Abstract A possibility is proposed to define the proper spacetime of a free nonzero-rest-mass m_0 particle based on the connection of its lasting proper time to an open sequence of natural numbers counting de Broglie time periods $(h/c^2)(m_0^{-1})$ [see R. Ferber, A Missing Link: What is Behind de Broglie's" Periodic Phenomenon"?, Foundations of Physics Letters 9, 575 (1996)]. It is suggested to define a set of two-directional intervals of the particle's proper space (proper distances) following the construction of positive and negative integers from the ordered pairs of the natural numbers, which belong to the sequence 1, 2, ..., n defining the elapsed interval of de Broglie time t_n. Corresponding to rational numbers, the ratios of proper distances to their common time interval t_n are referred to as proper velocities, which are expressed by a rational fraction of c without supposing any relation to an external system of coordinates. The particle's proper reference frame appears by resolving the t_n over proper spatial and temporal coordinates, which are connected in a wave-like process. The corresponding connection in energy representation takes the form of relativistic energy-momentum relation of the respective proper quantities, which reveals the existence, besides the rest energy term m_0c^2, of the term attributed to particle's proper space that is mainly determined by large proper distances and approaches with growing t_n the finite value of the order of m_0c^2.

Keywords proper spacetime · de Broglie proper time · proper time · construction of integers · proper space · energy of proper space

A Numbers-Based Approach to a Free Particle's Spacetime

R. Ferber

Received: date / Accepted: date

1 Introduction

It is hardly possible to imagine more fundamental concepts than time, space and numbers. The description of space and time is formalized by introducing a reference frame, which proved to be a basic issue in the development of physics. A rather foundational question may be posed regarding the most primitive object-—a free nonzero-rest-mass (let us suppose fundamental) particle—can its time and space be considered as existing per se or only in relation to some outer, externally defined reference frame? In other words, is it possible to introduce for a free, completely isolated entity-—a particle-—its proper spacetime by a system of spatial and temporal coordinates, which are defined by nothing else than the particle's intrinsic property-—its rest mass m_0, as well as by the constants of nature? This would add to the understanding of such a basic constituent of quantum mechanics (QM) as a free particle's wave function. The obvious challenging issue is that not only a particle's temporal, but also its spatial coordinate necessarily is included in wave-like behaviour has to be defined as proper, that is, without relation to any external reference frame, as it is considered in the special theory of relativity (STR).

Let us compare the starting points of description of a particle (an 'object') in STR and QM. The presentation of the STR in [1] begins by stating that, in order to describe an object, one needs a *system of reference*, by which "...we understand a system of coordinates serving to indicate the position of a particle in space, as well as clocks fixed in this system serving to indicate the time". Further, "... at each moment of time we can introduce a coordinate system rigidly linked to the moving clocks" and the time "...read by a clock moving with a given object is called the proper time for this object". It is, however, obvious that any 'rigid linking' necessarily involves interaction, which evidently contradicts the condition of remaining a free particle; clearly, the same contradiction relates to the spatial coordinate as well. One of the starting statements of QM is that an object is described by *its wave function*. Comparing the wording of STR and QM may

Department of Physics, University of Latvia, 19 Rainis blvd, Riga LV-1586, Latvia
E-mail: Ruvins.Ferbers@lu.lv

incline one to suggest that it is just the pronoun 'its' making the difference. Indeed, while the STR introduces an object (or its free motion) as essentially related to an external property, such as the transformation at changing systems of coordinates, QM introduces the wave function as a particle's proper feature. This however is not exactly the case, since the arguments of the wave function are likely supposed to be related to an external reference frame. To actualize a controversy, it is worth recalling a statement by Dirac [2], that the particle interferes only with itself, and this should mean, within its proper space (or spacetime). An attempt to address this controversy makes the core of the presented analysis.

The starting point is de Broglie's periodic phenomenon or the de Broglie frequency [3], which is uniquely determined in a particle's eigensystem of coordinates by its rest mass and the constants of nature combined as c^2/h. As mentioned in [4], this defines, for a free, nonzero-rest-mass particle an 'ideal' proper time scale as a sequence of 'marks' along the line of proper time, provided by a natural standard time unit—the de Broglie time period. The idea to create a clock that relies on a fundamental link between time and a particle's mass was recently realized in [5]. Association of proper time defined by de Broglie's periodic phenomenon with a sequence of natural numbers was suggested in [6]. Indeed, since evolving a particle's proper time as an uninterrupted ordered sequence of 'ideal' (with the precision of the constants of nature) de Broglie time units involves their uninterrupted counting, or *numbering*, such an 'ideal sequence' may be considered isomorphic with a growing open sequence of natural numbers 1, 2, ... , (some remarks on the long-lasting discussion of the connection between time and numbers may be found in [6] and references therein).

The key expressions of STR involve an *interval* of space and time. The question appears: is a non-contradictory definition of an interval of proper time for a particle, which is considered free of any external influence, at all possible? The interval in STR is defined by introducing the notion of event "... occurring in a certain material particle" and determined "by the place where it occurred and by the time when it occurred" [1]. In particular, an interval of proper time is defined by 'events' at two moments of time. In the suggested approach it is assumed that such moments of time are: (i) the moment of a particle's *present* existence (by 'existence' we will always mean present existence), and (ii) the moment of *start* (beginning, creation) of a particle's existence, without which the existence would last forever, thus making the notion of time meaningless. It is supposed that the present moment remains related to 1, while the start is related to a number n, whose running means the time flow. As a result, the elapsed interval of proper time between present and start is defined by a sequence 1, 2, ..., n of de Broglie time periods as a natural time unit.

The bottom line of the suggested approach lies in assuming that it is possible to define a two-directional interval of proper space following the construction of all possible *positive and negative integers* from the ordered pairs of natural numbers belonging to the sequence 1, 2, ..., n. Clearly, this requires passing from the de Broglie time unit to a Compton wavelength as a distance unit, which requires passing from c^2 to c, and, therefore, presumes a sign choice that justifies the appearance of positive and negative integers. Intervals of proper space scaled by their common interval of proper time are referred to as 'proper velocities', which are expressed by a rational fraction of c without any relation to an external reference frame. It is worth mentioning that such velocity's definition may remind one of a consequence of Newton's first law, according to which velocity is something that persists without cause, which means without any externally defined factor.

Similar to STR, the connection between intervals of particle's proper space and proper time appears in squared form. When such a connection is expressed in form of energy-momentum, the question appears-—is the term containing squared proper velocity related to what can be considered as some form of particle's proper space-related energy, which persists in addition to particle's rest energy m_0c^2? It will be shown that at large n the respective term approaches a finite value determined exclusively by the particle's rest mass and may be referred to as the spatial-energy of the particle's spacetime. The possible meaning of such energy is discussed, provided the main contribution to it comes from large intervals of proper space. The linearized expression connecting proper time and proper space appears in form of a wave-like process, straightforwardly bringing the QM connection between linear momentum and de Broglie wavelength as proper quantities. A particular example of two indistinguishable particles that remain free after common start instant, say, in a decay, will be considered: since their proper spacetime is constructed based on a common sequence of numbers 1, 2, ... , n, the distance and velocity may be regarded relative, while remaining proper. In conclusion it will be discussed to what extent the numbers-based introducing of a free entity's proper spacetime may provide a common, more primitive starting point for foundations of STR and QM.

2 Proper Time

2.1 Basic assumptions

Let us formulate the requirements, which would allow one to define the time course and an interval of proper time of a free, completely isolated, nonzero-rest-mass particle. Inevitably, this situation would require that the particle is able to function as a 'proper clock' by itself, without any externally imposed influence. This would require [6]:

- Existence of a 'natural' standard unit of proper time T_0 that is defined exclusively by an intrinsic characteristic of a particle—its rest mass m_0, as well as by the constants of nature;
- An uninterrupted numbering (counting) routine of T_0-units that is neither imposed nor influenced by any external factor, which is defined by (isomorphic with) a sequence of natural numbers 1,2, ...

The interval of proper time of a free particle may be defined based on the following assumptions:

- The particle's existence as a free individual entity means its existence at the *present* instant of time $\mathrm{P_I}$, which continues to correspond to 1 during the counting routine of the course of time;
- There exists a uniquely defined temporal beginning, or start instant S, which precedes $\mathrm{P_I}$, thus introducing the notion of the past, and therefore, of duration;
- Identifying any instant other than $\mathrm{P_I}$ and S would contradict the notion of *in principle* undisturbed (uninfluenced) existence of the particle.

Identifying two instants associated with the free entity, the present instant (P_I) and the start instant (S), defines the interval of particle's proper time $P_I S$. The particle's existence in time (particle's time course) occurs at fixed present instant P_I as evolving, from the present to past, of the interval $P_I S$. Let us analyze how the suggested requirements are related to basic concepts of physics and arithmetic.

2.2 A natural unit of proper time

For each particle with rest mass m_0, a natural unit of proper time T_0 is uniquely provided by de Broglie's periodic phenomenon. The latter postulates the existence of a proper frequency, or de Broglie frequency ν_0 [3] defined by particle's rest mass m_0 via a proportionality factor comprised of constants of nature:

$$T_0 = \nu_0^{-1} = (h/c^2)(m_0^{-1}). \tag{1}$$

The role of (1) in defining the proper time of a free m_0-particle is discussed in [4] and [6]; a somewhat related remark about preceding the notion of time by the notion of a periodic process can be found in Einstein's paper [7].

The accuracy of such a natural time unit T_0 is determined by the accuracy of the particle's rest mass m_0, as well as by the accuracy of the respective fundamental constants of nature. As such, the latter define (or, equivalently, are defined by) the fundamental properties of spacetime. Suppose that there is no particle whose rest mass m_0 is determined with the same accuracy as the constants of nature. This would mean that no unit of proper time, *i.e.*, no time could be determined with the accuracy of the constants of nature, and, consequently, the constants of nature would not be functioning properly as such. This argument suggests the existence of a particle with rest mass m_0 whose accuracy matches the accuracy of a constant of nature, which implies that it (the respective m_0-particle) is in this sense a fundamental particle. And indeed, the rest masses of fundamental particles are in the list of the constants of nature.

2.3 De Broglie proper time and connection to natural numbers

The necessity of counting time units presupposes a connection with numbers. It is worth calling to mind Dedekind's [8] consideration that (emphasis added) "... the whole arithmetic is a necessary, or at least natural, consequence of the *simplest arithmetic act*, that of *counting*, and counting itself is nothing else than the successive creation of the infinite series of positive integers in which each individual is defined by the one immediately preceding; the simplest act is the passing from an already formed individual to the consecutive new one to be formed".

Let us follow how the 'simplest arithmetic act' is manifested by de Broglie's periodic phenomenon (1). Accounting for the necessity of particle's temporal beginning (start), the periodicity postulated by (1) may equally mean [6] a reproduction of the T_0-unit or, equivalently, self-reproduction of the particle itself with de Broglie frequency ν_0. Since the particle's existence is (i) associated with the present and (ii) associated with de Broglie's periodic phenomenon (1), the reproduction has to be associated with the present instant P_I as well. Then, the uninterrupted, on-going proper time of a free particle is started by one T_0-unit at the start and continued by the 'simplest arithmetic act' of consequent continuous adding of T_0-units, which is equivalent to the generation of a sequence T_0, T_0T_0, ... , iT_0, ... Supposing the strict identity of the T_0-units, the sequence is isomorphic to the sequence of *natural numbers* (positive integers) 1, 2, ... , defined by the condition that every natural number i has a natural number successor SUC(i), and SUC(i) is $i+1$. The time, which is defined by the number of counts of de Broglie units T_0 determines a particle's *natural*, or *de Broglie proper time*.

Suppose that only a single T_0-unit corresponds to the present instant $\mathrm{P_I}$, which remains corresponding to one, $\mathrm{P_I} \leftrightarrow 1$ (this may as well be considered as a 'numbers-based' definition of the present—more discussion is postponed to Sec. 5.3). Then, at each reproduction of the present T_0-unit, the 'previous present' T_0-unit is added to the past. This brings the idea of uninterrupted, ongoing proper time as a continuous shifting of one (present) T_0-unit to form what is defined as 'past'. Since at the instant of temporal beginning, the present $\mathrm{P_I}$ coincides with the start S, the start can be labelled by one: $\mathrm{S} = \mathrm{S_I}$. The next act of reproduction means that the start does not correspond to the present any more, but is shifted by one T_0-unit to the past. Since the present and the start are the only identifiable instants of time, the $\mathrm{P_I} \leftrightarrow 1$ condition implies that the shifting renumbers the start, $\mathrm{S_1} \to \mathrm{S_2}$. Further, each consecutive act of reproduction of a T_0-unit at the present uniquely corresponds to a consecutive renumbering of starts as $\mathrm{S}_{i-1} \to \mathrm{S}_i$. Therefore, the sequence $\mathrm{S_1}$, $\mathrm{S_2}$, ... , uniquely corresponds to the sequence $1T_0$, $2T_0$, ... , isomorphic to the sequence of natural numbers 1, 2, ... Thus, we arrive at the idea of considering the particle's ongoing proper time as its *ageing* time. This ascribes the following meaning to particle's 'ageing': a consecutive reproduction of T_0-units at the present instant is equivalent to the respective shifting of particle's temporal beginning to the past. The number n of consecutive reproductions of a T_0-unit uniquely defines an *interval of de Broglie proper time* $\mathrm{P_I S}_n$:

$$\mathrm{P_I S}_n \equiv t_n = nT_0. \tag{2}$$

The continuity of a free particle's existence is equivalent to its continuous 'ageing' as a consecutive increasing of its interval of proper time as $\mathrm{P_I S}_n \to \mathrm{P_I S}_{n+1}$, which itself is isomorphic to the ongoing sequence of natural numbers SUC(n) $= n+1$. Such an uninterrupted binding of the present, throughout the past, to the start denotes the free particle's uninterrupted 'history'.

3 Proper Space

3.1 Basic assumptions

By intuition, it seems obvious to define particle's spatial coordinates with respect to an external reference frame. However, as already discussed, any externally imposed determination would violate the rigid condition of a completely isolated entity. A possibility is suggested to avoid this inconsistency by assuming that an interval of the free particle's proper space, or its 'proper distance'[1], can be defined

[1] This notion of a free entity's proper distance should not be confused with the proper distance between two events in the STR, nor with the notion of proper distance in cosmology, though some common features with the latter will be discussed further on.

based on the relation of particle's proper time t_n to numbers. The particle's proper distance thus introduced has to retain consistency with t_n defined by (2), which implies:

- The existence of a natural unit of an m_0-particle's proper distance that satisfies the same requirements as for the de Broglie unit of proper time T_0, namely, to be determined by nothing else but the particle's rest mass and fundamental constants of nature;
- For a given interval of the particle's proper time $t_n(2)$ that elapsed at the present moment $\mathrm{P_I}$, the particle's proper space has to be associated with the same present moment $\mathrm{P_I}$, which is the only meaningful moment of proper time besides the start, and the same sequence of natural numbers 1, 2, ... , n that defines t_n;
- It is possible to define (construct) the set of intervals of a free particle's proper space by constructing the set of positive and negative integers from the set of natural numbers belonging to the t_n-defining sequence 1, 2, ... , n.

In what follows we will define the set of intervals of proper space along with related proper velocities based on a realization of these requirements.

3.2 Proper space and its connection to the integers

An obvious option for defining a natural unit of proper space consistent with the suggested requirements is offered by multiplying the de Broglie time unit T_0 by c as the only fundamental constant of nature that possesses the distance over time dimension. Since T_0 (1) is defined by the coefficient that includes c^2/h, this presupposes, first, separating c^2 from h and then passing from c^2 to c, the latter implying a $\pm c$ choice of sign. As a result, a natural unit of proper space is

$$\sqrt{c}^2 T_0 = \pm c T_0 = \pm \frac{h}{m_0 c}, \tag{3}$$

its value being the Compton wavelength λ_C. It seems reasonable to connect the opposite signs in (3) with opposite directions of a distance unit. With respect to numbers, this relates to the appearance of positive and negative integers. Then, a particular integer n_x, positive or negative, multiplied by the distance unit value cT_0 may be associated with a particular *interval of proper space*, or *proper distance* x_n as

$$x_n = n_x c T_0. \tag{4}$$

The reason for the notations x_n and n_x is to relate them to a spatial coordinate, as well as to stress their relation to (origination from) the elapsed interval of the proper time t_n isomorphic to the sequence of natural numbers 1, 2, ... , n uniquely defined by n. Since there is no point in presupposing a plus or minus sign of the distance unit (3), n_x has to take all values of the set of integers that can be formed from the t_n-defining sequence 1, 2, ... , n. According to a constructive definition (see, for instance, [9]) the set of integers is formed from a given set of natural numbers by ordered pairs $(a, b) \in \{1, 2, ..., n; a \neq b\}$ as a result of subtracting b from a; this yields $n_x \in \{-n+1, ..., -2, -1, 1, 2, ..., n-1\}$. Note that the value $n_x = \pm n$ yielding $x_n = \pm n c T_0$ does not appear. For $n_x > 0$, the number of all possible pairs (a, b) that yield a particular n_x is $n - n_x$.

3.3 Proper velocity

Since all intervals of proper space x_n originate from the same elapsed interval of proper time t_n (2), it is feasible to explicitly include the latter in (4). After multiplying and dividing by n the right-hand-side of (4) and accounting for (2), one arrives at the expression

$$x_n = \frac{n_x}{n} c t_n. \tag{5}$$

Since x_n is proportional to t_n, it is appropriate to associate the proportionality coefficient with *proper velocity*:

$$v_{xn} = \frac{n_x}{n} c. \tag{6}$$

As can be seen, the proper velocity's value is straightforwardly defined as a fraction of c, the latter being its ultimate upper limit. According to the n_x-values, for a given t_n-defining n, the proper velocities vary from $v_{xn} = \pm c/n$ when $n_x = \pm 1$ to $v_{xn} = \pm c(n-1)/n$ when $n_x = \pm(n-1)$. Regarding connection to numbers, as follows from (4)–(6), for n running from 1 to infinity, the normalized over c, or fractional proper velocity $v_{xn}/c = n_x/n$ runs through all rational fractions $n_x/n \in \{(-n+1)/n, ..., -2/n, -1/n, 1/n, 2/n, ..., (n-1)/n\}$. Therefore, there is a correspondence between the construction of integers and rational fractions from the natural numbers on the one hand, and the respective proper distances and proper velocities on the other hand.

It is obvious that the meaning of a free particle's proper velocity thus introduced differs essentially from the conventional relative velocity referred to an external reference frame. In a particle's proper spacetime, the proper velocity and proper distance are defined conjointly as related to the same interval of proper time. The proper velocity has nothing to do with the speed of covering a distance—it may rather be understood as the speed of 'producing' (appearing, forming) a respective proper distance; more discussion on the meaning of proper velocity is postponed to the following sections.

As one may notice, for fixed t_n the connection between a particle's proper distance and its proper velocity $v_{xn} = t_n^{-1} x_n$ has the form of Hubble's law, with t_n^{-1} bearing the meaning of the 'present Hubble constant'. This analogy does not seem accidental for a completely isolated entity that falls under the definition of an isolated universe possessing a temporal beginning at S_n. The proper time t_n (2) would then acquire the meaning of the 'present age' of the particle's 'universe'.

3.4 Geometrical properties

As far as the geometrical properties of a free particle's proper space are concerned, the following aspects will be addressed: (i) connection of the interval of a particle's proper space to the segment of a straight line, and (ii) indications that proper space has to be three-dimensional. The segments of a straight line L are subject to the fundamental axiom (of the relationship between algebra and geometry), which predicates a one-to-one correspondence between the points of a segment and the real numbers $\mathbb{R}$. In order to assure the continuity of the straight line, the latter contains infinitely many points that correspond to no rational number:

"the straight line L is infinitely richer in point-individuals than the domain $\mathbb{Q}$ of rational numbers in number-individuals" [8], and thus a straight line possesses a correspondence to numbers if and only if the irrational numbers $\mathbb{I}$ are included. But irrational numbers are not expressed by any ratio of integer numbers, the relation of which to proper time, proper distance, and proper velocity has been outlined in the preceding discussion. The question arises if there could be anything that points beyond $\mathbb{Q}$ in the search for the correspondence of the interval of the free particle's proper space to numbers.

Suppose that the interval of proper space x_n (4) corresponds to a segment of the straight line L. Then, it has to be defined by the segment's ends $\mathrm{P_I}$ and $\mathrm{P_I}'$; we will denote x_n as $\mathrm{P_I P_I}'$. Since $\mathrm{P_I}$ and $\mathrm{P_I}'$ represent the same particle, they have to be in principle indistinguishable, therefore the x_n direction (sign) cannot change under the inversion of the segment's ends $\mathrm{P_I} \leftrightarrow \mathrm{P_I}'$, which, however, does not hold if a directed x_n behaves as a polar vector. This means that x_n, and therefore also the directed distance unit, has to behave as an axial vector that changes its direction at inversion. Such behaviour assumes a property related to rotation and therefore demands going beyond the one-dimensional space of a distance, *i.e.*, demands involving two more spatial dimensions related to opposite rotations. This requirement implies that the particle's proper space has to be *three-dimensional.*

The assumed rotation-related property suggests passing in (1) to a cyclic de Broglie frequency ω_0 and in (3) to a 'reduced' Compton wavelength $\lambda_C = \hbar/(m_0 c)$ as a distance unit, which is supposed to behave as an axial vector. Though these considerations will be kept in mind, in what follows the 'non-cyclic' units based on (1) and (3) will be used when it does not influence the analysis.

Let us return to the correspondence of an interval of particle's proper space to a straight line, which assumes that irrational numbers have to be related to the entity's proper spacetime, or its free motion. But if irrational numbers do not correspond to proper velocity, since the latter by definition corresponds to rational numbers, in what way might irrational numbers be connected to free motion? What remains is the aforementioned necessity of axial vector symmetry, which assumes a rotation-related property. Though the latter lacks any constructive definition, its geometrical description should be continuous, in the sense that there should be no 'empty cuts' in its correspondence to the points of a segment of a straight line. In other words, there may exist a correspondence of the state of free motion, including the rotational property, to all real (rational $\mathbb{Q}$ and irrational $\mathbb{I}$) numbers, that is, a correspondence to all points of an interval of particle's proper space to a segment of a straight line.

4 Proper spacetime

4.1 Basic assumptions

Introducing the interval of proper space $x_n = \mathrm{P_I P_I'}$ means that the instant of a particle's present existence (Sec. 2.1) has to be related simultaneously both to $\mathrm{P_I}$ and $\mathrm{P_I'}$. This can be regarded as 'splitting' the present $\mathrm{P_I}$ into a 'dual-present', $\mathrm{P_I} \to \mathrm{P_I P_I'}$, which means that along with de Broglie time $t_n = \mathrm{P_I S}_n$ (2), another interval of proper time $t_n' = \mathrm{P_I' S}_n$ has to appear. Thus, the present is connected to the start S_n by two intervals of proper time $\mathrm{P_I S}_n$ and $\mathrm{P_I' S}_n$. As both t_n and

t'_n relate to the same present and the same start S_n, they have to be determined by the same sequence of numbers 1, 2, ..., n. As a consequence, the only possible expression for t'_n is $t'_n = nT'_0$, in which the time unit T'_0 might differ from the de Broglie time unit T_0.

From a geometrical viewpoint, introducing proper distance means that a two-point present-past connection P_IS_n is now replaced by a three-point present-past connection $P_IP'_IS_n$. (It is noteworthy that three is the only number of points that coincides with the number of all possible straight line segments connecting the points). It may be assumed that the free particle's proper spacetime is formed as a resolution of the particle's 'ageing' or de Broglie time t_n into orthogonal spatial $x_n = P_IP'_I$ and temporal $t'_n = P'_IS_n$ coordinates. This presupposes the requirement that along with x_n (see Sec. 3.4), also t_n and t'_n can be considered to correspond to the respective straight-line segments. Their expression via numbers would be fully consistent if all three segments are dimensionless. In what follows, after addressing the mentioned requirements, the connection between the squared spatial and temporal intervals of particle's proper spacetime is proposed; their expression in energy form and the wave-like properties will be discussed in Sec. 5.

4.2 Dimensionless unit of a particle's spacetime

The possibility of defining a common dimensionless unit of proper time and proper space appears to be essential since any dimension externally imposed on a free entity does not seem to be strictly consistent with its completely isolated existence; such a unit would mean a direct correspondence of the particle's proper spacetime to numbers. The 'everything from a particle's rest mass' strategy exploited so far can be consistently applied by introducing a dimensionless natural spacetime unit, which is entirely defined by the particle's rest mass m_0 and the constants of nature. Obviously, de Broglie's periodic phenomenon (1) does not provide such a possibility. A fundamental phenomenon other than de Broglie's that connects m_0 to a quantity that possesses a time or space dimension, which is defined by constants of nature as fundamental as h and c, is gravitation, providing the expression for gravitational radius R_0 (see, for instance, [1])

$$R_0 = \frac{2Gm_0}{c^2}, \tag{7}$$

which includes the gravitational constant G. Here, R_0 is directly proportional to the particle's rest mass m_0, as distinct from the inverse proportionality, $\lambda_C \propto m_0^{-1}$ in Compton's distance unit (3) provided by de Broglie's periodic phenomenon.

The ratio of (3) and (7) yields a common dimensionless natural unit of the m_0-particle's proper time and proper space

$$\frac{\lambda_C}{R_0} = \frac{T_0}{R_0c^{-1}} = \frac{hc}{2G}m_0^{-2}. \tag{8}$$

It might be preferable to introduce the dimensionless standard unit via a recognizable Planck's mass $m_P = (\hbar c/G)^{1/2}$ as

$$\delta_0 = \left(\frac{m_P}{m_0}\right)^2 = \frac{\hbar c}{G}m_0^{-2}. \tag{9}$$

It is meaningful that δ_0 includes a product or ratio of all three fundamental constants of nature, namely of the general theory of relativity (GTR), QM, and the STR, as well as the scaling factor m_0^2 (the relation of the latter to gravitational and inertial mass is discussed in Sec. 6).

Since δ_0 is determined by the ratio of the particle's proper distance unit over its singularity size, δ_0 should be large enough to ensure a non-distorted unit to substantiate the requirement of correspondence to a straight line. Therefore, the necessary condition for the mass m_0 of a particle (a fundamental one, see Sec. 2.2) should be

$$\frac{m_P}{m_0} \gg 1. \tag{10}$$

And indeed, δ_0 is of the order of 10^{43} for an electron. The fact that for m_P comparable to m_0 one would have difficulties with correspondence to a straight line (see Sec. 3.4), and thus with the straightforward connection between proper time and proper space, may provide an argument for explaining a somewhat puzzling question (see, for instance, a remark in [10]): why is Planck's mass so huge when compared to the masses of fundamental particles, whereas Planck's units of time and length are so small when compared to the respective de Broglie time period and Compton wavelength?

4.3 Basic equation

Applying the dimensionless unit (9) to the m_0-particle's t_n and x_n defined by (2) and (4) yields their respective dimensionless analogues ${}^{DL}t_n = n\delta_0$ and ${}^{DL}x_n = n_x\delta_0$; obviously, $|{}^{DL}x_n| <{}^{DL} t_n$ since $|n_x| < n$. The condition $t'_n = nT'_0$ (Sec. 4.1) implies that ${}^{DL}t'_n = n\delta'_0$. The equivalence of positive and negative proper distances implies that the sought after connection has to include the squared value of spatial intervals, and thus also of the time intervals. Since $({}^{DL}t_n)^2$ is an ultimate invariant for all proper distances at a given n, namely, at a given particle 'age', it has to be invariant for each combination of $({}^{DL}x_n)^2$ and $({}^{DL}t'_n)^2$. Geometrically, attributing ${}^{DL}x_n$ to positive and negative integers n_x (4) constructed from the ${}^{DL}t_n$-defining set of natural numbers may be represented as attributing the segment ${}^{DL}x_n = \mathrm{P_I P'_I}$ to the respective positive and negative projections of ${}^{DL}t_n = \mathrm{P_I S}_n$, which are formed by ${}^{DL}t'_n = \mathrm{P'_I S}_n$ orthogonal to $\mathrm{P_I P'_I}$. This makes it possible to consider ${}^{DL}x_n$ and ${}^{DL}t'_n$ as the respective orthogonal spatial and temporal *proper coordinates*, which are produced by the respective resolution of ${}^{DL}t_n$, yielding the quadratic connection

$$({}^{DL}t_n)^2 = ({}^{DL}x_n)^2 + ({}^{DL}t'_n)^2. \tag{11}$$

The entire set of $({}^{DL}x_n, {}^{DL}t'_n)$-coordinates may be considered as introducing the *proper spacetime* of a free, t_n-'aged' particle (see the remark in Sec. 3.4 regarding the necessity of two more spatial dimensions). It is notable that ${}^{DL}t'_n$ and ${}^{DL}x_n$ appear from ${}^{DL}t_n$ as its only possible alternatives generated either by a *different* (diminished) number $|n_x|$ of the *same* unit δ_0, or by the *same* number n of a *different* (diminished) unit δ'_0. Expressing δ'_0 from (11) yields $\delta'_0 = \delta_0(1 - n_x^2/n^2)^{1/2}$, which corresponds to $m'_0 = m_0(1 - n_x^2/n^2)^{-1/2}$.

The basic equation (11) allows one to consider the de Broglie time as a vector $\mathbf{t_n}({}^{DL}x_n, {}^{DL}t'_n)$, $|\mathbf{t_n}| ={}^{DL} t_n$ in $({}^{DL}x_n, {}^{DL}t'_n)$-space. To explicitly follow the connection to numbers, let us expand $\mathbf{t_n}$ over the orthonormalized unit vectors $\boldsymbol{\chi} = \mathbf{x_n}/(n_x\delta_0)$, $|\mathbf{x_n}| =^{DL} x_n$, and $\boldsymbol{\tau'} = \mathbf{t'_n}/(n\delta'_0)$, $|\mathbf{t_n}'| =^{DL} t'_n$, $|\boldsymbol{\chi}| = |\boldsymbol{\tau'}| = 1$, which form the $\{\boldsymbol{\chi}, \boldsymbol{\tau'}\}$-basis. We get:

$$\mathbf{t_n} = \ ^{DL}x_n\boldsymbol{\chi} +^{DL} t'_n\boldsymbol{\tau'} = (\mathbf{t_n}\boldsymbol{\chi})\boldsymbol{\chi} + (\mathbf{t_n}\boldsymbol{\tau'})\boldsymbol{\tau'}. \tag{12}$$

Expressing the respective projections of $\mathbf{t_n}$ onto $\boldsymbol{\chi}$ and $\boldsymbol{\tau'}$ via $^{DL}t_n$ we get $^{DL}x_n = C_x^{DL}t_n$ and $^{DL}t'_n = C_{t'}^{DL}t_n$, where the distance-connected projection coefficient $C_x = (n_x/n) = v_{xn}/c$ is just the proper velocity normalized by c (5), while the t'_n-time connected projection coefficient is $C_{t'} = (1-n_x^2/n^2)^{1/2} = (1-v_{xn}^2/c^2)^{1/2}$. As can be seen, provided $C_x^2+C_{t'}^2 = 1$, C_x^2 and $C_{t'}^2$ acquire the meaning of the partition coefficients for resolving the de Broglie time over the respective orthogonal space-like and time-like coordinates of a particle's proper spacetime. Passing to the n-independent unit vector $\boldsymbol{\tau_I} = \mathbf{t_n}/nT_0$ with $|\boldsymbol{\tau_I}| = 1$ and expanding $\boldsymbol{\tau_I}$ over $\boldsymbol{\chi}$ and $\boldsymbol{\tau'}$ yields

$$\begin{aligned} \boldsymbol{\tau_I} &= (\boldsymbol{\tau_I}\boldsymbol{\chi})\boldsymbol{\chi} + (\boldsymbol{\tau_I}\boldsymbol{\tau'})\boldsymbol{\tau'} = C_x\boldsymbol{\chi} + C_{t'}\boldsymbol{\tau'}, \\ C_x^2 &+ C_{t'}^2 = |\boldsymbol{\tau_I}|^2 = 1, \end{aligned} \tag{13}$$

which means that $C_x = \boldsymbol{\tau_I}\boldsymbol{\chi}$ and $C_{t'} = \boldsymbol{\tau_I}\boldsymbol{\tau'}$ determine the $\boldsymbol{\tau_I}$-orientation $\boldsymbol{\tau_I}(\theta)$ in the $\{\boldsymbol{\chi}, \boldsymbol{\tau'_I}\}$-basis as $C_{x'} = \cos\theta$ and $C_{t'} = \sin\theta$.

5 The energy-momentum relation and the wave-like property of a particle's spacetime

5.1 Proper energy and momentum

Let us transform equation (11) to an invariant quantity—the energy. Coming back to de Broglie's unit (1) and accounting for (2) and (4), Eq. (11) can be written as

$$(nT_0)^2 = (n_xT_0)^2 + (nT'_0)^2. \tag{14}$$

In order to cancel the flow of time and to pass to energy-related quantities, let us multiply the terms of (14) by $h^2/(T_0T'_0n)^2$. We get:

$$(h\nu'_0)^2 = \frac{n_x^2}{n^2}(h\nu'_0)^2 + (h\nu_0)^2 = \frac{v_{xn}^2}{c^2}(h\nu'_0)^2 + (m_0c^2)^2, \tag{15}$$

where $\nu'_0 = 1/T'_0$. As can be seen, we have arrived at the ultimately invariant quantity of the energy-momentum relation in STR—the particle's squared rest energy $(h\nu_0)^2 = (m_0c^2)^2$, which has been transformed from the particle's proper temporal coordinate term $(t'_n)^2 = (nT'_0)^2$ of (14) or (11). This relates the m_0-particle's rest energy m_0c^2 to the *temporal coordinate* t'_n of its proper spacetime. Let us focus on the first right-hand-side term, which is transformed from the *spatial coordinate* $x_n^2 = (n_xT_0)^2$ in (14) or (11). As is seen from (14) and (15) it takes the form of the squared relativistic linear momentum term of energy-momentum relation in STR:

$$\frac{n_x^2}{n^2}\left(h\nu'_0\right)^2 = \left(\frac{n_x^2}{n^2}\right)\left(m_0^2c^4\right)\left(1-\frac{n_x^2}{n^2}\right)^{-1} = c^2v_{xn}^2m_0^2\left(1-\frac{v_{xn}^2}{c^2}\right)^{-1} \equiv p_{xn}^2c^2, \tag{16}$$

where a conventional notation $p_{xn}^2 = m_0^2 v_{xn}^2/(1 - v_{xn}^2/c^2)$ is used for the respective proper quantity, which is expressed by numbers. Following analogy of (15) to the STR energy-momentum formalism, $p_{xn}^2 c^2$ represents the difference between squared values of particle's full proper energy and its rest energy. As determined by the proper velocity referred to as the speed of 'producing' proper space (Sec. 3.3), $p_{xn}^2 c^2$ may be related to the *spatial component* of proper energy necessary for 'producing' a particular proper distance x_n, which is added to the *temporal component* $(m_0 c^2)^2$.

Let us determine the total $p_n^2 c^2$ value, which is summed over all proper distances that is, over all n_x for a fixed n defining an elapsed de Broglie time t_n. According to the construction of the set of integers from pairs of natural numbers belonging to the t_n-defining sequence 1, 2, ..., n (Sec. 3.2), it is necessary to perform a weighted summation of (16) accounting for the weight factor. For $n_x > 0$ the weight factor has to be taken as the number $n - n_x$ of all possible pairs $(a, b) \in [1, n]$ that yield a particular n_x for a fixed n, which is normalized by the sum of such possibilities over all n_x from 1 to $n - 1$, which is $n(n-1)/2$. As a result, after accounting for $n_x < 0$, we get

$$p_n^2 c^2 = \frac{4 m_0^2 c^4}{n(n-1)} \sum_{n_x=1}^{n-1} \frac{n_x^2}{n_x + n}. \tag{17}$$

As can be seen, the major contribution to $p_n^2 c^2$ is made by the terms with large n_x-values comparable to n, which correspond to large proper distances comparable to ct_n, therefore the proper momentum p_n is 'relativistic' in terms of the STR.

It is of decisive importance to check whether the particle's $p_n^2 c^2$ converges to a finite value for growing n, since it would be difficult to imagine a divergence of proper energy with growing n, which means with evolving de Broglie time t_n, for an isolated particle. And indeed, by finding the asymptotic limit of (17) as $n \to \infty$, one gets the convergence of $p_n^2 c^2$ to a finite value:

$$p^2 c^2 = \lim_{n \to \infty} p_n^2 c^2 = 2\,(2 \ln 2 - 1)\, m_0^2 c^4 \approx 0.7726 m_0^2 c^4. \tag{18}$$

which is determined exclusively by the rest mass, as should be expected for a free m_0-particle; note that it is distributed (delocalized) within large proper distances of the order of ct_n. Accounting for Eq. (15) and its connection to (11) and (14), the squared full energy of particle's proper spacetime acquires a value determined exclusively by its rest mass, being the sum of spatial-coordinate-related component (linear momentum term) (18) and temporal-coordinate-related component (rest energy term) $(m_0 c^2)^2$. Then, the full energy of the m_0-particle's proper spacetime expressed in terms of the momentum, or the m_0-particle's Hamiltonian [1] will possess a numerical value $\mathrm{H} = m_0 c^2 \sqrt{2(2\ln 2 - 1) + 1}$. The assumption that two other spatial dimensions (see Sec. 3.4) yield the same contribution into full proper energy as given by (18) would increase the respective term of the Hamiltonian by a factor of 3.

5.2 Wave-like properties

The periodic nature of de Broglie time t_n (1) presupposes that the connection between proper space and proper time should exhibit wave-like properties. The very nature of a wave process requires this connection to be linear. For this purpose, let us express $nT'_0 = t'_n$ in (14) as

$$t'_n = t_n \left(1 - \frac{n_x^2}{n^2}\right)^{1/2} = t_n \left(1 - \frac{v_{xn}^2}{c^2}\right)^{1/2}, \tag{19}$$

then multiply and divide the obtained expression by $(1 - v_{xn}^2/c^2)^{1/2}$. After explicitly including $v_{xn} = x_n/t_n$ in the numerator, we arrive at

$$t'_n = \frac{t_n - x_n v_{xn}/c^2}{\sqrt{1 - v_{xn}^2/c^2}}. \tag{20}$$

As can be seen, for a particular $v_{xn}/c = n_x/n$, time t_n and distance x_n enter the right-hand-side of (20) as a linear combination, thus providing a general condition for the argument of a wave-like process (see also [6]). Namely, the proper spatial coordinate x_n scaled by c^2/v_{xn} is linearly connected to the de Broglie time t_n in such a 'phase-matching' way that their difference, scaled by a factor $(1-v_n^2/c^2)^{1/2}$, remains equal to the temporal coordinate t'_n of particle's proper spacetime.

The phase of such a wave-like process is obtained after multiplying (20) by the cyclic de Broglie frequency $2\pi/T_0 = \omega_0$, which yields

$$\omega_0 t'_n = \frac{\omega_0(t_n - x_n v_{xn}/c^2)}{\sqrt{1 - v_{xn}^2/c^2}} = \omega'_0 t_n - k_{xn} x_n, \tag{21}$$

where $\omega_0/(1 - v_x^2/c^2)^{1/2} = \omega'_0 = 2\pi/T'_0$, while the factor

$$V_{xn} = c^2/v_{xn} = (n/n_x)c \tag{22}$$

can be identified as a *proper phase velocity*. Then, $\omega'_0/V_{xn} = k_{xn}$ acquires the meaning of the *proper wave number*, the respective proper de Broglie wavelength being

$$\Lambda_{xn} = 2\pi/k_{xn} = [h/(m_0 v_{xn})] \left(1 - v_{xn}^2/c^2\right)^{1/2}. \tag{23}$$

The comparison of (23) and (16) immediately reveals the connection of k_{xn} to the proper linear momentum $p_{xn} = \hbar k_{xn}$; it is meaningful that this appears just as a result of transforming (11) to the expressions containing the respective recognizable quantities. Though (21) and (23) have the form of a Lorentz transformation in the STR and of the wavelength of the de Broglie matter wave, respectively, they possess a different meaning since the spatial and temporal coordinates are defined as proper, in the particle's proper spacetime, without any relation to an external reference frame.

Let us follow the spreading of maximal value of proper distance $|x_n^{max}| = (n-1)cT_0$ with ongoing de Broglie time $t_n \to t_{n+1}$, or $\mathrm{P_I S}_n \to \mathrm{P_I S}_{n+1}$ (Sec. 2.3) in the wave-like process (20)–(23). For a three-dimensional particle's proper space (Sec. 3.4) the wave-like process may be considered as expanding of the radius of 'wave front' $r_{n-1} = |x_n^{max}|$ as $r_{n-1} \to r_n$, or $(n-1)cT_0 \to n(cT_0)$. As discussed in Sec. 2.3, the ongoing of t_n corresponds to a consequent periodic reproduction, at the present instant $\mathrm{P_I}$, of a T_0-unit of proper time, consequently, of a cT_0-unit of proper distance. In relation to integers, this corresponds to the periodic reproduction, at present, of $n_x = 1$. It can be seen that extending of r_n as $(n -$

$1)cT_0 \to ncT_0$ corresponding to $S_{n-1} \to S_n$ takes place at present; indeed, $n_x = 1$ corresponds to the proper velocity $v_{xn} = c/n$ (6), consequently, to the proper phase velocity $V_{xn} = nc$ (22), therefore the time interval taken for the wave front to expand equals $r_n/V_{xn} = T_0$, which means that the expansion occurs at the present. Along with that, the number $n - n_x$ of all possible pairs $(a, b) \in [1, n]$ yielding a particular positive value n_x is increased from $n - n_x$ to $(n + 1) - n_x$, or by 1. And since each such increasing occurs at present, the entire proper space belongs to the present, in agreement with the assumption in Sec. 3.1.

5.3 Duality and indistinguishability

Let us address the feasibility of combining the wave-like property of an m_0-entity's proper spacetime with its particle-related property, which means treating the wave-particle duality, within the number-based approach. The main issue is to preserve the feature of a 'point-like' particle while introducing the proper distance that is connected with proper time in a wave-like way. It is rather clear that possessing only a temporal proper coordinate t'_n while x_n remains zero is consistent with regarding the m_0-entity as 'purely' a particle, which agrees with the correspondence of the $({}^{DL}t'_n)^2$ term in (11) to the rest energy term $(h\nu_0)^2 = m_0^2c^4$ in (15). Note that this fits the orthogonality condition of proper temporal and spatial coordinates considered in Sec. 4. In a simplified way, one may say that according to Eq. (20) the m_0-entity is identified with a particle by t'_n in the left-hand-side and with a wave (wave's phase) by the right-hand-side, the equality sign standing for the equivalence of wave- and particle-like properties.

Conformity of temporal coordinate t'_n (14) with a 'point-like' particle's property means that since t'_n is defined by a sequence $1, 2, ..., n$, which is counting the T'_0-units, a unit of proper distance has to remain zero in each count. But a zero distance unit contradicts its definition (3) as a Compton wavelength λ_C, or, more precisely, as $\lambda_C = \hbar/(m_0c)$, see Sec. 3.4. This contradiction may be resolved by supposing that λ_C consists of *two one-half units* $\lambda_C = \pm[(1/2)\hbar/(m_0c) \pm (1/2)\hbar/m_0c]$. Then, the particle-related property associated with t'_n would have been provided by a summed to zero combination $\pm[(1/2)\hbar/(m_0c) - (1/2)\hbar/m_0c]$ whereas the wave-related property associated with x_n is provided by a summed to a Compton unit λ_C combination $\pm[(1/2)\hbar/(m_0c) + (1/2)\hbar/m_0c]$. Appearance of a factor $\pm(1/2)\hbar$ as a necessity to fit with the particle-related property seems meaningful. Keeping in mind a symmetry-based necessity for a distance unit to possess an axial vector property (Sec. 3.4), suggests the relevance of a $\pm(1/2)\hbar$ factor to a projection of spin-one-half angular momentum. This fully agrees with the latter being an inherent property of a fundamental (see a related remark in Sec. 2.2) nonzero-rest-mass particle. The fact that introducing the interval of proper space $x_n = \mathrm{P_I P_I}'$ presupposes the m_0-particle's 'dual' identification with *in principle* indistinguishable $\mathrm{P_I}$ and $\mathrm{P_I}'$ (Sec. 3.4) inclines one to identify with $\mathrm{P_I}$ and $\mathrm{P_I}'$ the respective opposite signs of spin-one-half projections.

Let us follow the relation of the particle-related property to numbers. Introducing an m_0-entity's de Broglie proper time t_n relies on the entity's *singular*-present $\mathrm{P_I}$ (Sec. 2), which it is reasonable to associate with a singular particle, while, when introducing proper distance one necessarily introduces '*dual*-present' $\mathrm{P_I}$ and $\mathrm{P_I}'$ (Sec. 4.1). The premise 'singular-present–singular-particle' implies the premise 'dual-present–dual-particle', which means that, at the present instant, the particle should be simultaneously identified with $\mathrm{P_I}$ and $\mathrm{P_I}'$. Indeed, if (i) a particle is associated with t'_n and (ii) identifying t'_n with *in principle* (Sec. 3.4) indistinguishable $\mathrm{P_I}$ or $\mathrm{P_I}'$ makes no sense, then t'_n, and together with it, the particle, has to be identified with the 'dual-present' $\mathrm{P_I}$ and $\mathrm{P_I}'$. This can be associated with particle's delocalization because the particle's 'dual-present' $\mathrm{P_I}$ and $\mathrm{P_I}'$ is ascribed to the entire set of proper distances $x_n = \mathrm{P_I P_I}'$. The question arises how the particle's 'dual-present' property agrees with the correspondence of its present existence to 1. Following the discussion in Sec. 2.3, a possible answer lies in tracing how exactly the ongoing proper time is ensured by continuous adding, at the present, of a T_0-unit to an already existing (and corresponding to the present) one T_0-unit, 'shifting' the latter to the past. It is essential that such adding presumes a simultaneous (within T_0) existence, at the present, of *both* the added T_0-unit and the previous T_0-unit, thus explaining the very possibility of a 'dual-present' $\mathrm{P_I}$ and $\mathrm{P_I}'$. One may notice that the same relates to constructing the, isomorphic to the ongoing time flow, sequence of natural numbers by a continuous, uninterrupted addition of 1 in $\mathrm{SUC}(i) = i + 1; i \in [1, n-1]$. Here, a unique symmetry occurs for $i = 1$ when passing from '1' to '1'+'1', which presumes the simultaneous existence of (consequently, relation of the present instant to) '1' *and* '1'+'1'. Indeed, it is not possible to distinguish whether the present corresponds to '1' or to '1'+'1': if the present only corresponds to '1', there is no ongoing sequence of natural numbers, therefore no time flow, whereas if it only corresponds to '1'+'1', it does not distinguish an individual entity. In other words, because of ongoing time, the premise "'1' corresponds to a *singular*-present" equivalently means that "'1' *and* '1'+'1' correspond to a *dual*-present". This justifies relating the particle's existence *both* to the single-present $\mathrm{P_I}$ and to the dual-present $\mathrm{P_I}$ and $\mathrm{P_I}'$, which, in fact, justifies the very idea that a free m_0-particle's proper space relates to the present.

Relation to numbers makes it possible to numerically assess the partition of an m_0-entity's particle-like and wave-like properties just by the respective values of the coefficients introduced by (13): $C^2_{t'} = 1 - (v_{xn}/c)^2 = 1 - (n_x/n)^2$ and $C^2_x = (v_{xn}/c)^2 = (n_x/n)^2$. In particular, the condition to be fully particle-like, that is to possess only a temporal proper coordinate t'_n would mean that $n_x = 0$, therefore $C^2_{t'} = 1$ and $t'_n = t_n$. Since $n_x = 0$ does not appear in the construction of integers from pairs of natural numbers $(a, b) \in \{1, 2, ..., n; a \neq b\}$, see Sec. 3.2, the smallest n_x-value is 1, which brings the Compton wavelength (3) as the minimal 'size'. It may be of interest to recall the STR notion of 'event' [1], which would break the condition of a fully isolated entity 'at the moment it occurs'. Following the discussion in Sec. 2.3, this means that, at the entity's present moment, the ongoing t_n-defining sequence $1, 2, ..., n$ 'collapses' into $n = 1, \mathrm{S}_n \to \mathrm{S}_1$. This can be considered as a 'collapse' of dual-present $\mathrm{P_I}$ and $\mathrm{P_I}'$ into a singular-present $\mathrm{P_I} \leftrightarrow \mathrm{S}_1$, which means the m_0-entity's 'collapse' into a 'pure' particle with the 'size' of a distance unit. If the particle can be considered free after the event, S_1 would identify a new temporal beginning and a restart the numbering of T_0-units. As far as C^2_x is concerned, associating the m_0-entity with a 'pure' wave without any particle-related property would mean $t'_n = 0$, therefore $n_x = n$, which, as already mentioned, is not possible to construct from pairs $(a, b) \in \{1, 2, ..., n; a \neq b\}$; thus, it is not possible to ascribe $v_{xn} = c$ to a nonzero-rest-mass particle.

It is of evident interest to consider the possibility of attributing the 'dual-present' $\mathrm{P_I}$ and $\mathrm{P_I}'$ to *two* particles. Let us begin by considering two identical (fundamental) m_0-particles, which are assumed to remain free after their common start (creation, temporal beginning), say, after a decay process. Since the present instant $\mathrm{P_I}$ is common for both particles at the start, $\mathrm{S}_1 = \mathrm{P_I}$, and the correspondence of $\mathrm{P_I}$ to 1 should remain common for a strictly isolated two-particle system, the sequence $1, 2, ..., n$ numbering common T_0-units now relates to both particles providing their common interval of proper time $t_n = nT_0$ from the common present $\mathrm{P_I}$ to the common start S_n. Let us suppose that, after elapsed t_n, the particles are separated by a distance $x_n = \mathrm{P_I P_I}'$. The question is: under what conditions can x_n be considered proper for both particles? To comply with the reasoning in Sec. 3.4, the ultimate condition should be the possibility to identify the particles with *in principle* indistinguishable dual-present $\mathrm{P_I}$ and $\mathrm{P_I}'$. And this becomes possible if and only if the particles themselves are in principle indistinguishable, which is exactly the case for a fundamental particle. Then, the set of (x_n, t'_n)-coordinates that define proper spacetime can be equivalently related to any of the particles, thus defining their common proper spacetime. This means that the distance $x_n = \mathrm{P_I P_I}'$, which corresponds to an integer n_x (4), as well as the respective proper velocity v_{xn} (6), can be considered relative, while remaining proper. The same relates to proper momentum p_{xn} (16) supplying an argument for the 'reality' of the proper energy (17) and (18). Identifying the proper reference frame with any of the particles would mean identifying the (x_n, t'_n)-coordinates with the 'other' particle in this reference frame. Since this means relating the temporal coordinate t'_n (19) to a particle that possesses (now also relative) velocity v_{xn}, one arrives at the 'time dilation' phenomenon in the STR. As is understandable, there is no need for any clock synchronization hypothesis since two indistinguishably-identical m_0-particles represent two ideally-equal clocks (see the remark in [6]), which are inevitably synchronized at the common start, thus providing a common t_n; the full equivalence of relating the system of reference to each of particles is obvious. Regarding the relation to the foundations of QM, one may notice that while the temporal coordinate t'_n (19) is diminished with respect to the de Broglie time t_n, the corresponding frequency $\nu'_0 = 1/T'_0$ is increased with respect to ν_0 in (1) as $\nu'_0 = \nu_0(1 - n_x^2/n^2)^{-1/2}$, see (14). As a consequence, increased is the respective QM momentum $h\nu'_0/c$. The fact that correspondence of the present instant to 1 now relates to both particles complies with possessing the one-entity-property, which points to the particles' entanglement. What is more, relating the opposite signs of one-half Compton units to the respective $\mathrm{P_I}$ and $\mathrm{P_I}'$ now means relating the opposite signs of spin-one-half projections to the respective particles. Then, if an 'event' (the 'collapse' of a *common* 1, 2, ..., n sequence into 1) occurs, at the present, to one of the particles breaking the condition of being free, it (the 'event') in fact occurs at the dual-present $\mathrm{P_I}$ and $\mathrm{P_I}'$ of *two* indistinguishable particles, and so it is in principle impossible to distinguish to which one the 'event' occurs. Since, at a common present instant, the particle-related time t'_n corresponding to the summed to zero combination $\pm[(1/2)\hbar/(m_0 c) - (1/2)\hbar/(m_0 c)]$ now relates to both particles, fixing, say, a $+(1/2)\hbar$ spin projection to one of the particles means simultaneous appearance of a $-(1/2)\hbar$ spin projection of the other particle at whatever proper distance x_n.

6 Concluding remarks

As has been proposed, proper spacetime may be ascribed to a free nonzero-rest-mass particle based on a connection to numbers of de Broglie time, which is defined by a numbered sequence of de Broglie time units T_0 (1). This, in a sense, conforms to the STR concept of proper time as read by the entity's proper clock based on a periodic process; as stated in [7], it does not fall into circular reasoning "...if one puts the concept of periodic occurrence ahead of the concept of time". Yet introducing a free particle's proper spacetime implies the necessity of introducing also its spatial coordinate as proper. This seems incompatible with the STR, according to which only the temporal coordinate may be considered as proper "...in a system of coordinates linked to the moving clocks" [1], while any spatial coordinate should be zero, in this sense identifying a point-like particle (one may notice however that a system of coordinates that involves only time may seem not to be fully consistent with the generality of the spacetime concept in the STR).

The proposed solution requires a careful analysis of the meaning of an m_0-particle's proper time. The meaning of the interval of de Broglie time t_n (2), which is isomorphic to a sequence of numbers $1, 2, ..., n$ differs essentially from the temporal coordinate in STR. It presumes a temporal beginning or 'start', which at $n = 1$ coincides with the 'present'. The proper time flows (lasts, passes) with increasing n, which is ascribed to the 'start', while the present remains fixed to 1 thus being ultimately attributed to the entity-individual. This conforms to the intuitive idea that an entity's existence means its *present* existence while time flows as 'ageing' by a continuous shifting of temporal beginning to the 'past'. As seen, the individual entity's de Broglie time conforms to the intuitive meaning of existential time, which is independent of any external factor or influence. Indeed, the de Broglie time unit T_0 is fully defined by the m_0-particle's rest mass; what is more, it can be reduced to the particle's specific number (9).

The premise of free particle's present existence not just in proper time but in its proper spacetime presumes that proper space relates to the particle's present as the only selected moment of time besides the start. This suggests that the proper space has to be defined exclusively based on the elapsed, at the present, interval of de Broglie time t_n, and so it has to be based on the t_n-defining sequence $1, 2, ..., n$. Since a distinctive property of a spatial interval x_n is to be bidirectional, its definition is reduced to a construction of positive and negative integers $n_x \in \{-n, ..., -2, -1, 1, 2, ..., n\}$. Clearly, as distinct from the point-like particle, the respective segment x_n has to be connected to the start not only by the de Broglie time interval t_n, but also by some other time interval t'_n. As defined by the same present and the same start, t'_n is defined by the same number n, numbering however the time period $T'_0 < T_0$, which is n_x-dependent and yields $t'_n < t_n$ (19). This means that defining proper space necessarily involves defining another kind of proper time, which is specific to a (squared) particular spatial interval x_n(4). Such a definition is unanimous supposing the orthogonality of x_n and t'_n; one may say that the set of x_n and t'_n is 'produced' from the de Broglie time t_n as its orthogonal projections. A form (19) of t'_n-dependence on $n_x^2/n^2 = v_{xn}^2/c^2$, where v_{xn} is referred to as proper velocity, suggests that t'_n acquires the property characteristic of the temporal coordinate in the STR. The 'absolute', existential meaning of the de Broglie time t_n as distinct from the proper temporal coordinate t'_n may add to the discussion on the inconsistency between the intuitive notion of time and the temporal component of four-dimensional relativistic time. One may recall the analysis made by Gödel [11], who found a cosmological solution of the GTR, though in the special case of rotating matter, which, as he stated in [12], proves that relativistic time contradicts the very existence of "an objective lapse of time (whose essence is that only the present really exists)" by demonstrating the 'absurd' possibility of 'time travel', say, to one's 'own past'.

Regarding the connection to the foundations of QM, a straightforward idea may be to connect an m_0-entity's de Broglie time t_n and coordinate time t'_n to the respective wave-like and particle-like properties. Indeed, the de Broglie time t_n enters the wave-like connection (20) between t_n and x_n, while t'_n is associated with possessing exclusively a particle's property. Recalling the premise [2] of a particle's interferences within its proper space may possibly suggest a way of experimental verification of the features characteristic to the present approach. One possibility might be based on some version of a double-slit experiment: if the particle 'sees' the distance between the 'slits' as proper, it should not exceed the maximal distance value related to the free entity's 'age' t_n, while the phase difference should include the same n for all x_n.

The number-connection of the proper spacetime of a free particle becomes complete provided existence of a common dimensionless natural unit of time and space composed from the particle's rest mast m_0 and the constants of nature. Say, t_n and x_n for an electron become defined by electron's specific number given by (9), which is multiplied by the respective integer. It is meaningful that this specific number is the product of two coefficients: the particle's rest mass, which determines the de Broglie frequency (1) and therefore, though designated as inertial mass, implies the basic quantum phenomenon, and the gravitational mass, which determines the gravitational radius. In general, these coefficients might have been totally different, and the very fact that they are reduced to a squared value m_0^2 of coinciding 'inertial-quantum' and gravitational masses demonstrates their unification into an 'inertial-quantum-gravitational' mass that determines or specifies the entity's proper spacetime. And such a unification seems justified by the very circumstance that the factor $\hbar c/G$ in (9), which scales m_0^2 to a dimensionless unit (to the particle's specific number), is composed of the three constants of nature, which are the respective fundamental constants of QM, the STR, and the GTR. It is notable that, if the existence of a dimensionless spacetime unit is considered as the starting point, the inclusion of m_0^2 offers a sign choice for m_0.

The most challenging issue is to comprehend the proper energy (18), which appears along with a proper spatial coordinate, in this sense being the 'cost of forming proper space'. According to the suggested estimate, this energy is comparable with the particle's rest energy m_0c^2 and, as follows from (17), is mainly determined by large proper distances, which are expanding with t_n. A rather inspiring question is: can this energy be associated with dark energy of the universe as a whole? One could mention a purely formal argument: since a fully isolated entity possesses the features of an isolated universe, the considerations regarding the particle's proper spacetime might be relevant. Applicability of present reasoning to a system of indistinguishable particles with a common start may justify merging of their proper spacetime into the proper spacetime of the universe as a whole (could it provide an argument for the hypothesis of a primordial particle as the end-constituent of matter?). One may notice that the proper-space-related particle energy is likely to fit such expected properties of the dark energy (see, for instance, [13]) as being recognized on the scale of large distances and being consistent with a homogeneous, observationally flat universe of very low density.

A direct involvement of numbers in defining a free m_0-particle's proper spacetime forms the core of the suggested approach. The assumption that formation of proper distances corresponds to the formation of positive and negative integers from natural numbers indicates that the de Broglie time interval defined by them is 'prior' to a spatial interval, just as natural numbers are prior to integers. Rational fractions may be related to the ratio of proper distance over proper time, thus making it possible to relate them to proper velocity, fractional with respect to c. What is more, there is a symmetry-based argument for the necessity of possessing a rotation-related property in order to define the proper distance, which points to the necessity for space to be three-dimensional and possibly bears indications for irrational numbers. The very fact that a definition of a particle's proper spacetime, which is fundamental for physics, is reduced to numbers, which form the basis of mathematics may contribute to the discussion of the 'unexplainable role' of mathematics in physics. A debatable epistemological issue concerns the 'reality' of numbers as mathematical objects (for an insight see, for instance, [14] and references therein): while the numbers are considered as not existing in spacetime, the usually agreed meaning of (physical) 'reality' would presume such existence. A possibility to comply with such a 'reality' would be to include the numbers into the definition of spacetime, and this is the core of the present analysis. Accordingly, the proposition of the 'reality' of numbers should not mean that they exist in spacetime but that they *define* (constructively determine) the spacetime; in such a way, the 'reality' of numbers is reduced to the 'reality' of spacetime.

Acknowledgements The author is indebted to Yakir Aharonov for encouraging comments during a discussion of the early ideas of the paper, to Janis Cirulis, Vyacheslavs Kashcheyevs, Florian Gahbauer, Mihails Belovs, Alexander Shostak, and Olga Nikolayeva for numerous helpful pieces of advice, as well as to Marcis Auzinsh for stimulating discussions on the fundamentals of quantum physics. Partial support from the COST Action MP1006 "Fundamental Problems in Quantum Physics" is gratefully acknowledged.

References

1. Landau, L.D. and Lifshitz, E.M.: The Classical Theory of Fields. Butterworth-Heinemann, Oxford (1996).
2. Dirac, P.A.M.: The Principles of Quantum Mechanics, Clarendon Press, 1958.
3. de Broglie, Louis: Recherches sur la théorie des quanta, Ann. Phys. **3**, 22–128 (1925). English translation is published at http://aflb.ensmp.fr/LDB-oeuvres/De_Broglie_Kracklauer.pdf
4. Penrose, R: Structure of Space-Time. Benjamin, New York (1968).
5. Lan, Shau-Yu, et al., Science 339, 554–557 (2013) DOI: 10.1126/science.1230767.
6. Ferber, R: A Missing Link: What is Behind de Broglie's "Periodic Phenomenon"? Foundations of Physics Letters **9**, 575–586 (1996).
7. Einstein, A.: Physics and Reality, J. Franklin Inst. **221**, 342–382 (1936).
8. Dedekind R.: Continuity and Irrational Numbers. In: Stephen Hawking, S. (ed) God Created the Integers, pp. 912–916. Running Press, Philadelphia (2005).
9. Feferman, S., The Number Systems. Foundations of Algebra and Analysis. Addison-Wesley, Reading, pp. 101–157 (1964).
10. Wilczek, F.: On Absolute Units, Physics Today, October 2005, 12–13.
11. Gödel, K.: An Example of a New Type of Cosmological Solutions of Einstein's Field Equations of Gravitation, Rev. Mod. Phys. 21, 447-450 (1949).

12. Gödel, K.: A Remark About the Relationship Between Relativity Theory and Idealistic Philosophy, 1949, in: P.A. Schilpp, ed. 1949, p. 557, Albert Einstein: philosopher-scientist. http://. 283403168925209589.weebly com/uploads/9/3/3/0/9330952/godel1949.pdf
13. Schneider, P.: Extragalactic Astronomy and Cosmology. An Introduction, Springer Verlag, Berlin-Heidelberg, pp. 455, 581 (2015).
14. Baker, A.: Mathematical Skepticism, Scientific Papers, University of Latvia. v739, Philosophy, 2008, pp. 7–21. ISBN 978-9984-45-059-9, https://www.lu.lv/materiali/apgads/raksti/739.pdf