon-\alpha}\}\rangle, & \text{if } \beta \in [\alpha, 1-\epsilon); \\ I^X, & \text{if } \beta \in [0,\alpha). \end{cases}$$

($\mathcal{T}^{\alpha,\epsilon}_\beta = \{M_1 \vee M_2 : M_1 \in \mathcal{T}_\alpha, M_2 \leq \frac{1-\epsilon-\beta}{1-\epsilon-\alpha}\}$ when $\beta \in [\alpha, 1-\epsilon)$).
It is easy to see that this topology is both lower semieven and strictly upper semieven, that is strictly even.
To complete the proof we have to show only that $\mathcal{T} = \inf_{\alpha\in[0,1)} \inf_{\epsilon\in(0,1-\alpha)} \mathcal{T}^{\alpha,\epsilon}$.
The inequality "$\leq$" is obvious because $\mathcal{T} \leq \mathcal{T}^{\alpha,\epsilon}$ for each $\alpha \in [0,1)$ and $\epsilon \in (0, 1-\alpha)$.
To show the converse inequality suppose that there exist $M \in I^X - \{0,1\}$ and $\gamma \in I$ such that $\mathcal{T}(M) < \gamma < \inf_{\alpha\in[0,1)} \inf_{\epsilon\in(0,1-\alpha)} \mathcal{T}^{\alpha,\epsilon}(M)$.
In this case $M \notin \mathcal{T}_\gamma$ and therefore $M \notin \mathcal{T}^{\gamma,\epsilon}_{1-\epsilon} = \mathcal{T}_\gamma$ for all $\epsilon \in (0, 1-\gamma)$, hence $\gamma = \inf_{\epsilon\in(0,1-\gamma)} \mathcal{T}^{\gamma,\epsilon}(M) \geq \inf_{\alpha\in[0,1)} \inf_{\epsilon\in(0,1-\alpha)} \mathcal{T}^{\alpha,\epsilon}(M)$. The obtained contradiction completes the proof. ∎

The property of eveness (strict eveness) is easily destroyed by different operations. In particular, as shown by the next example, these properties are not preserved by subspaces and hence moreover by preimages. Thus in the category **EFT** initial structures and subspaces generally do not exist.

Recall that if Y is a set, $(X,\mathcal{T})$ is a fuzzy topological space and $f: Y \longrightarrow X$ is a mapping, then the preimage $\mathcal{T}_Y$ of $\mathcal{T}$ is defined by the formula $\mathcal{T}_Y(M) = \sup\{\mathcal{T}(N) : M = f^{-1}(N)\}$ (see e.g. [Šo3]).
Notice that $\mathcal{T}_Y$ is obviously the initial fuzzy topology for f in the category **FT**.
If $(X,\mathcal{T})$ is a fuzzy topological space and $Y \subseteq X$, then the corresponding subspace is naturally defined as the pair $(Y,\mathcal{T}_Y)$ where $\mathcal{T}_Y$ is the initial fuzzy topology for the inclusion mapping $\iota: Y \longrightarrow (X,\mathcal{T})$. Explicitely: $\mathcal{T}_Y(M) = \sup\{\mathcal{T}(N) : M = Y \wedge N\}$ (see e.g. [Šo3]).

Example 1.13. Let (X,τ) be a Chang fuzzy topological space containing at least two points. Let Y be a proper crisp subset of X and let $P \in I^Y$ be such that $U|_Y \neq P$ for all $U \in \tau$. (In particular if τ is a crisp topology then as P one can take any non crisp fuzzy set).
Define Chang fuzzy topologies τ_α $\alpha \in I$, as follows:
For $\alpha \in [\frac{1}{2}, 1]$ let τ_α be τ;
For $\alpha \in [\frac{1}{4}, \frac{1}{2})$ let $\tau_\alpha = \langle \tau \cup \{P_s : s \in (0, 1-2\alpha]\}\rangle$ where the fuzzy set $P_s \in I^X$ is defined by $P_s(x) = \begin{cases} P(x), & \text{if } x \in Y; \\ s, & \text{otherwise.} \end{cases}$;
For $\alpha \in [0, \frac{1}{4})$ let $\tau_\alpha = \langle \tau_{\frac{1}{4}} \cup \{M : M \leq 1 - 4\alpha\}\rangle$.
For each $\alpha \in [0,1)$ it holds $\tau_\alpha = \bigwedge_{\alpha'<\alpha} \tau_{\alpha'}$. Indeed, this is obvious for $\alpha \neq \frac{1}{2}$. In case $\alpha = \frac{1}{2}$ $\tau_{\frac{1}{2}} = \tau$, but on the other hand it is clear that no fuzzy set P_s can belong to all $\tau_{\alpha'}$ simultaneously, and hence $\tau = \bigwedge_{\alpha'<\frac{1}{2}} \tau_{\alpha'}$.

Define a fuzzy topology $\mathcal{T}: I^X \longrightarrow I$ by setting $\mathcal{T}(M) = \vee\{\tau_\alpha \wedge \alpha : \alpha \in I\}$ for each $M \in I^X$. From (1.6) it follows that τ_α are just the α-level Chang fuzzy topologies $\mathcal{T}_\alpha$ of $\mathcal{T}$.
Since, obviously, $\tau_\alpha = \langle \bigvee_{\alpha'>\alpha} \tau_{\alpha'}\rangle$ for each $\alpha' > \alpha$, the fuzzy topology $\mathcal{T}$ is even.

Consider now the subspace $(Y, \mathcal{T}^Y)$ of $(X, \mathcal{T})$. It is easy to notice that $P \in \mathcal{T}^Y_{\frac{1}{2}}$. On the other hand, for all $\alpha' > \alpha$ $\mathcal{T}^Y_{\alpha'}$ is just the restriction of $\mathcal{T}_{\alpha'} = \tau$ to Y and hence $P \notin \mathcal{T}^Y_\alpha$ for $\alpha' > \alpha$. Thus $\mathcal{T}^Y$ is not even. ■

A special case when the preimage of an (strictly) even fuzzy topology is (strictly) even is described by the next

Proposition 1.14. The preimage of an even (strictly even) fuzzy topology under a surjection is even (resp. strictly even).

Proof: Let $f: X \longrightarrow (Y, \mathcal{T}^Y)$ be a surjection and for each $\alpha \in I$ let $\tau_\alpha = f^{-1}(\mathcal{T}^Y_\alpha)$, i.e. τ_α is the collection of all preimages of fuzzy sets belonging to $\mathcal{T}^Y_\alpha$. It is easy to see that τ_α is a Chang fuzzy topology and the mapping $\mathcal{T}: I^X \longrightarrow I$ defined by the formula $\mathcal{T}(M) = \wedge\{\tau_\alpha(M) \wedge \alpha : \alpha \in I\}$ is the fuzzy topology which is exactly the preimage of $\mathcal{T}^Y$ under f. Since f is a surjection, for each $N \in I^X$, $f^{-1}(N) \in \tau_\alpha$ iff $N \in \mathcal{T}^Y_\alpha$. It easily follows now that $\bigcap_{\alpha'<\alpha} \tau_{\alpha'} = \tau_\alpha$ for each $\alpha \in I$ and hence τ_α are exactly the α-level Chang fuzzy topologies of $\mathcal{T}$ (see 1.6). Thus to complete the proof one has to notice only that $\tau_\alpha = f^{-1}(\mathcal{T}^Y_\alpha) = f^{-1}(\langle \bigvee_{\alpha'>\alpha} \mathcal{T}^Y_{\alpha'}\rangle) = \langle \bigvee_{\alpha'>\alpha} f^{-1}(\mathcal{T}^Y_{\alpha'})\rangle = \langle \bigvee_{\alpha'>\alpha} \tau_{\alpha'}\rangle$. ■

Examples

1. The fuzzy discrete topology $\mathcal{T}_1(M) = 1 \quad \forall M \in I^X$ is strictly even.

2. The fuzzy indiscrete topology $\mathcal{T}_2(M) = \begin{cases} 1, & \text{if } M = 0 \text{ or } M = 1; \\ 0, & \text{otherwise.} \end{cases}$ is even but not strictly even.

3. For each $\epsilon \in (0,1]$ we consider for each $\alpha \in [0,\epsilon)$ the Chang fuzzy topology $\mathcal{T}^\epsilon_\alpha = \{0,1\} \cup \{M : M \leq (\frac{\alpha}{\epsilon})^c\}$ and for each $\alpha \in [\epsilon, 1]$ $\mathcal{T}^\epsilon_\alpha = \{0,1\}$.
The fuzzy topology generated by this system of Chang fuzzy topologies is $\mathcal{T}^\epsilon(M) = \begin{cases} 1, & \text{if } M = 0,1; \\ \epsilon(1 - \sup M), & \text{if } M \neq 0,1. \end{cases}$
It is easy to see that it is strictly even.

4. If (X, τ) is a Chang fuzzy topological space and $x_0 \in X$, for each $\alpha \in I$ we define $\Pi_\alpha = \tau \cup \{x_0^t : t \leq \alpha^c\}$ and $\tau_\alpha = \langle \Pi_\alpha \rangle = \{M \vee x_0^t : M \in \tau, t \leq \alpha^c\}$. One can easily prove that the fuzzy topology generated by this system of Chang fuzzy topologies is even but not strictly even.

5. If (X, τ) is a Chang fuzzy topological space, for each $\alpha \in I$ we define $\Pi_\alpha = \tau \cup \{M : M \leq \alpha^c\}$ and $\tau_\alpha = \langle \Pi_\alpha \rangle = \{M_1 \vee M_2 : M_1 \in \tau, M_2 \leq \alpha^c\}$. In this case, it is easy to see that $\tau_0 = \tau_d$ and, on the other hand, for each $\alpha \in I$, if $M \in \tau_\alpha$, there exist $M_1 \in \tau$ and $M_2 \leq \alpha^c$ such that $M = M_1 \vee M_2$. Therefore, $M_{\alpha'} = M_1 \vee (M_2 \wedge \alpha'^c) \in \tau_{\alpha'}$ for all $\alpha' > \alpha$ and $M = \sup\limits_{\alpha' > \alpha} M_{\alpha'}$ so $M \in \langle \bigcup\limits_{\alpha' > \alpha} \tau_{\alpha'} \rangle$.
Thus the fuzzy topology generated by this system of Chang fuzzy topologies is strictly even.

6. Let $\{(X, \tau_n)\}_{n \in \mathbb{N}}$ be a countable collection of Chang fuzzy topological spaces such that $\tau_n \subseteq \tau_{n+1}$ for all $n \in \mathbb{N}$ and $\langle \bigcup\limits_{n \in \mathbb{N}} \tau_n \rangle = \tau_d$, we can define the following:
For all $n \in \mathbb{N}$ and all $\alpha \in [\frac{1}{n+1}, \frac{1}{n}]$,
$\Pi_\alpha = \tau_n \cup \{M \in \tau_{n+1} : M \leq (n+1)(1 - n\alpha)\}$ and
$\tau_\alpha = \langle \Pi_\alpha \rangle = \{M_1 \vee M_2 : M_1 \in \tau_n, M_2 \in \tau_{n+1}, M_2 \leq (n+1)(1 - n\alpha)\}$.
(It is clear that for $\alpha = \frac{1}{n}$, $\tau_\alpha = \tau_n$ and for $\alpha = \frac{1}{n+1}$, $\tau_\alpha = \tau_{n+1}$).
We define $\tau_0 = \tau_d$ and it is easy to see that $\langle \bigcup\limits_{\alpha > 0} \tau_\alpha \rangle \supseteq \langle \bigcup\limits_{n \in \mathbb{N}} \tau_n \rangle = \tau_d$, therefore, $\langle \bigcup\limits_{\alpha > 0} \tau_\alpha \rangle = \tau_d$. Now, if $\alpha > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n+1} \leq \alpha < \frac{1}{n}$. Given $M \in \tau_\alpha$, there exist $M_1 \in \tau_{n+1}$ and $M_2 \in \tau_n$ such that $M_2 \leq (n+1)(1 - n\alpha)$ and $M = M_1 \vee M_2$.
Therefore, $M_{\alpha'} = M_1 \vee (M_2 \wedge (n+1)(1 - n\alpha')) \in \tau_{\alpha'}$ for all $\alpha' \in (\alpha, \frac{1}{n})$ and $M = \sup\limits_{\alpha' \in (\alpha, \frac{1}{n})} M_{\alpha'}$ so $M \in \langle \bigcup\limits_{\alpha' \in (\alpha, \frac{1}{n})} \tau_{\alpha'} \rangle \subseteq \langle \bigcup\limits_{\alpha' > \alpha} \tau_{\alpha'} \rangle$.
Hence the fuzzy topology generated by this system of Chang fuzzy topologies is strictly even.

2. Even fuzzy proximities

In [MŠo] a concept of a fuzzy proximity which is in accordance with our fuzzy topologies was considered. In this Section we consider a special kind of fuzzy proximities which we call even and show that in a natural way even fuzzy proximities correspond to even fuzzy topologies. For reader's convenience we reproduce here the main definitions and some constructions from [MŠo].

Definition 2.1. [Šo$_2$] By a fuzzy proximity on a set X we call a mapping $\delta: I^X \times I^X \longrightarrow I$ satisfying the following axioms ($M, N, N_1, N_2, P \in I^X$):

(FP1) $\delta(0,1) = 0$;

(FP2) $\delta(M,N) = \delta(N,M)$;

(FP3) $\delta(M,N_1) \vee \delta(M,N_2) = \delta(M, N_1 \vee N_2)$;

(FP4) $\delta(M,N) \geq \sup_{x \in X}(M + N - 1)(x)$;

(FP5) $\delta(M,N) \geq \inf\{\delta(M,P) \vee \delta(N,P^c) : P \in I^X\}$;

A pair (X,δ) where X is a set and δ is a fuzzy proximity on it is called a fuzzy proximity space.

Definition 2.2. [MŠo] A mapping $f: X \longrightarrow Y$ where (X,δ_X) and (Y,δ_Y) are fuzzy proximity spaces is called proximally continuous if $\delta_Y(f(M), f(N)) \geq \delta_X(M,N)$ for any $M, N \in I^X$.

Let FP denote the category the objects of which are fuzzy proximity spaces and the morphisms are proximally continuous mappings of such spaces.

A fuzzy proximity generates a fuzzy topology in the following way.

Let (X,δ) be a fuzzy proximity space, $M \in I^X$ and $\alpha \in (0,1]$. The α-closure of M (or the closure of M at the level α) is defined by the equality
$\mathrm{Cl}_\alpha(M) = (1 - \vee\{N : \delta(M,N) \leq \alpha^c\}) \vee M = (\wedge\{N^c : \delta(M,N) \leq \alpha^c\}) \vee M$.

Lemma 2.3. [MŠo] $\mathrm{Cl}_\alpha(M) = \widetilde{M}^\alpha$, where $\widetilde{M}^\alpha = (\vee\{x^\lambda : \delta(x^{\lambda^c}, M) > \alpha^c\}) \vee M$. ($x^\lambda$ denotes the fuzzy point with support $x \in X$ and value $\lambda \in I$: for technical reasons we do not exclude the case $\lambda = 0$ which corresponds to a degenerate fuzzy point $x^0 = 0$. We write $x^\lambda \in M$ iff $M(x) \geq \lambda$).

Proposition 2.4. [MŠo] For each $\alpha \in (0,1]$ the mapping $M \longrightarrow \mathrm{Cl}_\alpha(M)$ is an operator of fuzzy closure

Notation 2.5. [MŠo] Let $\sigma_\alpha = \{M \in I^X : M = \mathrm{Cl}_\alpha(M)\}$ and $\tau_\alpha = \{M \in I^X : M^c \in \sigma_\alpha\}$. It is known (and easy to verify) that τ_α is a Chang fuzzy topology on X.

Proposition 2.6. [MŠo] The fuzzy closure operator Cl_α is continuous along I in the following sense:

$$\forall \alpha \in (0,1] \quad \text{if} \quad \epsilon_n \to 0 \quad \text{and} \quad \epsilon_n > 0, \quad \text{then} \quad \bigvee_n \mathrm{Cl}_{\alpha - \epsilon_n}(M) = \mathrm{Cl}_\alpha(M).$$

Proposition 2.7. [MŠo] For each $\alpha \in (0,1]$, $\sigma_\alpha = \bigcap_{\alpha'<\alpha} \sigma_{\alpha'}$ and hence $\tau_\alpha = \bigcap_{\alpha'<\alpha} \tau_{\alpha'}$.

Definition 2.8. [MŠo] The fuzzy topology $\mathcal{T}_\delta : I^X \longrightarrow I$ defined by the equality $\mathcal{T}_\delta(M) = \sup\{\tau_\alpha(M) \wedge \alpha : \alpha \in I\}$ (see 1.6) is called the fuzzy topology generated by δ.

Theorem 2.9. [MŠo] If a mapping $f : (X, \delta_X) \longrightarrow (Y, \delta_Y)$ is proximally continuous, then the mapping $f : (X, \mathcal{T}_{\delta_X}) \longrightarrow (Y, \mathcal{T}_{\delta_Y})$ is continuous.

Thus by letting $\phi(X,\delta) = (X, \mathcal{T}_\delta)$ for every fuzzy proximity space (X,δ) and $\phi(f) = f : (X, \mathcal{T}_{\delta_X}) \longrightarrow (Y, \mathcal{T}_{\delta_Y})$ for every proximally continuous mapping $f : (X, \delta_X) \longrightarrow (Y, \delta_Y)$, a functor ϕ from the category **FP** into the category **FT** of fuzzy topological spaces is obtained.

Definition 2.10. Given a fuzzy proximity δ on a set X we say that it is *even* if it satisfies the following axiom:

(EFP) $\forall M \in I^X$ $\forall t \in [0,1)$ such that $M(x) < t^c$ and $0 < \delta(M, x^t) <$ we have that $\forall \eta > 0$ $\delta(M, x^t) - \delta(M, x^{t-\eta}) > 0$.

We say that δ is *strictly even* if it satisfies:

(SEFP) $\forall M \in I^X$ $\forall t \in [0,1)$ such that $M(x) < t^c$ and $0 < \delta(M, x^t)$ we have that $\forall \eta > 0$ $\delta(M, x^t) - \delta(M, x^{t-\eta}) > 0$.

The next theorem establishing relation between even fuzzy proximities and even fuzzy topologies is the main result in this section:

Theorem 2.11. The fuzzy topology T_δ generated by a fuzzy proximity δ is even iff δ is even.

Proof: Assume that δ is even. According to Theorem 1.10 we have to prove that for every $M \in I^X$ and every $\alpha \in (0,1)$ it holds $\bigwedge_{\epsilon>0} \mathrm{Cl}_{\alpha+\epsilon}(M) = \mathrm{Cl}_\alpha(M)$.
The inequality $\bigwedge_{\epsilon>0} \mathrm{Cl}_{\alpha+\epsilon}(M) \geq \mathrm{Cl}_\alpha(M)$ is obvious.
To show the converse inequality assume that $\bigwedge_{\epsilon>0} \mathrm{Cl}_{\alpha+\epsilon}(M)(x) > \mathrm{Cl}_\alpha(M)(x)$ for some $x \in X$.
Then there exists $\zeta \in I$ such that $x^\zeta \in \bigwedge_{\epsilon>0} \mathrm{Cl}_{\alpha+\epsilon}(M)$ but $x^\zeta \not\in \mathrm{Cl}_\alpha(M)$ and hence $x^\lambda \notin \mathrm{Cl}_\alpha(M)$ also for some $\lambda < \zeta$. However this means that $x^\lambda \notin M$, i.e., $M(x) < \lambda$ and $\delta(x^{\zeta^c}, M) \leq \delta(x^{\lambda^c}, M) \leq \alpha^c$.
Consider now the two posibilities:

- * If $\delta(x^{\lambda^c}, M) > 0$, taking into account that $\delta(x^{\lambda^c}, M) \leq \alpha^c$, $M(x) < \lambda$ and (EFP) holds we conclude that $\delta(x^{\eta^c}, M) < \alpha^c$ for every $\eta > \lambda$.
  In particular for $\eta = \frac{\lambda+\zeta}{2} < \zeta$, there exists $\epsilon > 0$ such that $\delta(x^{\eta^c}, M) < \alpha^c - \epsilon$.
  However this means that $x^\zeta \notin \mathrm{Cl}_{\alpha+\epsilon}(M)$ and so $x^\zeta \notin \bigwedge_{\epsilon>0} \mathrm{Cl}_{\alpha+\epsilon}(M)$ what contradicts our assumption.
- ** If $\delta(x^{\lambda^c}, M) = 0$ moreover $\delta(x^{\zeta^c}, M) = 0$.
  Take some $\epsilon_0 \in (0, \alpha^c)$, then $\delta(x^{\zeta^c}, M) < \alpha^c - \epsilon_0$ and hence in virtue of $x^\zeta \in \bigwedge_{\epsilon>0} \mathrm{Cl}_{\alpha+\epsilon}(M)$ it follows that $x^\eta \in M \leq \mathrm{Cl}_\alpha(M)$. However this again contradicts our assumption.

Thus a fuzzy topology generated by an even fuzzy proximity is even.

To prove the converse, assume that (EFP) is not valid. Then there exist $M \in I^X$, $t \in [0,1)$ and $x \in X$ such that $M(x) < t^c$ and $s < t$ such that $\delta(M, x^t) = \delta(M, x^s) \neq 1$.
Let $\delta(M, x^s) = \alpha^c$. Then, obviously, $\alpha \neq 0$ and for each $\beta > \alpha$ $\delta(M, x^s) > \beta^c$.
Now, if $\zeta = s^c$, then $x^\zeta \in \mathrm{Cl}_\beta(M)$ for any $\beta > \alpha$ and hence $x^\zeta \in \bigwedge_{\beta>\alpha} \mathrm{Cl}_\beta(M)$.
On the other hand, $x^\zeta \notin M$ (otherwise $M(x) \geq \zeta = s^c > t^c$). It is easy to see now that $x^\zeta \notin \mathrm{Cl}_\alpha(M)$. Indeed if $x^\zeta \in \mathrm{Cl}_\alpha(M)$ then $\delta(M, x^{\lambda^c}) > \alpha^c$ would hold for all $\lambda < \zeta$. However this is imposible because obviously $\delta(M, x^{\lambda^c}) = \alpha^c$ for all $\lambda^c \in [s, t]$.
Thus $\bigwedge_{\epsilon>0} \mathrm{Cl}_{\alpha+\epsilon}(M) \neq \mathrm{Cl}_\alpha(M)$ and hence τ is not even. ∎

In a similar way one can prove the following:

Theorem 2.11′. The fuzzy topology $\mathcal{T}_\delta$ generated by a fuzzy proximity δ is strictly even iff δ is strictly even.

Examples

1.- Define a fuzzy proximity δ_1 on a set X by setting
$\delta_1(M,N) = \sup_{x\in X}(M+N-1)(x)$ for all $M,N \in I^X$.
It is easy to verify that δ_1 is strictly even.
The corresponding fuzzy topology $\mathcal{T}_\delta$ is discrete, i.e. $\mathcal{T}_{\delta_1}(M) = 1$ for all $M \in I^X$.

2.- Let $\delta_2(M,N) = \begin{cases} 1, & \text{if } M \not\leq N^c; \\ 0, & \text{if } M \leq N^c. \end{cases}$, $M,N \in I^X$..
It is easy to see that the fuzzy proximity δ_2 is strictly even.
Notice also that $\delta_2 \geq \delta_1$ and $\mathcal{T}_{\delta_1}(M) = \mathcal{T}_{\delta_2}$.

3.- Define a fuzzy proximity $\mathcal{T}_{\delta_3}(M)$ on X by setting
$\delta_3(M,N) = \begin{cases} 1, & \text{if } M \neq 0 \text{ and } N \neq 0; \\ 0, & \text{otherwise}. \end{cases}$
One can easily notice that this fuzzy proximity is even but fails to be strictly even.
The generated fuzzy topology $\mathcal{T}_{\delta_3}$ is antidiscrete, i.e. $\mathcal{T}_{\delta_3}(1) = \mathcal{T}_{\delta_3}(0) = 1$ and $\mathcal{T}_{\delta_3}(M) = 0$ if $M \neq 0,1$.

4.- Let $\delta_4(M,N) = \sup(M \wedge N)$ for all $M,N \in I^X$.
One can easily notice that the fuzzy proximity δ_4 is not even.
The fuzzy topology generated by it can be defined by the formula
$\mathcal{T}_{\delta_4}(M) = \sup\{\tau_\alpha(M) \wedge \alpha : \alpha \in I\}$ where $\tau_\alpha = I^X$ if $\alpha \leq \frac{1}{2}$ and $\tau_\alpha = ([0,\alpha^c] \cup [\alpha,1])^X$ if $\alpha > \frac{1}{2}$.
Observe that $\mathcal{T}_{\delta_4}$ is not even; if $\alpha > \frac{1}{2}$ then $\mathcal{T}_{\delta_4}(c_\alpha) = \alpha$ but
$c_\alpha \notin (\bigcup_{\alpha'>\alpha} \tau_{\alpha'})$. Moreover in case X is infinite $\mathcal{T}_{\delta_4}$ is not even also at the level $\frac{1}{2}$; if $M(x) > \frac{1}{2}$ for all $x \in X$ and $\inf M = \frac{1}{2}$, then $\mathcal{T}(M) = \frac{1}{2}$, but, obviously, $M \notin (\bigcup_{\alpha'>\frac{1}{2}} \tau_{\alpha'})$.

5.- The fuzzy proximity $\delta_5(M,N)$ defined by $\delta_5(M,N) = \sup M \wedge \sup N$ for all $M,N \in I^X$ is also not even.
One can get convinced in this by noticing that the α-levels of the generated fuzzy topology $\mathcal{T}_{\delta_5}$ for $\alpha > \frac{1}{2}$ are given by the formula
$\tau_\alpha = [0,\alpha^c]^X \cup [\alpha,1]^X$ and hence $\tau_\alpha \neq (\bigcup_{\alpha'>\alpha} \tau_{\alpha'})$.

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H. Gutierrezs Garsia, A. Šostaks. Par gludām fāzi-topoloģijām.

Anotācija. Darbā ir ieviestas speciālas fāzi-topoloģijas (otrā autora nozīmē) - t.s. gludas fāzi-topoloģijas. Tiek pētītas dažas gludo fāzi-topoloģiju īpašības un to loma vispārīgā fāzi-topoloģisko telpu teorijā. Tiek apskatīts arī gludo fāzi-topoloģiju proksimālais analogs.

Х.Гутиеррез Гарсиа, А. Шостак. О гладких нечетких топологиях.

В заметке рассматриваются нечеткие топологии (в смысле второго автора) специального вида - т.н. гладкие нечеткие топологии. Исследуются некоторые свойства гладких нечетких топологий и обсуждается их роль в общей теории нечетких топологических пространств. Рассмотрен также близостный аналог гладких нечетких топологий.

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ON CLP-COMPACT AND COUNTABLY CLP-COMPACT SPACES

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SUMMARY. By a (countably) clp-compact space we call a topological space each clopen cover (resp. each countable clopen cover) of which contains a finite subcover. Obviously, all compact spaces and all connected spaces are clp-compact. The aim of this paper is to develop foundations of the theory of clp-compact and countably clp-compact spaces. Some relations of these spaces to other classes of topological spaces will be also discussed.

KEY WORDS: compactness, countable compactness, connectedness, zero-dimensionality, totally disconnectedness, clp-compactness, countable clp- compactness.

AMS subject classification: 54D30, 54D20, 54D05, 54G05.

This is the first one in a series of papers where we discuss such properties of topological spaces in which clopen (=closed and open) sets play the principal role.To be more precise we are interested in those topological properties which can be characterized in terms of clopen sets. Important examples of properties of such kind are connectedness and zero-dimensionality. Besides, developing this idea one can introduce also a series of new topological properties, depending only on clopen sets; some of these properties are, in our opinion, quite interesting and useful. This concerns, in particular, the properties of clp-compactness and countable clp-compactness, discussed in the present paper.

By a (countably) clp-compact space we call a topological space each clopen cover (resp. each countable clopen cover) of which contains a finite subcover. The aim of this paper is to develop foundations of the theory of clp–compact and countably clp–compact spaces. Some relations of these spaces to other classes of topological spaces will be also discussed.

Notice, that clp–compact spaces first appeared (under the name of cb–spaces) in the paper [6] written by the second author. That paper contained also some statements about these spaces. However most of the statements in [6] were given without proofs. The present work includes both results from [6] with proofs and, mostly, new results.

The structure of the paper is as follows. In Section 1 we study elementary properties of clp–compact spaces and discuss their relations to some other classes of topological spaces. The problem of products of clp–compact spaces is studied in Section 2. Countably clp–compact spaces are studied in Section 3. In Section 4 spaces of quasicomponents of clp–compact and countably clp–compact spaces are considered.

1. CLP–COMPACT SPACES: ELEMENTARY PROPERTIES

Replacing in the definition of the compact topological space open sets with clopen (=closed and open) sets we come to the concept of a clp–compact space.

(1.1) Definition. A topological space is called *clp–compact* if every its clopen cover (i.e. a cover with clopen sets) contains a finite subcover.

The concept of a clp-space obviously generalizes both compactness and connectedness

(1.2) Assertion. Every compact space is clp-compact.

(1.3) Assertion. Every connected space is clp-compact.

On the other hand, it is easy to construct examples showing that clp-compactness does not reduce to the properties of compactness and connectedness. One general method of constructing such examples is given in (1.10).

From the resemblance of the definitions of compactness and clp-compactness one can expect a certain analogy in the behaviour of these two properties.

How do matters stand in fact one can see from the statements (1.4)–(1.9) below.

(1.4) Proposition. If a topological space X is clp–compact and M is its clopen subspace then M is clp–compact, too.

Proof is obvious and therefore omitted.

(1.5) Remark. It is natural to call a subset M of a space X *clp–compact* if each cover of M with clopen sets in X has a finite subcover. Obviously if M is clp–compact subspace of X, then it is also a clp–compact set of X. However, as different from the propety of compactness, the converse does not hold. In particular, each subset of a connected space is , obviously, clp–compact.

One can easily prove also the following two statements:

(1.6) Proposition. If a topological space X is clp–compact and there exists a continuous mapping from X onto a space Y, then Y is also clp–compact.

(1.7) Proposition. A space X is clp–compact iff every system of its clopen subsets with the finite intersection property has non-empty intersection.

(1.8) Theorem. If a space Y is clp–compact and f is a mapping from a space X into Y with the following properties:

1) f is clopen, i.e. for every clopen subset U of X the image $f(U)$ is a clopen subset of Y;

2) for every point y from Y the preimage $f^{-1}(y)$ is a clp–compact subset of X;

then the space X is clp–compact, too.

Proof. Let $\mathcal{U} = \left\{U_i : i \in I\right\}$ be a clopen cover of the space X. Since for each $y \in Y$ the set $A_y = f^{-1}(y)$ is clp-compact in X, there exists a finite subfamily $\mathcal{U}_y$ of $\mathcal{U}$, covering A_y. Then, obviously $\cup\mathcal{U}_y =: B_y$ is a clopen set in X, containing the set A_y. We shall prove that for every $y \in Y$ there exists a clopen neighbourhood V_y such that $f^{-1}(V_y) \subset B_y$.

Really, the set $F_y := X \setminus B_y$ is clopen in X and therefore $f(F_y)$ is clopen in Y. Then the set $V_y = Y \setminus f(F_y)$ is also clopen and obviously it contains the point y .Besides,

$$F_y \cap f^{-1}(V_y) \subset f^{-1}\left(f\left(F_y\right)\right) \cap f^{-1}(V_y) = f^{-1}\left(f\left(F_y\right)\right) \cap V_y =$$

$= f^{-1}(\varnothing) = \varnothing$, i. e. $f^{-1}(V_y) \subset B_y$.

Since $\mathcal{V} = \left\{V_y : y \in Y\right\}$ is obviously a clopen cover of the clp–compact space Y, there exists a finite subcover $\mathcal{V}' = \left\{V_{y_1}, \dots, V_{y_n}\right\}$. Hence, taking into account that $f^{-1}\left(V_{y_k}\right) \subset B_{y_k}$ and $B_{y_k} = \cup\left\{U : U \in \mathcal{U}_{y_k}\right\}$ for all $k=1,\dots,n$ and each $\mathcal{U}_{y_k}$ is finite, we conclude that $\bigcup_{k=1}^{n} \mathcal{U}_{y_k}$ is a finite subcover of the given clopen cover $\mathcal{U}$ of the space X, i. e. X is clp–compact.

(1.9) Proposition. A direct sum $\oplus X_i$ of a family $\{X_i : i \in I\}$ of topological spaces is clp–compact iff $|I| < \aleph_0$ and each X_i is clp–compact. (Here $|A|$ denotes the cardinality of the set A.)

Proof is obvious and therefore omitted.

Basing on this fact it is easy to establish the following result which is useful for constructing new clp–compact spaces from old ones:

(1.10) **Proposition.** Let $(X,\mathcal{T})$ be a topological space and let $\mathcal{T}_A$ be the topology on X determined by the family $\mathcal{T}\cup\{A\}\cup\{X\setminus A\}$ as a subbase. Then $(X,\mathcal{T}_A)$ is clp–compact iff the subspaces $(A,\mathcal{T}|_A)$ and $(X\setminus A,\mathcal{T}|_{X\setminus A})$ of the space $(X,\mathcal{T})$ are clp–compact.

(Notice that clp–compactness of $(X,\mathcal{T}_A)$ obviously implies clp–compactness of $(X,\mathcal{T})$.)

We end this section with considering clp–compactness in connection with the properties of total disconnectedness and zero–dimensionality. (A space is called totally disconectedness if each point in it is an intersection of clopen sets . A space is called zero–dimensional if it has a base of clopen sets; no separation axioms are assumed unless additionally stated.) The significance of total disconnectedness in the theory of clp–compactness is in a certain sense analogous to the significance of the Hausdorff axiom in the theory of compactness. Moreover, for zero–dimensional spaces the properties of clp–compactness and compactness become equivalent.

(1.11) **Proposition.** A clp–compact subspace of a totally disconnected space is closed.

Proof. Let A be a clp–compact subspace in a totally disconnected space X and let $x\notin A$. By total disconnectedness of X for each point $y\in A$ there exists a clopen set U_y which contains y but does not contain x. By clp–compactness of A one can choose a finite number of points $y_1,\dots,y_n$ such that $A\subset U_{y_1}\cup\dots\cup U_{y_n} =: U_A$. Then obviously U_A is a clopen neighbourhood of A which does not contain x.

(1.12) Proposition. A zero-dimensional space is clp–compact iff it is compact.

Proof. The "if" part is obvious, (see also (1.2)). Conversely, assume that X is a zero–dimensional clp–compact space and let $\mathcal{U}$ be its open cover. Since X is zero–dimensional, each $U \in \mathcal{U}$ is a union of clopen sets, i. e. $U = \cup\left\{V_i : i \in I_U\right\}$ and hence $\mathcal{V} = \cup\left\{V_i : i \in \bigcup_{U \in \mathcal{U}} I_U\right\}$ is a clopen refinement of the cover $\mathcal{U}$. Since X is clp–compact, there exists a finite subcover $\mathcal{V}'$ of $\mathcal{V}$ and hence also a finite subcover $\mathcal{U}'$ of $\mathcal{U}$. Thus X is compact.

(1.13) Corollary. A Hausdorff zero–dimensional clp–compact space is normal.

(1.14) Corollary. If X is a clp–compact zero–dimensional space and A is its closed subset, then A is clp–compact as a subspace (and hence also as a subset) of X.

(1,15) Corollary. A continuous mapping from a clp–compact zero–dimensional space X into a zero–dimensional space Y is closed.

In the statement (1.12) the condition of zero–dimensionality can not be replaced by the condition of total disconnectedness. One can see this from the following example:

(1.16) Example. (A totally disconnected clp–compact non–compact space.)

Let $\mathcal{C}$ be the Cantor set and let $\mathcal{C} = \left\{ \mathcal{C}_\varsigma : \varsigma < c \right\}$ be a decomposition of $\mathcal{C}$ into continuum of its dense subsets $\mathcal{C}_\varsigma$ with cardinality c. Let $[0,1] = \left\{t_\varsigma : \varsigma < c\right\}$ and define a subset X of the product $\mathcal{C} \times [0,1]$ as $X = \cup\left\{ \mathcal{C}_\varsigma \times \left\{t_\varsigma\right\} : \varsigma < c\right\}$. It is easy to notice that this space has the desired properties.

2. PRODUCTS OF CLP–COMPACT SPACES

The fundamental feature of the property of compactness is its multiplicativity. Unfortunately, as it will be shown by Example 2.7 clp–compactness is not multiplicative: already the product of two clp–compact spaces may fail to be clp–compact. (Notice that in [6] it was erroneously claimed that clp–compactness is preserved by products.) However as it is established below, there are some positive results about products of clp -compact spaces in some special cases.

(2.1) Theorem. The product of a connected space and a clp–compact space is clp–compact.

Proof. Let X be a connected space and Y be clp–compact. Notice first of all that in this case each clopen subset W of the product $X \times Y$ looks like $X \times V$ where $V = p_y(W)$, i. e. V is the image of W under the projection $p_y: X \times Y \to Y$. Indeed, if $W \neq X \times V$, then there exists $y_0 \in Y$ such that $\left(X \times \{y_0\}\right) \cap W \neq \varnothing$ and $\left(X \times \{y_0\}\right) \cap \left(X \times Y \setminus W\right) \neq \varnothing$. However, this obviously contradicts the fact that X is connected.

Moreover, it is easy to notice that if $W = X \times V$ and W clopen, than V is a clopen subset of Y. Hence a clopen cover $\mathcal{U} = \left\{X_i \times V_i : i \in I\right\}$ of the product $X \times Y$ determines the clopen cover $\mathcal{V} = \left\{V_i : i \in I\right\}$ of Y. The statement of the theorem follows now easily from the fact of clp–compactness of the space Y.

(2.2) Corollary. The product of a connected space and a compact space is clp–compact.

(2.3) Theorem The product of a compact space and a clp–compact space is clp–compact.

Proof. Assume that X is a compact space and Y is a clp–compact space. Then the projection $p_y: X \times Y \to Y$ being a projection along a compact space transfers every clopen subset W of $X \times Y$ into a clopen set $V = p_y(W)$ in Y. Besides, all preimages of points under it are homeomorphic to the compact space X. Now the conclusion of the theorem follows directly from Theorem 1.8.

(2.4) **Remark.** Analysing the proofs of Theorem 2.1 and 2.3 one can easily notice that both of them were based on the fact that the projection $p_y: X \times Y \to Y$ along X is a clopen mapping, i. e. transfers a clopen set into clopen. (In case of Theorem 2.3 it was guaranteed by compactness of X and in case of Theorem 2.1 it was guranteed by connectedness of X.) Let us call a space X clp–projective if for every clp–compact space Y the projection $p_y: X \times Y \to Y$ is clopen. Now the following generalization and specification of Theorems 2.1 and 2.3 can be formulated and easily proved.

(2.5) **Theorem.** If X is a clp–projective clp–compact space then the product $X \times Y$ is clp–compact iff Y is clp–compact.

(2.6) **Problem.** Is the converse of (2.5) true? To be precise, is it true, that if the product $X \times Y$ is clp–compact for every clp–compact space Y, then X is clp–projective?

(2.7) **Example.** A clp–compact space X such that the product $X \times X$ is not clp–compact.

In [7] R. M. Stephenson has constructed a completely Hausdorff countably compact *U(i)* space X such that the product space $X \times X$ is not pseudocompact. We shall show that the same space X can be used also for our purposes. To make the paper self–contained, we shall reproduce here Stephenson's construction.

Let G be a subspace of the Stone–Čech compactification βN of the countable discrete space N which has the following properties: $N \subset G$; every infinite subset of βN has a limit point in G and there is an infinite closed subset D of the product $G\times G$ such that $D\subset N\times N$.(The existence of such a set G was established by Teresaka, see e. g. Theorem 9.15 in [5].)

Let B be the space whose points are those of βN and whose topology is the collection of all sets of the form $V\cup\left(W\cap\left(\beta N\setminus\left(G\setminus N\right)\right)\right)$ where V, W are open subsets of βN. Let $\mathcal{A}=\left\{0,1\right\}$ be the discrete space and let $X'=G\times\left\{0\right\}\cup B\times\left\{1\right\}$ be the subspace of the product space $P=B\times\mathcal{A}$. Consider the equivalence relation ρ on X' defined by the rule $\left(t,\alpha\right)\rho\left(s,\alpha'\right)$ if either $t=s$ and $\alpha=\alpha'$ or $t=s\in G\setminus N$. Now the space X is defined as the quotient space X'/ρ. We continue to use the symbols $\left(t,\alpha\right)$ for the points of X; thus $\left(t,0\right)=\left(t,1\right)$ for $t\in G\setminus N$.

To show that X is clp–compact consider a system $\mathcal{U}$ of clopen sets with finite intersection property. We need to show that there exists a point $(p,\alpha)\in\cap\left\{U:U\in\mathcal{U}\right\}$.

Since $N\times\mathcal{A}$ is a dense subset of X and $N\times\left\{0\right\}\cap N\times\left\{1\right\}=\varnothing$, for some $\alpha\in\mathcal{A}$ the system $\mathcal{U}'=\left\{U\cap N\times\left\{\alpha\right\}:U\in\mathcal{U}\right\}$ has the finite intersection property. It is easy to conclude from here that there exists an ultrafilter $\mathcal{M}$ on N such that $\mathcal{U}'$ is contained in the family $\left\{M\times\left\{\alpha\right\}:M\in\mathcal{M}\right\}$, and hence each member $U\in\mathcal{U}$ contains some set of the form $M\times\left\{\alpha\right\}$ where $M\in\mathcal{M}$.

Let p be the point of βN which has $\left\{Cl_{\beta N}M:M\in\mathcal{M}\right\}$ as a fundamental system of neighbourhoods. We shall show that $\left(p,\alpha\right)\in\cap\mathcal{U}$. To do this consider the following four possibilities:
(1) $p\in N$; (2) $p\in G\setminus N$; (3) $p\in\beta N\setminus G$, $\alpha=1$ and (4) $p\in\beta N\setminus G$, $\alpha=0$.

(1) In this case $p\in\cap\mathcal{M}$ and hence $\left(p,\alpha\right)\in\cap\mathcal{U}$.

(2) In this case $\left(p,0\right)=\left(p,1\right)$. Let V be an open neighbourhood of $\left(p,\alpha\right)$ in X. Besides, without loss of generality in this case we may

assume that V is open also in βN, and hence there exists $M_p \in \mathcal{M}$ such that $Cl_{\beta N} M_p \times \{\alpha\} \subset V$. However this means that V intersects each $U \in \mathcal{U}$ and therefore, taking into account that all U are clopen, $(p,\alpha) \in \cap \{U : U \in \mathcal{U}\}$.

(3) Let V be an open neighbourhood of $(p,1)$ in X. Without loss of generality we may assume in this case that $V \cap (B \times \{1\}) = (W \cap (\beta N \setminus G) \cup N) \times \{1\}$ for some open neighbourhood W of p in βN and hence there exists $M \in \mathcal{M}$ such that $M \times \{1\} \subset V$. However this means that V intersects each $U \in \mathcal{U}$ and hence again $(p,1) \in \cap \{U : U \in \mathcal{U}\}$.

(4) Let V be an open neighbourhood of $(p,1)$ in X. (Notice that the point $(p,0)$ does not exists in this case!) As in (3) it is clear that there exists $M \in \mathcal{M}$ such that $M \times \{1\} \subset V$.

Take an arbitrary set $U \in \mathcal{U}$. Then, from the maximiality of $\mathcal{M}$ it easily follows that the set $Y = \{n \in M : (n,0) \in U\}$ belongs to $\mathcal{M}$. Since $\mathcal{M}$ is, obviously, a free ultrafilter, the set Y is infinite, and therefore Y has a limit point $g \in G$ Thus $(g,1) = (g,0) \in \overline{Y} \times \{1\} \subset \overline{V} = V$. On the other hand, it is obvious that $(g,1) \in \overline{U} = U$. Thus $\overline{V} \cap U \neq \varnothing$. It follows from here that $(p,1) \in U$. (Indeed, otherwise $(p,1) \in X \setminus U = V$ is a clopen neighbourhood of $(p,1)$ such that $V \cap U = \varnothing$.) Thus again $(p,1) \in \cap \{U : U \in \mathcal{U}\}$.

To complete the proof, we have to show that $X \times X$ is not clp–compact.

By setting $f(s,t) = ((s,0),(t,0))$ we define a mapping f from the space $G \times G$ onto the closed subspace $(G \times \{0\}) \times (G \times \{0\})$ of the product $X \times X$. Besides, it is easy to notice, that f is a homeomorphism. Therefore, the image $f(D)$ of the set $D \subset N \times N \subset G \times G$ is a countable closed subset of $X \times X$. On the other hand, since, $f(D) \subset (N \times \{0\}) \times (N \times \{0\})$, $f(D)$ is also an open discrete subset of $X \times X$. From here it is clear that $X \times X$ is not clp-compact.

(2.8) Remark. Since in the above example the set $f(D)$ is countable, it is easy to notice that the product $X \times X$ is not countably clp–compact (see Section 3), too. Thus the product of two clp–compact spaces need not be even countably clp–compact.

3. COUNTABLY CLP–COMPACT SPACES

(3.1) Definition. A topological space X is called *countably clp-compact* if every its countable clopen cover (i. e. a countable cover with clopen sets) has a finite subcover.

The following two statements are obvious:

(3.2) Assertion Every countably compact space is countably clp-compact.

(3.3) Assertion. Every clp–compact space is countably clp–compact.

For spaces of countable weight, the converse is also true:

(3.4) Proposition. A space of the countable weight is compact iff it is countably clp–compact.

The following theorem presents different characterizations for the property of countable clp–compactness.

(3.5) Theorem. The following conditions are equivalent for a topological space X:

(1) X is countably clp–compact;

(2) every countable system of clopen subsets of X with finite intersection property has a non-empty intersection;

(3) every clopen disjoint cover of X is finite ;

(4) every discrete family of clopen sets is finite;

(5) for every continuous mapping $f: X \to N$ the image $f(X)$ is finite. (Here N is the countable discrete set.)

Proof. The equivalence $(1) \Leftrightarrow (2)$ is obvious.

To show the implication $(1) \Rightarrow (3)$ assume that $\mathcal{U} = \{U_\alpha : \alpha \in \mathcal{A}\}$ is a clopen disjoint infinite cover of X, where all U_α are non-empty, and let $\mathcal{A}_o$ be a countable subset of $\mathcal{A}$ and $V = \cup\{U_\alpha : \alpha \notin \mathcal{A}_o\}$ Then obviously $\{U_\alpha : \alpha \in \mathcal{A}_o\} \cup \{V\}$ is a countable clopen cover of X and hence there exists a finite subcover $\left\{U_{\alpha_1}, \ldots, U_{\alpha_n}, V\right\}$. However, this contradicts the assumption that all U_α are non–empty and $U_\alpha \cap U_{\alpha'} = \emptyset$ for $\alpha \neq \alpha'$.

$(3) \Rightarrow (4)$ Assume that $\{U_\alpha : \alpha \in \mathcal{A}\}$ is a discrete system of clopen sets in X, and let $V = X \setminus \cup\{U_\alpha : \alpha \in \mathcal{A}\}$. Then obviously, $\{U_\alpha : \alpha \in \mathcal{A}_o\} \cup \{V\}$ is a clopen disjoint cover of X. According to (3) it is finite, and hence the system $\{U_\alpha : \alpha \in \mathcal{A}\}$ is also finite.

$(4) \Rightarrow (3)$ is obvious, because a clopen disjoint cover of X is at the same time also a discrete family of clopen sets.

To show the implication $(3) \Rightarrow (1)$ consider a countable clopen cover $\mathcal{U} = \{U_n : n \in N\}$. By induction we obtain a disjoint countable cover $\mathcal{V} = \{V_1, \ldots, V_n, \ldots\}$ of X as follows:

$V_1 = U_1, V_2 = U_2 \setminus U_1, \ldots, V_n = U_n \setminus U_1 \cup \ldots \cup U_{n-1}$ for all $n \in N$. From (3) it follows that there exists $n_o \in N$ such that $V_n = \emptyset$ for all

$n \geq n_o$, and hence $\left\{U_1,...,U_{n_o}\right\}$ is a finite subcover of $\mathcal{U}$.

Finally, to complete the proof notice that the existence of a countable disjoint clopen cover of X is equivalent to the existence of a continuous mapping $f: X \to N$ with an infinite range, and hence $(3) \Leftrightarrow (5)$.

The class of countably clp–compact spaces is obviously hereditary with respect to clopen subspaces and is invariant under taking continuous images:

(3.6) **Proposition.** If X is a countably clp–compact space and Y is its clopen subspace, then Y is countably clp–compact, too.

(3.7) **Proposition.** If X is a countably clp–compact space and $f: X \to Y$ is a continuous surjection, then the space Y is countably clp–compact, too.

As shown in Section 1, in the realm of zero–dimensional spaces the properties of compactness and clp–compactness become equivalent (see Proposition 1.12). It is interesting to compare with this fact the following result:

(3.8) **Proposition.** If X is a zero–dimensional space, then X is countably clp–compact iff X is pseudocompact.

Proof. If X is not countably clp–compact, then according to Theorem 3.5 there exists a continuous unbounded mapping $f: X \to N$ and hence, moreover, a continuous unbounded mapping $f: X \to R$, i. e. X is not pseudocompact.

Conversely, let X be a non–pseudocompact space and let $f: X \to R$ be an continuous unbounded function. Then it is easy to construct a countable discrete family of open sets in X. Moreover, since X is zero–dimensional, these sets can be chooøen clopen. Hence according to Theorem 3.5 the space X is not countably clp–compact.

(3.9) **Proposition.** If X is a paracompact countably clp–compact strongly zero–dimensional T_1 space, then X is compact.

Proof. Let $\mathcal{U}$ be an open cover of X; since X is paracompact and regular (as a zero–dimensional space), there exist an open locally finite refinement $\mathcal{V} = \left\{V_\alpha : \alpha < \tau\right\}$ and a closed locally finite refinement $\mathcal{F} = \left\{F_\alpha : \alpha < \tau\right\}$ of $\mathcal{U}$ such that $F_\alpha \subset V_\alpha$ for every $\alpha < \tau$. (The sets constituting these refinements are indexed by all ordinals, less than some ordinal τ.) Now, since X is strongly zero–dimensional, for each $\alpha < \tau$ there exists a clopen set W_α such that $F_\alpha \subset W_\alpha \subset V_\alpha$. Then, obviously, $\mathcal{W} = \left\{W_\alpha : \alpha < \tau\right\}$ is also a clopen locally finite refinement of $\mathcal{U}$. By transfinite induction we construct a new cover $\left\{W'_\alpha : \alpha < \tau\right\}$, where $W'_\alpha = W_\alpha \setminus \bigcup_{\beta < \alpha} W_\beta$. Since $\left\{W_\alpha : \alpha < \tau\right\}$ is locally finite and the sets W_α are clopen, the sets W'_α are clopen, too. Thus $\left\{W'_\alpha : \alpha < \tau\right\}$ is a disjoint clopen cover of X refining $\mathcal{W}$, and hence also refining the original cover $\mathcal{U}$. According to Theorem 3.5, W'_α is in fact finite (i. e. all but a finite number of sets W'_α are empty). Hence there exists a finite subcover in $\mathcal{U}$, i. e. the space X is compact.

With respect to products the behaviour of countable clp–compactness has analogies with the behaviour of clp–compactness. In particular, patterned after the proof of Theorem 2.5 one can easily establish the following result:

(3.10) Theorem. If X is clp–projective and countably clp–compact, then the product $X \times Y$ is countably clp–compact, iff Y is countably clp–compact.

(3.11) Corollary. The product of a compact space and a countably clp–compact space is countably clp- compact.

(3.12) Corollary. The product of a connected space and a countably clp–compact space is countably clp–compact.

4.THE SPACES OF QUASICOMPONENTS OF CLP–COMPACT AND COUNTABLY CLP–COMPACT SPACES

In this section we shall establish correspondence between the properties of (countable) clp–compactness of a space X and certain compactness–type properties of the quasicomponent space $Q(X)$ (see Theorems 4.1, 4.4, 4.5).

Recall first that the quasicomponent A_x of a point x in a space X is defined as the intersection of all clopen sets containing x [2]. So $y \in A_x$ iff there are no clopen sets containing x but not y. Obviously, for different points $x, y \in X$ either $A_x = A_y$ or $A_x \cap A_y = \varnothing$. Therefore one can define an equivalence relation q on X by setting xqy iff $A_x = A_y$. Following [2] we endow the set $Q(X)$ of all quasicomponents of X with the topology $\mathcal{T}$, determined by the base consisting of all sets $U' = \{A : A \in Q(X), A \subset U\}$, where U is a clopen subset of X. It is clear that the quotient mapping $q: X \to Q(X)$ is continuous. Notice, however, that the topology $\mathcal{T}$ thus defined generally differs from the quotient topology on the set $Q(X)$.

(4.1) Theorem. The following statements are equivalent for a topological space:

(1) X is clp–compact;
(2) $Q(X)$ is clp–compact;
(3) $Q(X)$ is compact.

<u>Proof.</u> $(1) \Rightarrow (2)$ If X is clp–compact, then $Q(X)$ is also clp–compact as a continuous image of X (see Proposition 1.6).

$(2) \Rightarrow (3)$ Since $Q(X)$ is zero–dimensional (see e. g. [2]) and clp–compact, it is also compact by Proposition 1.12.

The implication $(3) \Rightarrow (2)$ is obvious.

To show the implication $(2) \Rightarrow (1)$ assume that X is not clp-compact, and let $\mathcal{U}$ be a clopen cover of X having no finite subcover. Since the quotient mapping $q: X \to Q(X)$ is , obviously , clopen , the system $\{q(U): U \in \mathcal{U}\}$ is a clopen cover of $Q(X)$. Besides, it can be easily seen that no finite subcover can exists in it, and hence $Q(X)$ is not clp–compact.

(4.2) Corollary. If $f: X \to Y$ is a surjective continuous mapping and the space $Q(X)$ is compact, then the space $Q(Y)$ is compact, too

In virtue of Proposition 1.8, Theorem 4.1 implies also the following:

(4.3) **Corollary.** If $f: X \to Y$ is a clopen continuous mapping, $Q(Y)$ is compact and for all points $y \in Y$ the spaces $Q(f^{-1}(y))$ are compact, then the space $Q(X)$ is compact, too.

In case when X is Hausdorff, the characterization of clp–compactness established in (4.1) admits a further specification:

(4.4) **Theorem.** A Hausdorff space X is clp–compact iff the space $Q(X)$ of its quasicomponents is homeomorphic to a Cantor set D^{τ} (for the appropriate cardinal number τ).

<u>Proof.</u> The "if" part is an immediate consequence of Theorem 3.1. Conversely, assume that X is a T_2–space, then according to [2, Theorem

5, p. 160] there exists a one-to-one mapping $f: Q(X) \to D^{\tau}$. On the other hand, according to Theorem 3.1, the space $Q(X)$ is compact and hence f is a homeomorphism.

Theorem 4.1 characterizing clp-compact spaces as those ones, whose spaces of quasicomponents are compact, allows also to estimate the cardinality of the family *Clp* (*X*) of all clopen subsets of a clp-compact space *X*:

(4.5) **Theorem.** If $|Clp(X)| \le \aleph_0$, then *X* is clp-compact. Conversely, if *X* is a clp-compact space and $w(Q(X)) \le \aleph_0$ (in particular, $w(X) \le \aleph_0$) then $|Clp(X)| \le \aleph_0$. (Here $w(X)$ denotes the weight of the space *X*, and $|A|$ stands for the cardinality of the set *A*.)

Proof. Assume that *X* is not clp-compact, then in virtue of (4.1) the space $Q(X)$ is not compact. We shall show that $Q(X)$ contains uncountably many clopen sets in this case. Indeed, consider the two possibilities:

(1) If $w(Q(X)) \le \aleph_0$, then the space $Q(X)$ is metrizable and hence there exists a countable discrete subset $\{x_1, \dots, x_n, \dots\}$ in $Q(X)$. Since the space $Q(X)$ is zero-dimensional, one can easily construct a discrete family of clopen neighbourhoods $U_1, \dots, U_n, \dots$ of points $x_1, \dots, x_n, \dots$ respectively. Now, taking unions of the sets $U_1, \dots, U_n, \dots$ in different combinations, we get exactly continuum different clopen sets in the space $Q(X)$, i. e. $|Clp(Q(X))| \ge c$.

(2) If $w(Q(X)) > \aleph_0$, then, in virtue of zero-dimensionality of $Q(X)$, it follows that $|Clp(Q(X))| \ge w(Q(X)) > \aleph_0$.

To complete the proof of the first part of the theorem, notice that, obviously $|Clp(X)| = |Clp(Q(X))|$.

Conversely assume that X is clp–compact and $w(Q(X)) \leq \aleph_o$. (The unequality $w(Q(X)) \leq \aleph_o$ is guaranteed also in case $w(X) \leq \aleph_o$ because $Q(X)$ is a continuous image of X and $Q(X)$ is compact.)

As $Q(X)$ is zero–dimensional, there exists a countable base in $Q(X)$ consisting of clopen sets: $\mathcal{B} = \{U_1, \ldots U_n, \ldots\}$. Applying Proposition 1.4 we conclude that each clopen set V in $Q(X)$ can be expressed as a union of a finite number of sets from $\mathcal{B}$, and hence the family of all clopen sets in $Q(X)$ is countable. Hence the number of clopen sets in X is countable, too.

(4.6) **Theorem.** A space X is countably clp–compact iff the space $Q(X)$ of its quasicomponents is pseudocompact.

Proof. Assume that X is not countably clp–compact, then there exists a countable clopen disjointr cover $\{U_1, \ldots, U_n, \ldots\}$ of X, where all U_n are non–empty. Let $V_n = q(U_n)$ where $q: X \to Q(X)$ is the quotient mapping. Then, obviously $\{V_1, \ldots, V_n, \ldots\}$ is a countable clopen disjoint cover of $Q(X)$ and all V_n are non–empty. It is easy to construct now a continuous unbounded mapping $f: Q(X) \to R$.

Conversely, if the space $Q(X)$ is not pseudocompact and is zero–dimensional, it is easy to construct in it a discrete countable family of clopen sets $V_1, \ldots, V_n, \ldots$. To complete the proof it is sufficient to notice that $q^{-1}(V_1), \ldots, q^{-1}(V_n), \ldots$ is a discrete family of clopen sets in X and to apply Theorem 3.5.

Since a zero–dimensional space X is homeomorphic to the space of its quasicomponents $Q(X)$, from the previous theorem we get the following:

(4.7) **Corollary.** A zero–dimensional space is pseudocompact iff it is countably clp–compact.

(4.8) **Remark.** As shown by L. Shapiro (private communication), there exists a zero-dimensional pseudocompact space X which fails to be countably compact. Obvious in this case $Q(X)=X$ and hence X is an example of a countably clp-compact space such that the space of quasicomponents $Q(X)$ is not countably compact.

(4.9) **Remark.** Note that separable meric spaces with compact spaces of quasicomponents were considered by H. Freudental [4] to construct a special kind of compactification (the so called λ–compactifications).

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О clp-компактных и счётно clp-компактных пространствах.

Аннотация. Топологическое пространство назовём (счётно) clp-компактным, если каждое его (счётное) покрытие открыто-замкнутыми множествами имеет конечное подпокрытие. Очевидно, компактные пространства и связные пространства являются примерами clp-пространств. В данной заметке развиваются основы теории clp-компактных и счётно clp-компактных пространств. Свойство (счётной) clp-компактности данного пространства X характеризуется посредством соответствующих свойств пространства Q(X) его квазикомпонент.

Par clp–kompaktām un sanumurējami clp–kompaktām telpām.

Anotācija. Par clp–kompaktu (sanumurējami clp–kompaktu) telpu tiek saukta tāda topoloģiska telpa, kuras katrs pārklājums ar reizē slēgtām un vaļējām kopām satur galīgu apakšparklājumu Acīmredzami, kompaktas, kā arī sakarīgas telpas ir clp–kompaktas telpas. Šis raksts ir veltīts clp–kompaktu (sanumurējami clp–kompaktu) telpu pamatteorijas izveidei, kā arī iztirzā šo telpu saistību ar citām topoloģisku telpu klasēm. Tiek dots clp–kompaktības (sanumurējami clp–kompaktības) raksturojums ar atbilstošo kvazikomponenšu telpu.

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ON CLP–LINDELÖF AND CLP–PARACOMPACT SPACES

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SUMMARY. By a clp–Lindelöf space we call a topological space each clopen cover of which contains a countable subcover, and by a clp–paracompact space we call a topological space each clopen cover of which contains a locally finite clopen refinement. The aim of this paper is to study basic properties of clp–Lindelöf and clp–paracompact spaces. Some relations of these spaces to other classes of topological spaces will be also discussed.

KEY WORDS. Lindelöfness, paracompactness, compactness, connectedness, Souslin property, zero–dimensionality, clp–compactness, clp–Lindelöfness, clp–paracompactness, clp–Souslin property.

AMS subject classification: 54D20, 54D18, 54C05.

This paper continues a series of works where we study topological properties defined by clopen covers, i. e. all elements of which are clopen (=closed and open) sets.

The present article considers clp–Lindelöf and clp–paracompact spaces which generalize the clp–compact spaces studied in our first paper [4].

By a clp–Lindelöf space we call a topological space each clopen cover of which contains a countable subcover, and by a clp–paracompact space we call a topological space each clopen cover of which contains a locally finite clopen refinement. The aim of this paper is to study basic properties of clp–Lindelöf and clp–paracompact spaces and to point out some relations of these spaces to other classes of topological spaces.

In Section 1 we study properties of clp–Lindelöf spaces. Clp–paracompact spaces are the subject of Section 2. In the last, Section 3, spaces of quasicomponents of clp–Lindelöf and clp–paracompact spaces are considered.

1. CLP–LINDELÖF SPACES

Replacing in the definition of a Lindelöf space open sets with clopen sets we come to the concept of a clp–Lindelöf space.

(1.1) Definition. A topological space is called *clp–Lindelöf* if every its clopen cover (i. e. a cover all elements of which are clopen sets) contains a countable subcover.

The concept of a clp–Lindelöf space is generalization of a clp–compact space on the one hand and of a Lindelöf space on the other:

(1.2) Assertion. Every clp–compact space is clp–Lindelöf.

(1.3) Assertion. Every Lindelöf space is clp–Lindelöf.

The Sorgenfrey line gives an example of a clp–Lindelöf space, which fails to be clp–compact. And the Niemytzki plane is a clp–Lindelöf, but not a Lindelöf space.

It is easy to see that the following statement holds:

(1.4) Proposition. A zero–dimensional space is clp–Lindelöf iff it is Lindelöf.

From the resemblance of the definitions of Lindelöf and clp–Lindelöf spaces one can expect a certain analogy in the properties of these spaces. The next statements establish how do the matters stand in fact:

(1.5) Proposition. If a space X is clp- Lindelöf and M is its clopen subspace then M is clp–Lindelöf, too.

(1.6) Proposition. If a space X is clp–Lindelöf and there exists a continuous mapping from X onto a space Y then Y is clp–Lindelöf, too.

(1.7) Proposition. The direct sum $X = \oplus X_i, i \in I$ of non-empty spaces X_i is clp–Lindelöf iff all $X_i, i \in I$, are clp–Lindelöf and the set I is countable.

Proofs of these statements are obvious and therefore omitted.

Recall that a system of sets is said to have the countable intersection property if the intersection of every its countable subsystem is not empty.

(1.8) Proposition. A space X is clp–Lindelöf iff every system of its clopen subsets with the countable intersection property has the non–empty intersection.

Proof. Let X be a clp–Lindelöf space and let $\mathcal{V} = \{V_i : i \in I\}$ be a system of its clopen subsets with the countable intersection property. Suppose that $\bigcap_{i \in I} V_i = \emptyset$. Then $\mathcal{U} = \{U_i : U_i = X \setminus V_i, i \in I\}$ is a clopen cover of the space X. Since X is clp–Lindelöf there exists a countable subcover $\{U_{i_1}, \ldots, U_{i_n}, \ldots\}$ and $X = \bigcup_{n \in N} U_{i_n} = \bigcup_{n \in N} \left(X \setminus V_{i_n}\right) = X \setminus \bigcap_{n \in N} V_{i_n}$. Hence it follows that $\bigcap_{n \in N} V_{i_n} = \emptyset$ but this contradicts the definition of $\mathcal{V}$.

Conversely, assume that every system of clopen subsets of a space X with the countable intersection property has the non–empty intersection and let $\mathcal{U} = \{U_i : i \in I\}$ be a clopen cover of the space X. Then the collection $\mathcal{V} = \{V_i : V_i = X \setminus U_i, i \in I\}$ is the system of clopen sets in X and besides $\bigcap_{i \in I} V_i = \emptyset$. Then there exists a countable subfamily $\{V_{i_1}, \ldots, V_{i_n}, \ldots\}$ with the empty intersection. It is clear that the corresponding sets $U_{i_1},\dots,U_{i_n},\dots$ make a countable subcover of $\mathcal{U}$.

(1.9) **Proposition.** If a space Y is clp–Lindelöf and f is a mapping from a space X into Y with following properties:

1) f is clopen, i. e. for every clopen subset of X its image is a clopen subset of Y,

2) for every point y from Y the preimage $f^{-1}(y)$ is a clp–compact subset of X,

then the space X is clp–Lindelöf, too.

Proof of this fact is similar to proof of Theorem 1.8 in [4] and therefore omitted.

As shown by the next example the property of clp–Lindelöfness is not multiplicative.

(1.10) **Example.** The Sorgenfrey line R_b is clp–Lindelöf but $R_b \times R_b$ is not clp–Lindelöf.

(1.11) **Proposition.** The product of a clp–Lindelöf space and a compact space is clp–Lindelöf.

(1.12) **Proposition.** The product of a clp–Lindelöf space and a connected space is clp–Lindelöf.

Proofs of the last two propositions are similar to the corresponding proofs of the propositions about clp–compact spaces (see [4], Section 2).

Specifying the standard terminology we say that a system $\mathcal{B} = \{B_t : t \in T\}$ of subsets of X is a refinement of another system $\mathcal{A} = \{A_s : s \in S\}$ of subsets of X if $\cup\mathcal{A} = \cup\mathcal{B}$ and for every $t \in T$ there exists $s_t \in S$ such that $B_t \subset A_{s_t}$.

In the sequel we shall need the following:

(1.13) Lemma. If $\mathcal{U}$ is a locally finite system of clopen sets of X then there exists a disjoint clopen (and hence also locally finite) refinement $\mathcal{U}'$ of $\mathcal{U}$.

<u>Proof.</u> Let $\mathcal{U} = \left\{U_i : i \in I\right\}$ be a locally finite system of clopen sets in X. Let $U_i' := U_i \setminus \bigcup_{j<i} U_j$. For each $i \in I$ the set U_i' is closed, because U_i is closed and $\bigcup_{j<i} U_j$ is an open set. On the other hand in virtue of local finitness of the system $\mathcal{U}$ it holds $\overline{\bigcup_{j<i} U_j} = \bigcup_{j<i} \overline{U_j} = \bigcup_{j<i} U_j$, and hence $\bigcup_{j<i} U_j$ is closed. Therefore for each $i \in I$ the set U_i' is open, too. It is easy to notice now that the family $\mathcal{U}' = \left\{U_i' : i \in I\right\}$ is a disjoint clopen refinement of the system $\mathcal{U}$.

(1.14) Proposition. Every locally finite system of non–empty clopen sets in a clp–Lindelöf space is countable.

<u>Proof.</u> Assume that $\mathcal{U} = \left\{U_i : i \in I\right\}$ is an uncountable locally finite system of non–empty clopen sets in a clp–Lindelöf space X. From Lemma 1.13 it follows that there exists a disjoint clopen locally finite refinement $\mathcal{U}' = \left\{U_j' : j \in J\right\}$ of $\mathcal{U}$. Besides, since $\mathcal{U}$ is locally finite and uncountable, it is easy to notice that the system $\mathcal{U}'$ is also uncountable and the set $\bigcup \mathcal{U}'$ is clopen. Therefore $\mathcal{V} = \mathcal{U}' \cup \left\{X \setminus \bigcup \mathcal{U}'\right\}$ is an uncountable disjoint clopen cover of X but this contradicts the fact that X is a clp–Lindelöf space.

(1.15) Corollary. Every discrete system of clopen sets of a clp–Lindelöf space is countable.

(1.16) **Proposition.** A space X is clp–Lindelöf iff every clopen cover of X contains a countable disjoint clopen refinement.

Proof. Let $\mathcal{U}$ be a clopen cover of a clp–Lindelöf space X. Then one can choose a countable clopen subcover $\left\{U_n : n \in N\right\} \subset \mathcal{U}$. Let $V_1 := U_1, \ldots, V_n := U_n \setminus \bigcup_{j=1}^{n-1} U_j, \ldots$. It is easy to see that the family $\mathcal{V} = \left\{V_n : n \in N\right\}$ thus obtained is a countable disjoint clopen cover of the space X refining $\mathcal{U}$.

The proof of the converse part is obvious.

(1.17) **Corollary.** A space X is clp–Lindelöf iff every its clopen cover contains a countable locally finite clopen refinement.

Replacing in the definition of a Souslin space open sets with clopen sets we come to the concept of a clp–Souslin space:

(1.18) **Definition.** A topological space is called *clp–Souslin* if the cardinality of each disjoint system of clopen disjoint non–empty subsets is countable.

As the next two examples show, the properties of clp–Souslinness and clp–Lindelöfness are incomparable (cf Corollary 1.15).

(1.19) **Example.** Let X be infinite, $p \in X$ be some fixed point and let $\mathcal{T} = \left\{X\right\} \cup \left\{U : p \notin U\right\} \cup \left\{V = X \setminus A, A \subset X, \left|A\right| < \aleph_0\right\}$. One can easily see that $\mathcal{T}$ is a topology on X and the space $\left(X, \mathcal{T}\right)$ is compact (and hence also clp–Lindelöf) but fails be clp–Souslin.

(1.20) **Example.** The space $R_b \times R_b$ (see Example 1.10) is clp–Souslin, but is not clp–Lindelöf.

2. CLP–PARACOMPACT SPACES

(2.1) **Definition.** A topological space is called *clp–paracompact* if every its clopen cover contains a locally finite clopen refinement.

(2.2) **Assertion.** Every clp–Lindelöf space is clp–paracompact.

Proof follows from Corollary 1.17.

(2.3) **Assertion.** Every paracompact space is clp–paracompact.

Clp–paracompact spaces can be characterized by the help of disjoint refinements:

(2.4) **Proposition.** A space X is clp–paracompact iff every its clopen cover contains a disjoint clopen refinement.

Proof of the "if" part follows from Lemma 1.13.
The proof of the converse part is obvious since every disjoint clopen refinement is locally finite.

From Propositions 1.16 and 2.4 follows:

(2.5) **Corollary.** Every clp–Lindelöf space is clp–paracompact.

(2.6) **Example.** Let X_α for every $\alpha < c$ be a connected non–paracompact space. It is easy to see that the space $\oplus\left\{X_\alpha, \alpha < c\right\}$ is clp–paracompact, but is neither clp–Lindelöf nor paracompact.

One can easily prove also the following two statements:

(2.7) **Proposition.** Every clp–paracompact clp–Souslin space is a clp–Lindelöf space.

(2.8) **Proposition.** If a topological space X is clp–paracompact and M is its clopen subspace then M is clp–paracompact, too.

(2.9) **Proposition.** If a space Y is clp–paracompact and f is a continuous mapping from a space X into Y with following properties:

1) f is clopen,

2) for every point y from Y the preimage $f^{-1}(y)$ is a clp–compact subset of X,

then the space X is clp–paracompact, too.

Proof. Notice first that since f is clopen, in virtue of Proposition 2.8 we can assume that the image of X is the whole space Y. Let $\mathcal{U} = \{U_i : i \in I\}$ be a clopen cover of the space X. Since for each $y \in Y$ the set $A_y = f^{-1}(y)$ is clp–compact in X there exists a finite subfamily $\mathcal{U}_y = \{U_i : i \in I_y\}$ of $\mathcal{U}$ covering A_y. Then, obviously, $\cup\mathcal{U}_y =: B_y$ is a clopen set in X.

Then recall the fact from Theorem 1.8 in [4] that for every $y \in Y$ there exists a clopen neighbourhood V_y such that $f^{-1}(V_y) \subset B_y$.

Since $\mathcal{V} = \{V_y : y \in Y\}$ a clopen cover of the clp–paracompact space Y there exists a locally finite clopen refinement $\{W_j : j \in J\}$. Then $\{f^{-1}(W_j) : j \in J\}$ is a clopen cover of the space X. For each $j \in J$ there exist $y_j \in Y$ and I_{y_j} such that $f^{-1}(W_j) \subset f^{-1}(V_{y_j}) \subset B_{y_j} = \bigcup_{i \in I_{y_j}} U_i$. It is easy to understand that the family $\{f^{-1}(W_j) \cap U_i : j \in J, i \in I_y, y \in Y\}$ is a locally finite clopen refinement of $\mathcal{U}$. Therefore X is clp–paracompact.

(2.10) Proposition. A direct sum $\oplus X_i$ of a family $\{X_i : i \in I\}$ of topological spaces is clp–paracompact iff each X_i is clp–paracompact.

The "only if" part follows by the help of Proposition 2.8.

Conversely, let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be a clopen cover of the space X. Then the family $\mathcal{V} = \{U_\alpha \cap X_i : \alpha \in A, i \in I\}$ is obviously also a clopen cover of X. Thus for each $i_o \in I$ $\mathcal{V}_{i_o} = \{U_\alpha \cap X_{i_o} : \alpha \in A\}$ is a clopen cover of a clp–paracompact space X_{i_o} and hence there exists a locally finite clopen refinement $\mathcal{V}'_{i_o}$ of $\mathcal{V}_{i_o}$.

Let $\mathcal{S} = \bigcup_{i \in I} \mathcal{V}'_i$. Obviously $\mathcal{S}$ is a clopen refinement of $\mathcal{U}$. To complete the proof notice that $\mathcal{S}$ is obviously locally finite.

(2.11) Proposition. A zero–dimensional space is clp–paracompact iff it is paracompact.

<u>Proof</u> is obvious.

(2.12) Example. Since, as one can easily notice $R_b \times R_b$ (see Example 1.10) is not clp–paracompact, it follows that the product of two clp–paracompact spaces need not be clp–paracompact.

Patterned after the proofs of the corresponding statements about clp–compact spaces [4], one can easily establish the following two special cases when the product of two spaces is clp–paracompact.

(2.13) Proposition. The product of a clp–paracompact space and a compact space is clp–paracompact.

(2.14) Proposition. The product of a clp–paracompact space and a connected space is clp–paracompact.

3. THE SPACES OF QUASICOMPONENTS OF CLP–LINDELÖF AND CLP–PARACOMPACT SPACES.

In this section we establish relations between the properties of clp–Lindelöfness and clp–paracompactness of a space X and certain properties of the space $Q(X)$ of its quasicomponents, see [2], see also [4], Section 4.

(3.1) Proposition. The following statements are equivalent for a topological space:

(1) X is clp–Lindelöf,

(2) $Q(X)$ is clp–Lindelöf,

(3) $Q(X)$ is Lindelöf.

Proof. $(1) \Rightarrow (2)$ If X is clp–Lindelöf then $Q(X)$ is also clp–Lindelöf as a continuous image of X (see Proposition 1.6).

$(2) \Rightarrow (3)$ Since $Q(X)$ is zero–dimensional and clp–Lindelöf it is also Lindelöf by Proposition 1.4.

$(3) \Rightarrow (1)$ Let $\mathcal{U} = \{U_i : i \in I\}$ be a clopen cover of X. Then $\mathcal{V} = \{q(U_i) : i \in I\}$, where $q: X \to Q(X)$ is the quotient mapping, is an open cover of the Lindelöf space $Q(X)$, therefore there exists a countable subcover $\mathcal{V}' = \left\{q\left(U_{i_n}\right) : n \in N\right\}$. Hence the family $\mathcal{U}' = \left\{U_{i_n} : n \in N\right\}$ is a countable clopen subcover of $\mathcal{U}$.

From Propositions 1.6 and 3.1 it follows:

(3.3) Corollary. If $f: X \to Y$ is a surjective continuous mapping and $Q(X)$ is a Lindelöf space then $Q(Y)$ is Lindelöf, too.

From Propositions 1.9 and 3.1 it follows:

(3.4) Corollary. If $f: X \to Y$ is a clopen continuous mapping $Q(Y)$ is Lindelöf and for all points $y \in Y$ the spaces $Q\left(f^{-1}(y)\right)$ are compact then the space $Q(X)$ is Lindelöf, too.

(3.5) Proposition. The following statements are equivalent for a topological space:

(1) X is clp–paracompact,

(2) $Q(X)$ is clp–paracompact,

(3) $Q(X)$ is paracompact.

<u>Proof.</u> $(1) \Rightarrow (2)$ Let X be clp–paracompact and let $\mathcal{V} = \{V_i : i \in I\}$ be a clopen cover of the space $Q(X)$. Then $\{q^{-1}(V_i) : i \in I\}$ is a clopen cover of the space X and hence there exists a disjoint clopen refinement $\mathcal{U} = \{U_j : j \in J\}$ of $\{q^{-1}(V_i) : i \in I\}$ (Proposition 2.4). It follows that $\{q(U_j) : j \in J\}$ is a disjoint open cover of the space $Q(X)$. Besides, clear that this cover also clopen and refines $\mathcal{V}$ and therefore $Q(X)$ is also a clp–paracompact space.

$(2) \Rightarrow (3)$ Since $Q(X)$ is zero–dimensional and clp–paracompact, by Proposition 2.11 it is also paracompact.

$(3) \Rightarrow (1)$ Let $\mathcal{U} = \{U_i : i \in I\}$ be a clopen cover of X. Then $\{q(U_i) : i \in I\}$ is an open cover of the paracompact space $Q(X)$, and hence there exists a locally finite open refinement $\mathcal{V} = \{V_j : j \in J\}$ of $\{q(U_i) : i \in I\}$. For each point $x \in X$ there exists an open neighbourhood M of $q(x)$ in $Q(X)$ which intersects only a finite number of elements of $\mathcal{V}$. Then also the open neighbourhood $q^{-1}(M)$ of x intersects a finite number of sets from $q^{-1}(\mathcal{V})=\{q^{-1}(V_j):j\in J\}$. To complete the proof it is sufficient to notice that $q^{-1}(\mathcal{V})$ is a clopen refinement of $\mathcal{U}$.

From Propositions 2.9 and 3.5 it follows:

(3.6) **Corollary.** If $f:X\to Y$ is a clopen continuous mapping, $Q(Y)$ is paracompact and for all points $y\in Y$ the spaces $Q(f^{-1}(y))$ are compact then the space $Q(X)$ is paracompact, too.

By the help of Propositions 3.1 and 3.5 we can easy prove some other properties of clp–Lindelöf and clp–paracompact spaces (3.11-3.14). However, first we have to develop an appropriate terminology.

Extending the notions of openness and closedness to the clp–situation we can introduce the following:

(3.7) **Definition.** A set $U\subset X$ is called *clp–open* if for every point $x\in U$ there exists a clopen set V such that $x\in V\subset U$.

(3.8) **Definition.** A set $U\subset X$ is called *clp–closed* if for every point $x\notin U$ there exists a clopen set V such that $x\in V$ and $V\cap U=\varnothing$.

Obviously, clp–open sets are open and clp–closed sets are closed. Clopen sets are both clp–open and clp–closed.

(3.9) **Definition.** A topological space is called *clp–normal* if every disjoint clp–closed sets A and B have disjoint clp–open neighbourhoods U_A and U_B.

(3.10) **Remark.** A zero–dimensional space is normal iff it is a clp–normal space. However, in general, the properties of normality and clp–normality are incomparable.

(3.11) **Proposition.** Every clp–paracompact space is clp–normal.

Proof. Let X be a clp–paracompact space then the space $Q(X)$ is paracompact. As, besides, $Q(X)$ is Hausdorff it follows that $Q(X)$ is a normal space, too (see [3]). Let A and B be disjoint clp–closed sets of X. It is easy to see that $q(A)$ and $q(B)$ are disjoint closed subsets in $Q(X)$. Then there exist disjoint open sets V_A and V_B such that $V_A \supset q(A)$ and $V_B \supset q(B)$. Then $U_A = q^{-1}(V_A) \supset A$ and $U_B = q^{-1}(V_B) \supset B$, $U_A \cap U_B = \emptyset$ and U_A, U_B are clopen subsets of X. Hence X is clp–normal.

From here and Corollary 2.5 it follows:

(3.12) **Corollary.** Every clp–Lindelöf space is clp–normal.

(3.13) **Proposition.** If a space X is clp–paracompact and contains a dense clp–Lindelöf subspace A then X is a clp–Lindelöf space.

Proof. Since the quotient mapping $q: X \to Q(X)$ is continuous it follows that $q(X) = q(\overline{A}) \subset \overline{q(A)}$ and therefore $\overline{q(A)} = Q(X)$. If the space X is clp–paracompact then $Q(X)$ is paracompact, $q(A)$ is a dense Lindelöf subspace of $Q(X)$ and therefore (see [3]) $Q(X)$ is a Lindelöf space. Hence by Proposition 3.1 X is a clp–Lindelöf space.

(3.14) **Corollary.** Every separable clp–paracompact space is clp–Lindelöf.

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О CLP–ЛИНДЕЛЁФОВЫХ И CLP–ПАРАКОМПАКТНЫХ ПРОСТРАНСТВАХ.

Аннотация. Топологическое пространство назовём clp–линделёфовым, если каждое его покрытие открыто-замкнутыми множествами имеет счетное подпокрытие; топологическое пространство назовем clp–паракомпактным, если в каждое его покрытие открыто-замкнутыми множествами можно вписать локально конечное измельчение. В данной работе изучаются свойства clp–линделёфовых и clp–паракомпактных пространств, а также исследуется связь этих свойств с некоторыми другими топологическими свойствами.

PAR CLP–LINDELOFA UN CLP–PARAKOMPAKTĀM TELPĀM.

Anotācija. Par clp–Lindeļofa telpu tiek saukta tāda topoloģiska telpa, kuras katrs pārklājums ar reizē slēgtām un vaļējām kopām satur sanumurējamu apakšpārklājumu. Un par clp–parakompaktu tiek saukta tāda telpa, kuras katrā pārklājumā ar reizē slēgtām un vaļējām kopām var ierakstīt lokāli galīgu slēgtu un vaļēju pārklājumu. Šis raksts veltīts clp–Lindeļofa un clp–parakompaktu telpu pamatīpašību izpētei, kā arī iztirzā šo telpu saistību ar citām topoloģisku telpu klasēm.

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CORRECTIONS TO MY PAPER "ON EQUIVARIANT HOMOTOPY TYPE"

S.V. Agheev

Abstract. The proof of the generalization of James-Segal's theorem presented in our paper [1] (Theorem A) goes off only under the additional assumption that the acting group is zero-dimensional. Nevertheless the statement itself is valid for an arbitrary (compact) group: the correct proof will be published in a forthcoming paper.
AMS subject classification: 57S15

As mentioned by S. Antonian, the G-map $f: X \dashrightarrow Y$ defined on page 41 of our paper [1] is continuous only for a zero-dimensional compact group G. Therefore, we have established the generalization of James-Segal's theorem in the following form:

Theorem. Let G be an arbitrary compact group with $\dim(G) = 0$, X and Y be metric G-ANE-spaces, and $g: X \to Y$ be a G-map. Then the following statements are equivalent:

(a) $g: X \to Y$ is a G-homotopy equivalence;

(b) for every closed subgroup H in G the map $g^H: X^H \to Y^H$ is a homotopy equivalence.

A complete generalization of this theorem (i.e. for an arbitrary compact group G) is proved by the author and will be published in the short run.

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S.V. Agejevs. Labojumi manam rakstam "On equivariant homotopy type".

Džeimsa-Segala teorēmas vispārinājuma pierādījums mūsu rakstā [1] ir pareizs tikai pie papildnosacījuma,ka darbojošamies grupa G ir nuldimensionāla.Tomēr pats rezultāts ir spēkā jebkurai (kompaktai) grupai G:tās pierādījums tiks publicēts mūsu nākamajā rakstā.

С.В. Агеев. Исправление к моей статье "On equivariant homotopy type".

Приведенное в нашей работе доказательство обобщенной теоремы Джеймса-Сегала в действительности справедливо лишь при дополнительном условии нульмерности действующей группы G . Тем не менее сам результат справедлив для произвольной (компактой) группы: соответствующее доказательство недавно получено автором и вскоре будет опубликовано.

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SOME GENERALIZATIONS OF W.A.KIRK'S FIXED POINT THEOREMS

Inese Bula

ABSTRACT. Some fixed point theorems for a family of nonexpansive mappings of a metric space are obtained. AMS SC 47H10

1. INTRODUCTION.

In a fixed point theory a great interest was roused by work of W.A.Kirk [1], where nonexpansive mappings on a subset with normal structure of a reflexive Banach space were examined. Later many mathematicians generalized this result for a commutative family of nonexpansive mappings. Also W.A.Kirk himself (together with L.P.Belluce) have generalized this result [2,3,4,5]. Completely this theorem for commutative family was generalized by T.C.Lim [6]. However the possibility to generalize the corollary of theorem remained an open question. In this work we generalize the corollary also.

Convexity is one of the most important set properties used in W.A.Kirk's theorems. Therefore only subsets of vector spaces are examined in these theorems. But on the other hand, see for example [7], it is also possible to define convexity of subsets of metric spaces by making use of closure operators. Convexity structure of a metric space is earlier defined by W.Takahashi [8]. But approach of closure operators is more general. Using systems of subsets which are stable under arbitrary intersections the convexity problem in a metric space is also examined by J.P.Penot [9] and W.A.Kirk [10]. It seems that these two approaches are equivalent. However, it will be shown later by example regarding c_0, approach of closure operators provides the results which are again more general than these results of J.P.Penot and W.A.Kirk. Besides we prove the existence of common fixed points for families of mappings. Previously it was not done.

2. BASIC DEFINITIONS AND FIXED POINT THEOREMS IN A BANACH SPACE.

DEFINITION 1. A convex set K of a Banach space X has normal structure if for each bounded and convex subset H of K with diam$H \neq 0$ there exists a point y such that

$$\sup \{ d(x,y) \mid x \in H \} < \operatorname{diam} H.$$

This concept was introduced by M.S.Brodskii and D.P.Milman [11], further this concept is analyzed in [12,13].

THEOREM 1 (W.A.Kirk, [1]). Suppose K is a nonempty, convex, bounded and closed subset of a reflexive Banach space X, and suppose K has normal structure. Then every nonexpansive mapping $f:K\to K$ has a fixed point.

COROLLARY. If in the Theorem the condition that K is bounded is replaced by the condition that the sequence $(f^n(p))_{n\in N}$ is bounded for some $p\in K$, then f has a fixed point.

DEFINITION 2. A family F of mappings $f:X\to X$ is called commutative if: $\forall f,g\in F$: $(f\circ g)(x)=(g\circ f)(x)$, $\forall x\in X$.

As mentioned in Introduction it was the Theorem 1 that was generalized by T.C.Lim [6] for a commutative family of nonexpansive mappings.

But generalization for the corollary is not given in [6].

THEOREM 2 ([14]). Suppose K is a nonempty, convex and closed subset of a reflexive Banach space X, and suppose K has normal structure. If for a commutative family F of nonexpansive mappings $f:K\to K$ there exists a point $p\in K$ such that the set $S=\{(f_1\circ f_2\circ\ldots\circ f_n)(p) \mid f_1,\ldots,f_n\in F \ \&\ n\in N\}$ is bounded, then F has a common fixed point: $\cap\{F\text{ix}f \mid f\in F\}\neq\emptyset$.

DEFINITION 3. A Banach space X is said to be strictly convex if all the points of the unit sphere of X are not inner points of straight lines in the unit ball.

W.A.Kirk and L.P.Belluce generalized Theorem 1 in [2] as following:

THEOREM 3 ([2]). Suppose X is a strictly convex reflexive Banach space and K is nonempty, convex, closed, bounded subset of X, and suppose K has normal structure. Suppose F is a commutative family of nonexpansive mappings $f:X\to X$. Then F has a common fixed point.

Combining the ideas of the Corollary and the previous Theorem 3 we shall prove:

THEOREM 4. Suppose X is a strictly convex reflexive Banach space and K is a nonempty, convex and closed subset of X, and suppose K has normal structure. Suppose $F=\{f_1,f_2,\dots,f_n\}$ is a commutative family of nonexpansive selfmaps of X. If there exists a point $p\in K$ such that the sequences $(f^n(p))_{n\in N}$ for every $f\in F$ are bounded, then F has a common fixed point.

▼ **Proof.**

By Corollary of Theorem 1 it is known that $Fixf_i\neq\emptyset, i=1,2,\dots,n$. Since X is strictly convex, Fixf is convex for nonexpansive mapping $f\in F$. Since $f_i, i=1,2,\dots,n$ are continuous, the sets $Fixf_i, i=1,2,\dots,n$ are closed.

Let us inductively prove that

$$FixF=\bigcap\{Fixf_i|i=1,2,\dots,n\}\neq\emptyset.$$

For n=1 the statement is true by the Corollary of Theorem 1. Assuming that $\bigcap\{Fixf_i|i=1,2,\dots,k\}\neq\emptyset$, let us prove that $\bigcap\{Fixf_i|i=1,2,\dots,k+1\}\neq\emptyset$. We denote $\{f_1,f_2,\dots,f_k\}$ by F′ and f_{k+1} by f. Since by assumption $f(x)=f(f_i(x))=f_i(f(x))$, it follows that $f(x)\in FixF'$ and hence $f:FixF'\to FixF'$. Let us prove that the mapping f has a fixed point in the set FixF′. The sets $Fixf_i, i=1,2,\dots,k$ are nonempty, closed and convex, therefore FixF′ is closed and convex being an intersection of closed and convex sets. We choose $z\in Fixf$ freely. The functional $\|z-y\|$, $y\in K$ is weakly lower semicontinuous and therefore attains its minimal value in each nonempty, closed and convex subset of the reflexive Banach space, consequently, also in FixF′. Since X is strictly convex for z there exists a unique nearest point $z_0\in FixF'$:

$$\|z-z_0\|=inf\{\|z-y\||y\in FixF'\}\leq$$
$$\leq\|z-f(z_0)\|=\|f(z)-f(z_0)\|\leq\|z-z_0\|.$$

We conclude that $f(z_0)=z_0$. Then $z_0\in FixF\neq\emptyset$. ▲

We remark that the previous result is not true for an infinite family of mappings.

3. GENERALIZATIONS IN A METRIC SPACE WITH A CLOSURE OPERATOR.

Further we act in a metric space X with a distance d. Let PX be the set of all subsets of X.

DEFINITION 3. A closure operator on X is a mapping $S: PX \to PX$ satisfying for each $A, B \in PX$ the following conditions :

1) $A \subset B \to S(A) \subset S(B)$;

2) $A \subset S(A)$,

3) $S(S(A)) = S(A)$.

DEFINITION 4. A closed operator S on X is said to be algebraic if for each $A \in PX$ and $x \in S(A)$ there exists a finite set $F \subset A$ such that $x \in S(F)$.

Let S be a closure operator on X. A subset A of X is said to be S-closed if $A = S(A)$. A space X is said to be S-compact if each centered system of S-closed subsets of X has a nonempty intersection. Note that intersection of S-closed subsets of X is S-closed. For more detailed applications of closure operators in fixed point theory see [7].

Let us denote

$A(x, f) := \bigcap \{A \in PX | x \in A \;\&\; A = S(A) \;\&\; f(A) \subset A\}, \forall f \in F, \forall x \in X (x \neq f(x))$.

$A(x) := \bigcap \{A \in PX | x \in A \;\&\; A = S(A) \;\&\; \forall f \in F: f(A) \subset A\}$.

In a metric space with a closure operator for commutative family we prove the following common fixed point

THEOREM 5 ([15]). Suppose (X, d) is a metric space, S is a closure operator on X and X is S-compact. Let each closed ball $B(x, r)$ $(x \in X,\ r \in \mathbb{R}_{++})$ be S-closed. Let F be a commutative family of nonexpansive selfmaps of X, such that the set of fixed points Fixf for every mappings $f \in F$ is S-closed. If there exists a point $y \in A(x, f)$ such that

$$\sup \{ d(y, z) / z \in A(x, f) \} < \operatorname{diam} A(x, f)$$

for every $f \in F$ and $x \in X$ $(x \neq f(x))$ (condition of "normal structure"), then F has a common fixed point.

Condition of normal structure is of a great importance in the results of W.A.Kirk and L.P.Belluce. Note that if a Banach space X has normal structure then the condition of Theorem 5 follows, but the converse is not true.

EXAMPLE. Consider the space

$c_0 = \{x = (x_1, x_2, \ldots, x_n \ldots) | x_n \in \mathbb{R}, n = 1, 2, \ldots \;\&\; \lim_{n\to\infty} x_n = 0\}$.

The space c_0 has not normal structure

$(\forall x \in c_0 : \|x\| := \sup\{x_n \mid n \in \mathbb{N}\})$.

For every $x \in B_+(0;1) := \{y \in c_0 \mid \|y\| \,\&\, \forall n \in \mathbb{N} : y_n \geq 0\}$ define $f(x) := 0$. Then $A(x,f) = \{tx \mid t \in [0;1]\} (x \neq 0)$ and for the point $y := \frac{x}{2}$ it holds $\sup\{\|y-z\| \mid z \in A(x,f)\} = \frac{\|x\|}{2} < \|x\| = diam A(x,f)$.

Inspired by theorems in [4, Th.3,7] we prove

THEOREM 6. Suppose (X,d) is a metric space and S is a closure operator. Let X is S-compact. Let each closed ball B(x,r) ($x \in X$, $r \in \mathbb{R}_{++}$) be S-closed. Let F be a family of continuous selfmaps of X satisfying:

1) $\exists q \in]0;1[\ \forall x, y \in X\ \forall f, g \in F$:

$d(f(x), g(y)) \leq \max\{d(x,y)\ \ q\,diam(A(x) \cup A(y))\}$;

2) $\forall x \in X (\exists v \in F : v(x) \neq x) \exists y \in A(x) : \sup\{d(y,z) \mid z \in A(x)\} < diam A(x)$.

Then F has a common fixed point.

▼ PROOF.

Using Zorn's Axiom and S-compactness of X we conclude that there exists a minimal nonempty S-closed and invariant under F subset M of X.

Let $a \in M$, and there exists $f \in F$ such that $f(a) \neq a$. Since $A(a) \subset M$, minimality of M implies that $M = A(a)$. By 2) there exists a point $y \in A(a) = M$ such that:

$$\sup\{d(y,z) \mid z \in A(a)\} =: r < diam A(a).$$

Consider the set $A := (\bigcap\{B(x,r) \mid x \in M\}) \cap M$. It is nonempty because $y \in A$ and it is S-closed as an intersection of S-closed sets. We prove that A is invariant under F. Let us assume that there exist $z \in A$ and $g \in F$ such that $g(z) \notin A$. Then there exists $w \in M$ such that $w \notin B(g(z), r)$. Hence $A_1 := B(g(z), r) \cap M$ is a proper subset of M. A_1 is invariant under F because for each $x \in A_1$ and $f \in F$ it holds:

$d(f(x), g(z)) \leq \max\{d(x,z); q\,diam(A(x) \cup A(z))\} \leq$
$\leq \max\{r; q\,diam M\} = r$ ($z \in A$ and $x \in A_1 \subset M$).

The set A_1 is nonempty ($g(z) \in A_1$) and S-closed. By minimality of M it is clear that $M = A_1$. Hence $f(A) \subset A$ $\forall f \in F$. By minimality of M obviously $M = A$. However $diam A \leq r < diam A(a) = diam M$. The obtained contradiction completes the proof ▲

We prove in a metric space with a closure operator a theorem similar to the theorems from [4,7].

THEOREM 7 ([16]). Suppose (X,d) is a metric space and S is an algebraic closure operator on X. Suppose $\overline{S(A)}=S(\overline{S(A)})=:S'(A)$ for each $A\in PX$ and X is S'-compact. Let each closed ball B(x,r) ($x\in X$, $r\in\mathbf{R}_{++}$) be S-closed. If F is a family of continuous selfmaps of X satisfying:

1) $\exists q\in]0;1[\ \forall x,y\in X\ \forall f,g\in F:$

$d(f(x),g(y))\le\max\{d(x,y);q\,diam(A(x)\cup A(y))\}$;

2) $\forall x\in X(\exists v\in F: v(x)\ne x)\exists y\in A(x):$

$\sup\{\inf\{\sup\{d(y,f^m(x))|m\ge n\}\ n\in Z^+\}\ f\in F\}<diamA(x)$,

then F has a common fixed point.

In Theorem 6 in [-] W.A.Kirk examines a more general situation: when there exists an integer N such that f^N has diminishing orbital diameters on X. A similar theorem is true in a metric space with a closure operator for one nonexpansive mapping and also for a family of mappings.

THEOREM 8. Suppose (X,d) is a metric space and S is an algebraic closure operator on X. Suppose $\overline{S(A)}=S(\overline{S(A)})=:S'(A)$ for each $A\in PX$ and X is S'-compact. Let each closed ball B(x,r) ($x\in X$, $r\in\mathbf{R}_{++}$) be S-closed. If F is family of selfmaps of X satisfying:

1) $\exists t\in]0;1[\ \forall x,y\in X\forall f,g\in F:$

$d(f(x),g(y))\le\max\{d(x,y);t\,diam(A(x)\cup A(y))\}$;

2) $\exists N\in\mathbf{N}\ \forall x\in X(\exists v\in F: v(x)\ne x)\exists y\in A(x):$

$\sup\{\inf\{\sup\{d(y,f^m(x))|m\ge n\}\ n\ge N\}\ f\in F\}<diamA(x)$,

then F has a common fixed point.

Proof.

Using Zorn's Axiom and S'-compactness of X we conclude that there exists a minimal subset M of X such that:

1) $M\ne\emptyset$;

2) $M=S'(M)$;

3) $f(M)\subset M, \forall f\in F$.

By Theorem 7 the family $F^N=\{f^N|f\in F\}$ has a common fixed point $x^*\in M$. We shall prove that x^* is common fixed point for family F.

We note that $M=A(x^*)$ by minimality of M.

Let there exist $f\in F$ such that $f(x^*)\ne(x^*)$. By 2) there exists a point $y\in A(x^*)$ such that:

$q=\sup\{\inf\{\sup\{d(y,f^m(x))|m\ge n\}\ n\ge N\}f\in F\}<diamA(x^*)$.

Let $A_0:=\{x^*,f(x^*),\dots,f^{N-1}(x^*)|\forall f\in F\}$.

Then $q=\sup\{d(y,z)|z\in A_0\}<diamA(x^*)$

Let $r\in]\max\{q;t\,diamA(x^*)\};diamA(x^*)[$, then $S(A_0)\subset B(y,r)$.

We consider the set $A:=(\bigcap\{B(z,r)|z\in M\})\cap M$. Then:

1) $A\ne\emptyset$ because $y\in A$;

2) A is S-closed as an intersection of S-closed sets;

3) A is invariant under F. Really, if there exist $u \in A$ and $g \in F$ such that $g(u) \notin A$, then $B(g(u),r)$ is a proper subset of M. Set $B(g(u),r) \cap M$ is S-closed, nonempty ($g(u) \in B(g(u),r) \cap M$) and invariant under F. Indeed, choosing arbitrary $z \in B(g(u),r) \cap M$ and $h \in F$ we have:

$d(g(u),h(z)) \leq \max\{d(u,z); t\,diam(A(u) \cup A(z))\} \leq$
$\leq \max\{r; t\,diamM\} = r.$

The minimality of M implies: $M = B(g(u),r) \cap M$. The obtained contradiction proves 3).

Therefore by minimality of M it follows that M=A. But diamA<r<diamM. The obtained contradiction shows that initial assumption is not true.▲

Our article doesn't answer to many open questions formulated in [17], where situation in details is examined in Banach spaces. We hope that our article will be useful for further generalizations in a metric space using closure operators of many other theorems valid in a Banach space.

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ERROR ESTIMATES OF THE APPROXIMATION BY SMOOTHING SPLINES

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Summary. In this paper we study the problem of fitting the given measurments of an unknown function by means of smoothing piecewise linear splines, which minimize a certain combination of smoothness and goodness of the fit. We analyse the influence of a smoothing parameter on the convergence of the fitting algorithm. The exact error estimates on W^1_p for $p=1,2,\infty$ are obtained. AMS SC 41A15,65D10.

0. INTRODUCTION

Let $\Delta=(0=t_1<t_2<\ldots<t_n=1)$ for some integer $n>0$ be a given partition of the interval $[0,1]$ with equidistant knots. Suppose that $z_1,z_2,\ldots,z_n$ are the measurments on the values of an unknown function f at the knots of Δ.

By s we denote a piecewise linear continuous function over the grid Δ (that is the first degree spline), which interpolates f in the sense $s(t_i)=z_i$. The classic result of Holladay show that the interpolating spline s appears as the unique solution of the problem

$$\int_0^1\left[s'(t)\right]^2dt=\min\left\{\int_0^1\left[x'(t)\right]^2dt\,\middle|\,x\in H^1,\ x(t_i)=z_i,\ i=1,2,\ldots,n\right\} \quad (1)$$

on Sobolev space H^1 of the absolutely continuous functions.

In many applications the strong interpolation conditions are not adequate, since the given data are affected by errors. Therefore, the notion of a smoothing spline s_ρ was defined in [1], [2] as the solution of the problem

$$\int_0^1\left[s'_\rho(t)\right]^2dt+\rho\sum_{i=1}^n\left[s_\rho(t_i)-z_i\right]^2=\min_{x\in H^1}\int_0^1\left[x'(t)\right]^2dt+\rho\sum_{i=1}^n\left[x(t_i)-z_i\right]^2. \quad (2)$$

Here $\rho>0$ is a given smoothing parameter, which balances goodness of the fit against smoothness. The solution of this optimization problem turns out to be the first degree spline and this can be found as the solution of the system of linear equations (see. e.g. [3]).

The main purpose of this paper is to estimate the error of smoothing by the first degree splines and to analise on this basis the influence of the parameter ρ on the convergence of the fitting algorithm.

1. MAIN RESULT

By L_p, $1 \le p \le \infty$, we denote the usual Lebesgue space of p-integrable functions on [0,1], equipped with the norm

$$\|f\|_p = \begin{cases} \left(\int_0^1 |f(t)|\,dt\right)^{1/p}, & \text{when } p < \infty, \\ \sup_{t\in[0,1]} \text{vrai } |f(t)|, & \text{when } p = \infty. \end{cases}$$

Let W_p^1 be the space of absolutely continuous functions f with derivative $f' \in L_p$ such, that $\|f'\|_p \le 1$. We denote

$$R_{\rho,n,p} = \sup_{f\in W_p^1} \|f - s_\rho(f)\|_\infty,$$

where $s_\rho(f)$ is the unique spline associated with f, that is the solution of the smoothing problem (2) for $z_i = f(t_i)$.

We may now state our main result as follows

Theorem. For an even n we have

$$R_{\rho,n,1} = \frac{1+\rho h}{1+\rho h/2+\sqrt{\rho h(1+\rho h/4)}}\,(1-\beta_1(\rho h,n)), \tag{3}$$

$$R_{\rho,n,2} = \sqrt{h}\,\frac{\sqrt[4]{4+\rho h}}{2\sqrt[4]{\rho h(4+\rho h)}}\,(1-\beta_2(\rho h,n))^{1/2}, \tag{4}$$

$$R_{\rho,n,\infty} = h\,\frac{\sqrt{4+\rho h}}{2\sqrt{\rho h}}\,(1-\beta_\infty(\rho h,n)), \tag{5}$$

where $h=1/(n-1)$,

$$c = 1+\rho h/2+\sqrt{\rho h(1+\rho h/4)}, \tag{6}$$

$$\beta_1(\rho h,n) = \frac{c^2-1}{c^{2n}-1}, \qquad \beta_\infty(\rho h,n) = \frac{2}{c^{n/2}+1},$$

$$\beta_2(\rho h,n) = \frac{(C^{2n}+1)C^n(2n+1)\rho h(4+\rho h)}{(2+\rho h)(C^{2n}-1)^2} - \frac{(C^n+1)\sqrt{\rho h(4+\rho h)}}{(C^{2n}-1)}$$

Proof of the theorem is based on the following results.

2. INTEGRAL REPRESENTATION OF THE ERROR

Our approach follows [4], where one smoothing problem was investigated by introduction of a special basis. Let us denote by $s_{\rho,i}$ the unique first degree smoothing spline associated with the vector $e_i=(\delta_{i1},\delta_{i2},\ldots,\delta_{in})$, where δ_{ik} is the Kronecker symbol. Then $s_{\rho,1},s_{\rho,2},\ldots,s_{\rho,n}$ form a basis in the space of piecewise linear continuous functions over the grid Δ, and the solution s_ρ of the smoothing problem (2) has an expansion

$$s_\rho(t) = \sum_{i=1}^{n} z_i s_{\rho,i}(t).$$

The object of this section is to obtain the integral representation of the error $f-s_p(f)$ for $f\in W_p^1$. Taking into account that the smoothing operator s_p is linear and exact for constant functions, we may assume that $f(0)=0$. Thus

$$f(t) = \int_0^1 f'(\tau)\varphi(t,\tau)d\tau, \quad t\in[0,1],$$

where $\varphi(t,\tau)=e(t-\tau)$, e is the unit Heaviside function. Therefore, we can write

$$s_\rho(f,t) = \sum_{i=1}^{n} s_{\rho,i}(t)\int_0^1 f'(\tau)\varphi(t_i,\tau)d\tau =$$

$$= \int_0^1 f'(\tau)\sum_{i=1}^{n} s_{\rho,i}(t)\varphi(t_i,\tau)d\tau = \int_0^1 f'(\tau)s_\rho(\varphi(.,\tau),t)d\tau,$$

from which we have

$$f(t)-s_\rho(f,t) = \int_0^1 f'(\tau)K_\rho(t,\tau)d\tau, \tag{7}$$

where

$$K_\rho(t,\tau) = \varphi(t,\tau)-s_\rho(\varphi(.,\tau),t).$$

For further investigation of the error it is useful to transform the expression of the kernel K_ρ

$$K_\rho(t,\tau) = \begin{cases} 1 - \sum_{k=m}^{n} s_{\rho,k}(t), \text{ when } \tau \in]t_{m-1},t_m]\cap[a,t], \\ -\sum_{k=m}^{n} s_{\rho,k}(t), \text{ when } \tau \in]t_{m-1},t_m]\cap]t,b]. \end{cases}$$

Since $\sum_{k=1}^{n} s_{\rho,k}(t) = 1$, we conclude that

$$K_\rho(t,\tau) = \begin{cases} u_{\rho,m-1}(t), \text{ when } \tau \in]t_{m-1},t_m]\cap[a,t], \\ -v_{\rho,m}(t), \text{ when } \tau \in]t_{m-1},t_m]\cap]t,b], \end{cases} \tag{8}$$

where

$$u_{\rho,m}(t) = \sum_{k=1}^{m} s_{\rho,k}(t), \quad v_{\rho,m}(t) = \sum_{k=m}^{n} s_{\rho,k}(t). \tag{9}$$

From (7) we can arrive at the error bound

$|f(t)-s_\rho(f,t)|\le\|K_\rho(t,.)\|_q$, $\quad 1/p + 1/q = 1$.

One can easily see that this estimate is exact on W_p^1. This completes the proof of the following lemma.

Lemma 1. Let $s_\rho(f)$ be the first degree smoothing spline associated with $f \in W_p^1$, $1\le p\le\infty$, then for any $t\in[0,1]$ we have

$$\sup_{f\in W_p^1} |f(t)-s_\rho(f,t)| = \|K_\rho(t,.)\|_q ,$$

where $1/p + 1/q = 1$ and K_ρ is defined by (8).

3. BASIS SPLINES

For further analysis of the error it is necessary to investigate functions $s_{\rho,k}$, $u_{\rho,m}$, $v_{\rho,m}$. We begin with Lemma 2, concerning the values of the basis splines $s_{\rho,k}$.

<u>Lemma 2</u>. For $k=1,2,\dots,n$ and $i=1,2,\dots,n$ we have

$$s_{\rho,k}(t_i)=\mathcal{D}_n^{-1}\rho h d_{\min(i,k)} d_{n+1-\max(i,k)} ,$$

where $(d_j)_{j\in N}$ is the recurrent sequence defined by

$$d_1=1,\ d_2=1+\rho h,\ d_j=(2+\rho h)d_{j-1}-d_{j-2}\ ,\ j\geq 3, \qquad (10)$$

$\mathcal{D}_n=d_{n+1}-d_n$.

<u>Proof</u>. If s_ρ is a piecewise linear function over the grid Δ, then there exist coefficients α, λ_1, $\lambda_2,\dots,\lambda_n$ such that

$$s_\rho(t)=\alpha+\sum_{j=1}^{n}\lambda_j(t-t_j)\varphi(t,t_j). \qquad (11)$$

By using the expansion (11) the smoothing problem (2) can be reduced to solving the system of linear equations

$$\text{a)}\ \sum_{j=1}^{n}\lambda_j=0 \qquad \text{b)}\ s_\rho(t_j)-\lambda_j\rho^{-1}=z_j\ ,\ j=1,2,\dots,n. \qquad (12)$$

From (11) and (12a) follows that

$$s_\rho(t_{i+1})-s_\rho(t_i)=h\sum_{j=1}^{i}\lambda_j\ ,\ i=1,2,\dots,n-1.$$

Hence

$$\begin{cases}\lambda_1=h^{-1}(s_\rho(t_2)-s_\rho(t_1)),\\ \lambda_i=h^{-1}(s_\rho(t_{i-1})-2s_\rho(t_i)+s_\rho(t_{i+1})),\ i=2,3,\dots,n-1,\\ \lambda_n=h^{-1}(s_\rho(t_{n-1})-s_\rho(t_n)).\end{cases} \qquad (13)$$

Substituting (13) into (12b) we get

$$\begin{cases}h\rho s_\rho(t_1)+s_\rho(t_1)-s_\rho(t_2)=h\rho z_1\ , & (14)\\ h\rho s_\rho(t_i)-s_\rho(t_{i-1})+2s_\rho(t_i)-s_\rho(t_{i+1})=h\rho z_i\ ,\ i=2,3,\dots,n-1, & (15)\\ h\rho s_\rho(t_n)-s_\rho(t_{n-1})+s_\rho(t_n)=h\rho z_n\ . & (16)\end{cases}$$

Now we can consider the basis spline $s_{\rho,k}$, $1<k\leq n$. It means that now $z_i=\delta_{ik}$, $i=1,2,\dots,n$. If $k\geq 2$, then for $i\leq k-1$ from (14) and (15) we obtain

$$s_{\rho,k}(t_2)=(1+\rho h)s_{\rho,k}(t_1),$$

$$s_{\rho,k}(t_i)=(2+\rho h)s_{\rho,k}(t_{i-1})-s_{\rho,k}(t_{i-2}),\ i=3,2,\dots,k.$$

Therefore

$$s_{\rho,k}(t_i)=d_i s_{\rho,k}(t_1),\ i=1,2,\dots,k. \qquad (17)$$

Similar arguments show that if $k\leq n-1$, then for $i\geq k+1$ from

(15) and (16) it follows

$$s_{\rho,k}(t_i)=d_{n+1-i}s_{\rho,k}(t_n),\ i=k,k+1,\dots,n. \qquad (18)$$

Let now $2\le k\le n-1$. Comparing (17) with (18) for $i=k$, we get $s_{\rho,k}(t_1)d_k=s_{\rho,k}(t_n)d_{n+1-k}$. Taking into account this equality, by substituting the values $s_{\rho,k}(t_{k-1})$, $s_{\rho,k}(t_k)$ and $s_{\rho,k}(t_{k+1})$, received from (17) and (18), into (15) we get

$$\text{a) } s_{\rho,k}(t_1)=\rho h d_{n+1-k}\mathcal{D}^{-1}_{n,k}, \quad \text{b) } s_{\rho,k}(t_n)=\rho h d_k\mathcal{D}^{-1}_{n,k}, \qquad (19)$$

where $\mathcal{D}_{n,k}=d_k d_{n+2-k}-d_{k-1}d_{n-k+1}$. Finally from (17), (18) and (19) we can write

$$s_{\rho,k}(t_i)=\rho h\mathcal{D}^{-1}_{n,k}d_{\min(i,k)}d_{n+1-\max(i,k)},\ i=1,2,\dots,n. \qquad (20)$$

For $k=1$ (respectively for $k=n$) the equality (20) follows from (18) and (19b) (respectively, from (17) and (19a)), this latter can be obtained by substituting (18) for $i=n$ into (16) (respectively, (17) for $i=1$ into (14)).

To complete the proof, we show that $\mathcal{D}_{n,k}$ does not depend on k because

$$\mathcal{D}_{n,k+1}=d_{k+1}d_{n-k+1}-d_kd_{n-k}=((2+\rho h)d_k-d_{k-1})d_{n-k+1}-$$

$$-d_k((2+\rho h)d_{n-k+1}-d_{n+2-k})=d_kd_{n+2-k}-d_{k-1}d_{n-k+1}=\mathcal{D}_{n,k},\ k\ge 1.$$

Hence for $k\ge 1$

$$\mathcal{D}_{n,k}=\mathcal{D}_{n,n}=d_n(1+\rho h)-d_{n-1}=d_{n+1}-d_n.$$

Remark. It is useful for the future to know the expressions for d_j. For the roots c_1 and c_2 of the characteristic equation of the recurrent relation (10) we have $c_1=c^{-1}$, $c_2=c$, where c is defined by (6). Therefore,

$d_j=\gamma_1c_1^{j-1}+\gamma_2c_2^{j-1}$, where values γ_1 and γ_2 can be received by using conditions $d_1=1$, $d_2=\rho h+1$. We note that $c_i=(4+\rho h)\gamma_i^2$, $i=1,2$. Thus

$$d_j = \frac{c^{j-1/2}+c^{-j+1/2}}{\sqrt{4+\rho h}} \quad , \; j=1,2,\dots . \tag{21}$$

Lemma 3. For the splines $u_{\rho,m}$ and $v_{\rho,m}$, $m=1,2,\dots,n$, defined in (9), we have

$$u_{\rho,m}(t_i)=\mathcal{D}_m \mathcal{D}_n^{-1} d_{n+1-i} \; , \; i\le m,$$

$$v_{\rho,m}(t_i)=\mathcal{D}_{n+1-m} \mathcal{D}_n^{-1} d_i \; , \; i\ge m. \tag{22}$$

Proof. From the previous lemma follows that

$$u_{\rho,m}(t_i)=\rho h \mathcal{D}_n^{-1} d_{n+1-i} \sum_{k=1}^{m} d_k \; , \; i\le m.$$

$$v_{\rho,m}(t_i)=\rho h \mathcal{D}_n^{-1} d_i \sum_{k=m}^{n} d_{n+1-k} = \rho h \mathcal{D}_n^{-1} d_i \sum_{k=1}^{n+1-m} d_k \; , \; i\ge m.$$

By using (21), direct calculations of the sum

$$\sum_{k=1}^{l} d_k = \frac{1}{\sqrt{4+\rho h}} \sum_{k=1}^{l} (c^{k-1/2}+c^{-k+1/2}) = (d_{l+1}-d_l)/\rho h = \mathcal{D}_l/\rho h$$

give the final result.

4. PROOF OF THEOREM

From Lemma 1 follows that

$$R_{\rho,n,p} = \max_{t\in[0,1]} \|K_\rho(t,.)\|_q \; , \; 1/p + 1/q = 1 \tag{23}$$

So, to prove the theorem we need the norms $\|K_\rho(t,.)\|_q$ for $q=1,2,\infty$.

We turn now to the problem of investigation of $K_\rho(t,.)$. Let us suppose that t is a fixed point in $[t_{i-1},t_i]$, $2\le i\le n$. According to (8) the function $K_\rho(t,.)$ is piecewise constant over the grid $\Delta\cup(t)$. So, it is easy to see that

$$\|K_\rho(t,.)\|_1 = h \sum_{m=1}^{i-2} u_{\rho,m}(t) + (t-t_{i-1}) u_{\rho,i-1}(t) +$$

$$+ (t_i - t) v_{\rho,i}(t) + h \sum_{m=i+1}^{n} v_{\rho,m}(t).$$

$$\|K_\rho(t,.)\|_2=\Big[h\sum_{m=1}^{i-2}u^2_{\rho,m}(t)+(t-t_{i-1})u^2_{\rho,i-1}(t)+(t_i-t)v^2_{\rho,i}(t)+$$

$$+h\sum_{m=i+1}^{n}v^2_{\rho,m}(t)\Big]^{1/2},$$

$$\|K_\rho(t,.)\|_\infty=\max(u_{\rho,1}(t),\dots,u_{\rho,i-1}(t),v_{\rho,i}(t)\dots,v_{\rho,n}(t))$$

(there $\sum_{m=l}^{k}\alpha_m=0$ if $l>k$, independently of α_m).

Taking into account that functions $u_{\rho,m}$ and $v_{\rho,m}$ are linear on $[t_{i-1},t_i]$ and using the equalities

$$\max_{m=1,\dots,k}\mathcal{D}_m=\mathcal{D}_k\ ,\ \sum_{m=1}^{k}\mathcal{D}_m=d_{m+1}-1,\ \ \sum_{m=1}^{k}\mathcal{D}_m^2=\Big[\mathcal{D}_{2k}-(2k+1)\rho h\Big]/(4+\rho h)$$

by Lemma 3 we obtain that

$$\|K_\rho(t,.)\|_q=h^{1/q}\mathcal{D}_n^{-1}g_{q,i}(\omega),\ q=1,2,\infty, \tag{24}$$

where $\omega=(t-t_{i-1})/h$,

$$g_{1,i}(\omega)=(d_i\omega+d_{i-1}(1-\omega)-1)(d_{n+1-i}\omega+d_{n+2-i}(1-\omega))+$$
$$+(d_i\omega+d_{i-1}(1-\omega))(d_{n+1-i}\omega+d_{n+2-i}(1-\omega)-1),$$

$$g^2_{2,i}(\omega)=\Big[\mathcal{D}^2_{i-1}\omega+(\mathcal{D}_{2i-4}-(2i-3)\rho h)/(4+\rho h)\Big]\Big[d_{n+1-i}\omega+d_{n+2-i}(1-\omega)\Big]^2+$$

$$+\Big[\mathcal{D}^2_{n+1-i}(1-\omega)+(\mathcal{D}_{2n+1-2i}-(2n+1-2i)\rho h)/(4+\rho h)\Big]\Big[d_i\omega+d_{i-1}(1-\omega)\Big]^2,$$

$$g_{\infty,i}(\omega)=\max\Big\{\mathcal{D}_{i-1}(d_{n+1-i}\omega+d_{n+2-i}(1-\omega)),\mathcal{D}_{n+1-i}(d_i\omega+d_{i-1}(1-\omega))\Big\}.$$

According to (23) and (24), to get (3)-(5) we need the maximal values of $g_{q,i}(\omega)$, $q=1,2,\infty$, for $\omega\in[0,1]$ and $i=2,3,\dots,n$:

$$k_{\rho,n,p}=h^{1/q}\mathcal{D}_n^{-1}\max_{i=2,\dots,n}\ \max_{\omega\in[0,1]}g_{q,i}(\omega),\ 1/p+1/q=1, \tag{25}$$

Let us start with $q=1$. By investigation of the first and the second derivatives of $g_{1,i}$ we arrive at the conclusions

a) $g''_{1,i}(\omega)=-4\mathcal{D}_{i-1}\mathcal{D}_{n+1-i}<0$,

b) $g'_{1,i}(\omega)=0 \Rightarrow \omega=\omega_{1,i}=1/2+(1/\mu_{n+2-i}-1/\mu_i)/4$,

where $\mu_k=(d_k-d_{k-1})/(2d_k-1)=(1-(2d_{k-1}-1)/(2d_k-1))/2$.

The estimate $0<\mu_k<1/2$ garantees that $\omega_{1,i}\in[0,1]$. Therefore

$$\max_{\omega\in[0,1]} g_{1,i}(\omega)=g_{1,i}(\omega_{1,i})=(\zeta_i^2-1)/2, \qquad (26)$$

where we denoted

$$\zeta_i=D_{i-1}^{1/2}D_{n+1-i}^{-1/2}(d_{n+1-i}-1/2)+D_{i-1}^{-1/2}D_{n+1-i}^{1/2}(d_{i-1}-1/2). \qquad (27)$$

In order to estimate ζ_i for $i=2,3,\dots,n$ we transform (27) to

$$\zeta_i=\frac{2(c^n-c^{-n})-(c^{i-1}-c^{-i+1})-(c^{n+1-i}-c^{-n-1+i})}{2(c^{i-1}-c^{-i+1})^{1/2}(c^{n+1-i}-c^{-n-1+i})^{1/2}}.$$

By differentiating ζ_i (as a function of $x=i$, $x\in[2,n]$)

$$\zeta_i'=(c^{i-1}-c^{-i+1})^{-3/2}(c^{n+1-i}-c^{-n-1+i})^{-3/2}\eta_i \ln c/2,$$

where $\eta_i=(c^{n+1-i}-c^{-n-1+i})+(c^{i-1}-c^{-i+1})-2(c^{n+2-2i}-c^{-n-2+2i})$,

we obtain that $\zeta'_{i_*}=0$ for $i_*=1+n/2$.

Since $\eta_i'<0$, we conclude that ζ_{i_*} is the maximal value of ζ_i.

Taking into account (25) and (26) we get the final result

$$R_{\rho,n,\infty}=hD_n^{-1}g_{1,i_*}(\omega_{1,i_*})=hD_n^{-1}D_{n/2}(D_{n/2}-2)/2=$$

$$=h\frac{\sqrt{4+\rho h}}{2\sqrt{\rho h}}\left[1-2/(c^{n/2}+1)\right].$$

In a similar way we investigate $g_{2,i}$. Analysing the first, the second and the third derivatives of $g^2_{2,i}$ on $[0,1]$:

a) $(g^2_{2,i})'''(\omega)=0$

b) $(g^2_{2,i})''(\omega)=(g^2_{2,i})''(1/2)=-(v_i+v_{n+2-i})$, where

$$v_k=D_{k-1}^2\left[2D_{n+3-2k}-D^2_{n+1-k}-2(D_{2n+2k}-\rho h(2n+1-2k))/(4+\rho h)\right]>0,$$

c) $(g^2_{2,i})'(\omega)=0 \Rightarrow \omega=\omega_{2,i}=$

$$=\frac{1}{2}+\frac{2\rho h}{4+\rho h}\,\frac{1D_{2n+3-2i}(n+2-i)D_{2i-1}-(4+\rho h)(D^2_{i-1}-D^2_{n+1-i})/2\rho h}{v_i+v_{n+2-i}},$$

we conclude that

$$\max_{\omega\in[0,1]} g_{2,i}(\omega)=g_{2,i}(\omega_{2,i}).$$

It is easy to see that $g_{2,i_*}(\omega_{2,i_*})$, when $i_*=1+n/2$, is the maximal value of $g_{2,i}(\omega_{2,i})$ for $i=2,3,\dots,n$. By (25) this proves that

$$R_{\rho,n,2}=\sqrt{h}\ \mathcal{D}_n^{-1}g_{2,i_*}(\omega_{2,i_*})=\sqrt{h}\ \mathcal{D}_n^{-1}g_{2,1+n/2}(1/2).$$

Direct calculations of the value $g_{2,1+n/2}(1/2)$ give the final result (4).

Let now $q=\infty$. Taking into account the obvious equality

$$\max_{\omega\in[0,1]} g_{\infty,i}(\omega)=\max(\mathcal{D}_{i-1}d_{n+2-i},\ \mathcal{D}_{n+1-i}d_i)$$

from (25) we get

$$R_{\rho,n,1}=\mathcal{D}_n^{-1}\max_{i=2,\dots,n}(\mathcal{D}_{i-1}d_{n+2-i},\mathcal{D}_{n+1-i}d_i)=\mathcal{D}_n^{-1}\max_{i=1,\dots,n-1}\mathcal{D}_i d_{n+1-i}.$$

By mathematical induction we prove that the sequence $(\mathcal{D}_i d_{n+1-i})_{1\le i\le n}$ increases. The inequality $\mathcal{D}_i d_{n+1-i}>\mathcal{D}_{i-1}d_{n+2-i}$, is equivalent to

$$\frac{\mathcal{D}_i}{\mathcal{D}_{i-1}}>\frac{d_{n+2-i}}{d_{n+1-i}} \tag{28}$$

which for $i=1$ is obvious. Under assumption that (28) is true for $i=k$, we obtain that it holds also for $i=k+1$

$$\frac{\mathcal{D}_{k+1}}{\mathcal{D}_k}-\frac{d_{n+1-k}}{d_{n-k}}=\left[(2+\rho h)-\frac{\mathcal{D}_{k-1}}{\mathcal{D}_k}\right]-\left[(2+\rho h)-\frac{d_{n-1-k}}{d_{n-k}}\right]=$$

$$=\left[\frac{d_{n-1-k}}{d_{n-k}}-\frac{d_{n+1-k}}{d_{n+2-k}}\right]+\left[\frac{d_{n+1-k}}{d_{n+2-k}}-\frac{\mathcal{D}_{k-1}}{\mathcal{D}_k}\right]=$$

$$=\frac{\rho h}{d_{n-k}d_{n+1-k}}+\frac{\rho h}{d_{n+1-k}d_{n+2-k}}+\left[\frac{d_{n+1-k}}{d_{n+2-k}}-\frac{\mathcal{D}_{k-1}}{\mathcal{D}_k}\right]>0.$$

Therefore

$$R_{\rho,n,1}=\mathcal{D}_n^{-1}\mathcal{D}_{n-1}d_2=\frac{1+\rho h}{c}\left[1-\frac{c^2-1}{c^{2n}-1}\right].$$

Thus, the theorem is proved.

5. CONCLUSION

Analysing the results of this paper we want to point out the following.

1) Since $\beta_p(\rho h, n) > 0$, $p=1,2,\infty$, from (3)-(5) we get the inequalities

$$R_{\rho,n,1} \le \frac{1 + \rho h}{1+\rho h/2 + \sqrt{\rho h(1+\rho h/4)}} ,$$

$$R_{\rho,n,2} \le \sqrt{h}\ \frac{\sqrt{2+\rho h}}{2^{4}\sqrt{\rho h(4+\rho h)}} , \quad R_{\rho,n,\infty} \le h\,\frac{\sqrt{4+\rho h}}{2\sqrt{\rho h}} .$$

2) It can be shown that as $\rho \to \infty$, the values $\beta_p(\rho h, n)$, $p=1,2,\infty$, monotone increase to 1. It means that as $\rho \to \infty$, the errors $R_{\rho,n,p}$ monotone decrease to the errors $R_{n,p}$ of the interpolation

$R_{n,1} = 1$, $R_{n,2} = h^{1/2}/2$, $R_{n,\infty} = h/2$.

3) For a fixed ρ when $n \to \infty$, the smoothing algorithm convergens on W^1_2 as $h^{1/4}$, and on W^1_∞ as $h^{1/2}$.

4) If $\rho = O(1/n)$, than the order of the convergence of the smoothing algorithm on W^1_p, $p=1,2,\infty$, equals to the one of the interpolating algorithm.

It is useful to take into account those remarks when choosing the smoothing parameter ρ.

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С. Асмусс. Погрешность приближения функций сглаживающими сплайнами.

Аннотация. В данной статье рассматривается задача сглаживания исходных данных кусочно-линейными сплайнами, которые минимизирует взвешенную сумму функционалов гладкости и интерполяции. Анализируется влияние параметра сглаживания на сходимость алгоритма. Получены точные на классах W^1_p, p=1,2,∞, оценки погрешности. УДК 517.5.

S. Asmuss. Nogludinošas aproksimācijas ar splainiem kļūdas novērtējumi.

Anotācija. Šajā rakstā aplūkota funkciju aproksimācija ar nogludinošiem splainiem, kuri minimizē kādu nogludinoša un interpolējoša funkcionāla kombināciju. Analizēta nogludinoša parametra ietekme uz algoritma konverģenci. Iegūti precīzi kļūdas novērtējumi W^1_p klasē pie p=1,2,∞.

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THE METHOD OF SUCCESSIVE APPROXIMATIONS
FOR SOLUTION OF A BOUNDARY VALUE PROBLEM
FOR THIRD ORDER DIFFERENTIAL EQUATION
WITH FUNCTIONAL BOUNDARY CONDITIONS

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Abstract. Sufficient conditions are given for the convergence of the method of successive approximations.
1991 MSC 34B99

Consider the boundary value problem:

$$x'''=f(t,x,x',x''), \tag{1}$$

$$l_i x(\cdot)=r_i, \quad i=1,2,3, \tag{2}$$

where $f\in Car(I\times\mathbb{R}^3,\mathbb{R})$, $l_i: AC^2(I,\mathbb{R})\to\mathbb{R}$, l_i - linear continuous functionals, $r_i\in\mathbb{R}$, $i=1,2,3$, $I=[a,b]$, $-\infty<a<b<+\infty$, $Car(I\times\mathbb{R}^3,\mathbb{R})$ denote the set of functions $f: I\times\mathbb{R}^3\to\mathbb{R}$, satisfying the Caratheodory conditions [1], $AC^2(I,\mathbb{R})$ - the set of differentiable functions wiith absolutely continuous second order derivatives, $C^2(I,\mathbb{R})$ - twice continuously differentiable functions.

In the work sufficient conditions are given for the method of successive approximations to be convergent when solving the problem (1), (2). Similar results are stated also in the work [2] for three-poiint BVP.

We assume that f satisfies the Lipschitz condition. For any (x_1,x_2,x_3), $(y_1,y_2,y_3)\in\mathbb{R}^3$ and almost everywhere in I holds:

$$|f(t,x_1,x_2,x_3)-f(t,y_1,y_2,y_3)|\le$$
$$\le k(t)|x_1-y_1|+l(t)|x_2-y_2|+m(t)|x_3-y_3|,$$

where functions $t\to k(t)$, $t\to l(t)$, $t\to m(t)$ are positive, bounded a.e. in I. Denote

$$K=\operatorname{vrai\,sup}_{t\in I} k(t),\quad L=\operatorname{vrai\,sup}_{t\in I} l(t),\quad M=\operatorname{vrai\,sup}_{t\in I} m(t).$$

Suppose that the homogeneous BVP

$$x'''=0,\quad l_i x(\cdot)=0,\quad i=1,2,3$$

has only the trivial solution and denote by $G(t,x)$ a corresponding Green's function. Then the matrix

$$A=\begin{pmatrix} l_1(1) & l_1(t) & l_1\left(\frac{t^2}{2}\right) \\ l_2(1) & l_2(t) & l_2\left(\frac{t^2}{2}\right) \\ l_3(1) & l_3(t) & l_3\left(\frac{t^2}{2}\right) \end{pmatrix}$$

has an inverse A^{-1}.

Denote by $t\to K_0(t)$ the scalar product of three-dimensional vectors $\left(1,t,\frac{t^2}{2}\right)$, $A^{-1}\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$.

Set for any $t\in I$

$$Q(t)=|f(t,k_0(t),k_0(t),k_0''(t)|$$

and suppose that $q=\operatorname{vrai\,sup}_{t\in I} Q(t)\in R$.

Denote

$$\int_a^b |G(t,s)|ds = g_0(t),$$

$$\int_a^b |G_t(t,s)|ds = g_1(t),$$

$$\int_a^b |G_{tt}(t,s)|ds = g_2(t).$$

Theorem. Let the condition

$$\max_{t\in I}(Kg_0(t)+Lg_1(t)+Mg_2(t))=g<1.$$

Then the BVP (1), (2) has a unique solution.

Proof. Let $C^3(I,R)$ be a linear normed space equipped with the norm

$$\|x\| = \max_{t\in I} (K|x(t)|+L|x'(t)|+M|x''(t)|). \tag{3}$$

Define successive approximations

$$x_0(t)=k_0(t),$$

$$x_{n+1}(t) = \int_a^b G(t,s)f(s,x_n(s),x_n'(s),x_n''(s))ds + k_0(t)$$

for $n=0,1,2,\dots$.

Let estimate $\|x_1-x_0\|$. We have

$$|x_1(t)-x_0(t)| \le q \int_a^b |G(t,s)|ds = qg_0(t),$$

$$|x_1'(t)-x_0'(t)| \le q \int_a^b |G_t(t,s)|ds = qg(t),$$

$$|x_1''(t)-x_0''(t)| \le q \int_a^b |G_{tt}(t,s)|ds = qg_2(t).$$

Hence

$$\|x_1-x_0\| = \max_{t\in I} \Big(K|x_1(t)-x_0(t)|+L|x_1'(t)-x_0'(t)|+$$

$$+M|x_1''(t)-x_0''(t)|\Big) \le \max_{t\in I} \Big(Kqg_0(t)+Lqg_1(t)+Mqg_2(t)\Big) = qg.$$

Estimate $\|x_2-x_1\|$ making use of the Lipschitz condition. We have

$$|x_2(t)-x_1(t)| \le \int_a^b |G(t,s)|\cdot|f(s,x_1(s),x_1'(s),x_1''(s))-$$

$$-f(s,x_0(s),x_0'(s),x_0''(s)|ds \le$$

$$\le \int_a^b |G(t,s)|\Big(k(s)|x_1(s)-x_0(s)|+l(s)|x_1'(s)-x_0'(s)|+$$

$$+m(s)|x_1''(s)-x_0''(s)|\Big)ds \le \int_a^b |G(t,s)|\Big(K|x_1(s)-x_0(s)|+$$

$$+L|x_1'(s)-x_0'(s)|+M|x_1''(s)-x_0''(s)|\Big)ds \le$$

$$\le \|x_1-x_0\|\cdot g_0(t) \le qgg_0(t).$$

Analogously

$$|x_2'(t)-x_1'(t)| \le qgg_1(t),$$

$$|x_2''(t)-x_1''(t)| \le qgg_2(t),$$

i.e.

$$\|x_2-x_1\| = \max_{t\in I}\Big(K|x_2(t)-x_1(t)|+L|x_2'(t)-x_1'(t)|+$$

$$+M|x_2''(t)-x_1''(t)|\Big) \le \max_{t\in I}\Big(Kqgg_0(t)+Lqgg_1(t)+Mqgg_2(t)\Big) \le$$

$$\le qg\max_{t\in I}\Big(Kg_0(t)+Lg_1(t)+Mg_2(t)\Big) = qg^2.$$

By induction we have for $n=0,1,\ldots$

$$\|x_{n+1}-x_n\| \le qg^n.$$

Let us show that the sequence $n\to x_n$ is fundamental. We have

$$\|x_{n+p}-x_n\| \le \|x_{n+p}-x_{n+p-1}\| +$$

$$+\|x_{n+p-1}-x_{n+p-2}\| + \ldots + \|x_{n+1}-x_n\| \le$$

$$\le qg^{n+p-1}+qg^{n+p-2}+\ldots+qg^n = \frac{q(g^n-g^{n+p})}{1-g} \le \frac{qg^n}{1-g}.$$

Hence fundamentality of the sequence $n\to x_n$ follows and, in view of the completness of $C^2(I,\mathbb{R})$ with respect to the norm (3), it converges to an element $y\in C^2(I,\mathbb{R})$ such that

$$y(t) = \int_a^b G(t,s)f(s,y(s),y'(s),y''(s))ds + k_0(t).$$

This means that $y\in AC^2(I,\mathbb{R})$ and therefore solves the BVP (1), (2).

Show the uniqueness of y. Let y_1 be another solution of the problem (1), (2). Then

$$\|y-y_1\| \le \max_{t\in I}\Big(K\int_a^b |G(t,s)|\cdot|f(s,y(s),y'(s),y''(s))-$$

$$-f(s,y_1(s),y_1'(s),y_1''(s))|ds +$$

$$+ L\int_a^b |G_t(t,s)|\cdot|f(s,y(s),y'(s),y''(s))-$$

$$-f(s,y_1(s),y_1'(s),y_1''(s))|ds +$$

$$+ M\int_a^b |G_{tt}(t,s)|\cdot|f(s,y(s),y'(s),y''(s))-$$

$$-f(s,y_1(s),y_1'(s)\ y_1''(s))|ds \le \|y-y_1\|q < \|y-y_1\|.$$

The contradiction obtained proved the theorem.

Remark. An analogous assertion is valid for n-th order equations.

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В. Пономарев. Метод последовательных приближений для решения краевой задачи для дифференциального уравнения третьего порядка с функциональными граничными условиями.
Аннотация. Приводятся достаточные условия для сходимости метода последовательных приближений к решению краевой задачи.
УДК 517.927

V.Ponomarjovs. Pakāpenisko tuvinājumu metode robežproblēmas atrisināšanai trešās kārtas diferenciālvienādojumam ar funkcionāliem robežnosacījumiem.
Anotācija. Doti pietiekamie nosacījumi pakāpenisko tuvinājumu metodes konverģencei uz robežproblēmas atrisinājumu.

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Some personal reflections on ICM94

The ICM (the International Congress of Mathematicians) was held in Zurich (Switzerland) from August 3 till August 11. This was the 22 Congress and it had some important features.

First of all there was a very large number of participan's from East European countries and Russia (if compared with the situation at the previous Congresses). This can be explained by two reasons. First, now there are no artificial, administrative or political obstacles in these countries and people are free to move abroad. Second, the Organizing Committee of the Congress had carried out an enormous work in order to obtain financial help for participants from these countries.

Especially I would like to emphasize the understanding of the importance of science, mathematics in particular, and necessity to support it financially,which was manifested by the Switzerland Government, by the authorities of the Zurich Canton and by the business (especially, by banks).

And, on the second hand, with this attitude towards mathematics quite good relates the opinion expressed at the Congress that the scientific (and, in particular mathematical among them) community must take care to explain to the wide sections of the population about the role of the science and the possible benefits which the society can get out from it.

One of the real steps in this direction was the fact that at the ICM94 there was organized a special section: applications of mathematics in the science (in fact this meant applications of mathematics of the needs of the society).

The financial help, offered by the Organizing Committee of the Congress, by the Soros Foundation of Latvia and by local sponsors, enabled 8 Latvian mathematicians to participate at the Congress These mathematicians represented all the main centers of mathematics in Latvia: the University of Latvia, the Institute of Mathematics and Computer Science, the Institute of Mathematics and the Riga Technical University - and from the Latvia University of Agriculture.

Unfortunately, because of some lack of information and a certain passivity from some colleagues the investigations of Latvian mathematicians in the field of applications of mathematics were not enough represented at the Congress.

The working structure of the Congress: Plenary Addresses (they were 16), Section Lectures (about 170) and Short Communications in the form of posters (about 1000) seems to be adequate to achieve goals of the Congress. The plenary addresses were aimed to give an insight into the most important problems and current trends in mathematics, the section lectures - mainly to present surveys of the recent development of actual problems in a given field of mathematics while the purpose of the short communications was to give an idea about the latest results.

Unfortunately a part of lecturers (mainly in sections) did not succeeded these purposes. Some lectures were full of technical details lacking clear exposition of the place of the discussed subject among other investigations.

From my professional point of view I would like to say that I have got a feeling that now the theory of nonlinear partial differential equations is being developed at the boundaries of the present knowledge. And, by making small steps in various directions the search of new ideas, new statements of problems is going on. In particular, to meet the demands of nonlinearities one can notice the tendency to work in more general spaces of functional arguments. It seems that the traditional (and handy or convenient to use) spaces, for instance standard Sobolev spaces, are not adequate enough to deal with new nonlinear problems. Quite often and fruitful is used the approach: by considering non-trivial refined examples to describe how far the existing ideas, methods, etc. could be applied.

Particularly one could come to these conclusions from the lectures given by Fields medalists. As it is well known the Fields Medals is a certain analogue of the Nobel prize presented to mathematicians. These medals are being awarded to young, i.e., below 40 years mathematicians for their outstanding achievements in mathematics.

This time the following mathematicians were awarded by the Fields Medal: P.-L.Lions (Universite Paris-Dauphine) for his investigations in the theory of nonlinear partial differential equations; J.Bourgain (University of Illinois, Institut des Hautes Etudes Scientifiques, Princeton) for the investigations in the theory of finite dimensional Banach spaces; J.-C.Yoccoz (Universite Paris-Sud) for the investigations in dynamic systems; E.Zelmanov (University of Madison) for the investigations in the group theory.

The special Rolf Nevanlinna Prize in the field of applications of mathematics in informatics was awarded to A.Widgerston (Hebrew University) for his investigations in the field of the complexity theory.

For me it was very interesting that one of the Fields Medals was awarded for investigations which were close to my profesional field of work.

At the closing ceremony of the Congress the new President of the International Union of Mathematicians as well as the place and time of the next International Congress of Mathematicians were anounced.

Now the President of the IUM is professor David Mumford from the Harvard University and the next Congress will be held in the August of 1998 in Berlin.

In the conclusion I would like to say that the Congress was very valuable for me from the point of view of the general mathematical education and particularly it extended my outlook on mathematics on the whole.

U.Raitums