Uniquely Hamiltonian Graphs

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A TALK IN THREE PARTS

1. Definitions and Prehistory
2. Sheehan’s Conjecture
3. UH3 graphs
The dodecahedron is a \textit{cubic} and \textit{planar} graph.
A cycle is a circular sequence of vertices \((v_0, v_1, \ldots, v_{k-1})\), each adjacent to the next.
A *Hamilton cycle* is a cycle that uses all of the vertices of the graph.
Sir William Rowan Hamilton (1805–1865)

Famously invented *quaternions*, but also the “*Icosian Game*”. 
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*(Also available in handy portable travel-set)*
4CC – THE FOUR-COLOUR CONJECTURE

Conjecture (Guthrie, 1850s)
The *faces* of a *cubic planar graph* can be coloured with 4 colours, so that neighbouring “countries” never have the same colour.
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THE FOUR-COLOUR CONJECTURE

The four-colour conjecture:

- dominated graph theory until the 1970s
- consumed numerous academic careers
- catalysed the introduction of a vast range of tools
- caused a furore when resolved in 1976
Peter Guthrie Tait (1831–1901)

Tait’s Conjecture
Every 3-connected cubic planar graph has a Hamilton cycle.

This conjecture is stronger than the 4CC — a Hamilton cycle can be used to find a 4-colouring of the faces.
HAMILTON CYCLE TO FACE-COLOURING
Tutte — The modest giant of combinatorics
Tutte disproves Tait
But why is this non-Hamiltonian?

Look at one of the three identical pieces of the graph.

Case analysis shows no Hamilton path connecting the red vertices.
Smith’s result

Theorem (Smith)
Any edge in a cubic graph lies in an even number of Hamilton cycles.

In this example, each rim edge lies in 4 Hamilton cycles, and each spoke edge lies in 2.

So a Hamiltonian cubic graph has at least three Hamilton cycles.
Part II: Sheehan’s Conjecture
**Uniquely Hamiltonian graphs**

A graph is *uniquely Hamiltonian* if it has *exactly one* Hamilton cycle.

Vertices of degree 2 are cheating (at least, uninteresting).

We want *uniquely hamiltonian* graphs with *minimum degree* at least 3, or *UH3 graphs* for short.
Which graphs *can*, or *cannot*, be UH3 graphs?

Over the last decades, a steady trickle of papers have provided partial answers . . .

. . . but major questions remain unresolved.
Sheehan’s Conjecture

The most famous is Sheehan’s conjecture.

Conjecture (John Sheehan, 1975)
There are no uniquely hamiltonian 4-regular graphs
Progress to date is largely due to these two eminent mathematicians.

Carsten Thomassen  
Andrew Thomason
A wonderful result that takes the most *modest of ingredients*: “Any graph has an even number of vertices of odd degree” and turns it into something with far-reaching consequences.
Take a cubic graph with a Hamilton cycle through an edge $e = xy$, and then delete $e$. 
Change the Hamilton path by using the “other edge” through $y$. 
Now two choices — one reverses what we just did, the other moves to a new Hamilton path.

When can this process end?
Why lollipop?
Thomassen adapted Thomason’s lollipop method to a

- *(Thomason)* No odd-regular UH-graphs
- *(Thomassen)* No \( \geq 300 \)-regular UH-graphs
- *(Haxell, Seamone, Verstraëte)* No \( \geq 22 \) regular UH-graph

Also *no solution* to Sheehan’s conjecture!
A UH4 graph!

Sheehan’s conjecture seems *almost* self-evidently true . . .

. . . but Herb Fleischner has constructed a *non-regular* UH4 graph.
**Another Thomassen Conjecture**

**Conjecture (Thomassen)**

In a *hamiltonian graph* $G$ of minimum degree at least 3, there exists at least one edge $e$ such that both the *deletion* $G \setminus e$ and *contraction* $G/e$ are hamiltonian.

![Graph](image-url)
Our interest

**Conjecture (Thomassen)**
A Hamiltonian graph has *no chromatic roots* in the interval \((1, 2)\).

Fengming and I have several *more precise* conjectures about exactly which graphs have no chromatic roots in \((1, 2)\).
MORE THAN ONE Hamilton cycle

Pick $e$ lying in the yellow cycle, but not the red.

- The red Hamilton cycle is a Hamilton cycle for $G\setminus e$.
- The yellow Hamilton cycle is a Hamilton cycle for $G/e$. 
EXAMPLES OF UH3 GRAPHS

Viesākām 1, 2, 0, viņš parādē 4, arīm - 3.

Lai sāktu viņas un automorfinus.

Vienādus metrikās automorfinus.

šimetrījas (10) (27) (35) (4,5) (6,7) (8,9) (9,8) (6,5) (1,2) (12) (62) (58)

9. 9.

19. 10. 78

Document generously supplied by Dainis Zeps, academic “grandson” of Grinbergs
Emanuels Grinbergs (1912–1982)

- Latvian polymath with 2 PhDs
- Famous for *Grinberg’s theorem*
- Graph found 1979, published 1986

Often known by “westernized” name *Grinberg*.

But really Grinberg’s theorem *should be* Grinbergs’ theorem.
Redrawing Grinbergs’ graph we get something rather familiar.
In the *half-graph*, there is

- *exactly one* Hamilton *path* from the yellow to blue,
- *exactly zero* Hamilton paths from yellow to green
Dainis Zeps rescued this construction from Grinbergs’ archives.

A 3-preparation is a triple of vertices \((x, y, z)\) with

- A unique Hamilton path from \(x\) to \(y\), and
- No Hamilton path from \(x\) to \(z\).
Start with a cycle of desired length.

Systematically add *chords* (edges not in the cycle) so that

- No *additional* Hamilton cycles are created
- No chord joins two vertices of degree *greater than* 3
Amazingly, working by hand, Grinbergs missed out by just one edge on finding the unique smallest UH3 graph.
Dainis Zeps has worked out what the conditions are in this case.

A *4-preparation* is a quadruple of vertices \((x, y, a, b)\) with:

- A unique Hamilton path from \(x\) to \(y\).
- An edge between \(x\) and \(a\).
- No Hamilton paths between any two of \(\{y, a, b\}\).
Many more small UH3 graphs

Dainis Zeps is classifying the \(k\)-preparations involved.

The graph \(P \setminus v\) is a recurring theme.
SO HOW’S THOMASSEN’S CONJECTURE?

Our potential counterexample has a *near-Hamilton* cycle (that is, missing just one vertex).

A *chord* from the *missed vertex* satisfies the conjecture.
So now we are asking for more — we need

- A UH3 graph on \( n \) vertices . . .
- . . . but also with no \( n - 1 \) cycles.

*First idea:* A *bipartite* graph on an *even* number of vertices has no odd cycles, so no \( n - 1 \) cycles.
We prove that a bipartite uniquely Hamiltonian graph has a vertex of degree 2 in each color class. As consequences, every bipartite Hamiltonian graph of minimum degree $d$ has at least $2^{1-d}d!$ Hamiltonian cycles, and every bipartite Hamiltonian graph of minimum degree at
Can we find a graph and 3 vertices \((x, y, z)\) such that

- Unique Hamilton path from \(x\) to \(y\)
- No Hamilton path from \(x\) to \(z\)
- No near-Hamilton path from \(x\) to \(y\)

... but none found so far.
Planar Graphs

- Kratochvil & Zeps prove that Hamiltonian planar triangulations have at least 4 Hamilton cycles.

- Hakimi, Schmeichel & Thomassen find infinite family of planar triangulations with exactly 4 Hamilton cycles.

Conjecture (Bondy / Jackson)
There are no planar UH3 graphs.
THANK YOU FOR LISTENING